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Abstract

In this paper, we provide a new solution in a cooperative game with a coalition structure. The collective value of a player is defined as the summation of equal division of pure surplus which his coalition obtained from the coalitional bargaining and his Shapley value for the internal coalition. The weighted Shapley value applied for a game played by coalitions with coalition-size weights, is assigned to each coalition, reflecting the size-asymmetries among coalitions. On the surface, this solution appears to lie in the very different line from existing studies, but we show that the collective values matches endogenous and exogenous interpretations of coalition structures. In addition to the potential function which derives the solution, two axiomatic characterizations of the collective value are also presented.

JEL Classification Numbers: C70; C71
Keywords: coalition structure; two step Shapley value; weighted Shapley value; coalition-size weights

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1 Introduction

This paper studies a distribution rule of a cooperative surplus among players when they already partition themselves into 'coalitions' before realizing cooperation. A distribution rule, a solution concept in a framework of a cooperative game with a coalition structure, considered in this paper departs from the existing solution concepts in two major directions. One is to take into account the mutual-aid tendency of groups or generous reallocation among members in the internal cooperation.[1] This point is expressed through a two-step approach introduced by Kamijo (2007). The other is to treat the asymmetric sizes of coalitions as a factor affecting the bargaining outcome. From the theoretical point of view, Kalai (1977) and Thomson (1986) show that in the context of bargaining problems, purely replications of players generate the size-dependent asymmetric weights of the Nash solution. On the other hand, from an empirical point of view, Metcalf, Wadsworth, and Ingram (1993) reported that in the observations of British manufacturing industry, strike incidence rose with the size of bargaining group, and it is known that the strike activity affects the bargaining outcome between employers and employees.[2]

The Aumann and Dreze's (1974) value and the Owen's (1977) coalitional value, two traditional solution concepts in cooperative games with coalition structures and each of which is an extension of the Shapley value to a cooperative game with a coalition structure, do not satisfy both requirements mentioned above. On the one hand, both solutions give nothing to a player with no effective contribution whatever cooperation relationship he belongs to. Thus, according to these solution concepts, it does not happen that such player receives some portion of the cooperation surplus from his coalition due to a strong position of his coalition, thus these solutions not having an essence of mutual assistance within the internal members. On the other hand, these solutions treat two distinct coalitions equally even if these are different in their sizes. As pointed out by Hart and Kurz (1983) and Winter (1992), a solution concept of a cooperative game with a coalition structure assumes the two levels interaction among players, *i.e.*, interactions inter- and intra- coalitions. In fact, the Owen's coalitional value satisfies the condition that the sum of the coalitional values of the players in a coalition coincides with the Shapley value of the coalition obtained from the game which is played by inter-coalitions. Thus, the coalitional value well describes a two levels interaction but not reflects an asymmetry in the interaction among coalitions pointed out by Kalai and Samet (1987) and Levy and McLean (1989), caused by the sizes of the coalitions.

The definition of our new solution concept, named a *collective value*, is established relying on a two-step bargaining process among players, a bargaining inter-coalitions in the first step and a bargaining intra-coalition in the second, and generous reallocation tendency among the internal members. In the first step, each coalition obtains its *weighted* Shapley value applied for a game among coalitions. The pure surplus of a coalition in the first step bargaining (its weighted Shapley value obtained from the first step minus the worth of the coalition) is divided equally among players in the coalition. In the second step, players in the coalition receive their Shapley value applied for their own internal game. Thus, the collective value gives the sum of the payoffs in the first step and the second to each player. This definition means that

[1] Such tendency of groups is examined and explained in various contexts. Kropotkin (1972) explains this from a human evolution in the struggle for life. In a context of rent-seeking problem among two groups, Noh (1999) demonstrates that the group members can agree with egalitarian-like sharing rule among them to resolve a free rider problem in the group. Researchers in community psychology argue that recent development of a number of mutual assistance organizations is due to divergent stressful situations around ourselves (Levine 1988). Further, in the study of labor-management, reasons for and usefulness of profit sharing among employer and employees are examined (FitzRoy and Kraft 1986, 1987; Drago and Turnbull 1988; Kandel and Lazear 1992).

[2] One reason is that most union power is partly derived from the threat of the strike (Ashenfelter and Johnson 1969).

the collective value is involved with the egalitarian solution as well as the Shapley value: the egalitarian solution is used for the bargaining surplus of a coalition and the Shapley value for the worth of the coalition.

On the surface, our solution concept appears to lie in the very different line of research from existing studies. However, the collective value matches endogenous and exogenous interpretations of coalition structures. Aumann and Dreze (1974) consider that the existing coalition structure arises from the *endogenous* formation of coalitions, given the game itself. They consider that lack of the superadditivity of the game leads to the formation of coalition structures. Here, we provide a different condition, a *quasi-partnership decomposition*, which is also considered as a reason of forming coalition structures and show that the collective value is consistent with this this condition. Furthermore, in the line of Myerson (1977, 1980), a coalition structure can be considered as *exogenously* given communication restriction among players. We introduce a new interpretation of the coalition structure as restriction of communication among players and show that the collective value coincides with the Shapley value applied for the game appropriately derived from the original game. Thus, the collective value is consistent with these interpretations of coalition structures.

Further, with the aid of research by Calvo and Santos (1997) and Bilbao (1998) on potential theory in cooperative games with communication restriction, we obtain a potential function for games with coalition structures, which is quite different from the one of Winter (1992). The collective value is expressed as the marginal contribution relative to this potential function. The potential function behind the solution concept inspires one of its properties similar to the balanced contributions of the Shapley value. We show that this property, called a *collective balanced contributions*, with some moderate additional conditions characterizes our solution. An axiomatization by the *additivity* axiom is also presented.

The rest of this paper is organized as follows. In the next section we give the basic notations and definitions used in this paper. The exact definition of a new solution concept is provided in Section 3. In Section 4, other expressions and interpretations of the solution are explained. In Section 5, we show that the collective value admits a potential function. Axiomatic characterizations of this solution are given in Section 6. Section 7 gives concluding remarks.

2 Preliminary

A *cooperative game* or a simply *game* is a pair (N,v) where $N=\{1,\dots,n\}$ is a finite set of players and $v:2^N\to\mathbb{R}$ is a characteristic function with $v(\emptyset)=0$. A subset S of N is called a *coalition* and $v(S)$ is the *worth* of coalition S. The set of all the games is denoted by $\mathbf{G}$. We use the short-cut notations of $S-i$ and $S\cup i$ instead of $S\setminus\{i\}$ and $S\cup\{i\}$ for convenience. Given $(N,v)\in\mathbf{G}$ and a coalition S, we denote the subgame of (N,v) to S by (S,v) if there is no risk of confusion.

A game (N,v) is superadditive if for any two coalitions S and T with $S\cap T=\emptyset$, $v(S\cup T)\geqq v(S)+v(T)$. A game (N,v) is zero-monotonic if for any player i and for any coalition $S\subseteq N-i$, $v(S\cup i)\geqq v(S)+v(\{i\})$. A superadditive game is, of course, zero-monotonic but the inverse is not true.

Player $i\in N$ is a *null player* if $v(S\cup i)=v(S)$ for any $S\subseteq N-i$ and a *dummy player* if $v(S\cup i)=v(S)+v(\{i\})$ for any $S\subseteq N-i$. Clearly a null player is also dummy but the converse does not hold. It is said that $i\in N$ and $j\in N$ are *symmetric* in (N,v) if $v(S\cup i)=v(S\cup j)$ for any $S\subseteq N\setminus\{i,j\}$ and $i\in N$ and $j\in$ are symmetric in (T,v), $T\subseteq N$, if $v(S\cup i)=v(S\cup j)$ for any $S\subseteq T\setminus\{i,j\}$.

Assuming that the grand coalition N will be formed, the question arises how to divide the worth $v(N)$ among the players. Thus, a solution of a game is a function ϕ which assigns to every game $(N,v)\in\mathbf{G}$ a payoff vector $\phi(N,v) = (\phi_i(N,v))_{i\in N} \in \mathbb{R}^N$ that satisfies $\sum_{i\in N}\phi_i(N,v) \leq v(N)$. If ϕ always distributes just $v(N)$ to the players, it is called an efficient solution.

A well-known solution was provided by Shapley (1953b). Let $\theta : N \to N$ denote a permutation on N and $\Theta(N)$ denote a set of all the permutations on N. A permutation θ is identified as an order $(i_1,\ldots,i_n)$ on N if $\theta(j) = k$ implies $i_k = j$, and *vice versa*. A set of players preceding to i at order θ is $A_i^\theta = \{j \in N : \theta(j) < \theta(i)\}$. A marginal contribution of player i at order θ in (N,v) is defined by $m_i^\theta(N,v) = v(A_i^\theta \cup i) - v(A_i^\theta)$. The Shapley value Sh of (N,v) is defined as follows:

$$\mathrm{Sh}_i(N,v) = \frac{1}{|\Theta(N)|}\sum_{\theta\in\Theta(N)} m_i^\theta(N,v), \text{ for all } i \in N,$$

where $|\cdot|$ represents the cardinality of the set. Thus, the Shapley value is an average of marginal contribution vectors where each order $\theta \in \Theta(N)$ occurs in an equal probability, that is, $1/|\Theta(N)|$.

The Shapley value is characterized by the four properties: (i) efficiency, (ii) additivity, (iii) symmetry and (iv) null player. Let ϕ be a solution on $\mathbf{G}$. The efficiency requires that the solution distributes the worth of the grand coalition to the players. The additivity is that for any two games (N,v) and (N,v'), $\phi(N,v+v') = \phi(N,v) + \phi(v')$ holds where the additive game $v+v'$ is defined by $(v+v')(S) = v(S)+v'(S)$ for any $S \subseteq N$. The symmetry means that two symmetric players in (N,v) receive the equal payoffs, thus, $\phi_i(N,v) = \phi_j(N,v)$ holds whenever i and j are symmetric in (N,v). The null players axiom is that the null player always obtains nothing.

In various applications of cooperative games, it seems to be natural that players partitions themselves into some ‘coalitions’ such as labor union, syndicate of firms, customs unions in international economics, and so on. Such coalitions form a coalition structure $\mathscr{C} = \{C_1,\ldots,C_m\}$, which is partition of N, *i.e.*, it holds that $C_k \cap C_h = \emptyset$ for any k and any h with $k \neq h$ and $\bigcup_{k=1}^m C_k = N$. Such a situation, called a cooperative game with a coalition structure, is first considered by Aumann and Dreze (1974) and developed by a number of authors. A counterpart of the Shapley value for such games was defined by Owen (1977) and given the axiomatic foundation from the viewpoint of coalition formation by Hart and Kurz (1983).

A *game with a coalition structure* is a triple $(N,v,\mathscr{C})$ where (N,v) is a game and $\mathscr{C} = \{C_1,\ldots,C_m\}$ is a coalition structure. We usually use notation $M = \{1,\ldots,m\}$ to denote the set of coalitional indices in $\mathscr{C}$. The set of all the games with coalition structures is denoted by $\mathbf{G}^c$. An order $\theta \in \Theta(N)$ is *consistent* with $\mathscr{C}$ if for any $i \in C_h \in \mathscr{C}$ and $j \in C_h \in \mathscr{C}$ and $k \in N$, $\theta(i) < \theta(k) < \theta(j)$ implies that player k also belongs to coalition C_h, that is, $k \in C_h$. Thus, in the consistent order, players line up in a way that players in the same coalition are side-by-side. A set of all the orders on N consistent with $\mathscr{C}$ is denoted by $\Theta(N,\mathscr{C})$. Then, Owen’s (1977) coalitional value CV is an average of player’s marginal contributions when all the orders consistent with $\mathscr{C}$ occur with equal probability, being defined by,

$$\mathrm{CV}_i(N,v,\mathscr{C}) = \frac{1}{|\Theta(N,\mathscr{C})|}\sum_{\theta\in\Theta(N,\mathscr{C})} m_i^\theta(N,v), \text{ for each } i \in N.$$

Thus, according to the coalitional value, players in N appear in a way that the players in the same coalition appear successively. In other words, first coalitions enter subsequently in a random order and within each coalition the players enter subsequently in a random order.

An *external game* or a game played by the (representatives of the) coalitions $(M,v_{\mathscr{C}})$ is defined by $M =$

$\{1,\ldots,m\}$ and $v_{\mathscr{C}}(H) = v(\bigcup_{k\in H} C_k)$ for each $H \subseteq M$.[3] For external game $(M, v_{\mathscr{C}})$, the Owen's coalitional value satisfies the following: for any $C_k \in \mathscr{C}$,

$$\sum_{i\in C_k} \mathrm{CV}_i(N, v, \mathscr{C}) = \mathrm{CV}_k(M, v_{\mathscr{C}}, \{M\}).$$

This property is called the intermediate game property. The coalitional value is characterized by efficiency, additivity, null player property, the intermediate game property and the restricted equal treatment property which requires that if two players in $C_k \in \mathscr{C}$ are symmetric in (N, v), the two players should receive the equal payoff (see Owen 1977 and Peleg and Sudhölter 2003). Here, the first three axioms are the ones which are naturally extended to a game with a coalition structure. However, the null player property in this case may be considered to be a bit strong requirement because it means that the null player gets nothing even though the coalition he belongs to is in very strong position. Thus, the coalitional value does not reflect a function of the formed coalition as system of mutual assistance. In Section 6, we provide a weaker version of the null player property in a game with a coalition structure to characterize our new solution, which is defined in the next section.

3 A new solution concept

As motivated by Hart and Kurz (1983) and Winter (1989), the coalition in $\mathscr{C}$ can be seen as a pressure group for the division of $v(N)$. So, van den Brink and van der Laan (2005) stated (p195):

> to divide the worth of the grand coalition over all players, first this worth is distributed over the coalitions in the a priori given coalition structure, and then the payoff assigned to a coalition is distributed over its players.

The Owen's coalitional value describes the above two level interactions, which are an interaction among coalitions and a one among players within a coalition, and has the consistent relation with the Shapley value's allocation. The coalitional value satisfies

$$\sum_{i\in C_k} \mathrm{CV}_i(N, v, \mathscr{C}) = \mathrm{CV}_k(M, v_{\mathscr{C}}, \{M\}) = \mathrm{Sh}_k(M, v_{\mathscr{C}})$$

for any $C_k \in \mathscr{C}$ because the coalitional value satisfies the intermediate game property and the coalitional value for a game with the grand coalition structure coincides with the Shapley value for the game.

There is an asymmetry of players in external game $(M, v_{\mathscr{C}})$ since players in the game represent the coalitions which may be different in size. In such a situation, the *weighted* Shapley value (Shapley 1953a) can be appropriate to deal with such asymmetries. Kalai and Samet stated in their paper (Kalai and Samet 1987, p221) as follows:

> It is important for applications in which the players themselves are, or are representing, groups of individuals. Such is the case for example when the players are parties, cities, or management boards. ... A natural candidate for a solution is the weighted Shapley value where the players are weighted by the size of the constituencies they stand for.

[3] This game is referred to as an intermediate game in Peleg and Sudhölter (2003) and as a quotient game in Owen (1977).

As the following definition will show, a new solution concept presented in this paper is the very solution that reflects such a viewpoint in addition to the two level interactions.

Let $w_i(>0)$ denote a positive weight for a player $i \in N$. Given a collection of positive weights $w = (w_i)_{i\in N}$, let $\mu^w(.)$ represent a probability measure on $\Theta(N)$ such that for an order $\theta = (i_1,\ldots,i_n)$, $\mu^w(\theta) = \Pi_{j=1}^{n} \frac{w_{i_j}}{\sum_{k=1}^{j} w_{i_k}}$.[4] The w-weighted Shapley value Sh^w for $(N,v) \in \mathbf{G}$ is

$$\mathrm{Sh}_i^w(N,v) = \sum_{\theta\in\Theta(N)} \mu^w(\theta)\, m_i^\theta(N,v)$$

for any $i \in N$.[5]

Clearly, if $w_i = w_j$ for any $i \in N$ and for any $j \in N$, the w-weighted Shapley value coincides with the Shapley value. Given coalition T of N, let (N,u_T) denote a T-unanimity game defined by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ otherwise. It is easily checked that $\mathrm{Sh}_i^w(N,u_T) = \frac{w_i}{\sum_{j\in T} w_j}$ if $i \in T$ and $\mathrm{Sh}_i^w(N,u_T) = 0$ otherwise.

A solution concept ψ^γ in a game with a coalition structure is defined in the following.

Definition 1. *For $(N,v,\mathscr{C}) \in \mathbf{G}^c$, let w_k denote the weight for $k \in M$ such that $w_k = |C_k|$ and $w = (w_k)_{k\in M}$. Then, the collective value ψ^γ for $(N,v,\mathscr{C})$ is defined by*

$$\psi_i^\gamma(N,v,\mathscr{C}) = \frac{\mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k,v)$$

for any $i \in C_k \in \mathscr{C}$.

The definition of the collective value shows the relation with a two step approach introduced in Kamijo (2007): the first step is a negotiation among coalitions for the division of $v(N)$ and the second step is a negotiation among players for the division of the assignment of the coalition from the first step. The bargaining surplus of the coalition from the first step, $\mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - v(C_k)$, is equally divided among its members. Moreover they obtain the Shapley value for their own game in the second step, $\mathrm{Sh}(C_k,v)$. Thus, this expression indicates that ψ^γ has a flavor of egalitarian rule in addition to the Shapley value: the egalitarian solution for the bargaining surplus of his coalition and the Shapley value for the worth of the coalition. As the result of this egalitarian part, ψ^γ does not satisfy the usual null player axiom but the weaker version of null player axiom. This point is considered in section 6 to characterize the collective value by the additivity axiom.

The particular difference from Owen (1977) and Kamijo (2007) is that in the definition above, each coalition, say C_k, receives $\mathrm{Sh}_k^w(M,v_{\mathscr{C}})$, *i.e.*, the w-weighted Shapley value of the external game, instead of the usual Shapley value. Further, the weights are the sizes of each coalition, *i.e.*, $w_k = |C_k|$ for each $C_k \in \mathscr{C}$.

[4] To obtain this probability, consider the following model of choosing an order $(i_1,\ldots,i_n)$. First, a player in N, say i_n, is randomly selected, due to a probability distribution such that the probability for a player to be selected is proportional to his weight and put in the last of the order. Next, another player i_{n-1} is selected by the same process for $n-1$ players and put in the second last of the order. Repeating the same process by $n-2$ times, we have an order $(i_1,\ldots,i_n)$ and the probability of occurrence of this order is this formula.

[5] Kalai and Samet (1987, 1988) generalized positive weights to a weight system which is a pair of weights and an ordered partition on N in order to allow a weight of zero for some of the players.

From the definition, it easily confirmed that ψ^{γ} satisfies

$$\sum_{i\in C_k} \psi_i^{\gamma}(N,v,\mathscr{C}) = \mathrm{Sh}_k^w(M,v_{\mathscr{C}}),$$

reflecting the asymmetries in the sizes of coalitions.

One may consider that the definition of the collective value is a bid strange because it applies inconsistent treatment between a negotiation among coalitions and a negotiation within a coalition. This is not true, however; rather the collective value treats the two types of bargaining in consistent manner in terms of the players' sizes because in the subgame (C_k,v), each player in C_k has an equal size and the weighted Shapley value with equal weights among the players coincides with the Shapley value, that is, $\mathrm{Sh}^w(C_k,v) = \mathrm{Sh}(C_k,v)$, given $w_i = 1$ for all $i \in C_k$.

To obtain a better understanding on a two-step interpretation of ψ^{γ}, we introduce a "redistribution game" defined below. Let ϕ be a solution on $\mathbf{G}$ and $C_k \in \mathscr{C}$. Define a function $v^r(.|\phi) : 2^{C_k} \to \mathbb{R}$ by, for all $S \subseteq C_k$,

$$v^r(S|\phi) = \begin{cases} \phi_k(M,v_{\mathscr{C}}) & \text{if } S = C_k, \\ v(S) & \text{otherwise.} \end{cases}$$

A game $(C_k, v^r(.|\phi))$ is called a *redistribution game* for C_k over the coalitional bargaining surplus at distribution rule ϕ. Let $(N,v,\mathscr{C}) \in \mathbf{G}^c$, $C_k \in \mathscr{C}$, and $M = \{k : C_k \in \mathscr{C}\}$. Let $w = (w_k)_{k\in M}$ with $w_k = |C_k|$ for any $k \in M$. The following theorem is easily derived from the definition of ψ^{γ}.

Theorem 1. *For $C_k \in \mathscr{C}$ and for $i \in C_k$,*

$$\psi_i^{\gamma}(N,v,\mathscr{C}) = \mathrm{Sh}_i(C_k, v^r(.|\mathrm{Sh}^w)).$$

Proof. Define (C_k,u) by $u(S) = \mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - v(C_k)$ if $S = C_k$ and $u(S) = 0$ otherwise. Then, $v^r(.|\mathrm{Sh}^w) = v+u$. Since the Shapley value satisfies the additivity,

$$\mathrm{Sh}_i(C_k, v^r(.|\mathrm{Sh}^w)) = \mathrm{Sh}_i(C_k,v) + \mathrm{Sh}_i(C_k,u).$$

Furthermore, since the Shapley value satisfies the symmetry and the efficiency, $\mathrm{Sh}_i(C_k,u) = \frac{\mathrm{Sh}_k^w(M,v_{\mathscr{C}})-v(C_k)}{|C_k|}$. □

Remark 1. *The Owen's coalitional value is also described as the Shapley value for the other type of redistribution game. For $C_k \in \mathscr{C}$, $(C_k, v^c(.|\phi))$ is defined by $v^c(S|\phi) = \phi_k(M, v_{\mathscr{C}}^S)$ for all $S \subseteq C_k$ where $(M, v_{\mathscr{C}}^S)$ is a game played by coalitions with C_k being replaced by $S \subset C_k$. That is, $v_{\mathscr{C}}^S(H) = \bigcup_{h\in H} C_h$ if $k \notin H$ and $v_{\mathscr{C}}^S(H) = \bigcup_{h\in H\setminus\{k\}} C_h \cup S$ if $k \in H$. Then, $\mathrm{CV}_i(N,v,\mathscr{C}) = \mathrm{Sh}_i(C_k, v^c(.|\mathrm{Sh}))$ holds (see, Owen 1977 and Winter 1992).*

Remark 2. *Kamijo (2006, 2007) considers another two-step Shapley value in which the Shapley value is applied for both intra- and inter- coalitions. In other words, not the weighted Shapley value but the usual Shapley value is applied to the external game. Thus, a solution ψ^{δ} on $\mathbf{G}^c$ is defined by*

$$\psi_i^{\delta}(N,v,\mathscr{C}) = \mathrm{Sh}_i(C_k, v^r(.|\mathrm{Sh})) = \frac{\mathrm{Sh}_k(M,v_{\mathscr{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k,v)$$

for any $C_k \in \mathscr{C}$ and for any $i \in C_k$.

4 Interpretations of the value and coalition structures

In this section, we consider an endogenous interpretation and an exogenous interpretation of coalition structures and show that the collective value fits these interpretations.

4.1 A value on $\mathscr{C}$-communication restricted situation

An exogenous interpretation of coalition structures is that they represent the some kinds of constraint on communication among players (see Aumann and Dreze 1974). Myerson (1977) considers a situation that a communication between players is restricted on an undirected graph of N (such game is called a graph-restricted game). Myerson (1980) considers more generalized situation that there is a sequence of conferences in which players communicate with each other and this communication restriction is expressed as the hyper-graph on N. Since Myerson's works, there are various kinds of research on games with restriction or constraint on communication among players (*e.g.*, a permission structure by Gilles, Owen, and van den Brink 1992; restricted coalitions by Derks and Peters 1993; a weighted hyper-graph by Amer and Carreras 1995, 1997; a probabilistic graph by Calvo, Lasaga, and van den Nouweland 1999; a partition system by Bilbao 1998).

Along this line of research, Aumann and Dreze's (1974) value, which is defined by $\mathrm{AD}_i(N,v,\mathscr{C}) = \mathrm{Sh}(C_k,v)$ for all $i \in C_k \in \mathscr{C}$, assumes a situation that a coalition structure describes a communication restriction such that players in the same coalition communicate with each other, but each coalition is physically separated. This situation is also described as the graph such that each maximal component of the graph corresponds to a coalition in the coalition structure and each subgraph on the component is a complete graph. Thus, Aumann and Dreze's value coincides with the Myerson value for such a graph situation. However, this interpretation of coalition structure does not fit the view that players form coalitions for the division of $v(N)$ since Aumann and Dreze's value does not satisfy the efficiency but the relative efficiency ($\sum_{i\in N} \mathrm{AD}_i(N,v,\mathscr{C}) = \sum_{k\in M} v(C_k)$). This motivates another view of communication restriction by a coalition structure below.

Given a coalition structure $\mathscr{C}$ on N, assume that $\mathscr{C}$ represents the communication restricted situation as follows:

(i) players in the same coalition $C_k \in \mathscr{C}$ can freely communicate with each other, and

(ii) players in C_k can communicate with players in the other coalitions if there is a permission of all the players in C_k.

Condition (i) means that players in any sub coalition $S \subseteq C_k \in \mathscr{C}$ can communicate with each other and thus obtain their worth of coalition, $v(S)$. In addition to (i), (ii) implies that there is a possibility of cooperation among players in the different coalitions. This is possible only if all the players in the relevant coalitions agree. Let $i \in C_k$ and $C_h \in \mathscr{C}, C_h \neq C_k$. While C_k and C_h obtain their worth $v(C_k \cup C_h)$, $C_k - i$ and C_h obtain the sum of $v(C_k - i)$ and $v(C_h)$ because there is no permission by player i or there is no permission of the party which the coalition represents and which requires the unanimous agreement.[6]

[6]Carreras (1992) refers the similar restriction of coalition as "voting discipline" in the context of simple games.

Definition 2. *Let $(N,v,\mathscr{C}) \in \boldsymbol{G}^c$. $\mathscr{C}$-communication restricted game $(N,v^{\mathscr{C}})$ is defined as follows. For all $S \subseteq N$,*

$$v^{\mathscr{C}}(S) = v(\bigcup_{C_k \in \mathscr{C}(S)} C_k) + \sum_{T \in \mathscr{C}^0(S)} v(T)$$

where $\mathscr{C}(S) = \{C_k \in \mathscr{C} : C_k \subseteq S\}$ and $\mathscr{C}^0(S) = \{C_k \cap S : C_k \cap S \neq C_k, C_k \in \mathscr{C}\}$.

Then, ψ^{γ} is interpreted as a value on the $\mathscr{C}$-communication restricted game.

Theorem 2. *Let $(N,v,\mathscr{C}) \in \boldsymbol{G}^c$. For $i \in N$,*

$$\psi_i^{\gamma}(N,v,\mathscr{C}) = \mathrm{Sh}_i(N,v^{\mathscr{C}}).$$

Proof. Take any order $\theta \in \Theta(N)$. Let $\theta[C_k]$ denote an order on C_k induced from θ such that for any $i, j \in C_k$, $\theta[C_k](i) < \theta[C_k](j)$ exactly if $\theta(i) < \theta(j)$, and let θ_M denote an order on M induced from θ such that for any $k,h \in M$, $\theta_M(k) < \theta_M(h)$ exactly if there is a player $i \in C_h$ such that $\theta(j) < \theta(i)$ for all $j \in C_k$. According to marginal contributions in $\mathscr{C}$-communication restricted game $v^{\mathscr{C}}$ at order θ, $i \in C_k$ obtains, when i is not the last in the order $\theta[C_k]$,

$$v(A_i^{\theta[C_k]} \cup i) - v(A_i^{\theta[C_k]}),$$

and when i is the last in the order, he obtains

$$[v(A_k^{\theta_M} \cup C_k) - v(A_k^{\theta_M})] - v(C_k) + [v(C_k) - v(C_k - i)].$$

Because in the situation that each $\theta \in \Theta(N)$ occurs in equal probability, $\theta[C_k]$ coincides with one order on C_k in probability $1/|\Theta(C_k)|$, thus irrelevant to the selection of the order, and each $i \in C_k$ has a equal probability to be the last, it suffices to show that $\sum_{i \in C_k} \mathrm{Sh}_i(N,v^{\mathscr{C}}) = \mathrm{Sh}_k^w(M,v_{\mathscr{C}})$.

We denote by $\mathrm{Prob}(\cdot)$ the probability that some phenomena happen in the situation that each $\theta \in \Theta(N)$ occurs in equal probability $1/|\Theta(N)|$. We will show that for any given order $\sigma \in \Theta(M)$, $\mathrm{Prob}(\theta_M = \sigma)$ is $\mu^w(\sigma)$ where $w_k = |C_k|$ for each $k \in M$. For simplifying explanation, let $\sigma = (\sigma_1, \ldots, \sigma_m)$ be $(1, \ldots, m)$. First, we consider the probability that $\theta_M(m)$ coincides with $\sigma_m = m$, that is, $\mathrm{Prob}(\theta_M(m) = m)$. Since this probability is equal to the probability that some player in C_m is the last position in order θ, we obtain

$$\mathrm{Prob}\left(\theta_M(m) = m\right) = \frac{|C_m|}{|N|} = \frac{w_m}{\sum_{h \in M} w_h}.$$

Further, assume that $\mathrm{Prob}\left(\theta_M(h) = h, h = k+1, \ldots, m\right) = \Pi_{h=k+1}^m \frac{w_h}{\sum_{h'=1}^h w_{h'}}$. Then, given the condition that $\theta_M(h) = h, h = k+1, \ldots, m$, the conditional probability that $\theta_M(k)$ coincides with k is

$$\frac{|C_k|}{\sum_{h'=1}^k |C_{h'}|} = \frac{w_k}{\sum_{h'=1}^k w_{h'}},$$

because this probability is equal to the probability that some $i \in C_k$ is the last player in the order that players in $C_h, h = k+1,\ldots,m$ are extracted from. Thus,

$$\begin{aligned}\text{Prob}\,(\theta_M(h) = h, h = k,\ldots,m) &= \text{Prob}\,(\theta_M(h) = h, h = k+1,\ldots,m)\\ &\quad \times \text{Prob}\,(\theta_M(k) = k \mid \theta_M(h) = h, h = k+1,\ldots,m)\\ &= \Pi_{h=k}^{m} \frac{w_h}{\sum_{h'=1}^{h} w_{h'}}.\end{aligned}$$

Therefore, repeating this, we obtain $\text{Prob}(\theta_M = \sigma) = \Pi_{h=1}^{m} \frac{w_h}{\sum_{h'=1}^{h} w_{h'}} = \mu^w(\sigma)$. □

Remark 3. *ψ^δ is also considered as a value for $\mathscr{C}$-communication restricted game $(N, v^{\mathscr{C}})$. However, we use the weighted Shapley value instead of the usual Shapley value. Given $(N, v, \mathscr{C}) \in \boldsymbol{G}^c$, let $w = (w_i)_{i\in N}$ be such that $w_i = \frac{1}{|C_k|}$ for $i \in C_k \in \mathscr{C}$. Then,*

$$\psi^g(N, v, \mathscr{C}) = \text{Sh}^w(N, v^{\mathscr{C}}).$$

See Kamijo (2007).

The communication situation considered in this sub section can be seen as a partition system of Bilbao (1998). A pair $(N, \mathscr{F})$, where $\mathscr{F} \subseteq 2^N$, is called a partition system if (P1) $\emptyset \in \mathscr{F}$ and for all $i \in N$, $\{i\} \in \mathscr{F}$, and (P2) for all $S \subseteq N$, the maximal components of S by $\mathscr{F}$, which are defined by $\{T \subset S : T \in \mathscr{F}$ and $\neg\exists T' \in \mathscr{F}$ such that $T \subsetneq T' \subseteq S\}$, form a partition of S (see definition 1 of his paper). Given a coalition structure $\mathscr{C}$, define $\mathscr{F}^{\mathscr{C}}$ by

$$\mathscr{F}^{\mathscr{C}} = \bigcup_{C_k\in\mathscr{C}} 2^{C_k} \cup \{\bigcup_{k\in L} C_k \ : L \subseteq M\}.$$

Then, $(N, \mathscr{F}^{\mathscr{C}})$ becomes a partition system.

Given a partition system $(N, \mathscr{F})$, the restricted game $(N, v^{\mathscr{F}})$ is defined by

$$v^{\mathscr{F}}(S) = \sum_{T\in\Pi_S} v(T),$$

where Π_S is a partition of S which the maximal feasible subsets of S on $\mathscr{F}$, are called components, form. By the definition of $v^{\mathscr{C}}$, we have the following proposition.

Proposition 1. $(N, v^{\mathscr{C}}) = (N, v^{\mathscr{F}^{\mathscr{C}}})$ *holds.*

Proof. For any $S \subseteq N$,

$$\mathscr{C}(S) \cup \mathscr{C}^0(S)$$

is a partition of S and $\mathscr{C}(S) \cup \mathscr{C}^0(S) = \Pi(S)$ holds. By the definitions of $v^{\mathscr{C}}$ and $v^{\mathscr{F}^{\mathscr{C}}}$, $v^{\mathscr{C}} = v^{\mathscr{F}^{\mathscr{C}}}$ holds. □

4.2 An endogenous interpretation of a coalition structure

Aumann and Dreze (1974) consider that one of the transparent explanations for the formation of coalition structures from games themselves is by the lack of the superadditivity (see the discussion of their paper).

However, from the viewpoint that players form coalition structures for the bargaining of division of $v(N)$, we have to introduce another endogenous argument for the formation of coalition structures.

Let $(N,v) \in \mathbf{G}$. A coalition S is called a partnership in (N,v) if for any $T \subsetneq S$ and for any $R \subseteq N \setminus S$, $v(T \cup R) = v(R)$. Further, S is called a quasi-partnership in (N,v) if for any $T \subsetneq S$ and for any $R \subseteq N \setminus S$, $v(T \cup R) = v(T) + v(R)$. Thus, players in a quasi-partnership coalition T seem to have some rationale to act together.

Let (N,v) be a game and $\mathscr{C}$ be a coalition structure on N. Then, $\mathscr{C}$ is called a *quasi-partnership decomposition* with respect to v if every $C_k \in C$ is a quasi-partnership in (N,v). The next theorem indicates that ψ^γ is consistent with this endogenous view of the coalition structure and the allocation of the Shapley value.

Theorem 3. *Let $(N,v,\mathscr{C}) \in \mathbf{G}^c$ and $w = (w_k)$ such that $w_k = |C_k|$. If $\mathscr{C}$ is a quasi-partnership decomposition with respect to v, then*

$$\mathrm{Sh}(N,v) = \psi^\gamma(N,v,\mathscr{C}).$$

Proof. If $(N,v) = (N,v^{\mathscr{C}})$, Theorem 2 implies that $\mathrm{Sh}(N,v) = \mathrm{Sh}(N,v^{\mathscr{C}}) = \psi^\gamma(N,v,\mathscr{C})$. Thus, it suffices to show $(N,v) = (N,v^{\mathscr{C}})$. For any $S \subseteq N$,

$$\begin{aligned} v^{\mathscr{C}}(S) &= v(\bigcup_{C_k \in \mathscr{C}(S)} C_k) + \sum_{T \in \mathscr{C}^0(S)} v(T) \\ &= v(S) \end{aligned}$$

where the first equality is by the definition of $v^{\mathscr{C}}$ and the second is by the quasi-partnership of $C_k \in \mathscr{C}$. □

5 A potential function for games with coalition structures

Hart and Mas-Colell (1989) are the first to introduce a concept of a potential to cooperative game theory and show that a potential for a game exists (with an additional condition of the normalization, it is unique) and it derives the Shapley value. After Hart and Mas-Colell, the concept of potential was introduced to a non-cooperative game by Monderer and Shapley (1996) and has been considered for a cooperative game with several frameworks such as a game with a coalition structure by Winter (1992), a partition system by Bilbao (1998), a finite type continuum by Calvo and Santos (1997). Calvo and Santos (1997) also characterized the family of solutions which admitted a potential function.

Let P denote a real valued function on $\mathbf{G}$ which is normalized to $P(\emptyset,v) = 0$. Given $(N,v) \in \mathbf{G}$ and $i \in N$, define a marginal contribution of player i relative to P by

$$D_iP(N,v) = P(N,v) - P(N-i,v).$$

Thus, this marginal contribution is the difference of two situations measured by P which player i is there and he leaves. Function P is called a *potential* for games if it satisfies

$$v(N) = \sum_{i \in N} D_iP(N,v)$$

for any $(N,v) \in \mathbf{G}$. Thus, a potential function is such that the allocation of marginal contributions (according to the potential function) always adds up exactly to the worth of the grand coalition. Hart and Mas-Colell (1989) show (in theorem A, p591) that (i) potential function P is uniquely determined, and (ii) the marginal contribution vector relative to the potential coincides with the Shapley value payoff vector, *i.e.*, $D_iP(N,v) = \mathrm{Sh}_i(N,v)$ for all $i \in N$.

They also consider a non-symmetric generalization of a potential approach. Let $w = (w_i)_i$ be a collection of the positive weights and P^w denote a real-valued function on $\mathbf{G}$ with $P^w(\emptyset, v) = 0$. Function P^w is called a *w-weighted potential* if it satisfies

$$v(N) = \sum_{i \in N} w_i D_i P(N,v)$$

for any $(N,v) \in \mathbf{G}$. They show (in theorem 5.2, p603) that (i) w-weighted potential function P^w is uniquely determined, and (ii) the marginal contribution relative to the potential multiplied by the corresponding weight coincides with the w-weighted Shapley value, *i.e.*, $w_i D_i P(N,v) = \mathrm{Sh}_i^w(N,v)$ for any $i \in N$.

According to Calvo and Santos (1997) and Bilbao (1998), a potential function for a game with a restricted communication is Hart and Mas-Colell's potential function (hereafter, the HM potential function. Similarly we use the term, the HM w-weighted potential function.) for the corresponding game which is appropriately defined to reflect the restriction on communication. Thus, the next theorem is an immediate consequence of Theorem 2.

Theorem 4. *Let $P : \boldsymbol{G} \to \mathbb{R}$ denote the HM potential function. Then, given any $(N,v,\mathscr{C})$,*

$$\psi_i^\gamma(N,v,\mathscr{C}) = D_iP(N,v^{\mathscr{C}}) = P(N,v^{\mathscr{C}}) - P(N-i,v^{\mathscr{C}})$$

for any $i \in N$.

Bilbao (1998) also shows (in theorem 2, p135) that given a partition system $(N,\mathscr{F})$, for $S \notin \mathscr{F}$,

$$P(S,v^{\mathscr{F}}) = \sum_{T \in \Pi_S} P(T,v^{\mathscr{F}}).$$

Hence, this result together with $(N,v^{\mathscr{C}}) = (N,v^{\mathscr{F}^{\mathscr{C}}})$ by Proposition 1 implies that for $i \in C_k \in \mathscr{C}$,

$$D_iP(N,v^{\mathscr{C}}) = P(N,v^{\mathscr{C}}) - P(N \setminus C_k,v^{\mathscr{C}}) - P(C_k - i,v^{\mathscr{C}}). \tag{1}$$

The next proposition gives another formula of $P(N,v^{\mathscr{C}})$ which seems to describe the restriction of communication by $\mathscr{C}$ well and which is specific expression of the potential for the particular subclass of games with permission systems, which is different from the class Bilbao (1998) mainly considers.

Proposition 2. *Let $(N,v,\mathscr{C}) \in \boldsymbol{G}^c$ and $M = \{k : C_k \in \mathscr{C}\}$. Define a game (M,u) by*

$$u(L) = v(\bigcup_{k \in L} C_k) - \sum_{k \in L} v(C_k) + \sum_{k \in L} |C_k| P(C_k,v)$$

for each $L \subseteq M$. Then,

$$P(N,v^{\mathscr{C}}) = P^w(M,u),$$

where P^w is the HM w-weighted potential function and $w = (w_k)_{k \in M}$ is such that $w_k = |C_k|$ for any $k \in M$.

Proof. Let $(N,v,\mathscr{C}) \in \mathbf{G}^c$ be given. Put $M = \{k : C_k \in \mathscr{C}\}$. The proof proceeds by the way of mathematical induction of the number of $|M|$. For any $C_k \in \mathscr{C}$,

$$P(C_k, v^{\mathscr{C}}) = P(C_k, v) = \frac{1}{|C_k|} u(\{k\}) = P^w(\{k\}, u),$$

where the first equality is by $(C_k, v^{\mathscr{C}}) = (C_k, v)$, the second is by the definition of u, and the last is by the definition of the HM w-weighted potential function and $w_k = |C_k|$.

Assume that for any $L \subsetneq M$ and $N' = \bigcup_{k \in L} C_k$, $(N', v^{\mathscr{C}}) = P^w(L, u)$ holds. We consider the case for $(N, v^{\mathscr{C}})$. By the definition of the potential function,

$$\begin{aligned}
v(\bigcup_{k\in M} C_k) &= v(N) \\
&= \sum_{i \in N} D_i P(N, v^{\mathscr{C}}) \\
&= \sum_{k\in M} \sum_{i \in C_k} \Big(P(N, v^{\mathscr{C}}) - P(N \setminus C_k, v^{\mathscr{C}}) - P(C_k - i, v^{\mathscr{C}}) \Big) \\
&= \sum_{k \in M} |C_k| \Big(P(N, v^{\mathscr{C}}) - P(N \setminus C_k, v^{\mathscr{C}}) \Big) \\
&\quad - \sum_{k\in M} |C_k| P(C_k, v^{\mathscr{C}}) + \sum_{k \in M} \sum_{i \in C_k} (P(C_k, v^{\mathscr{C}}) - P(C_k - i, v^{\mathscr{C}})) \\
&= \sum_{k \in M} |C_k| \Big(P(N, v^{\mathscr{C}}) - P(N \setminus C_k, v^{\mathscr{C}}) \Big) \\
&\quad - \sum_{k\in M} |C_k| P(C_k, v^{\mathscr{C}}) + \sum_{k \in M} \sum_{i \in C_k} \mathrm{Sh}_i(C_k, v^{\mathscr{C}}) \\
&= \sum_{k \in M} |C_k| \Big(P(N, v^{\mathscr{C}}) - P(N \setminus C_k, v^{\mathscr{C}}) \Big) - \sum_{k\in M} |C_k| P(C_k, v) + \sum_{k\in M} v(C_k),
\end{aligned}$$

where the third equality is by Equation (1), the second last equality is by the result of the HM potential function, and the last is by the efficiency of the Shapley value and $(C_k, v^{\mathscr{C}}) = (C_k, v)$. Hence we obtain

$$\sum_{k \in M} |C_k| \Big(P(N, v^{\mathscr{C}}) - P(N \setminus C_k, v^{\mathscr{C}}) \Big) = v(\bigcup_{k\in M} C_k) + \sum_{k\in M} |C_k| P(C_k, v) - \sum_{k\in M} v(C_k),$$

By the assumption of the induction and the definition of u, this is equivalent to

$$\sum_{k\in M} w_k \Big(P(N, v^{\mathscr{C}}) - P^w(M \setminus \{k\}, u) \Big) = u(M),$$

where $w_k = |C_k|$ for any $k \in M$. Therefore the uniqueness of the weighted potential implies $P(N, v^{\mathscr{C}})$ must be $P^w(M, u)$. □

6 Axiomatic characterizations

6.1 Collective balanced contributions

The balanced contributions property for the Shapley value was first considered by Myerson (1980). It means that any two players' marginal contributions to the other measured by the Shapley value balance. In other words, the Shapley value satisfies, given two players $i \in N$ and $j \in N$, $\mathrm{Sh}_i(N,v) - \mathrm{Sh}_i(N-j,v) = \mathrm{Sh}_j(N,v) - \mathrm{Sh}_j(N-i,v)$. Myerson (1980) showed that the efficiency and this property characterize the Shapley value.

Extensions of the balanced contributions to a game with a coalition structure is considered by Calvo, Lasaga, and Winter (1996). They introduce two counterparts of the balanced contributions to that case and show that a unique efficient solution on $\mathbf{G}^c$ satisfying these two properties is the Owen's coalitional value. These two are:

Individual Balanced Contributions: For $i \in C_k$ and $j \in C_k$, $C_k \in \mathscr{C}$,

$$\psi_i(N,v,\mathscr{C}) - \psi_i(N-j,v,\mathscr{C}-j) = \psi_j(N,v) - \psi_j(N-i,v,\mathscr{C}-i)$$

where $\mathscr{C} - i = \mathscr{C} \setminus \{C_k\} \cup \{C_k - i\}$.

Coalitional Balanced Contributions: For $C_k \in \mathscr{C}$ and for $C_h \in \mathscr{C}$,

$$\sum_{i \in C_k} (\psi_i(N,v,\mathscr{C}) - \psi_i(N \setminus C_h, v, \mathscr{C} \setminus \{C_h\})) = \sum_{i \in C_h} (\psi_i(N,v,\mathscr{C}) - \psi_i(N \setminus C_k, v, \mathscr{C} \setminus \{C_k\})).$$

However, we introduce different extensions of the balanced contributions for games with coalition structures. One is just the same requirement as the condition for the Shapley value, and the other is easily interpreted.

Balanced Contributions: For $i \in N$ and $j \in N$,

$$\psi_i(N,v,\{N\}) - \psi_i(N-j,v,\{N-j\}) = \psi_j(N,v,\{N\}) - \psi_j(N-i,v,\{N-i\}).$$

Collective Balanced Contributions: If $|\mathscr{C}| \geqq 2$, for every $i \in C_k \in \mathscr{C}$ and for every $j \in C_h \in \mathscr{C}$, $C_k \neq C_h$,

$$\psi_i(N,v,\mathscr{C}) - \psi_i(N \setminus C_h, v, \mathscr{C} \setminus \{C_h\}) = \psi_j(N,v,\mathscr{C}) - \psi_j(N \setminus C_k, v, \mathscr{C} \setminus \{C_k\}).$$

Since $(N,v,\{N\})$ is looked as the same situation as (N,v),[7] Balanced Contributions is the same condition as the one which the Shapley value satisfies, and thus we use the same name. Collective Balanced Contributions requires that 'my group's contribution for your payoff measured by the solution balances with your group's contribution for my payoff measured by the solution.'

On the relationship between our axioms and ones of Calvo, Lasaga, and Winter (1996), Individual Balanced Contributions implies Balanced Contributions. Collective Balanced Contributions induces Coalitional Balanced Contributions only if $|C_k| = |C_h|$. However, in general, there is no general relationship

[7] In fact, all the values for games with coalition structures considered in this paper, AD, CV, ψ^δ, and ψ^γ, for $(N,v,\{N\})$ coincide with the Shapley value for (N,v).

between Collective Balanced Contributions and Coalitional Balanced Contributions. The next proposition shows that ψ^γ satisfies Balanced Contributions and Collective Balanced Contributions instead of Individual Balanced Contributions and Coalitional Balanced Contributions.

Proposition 3. *ψ^γ satisfies Balanced Contributions and Collective Balanced Contributions.*

Proof. First consider the case of $|\mathscr{C}| = 1$. By definition of ψ^γ, $\psi^\gamma(N, v, \{N\}) = \mathrm{Sh}(N, v)$ holds. We obtain the desired result because of the result of Myerson (1980).

Next we consider the case of $|\mathscr{C}| \geqq 2$. Note that by the definition of $\mathscr{C}$-communication restricted game, $(N \setminus C_k, v^{\mathscr{C}})$ which is a subgame of $(N, v^{\mathscr{C}})$ on $N \setminus C_k$, coincides with $(N \setminus C_k, v^{\mathscr{C} \setminus \{C_k\}})$ which is $\mathscr{C} \setminus \{C_k\}$-restricted game for game $(N \setminus \{C_k\}, v, \mathscr{C} \setminus \{C_k\})$. By Theorem 2, the property of the HM potential function and Equation (1)

$$
\begin{aligned}
\psi_i^\gamma(N,v,\mathscr{C}) - &\psi_i^\gamma(N \setminus C_h, v, \mathscr{C} \setminus \{C_h\}) \\
&= \mathrm{Sh}_i(N, v^{\mathscr{C}}) - \mathrm{Sh}_i(N \setminus C_h, v^{\mathscr{C}}) \\
&= P(N, \mathscr{C}) - P(N \setminus C_k, v^{\mathscr{C}}) - P(C_k - i, v^{\mathscr{C}}) \\
&\quad - \Big(P(N \setminus C_h, v^{\mathscr{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathscr{C}}) - P(C_k - i, v^{\mathscr{C}}) \Big) \\
&= P(N, v^{\mathscr{C}}) - P(N \setminus C_k, v^{\mathscr{C}}) - \Big(P(N \setminus C_h, v^{\mathscr{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathscr{C}}) \Big) \\
&= P(N, v^{\mathscr{C}}) - P(N \setminus C_h, v^{\mathscr{C}}) - \Big(P(N \setminus C_k, v^{\mathscr{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathscr{C}}) \Big) \\
&= P(N, v^{\mathscr{C}}) - P(N \setminus C_h, v^{\mathscr{C}}) - P(C_h - j, v^{\mathscr{C}}) \\
&\quad - \Big(P(N \setminus C_k, v^{\mathscr{C}}) - P(N \setminus (C_k \cup C_h), v^{\mathscr{C}}) - P(C_h - j, v^{\mathscr{C}}) \Big) \\
&= \mathrm{Sh}_j(N, v^{\mathscr{C}}) - \mathrm{Sh}_j(N \setminus C_k, v^{\mathscr{C}}) \\
&= \psi_j^\gamma(N, v, \mathscr{C}) - \psi_j^\gamma(N \setminus C_k, v, \mathscr{C} \setminus \{C_k\}).
\end{aligned}
$$

□

This proposition means that, by the definition of ψ^γ, for every $C_k \in \mathscr{C}$ and for every $C_h \in \mathscr{C}$,

$$
\frac{\mathrm{Sh}_k^w(M, v_{\mathscr{C}}) - \mathrm{Sh}_k^w(M \setminus \{h\}, v_{\mathscr{C}})}{|C_k|} = \frac{\mathrm{Sh}_h^w(M, v_{\mathscr{C}}) - \mathrm{Sh}_h^w(M \setminus \{k\}, v_{\mathscr{C}})}{|C_h|}.
$$

This is the special case of the properties of the w-weighted Shapley value: For $(N, v) \in \mathbf{G}$, its weight $(w_i)_{i \in N}$, and for every $i, j \in N$,[8]

$$
\frac{\mathrm{Sh}_i^w(N, v) - \mathrm{Sh}_i^w(N \setminus \{j\}, v)}{w_i} = \frac{\mathrm{Sh}_j^w(N, v) - \mathrm{Sh}_j^w(N \setminus \{i\}, v)}{w_j}.
$$

Next theorem shows that Balanced Contributions and Collective Balanced Contributions are almost sufficient to characterize ψ^γ.

[8]This property is pointed out in Hart and Mas-Colell (1989) and Amer and Carreras (1997).

Theorem 5. *ψ^γ is a unique efficient solution satisfying the following two properties:*

(i) Balanced Contributions.
(ii) Collective Balanced Contributions.

Proof. We have known that ψ^γ satisfies the efficiency, Balanced Contributions and Collective Balanced Contributions. Hence we will show the converse.

Let ψ be an efficient solution satisfying these two axioms. Fix $(N,v,\mathscr{C}) \in \mathbf{G}^c$. We first show that ψ coincides with the Shapley value when $|\mathscr{C}| = 1$ or n. When $|\mathscr{C}| = n$, Collective Balanced Contributions coincides with the balanced contributions. Because of the result of Myerson (1980), $\psi(N,v,[N]) = \mathrm{Sh}(N,v)$. Moreover, by Balanced Contributions, the same argument means that $\psi(N,v,\{N\}) = \psi(N,v,[N]) = \mathrm{Sh}(N,v)$.

Next we show the following claims.

Claim 1: For all $C_k \in \mathscr{C}$,

$$\sum_{i \in C_k} \psi_i(N,v,\mathscr{C}) = |C_k| D_k P^w(M, v_{\mathscr{C}}) \tag{2}$$

where P^w is the HM w-weighted potential function with weight vector $w = (w_k)_{k \in M}$ such that $w_k = |C_k|$ for each $k \in M$.

Let $(C_k, v, \{C_k\})$ be a subgame of $(N,v,\mathscr{C})$ to coalition C_k. Then the left hand side of (2) is

$$\sum_{i \in C_k} \psi_i(C_k, v, \{C_k\}) = v(C_k)$$

by the efficiency of ψ. The right hand side of (2) is

$$|C_k| D_k P^w(\{k\}, v_{\mathscr{C}}) = |C_k| \frac{v(C_k)}{|C_k|} = v(C_k).$$

Thus, condition (2) holds true for any subgame $(C_k, v, \{C_k\})$ of $(N,v,\mathscr{C})$.

We assume that (2) is satisfied for any $(N', v, \mathscr{C}')$ such that $L \subsetneq M$, $N' = \cup_{k \in L} C_k$ and $\mathscr{C}' = \{C_k : k \in L\}$. We now show that it holds true for $(N,v,\mathscr{C})$.

Condition (2) is equivalent to

$$\sum_{i \in C_k} \psi_i(N,v,\mathscr{C}) = |C_k| (P^w(M, v_{\mathscr{C}}) - P^w(M \setminus \{k\}, v_{\mathscr{C}})).$$

Then we obtain

$$P^w(M, v_{\mathscr{C}}) = \frac{\sum_{i \in C_k} \psi_i(N,v,\mathscr{C})}{|C_k|} + P^w(M \setminus \{k\}, v_{\mathscr{C}}).$$

We show that $\frac{\sum_{i \in C_k} \psi_i(N,v,\mathscr{C})}{|C_k|} + P^w(M \setminus \{k\}, v_{\mathscr{C}})$ is constant for every $k \in M$. Take any $C_k \in \mathscr{C}$ and $C_h \in \mathscr{C}, C_k \neq C_h$. Then,

$$\frac{\sum_{i \in C_k} \psi_i(N,v,\mathscr{C})}{|C_k|} + P^w(M \setminus \{k\}, v_{\mathscr{C}}) - \left(\frac{\sum_{j \in C_h} \psi_j(N,v,\mathscr{C})}{|C_h|} + P^w(M \setminus \{h\}, v_{\mathscr{C}}) \right)$$

equals

$$\frac{\sum_{i\in C_k}\psi_i(N,v,\mathscr{C})}{|C_k|}-\frac{\sum_{j\in C_h}\psi_j(N,v,\mathscr{C})}{|C_h|}+\Big(P^w(M\setminus\{k\},v_{\mathscr{C}})-P^w(M\setminus\{h\},v_{\mathscr{C}})\Big). \tag{3}$$

The bracketed terms in (3) equals

$$P^w(M\setminus\{k\},v_{\mathscr{C}})-P^w(M\setminus\{k,h\},v_{\mathscr{C}})-\big(P^w(M\setminus\{h\},v_{\mathscr{C}})-P^w(M\setminus\{k,h\},v_{\mathscr{C}})\big).$$

By the definition of operator D and the assumption,

$$\begin{aligned}&=D_hP^w(M\setminus\{k\},v_{\mathscr{C}})-D_kP^w(M\setminus\{h\},v_{\mathscr{C}})\\&=\frac{\sum_{j\in C_h}\psi_j(N\setminus C_k,v,\mathscr{C}\setminus\{C_k\})}{|C_h|}-\frac{\sum_{i\in C_k}\psi_i(N\setminus\{C_h\},v,\mathscr{C}\setminus\{C_h\})}{|C_k|}.\end{aligned}$$

Substitute the above for the bracketed terms in (3), and we obtain

$$\frac{\sum_{i\in C_k}\psi_i(N,v,\mathscr{C})}{|C_k|}-\frac{\sum_{j\in C_h}\psi_j(N,v,\mathscr{C})}{|C_h|}+\frac{\sum_{j\in C_h}\psi_j(N\setminus C_k,v,\mathscr{C}\setminus\{C_k\})}{|C_h|}-\frac{\sum_{i\in C_k}\psi_i(N\setminus\{C_h\},v,\mathscr{C}\setminus\{C_h\})}{|C_k|}.$$

Note that by Collective Balanced Contributions, $\psi_i(N,v,\mathscr{C})-\psi_i(N\setminus\{C_h\},v,\mathscr{C}\setminus\{C_h\})=\psi_j(N,v,\mathscr{C})-\psi_j(N\setminus\{C_k\},v,\mathscr{C}\setminus\{C_k\})$ is constant for every $i\in C_k$ and for every $j\in C_h$. Hence the above expression is zero and thus, (3) equals zero.

Therefore for some real number K,

$$\frac{\sum_{i\in C_k}\psi_i(N,v,\mathscr{C})}{|C_k|}+P^w(M\setminus\{k\},v_{\mathscr{C}})=K$$

holds true for any $k\in M$.

Then by efficiency of ψ, we obtain that

$$v_{\mathscr{C}}(M)=v(N)=\sum_{k\in M}\sum_{i\in C_k}\psi_i(N,v,\mathscr{C})=\sum_{k\in M}|C_k|(K-P^w(M\setminus\{k\},v_{\mathscr{C}}))$$

Therefore K is exactly the HM weighted potential function $P^w(M,v_{\mathscr{C}})$ because of its uniqueness.

Next we show the following claim.

Claim 2: $\psi_i(N,v,\mathscr{C})=\bar{C}+\psi_i(C_k,v,\{C_k\})$ for every $i\in C_k$ where $\bar{C}$ is a constant real number.

We prove Claim 2 by the induction on the cardinality of $\mathscr{C}$. When $|\mathscr{C}|=1$, this is obvious because we simply put $\bar{C}=0$.

Assume that the claim holds true when the number of elements in $\mathscr{C}$ is less than $m\,(m\geqq 2)$. For $(N,v,\mathscr{C})$ such that $|\mathscr{C}|=m$, by Collective Balanced Contributions, given $C_h\in\mathscr{C}$, we have

$$\psi_i(N,v,\mathscr{C})-\psi_i(N\setminus\mathscr{C}_h,v,\mathscr{C}\setminus\{C_h\})=\bar{C_1}\quad\text{for every } i\in C_k, C_k\neq C_h$$

By the assumption of the induction, the left hand side of the above equation is

$$\psi_i(N,v,\mathscr{C}) - (\bar{C_2} + \psi_i(C_k,v,\{C_k\})),$$

where $\bar{C_2}$ is constant for all $i \in C_k$. Therefore we obtain

$$\psi_i(N,v,\mathscr{C}) = \bar{C_1} + \bar{C_2} + \psi_i(C_k,v\{C_k\}) = \bar{C} + \psi_i(C_k,v,\{C_k\}).$$

This is the desired result.

By Claim 1, we know that the summation of $\psi_i(N,v,\mathscr{C})$ over $i \in C_k$ is exactly $|C_k| D_k P^w(M,v_{\mathscr{C}}) = \mathrm{Sh}_k^w(M,v_{\mathscr{C}})$. Then we conclude that

$$\bar{C} = \frac{\mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - \sum_{i\in C_k} \psi_i(C_k,v,\{C_k\})}{|C_k|} = \frac{\mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - v(C_k)}{|C_k|}$$

by efficiency of ψ. Therefore if $\psi_i(C_k,v,\{C_k\})$ is uniquely determined, $\psi(N,v,\mathscr{C})$ is also determined. However when $|\mathscr{C}| = 1$, we have shown that ψ equals the Shapley value Sh. Hence we obtain

$$\psi_i(N,v,\mathscr{C}) = \frac{\mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - v(C_k)}{|C_k|} + \mathrm{Sh}_i(C_k,v).$$

□

As in the proof of Theorem 5, Balanced Contributions is necessary only to prove that if $\mathscr{C} = \{N\}$, the solution coincides with the Shapley value for (N,v). Thus the following corollaries also hold.

Corollary 1. *ψ^γ is a unique efficient solution satisfying the following two properties:*

(i) $\psi(N,v,\{N\}) = \mathrm{Sh}(N,v)$ for all $(N,v) \in \Gamma$.
(ii) Collective Balanced Contributions.

Corollary 2. *ψ^γ is a unique efficient solution satisfying the following two properties:*

(i) Coincidence between the Grand and the Singleton Coalition Structure: For all $(N,v) \in \Gamma$, $\psi(N,v,\{N\}) = \psi(N,v,[N])$,
(ii) Collective Balanced Contributions.

6.2 Additivity

In this subsection, we provide an axiomatization of ψ^γ through the additivity axiom. Let ψ be a solution on $\mathbf{G}^c$. Let $(N,v,\mathscr{C}),(N,v',\mathscr{C}) \in \mathbf{G}^c$.

Theorem 6. *ψ^γ is a unique efficient solution on $\mathbf{G}^c$ satisfying the following four axioms.*

(i) Additivity: $\psi(N,v,\mathscr{C}) + \psi(N,v',\mathscr{C}) = \psi(N,v+v',\mathscr{C})$, where $(v+v')(S) = v(S)+v'(S)$ for all $S \subseteq N$.

(ii) Equal Power of Partnership Members: If $T \subseteq N$ is a partnership in (N,v) and $M' = \{k \in M : C_k \cap T \neq \emptyset\}$ is also a partnership in $(M,v_{\mathscr{C}})$, then $\psi_i(N,v,\mathscr{C}) = \psi_j(N,v,\mathscr{C})$ for any $i,j \in T$.

(iii) Strong Restricted Equal Treatment Property: If $i \in C_k$ and $j \in C_k$ are symmetric in (C_k, v), then $\psi_i(N,v,\mathscr{C}) = \psi_j(N,v,\mathscr{C})$.

(iv) Coalition Structure-Null Player: If C_k is a dummy coalition (i.e., k is a dummy player in $(M, v_{\mathscr{C}})$) and $i \in C_k$ is a null player in (N,v), then $\psi_i(N,v,\mathscr{C}) = 0$. .

Equal Power of Partnership Members says that all the members of partnership T obtain the equal payoff if its projection on M is also a partnership in $(M, v_{\mathscr{C}})$. The *extended* Shapley value defined by $\overline{\text{Sh}}(N,v,\mathscr{C}) = \text{Sh}(N,v)$ satisfies Equal Power of Partnership Members because all the players in T are symmetric in (N,v) and the Shapley value assigns equal payoff to symmetric players.

Strong Restricted Equal Treatment and Coalition Structure-Null Player are the axioms introduced in Kamijo (2007) to characterize ψ^{δ} (see Remark 2 of this paper). Strong Restricted Equal Treatment is stronger than the restricted equal treatment property which both the extended Shapley value and the Owen's coalitional value satisfy. Coalition Structure-Null Player is weaker than the usual null player axiom. Thus the extended Shapley value satisfies the all the properties except for Strong Restricted Equal Treatment and the Owen's coalitional value does not satisfies Equal Power of Partnership Members and Strong Restricted Equal Treatment.

The next lemma is from Kalai and Samet (1987).

Lemma 1. *Let $w \in \mathbb{R}^N_{++}$ be a weight vector of N. If T is a partnership in (N,v), then $\text{Sh}^w_i(N,v)/w_i = \text{Sh}^w_j(N,v)/w_j$ for all $i, j \in T$.*

Proof. See the proof of Theorem 2 of Kalai and Samet (1987). □

Lemma 2. *Let ϕ be a solution on $\boldsymbol{G}$ such that it satisfies the symmetry and the null player axioms. Let ψ be a two step solution on $\boldsymbol{G}^c$ defined by*

$$\psi_i(N,v,\mathscr{C}) = \frac{\text{Sh}^w_k(M,v_{\mathscr{C}}) - v(C_k)}{|C_k|} + \phi_i(C_k,v).$$

for all $(N,v,\mathscr{C}) \in \boldsymbol{G}^c$ and for all $i \in C_k \in \mathscr{C}$, where $w = (w_k)_{k \in M}$ is such that $w_k = |C_k|$ for all $k \in M$. Then, ψ satisfies Equal Power of Partnership Members.

Proof. Let $T \subseteq N$ be a partnership in (N,v) and $M' = \{k \in M : C_k \cap T \neq \emptyset\}$ be also a partnership in $(M, v_{\mathscr{C}})$. Suppose $|M'| \geqq 2$. Let $k \in M'$. Since T is a partnership in (N,v), $v(S \cup C) = v(S)$ for any $S \subseteq C_k \setminus T$ and $C \subseteq T \cap C_k \subsetneq T$. Thus, for any $S \subseteq C_k$, $v(S) = v(S \cap (C_k \setminus T))$ and thus, any $i \in T \cap C_k$ is a null player in subgame (C_k, v). So $\varphi_i(C_k,v) = 0$ for any $i \in T \cap C_k$ since φ satisfies the null player axiom. Because $w_k = |C_k|$ for any $k \in M$, $\frac{\text{Sh}^w_k(M,v_{\mathscr{C}})}{|C_k|} = \frac{\text{Sh}^w_h(M,v_{\mathscr{C}})}{|C_h|}$ for any $k, h \in M'$ by Lemma 1. By the partnership of M' in $(M, v_{\mathscr{C}})$, $v_{\mathscr{C}}(\{k\}) = v(C_k) = 0$ for each $k \in M'$. Thus $\psi_i(N,v,\mathscr{C}) = \psi_j(N,v,\mathscr{C})$ holds for any $i, j \in T$.

Suppose $|M'| = 1$ and let $k \in M'$. Since $T \subseteq C_k$ is a partnership in (N,v), all the players in T are symmetric in (N,v) and, of course, they are symmetric in (C_k,v). Thus $\text{Sh}_i(C_k,v)$ is constant over $i \in T$. Hence ψ satisfies Equal Power of Partnership Members. □

Proof of Theorem 6. From Lemma 2, we have known that ψ^{γ} satisfies Equal Power of Partnership Member since the Shapley value satisfies the symmetry and the null player axioms. Furthermore, it is obvious that it satisfies axioms (i) to (iii) by its definition.

Next we show the converse part. Let ψ be an efficient solution on $\mathbf{G}^c$ which satisfies axioms (i) to (iv). Let $(N,v,\mathscr{C}) \in \mathbf{G}^c$. Since ψ satisfies Additivity, it is sufficient to show that $\psi(N,cu_T,\mathscr{C})$ is uniquely determined for any $T \subseteq N$, where $c \in \mathbb{R}$ and cu_T is a scalar multiple of u_T by c. Let $D = \{k \in M : C_k \cap T \neq \emptyset\}$. Since $C_k \in \mathscr{C}$, $k \notin D$, is a dummy coalition and $i \in C_k$ is a null player, $\psi_i(N,cu_T,\mathscr{C}) = 0$ by Coalition Structure-Null Player. Thus, efficiency means that $\sum_{k \in D} \sum_{i \in C_k} \psi_i(N,cu_T,\mathscr{C}) = c$.

Clearly T is a partnership in (N,cu_T) and D is also a partnership in $(M,(u_T)_{\mathscr{C}})$. Therefore $\psi_i(N,cu_T,\mathscr{C}) = \psi_j(N,cu_T,\mathscr{C})$ for all $i,j \in T$ by Equal Power of Partnership Members.

Case a: $|D| = 1$. Let $k \in D$. Since C_k is a dummy coalition and $i \in C_k \setminus T$ is a null player, $\psi_i(N,cu_T,\mathscr{C}) = 0$ by Coalition Structure-Null Player. Thus, $\psi_i(N,cu_T,\mathscr{C}) = \frac{c}{|T|}$.

Case b: $|D| \geqq 2$. For each $C_k \in \mathscr{C}$, $k \in D$, $i \in C_k$ and $j \in C_k$ are symmetric in (C_k,v). Therefore $\psi_i(N,cu_T,\mathscr{C}) = \psi_j(N,cu_T,\mathscr{C})$ by Strong Restricted Equal Treatment Property. Moreover $\psi_i(Ncu_T,\mathscr{C}) = \psi_j(N,cu_T,\mathscr{C})$ for $i \in T \cap C_k$ and for $j \in T \cap C_h$. As a result, for any $i \in \cup_{k \in D} C_k$, $\psi_i(N,cu_T,\mathscr{C}) = \frac{c}{\sum_{h \in D} |C_h|}$. □

Remark 4. *The efficiency of a solution is derived from the four axioms in Theorem 6. In fact, consider a solution ψ satisfying these four. The main logic is similar to Theorem 8.1.3 of Peleg and Sudhölter (2003). Let (N,v^0) be zero-game such that $v^0(S) = 0$ for any $S \subseteq N$ and $\mathscr{C}$ be a coalition structure on N. Then, $\psi(N,v^0,\mathscr{C})$ must be $0^N \in \mathbb{R}^N$ by Coalition Structure-Null Player. Let $(N,v,\mathscr{C}) \in \boldsymbol{G}^c$. By Additivity, $\psi(N,v,\mathscr{C}) + \psi(N,-v,\mathscr{C}) = \psi(N,v-v,\mathscr{C}) = \psi(N,v^0,\mathscr{C}) = 0^N$ and thus, $\psi(N,v,\mathscr{C}) = -\psi(N,-v,\mathscr{C})$ holds. Since the payoff proposed by a solution must be feasible, $\sum_{i \in N} \psi_i(N,v,\mathscr{C}) \leqq v(N)$ and $\sum_{i \in N} \psi_i(N,v,\mathscr{C}) = -\sum_{i \in N} \psi_i(N,-v,\mathscr{C}) \geqq -(-v(N))$. Thus, $\sum_{i \in N} \psi_i(N,v,\mathscr{C}) = v(N)$ holds.*

Example 1. *The following solutions show the independence of each axiom from the others (except the efficiency) in Theorem 6. Let $(N,v,\mathscr{C}) \in \boldsymbol{G}^c$.*

(i) Consider a solution ψ^n defined by

$$\frac{\mathrm{Sh}_k^w(M,v_{\mathscr{C}}) - v(C_k)}{|C_k|} + \mathrm{Nu}_i^n(C_k,v)$$

where $w \in \mathbb{R}_{++}^M$ is such that $w_k = |C_k|$ and Nu^n is the nucleolus introduced by Schmeidler (1969). Since φ^n satisfies the symmetry and the null player axioms, ψ^n satisfies Equal Power of Partnership Member by Lemma 2. Thus, ψ^n satisfies Strong Restricted Equal Treatment, Equal Power of Partnership Members and Coalition Structure-Null Player since Nu *satisfies the symmetry and the null player axioms, but the additivity since φ^n does not satisfy the additivity.*

(ii) The extended Shapley value satisfies all the axioms except for Strong Restricted Equal Treatment.

(iii) ψ^δ is characterized by Additivity, Strong Restricted Equal Treatment, Coalition Structure-Null Player and Coalitional Symmetry which is defined by, if $k \in M$ and $h \in M$ are symmetric in $(M,v_{\mathscr{C}})$, then $\sum_{i \in C_k} \psi_i(N,v,\mathscr{C}) = \sum_{i \in C_h} \psi_i(N,v,\mathscr{C})$. Since ψ^δ and ψ^γ are the different solutions, Equal Power of Partnership Members is independent of the other axioms.

(iv) The egalitarian solution defined by $\psi_i^e(N,v,\mathscr{C}) = \frac{v(N)}{|N|}$ for all $i \in N$ satisfies all the axioms except for Coalition Structure-Null Player.

7 Concluding remarks

Recently, Vidal-Puga (2005) considered another value on games with coalition structures from a viewpoint of non-cooperative bargaining among the players. This solution also satisfies the condition that the sum of the payoffs of the players in C_k coincides with the weighted Shapley value of player k for the external game with coalition-size weights. Vidal-Puga (2005) states that a generation of coalition size weights is due to "right to talk" of players. In contrast, in this paper, we show that the generation of coalition size weights is due to communication restriction by coalitions.

Finally, our solution can be extended to games with levels structures introduced by Winter (1989). Levels structure on N is a finite sequence of coalition structures, $\mathscr{C}^0, \dots, \mathscr{C}^l$ with $\mathscr{C}^0 = [N]$ and $\mathscr{C}^l = \{N\}$ such that if $k < h$, $\mathscr{C}^k$ is a finer coalition structure than $\mathscr{C}^h$. Consider the levels structure for six person game described by Table 1. Then, the payoff for player 1 is calculated as the following way.

level		coalition structure
3	$\mathscr{C}^3$	$\{\{1,2,3,4,5,6\}\}$
2	$\mathscr{C}^2$	$\{\{1,2,3\},\{4,5\},\{6\}\}$
1	$\mathscr{C}^1$	$\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$
0	$\mathscr{C}^0$	$\{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$

Table 1: Levels structure on $N = \{1,2,3,4,5,6\}$

First, in level $\mathscr{C}^2$, coalitions $\{1,2,3\}$, $\{4,5\}$ and $\{6\}$ bargain for the division of $v(N)$. As a result, coalition $\{1,2,3\}$ obtains $\mathrm{Sh}^w_1(M^2_1, v_{\mathscr{C}^2})$ where $M^2_1 = \{1,2,3\}$ and $w_1 = 3$, $w_2 = 2$ and $w_3 = 1$. Then, player 1 receives his dividend for this bargaining surplus, that is, $\frac{\mathrm{Sh}^w_1(M^2_1, v_{\mathscr{C}^2}) - v(\{1,2,3\})}{|\{1,2,3\}|}$. Next, in level $\mathscr{C}^1$, coalitions $\{1,2\}$ and $\{3\}$ bargain for the division of $v(\{1,2,3\})$ and $\{1,2\}$ obtains $\mathrm{Sh}^w_1(M^1_1, v_{\mathscr{C}^1})$ where $M^1_1 = \{1,2\}$ and $w_1 = 2$ and $w_2 = 1$. Player 1 receives $\frac{\mathrm{Sh}^w_1(M^1_1, v_{\mathscr{C}^1}) - v(\{1,2\})}{|\{1,2\}|}$. Finally, in level $\mathscr{C}^0$, players 1 and 2 bargain for the division of $v(\{1,2\})$ and player 1 obtains $\mathrm{Sh}_1(\{1,2\}, v)$. Therefore, the payoff for player 1 is

$$\frac{\mathrm{Sh}^w_1(M^2_1, v_{\mathscr{C}^2}) - v(\{1,2,3\})}{|\{1,2,3\}|} + \frac{\mathrm{Sh}^w_1(M^1_1, v_{\mathscr{C}^1}) - v(\{1,2\})}{|\{1,2\}|} + \mathrm{Sh}_1(\{1,2\}, v).$$

Generally, let $(N, v, \mathscr{L})$ be a game with levels structure where $(N, v) \in \mathbf{G}$ and $\mathscr{L} = \{\mathscr{C}^0, \dots, \mathscr{C}^l\}$ is a levels structure on N. For each $k = 0, \dots, l$, let $\mathscr{C}^k = \{C^k_1, \dots, C^k_{m_k}\}$ and $M^k = \{1, \dots, m_k\}$. For given $i \in N$, let $i(k)$ denote a coalitional index of coalition of level k which player i belongs to, *i.e.*, $C^k_{i(k)} \in \mathscr{C}^k$ and $i \in C^k_{i(k)}$. Further, put $M^k_i = \{h \in M^k \ : C^k_h \subseteq C^{k+1}_{i(k+1)}\}$ and $w^k_h = |C^k_h|$ for all $h \in M^k_i$. Of course, $i(k) \in M^k_i$. Let $(M^k_i, v_{\mathscr{C}^k})$ be a subgame of $(M^k, v_{\mathscr{C}^k})$ on M^k_i.

Definition 3. *A value ψ^{γ} for $(N,v,\mathscr{L})$ is defined by*

$$\psi_i^{\gamma}(N,v,\mathscr{L}) = \sum_{k=0}^{l-1} \frac{\mathrm{Sh}_{i(k)}^{w^k}(M_i^k, v_{\mathscr{C}^k}) - v(C_{i(k)}^k)}{|C_{i(k)}^k|} + v(\{i\})$$
$$= \sum_{k=1}^{l-1} \frac{\mathrm{Sh}_{i(k)}^{w^k}(M_i^k, v_{\mathscr{C}^k}) - v(C_{i(k)}^k)}{|C_{i(k)}^k|} + \mathrm{Sh}_i(C_{i(1)}^1, v)$$

for all $i \in N$.

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