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Robust Sequential Implementation and Maskin Monotonicity

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Robust Sequential Implementation and Maskin Monotonicity [*]

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Abstract

We study social choice functions that are robustly implemented in sequential equilibria of extensive-form games on all common prior type spaces. We show that the combination of strategy-proofness, Maskin monotonicity, and no veto power is a sufficient condition for these implementation. Unlike robust implementation by normal-form games, desirable social choice functions can be implemented even in single-peaked voting or quasi-linear preferences environments.

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1 Introduction

This paper considers the problem of fully implementing a social choice function (SCF) in incomplete information environments. In the spirit of "Wilson doctrine" (Wilson [1987]), we consider full implementation where mechanisms do not rely on the features of agents' common knowledge about probability distributions on the agents' types in the same way of "robust implementation" by Bergemann and Morris [2005a,b]. We examine implementation by extensive-form games, where the mechanism designers construct extensive-form games as their mechanisms not only normal-form games. Until now, robust implementation has been studied only in implementation by normal-form games. However, in the literature of full implementation, it is known that if designers can design extensive-form games, then the larger class of SCFs can be implemented. Our purpose is to determine whether this valuable property is applicable in the case of robust implementation.

In the literature of full implementation with complete information Maskin [1999] analyses implementation problems where an SCF must be attained by all Nash equilibria refereed to as "Nash implementation". He provides a necessary and almost sufficient

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condition for Nash implementation, known as *Maskin monotonicity*. Since Maskin's formulation assumes that the agents have perfect information, the researches has been dealing with implementation theory with incomplete information (see Postlewaite and Schmeidler [1986], Palfrey and Srivastava [1989], Jackson [1991]). In incomplete information environments, the agents' types are private information and they have beliefs about the other agents' types.

In the literature of incomplete information environments, the designers has been allowed to use the information what beliefs the agents had. Many researchers have criticized these assumptions about information since, in realistic implementation problems, the mechanism designers may not have the detail information of the agents' prior beliefs. In a novel approach, Bergemann and Morris [2005b,a] propose a new implementation concept, referred to as robust implementation. Robust implementation requires that the implementation result of a mechanism must be robust to the assumption of the information about the agents. They show that strategy-proofness is a necessary condition for robust implementation of SCFs with private values.[1] In addition, Saijo et al. [2007] reveal a necessary and sufficient condition for robust implementation. Strategy-proofness in itself does not imply the negative results for social decision in some economic problems such as auction environments or single-peaked preferences environments. However, Saijo et al. [2007] reveal that robust implementation is stronger than strategy-proofness; in fact, no SCFs can be robustly implemented in the above two environments.

On the other hand, implementation by extensive-form games is an well-known method to fully implement the larger class of SCFs than implementation by normal-form games. If the agents have sequential rationality and the designer can design the mechanisms as extensive-form games, the designer can exclude undesirable equilibria by the equilibrium refinement and fully implement the larger class of SCFs.[2] Our purpose is to study robust sequential implementation; it is robust implementation in sequential equilibria, which is a common solution concept of extensive-form games developed by Kreps and Wilson [1982]. It corresponds to robust implementation in Bergemann and Morris [2005a], which alternatively employs the Bayesian equilibria of normal-form games. In the literature of extensive-form implementation with incomplete information, many of the studies revealed sufficient conditions on SCFs to be implemented by extensive-form games by restricting their analyses to the environments such as economic or private values environments.

Some researchers point out that implementation by extensive-form games is more suitable in order not to depend on the agents' prior beliefs than normal-form games. Bergin and Sen [1998] emphasize that, in extensive-form implementation, the designer can use posterior beliefs even if the SCF cannot be implemented by using only prior beliefs. Duggan [1998] analyzes, with quasi-linear and private values preferences. a sufficient condition to implement the designer's second best outcomes in sequential

[1] Bergemann and Morris [2005b] show that ex post incentive compatibility is necessary for *partial implementation* with robustness, which requires that there exists at least one equilibrium that achieves the SCF. In private value environments, ex post incentive compatibility is equivalent to strategy-proofness.

[2] In the literature of implementation by extensive-form games, Moore and Repullo [1988] and Abreu and Sen [1990] study implementation with complete information and Baliga [1999] and Brusco [2006] study implementation with incomplete information.

equilibria on all the agents' prior beliefs when the designer permits arbitrarily small perturbations of his SCF. their approaches are actually independent of robust implementation but their aim is the same as ours. We would like to answer the question whether extensive-form implementation is robust to the assumption on information.

We provide a relatively simple sufficient condition for robust sequential implementation by restricting our analysis to the case of private values; an SCF can be robustly sequentially implemented if the SCF satisfies strategy-proofness, Maskin monotonicity, and no veto power. This sufficient condition implies that robust sequential implementation is significantly weaker than robust implementation by normal form games. The mechanism in the proof of sufficiency theorem is similar to those of the studies in this literature; it is composed of at most two rounds of announcements by the agents and needs only one round in any equilibrium paths. [3] We consider the implication of our sufficient condition in two kinds of economically important environments: the environments with single-peaked preferences and quasi-linear preferences. We show that, with single-peaked preferences, the median voter rule environments can be robustly sequentially implemented, and, with quasi-linear preferences, there exists at least one surplus-maximizing SCF that can be robustly sequentially implemented. These results contrast with the previously described results of robust implementation with normal-form games.

The paper is organized as follows. Section 2 discusses the basic notation and defines robust implementation by extensive-form games. Section 3 provides a sufficient condition for this implementation and considers our sufficient condition in single-peaked preference environments and quasi-linear preferences environments. Section 4 concludes the paper.

2 Notation and Definitions

We consider a finite set of agents, $\mathcal{N} \equiv \{1, 2, \ldots, N\}$. Let $N \geq 3$. X is the set of *outcomes*, the objects of social choice. For each $i \in \mathcal{N}$, agent i's utility function is $u_i : X \to \mathbb{R}$, and the class of the possible utility functions of agent i is U_i. Let $u = (u_1, u_2, \ldots, u_N)$ and $U \equiv \times_{i \in \mathcal{N}} U_i$. U is a countable set. It should be clear that preferences depend only on an agent's own information and not those of other agents. We assume that preferences satisfy the von Neumann and Morgenstern axioms.

An SCF is a function $f : U \to X$, which associates with each $u \in U$ a unique social optimal $f(u)$ in X.

A Type space

$t_i \in T_i$ is agent i's type, where T_i is a countable set for each i. Let $T = \times_{i \in N} T_i$. A type of agent i includes a description of his preference. Thus, there is a function $\hat{u}_i : T_i \to U_i$ with $\hat{u}_i(t_i)$ being agent i's utility function when his type is t_i. Let $\hat{u}(t) = (\hat{u}_1(t_1), \hat{u}_2(t_2), \ldots, \hat{u}_N(t_N))$. A type of agent i also includes a description of his beliefs

[3] In this regard, our mechanism is not a multi-stage game with complete information unlike many of the studies. Brusco [1995] and Brusco [2006] also uses the mechanism that is not a multi-stage game in his sufficiency theorem.

about the types of the other agents; we describe $\hat{\pi}_i(t_i)[t_{-i}]$ as the probability assigned by type t_i to the other agents' type profile t_{-i}. Now, a *type space* is a collection:

$$\mathcal{T} = (T_i, \hat{u}_i, \hat{\pi}_i)_{i \in N}.$$

We assume that a type space $\mathcal{T} = (T_i, \hat{u}_i, \hat{\pi}_i)_{i \in N}$ satisfies the *common prior assumption*.[4] That is, there exists $p \in \Delta(T)$ such that, for each i and t_i,

$$\sum_{t'_{-i} \in T_{-i}} p(t_i, t'_{-i}) > 0$$

and

$$\hat{\pi}_i(t_i)[t_{-i}] = \frac{p(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}} p(t_i, t'_{-i})}.$$

Note that there is generally no relation between a preference type set U and a type set T, which is defined by a type space $\mathcal{T} = (T_i, \hat{u}_i, \hat{\pi}_i)_{i \in N}$; It depends on the property of a type space $\mathcal{T}$. That is, there may exist a profile $u \in U$ such that $\hat{u}(t) \neq u$ for each $t \in T$ and there may exist a distinct pair of type profiles $t, t' \in T$ such that $\hat{u}(t) = \hat{u}(t')$.[5]

The concept of a type space helps us to analyze the implementation problems significantly robust to the assumption on the agents' private information, as we will see later.

Extensive-form games and equilibrium

A mechanism $\mathcal{M}$ is an extensive-form game with incomplete information. A detailed description of extensive-form game is provided in Selten [1975].[6] $h \in H$ is a history in the $\mathcal{M}$, where H constitutes the set of histories. The set of terminals is a subset $\overline{H} \in H$. An outcome function assigns an element in X to each terminal. For agent i, an information set $I_i \in \mathcal{I}_i$ is a subset of H; each non-terminal history h is an element of exactly one information set of some agent. Each information set of an agent identifies the set of histories that are indistinguishable for the agent when he reaches one of the histories. An action set is assigned to each information set of the agents.

Given a type space $\mathcal{T} = (T_i, \hat{u}_i, \hat{\pi}_i)_{i \in N}$, a pair $(\mathcal{M}, \mathcal{T})$ denotes an (extensive-form) game.

Let $\sigma_i \in \Sigma_i$ be a behavior strategy of agent i; σ_i assigns a probability distribution on the action set to each pair $(I_i, t_i) \in \mathcal{I}_i \times T_i$. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$.

μ_i is a belief system of agent i. It assigns to each pair $(I_i, t_i) \in \mathcal{I}_i \times T_i$ a probability distribution on $I_i \times T_{-i}$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$.

We state "a belief profile μ is *Bayes consistent* with a strategy profile σ" if and only if beliefs are updated from an information set to the next information set by using Bayes' rule whenever possible.

[4] Although we require the common prior in accordance with the standard definition of sequential equilibrium, our sufficient theorem can be obtained with larger type spaces. In fact, the proof does not depend the commonness of the agents' prior distributions.

[5] Bergemann and Morris [2005b] and Bergemann and Morris [2005a] provides a more detailed analysis of type spaces

[6] For more details of an extensive-form mechanism, refer to Bergin and Sen [1998].

We write $\mathrm{E}(\hat{u}_i)[\sigma|\mu_i(I_i, t_i), (I_i, t_i)]$ as the expected utility of type t_i with belief μ_i when information set I_i is attained and strategy profile σ are employed by the agents.

A *sequential equilibrium (or equilibrium) assessment* (σ, μ) on game $(\mathcal{M}, \mathcal{T})$ is a pair of a strategy profile and belief profile that satisfies the following two conditions:

Sequential rationality For each $i \in \mathcal{N}$, $I_i \in \mathcal{I}_i$, $t_i \in T_i$, and $\sigma'_i \in \Sigma_i$,

$$\mathrm{E}(\hat{u}_i)[\sigma|\mu_i(I_i, t_i), I_i, t_i] \geq \mathrm{E}(\hat{u}_i)[(\sigma'_i, \sigma_{-i})|\mu_i(I_i, t_i), I_i, t_i].$$

Consistency There exists a sequence $(\sigma^n)_{n=1}^{\infty}$ of proper mixed strategies converging to σ with Bayes consistent beliefs $(\mu^n)_{n=1}^{\infty}$ converging to μ.

A proper mixed strategy is a strategy that assigns positive probabilities to all actions in each action set at each pair $(I_i, t_i) \in \mathcal{I}_i \times T_i$.

Let $\mathrm{SEO}((\mathcal{M}, \mathcal{T}), t)$ be the set of sequential equilibrium outcomes of game $(\mathcal{M}, \mathcal{T})$ at type t.[7]

Robust sequential implementation

Note that our definitions about an extensive-form mechanism and sequential equilibrium are basically equivalent to the definitions in the traditional studies of extensive-form implementation, except we describe them with a concept of a *type space* expressly. In the traditional notation, a type space is treated as given; the description of what type set each agent has and what beliefs each agent has on the other agents' type profiles are treated as a part of the environment. On the other hand, we separate the descriptions from the environments and define a type space expressly.

The definition of *non-robust* extensive-form implementation can be described with a concept of a type space as follows:

Definition 1 (Sequential Implementation)**.** *A mechanism $\mathcal{M}$ sequentially implements an SCF f on a type space $\mathcal{T} = (T_i, \hat{u}_i, \hat{\pi}_i)_{i \in N}$ if and only if $SEO((\mathcal{M}, \mathcal{T}), t) = f(\hat{u}(t))$ for each $t \in S$.*

This definition corresponds to the implementation using sequential equilibria in Bergin and Sen [1998] and Baliga [1999].[8] It requires full implementation; every sequential equilibrium must attain the SCF. And, it is not robust; whether a mechanism $\mathcal{M}$ sequentially implements an SCF depends on the type space, since a type space influences the game that the agents will play and the sequential equilibrium of the game.

Next, we consider robust implementation.

Definition 2 (Robust Sequential Implementation on Common Prior Type Spaces)**.** *A mechanism $\mathcal{M}$ robustly sequentially implements an SCF f if and only if $\mathcal{M}$ sequentially*

[7] Although we define implementation by employing sequential equilibrium as the solution concept, our results are also satisfied by employing slightly weak solution concepts such as weak perfect Bayesian equilibrium Mas-Colell et al. [1995].

[8] Note that implementation is required only if type profiles is an element in S. That is, the designer wants to implement at type profile $t \in T$ only if t is considered possible by an agent in N. Our definition corresponds to *robust Bayesian implementation* in Saijo et al. [2007].

implements f on all type spaces. An SCF f is robustly sequentially implementable if and only if there exists a mechanism that robustly sequentially implements f on all type spaces.

Robust sequential implementation requires that the implementation result is robust on the assumption of the type space.Robust implementation is stronger than implementation with all prior beliefs on preference type set U. Since each type space determines a type set independently of U, robust implementation can consider more general structures of private information than U.

3 Implementation and Monotonicity

Now, we introduce some conditions associated with robust implementability.

SP requires that truth-telling is a dominant strategy in a direct revelation game for each agent.

Definition 3 (Strategy-Proofness)**.** *An SCF f is strategy-proof (SP), if for each $u \in U$, $i \in N$ and $u'_i \in U_i$,*

$$u_i(f(u)) \geq u_i(f(u'_i, u_{-i})).$$

SP is also a necessary condition for robust sequential implementation. The reason is that SP is a necessary condition for *partial* robust implementation in Bayesian equilibria (see Bergemann and Morris [2005b]) and sequential equilibrium is a refinement of Bayesian equilibrium.

Remark 1. *If an SCF f can be robustly sequentially implemented, then f satisfies SP.*

Maskin [1999] provides that Maskin monotonicity as a necessary condition for *Nash implementation* – implementation in Nash equilibria with complete information. In our environments, Maskin monotonicity is described as follows.

Definition 4 (Maskin Monotonicity (Maskin [1999]))**.** *An SCF f is Maskin monotonic (MM) if and only if, for each $u, u' \in U$,*

$$\forall i \in N,\ \forall x \in X:\ u_i(f(u)) \geq u_i(x) \quad \Longrightarrow \quad u'_i(f(u)) \geq u'_i(x),$$

then,

$$f(u) = f(u').$$

change In other words, Maskin monotonicity implies the following; if an SCF chooses an outcome x when the preference profile is u and any agents does not newly strictly prefer any outcome to x when the preference profile change into u', then the SCF remains to choose x at profile u'.

For our purposes, it is useful to state the above definition in its contrapositive form. That is, an SCF f satisfies MM if and only if, for each $u, u' \in U$ with $f(u) \neq f(u')$, there exists i and x such that

$$u_i(x) > u_i(f(u')) \quad \& \quad u'_i(f(u')) \geq u'_i(x). \tag{1}$$

Given an SCF satisfying MM, we state "$u' \in U$ is an unacceptable deception of u" if $f(u) \neq f(u')$. Let $UD(u) \subseteq U$ be the set of "unacceptable deceptions of u". For each $u, u' \in U$ with $u' \in UD(u)$, $i(u, u') \in \mathcal{N}$ and $x(u, u') \in X$ denote the agent and the outcome satisfying equation (1).

Maskin [1999] shows that an SCF can be implemented in Nash equilibria if the SCF satisfies MM and *no veto power*. Let $B_i(u)$ be the set of best outcomes of u_i on X.

Definition 5 (No Veto Power (Maskin [1999]))**.** *An SCF f satisfies no veto power if and only if, for each u and a,*

$$x \in B_j(u) \text{ for each } j \neq i \quad \Longrightarrow \quad f(u) = x.$$

No veto power requires that if an outcome is considered the best outcome for $N-1$ agents, then the outcome is chosen by the SCF. No veto power is regarded as for a weak condition; in fact, any SCF satisfies no veto power with many economic environments including the cases with quasi-linear preferences (we will see later in this section).

The following theorem provides a sufficient condition for robust sequential implementation.

Theorem 1. *If an SCF f satisfies SP, MM, and no veto power, then f can be robustly sequentially implemented.*

Proof. The proof is provided in the Appendix. □

The proof is constructive. The mechanism in the proof is a game with *one round of signaling* (Bergin and Sen [1998]), where each agent announces his own preference type at the first stage and any equilibria on any type spaces do not go beyond the first stage. SP assures that truth telling at the first stage is an equilibrium strategy in the mechanism on any type spaces. And by MM and no veto power, it can be shown that all equilibria attain the SCF on any type spaces. Intuitively, MM has the following effect on the mechanism; if the true preference profile is $u \in U$ but the agents report u' satisfying $f(u) \neq f(u')$ at the first stage, then agent $i(u, u')$ must strictly prefer to confess the deception and achieve $x(u, u')$ instead of $f(u')$ at a second stage, and it implies that some agent prefers to move the game to the second stage.

Our approach is linked to that of Baliga [1999], which shows that if an SCF satisfies three conditions referred to as incentive compatibility, preference reversal, and the economic environment, then the SCF is sequentially implementable. Incentive compatibility corresponds to SP in our theorem and preference reversal corresponds to MM in our theorem. No veto power necessarily holds in the economic environment. We use MM instead of preference reversal to assure that moving the game to the second stage achieves the same outcome as the outcome achieved by not moving the second stage even if no agent confesses in any second stages.

There are some studies about the relation between SP and MM. For one thing, if an SCF satisfies SP and non-bossy, then the SCF satisfies MM (see, for example, lemma 2 of Barbera and Jackson [1995]). Thus, we obtain the following corollary.

Definition 6 (Non-Bossiness[9]). *An SCF f is non-bossy if and only if, for each $u \in U$, $i \in N$, and $u'_i \in U_i$,*

$$u_i(f(u)) = u_i(f(u'_i, u_{-i})) \implies f(u) = f(u').$$

Corollary 1. *If an SCF satisfies SP, no veto power, and non-bossiness, then the SCF can be robustly sequentially implemented.*

This corollary can be used for the SCFs that satisfy SP and non-bossiness; as an example, we consider the median voter rule with single-peaked preferences environments.

Single-peaked voting

The set of alternative is $X = [0, 1]$. Each preference $u_i \in U_i$ of $i \in \mathcal{N}$ is single-peaked; for each $u_i \in U_i$, there exists a peak $p(u_i) \in X$ such that u_i is strictly increasing on $[0, p(u_i)]$ and strictly decreasing on $[p(u_i), 1]$.

In the case where N is odd, *the median voter rule* is the leading example of SCFs. The median voter rule assigns to each profile $u \in U$ the $\frac{N+1}{2}$-th largest peak of the profile. It is known that the median voter rule satisfies SP, Pareto efficiency, and non-bossiness (see Barbera and Jackson [1995]).

By Saijo et al. [2007], it is known that the median voter rule is not robustly implemented by normal-form games. However, since the median voter rule obviously satisfies no veto power, it follows from corollary 1 that the median voter rule can be robustly sequentially implemented.

Quasi-linear preferences

Next, we consider environments with quasi-linear preferences. In these environments, the set of outcomes can be described as

$$X \equiv \{(y, m_1, m_2, \ldots, m_N) | y \in Y, m_i \in \mathbb{R}\ \forall i\},$$

where y is a social decision and m_i is a transfer to i. Let $m \equiv (m_1, m_2, \ldots, m_N)$. In addition, for each preference $u_i \in U_i$, there exists a *valuation function* $v_i : Y \to \mathbb{R}$ satisfying

$$u_i(y, m) = v_i(y) + m_i.$$

Thus, we can directly describe a valuation function v_i as the alternative of the corresponding utility function u_i. Let the set of the valuation functions be V_i for each i. An SCF f can be described as a function on V. An SCF f can be described as a pair of two functions, y^f (on Y) and m^f (on $\mathbb{R}$): $f(v) = (y^f(v), m^f(v))$ for each v. An SCF f is *surplus-maximizing* if y^f maximizes the *social surplus*, described as $\sum_i v_i(y) - c(y)$, where $c(y)$ is the cost of taking a social decision y.

Auction is an important example of environments with quasi-linear preferences. By Saijo et al. [2007], it is showed that, in auction, any surplus-maximizing SCF does not

[9] Our definition is the same as the definition of Saijo et al. [2007].

robustly implemented by normal-form games. However, we can show that there exist efficient SCFs robustly implemented by sequential equilibria; an SCF associated with second price auction is an example. More generally, robust sequential implementation and surplus-maximizing problems are generally compatible in environments with quasi-linear preferences.

Now, we say "an SCF f is associated with a Groves mechanism" if f is surplus-maximizing and m^f satisfies the following conditions: for each i and v,

$$m_i^f(v) = \sum_{j \neq i} v_j(y^f(v)) - c(y^f(v)) + \gamma_i(v_{-i}),$$

where γ is an arbitrary function on V_{-i}. It is known that Groves mechanism satisfies SP (see Groves [1973], Groves and Loeb [1975]). Now, we can obtain the following lemma.

Lemma 1. *There exists a SCF associated with a Groves mechanism and satisfies MM in environments with quasi-linear preferences.*

Proof. The proof is provided in the Appendix □

In our environments, since each agent has no best outcome, no veto power is always satisfied. From proposition 2 and lemma 1, we obtain the following proposition.

Proposition 1. *In quasi-linear preferences environments, there exists an SCF that is surplus-maximizing and robustly sequentially implemented.*

4 Concluding remarks

This paper analyses robust implementation as sequential equilibria on all common prior type spaces. We show that an SCF can be robustly sequentially implemented if the SCF satisfies strategy-proofness, Maskin monotonicity, and no veto power. SCFs can be implemented in single-peaked voting and quasi-linear preferences environments. Since it is known that in these environments, robust implementation with normal-form games is not functional, we find that implementation with extensive-form games is an effective measure to satisfy incentive constraints even in the case of robust implementation.

Appendix

Proof of proposition 1. Let an SCF f satisfy no veto power, SP and MM. Since f satisfies MM, for each pair of profiles $u, u' \in U$ with $f(u) \neq f(u')$, there exist an agent $j(u,u')$ and an outcome $x(u,u') \in X$ such that

$$u_{j(u,u')}(f(u)) \geq u_{j(u,u')}(x(u,u')) \quad \& \quad u'_{j(u,u')}(f(u)) > u'_{j(u,u')}(x(u,u')).$$

We now show that the following mechanism $\mathcal{M}$ robustly sequentially implements f.

Main stage: Each agent i announces a list $(u_i, n_i^1, u', x_i) \in U_i \times \mathbf{N}_+ \times U \times X$ (where $\mathbf{N}_+$ is the set of non-negative integers). We term *the announced preference profile* (at the Main-stage) as the profile of the first components of the agents' announcements.

Case 1 If (i) the announced preference profile is u, (ii) there exists $i \in \mathcal{N}$ announcing (u_i, n_i^1, u', x) such that $n_i^1 > 0$ and $f(u) \neq f(u')$, and (iii) each $j \neq i$ announces $n_j^1 = 0$, then go to sub-stage (u, u').

Case 2 If the announced preferences are u, and either

- all members announce 0 as the second components or
- there exists i announcing (u_i, n_i^1, u', x_i) such that $n_i^1 > 0$ and $f(u) = f(u')$ and each $j \neq i$ announces $n_j^1 = 0$,

then $f(u)$ is implemented.

Case 3 Otherwise, x_{i^*} is implemented, where $i^* \equiv \min \arg\max_j \{n_j^1\}$.

Sub-stage (u, u') Only two agents play at this sub-stage: agent i announcing $n_i^1 > 0$ at the main stage and agent $j(u, u')$. In what follows, we simply write j instead of $j(u, u')$. In the sub-stage, agent j does not be informed about what number agent i announces at the main stage. In this sub-stage, each agent $k \in \{i, j\}$ announces a non-negative integer n_k^2.

Case 1 If $n_j^2 = 0$, then $f(u)$ is implemented.

Case 2 If $n_i^1 = 1$ and $n_j^2 > 0$, then $x(u, u')$ is implemented.

Case 3 Otherwise, $B_{k^*}(u_{k^*})$ is implemented, where $k^* \equiv \min \arg\max_{k \in \{i,j\}} n_k^2$ and $u_{k_i^*}$ is the first component of the announce in the first stage.

Fix a type space $\mathcal{T} = (T_i, \hat{u}_i, \hat{\pi}_i)_{i \in N}$. We first show that any sequential equilibrium on game $(\mathcal{M}, \mathcal{T})$ attains the SCF f if there exists an equilibrium in the game.

Claim 1: For each $t \in S$, $\mathrm{SEO}((\mathcal{M}, \mathcal{T}), t) \subseteq f(\hat{u}(t))$.

We first consider a condition for (σ, μ) in each sub-stage. Fix an information set of agent i where a type t_i of i announced $n_i^1 > 1$ and the game reached a sub-stage (u, u').

Step 1: *If μ assigns a positive probability to a history (h, t) in the information set, then σ must achieve either $f(u)$ or $B(t_i)$ after the history. Additionally, if $\hat{u}_j[t_j](x(u, u')) > \hat{u}_j[t_j](f(u))$ for $j \equiv j(u, u')$, then $x \in B(\hat{u}_i(t_i))$ must be achieved.*

In mechanism $\mathcal{M}$, when j chooses $n_j^2 > 0$, i can obtain $B_i(\hat{u}_i(t_i))$ by choosing $n_i^2 > n_j^2$. Thus, since $(\sigma\mu)$ satisfies sequential rationality and u has a lower bound on X, if $\sigma_j(t_j)$ chooses $n_j^2 > 0$ with a positive probability at the sub-stage, σ must obtain $B_i(\hat{u}_i(t_i))$ by choosing sufficiently large n_j^2 when $\sigma_j(t_j)$ indeed chooses $n_j^2 > 0$. On the other hand, if j indeed chooses $n_j^2 = 0$, then mechanism $\mathcal{M}$ implements $f(u)$ independently of the choice of i. Thus, either $f(u)$ or $B(t_i)$ is achieved after the history.

Next consider the case where $\hat{u}_j[t_j](x(u, u')) > \hat{u}_j[t_j](f(u))$. In this case, choosing $n_j^2 = 0$ is not a best response for j to any choice of i since, for each n_i^2, j can obtain, by choosing $n_j^2 > n_i^2$, $B_j(\hat{u}_j(t_j))$ or $x(u,u')$, which satisfy $\hat{u}_i[t_i](B_i(\hat{u}_i(t_i))) \geq \hat{u}_i[t_i](x(u,u')) > \hat{u}_i[t_i](f(u))$. Thus, since $(\sigma.\mu)$ satisfies sequential rationality and $u(X)$ has lower bound, $\sigma_j(t_j)$ does not assign any positive probability to $n_j^2 = 0$ and σ must obtain $B_i(\hat{u}_i(t_i))$ in the case.

Next, we consider a condition of (σ,μ) related to the main stage.

Step 2: *If $\sigma(t^*)$ achieves $x^* \neq f(\hat{u}(t^*))$ at a type $t^* \in S$ with a positive probability, then $\sigma_i(t_i^*)$ assigns probability 1 to $n_i^1 > 0$ at the main stage.*

By no vet power, there exists i such that $x^* \notin B(\hat{u}_i(t_i^*))$. Fix such i and x^*. Assume by contradiction that $\sigma_i(t_i^*)$ chooses $n_i^1 = 0$ with a positive probability at the main stage.

Since (σ,μ) satisfies sequential rationality and u has a lower bound on X, $\sigma(t^*)$ must achieve $B(t_i^*)$ when $\sigma_k(t_k^*)$ of some $k \neq i$ indeed chooses $n_k^1 > 0$. From $x^* \notin B_i(t_i^*)$, it follows that x^* is implemented when, for each $k \neq i$, $\sigma_k(t_k^*)$ indeed chooses $n_k^1 = 0$. Then, x^* is implemented as the result of either Case 1 of the main stage or Case 1 of the sub-stage after t_i^* chooses $n_i^1 > 0$. From Step 1 and $x^* \notin B_i(t_i^*)$, it follows that $\sigma(t^*)$ announces, with a positive probability, a preference profile $u \in U$ with $x^* = f(u)$ at the main stage. Thus, when $\sigma_i(t_i^*)$ indeed chooses $n_i^1 = 0$, $\sigma(t^*)$ achieves $f(u)$ with a positive probability as the result of case 1 in the main stage. However, since u has a lower bound on X, Step 1 implies that type t_i^* can strictly improve his payoff by choosing a sufficiently large n_i^1 in the main stage. It contradicts the assumption that (σ,μ) satisfies sequential rationality. Thus, $\sigma_i(t_i^*)$ must choose $n_i^1 > 0$ at the main stage.

Since (σ,μ) satisfies sequential rationality and u has a lower bound on X, Step 2 implies that $x^* \in B(t_k^*)$ for each $k \neq i$. However, from no veto power, it implies $x^* = f(\hat{u}(t^*))$ and contradicts the assumption. Thus, (σ,μ) achieves only $f(\hat{u}(t))$ at each $t \in S$ if an sequential equilibrium exists and Claim 1 is proved.

We next show that there exists an equilibrium on game $(\mathcal{M},\mathcal{T})$.

Claim 2: The following assessment σ,μ is an sequential equilibrium on game $(\mathcal{M},\mathcal{T})$.

- Each type t_i of each agent i announces $(\hat{u}_i(t_i), 0, u, x)$ at the main stage, where $u \in U$ and $x \in X$ is arbitrarily chosen.
- At sub-stage (u,u'), for $j = j(u,u')$, each type t_j chooses $n_j^2 = 0$ if $\hat{u}_j[t_j](f(u)) \geq \hat{u}_j[t_j](x(u,u'))$, and $n_j^2 1$ otherwise.
- At sub-stage (u,u') after t_i chooses $n_i^1 = 1$, t_i chooses $n_i^2 = 0$.
- At sub-stage (u,u') after t_i chooses $n_i^1 > 1$, t_i chooses $n_i^2 = 2$.
- At sub-stage (u,u'), for $j = j(u,u')$, $\mu_k(t_k)$ of each type t_k assigns positive probability's only to histories where agent i chooses $n_i^1 = 1$.

First, we consider sequential rationality. Since f satisfies SP, each agent cannot improve his payoff by choosing another payoff type at the main stage. If an agent i

deviates at the main-stage and chooses $n_i^1 > 0$ and the game goes to a sub-stage (u, u'), since each $t_j \in T_j$ of $j = j(u, u')$ always announces his true preference type in the main stage, it follows that $\hat{u}_j[t_j](f(u)) \geq \hat{u}_j[t_j](x(u, u'))$ and he chooses $n_j^2 = 0$. Thus, agent i cannot affect the implemented outcome by deviating and chooses $n_i^1 > 0$ at the main-stage.

At sub-stage (u, u') after agent i announces $n^i > 0$, since each $t_j \in T_j$ of $j = j(u, u')$ believes agent i chooses $n_i^1 = 1$ at the main stage, he cannot improve his (expected) payoff by changing the number. On the other hand, for t_i announcing $n_i^1 = 1$ at the main stage, his choice at sub-stage has no effect on the outcome. For t_i at the sub-stage after he chooses $n_i^i > 1$, since t_j chooses 0 or 1, he can obtain a best outcome by choosing 2. Therefore, this assessment is sequential rational.

Finally, we show this assessment satisfies consistency. Assume a sequence of perfect mixed strategies $(\sigma^n)_{n=1}^{\infty}$ converging to σ such that for each type of each agent i, the probability choosing $n_i^1 = 1$ at the main stage sufficiently faster converges to zero than the summation of the probability choosing all $n_i^1 > 1$. Since the sequence of belief systems $(\mu^n)_{n=1}^{\infty}$ associated with $(\sigma^n)_{n=1}^{\infty}$ converges to μ, μ is consistent. Then, (σ, μ) is an equilibrium. □

Proof of lemma 1. Let $\mathrm{SM}(v) \subseteq Y$ be the set of surplus maximizing social decisions at v. By the assumption, $\mathrm{SM}(v) \neq \emptyset$ for each $v \in V$.

Now, we introduce a pairwise relation $\succ_y$ on U for each $y \in Y$ as the following: $v \succ_y v'$ if and only if, for each i and $y' \in Y$,

$$v_i'(y) - v_i'(y') \geq v_i(y) - v_i(y'). \tag{2}$$

A relation $v \succ_y v'$ is equivalent to the fact that, for each $i \in \mathcal{N}$ and $x, x' \in X$ with $x \equiv (y, m)$ and $x' \equiv (y', m')$,

$$v_i(y) + m_i \geq v_i(y') + m_i' \quad \Longrightarrow \quad v_i'(y) + m_i \geq v_i'(y') + m',$$

Then, an SCF $f = (y^f, m^f)$ satisfies MM if and only if for each u, u', whenever $u \succ_{y^f(u)} u'$, $f(u) = f(u')$. In the following, we write " an social decision rule y^f is Maskin monotonic" if and only if for each u, u', whenever $u \succ_{y^f(u)} u'$, $f(u) = f(u')$. It is sufficient for the proof of Lemma 1 to show that there exists a Maskin monotonic and surplus maximizing social decision rule y^f. Let us consider the following clam.

Claim 3: For each $v, v' \in V$ and $y \in Y$ if $y \in E(v)$ and $v \succ_y v'$, then $y \in E(v') \subseteq E(v)$.

Since $y \in E(v)$ implies that $\sum_i v_i(y) - c(y) \geq \sum_i v_i(y') - c(y')$ for each $y' \in Y$ and $v \succ_y v'$ implies that $\sum_i (v_i'(y) - v_i'(y')) \geq \sum_i (v_i(y) - v_i(y'))$,

$$\sum_i v_i'(y) - c(y) \geq \sum_i v_i'(y') - c(y') \quad \text{for each } y' \in Y.$$

Then, $y \in E(v')$. Since $y' \in E(v')$ implies that $\sum_i v_i'(y) - c(y) = \sum_i v_i'(y') - c(y')$ and $v \succ_y v'$ implies that $\sum_i (v_i'(y) - v_i'(y')) \geq \sum_i (v_i(y) - v_i(y'))$, it follows that $\sum_i v_i(y') - c(y') \geq$

$\sum_i v_i(y) - c(y)$. From $y \in E(v)$, it follows that $y' \in E(u)$. Thus, $E(v') \subseteq E(v)$ and Claim 3 is proved.

Now, define a social decision rule y^f in the following way: let $n = 1$ and $\bar{V}^1 \equiv V$, then,

Step 1: Choose a payoff type profile $v^n \in \bar{V}^n$ and $y^n \in E(v^n)$ arbitrarily.

Step 2: Let $y^f(v) = y^n$ for each $v \in V^n \equiv \{v \in \bar{V}^n | y^n \in E(v)\}$.

Step 3: Define $\bar{V}^{n+1} \equiv \bar{V}^n \setminus V^n$.

Step 4: If $\bar{V}^{n+1} \neq \emptyset$, then increase n to $n+1$ and return to Step 1.

Since V is a countable set, a social decision rule y^f can be defined in this way. It is clear that y^f is surplus-maximizing. If a pair of $v \in V^n$ and $v' \in V^{n'}$ satisfies $v \succ_{y^f(v)} v'$, since Claim 3 implies $y^f(v) \in E(v') \subseteq E(v)$, it follows that $n = n'$. Thus, y^f is Maskin monotonic. □

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