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Difference between the Position Value and the Myerson Value is Due to the Existence of Coalition Structures

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Abstract

The paper characterizes the position value and the Myerson value for communication situations. The position value is originally defined as the sum of half the Shapley value of the *link games* but is also represented by the Shapley value of a modification of the link games. That modification is called the *divided link games*. In addition, by constructing coalition structures of the divided links from original communication situations, two types of the Shapley value (the Owen value/the two-step Shapley value) of the divided link games with coalition structures coincide with the Myerson value of the original communication situations. Thus, the difference between the two values is due to the existence of a coalition structure.

Keywords: position value; Myerson value; coalition structure
JEL classification: C71

1 Introduction

The influence of cooperation and communication between agents in economic or social problems is well described by communication situations introduced by Myerson (1977). He defined an allocation rule for communication situations called the Myerson value. An alternative allocation rule called the position value was given by Borm et al. (1992). Both allocation rules were subsequently characterized axiomatically in several ways. Two axiomatic characterizations of the Myerson value were provided by Myerson (1977, 1980), while an axiomatic characterization of the position value was provided by Slikker (2005).[1] Non-axiomatically, Casajus (2007) characterized the position value by the Myerson value of a modification of communication situations called the *link agent form*.

In this paper, we provide unified and *non-axiomatic* characterizations of the two allocation rules. As in the case of the definitions of the two allocation rules, in our characterizations, each allocation rule is characterized by (the sum of) the Shapley value of a game obtained from the original communication situation. For each allocation rule, the modified games, obtained from the same communication situation, is the same, irrespective of the existence of a coalition structure. For the position value, the modified game is called divided link game. The position value is represented as the sum of the Shapley value of the divided link game, whereas, for the Myerson value, the modified game is called divided link game with a coalition structure. The Myerson value is represented as the sum of the Shapley value of the divided link game with a coalition structure. There are two types of the Shapley value of the games with coalition structures: the Owen value, introduced by Owen (1977); and the two-step Shapley value, introduced by Kamijo (2007). In our modification, both values characterize the Myerson value; thus, the difference in the characterizations of the two allocation rules is simply the existence of coalition structures.

The paper is organized as follows. Basic definitions and notations are given in the next section. Characterizations of the position value and the Myerson value is given in Section 3 and 4, respectively,

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[1] Borm et al. (1992) axiomatically characterized both values in a restricted class.

and another characterization of the position value is given in Section 5. Additional remarks are provided in Section 6.

2 Preliminaries

A *cooperative game with transferable utility* or simply a *game* is a pair (N, v) where $N = \{1, \ldots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function with $v(\emptyset) = 0$. Let $|N| = n$ where $|\cdot|$ represents the cardinality of the set. A subset S of N is called a *coalition* and $v(S)$ is the *worth* of the coalition. A game (N, v) is zero-normalized if for any $i \in N$, $v(\{i\}) = 0$. In this paper, we consider only zero-normalized games. A set of all zero-normalized games is denoted by $\mathcal{G}$. Assuming that the grand coalition N will be formed, the problem of allocation of the worth $v(N)$ among the players arises. One of the widely used allocation rules of the games is the Shapely value, introduced by Shapley (1953). Let π be a permutation on N and $\Pi(N)$ be a set of all permutations on N. Given π and for any $i \in N$, a player $j \in N$ that satisfies $\pi(j) < \pi(i)$ is called a *predecessor* of i in π. Let

$$m_i^\pi(N, v) = v(\{j \in N | \pi(j) \leq \pi(i)\}) - v(\{j \in N | \pi(j) < \pi(i)\})$$

be *i's marginal contributions* for π in (N, v). Given a game $(N, v) \in \mathcal{G}$, the *Shapley value* $\phi(N, v) = (\phi_1(N, v), \phi_2(N, v), \ldots, \phi_n(N, v))$ is defined by

$$\phi_i(N, v) = \frac{1}{|\Pi(N)|} \sum_{\pi \in \Pi(N)} m_i^\pi(N, v),$$

for any $i \in N$.

Given a player set N, let $C = \{C_1, C_2, \ldots, C_m\}$ be a coalition structure of N, that is, for any $C_h, C_k \in C$ with $h \neq k$, $C_h \cap C_k = \emptyset$ and $\bigcup_{C_h \in C} C_h = N$. A triple (N, v, C) is a *game with a coalition structure*, and a set of all games with coalition structures is denoted by $\mathcal{GC}$. Generalizations of the Shapley value to games with coalition structures has been presented by Owen (1977) and Kamijo (2007), called the Owen value and the two-step Shapley value, respectively.

In the Owen value, the set of permutations on N is restricted by the coalition structure. A permutation $\pi \in \Pi(N)$ is *consistent with* C if for any $i, j, k \in N$ with $\pi(i) < \pi(j) < \pi(k)$ and $i, k \in C_h \in C$, then $j \in C_h$. In other words, π is consistent with C if players in the same element of the coalition structure appear successively in π. Let $\Sigma(N, C)$ be a set of all permutations on N, which is consistent with C. Given a game $(N, v, C) \in \mathcal{GC}$, the *Owen value* $\psi(N, v, C) = (\psi_1(N, v, C), \psi_2(N, v, C), \ldots, \psi_n(N, v, C))$ is defined by

$$\psi_i(N, v, C) = \frac{1}{|\Sigma(N, C)|} \sum_{\pi \in \Sigma(N, C)} m_i^\pi(N, v),$$

for any $i \in N$.

In the two-step Shapley value, a game $(N, v, C) \in \mathcal{GC}$ is treated as having two steps. For the first step, we consider the games restricted within each element of the coalition structure. Players in $C_h \in C$ participate in a game $(C_h, v|_{C_h})$ where $v|_{C_h}$ is a restriction of v on C_h, that is, $v|_{C_h}(S) = v(S)$ for any $S \subseteq C_h$. For the second step, we consider the game between elements of the coalition structure. The game is called the *intermediate game* or the *quotient game* and is defined as a pair (M, v_M) where M is a set of indices of the coalition structure C and $v_M(S) = v(\bigcup_{h \in S} C_h)$ for any $S \subseteq M$. Given a game $(N, v, C) \in \mathcal{GC}$, the *two-step Shapley value* $\chi(N, v, C) = (\chi_1(N, v, C), \chi_2(N, v, C), \ldots, \chi_n(N, v, C))$ is defined by

$$\chi_i(N, v, C) = \phi_i(C_h, v|_{C_h}) + \frac{\phi_h(M, v_M) - v(C_h)}{|C_h|},$$

for any $i \in N$ with $i \in C_h \in C$.

Given a player set N, the bilateral communication channels between the players in N are described by a graph $L \subseteq \{\{i, j\} | i, j \in N, i \neq j\}$. Each element of the graph represents a communication channel between the two players and is called a *link*. For convenience, each link is represented as ℓ instead of $\{i, j\}$ when there is no possibility of confusion. Given a graph L, if there exists a finite sequence of players $i_1, \ldots, i_H$ such that $i_1 = i, i_H = j$ and $\{i_h, i_{h+1}\} \in L$ for any $h = 1, \ldots, H-1$, then i *is connected to* j in the graph. Given a graph L, let

$$N/L = \{\{j \in N | i \text{ is connected to } j \text{ in } L\} \cup \{i\} | i \in N\}.$$

N/L represents the collection of sets of connected players in L. For any $S \subseteq N$, let $L(S) = \{\{i, j\} \in L | i, j \in S\}$ which is a restriction of L on S. By $L(S)$, S/L is defined in the same manner as N/L, that is,

$$S/L = \{\{j \in S | i \text{ is connected to } j \text{ in } L(S)\} \cup \{i\} | i \in S\}.$$

A triple (N, v, L) is called a *communication situation* and a set of all communication situations is denoted by $\mathcal{CS}$. Allocation rules widely used in communication situations are the Myerson value introduced by Myerson (1977) and the position value introduced by Borm et al. (1992). Both are derived from the Shapley value of games obtained from original communication situations.

In the Myerson value, a communication situation (N, v, L) is modified as a *graph-restricted game* (N, v^L), where the function $v^L : 2^N \to \mathbb{R}$ is defined by

$$v^L(S) = \sum_{T \in S/L} v(T),$$

for any $S \subseteq N$. This definition reflects the restriction of cooperation in a graph, that is, in a coalition S, only those players who are connected with each other in $L(S)$ can cooperate in the coalition. Given a communication situation $(N, v, L) \in \mathcal{CS}$, the *Myerson value* $\mu(N, v, L) = (\mu_1(N, v, L), \mu_2(N, v, L), \ldots, \mu_n(N, v, L))$ is defined by

$$\mu_i(N, v, L) = \phi_i(N, v^L),$$

for any $i \in N$.

On the other hand, in the position value, the role of each link in a graph is emphasized more than it is in the Myerson value. For the position value, a communication situation (N, v, L) is transformed into a *link game* (L, w) where the function $w : 2^L \to \mathbb{R}$ is defined by

$$w(L') = v^{L'}(N),$$

for any $L' \subseteq L$. The function w represents the worth of sets of links. Given a communication situation $(N, v, L) \in \mathcal{CS}$, the *position value* $\tau(N, v, L) = (\tau_1(N, v, L), \tau_2(N, v, L), \ldots, \tau_n(N, v, L))$ is defined by

$$\tau_i(N, v, L) = \sum_{\ell \in L_i} \frac{1}{2} \phi_\ell(L, w),$$

where $L_i = \{\{i, j\} \in L | j \in N\}$, for any $i \in N$.

3 Characterization of the position value

From this point on, we consider only graphs in which all players in N have at least one link.[2]

In the position value, the Shapley value that a link ℓ receives in the link games is equally divided between two players who form the link. This definition reflects the assumption that each link is composed of two players cooperation, and the contributions of the two players toward maintaining the link are considered to be the same. If we divide each link into two in advance and define a game on the set of divided links, the position value of the original communication situation is represented by the Shapley value of the game.

Given a graph L, we divide each link $\{i, j\}$ into two as ij and ji. A divided link ij can be interpreted as an unilateral communication channel from i to j. Let $D = \{ij, ji | \{i, j\} \in L\}$ be a set of all divided

[2]This restriction is just a simplification of the discussion hereafter. Because of *component decomposability* and *component efficiency* (see van den Nouweland (1993)), players who form no link have no effect on the others and obtain the value of a stand-alone coalition in both the Myerson value and the position value. Thus, if there exist some players who form no link, the deletion of such players from a communication situation is not significant, and the restricted situation can be considered on only the set of players who have at least one link.

links in L. An element of D is also represented as d for convenience. As in the link games, a pair (D, u) is a *divided link game* where $u : 2^D \to \mathbb{R}$ is defined by,

$$u(D') = w(\{\{i,j\} | ij \in D' \text{ and } ji \in D'\}),$$

for any $D' \subseteq D$.

The following is an example of the divided link game.

Example 1. *Let* $N = \{1, 2, 3\}$, $v(S) = 0$ *if* $|S| = 1$, $v(S) = 1$ *if* $|S| = 2$, $v(N) = 4$ *and* $L = \{\{1, 2\}, \{2, 3\}\}$. *Then,* $D = \{12, 21, 23, 32\}$ *and*

$$u(D') = \begin{cases} 4 & \text{if } D' = D \\ 1 & \text{if } D' = \{12, 21\}, \{23, 32\}, \{12, 21, 23\}, \{12, 21, 32\}, \{12, 23, 32\}, \{21, 23, 32\} \\ 0 & \text{otherwise} . \end{cases}$$

For the divided link games, the following holds.

Theorem 1. *For any* $(N, v, L) \in \mathcal{CS}$ *and any* $i \in N$,

$$\tau_i(N, v, L) = \sum_{d \in D_i} \phi_d(D, u),$$

where $D_i = \{ij \in D | j \in N\}$.

Proof. Let $f : \Pi(D) \to \Pi(L)$ be defined as follows: For any $\{i, j\}, \{i', j'\} \in L$, $(f(\pi))(\{i, j\}) < (f(\pi))(\{i', j'\})$ if and only if

$$\pi(ij) < \pi(i'j') \text{ and } \pi(ji) < \pi(i'j') \qquad \text{or} \qquad \pi(ij) < \pi(j'i') \text{ and } \pi(ji) < \pi(j'i').$$

By definition, in $f(\pi) \in \Pi(L)$, a link $\{i, j\}$ precedes a link $\{i', j'\}$ when its two parts ij and ji precede either $i'j'$ or $j'i'$ in $\pi \in \Pi(D)$.

For any $\pi \in \Pi(D)$ and any $ij \in D$, if ji is a predecessor of ij in π then $m^{\pi}_{ij}(D, u) = m^{f(\pi)}_{\{i,j\}}(L, w)$. Similarly, if ij is a predecessor of ji in π, $m^{\pi}_{ij}(D, u) = 0$. Thus,

$$m^{\pi}_{ij}(D, u) + m^{\pi}_{ji}(D, u) = m^{f(\pi)}_{\{i,j\}}(L, w).$$

Since f is a onto mapping and $|\Pi(D)| \geq |\Pi(L)|$, some different permutations on D are mapped onto the same permutation on L. Let $\sigma \in \Pi(L)$. The number of $\pi \in \Pi(D)$ that satisfies $f(\pi) = \sigma$ is $\frac{|\Pi(D)|}{|\Pi(L)|}$.

Therefore,

$$\frac{1}{|\Pi(D)|} \sum_{\pi \in \Pi(D)} \Big(m^{\pi}_{ij}(D, u) + m^{\pi}_{ji}(D, u) \Big) = \frac{1}{|\Pi(L)|} \sum_{\sigma \in \Pi(L)} m^{\sigma}_{\{i,j\}}(L, w),$$

or,

$$\phi_{ij}(D, u) + \phi_{ji}(D, u) = \phi_{\{i,j\}}(L, w).$$

By the definition of u, for any $D' \subseteq D \backslash \{ij, ji\}$, $u(D' \cup \{ij\}) = u(D') = u(D' \cup \{ji\})$. This implies that ij and ji are symmetric in (D, u). By symmetry of the Shapley value, $\phi_{ij}(D, u) = \phi_{ji}(D, u)$. Thus,

$$\phi_{ij}(D, u) = \phi_{ji}(D, u) = \frac{1}{2} \phi_{\{i,j\}}(L, w),$$

which implies,

$$\sum_{d \in D_i} \phi_d(D, u) = \sum_{\ell \in L_i} \frac{1}{2} \phi_\ell(L, w) = \tau_i(N, v, L). \quad \square$$

4 Characterizations of the Myerson value

Next, we mention the relation between the Myerson value of communication situations and divided link games. By definition, $D_i \cap D_j = \emptyset$ for any $i, j \in N$ with $i \neq j$ and $\bigcup_{i \in N} D_i = D$. Consequently, $\{D_1, D_2, \ldots, D_n\}$ is a coalition structure on D. A triple $(D, u, \{D_1, D_2, \ldots, D_n\})$ is a *divided link game with a coalition structure*. The Myerson value of communication situations is represented by the Owen value of the divided link games with coalition structures.

Theorem 2. *For any $(N, v, L) \in \mathcal{CS}$ and any $i \in N$,*

$$\mu_i(N, v, L) = \sum_{d \in D_i} \psi_d(D, u, \{D_1, D_2, \ldots, D_n\}).$$

Proof. By the intermediate game property of the Owen value (see p.231 of Peleg and Sudhölter (2003)),

$$\sum_{d \in D_i} \psi_d(D, u, \{D_1, D_2, \ldots, D_n\}) = \phi_i(M, u_M),$$

where $M = \{1, \ldots, n\}$ is a set of indices of the coalition structure $\{D_1, D_2, \ldots, D_n\}$ and for any $S \subseteq M$, $u_M(S) = v(\bigcup_{j \in S} D_j)$.

Now, by the construction of the coalition structure, $M = N$. For any $S \subseteq M = N$,

$$u_M(S) = u(\bigcup_{j \in S} D_j) = w(L(S)) = v^{L(S)}(N) = v^{L(S)}(S) + \sum_{j \neq S} v(\{j\}) = v^L(S),$$

where the last equality holds since v is zero-normalized. Therefore, $(M, u_M) = (N, v^L)$ and hence for any $i \in N$,

$$\mu_i(N, v, L) = \phi_i(N, v^L) = \phi_i(M, u_M) = \sum_{d \in D_i} \psi_k(D, u, \{D_1, D_2, \ldots, D_n\}). \quad \square$$

Similarly, the Myerson value is represented by the two-step Shapley value of the divided link games with coalition structures.

Theorem 3. *For any $(N, v, L) \in \mathcal{CS}$ and any $i \in N$,*

$$\mu_i(N, v, L) = \sum_{d \in D_i} \chi_d(D, u, \{D_1, D_2, \ldots, D_n\}).$$

Proof. By definition, it is obvious that the two-step Shapley value satisfies the intermediate game property. The proof is the same as the proof of Theorem 2; hence we omit it. $\square$

5 Another characterization of the position value

In the definition of u, a link $\{i, j\}$ is formed when both ij and ji exist. If we change the situation to "a link $\{i, j\}$ is formed when either ij or ji exists," what will change? Let $\hat{u} : 2^D \to \mathbb{R}$ be defined by,

$$\hat{u}(D') = w(\{\{i, j\} | ij \in D' \text{ or } ji \in D'\}),$$

for any $D' \subseteq D$. This change corresponds to the situation in which we duplicate each link and allocate it to each player who form the link. Thus, we call a pair $(D, \hat{u})$ a *duplicated link game*.

Example 2. *Let (N, v, L) be the same as Example 1. Then,*

$$\hat{u}(D') = \begin{cases} 4 & \textit{if } D' \supseteq \{12, 23\}, \{12, 32\}, \{21, 23\} \textit{ or } \{21, 32\} \\ 1 & \textit{if } D' = \{12\}, \{21\}, \{23\}, \{32\}, \{12, 21\} \textit{ or } \{23, 32\} \\ 0 & \textit{if } D' = \emptyset. \end{cases}$$

As in Theorem 1, the position value of the communication situations is also represented by the Shapley value of the duplicated link games.

Theorem 4. *For any* $(N, v, L) \in \mathcal{CS}$ *and any* $i \in N$*,*

$$\tau_i(N, v, L) = \sum_{d \in D_i} \phi_d(D, \hat{u}).$$

Proof. Almost the same proof of Theorem 1 is applicable. Let $g : \Pi(D) \to \Pi(L)$ be defined as follows: For any $\{i, j\}, \{i', j'\} \in L$, $(g(\pi))(\{i, j\}) < (g(\pi))(\{i', j'\})$ if and only if

$$\pi(ij) < \pi(i'j') \text{ and } \pi(ij) < \pi(j'i') \qquad \text{or} \qquad \pi(ji) < \pi(i'j') \text{ and } \pi(ji) < \pi(j'i').$$

By definition, in $g(\pi) \in \Pi(L)$, a link $\{i, j\}$ precedes a link $\{i', j'\}$ when either of its two parts ij and ji precede both $i'j'$ and $j'i'$ in $\pi \in \Pi(D)$.

For any $\pi \in \Pi(D)$ and any $ij \in D$, if ji is a predecessor of ij in π then $m^{\pi}_{ij}(D, \hat{u}) = 0$. Similarly, if ij is a predecessor of ji in π, $m^{\pi}_{ij}(D, \hat{u}) = m^{g(\pi)}_{\{i,j\}}(L, w)$. Thus,

$$m^{\pi}_{ij}(D, \hat{u}) + m^{\pi}_{ji}(D, \hat{u}) = m^{g(\pi)}_{\{i,j\}}(L, w),$$

and by an argument similar to the proof of Theorem 1,

$$\phi_{ij}(D, \hat{u}) + \phi_{ji}(D, \hat{u}) = \phi_{\{i,j\}}(L, w).$$

By the definition of $\hat{u}$, for any $D' \subseteq D \backslash \{ij, ji\}$, $\hat{u}(D' \cup \{ij\}) = w(\{\{h, k\} \in L | hk \in D' \text{ or } kh \in D'\} \cup \{i, j\}) = \hat{u}(D' \cup \{ji\})$. This implies that ij and ji are symmetric in $(D, \hat{u})$ and $\phi_{ij}(D, \hat{u}) = \phi_{ji}(D, \hat{u})$. Thus,

$$\phi_{ij}(D, \hat{u}) = \phi_{ji}(D, \hat{u}) = \frac{1}{2}\phi_{\{i,j\}}(L, w),$$

which implies

$$\sum_{d \in D_i} \phi_d(D, \hat{u}) = \sum_{\ell \in L_i} \frac{1}{2}\phi_\ell(L, w) = \tau_i(N, v, L). \quad \square$$

In the duplicated link games, however, the results that correspond to Theorem 2 and 3 are not obtained, that is, the Myerson value of the communication situations is not represented by the sum of the Owen value or the two-step Shapley value. For instance, in the case of Example 2, $\mu_i(N, v, L) = (\frac{7}{6}, \frac{10}{6}, \frac{7}{6})$ but $\sum_{d \in D_1} \psi_d(D, \hat{u}, \{D_1, D_2, D_3\}) = \sum_{d \in D_1} \chi_d(D, \hat{u}, \{D_1, D_2, D_3\}) = \frac{5}{6}$, $\sum_{d \in D_2} \psi_d(D, \hat{u}, \{D_1, D_2, D_3\}) = \sum_{d \in D_2} \chi_d(D, \hat{u}, \{D_1, D_2, D_3\}) = \frac{14}{6}$ and $\sum_{d \in D_3} \psi_d(D, \hat{u}, \{D_1, D_2, D_3\}) = \sum_{d \in D_3} \chi_d(D, \hat{u}, \{D_1, D_2, D_3\}) = \frac{5}{6}$. It is not clear whether the allocation rule for communication situations is represented by the Owen value or the two-step Shapley value of the duplicated link games with coalition structures.

6 Some remarks

For any $\alpha \in [0, 1]$, let

$$u^\alpha = \alpha u + (1 - \alpha)\hat{u}.$$

Then, by linearity of the Shapley value, the following is obtained as a corollary of Theorems 1 and 4:

Corollary 1. *For any* $(N, v, L) \in \mathcal{CS}$*, any* $\alpha \in [0, 1]$ *and any* $i \in N$*,*

$$\tau_i(N, v, L) = \sum_{d \in D_i} \phi_d(D, u^\alpha).$$

Next, we mention the coincidence between the position value and the Myerson value. By definition, both the Owen value and the two-step Shapley value coincide with the Shapley value if a coalition structure is either the coarsest or the finest, that is, if the coalition structure is composed of the grand coalition or singletons. In our settings, the coalition structure D is the coarsest if and only if $D = \emptyset$, and is the finest if and only if each player has at most one link. This implies the following as a corollary to Theorems 1, 2, and 3:

Corollary 2. *The position value coincides with the Myerson value for any communication situation if and only if each player has at most one link.*

Lastly, we mention a generalization of our results. Communication situations were generalized as *hypergraph communication situations* by van den Nouweland et al. (1992). They extended both the Myerson value and the position value to the hypergraph communication situations. We can extend our results to hypergraph communication situations in a similar way, that is, the position value of hypergraph communication situations is represented as the sum of the Shapley value of the *divided or duplicated hyperlink games* and the Myerson value of hypergraph communication situations is represented as the sum of the Owen value/the two-step Shapley value of the *divided hyperlink games with coalition structures.*

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