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Relationship among Solutions of Cooperative Game under Incomplete Information

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Relationship among Solutions of Cooperative Game under Incomplete Information *

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Abstract

We studied a strategic cooperative game model under incomplete information. We define the notation of the new Bayesian incentive compatibility and some solutions under the incentive compatibility. Moreover, we study the relationships between the solutions. This paper claims the robustness of the solution with respect to various predictions. This is because it is possible to compare a solution defined by one prediction with a solution defined by another prediction by observing the relationships between them.

Keywords: Bayesian game, $\alpha(\beta)$-incentive compatibility, Ex ante $\alpha(\beta)$-core

JEL classification numbers: D82, C71

1 Introduction

The purpose of this paper is to analyze the cooperative behavior of players in games with incomplete information when both the externality and incentive constraints are taken into consideration. Specifically, this paper analyzes the ex ante α and β-cores of a Bayesian game. The α and β-cores, proposed by Aumann and Peleg (1964), are those of the solution concepts to study players' cooperation. We consider a situation in which the players make an agreement on their strategies at the ex ante period, i.e., players make a contract before they have their private information.

In an economy with incomplete information, a cooperative game theoretic approach was initiated by Wilson (1978). In early research in this area, much attention was given to the measurability of strategies. This requires that each player's strategy is consistent with his or her information structure. The additional concept, termed Bayesian incentive compatibility, was incorporated into cooperative games by Ichiishi and Idzik (1996). Roughly speaking, this condition implies that no player can improve upon his or her payoff even if he or she lies about his or her own type. In exchange economies with incomplete information, Vohra (1999) and Forges

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and Minelli (2000) investigated an incentive compatible core. Forges (2004) studied assignment games with incomplete information. Their researches studied the non emptiness of the ex ante incentive compatible core.

Most of the research done deals with situations in which the actions of players outside a coalition do not explicitly impact the payoff of any player in the coalition. In economics, however, there is a situation in which an action by the outsiders influences some player's payoff in the coalition, for example a public good economy, a common pool resource problem, and an economy with externality. We need to consider such situations; such studies in cooperative games with incomplete information have barely been conducted. Yannelis (1991) considered strategic form games with incomplete information and proved the non emptiness of the ex ante α-core. While he considered a measurability, he did not impose incentive constraints. Yusuke et al. (2006) considered both conditions: measurability and incentive constraints. In order to do this, they extended the Bayesian incentive compatibility. They defined the α-incentive compatibility, which is an extension of the Bayesian incentive compatibility, and the ex ante α-incentive compatible core. Moreover, they gave sufficient conditions under which the ex ante α-incentive compatible core was non-empty.

In this paper, we define new concepts termed β-incentive compatibility and ex ante β-incentive compatible core. We show the relationship between the ex ante α-incentive compatible core and the ex ante β-incentive compatible core. Moreover, we define the concepts of variation of α-incentive compatible core and β-incentive compatible core and show the relationship between the four solutions. Finally, this paper claims the robustness of the solution with respect to various predictions. This is because it is possible to compare a solution defined by one prediction with a solution defined by another prediction by observing the relationships between them.

The subsequent section describes the model and defines the α and β-incentive compatibilities, which is an extension of the Bayesian incentive compatibility, and the ex ante α and β-incentive compatible cores. Section 3 presents the results of this paper; section 4, the concluding remarks.

2 The model

Let $N := \{1, \ldots, n\}$ be a set of players. The nonempty subset of N is denoted by $\mathcal{N} := 2^N \setminus \{\emptyset\}$. Each element of $\mathcal{N}$ is termed a coalition. Let us now define a Bayesian game as follows.

Definition 2.1. A *Bayesian game* is a list of specified data, $(\{C^i, T^i, u^i\}_{i\in N}, p)$, where

(1) $C^i \subset \mathbb{R}^{m_i}$ is a set of alternatives for player i,

(2) T^i is a finite set of player i's types,

(3) $u^i : \prod_{j\in N} C^j \times \prod_{j\in N} T^j \to \mathbb{R}$ is player i's payoff function, and

(4) p is the objective probability on $\prod_{i\in N} T^i$.

For notational convenience, let $C^S := \prod_{i\in S} C^i$ and $T^S := \prod_{i\in S} T^i$ for all $S \in \mathcal{N}$. Each element of these sets is denoted by c^S and t^S, respectively.

Considering the cooperation among S, we assume that all feasible collective decisions of S can be represented as the outcomes of its members' feasible strategy bundles, which is defined as follows. A strategy of player j specifies his or her choice contingent upon his or her type, that is, $x^j : T^j \to C^j$. The set of all strategies of player j is denoted by $X^j := \{x^j | x^j : T^j \to C^j\}$. In the same manner, we define $X^S := \prod_{j\in S} X^j$ for each $S \in \mathcal{N}$. Let $x^j(T^j)$ be the range of strategy x^j.

Subsequently, we introduce the concept of feasible strategies. Kamishiro et al.(2006) defined feasible strategies as those that satisfy incentive constraints, which they termed α-incentive compatibility. α-incentive compatibility is defined as follows.

Definition 2.2 (Kamishiro et al.(2006)). A strategy bundle $x^S \in X^S$ of coalition S satisfies the *α-incentive compatibility* if, for all $x^{N\setminus S} \in X^{N\setminus S}$, $j \in S$, $t^j \in T^j$, and $c^j \in x^j(T^j)$, the inequality

$$E[u^j(x^S, x^{N\setminus S})|t^j] \geq E[u^j(c^j, x^{S\setminus\{j\}}, x^{N\setminus S})|t^j]$$

holds, where $E[u^j(x)|t^j]$ is the conditional expected payoff for player j given t^j, which is defined as $\sum_{t^{N\setminus\{j\}} \in T^{N\setminus\{j\}}} p(t^{N\setminus\{j\}}|t^j)u^j(x^1(t^1), ..., x^n(t^n), t^1, ..., t^n)$.

This condition implies that no player in coalition S has an incentive to opt for another alternative regardless of the strategy bundle that the complementary coalition $N \setminus S$ chooses. Note that we restrict the choice set to $x^j(T^j)$. That is because if player j chooses an alternative which does not belong to $x^j(T^j)$, then his or her action implies that rule x^S has been broken. In this sense, a feasible strategy bundle can be considered as a contract among the members of a coalition.

The set of strategy bundles for coalition S that satisfy the α-incentive compatibility is denoted as

$$F^S_\alpha := \{x^S \in X^S | x^S \text{ satisfies the } \alpha\text{-incentive compatibility}\}.$$

We term this set α feasible strategy.

Subsequently, we introduce a new concept of feasible strategies and define feasible strategies as those that satisfy the incentive constraints, which we term β-incentive compatibility. β-incentive compatibility is defined as follows.

Definition 2.3. A strategy bundle $x^S \in X^S$ of coalition S satisfies the *β-incentive compatibility* if, given some $\bar{x}^{N\setminus S} \in X^{N\setminus S}$, for any $j \in S$, $t^j \in T^j$, and $c^j \in x^j(T^j)$, the inequality

$$E[u^j(x^S, \bar{x}^{N\setminus S})|t^j] \geqslant E[u^j(c^j, x^{S\setminus\{j\}}, \bar{x}^{N\setminus S})|t^j]$$

holds, where $E[u^j(x)|t^j]$ is the conditional expected payoff for player j given t^j, which is defined as $\sum_{t^{N\setminus\{j\}} \in T^{N\setminus\{j\}}} p(t^{N\setminus\{j\}}|t^j)u^j(x^1(t^1), ..., x^n(t^n), t^1, ..., t^n)$.

Given some $\bar{x}^{N\setminus S}$, the set of strategy bundles for coalition S that satisfy the β-incentive compatibility is denoted as

$$F^S_\beta(\bar{x}^{N\setminus S}) := \{x^S \in X^S | x^S \text{ satisfies the } \beta\text{-incentive compatibility}\}.$$

We term this set β feasible set.

Since both feasible strategies are the same in a grand coalition, we denote $F^N := F^N_\alpha = F^N_\beta$.

Proposition 2.1. *The Nash equilibrium of a Bayesian game is a Bayesian incentive compatibility strategy in the grand coalition.*

By Proposition 2.1, Bayesian Nash equilibrium strategies are always an element of the feasible strategy of the grand coalition. Moreover, we present the relation of inclusion between then α feasible strategy and β feasible strategy. This relationship is evident from their definitions. However, the following lemmas suggest that the α feasible strategy becomes a stronger condition than the β feasible strategy.

Lemma 2.2. *For all coalition S,*

$$F^S_\alpha \subseteq F^S_\beta(x^{N\setminus S}).$$

Lemma 2.3. *For any coalition* S,

$$F_\alpha^S = \bigcap_{x^{N\setminus S} \in X^{N\setminus S}} F_\beta^S(x^{N\setminus S}).$$

We suppose that each coalition chooses a strategy or a contract at the ex ante stage. Thus, each player's payoff should be measured by the ex ante expected one. The ex ante expected payoff of player j is given by

$$\begin{aligned} E[u^j(x^S, x^{N\setminus S})] &= \sum_{t^j \in T^j} p_j(t^j) E[u^j(x^S, x^{N\setminus S})|t^j] \\ &= \sum_{t \in T} p(t) u^j(x^1(t^1), ..., x^n(t^n), t^1, ..., t^n), \end{aligned}$$

where t_j is the marginal probability of T^j.

In the same manner as Aumann and Peleg (1960), Kamihsiro et al. (2006) proposed a core concept that was an extension of the core of exchange economies. Considering the externality and the incentive constraints, their core concept can also be regarded as an extension of Yannelis (1991). Kamishiro et al. (2006) adopted $X^{N\setminus S}$ as the outsiders' feasible strategies in the definition because they considered the case of the pessimistic expectation with respect to the outsiders' actions.

Definition 2.4 (Kamishiro et al. (2006)). A strategy bundle $x^* \in F^N$ is an *ex ante* α *incentive compatible core strategy* if $S \in \mathcal{N}$ and $x^S \in F_\alpha^S$ do not exist such that for all $x^{N\setminus S} \in X^{N\setminus S}$,

$$E[u^j(x^S, x^{N\setminus S})] > E[u^j(x^*)]$$

holds for every $j \in S$.

Definition 2.4 suggests that no player in coalition S has an incentive to opt for another alternative given some strategy bundle $\bar{x}^{N\setminus S}$ that the complementary coalition $N \setminus S$ chooses. Note that we restrict the choice set to $x^j(T^j)$. This is because if player j chooses an alternative that does not belong to $x^j(T^j)$, then his or her action implies that the rule x^S has been broken. In this sense, a feasible strategy bundle can be considered as a contract among the members of a coalition.

Similar to the α incentive compatible core strategy, we propose another core concept that is an extension of the core of exchange economies. Considering the externality and the incentive constraints, we can also regard our core concept as an extension of Kamishiro et al. (2006). Here, we would like to consider the case of the optimistic expectation with respect to the outsiders' actions.

Definition 2.5. A strategy bundle $x^* \in F^N$ is an *ex ante* β *incentive compatible core strategy* if $S \in \mathcal{N}$ and $x^S \in F_\beta^S(\bar{x}^{N\setminus S})$ do not exit such that for some $\bar{x}^{N\setminus S} \in X^{N\setminus S}$,

$$E[u^j(x^S, \bar{x}^{N\setminus S})] > E[u^j(x^*)]$$

holds for every $j \in S$.

In Definition 2.5, deviating strategies are restricted to the β feasible strategy. This is because the concept of the ex ante β incentive compatible core strategy is a strategy where in the payoff does not improve even if it is an appropriate reaction to any possible strategy choice of the outsiders. Thus, the strategy is a stable strategy of the grand coalition such that for some outsiders' strategies no coalition can improve the payoff.

We again consider the implications of both definitions. Coalition S chooses from the $\alpha(\beta)$ feasible strategy. On the other hand, the outsiders $N \setminus S$ choose a strategy that belongs to $X^{N\setminus S}$. When the players in a deviating coalition predict the outsiders' feasible strategies to be those that belong to the $\alpha(\beta)$ feasible strategy, a new concept of incentible core. Now, we define the new solutions when the solutions are limited to the Bayesian incentive compatibility for strategies that the outsiders can adopt.

Definition 2.6. A strategy bundle $x^* \in F^N$ is an *ex ante $\hat{\alpha}$ incentive compatible core strategy* if $S \in \mathcal{N}$ and $x^S \in F^S_\alpha$ do not exist such that for all $x^{N\setminus S} \in F^{N\setminus S}_\beta(x^S)$,

$$E[u^j(x^S, x^{N\setminus S})] > E[u^j(x^*)]$$

holds for every $j \in S$.

Definition 2.6 differs from Definition 2.4 in that the former is restricted to a Bayesian incentive compatibility for strategies that the outsiders can adopt. Thus, players in a deviating coalition predict that outsiders can only adopt a strategy that satisfies a Bayesian incentive compatibility. Note that we consider the β feasible strategy as a feasible strategy for the outsiders.

In same manner, we define the new β-core strategy when the solutions are limited to a Bayesian incentive compatibility for strategies that the outsiders can adopt.

Definition 2.7. A strategy bundle $x^* \in F^N$ is an *ex ante $\hat{\beta}$ incentive compatible core strategy* if $S \in \mathcal{N}$ and $x^S \in F^S_\beta(\bar{x}^{N\setminus S})$ do not exist such that for some $\bar{x}^{N\setminus S} \in F^{N\setminus S}_\alpha$,

$$E[u^j(x^S, \bar{x}^{N\setminus S})] > E[u^j(x^*)]$$

holds for every $j \in S$.

In Definition 2.7, we consider the α feasible a strategy as feasible strategy for the outsiders.

Note. The core strategy that is not limited to a Bayesian incentive compatibility for the feasible strategies for the outsiders can be interpreted in the following manner. In this case, players belonging to the set of outsiders can adopt strategies separately. However, in other cases, it implies that they can only adopt a joint strategy. Therefore, the concept of the ex ante $\alpha(\beta)$ incentive compatible core strategy is weaker than that of the ex ante $\hat{\alpha}(\hat{\beta})$ incentive compatible core strategy.

Moreover, we consider an ex ante strong Nash equilibrium as an acceptable concept for the β feasible strategy. This equilibrium is a strategy bundle of the grand coalition such that no coalition, having passive perception for outsiders' actions, can improve upon it. Thus, we consider that when players belonging to the set of outsiders do not change the strategy, the players in coalition S can choose a strategy. Therefore, we consider that we can apply the β feasible strategy as a feasible strategy of this solution.

Definition 2.8. A strategy bundle $x^* \in F^N$ is an *ex ante incentive compatible strong Nash strategy* if $S \in \mathcal{N}$ and $x^S \in F^S_\beta(x^{*N\setminus S})$ do not exist such that

$$E[u^j(x^S, x^{*N\setminus S})] > E[u^j(x^*)]$$

holds for every $j \in S$.

Ichiishi and Idzik (1996) considered strategic form games with incomplete information and proved the non emptiness of the ex ante strong Nash strategy.

We describe each set of the solutions defined here as $\mathbb{C}^\alpha_E, \mathbb{C}^\beta_E, \mathbb{C}^{\hat{\alpha}}_E, \mathbb{C}^{\hat{\beta}}_E, SE_E$. In next section, we study the relations of inclusion between their solutions.

3 Main Results

In section 2, we gave the various concepts of solutions. In this section, we analyze the relations of inclusion between those solutions.

The relations of inclusion between $\mathbb{C}^{\alpha}$ and $\mathbb{C}^{\beta}$ and SE under complete information are, in general as follows.

$$SE \subseteq \mathbb{C}^{\beta} \subseteq \mathbb{C}^{\alpha}.$$

Next, we analyze the relationships between the new solutions that have been defined in this paper. Under incomplete information, the relationships is similar to that under complete information.

Proposition 3.1.

$$SE_E \subseteq \mathbb{C}^{\beta}_{E} \subseteq \mathbb{C}^{\alpha}_{E}.$$

The relationship $\mathbb{C}^{\beta}_{E} \subseteq \mathbb{C}^{\alpha}_{E}$ exists even when the solutions are limited to a Bayesian incentive compatibility for the set of strategies that the outsiders can adopt. This fact is described by the following corollary.

Corollary 3.2. *The set of the ex ante $\hat{\alpha}$ incentive compatible core strategies includes the set of the ex ante $\hat{\beta}$ incentive compatible core strategies.*

$$\mathbb{C}^{\hat{\beta}}_{E} \subseteq \mathbb{C}^{\hat{\alpha}}_{E}.$$

However, in general, it can not show a similar relationship with regard to the ex ante incentive compatible strong Nash strategy. We study under what condition the relationship $SE_E \subseteq \mathbb{C}^{\hat{\beta}}_{E}$ holds.

To do so, we define the following strategy set. For any coalition $S \in \mathcal{N}$,

$$Id^S := \left\{ x^S \;\middle|\; \substack{\forall\, j \in S,\\ x^S : T \to c^j} \right\}.$$

The set of these strategies is a set of identical strategies for each coalition. Identical strategies suggest that each player takes this same action for any type.

The condition under which $SE_E \subseteq \mathbb{C}^{\hat{\beta}}_{E}$ holds is described by the following proposition.

Proposition 3.3. *If $F^N = Id^N$, then, $SE_E \subseteq \mathbb{C}^{\hat{\beta}}_{E}$.*

Proof. We show that for elements $\bar{x} \in SE_E$, $\bar{x}$ belongs to $\mathbb{C}^{\hat{\beta}}_{E}$. We consider any element $\bar{x} \in SE_E$. By the definition of the ex ante incentive compatible strong Nash strategy, $\bar{x} \in F^N$,

$$\forall\, S \in \mathcal{N} : \;\forall\, x^S \in F^S_{\beta}(\bar{x}^{N \setminus S}) : \;\exists\, j \in S :$$

$$E[u^j(x^S, \bar{x}^{N \setminus S})] \leqslant E[u^j(\bar{x})].$$

Since the strategy $\bar{x}^{N \setminus S}$ belongs to F^N, this strategy is an element of the identical strategies. This is because an identical strategies the Bayesian incentive compatibility, $\bar{x}^{N \setminus S} \in F^{N \setminus S}_{\alpha}$. Therefore, by outsiders adopting the strategy $\bar{x}^{N \setminus S}$, regardless of the strategy any deviating coalition adopts, a player in the coalition can not improve his or her payoff at $\bar{x}$. We follow that $\bar{x} \in \mathbb{C}^{\hat{\beta}}_{E}$. Q.E.D.

Proposition 3.3 states that the set of the ex ante incentive compatible strong Nash strategies is included by $\mathbb{C}_E^{\hat{\beta}}$ if the set of the feasible strategies for a grand coalition is equivalent to the set of identical strategies. Thus, if the set of feasible strategy for a grand coalition are equivalent to the set of identically strategy, players who predict outsiders' actions with passive can deviate easier than players who predict that the outsiders can adopt only $x^{N\setminus S} \in F_\alpha^{N\setminus S}$.

By Corollary 3.2 and Proposition 3.3, under the condition $F^N = Id^N$, we can derive the following proposition.

Proposition 3.4. *If* $F^N = Id^N$, *then*

$$SE_E \subseteq \mathbb{C}_E^{\hat{\beta}} \subseteq \mathbb{C}_E^{\hat{\alpha}}.$$

Next, we study the condition under which $\mathbb{C}_E^\beta = \mathbb{C}_E^\alpha$ holds. In the case of complete information, the condition was demonstrated by Nakayama (1998). We extend the condition to the case of incomplete information.

We define the following set.

$$BP_j^{N\setminus S}(x^S) := \left\{ x^{N\setminus S} \in X^{N\setminus S} \mid E[u^j(x^S, x'^{N\setminus S})] \geqslant E[u^j(x^S, x^{N\setminus S})];\ \forall\, x'^{N\setminus S} \in X^{N\setminus S} \right\}.$$

Set $BP_j^{N\setminus S}(x^S)$ is the set of the outsiders'$(N \setminus S)$ strategies that when a coalition S adopts strategy x^S, give player j in the coalition S minimum payoff.

The condition under which $\mathbb{C}_E^\beta = \mathbb{C}_E^\alpha$ holds is descried by the following proposition.

Proposition 3.5. *If*

$$\forall\, S \in \mathcal{N} :\ \exists\, j \in S :$$
$$\bigcap_{x^S \in X^S} BP_j^{N\setminus S}(x^S) \neq \emptyset,$$

then,

$$\mathbb{C}_E^\beta = \mathbb{C}_E^\alpha.$$

Proof. We show the equivalence relation. To do so, we need to show the relation of inclusion at both sides. $\mathbb{C}_E^\beta \subseteq \mathbb{C}_E^\alpha$ follows from proposition 3.1.

Thus, to complete the proof, we need to show the reverse relation of inclusion. Hence, we show that any elements of $\mathbb{C}_E^\alpha$ belong to $\mathbb{C}_E^\beta$. Let $\bar{x}$ to be any element of $\mathbb{C}_E^\alpha$. By the definition,

$$\forall\, S \in \mathcal{N} :\ \forall x^S \in F_\alpha^S :\ \exists\, x^{N\setminus S} \in X^{N\setminus S} :\ \exists\, j \in S :$$
$$E[u^j(x^S, x^{N\setminus S})] \leqslant E[u^j(\bar{x})].$$

We consider an element $\tilde{x}^{N\setminus S} \in \bigcap_{x^S \in X^S} BP_j^{N\setminus S}(x^S) \neq \emptyset$.

By the definition of $BP_j^{N\setminus S}(x^S)$,

$$\begin{aligned} E[u^j(\bar{x})] &\geqslant E[u^j(x^S, x^{N\setminus S})], \\ &\geqslant E[u^j(x^S, \tilde{x}^{N\setminus S})]. \end{aligned}$$

Therefore, if the players belonging to the set of outsiders adopt $\tilde{x}^{N\setminus S}$, then a player$j$'s payoff cannot improve it regardless of the strategy that coalition S adopts. We follow $\bar{x} \in \mathbb{C}_E^\beta$. Q.E.D.

Proposition 3.5 states that when a coalition deviates, if the strategy that belongs to the set of feasible strategies for the outsiders can provide some players in the coalition minimum payoff, then $\mathbb{C}_E^{\alpha}$ is equal to $\mathbb{C}_E^{\beta}$ regardless of the prediction (optimistic or pessimistic) that the players in a deviating coalition consider with regard to the outsiders.

Next, we consider a condition under which $\mathbb{C}_E^{\hat{\beta}} = \mathbb{C}_E^{\hat{\alpha}}$ holds. In this case, only the above condition can be proved. This is because strategy $BP_j^{N \setminus S}(x^S)$ may not satisfy the condition of a Bayesian incentive compatibility.

Hence, we need to include the following condition.

Corollary 3.6. *If*

$$\forall\, S \in \mathcal{N} :\ \exists\, j \in S :$$

$$\left(\bigcap_{x^S \in X^S} BP_j^{N \setminus S}(x^S) \right) \bigcap Id^{N \setminus S} \neq \emptyset,$$

then,

$$\mathbb{C}_E^{\hat{\beta}} = \mathbb{C}_E^{\hat{\alpha}}.$$

Proof. Based on the condition, $BP_j^{N \setminus S}(x^S)$ belongs to the set of identically strategies. Therefore, considering this strategy, we can prove this corollary in the same manner in which Proposition 3.5 proved. Q.E.D.

We separately studied the relationship between the solutions without a Bayesian incentive constrain for outsiders and with a Bayesian incentive constrain for outsiders. We are sent all the relationships between their solutions.

First, we consider a condition under which $\mathbb{C}_E^{\hat{\alpha}} = \mathbb{C}_E^{\hat{\alpha}}$ holds.

Proposition 3.7. *If*

$$\forall\, S \in \mathcal{N} :\ \exists\, j \in S :$$

$$\left(\bigcap_{x^S \in X^S} BP_j^{N \setminus S}(x^S) \right) \bigcap Id^{N \setminus S} \neq \emptyset,$$

then, $\mathbb{C}_E^{\alpha} = \mathbb{C}_E^{\hat{\alpha}}$.

Proof. We must show the relation of inclusion at both sides.

Based on the definition of these solutions,

$$\mathbb{C}_E^{\hat{\alpha}} \subseteq \mathbb{C}_E^{\alpha}.$$

Hence, we show that the relation of inclusion at the other sides

$$\mathbb{C}_E^{\hat{\alpha}} \supseteq \mathbb{C}_E^{\alpha}.$$

Let $\bar{\boldsymbol{x}}$ be any element of $\mathbb{C}_E^{\alpha}$. Then, by the definition of $\mathbb{C}_E^{\alpha}$,

$$\forall\, S \in \mathcal{N} :\ \forall x^S \in\ F_\alpha^S :\ \exists\, x^{N \setminus S} \in F^{N \setminus S} :\ \exists\, j \in S :$$

$$E[u^j(x^S, x^{N \setminus S})] \leqslant E[u^j(\bar{x})].$$

Since element $\tilde{x}^{N \setminus S}$ of $\left(\bigcap_{x^S \in X^S} BP_j^{N \setminus S}(x^S) \right) \bigcap Id^{N \setminus S}$ is an identical strategy, the strategy satisfies the condition of a Bayesian incentive compatibility.

Thus, the strategy belongs to $F_\beta^{N\setminus S}(x^S)$.

Moreover, by the definition of $BP_j^{N\setminus S}(x^S)$,

$$\begin{aligned} E[u^j(\bar{x})] &\geqslant E[u^j(x^S, x^{N\setminus S})], \\ &\geqslant E[u^j(x^S, \tilde{x}^{N\setminus S})]. \end{aligned}$$

Given the above, regardless of the strategy that a coalition adopts, if the players belonging to the set of outsiders adopt $\tilde{x}^{N\setminus S}$, some players in the coalition can not their improve payoff of $\bar{x}$. Thus, $\bar{x}$ belongs to $\mathbb{C}_E^{\hat{\alpha}}$. Q.E.D.

Proposition 3.8. *If*

$$\forall\ S \in \mathcal{N}:\ \exists\ j \in S:$$

$$\left(\bigcap_{x^S \in X^S} BP_j^{N\setminus S}(x^S)\right) \bigcap Id^{N\setminus S} \neq \emptyset,$$

then, $\mathbb{C}_E^{\beta} = \mathbb{C}_E^{\hat{\beta}}$.

Proof. Simmiler to Proposition 3.7, we can prove this proposition. Q.E.D.

Based on these propositions, we can derive the following corollary.

Corollary 3.9. *If*

$$\forall\ S \in \mathcal{N}:\ \exists\ j \in S:$$

$$\left(\bigcap_{x^S \in X^S} BP_j^{N\setminus S}(x^S)\right) \bigcap Id^{N\setminus S} \neq \emptyset,$$

then, $\mathbb{C}_E^{\alpha} = \mathbb{C}_E^{\hat{\alpha}} = \mathbb{C}_E^{\beta} = \mathbb{C}_E^{\hat{\beta}}$.

$\left(\bigcap_{x^S \in X^S} BP_j^{N\setminus S}(x^S)\right) \bigcap Id^{N\setminus S} \neq \emptyset$ is to some extent a new condition. This fact suggests that there is a significant difference between the case of the set of the outsiders' strategies without a Bayesian incentive compatibility and the case of the set of the outsiders' strategies with a Bayesian incentive compatibility.

There is a interesting relationship between $\mathbb{C}_E^{\hat{\alpha}}$ and $\mathbb{C}_E^{\beta}$. We present an example such that $\mathbb{C}_E^{\hat{\alpha}} \subset \mathbb{C}_E^{\beta}$. This relation does not exist under the complete information game.

Example 3.1.

- The set of player: $N := \{1, 2, 3\}$.
- The set of choices for player j: $C^i := \{\alpha_i, \beta_i\} \quad \forall i \in N$.
- The set of types for player j: $T^j = \begin{cases} \{t_1^j, t_2^j\} & \text{if } j = 1, 2, \\ \{t_0^j\} & \text{if } j = 3\ . \end{cases}$

Table 1: On G_{110}, player 3 plays α_3

	α_2	β_2
α_1	$2,1,2$	$3,0,0$
β_1	$3,0,0$	$1,2,1$

Table 2: On G_{110}, player 3 plays β_3

	α_2	β_2
α_1	$1,2,1$	$0,0,4$
β_1	$2,0,1$	$3,1,2$

Table 3: On G_{210}, player 3 plays α_3

	α_2	β_2
α_1	$1,2,1$	$0,0,0$
β_1	$0,0,1$	$1,0,1$

Table 4: On G_{210}, player 3 plays β_3

	α_2	β_2
α_1	$1,1,1$	$0,0,0$
β_1	$3,0,0$	$3,1,0$

Table 5: On G_{120}, player 3 plays α_3

	α_2	β_2
α_1	$1,4,2$	$3,0,3$
β_1	$0,0,0$	$2,1,0$

Table 6: On G_{120}, player 3 plays β_3

	α_2	β_2
α_1	$1,1,0$	$2,0,0$
β_1	$1,0,1$	$3,1,0$

Table 7: On G_{220}, player 3 plays α_3

	α_2	β_2
α_1	$1,3,1$	$1,0,2$
β_1	$0,0,3$	$3,2,1$

Table 8: On G_{220}, player 3 plays β_3

	α_2	β_2
α_1	$1,3,0$	$3,0,1$
β_1	$1,0,0$	$1,1,2$

- For each player's type, his or her payoff is as follows. Where G_{ijk} implies that the type of player 1 is i, a type of player 2 is j, and the type of player 3 is k.

The simultaneous distribution for the players is as follows.

$$P(t^1_1,t^2_1,t^3_0)=1/8,\quad P(t^1_1,t^2_2,t^3_0)=3/8.$$
$$P(t^1_2,t^2_1,t^3_0)=3/8,\quad P(t^1_2,t^2_2,t^3_0)=1/8.$$

$$\begin{aligned}\mathbb{C}^{\beta}_{E}=\{\quad x^{\{1,2,3\}}_{1}&=((\alpha_1,\alpha_1),(\alpha_2,\alpha_2),\alpha_3),x^{\{1,2,3\}}_{16}=((\beta_1,\beta_1),(\beta_2,\beta_2),\alpha_3),\\ x^{\{1,2,3\}}_{32}&=((\beta_1,\beta_1),(\beta_2,\beta_2),\beta_3)\}.\end{aligned}$$

$$\mathbb{C}^{\hat{\alpha}}_{E}=\{x^{\{1,2,3\}}_{1}=((\alpha_1,\alpha_1),(\alpha_2,\alpha_2),\alpha_3),x^{\{1,2,3\}}_{16}=((\beta_1,\beta_1),(\beta_2,\beta_2),\alpha_3)\}.$$

Hence, we consider a condition under which $\mathbb{C}^{\hat{\alpha}}_{E}\subseteq\mathbb{C}^{\beta}_{E}$.
Under the following condition, $\mathbb{C}^{\hat{\alpha}}_{E}\subseteq\mathbb{C}^{\beta}_{E}$ holds.

Proposition 3.10. *If* $\forall\ S\in\mathcal{N}:\ \exists\ j\in S:\bigcap_{x^S\in X^S}BP^{N\setminus S}_{j}(x^S)\neq\emptyset$, *then* $\mathbb{C}^{\hat{\alpha}}_{E}\subseteq\mathbb{C}^{\beta}_{E}$.

Proof. We would like to prove that an elements in $\mathbb{C}^{\hat{\alpha}}_{E}$ belong to $\mathbb{C}^{\beta}_{E}$.
Let $\bar{x}$ be any element in $\mathbb{C}^{\hat{\alpha}}_{E}$. This strategy is, by the definition of $\mathbb{C}^{\hat{\alpha}}_{E}$,

$$\forall\ S\in\mathcal{N}:\ \forall x^S\in F^S_{\alpha}:\ 0\exists x^{N\setminus S}\in F^{N\setminus S}_{\beta}(x^S):\exists j\in S:$$

$$E[u^j(x^S, x^{N\backslash S})] \leqslant E[u^j(\bar{x})].$$

Hence, we considers an element $\tilde{x}^{N\backslash S} \in \bigcap_{x^S \in X^S} BP_j^{N\backslash S}(x^S)$,

$$\begin{aligned} E[u^j(\bar{x})] &\geqslant E[u^j(x^S, x^{N\backslash S})], \\ &\geqslant E[u^j(x^S, \tilde{x}^{N\backslash S})]. \end{aligned}$$

Therefore, when the players belonging to the set of the outsiders adopt $\tilde{x}^{N\backslash S}$), regardless of the strategy adopted by a deviating coalition, some players in the coalition can not improve their payoff.

We consider $\bar{x} \in \mathbb{C}_E^{\beta}$. Q.E.D.

Proposition 3.10 suggests that if there exists an outsider's strategy that provides some players in a deviating coalition with minimum payoff, the ex ante $\hat{\alpha}$ incentive compatible core strategy is included by the ex ante β incentive compatible core strategy. Note that under this condition, we cannot show this reverse relation. Because, $BP_j^{N\backslash S}(x^S)$ may not satisfy the condition of a Bayesian incentive compatibility.

4 Conclusions and Remarks

This paper defines some new solutions under the model of Kamishiro et al. (2006) and presents all relationships between each solution.

In this paper, the most interesting result is the fact that $\mathbb{C}_E^{\hat{\alpha}}$ is included by $\mathbb{C}_E^{\beta}$. This result including the case of complete information, has never observed in the relationship between α-core and β-core. This result depends on the manner of prediction. In Ichiishi (1997), the α-core strategy is a stable strategy having a pessimistic expectation for the opposite players, and the β-core strategy is a stable strategy having an optimistic expectation for the opposite players. In those solutions, the most optimistic strategy is $\mathbb{C}_E^{\hat{\beta}}$, the most pessimistic strategy is $\mathbb{C}_E^{\alpha}$. $\mathbb{C}_E^{\hat{\alpha}}$ and $\mathbb{C}_E^{\beta}$ are in the middle of $\mathbb{C}_E^{\hat{\beta}}$ and $\mathbb{C}_E^{\alpha}$. Whether a solution is pessimistic or optimistic depends on the condition of a model. If the model satisfies the condition presented in this paper, $\mathbb{C}_E^{\hat{\alpha}}$ is a more optimistic solution than $\mathbb{C}_E^{\beta}$.

We can suggest some direction for further research. First, more general conditions of relationships between these solutions are needed. Since our condition is a strong condition, we should consider a weaker condition as well. Second, a study of the α core in an interim period is required. In this research the exchange of information among players also needs to be taken into consideration more carefully.

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