21COE-GLOPE

21COE-GLOPE Working Paper Series

Q-Anonymity and preference continuity

Kohei Kamaga and Takashi Kojima

Working Paper No. 36

$\mathcal{Q}$-Anonymity and preference continuity*

Kohei Kamaga† Takashi Kojima‡

December 18, 2007

Abstract In a recent paper published in *Social Choice and Welfare* (27 (2006) 327-339), Banerjee characterized extensions of the Suppes-Sen grading principle and the Basu-Mitra utilitarian relation defined on infinite utility streams with the axiom of $\mathcal{Q}$-Anonymity and discussed the relative merits of the extended utilitarian relation. On the other hand, Asheim and Tungodden (Economic Theory 24: 221-230, 2004) used conditions of Preference Continuity to characterize leximin and utilitarianism. We characterize extensions of the Asheim-Tungodden leximin and utilitarian relations with $\mathcal{Q}$-Anonymity, compare the rankings by the extended overtaking criteria with those by the extended simplified criteria and discuss their relative merits.

Keywords: $\mathcal{Q}$-Anonymity; Preference continuity; Overtaking criterion; Leximin; Utilitarianism; Simplified criterion

1 Introduction

In a recent paper, Banerjee (2006) characterized extensions of the Suppes-Sen grading principle and the Basu-Mitra utilitarian relation defined on infinite utility streams with $\mathcal{Q}$-Anonymity and argued that the rankings by the extended utilitarian relation are far more acceptable than those by the catching up relation[1] or the Basu-Mitra utilitarian relation.

On the one hand, Asheim and Tungodden (2004) used Preference Continuity to characterize leximin and utilitarianism. The Asheim-Tungodden leximin relation is more complete than a leximin relation characterized by Bossert et al. (2007) and so is the Asheim-Tungodden utilitarian relation than the Basu-Mitra utilitarian relation, that is, an overtaking criterion is more complete than the corresponding simplified criterion.

We characterize extensions of the Asheim-Tungodden leximin and utilitarian relations with $\mathcal{Q}$-Anonymity and argue that the rankings by the extended

*Very preliminary. Please do not quote without the authors' permission.

†Graduate School of Economics, Waseda University, Shinjuku, Tokyo 169-8050, Japan (E-mail: k-kmg@ruri.waseda.jp)

‡Graduate School of Economics, Waseda University, Shinjuku, Tokyo 169-8050, Japan (E-mail: tkojima@ruri.waseda.jp)

[1] Banerjee (2006) referred to this relation as the overtaking relation.

overtaking criteria are more complete than those by the extended simplified criteria.

The structure of the paper is as follows. In Section 2, we present the basic definitions. Section 3 discusses the incompatibleness of $\mathcal{Q}$-Anonymity and Strong Preference Continuity. In Section 4, we consider the compatibility of $\mathcal{Q}$-Anonymity and Weak Preferecne Continuity. Section 5 discusses the relative merits of the extended overtaking criteria and concludes the analysis.

2 Basic definitions

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ the set of all natural numbers. Let $X = \mathbb{R}^{\mathbb{N}}$ be the domain of infinite utility streams. A typical element of X is an infinite-dimensional vector $x = (x_1, x_2, \dots)$. For all $x \in X$ and all $n \in \mathbb{N}$, we denote $(x_1, \dots, x_n)$ by x^{-n} and $(x_{n+1}, x_{n+2}, \dots)$ by x^{+n}. Thus for all $x \in X$ and all $n \in \mathbb{N}$, we can write $x = (x^{-n}, x^{+n})$.

A social welfare relation (SWR) is a binary relation $\succsim$ on X which is reflexive and transitive (a quasi-ordering). We write, as usual, $x \succ y$ if $x \succsim y$ holds but $y \succsim x$ does not and $x \sim y$ if $x \succsim y$ and $y \succsim x$ both hold. A SWR $\succsim_A$ is a subrelation to a SWR $\succsim_B$ if (a) $x \succ_A y \Rightarrow x \succ_B y$ and (b) $x \sim_A y \Rightarrow x \sim_B y$. We write $\succsim_A \equiv \succsim_B$ if two SWRs $\succsim_A$ and $\succsim_B$ are subrelations to each other.

A permutation is a bijection on $\mathbb{N}$. We denote the set of all permutations by $\mathcal{P}$. A finite permutation is a permutation π such that there exists $\bar{n} \in \mathbb{N}$ with $\pi(n) = n$ for all $n > \bar{n}$. The set of all finite permutations is denoted by $\mathcal{F}$.

We are concerned with fixed step permutations. Let $\mathcal{Q} = \{\pi \in \mathcal{P} :$ there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\pi(\{1, \dots, nk\}) = \{1, \dots, nk\}\}$. For all $x \in X$ and all $\pi \in \mathcal{P}$, we denote $(x_{\pi(1)}, x_{\pi(2)}, \dots)$ by $\hat{\pi}(x)$.

Negation of a statement is indicated by the logical quantifier $\neg$. For all $x, y \in X$, we write $x \geqslant y$ if for all $i \in \mathbb{N}$, $x_i \geq y_i$ and $x > y$ if $x \geqslant y$ and $x \neq y$.

The following two axioms are imposed on the SWRs.

Strong Pareto *For all $x, y \in X$, if $x > y$, then $x \succ y$.*

$\mathcal{Q}$-Anonymity *For all $x \in X$ and all $\pi \in \mathcal{Q}$, $\hat{\pi}(x) \sim x$.*

3 Impossibility

In this section, we discuss the incompatibleness of $\mathcal{Q}$-Anonymity and Strong Preference Continuity.

Strong preference continuity *For all $x, y \in X$, if (a) there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $(x^{-n}, y^{+n}) \succsim y$ and (b) for all $\bar{n} \in \mathbb{N}$, there exists an integer $n \geq \bar{n}$ such that $(x^{-n}, y^{+n}) \succ y$, then $x \succ y$.*

3.1 Propositions

Proposition 1 *There exists no SWR $\succsim$ satisfying Strong Pareto, $\mathcal{Q}$-Anonymity and Strong Preference Continuity.*

Proof Suppose not. Assume that $\succsim$ satisfies Strong Pareto, $\mathcal{Q}$-Anonymity and Strong Preference Continuity. Let $x = (1, 0, 1, 0, \dots)$ and $y = (0, 1, 0, 1, \dots)$. $\mathcal{Q}$-Anonymity of $\succsim$ implies that for all $n \in \mathbb{N}$, $(x^{-2n}, y^{+2n}) \sim y$ and $(x^{-(2n-1)}, y^{+(2n-1)}) \sim (x_1, y^{+1})$. Since $\succsim$ satisfies Strong Pareto, $(x_1, y^{+1}) \succ y$. Transitivity of $\succsim$ implies that for all $n \in \mathbb{N}$, $(x^{-(2n-1)}, y^{+(2n-1)}) \succ y$. By Strong Preference Continuity of $\succsim$, we have $x \succ y$, which contradicts $x \sim y$ implied by $\mathcal{Q}$-Anonymity of $\succsim$. □

Basu and Mitra (2007) used the axiom of Strong Consistency in their characterization of the catching up SWR. Denoting $(0, 0, \dots)$ by o, this axiom is stated as follows:

Strong consistency *For all $x, y \in X$*

(a) If there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $(x^{-n}, o) \succsim (y^{-n}, o)$, then $x \succsim y$

(b) If (i) there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $(x^{-n}, o) \succsim (y^{-n}, o)$ and (ii) for all $\bar{n} \in \mathbb{N}$, there exists an integer $n \geq \bar{n}$ such that $(x^{-n}, o) \succ (y^{-n}, o)$, then $x \succ y$.

We can also show the incompatibleness of $\mathcal{Q}$-Anonymity and Strong Consistency.

Proposition 2 *There exists no SWR $\succsim$ satisfying Strong Pareto, $\mathcal{Q}$-Anonymity and Strong Consisteny.*

Proof Suppose not. Assume that $\succsim$ satisfies Strong Pareto, $\mathcal{Q}$-Anonymity and Strong Consistency. Let $x = (1, 0, 1, 0, \dots)$ and $y = (0, 1, 0, 1, \dots)$. $\mathcal{Q}$-Anonymity of $\succsim$ implies that for all $n \in \mathbb{N}$, $(x^{-2n}, o) \sim (y^{-2n}, o)$ and $(x^{-(2n-1)}, o) \sim (y^{-(2n+1)}, o)$. Since $\succsim$ satisfies Strong Pareto, for all $n \in \mathbb{N}$, $(x^{-(2n+1)}, o) \succ (x^{-(2n-1)}, o)$. Transitivity of $\succsim$ implies that for all $n \in \mathbb{N}$, $(x^{-(2n+1)}, o) \sim (y^{-(2n+1)}, o)$. By Strong Consistency of $\succsim$, we have $x \succ y$, which contradicts $x \sim y$ implied by $\mathcal{Q}$-Anonymity of $\succsim$. □

3.2 Examples

Consider the following two SWRs characterized by Asheim and Tungodden (2004).

Example 1 Consider a leximin relation called the *S-leximin* relation. We first introduce the usual leximin ordering on $\mathbb{R}^n$. For all $x \in X$ and all $n \in \mathbb{N}$, let $(x^{-n}_{(1)}, \dots, x^{-n}_{(n)})$ denote a non-decreasing permutation of x^{-n}, that is, $x^{-n}_{(1)} \leq \cdots \leq x^{-n}_{(n)}$, ties being broken arbitrarily. Then we can define the usual leximin ordering on $\mathbb{R}^n$ as follows: For all $x^{-n}, y^{-n} \in \mathbb{R}^n$

$$x^{-n} \succsim^n_L y^{-n} \text{ holds if and only if } (x^{-n}_{(1)}, \dots, x^{-n}_{(n)}) = (y^{-n}_{(1)}, \dots, y^{-n}_{(n)}) \text{ or there exists an integer } k < n \text{ such that } (x^{-n}_{(1)}, \dots, x^{-n}_{(k)}) = (y^{-n}_{(1)}, \dots, y^{-n}_{(k)}) \text{ and } x^{-n}_{(k+1)} > y^{-n}_{(k+1)}.$$

Using $\succsim_L^n$, we can define S-Leximin as follows: For all $x, y \in X$

$x \succsim_{Ls} y$ holds if and only if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $(x^{-n}_{(1)}, \ldots, x^{-n}_{(n)}) = (y^{-n}_{(1)}, \ldots, y^{-n}_{(n)})$ or there exists a positive integer $k < n$ such that $(x^{-n}_{(1)}, \ldots, x^{-n}_{(k)}) = (y^{-n}_{(1)}, \ldots, y^{-n}_{(k)})$ and $x^{-n}_{(k+1)} > y^{-n}_{(k+1)}$.

Let $x = (1, 0, 1, 0, \ldots)$ and $y = (0, 1, 0, 1, \ldots)$. Then we have $x \succ_{Ls} y$, which contradicts $x \sim y$ implied by $\mathcal{Q}$-Anonymity. $\square$

Example 2 Consider a utilitarian relation called the *catching up* relation: For all $x, y \in X$

$x \succsim_C y$ holds if and only if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $\sum_{i=1}^n x_i \geq \sum_{i=1}^n y_i$.

Let $x = (1, 0, 1, 0, \ldots)$ and $y = (0, 1, 0, 1, \ldots)$. Then we have $x \succ_C y$, contradicting $x \sim y$ implied by $\mathcal{Q}$-Anonymity. $\square$

4 Possibility

In this section, we consider the compatibility of $\mathcal{Q}$-Anonymity and Weak Preference Continuity.

4.1 Overtaking criterion

For all $n \in \mathbb{N}$, let $\succsim_\xi^n$ a reflexive, complete and transitive binary relation (an ordering) on $\mathbb{R}^n$ satisfying the following three properties: For all $x^{-n}, y^{-n} \in \mathbb{R}^n$

(α) If $x^{-n} > y^{-n}$, then $x^{-n} \succ_\xi^n y^{-n}$

(β) If $(x^{-n}_{(1)}, \ldots, x^{-n}_{(n)}) = (y^{-n}_{(1)}, \ldots, y^{-n}_{(n)})$, then $x^{-n} \sim_\xi^n y^{-n}$

(γ) For all $\delta \in \mathbb{R}$, $(x^{-n}, \delta) \succsim_\xi^{n+1} (y^{-n}, \delta)$ if and only if $x^{-n} \succsim_\xi^n y^{-n}$.

Using $\succsim_\xi^n$, we can define an overtaking criterion on X as follows: For all $x, y \in X$

$x \succ_\xi y$ holds if and only if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $x^{-n} \succ_\xi^n y^{-n}$ and
$x \sim_\xi y$ holds if and only if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $x^{-n} \sim_\xi^n y^{-n}$.

We now need to show that $\succsim_\xi$ is a SWR. This is proved in Lemma 1.

Lemma 1 $\succsim_\xi$ *is a SWR.*

Proof Reflexivity of $\succsim_\xi$ follows from the fact that $\succsim_\xi^n$ is reflexive. To check transitivity, let $x \succsim_\xi y$ and $y \succsim_\xi z$. By definition, there exist $\bar{n}, \bar{n}' \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, either $x^{-n} \succ_\xi^n y^{-n}$ or $x^{-n} \sim_\xi^n y^{-n}$, and for all integers $n' \geq \bar{n}'$, either $y^{-n'} \succ_\xi^{n'} z^{-n'}$ or $y^{-n'} \sim_\xi^{n'} z^{-n'}$. Let $\bar{N} = \max\{\bar{n}, \bar{n}'\}$. Then by definition, we distinguish the four cases which cover all possiblties: For all integers $N \geq \bar{N}$, (a) $x^{-N} \succ_\xi^N y^{-N}$ and $y^{-N} \succ_\xi^N z^{-N}$, (b) $x^{-N} \succ_\xi^N y^{-N}$ and $y^{-N} \sim_\xi^N z^{-N}$, (c) $x^{-N} \sim_\xi^N y^{-N}$ and $y^{-N} \succ_\xi^N z^{-N}$ and (d) $x^{-N} \sim_\xi^N y^{-N}$ and $y^{-N} \sim_\xi^N z^{-N}$. Transitivity of $\succsim_\xi^N$ implies that for all integers $N \geq \bar{N}$, either $x^{-N} \succ_\xi^N z^{-N}$ or $x^{-N} \sim_\xi^N z^{-N}$. From the definition of $\succsim_\xi$, we obtain $x \succsim_\xi z$. □

Moreover, $\succsim_\xi$ satisfies the following two axioms.

Finite Anonymity *For all $x \in X$ and all $\pi \in \mathcal{F}$, $\hat{\pi}(x) \sim x$.*

Weak preference continuity *For all $x, y \in X$, if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $(x^{-n}, y^{+n}) \succ y$, then $x \succ y$.*

Lemma 2 *$\succsim_\xi$ satisfies Finite Anonymity.*

Proof Let $x \in X$ and $\pi \in \mathcal{F}$. By definition, there exists $\bar{n} \in \mathbb{N}$ such that $(\hat{\pi}(x))^{+\bar{n}} = x^{+\bar{n}}$. By the property (β), for all integers $n \geq \bar{n}$, $(\hat{\pi}(x))^{-n} \sim_\xi^n x^{-n}$. From the definition of $\succsim_\xi$, we obtain $\hat{\pi}(x) \sim_\xi x$. □

Lemma 3 *$\succsim_\xi$ satisfies Weak Preference Continuity.*

Proof Assume that there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, (x^{-n}, y^{+n}) $\succ_\xi y$. By definition, there exists $\bar{n}' \in \mathbb{N}$ such that for all integers $n' \geq \bar{n}'$

$$\begin{cases} \text{(a) } x^{-n'} \succ_\xi^{n'} y^{-n'} & \text{if } n' \leq n \\ \text{(b) } (x^{-n}, y_{n+1}, \ldots, y_{n'}) \succ_\xi^{n'} y^{-n'} & \text{otherwise.} \end{cases}$$

In the case (b), since $\succsim_\xi^{n'}$ satisfies the property (γ), we have $x^{-n} \succ_\xi^n y^{-n}$. Hence in both cases, from the definition of $\succ_\xi$, we obtain $x \succ_\xi y$. □

Using the SWR $\succsim_\xi$, we can define an extension of $\succsim_\xi$ as follows:[2] For all $x, y \in X$

$x \succ_{Q\xi} y$ holds if and only if there exist $\pi, \rho \in \mathcal{Q}$ such that $\hat{\pi}(x) \succ_\xi \hat{\rho}(y)$ and
$x \sim_{Q\xi} y$ holds if and only if there exists $\pi \in \mathcal{Q}$ such that $\hat{\pi}(x) \sim_\xi y$.[3]

We now need to show that $\succsim_{Q\xi}$ is a SWR. This is proved in Lemma 4.

---

[2] Banerjee (2006) defined extensions of the Suppes-Sen grading principle and the Basu-Mitra utilitarian relation as follows: For all $x, y \in X$

$x \succsim_{Q\zeta} y$ holds if and only if there exists $\pi \in \mathcal{Q}$ such that $\hat{\pi}(x) \succsim_\zeta y$

where $\succsim_\zeta$ denotes the Suppes-Sen grading principle or the Basu-Mitra utilitarian relation.

[3] Reflexivity of $\succsim_\xi$ implies $\mathcal{Q}$-Anonymity of $\succsim_{Q\xi}$.

Lemma 4 $\succsim_{Q\xi}$ *is a SWR.*

We first prove the following two lemmas which are used to prove Lemma 4.

Lemma 5 $\succsim_{Q\xi}$ *satisfies quasi-transitivity, that is, for all* $x, y, z \in X$*, if* $x \succ_{Q\xi} y$ *and* $y \succ_{Q\xi} z$*, then* $x \succ_{Q\xi} z$*.*

Proof Assume that $x \succ_{Q\xi} y$ and $y \succ_{Q\xi} z$. By definition, there exist $\pi, \rho, \sigma, \tau \in \mathcal{Q}$ such that $\hat{\pi}(x) \succ_\xi \hat{\rho}(y)$ and $\hat{\sigma}(y) \succ_\xi \hat{\tau}(z)$. Since $\pi, \rho, \sigma, \tau \in \mathcal{Q}$, there exist $p, r, s, t \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\pi(\{1, \ldots, np\}) = \{1, \ldots, np\}$, $\rho(\{1, \ldots, nr\}) = \{1, \ldots, nr\}$, $\sigma(\{1, \ldots, ns\}) = \{1, \ldots, ns\}$ and $\tau(\{1, \ldots, nt\}) = \{1, \ldots, nt\}$. Now, since $\hat{\pi}(x) \succ_\xi \hat{\rho}(y)$ and $\hat{\sigma}(y) \succ_\xi \hat{\tau}(z)$, there exist $\bar{\ell}, \bar{m} \in \mathbb{N}$ such that $\bar{\ell} = np = n'q, \bar{m} = n''r = n'''s$, for all integers $\ell \geq \bar{\ell}$, $(\hat{\pi}(x))^{-\ell} \succ_\xi^\ell (\hat{\rho}(y))^{-\ell}$ and for all integers $m \geq \bar{m}$, $(\hat{\sigma}(y))^{-m} \succ_\xi^m (\hat{\tau}(z))^{-m}$. Let $\bar{N}$ be a common multiple of $\bar{\ell}$ and $\bar{m}$. Then for all integers $N \geq \bar{N}$, $(\hat{\pi}(x))^{-N} \succ_\xi^N (\hat{\rho}(y))^{-N}$ and $(\hat{\sigma}(y))^{-N} \succ_\xi^N (\hat{\tau}(z))^{-N}$. It follows from the choice of $\bar{N}$ and the property (β) of $\succsim_\xi^{nN}$ that for all $n \in \mathbb{N}$, $(\hat{\pi}(x))^{-n\bar{N}} \succ_\xi^{n\bar{N}} (\hat{\rho}(y))^{-n\bar{N}} \sim_\xi^{n\bar{N}} (\hat{\sigma}(y))^{-n\bar{N}} \succ_\xi^{n\bar{N}} (\hat{\tau}(z))^{-n\bar{N}}$. Transitivity of $\succsim_\xi^{n\bar{N}}$ implies that for all $n \in \mathbb{N}$, $(\hat{\pi}(x))^{-n\bar{N}} \succ_\xi^{n\bar{N}} (\hat{\tau}(z))^{-n\bar{N}}$. We show that there exist $\pi', \tau' \in \mathcal{Q}$ such that for all integers $N \geq \bar{N}$, $(\hat{\pi}'(x))^{-N} \succ_\xi^N (\hat{\tau}'(z))^{-N}$, that is, $\hat{\pi}'(x) \succ_\xi \hat{\tau}'(z)$. We can construct π' and τ' as follows: If for all integers $N \geq \bar{N}$, $(\hat{\pi}(x))^{-N} \succ_\xi^N (\hat{\tau}(z))^{-N}$, we are done. So assume that there exists $i \in \{n\bar{N}+1, \ldots, (n+1)\bar{N}-1\}$ such that $\neg((\hat{\pi}(x))^{-i} \succ_\xi^i (\hat{\tau}(z))^{-i})$ and (by the properties (α) and (γ) of $\succsim_\xi^i$) $(\hat{\pi}(x))_i < (\hat{\tau}(z))_i$. Then there must exist $j \in \{i+1, \ldots, (n+1)\bar{N}\}$ such that $(\hat{\pi}(x))^{-j} \succ_\xi^j (\hat{\tau}(z))^{-j}$ and (by the properties (α) and (γ) of $\succsim_\xi^j$) $(\hat{\pi}(x))_j > (\hat{\tau}(z))_j$ since $(\hat{\pi}(x))^{-(n+1)\bar{N}} \succ_\xi^{(n+1)\bar{N}} (\hat{\tau}(z))^{-(n+1)\bar{N}}$. Let $\upsilon_1 \in \mathcal{F} \subset \mathcal{Q}$ be a permutation such that $\hat{\upsilon}_1^2(e^i) = \hat{\upsilon}_1(e^j) = e^i$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $\hat{\upsilon}_1(e^k) = e^k$. Then (by using the same argument repeatedly if necessary) there exists a positive integer $k \leq \bar{N}$ such that for all integers $N \geq \bar{N}$, $(\hat{\upsilon}_k(\ldots(\hat{\upsilon}_1(\hat{\pi}(x)))))^{-N} \succ_\xi^N (\hat{\upsilon}_k(\ldots(\hat{\upsilon}_1(\hat{\tau}(z)))))^{-N}$. Using the fact that $\upsilon_k \circ \cdots \circ \upsilon_1 \circ \pi, \upsilon_k \circ \cdots \circ \upsilon_1 \circ \tau \in \mathcal{Q}$, from the definition of $\succsim_{Q\xi}$, we obtain $x \succ_{Q\xi} z$. $\square$

Lemma 6 *For all* $x, y \in X$*,* $x \sim_\xi y$ *if and only if for all* $\pi \in \mathcal{Q}$*,* $\hat{\pi}(x) \sim_\xi \hat{\pi}(y)$*.*

Proof (only if part) Assume $x \sim_\xi y$. Since $\pi \in \mathcal{Q}$, there exists $k \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\pi(\{1, \ldots, nk\}) = \{1, \ldots, nk\}$. Now, since $x \sim_\xi y$, there exists $\bar{N} \in \mathbb{N}$ such that $\bar{N} = nk$ and for all integers $N \geq \bar{N}$, $x^{-N} \sim_\xi^N y^{-N}$. Since $\succsim_\xi^N$ satisfies the property (γ), we have $x^{+\bar{N}} = y^{+\bar{N}}$. It follows from the choice of $\bar{N}$ and the property (β) of $\succsim_\xi^{\bar{N}}$ that $(\hat{\pi}(x))^{-\bar{N}} \sim_\xi^{\bar{N}} (\hat{\pi}(y))^{-\bar{N}}$ and $(\hat{\pi}(x))^{+\bar{N}} = (\hat{\pi}(y))^{+\bar{N}}$. Since $\succsim_\xi^N$ satisfies the property (γ), for all integers $N \geq \bar{N}$, $(\hat{\pi}(x))^{-N} \sim_\xi^N (\hat{\pi}(y))^{-N}$. From the definition of $\succsim_\xi$, we obtain $\hat{\pi}(x) \sim_\xi \hat{\pi}(y)$.

(if part) Assume $\hat{\pi}(x) \sim_\xi \hat{\pi}(y)$. Using the fact that $\pi^{-1} \in \mathcal{Q}$ and the "only if" part of the lemma, we obtain $x \sim_\xi y$. $\square$

Proof of Lemma 4 Reflexivity of $\succsim_{Q\xi}$ follows from the fact that $\iota \in \mathcal{Q}$ and $\succsim_\xi$ is reflexive. To check transitivity, we consider the following four cases which cover all possibilities: (a) $x \succ_{Q\xi} y$ and $y \succ_{Q\xi} z$, (b) $x \succ_{Q\xi} y$ and $y \sim_{Q\xi} z$, (c) $x \sim_{Q\xi} y$ and $y \succ_{Q\xi} z$ and (d) $x \sim_{Q\xi} y$ and $y \sim_{Q\xi} z$.

(a) $x \succ_{Q\xi} y$ and $y \succ_{Q\xi} z$: In this case, by Lemma 5, we obtain $x \succ_{Q\xi} z$.

(b) $x \succ_{Q\xi} y$ and $y \sim_{Q\xi} z$: In this case, by definition, there exist $\pi, \rho, \sigma \in \mathcal{Q}$ such that $\hat{\pi}(x) \succ_{\xi} \hat{\rho}(y)$ and $\hat{\sigma}(y) \sim_{\xi} z$. Using Lemma 6 and the fact that $\sigma^{-1} \in \mathcal{Q}$, we have $y \sim_{\xi} \hat{\sigma}^{-1}(z)$. Again, using Lemma 6 and the fact that $\rho \circ \sigma^{-1} \in \mathcal{Q}$, we have $x \succ_{\xi} \hat{\rho}(y) \sim_{\xi} \hat{\rho}(\hat{\sigma}^{-1}(z))$. Transitivity of $\succsim_{\xi}$ implies $x \succ_{\xi} \hat{\rho}(\hat{\sigma}^{-1}(z))$. From the definition of $\succsim_{Q\xi}$, we obtain $x \succ_{Q\xi} z$.

(c) $x \sim_{Q\xi} y$ and $y \succ_{Q\xi} z$: In this case, by definition, there exist $\pi, \rho, \sigma \in \mathcal{Q}$ such that $\hat{\pi}(x) \sim_{\xi} y$ and $\hat{\rho}(y) \succ_{\xi} \hat{\sigma}(z)$. Using Lemma 6 and the fact that $\pi \circ \rho \in \mathcal{Q}$, we have $\hat{\pi}(\hat{\rho}(x)) \sim_{\xi} \hat{\rho}(y) \succ_{\xi} \hat{\sigma}(z)$. Transitivity of $\succsim_{\xi}$ implies $\hat{\pi}(\hat{\rho}(x)) \succ_{\xi} \hat{\sigma}(z)$. From the definition of $\succsim_{Q\xi}$, we obtain $x \succ_{Q\xi} z$.

(d) $x \sim_{Q\xi} y$ and $y \sim_{Q\xi} z$: In this case, by definition, there exist $\pi, \rho \in \mathcal{Q}$ such that $\hat{\pi}(x) \sim_{\xi} y$ and $\hat{\rho}(y) \sim_{\xi} z$. Using Lemma 6 and the fact that $\pi \circ \rho \in \mathcal{Q}$, we have $\hat{\pi}(\hat{\rho}(x)) \sim_{\xi} \hat{\rho}(y) \sim_{\xi} z$. Transitivity of $\succsim_{\xi}$ implies $\hat{\pi}(\hat{\rho}(x)) \sim_{\xi} z$. From the definition of $\succsim_{Q\xi}$, we obtain $x \sim_{Q\xi} z$. □

Theorem 1 *If a SWR $\succsim$ satisfies $\mathcal{Q}$-Anonymity and all the axioms that characterizes $\succsim_{\xi}$, then $\succsim_{Q\xi}$ is a subrelation to $\succsim$.*

Proof Assume that a SWR $\succsim$ satisfies $\mathcal{Q}$-Anonymity and all the axioms that characterizes $\succsim_{\xi}$. To prove that $\succsim_{Q\xi}$ is a subrelation to $\succsim$, we have to establish (a) $x \succ_{Q\xi} y \Rightarrow x \succ y$ and (b) $x \sim_{Q\xi} y \Rightarrow x \sim y$. Recall that the inverse of P in $\mathcal{Q}$ is denoted by π^{-1}.

(a) Let $x \succ_{Q\xi} y$. By definition, there exists $\pi \in \mathcal{Q}$ such that $\hat{\pi}(x) \succ_{\xi} y$. Since $\succsim_{\xi}$ is a subrelation to $\succsim$, $\hat{\pi}(x) \succ y$. Since $\succsim$ satisfies $\mathcal{Q}$-Anonymity, $x = \hat{\pi}^{-1}(\hat{\pi}(x)) \sim \hat{\pi}(x) \succ y$ and by transitivity, $x \succ y$.

(b) Let $x \sim_{Q\xi} y$. By definition, there exists $\pi \in \mathcal{Q}$ such that $\hat{\pi}(x) \sim_{\xi} y$. Since $\succsim_{\xi}$ is a subrelation to $\succsim$, $\hat{\pi}(x) \sim y$. Since $\succsim$ satisfies $\mathcal{Q}$-Anonymity, $x = \hat{\pi}^{-1}(\hat{\pi}(x)) \sim \hat{\pi}(x) \sim y$ and by transitivity, $x \sim y$. □

4.2 Two versions of the overtaking criteria

Following Asheim and Tungodden (2004), define the following two SWRs. Using $\succsim_L^n$, we first define a leximin relation called the *W-leximin* relation: For all $x, y \in X$

$x \succ_{Lw} y$ holds if and only if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$,
$$x^{-n} \succ_L^n y^{-n}, \text{ and}$$
$x \sim_{Lw} y$ holds if and only if there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$,
$$(x_{(1)}^{-n}, \ldots, x_{(n)}^{-n}) = (y_{(1)}^{-n}, \ldots, y_{(n)}^{-n}).$$

Hammond equity *For all $x, y \in X$ and all $i, j \in \mathbb{N}$, if $y_i < x_i < x_j < y_j$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$, then $x \succsim y$.*

Proposition 3 (Asheim and Tungodden (2004), Proposition 2) *A SWR $\succsim$ satisfies Strong Pareto, Finite Anonymity, Weak Preference Continuity and Hammond Equity if and only if $\succsim_{Lw}$ is a subrelation to $\succsim$.*

Using the SWR $\succsim_{Lw}$, we can define an extension of the W-leximin relation as follows: For all $x, y \in X$

$$x \succ_{QLw} y \text{ holds if and only if there exist } \pi, \rho \in \mathcal{Q} \text{ such that } \hat{\pi}(x) \succ_{Lw} \hat{\rho}(y) \text{ and}$$
$$x \sim_{QLw} y \text{ holds if and only if there exists } \pi \in \mathcal{Q} \text{ such that } \hat{\pi}(x) \sim_{Lw} y.$$

Theorem 2 *A SWR $\succsim$ satisfies Strong Pareto, $\mathcal{Q}$-Anonymity, Weak Preference Continuity and Hammond Equity if and only if $\succsim_{QLw}$ is a subrelation to $\succsim$.*

Proof (only if part) By Theorem 1, a SWR $\succsim$ satisfies the four axioms of the theorem statement only if $\succsim_{QLw}$ is a subrelation to $\succsim$.

(if part) Assume that $\succsim_{QLw}$ is a subrelation to $\succsim$.

(Strong Pareto) Suppose that $x, y \in X$ are such that $x > y$. Since $\succsim_{Lw}$ satisfies Strong Pareto, $x \succ_{Lw} y$. From the definition of $\succ_{QLw}$, we have $x \succ_{QLw} y$. Since $\succsim_{QLw}$ is a subrelation to $\succsim$, we obtain $x \succ y$.

($\mathcal{Q}$-Anonymity) Let $\pi \in \mathcal{Q}$. By definition, $\pi^{-1}, \pi^{-1} \circ \pi \in \mathcal{Q}$. Since $\succsim_{Lw}$ is reflexive, $\hat{\pi}^{-1}(\hat{\pi}(x)) = x \sim_{Lw} x$. By definition, $\hat{\pi}(x) \sim_{QLw} x$. Since $\succsim_{QLw}$ is a subrelation to $\succsim$, we obtain $\hat{\pi}(x) \sim x$.

(Weak Preference Continuity) Suppose that $x, y \in X$ are such that there exists $\bar{n} \in \mathbb{N}$ with for all integers $n \geq \bar{n}$, $(x^{-n}, y^{+n}) \succ y$. Since $\succsim_{QLw}$ is a subrelation to $\succsim$, $\succsim_{Lw}$ is a subrelation to $\succsim_{QLw}$, and $\succsim_{Lw}$ is complete for comparisons between (x^{-n}, y^{+n}) and y, this implies that there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, $(x^{-n}, y^{+n}) \succ_{Lw} y$. By definition, this entails that $x \succ_{Lw} y$, which in turn implies $x \succ y$ since $\succsim_{Lw}$ is a subrelation to $\succsim_{QLw}$ and $\succsim_{QLw}$ is a subrelation to $\succsim$. Thus, we have established that $\succ$ satisfies Weak Preference Continuity.

(Hammond Equity) Suppose that $x, y \in X$ and $i, j \in \mathbb{N}$ are such that $y_i < x_i < x_j < y_j$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $x_k = y_k$. Let $I = \max\{i, j\}$. Then for all integers $n \geq I$, $x^{-n} \succsim^n_{Lw} y^{-n}$. By definition, we have $x \succsim_{Lw} y$ and since $\succsim_{Lw}$ is a subrelation to $\succsim_{QLw}$ and $\succsim_{QLw}$ is a subrelation to $\succsim$, $x \succsim y$. □

Following Banerjee (2006), we can strengthen the conclusion of Theorem 2 further. We denote the set of all SWRs satisfying Strong Pareto, $\mathcal{Q}$-Anonymity, Weak Preference Continuity and Hammond Equity by Ξ and consider the following binary relation on X: For all $x, y \in X$

$$x \succsim^* y \text{ holds if and only if for all } \succsim \in \Xi,\ x \succsim y.$$

We can now prove

Theorem 3 *$\succsim^*$ is a SWR satisfying Strong Pareto, $\mathcal{Q}$-Anonymity, Weak Preference Continuity and Hammond Equity. Moreover, $\succsim^* \equiv \succsim_{QLw}$.*

The proof is omitted for the sake of brevity.

Next, we define a utilitarian relation called the *overtaking* relation: For all $x, y \in X$

$$x \succ_O y \text{ holds if and only if there exists } \bar{n} \in \mathbb{N} \text{ such that for all integers } n \geq \bar{n},$$
$$\sum_{i=1}^n x_i > \sum_{i=1}^n y_i \text{ and}$$
$$x \sim_O y \text{ holds if and only if there exists } n \in \mathbb{N} \text{ such that } \sum_{i=1}^n x_i = \sum_{i=1}^n y_i.$$

2-Generation unit comparability *For all $x, y, z \in X$ and all $i, j \in \mathbb{N}$, if $x \succsim y$ and for all $k \in \mathbb{N} \setminus \{i, j\}$, $z_k = 0$, then $(x + z) \succsim (y + z)$.*

Proposition 4 (Asheim and Tungodden (2004), Proposition 5) *A SWR $\succsim$ satisfies Strong Pareto, Finite Anonymity, Weak Preference Continuity and 2-Generation Unit Comparability if and only if $\succsim_{QO}$ is a subrelation to $\succsim$.*

Using the SWR $\succsim_O$, we can define an extension of the overtaking relation as follows: For all $x, y \in X$

$$x \succ_{QO} y \text{ holds if and only if there exist } \pi, \rho \in \mathcal{Q} \text{ such that } \hat{\pi}(x) \succ_O \hat{\rho}(y) \text{ and}$$
$$x \sim_{QO} y \text{ holds if and only if there exists } \pi \in \mathcal{Q} \text{ such that } \hat{\pi}(x) \sim_O y.$$

Theorem 4 *A SWR $\succsim$ satisfies Strong Pareto, $\mathcal{Q}$-Anonymity, Weak Preference Continuity and 2-Generation Unit Comparability if and only if $\succsim_{QO}$ is a subrelation to $\succsim$.*

Proof (only if part) By Theorem 1, a SWR $\succsim$ satisfies the four axioms of the theorem statement only if $\succsim_{QO}$ is a subrelation to $\succsim$.

(if part) Assume that $\succsim_{QO}$ is a subrelation to $\succsim$. Arguments similar to those used in the only-if part of the proof of Theorem 2 establish that $\succsim$ satisfies Strong Pareto, $\mathcal{Q}$-Anonymity and Weak Preference Continuity.

(2-Generation Unit Comparability) Suppose that $x, y, z \in X$ and $j, k \in \mathbb{N}$ are such that $x \succsim y$, for all $i \in \mathbb{N} \setminus \{j, k\}$, $z_i = 0$. Since $\succsim_{QO}$ is a subrelation to $\succsim$ and $\succsim_O$ is a subrelation to $\succsim_{QO}$, this implies that there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, either $\sum_{i=1}^n x_i > \sum_{i=1}^n y_i$ or $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. By definition, this entails that there exists $\bar{n} \in \mathbb{N}$ such that for all integers $n \geq \bar{n}$, either $\sum_{i=1}^n (x_i + z_i) > \sum_{i=1}^n (y_i + z_i)$ or $\sum_{i=1}^n (x_i + z_i) = \sum_{i=1}^n (y_i + z_i)$, which in turn implies $x \succsim y$ since $\succsim_O$ is a subrelation to $\succsim_{QO}$ and $\succsim_{QO}$ is a subrelation to $\succsim$. Thus, we have established that $\succsim$ satisfies 2-Generation Unit Comparability. □

Again following Banerjee (2006), the characterization result can be strengthened further. Let Ξ' denote the set of all SWRs satisfying Strong Pareto, $\mathcal{Q}$-Anonymity, Weak Preference Continuity and 2-Generation Unit Comparability and consider the following binary relation on X: For all $x, y \in X$

$$x \succsim' y \text{ holds if and only if for all } \succsim \in \Xi',\ x \succsim y.$$

Theorem 5 *$\succsim'$ is a SWR satisfying Strong Pareto, $\mathcal{Q}$-Anonymity, Weak Preference Continuity and 2-Generation Unit Comparability. Moreover, $\succsim' \equiv \succsim_{QO}$.*

The proof is omitted for the sake of brevity.

5 Comparison with the overtaking and $\mathcal{Q}$-simplified criteria

In this section, we compare the rankings by the $\mathcal{Q}$-overtaking criteria with those by the overtaking criteria and the $\mathcal{Q}$-simplified criteria. We will consider a class of examples for which it is argued that the rankings by the $\mathcal{Q}$-overtaking criteria are more complete than those by the overtaking criteria and the $\mathcal{Q}$-simplified criteria. Throughout this section, let π be the permutation defined as follows:

$$\pi(n) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{otherwise.} \end{cases}$$

It is easy to check that for all $n \in \mathbb{N}$, $\pi(\{1, \ldots, 2n\}) = \{1, \ldots, 2n\}$. This shows that $\pi \in \mathcal{Q}$.

We first provide a class of examples to illustrate the relative merits of the $\mathcal{Q}$-overtaking relation.

Example 3 Consider the following two utility streams x and y:

$$\begin{aligned} x &= (1, 0, 1, 0, 1, 0, \ldots) \\ y &= (0, 1, 0, 1, 0, 1, \ldots). \end{aligned} \tag{1}$$

We will compare the ranking of x and y made by the $\mathcal{Q}$-overtaking relation with that by the overtaking relation. Note that in the pair defined in (1), for all odd numbers n, $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$ and for all even numbers n, $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. By definition, the overtaking relation declares x and y as non-comparable and using the definition of the catching up relation, we get $x \succ_C y$. Now, $\hat{\pi}(x) = y$ and hence, $\hat{\pi}(x) \sim_O y$. By definition, $x \sim_{QO} y$. □

Example 4 Consider the following two utility streams x and y:

$$\begin{aligned} x &= (\tfrac{1}{2}, 0, 1, 0, 1, 0, \ldots) \\ y &= (0, 1, 0, 1, 0, 1, \ldots). \end{aligned} \tag{2}$$

We will compare the ranking of x and y made by the $\mathcal{Q}$-overtaking relation with that by the overtaking relation. Note that in the pair defined in (2), for all odd numbers n, $\sum_{i=1}^{n} x_i > \sum_{i=1}^{n} y_i$ and for all even numbers n, $\sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$. By definition, the catching up relation declares x and y as non-comparable. Now, $\hat{\pi}(y) > x$ and hence, $\hat{\pi}(y) \succ_O x$. By definition, $y \succ_{QO} x$. □

Next, we introduce two versions of the simplified criterion: The Basu-Mitra utilitarian relation and the leximin relation characterized by Bossert et al. (2007).

The Basu-Mitra utilitarian relation is defined as follows: For all $x, y \in X$

$$x \succsim_U y \text{ holds if and only if there exists } n \in \mathbb{N} \text{ such that } \left(\textstyle\sum_{i=1}^{n} x_i, x^{+n}\right) \geqslant \left(\textstyle\sum_{i=1}^{n} y_i, y^{+n}\right).$$

Using the SWR $\succsim_U$, we can define the $\mathcal{Q}$-utilitarian relation characterized by Banerjee (2006) as follows:[4] For all $x, y \in X$

$$x \succsim_{QU} y \text{ holds if and only if there exists } \pi \in \mathcal{Q} \text{ such that } \hat{\pi}(x) \succsim_U y.$$

[4]An alternative characterization of this extended SWR was provided in Kamaga and Kojima (2007).

Next, using $\succsim_L^n$, we define the leximin relation characterized by Bossert et al. (2007) as follows: For all $x, y \in X$

$$x \succsim_L y \text{ holds if and only if there exists } n \in \mathbb{N} \text{ such that } x^{-n} \succsim_L^n y^{-n} \text{ and } x^{+n} \geqslant y^{+n}.$$

Using the SWR $\succsim_L$, we can define an extension of the leximin relation as follows:[5] For all $x, y \in X$

$$x \succsim_{QL} y \text{ holds if and only if there exists } \pi \in \mathcal{Q} \text{ such that } \hat{\pi}(x) \succsim_L y.$$

We now consider an example to illustrate the relative merits of the $\mathcal{Q}$-overtaking relation.

Example 5 (Banerjee (2006), Example 3) Consider the following two utility streams x and y:

$$x = (1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2^3}, \tfrac{1}{2^3}, \tfrac{1}{2^5}, \dots)$$
$$y = (1, 1, \tfrac{1}{2^2}, \tfrac{1}{2^2}, \tfrac{1}{2^4}, \tfrac{1}{2^4}, \dots).$$

As Banerjee (2006) discussed, the $\mathcal{Q}$-utilitarian relation declares x and y to be non-comparable. However, since for all integers $n \geq 2$, $\sum_{i=1}^n y_i > \sum_{i=1}^n x_i$, we have $y \succ_O x$ which is compatible with Banerjee (2006)'s observation. Since $\succsim_O$ is subrelation to $\succsim_{QO}$, we also have $y \succ_{QO} x$.

Moreover, as Banerjee (2006) showed, it is impossible to achieve Pareto dominance after some finite generation with infinite permutation matrices in the class $\mathcal{Q}$. So the $\mathcal{Q}$-leximin relation also declares x and y to be non-comparable. However, $y \succ_{Lw} \hat{\pi}(x)$. By definition, $y \succ_{QLw} x$.[6] □

Example 6 Consider the following two utility streams x and y:

$$x = (1, 1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{9}, \tfrac{1}{27}, \dots)$$
$$y = (1, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{2}{27}, \tfrac{2}{27}, \dots).$$

One can generate the utility stream x in the following way: $x_1 = 1$ and for all integers $n \geq 2$

$$x_n = \begin{cases} \frac{3}{\sqrt{3}^n} & \text{if } n \text{ is even} \\ \frac{\sqrt{3}}{\sqrt{3}^n} & \text{otherwise.} \end{cases}$$

Similarly, $y_1 = 1$ and for all integers $n \geq 2$

$$y_n = \begin{cases} \frac{2}{\sqrt{3}^n} & \text{if } n \text{ is even} \\ \frac{2\sqrt{3}}{\sqrt{3}^n} & \text{otherwise.} \end{cases}$$

Clearly, x and y are non-comparable according to the W-leximin relation, since for all even numbers n, $\min\{x_1, \dots, x_n, y_1, \dots, y_n\} = y_n$ and for all odd numbers

[5] This extended SWR was characterized in Kamaga and Kojima (2007).

[6] Note that there exists no $\rho \in \mathcal{Q}$ satisfying $\hat{\rho}(y) \succ_{Lw} x$ in this example.

n, $\min\{x_1, \dots, x_n, y_1, \dots, y_n\} = x_n$. Moreover, x and y are non-comparable according to the overtaking relation, since for all even numbers n, $\sum_i^n x_i > \sum_i^n y_i$ and for all odd numbers n, $\sum_i^n x_i = \sum_i^n y_i$. However, $x \succ_O \hat{\pi}(y)$ and $\hat{\pi}(y) \succ_{Lw} x$. By definition, $x \succ_{QO} y$ and $y \succ_{QLw} x$.[7] $\square$

We now discuss a potential drawback of two versions of the $\mathcal{Q}$-overtaking criteria. Example 7 presents an example in which two versions of the $\mathcal{Q}$-overtaking criteria fail to compare them.

Example 7 (Lauwers (1997, p. 230)) Consider the following two utility streams x and y:

$$x = (\overbrace{\tfrac{1}{2}}^{1}, \overbrace{0,0}^{2}, \overbrace{0,0,1}^{3}, \overbrace{0,0,0,0}^{4}, \overbrace{0,0,0,0,1}^{5}, \overbrace{0,0,0,0,0,0}^{6}, 0,0,0,\dots)$$
$$y = (\underbrace{0}_{1}, \underbrace{0,1}_{2}, \underbrace{0,0,0}_{3}, \underbrace{0,0,0,1}_{4}, \underbrace{0,0,0,0,0}_{5}, \underbrace{0,0,0,0,0,1}_{6}, 0,0,0,\dots).$$

One can generate the utility stream x in the following way: $x_1 = \frac{1}{2}$ and for all integers $n \geq 2$

$$x_n = \begin{cases} 1 & \text{if } n = k(2k-1) \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly

$$y_n = \begin{cases} 1 & \text{if } n = k(2k+1) \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

There exist no $\pi, \rho \in \mathcal{Q}$ satisfying $\hat{\pi}(x) \succsim_\xi \hat{\rho}(y)$ or $\hat{\pi}(y) \succsim_\xi \hat{\rho}(x)$, since for all $n \in \mathbb{N}$, $x^{-n(2n-1)} \succ_\xi^{n(2n-1)} y^{-n(2n-1)}$, $y^{-n(2n+1)} \succ_\xi^{n(2n+1)} x^{-n(2n+1)}$, $x^{-(n+1)(2n-1)} \succ_\xi^{(n+1)(2n-1)} y^{-(n+1)(2n-1)}$ and so forth, where $\succsim_\xi$ denotes the W-leximin or overtaking relation. Consequently, x and y are non-comparable according to two versions of the $\mathcal{Q}$-overtaking criteria. Note that in order to extend two versions of the $\mathcal{Q}$-overtaking criteria to complete orderings, one has to judge such types of utility streams. $\square$

References

[1] Asheim GB, Tungodden B (2004) Resolving distributional conflicts between generations. Econ Theory 24:221-230

[2] Banerjee K (2006) On the extension of the utilitarian and Suppes-Sen social welfare relations to infinite utility streams. Soc Choice Welfare 27:327-339

[3] Basu K, Mitra T (2007) Utilitarianism for infinite utility streams: a new welfare criterion and its axiomatic characterization. J Econ Theory 133:350-373

[7] Note that there exists no $\rho \in \mathcal{Q}$ satisfying $\hat{\rho}(x) \succ_O y$ in this example.

[4] Bossert W, Sprumont Y, Suzumura K (2007) Ordering infinite utility streams. J Econ Theory 135:579-589

[5] Kamaga K, Kojima T (2007) $\mathcal{Q}$-anonymous social welfare relations on infinite utility streams. 21COE-GLOPE Working Paper #25, Waseda University

[6] Lauwers L (1997) Infinite utility: insisting on strong monotonicity. Aust J Philos 75:222-233

[7] Mitra T, Basu K (2005) On the existence of Paretian social welfare relations for infinite utility streams with extended anonymity. In: Roemer J, Suzumura K (eds) Intergenerational equity and sustainability. Palgrave, London (forthcoming)