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Error Estimates of a Pressure-Stabilized Characteristics Finite Element Scheme for the Navier-Stokes Equations

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Abstract

A pressure-stabilized characteristics finite element scheme for the Navier-Stokes equations proposed by the authors is mathematically analyzed. Stability and convergence results with optimal error estimates for the scheme with a natural stabilization parameter are proved. The scheme can deal with convection-dominated problems and leads to a symmetric coefficient matrix of the system of linear equations. A cheap P1/P1 finite element is employed and the degrees of freedom are smaller than that of conventional elements for the equations, e.g., P2/P1. Therefore, the scheme is efficient especially for three dimensional problems. Two and three dimensional numerical results are shown to recognize the theoretical convergence order.

Keywords Error estimates · The finite element method · The method of characteristics · Pressure-stabilization · The Navier-Stokes equations

Mathematics Subject Classification (2000) 65M12 · 65M60 · 65M25 · 76D07

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1 Introduction

A combined finite element scheme with a pressure-stabilization and a characteristics method for the Navier-Stokes equations has been proposed by us [20, 22]. In this paper we prove stability and optimal error estimates for the scheme with a natural stabilization parameter.

The system of the Navier-Stokes equations is one of the most important basic models in flow dynamics and is often employed in scientific computation. In order to solve convection-dominated flow problems many ideas have been proposed, e.g., upwind methods [1,4,7,14,18,19,32,34], characteristics(-based) methods [3,10,12, 13,20–23,25–28] and so on.

We focus on the characteristics finite element (C-FE) methods, which include less numerical diffusion among them and such a common advantage that the resulting matrix of the system of linear equations is symmetric. The advantage enables us to use efficient linear iterative solvers for symmetric matrices, i.e., MINRES, CR and so on [2, 29]. A C-FE scheme for the Navier-Stokes equations has been originally proposed in [25] and error estimates of the form $O(\Delta t+h^m+(h^{m+1}/\Delta t))$ have been proved. In [31] optimal error estimates of $O(\Delta t+h^m)$ have been proved for a C-FE scheme of first order in time for the Navier-Stokes equations. In [3] a C-FE scheme for the Navier-Stokes equations of high order in time has been presented and optimal error estimates of $O(\Delta t^2+h^m)$ have been proved. In these schemes it is supposed that the pair of finite element spaces for the velocity and the pressure satisfies the conventional inf-sup condition [16], e.g., P2/P1 (Hood-Taylor) finite element, which leads to large degrees of freedom.

In [20,22] we have proposed a stabilized C-FE scheme for the Navier-Stokes equations by combining a pressure-stabilization [6] and a characteristics method of first order in time, while its theoretical analysis was not completed at that time. To the best of our knowledge the scheme is the first stabilized C-FE scheme for the Navier-Stokes equations. The characteristics method works well for convection-dominated problems, and the pressure-stabilization is employed for the use of the cheap P1/P1 finite element. The scheme is symmetric by virtue of the characteristics method and we can use efficient linear iterative solvers for symmetric matrices. Since the resulting matrix is identical with respect to the time step, it is enough to make the matrix only once at the beginning. As for a corresponding stabilized C-FE scheme for the Oseen equations we have recently proved essentially unconditional stability as well as convergence with optimal error estimates of $O(\Delta t+h)$ in [24].

In this paper we prove conditional stability and optimal error estimates of $O(\Delta t+h)$ for the scheme proposed in [20, 22] with a natural stabilization parameter. The nonlinearity of the Navier-Stokes equations is overcome by mathematical induction, which has been developed for C-FE schemes in [3, 31]. The key issue of the proof of stability and convergence is how to estimate the essential supremum norm of first derivatives of the numerical velocity, and the estimate is more delicate than those in [3, 31] because the P1/P1 finite element has only the first order interpolation property for the first derivatives. The condition on time increment for the stability and convergence to be proved is the same as that of [31]. Consequently, the scheme leads to efficient computation especially in 3D as well as mathematical reliability with the optimal error estimates.

Let m be a non-negative integer and Ω be a domain in $\mathbb{R}^d (d=2,3)$. We use the Sobolev spaces $W^{m,\infty}(\Omega)$, $H^m(\Omega)$ and $H^1_0(\Omega)$ as well as $C^m(\bar{\Omega})$. For any normed space X with norm $\|\cdot\|_X$, we define function spaces $C^m([0,T];X)$ and $H^m(0,T;X)$

consisting of X-valued functions in $C^m([0,T])$ and $H^m(0,T)$, respectively. We use the same notation $(\cdot,\cdot)$ to represent the $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions. The norms in $W^{m,\infty}(\Omega)^d$ and $H^m(\Omega)^d$ are simply denoted as

$$\|\cdot\|_{m,\infty} \equiv \|\cdot\|_{W^{m,\infty}(\Omega)^d}, \quad \|\cdot\|_m \equiv \|\cdot\|_{H^m(\Omega)^d},$$

and the notation $\|\cdot\|_m$ is employed not only for vector-valued functions but also for scalar-valued ones. $L^2_0(\Omega)$ is a subspace of $L^2(\Omega)$ defined by

$$L^2_0(\Omega) \equiv \{q \in L^2(\Omega);\ (q,1) = 0\}.$$

We often omit $[0,T]$, Ω and/or d if there is no confusion, e.g., $C^0(H^1)$ in place of $C^0([0,T];H^1(\Omega)^d)$. For t_0 and $t_1 \in \mathbb{R}$ we introduce function spaces

$$Z^m(t_0,t_1) \equiv \{v \in H^j(t_0,t_1;H^{m-j}(\Omega)^d);\ j = 0,\cdots,m,\ \|v\|_{Z^m(t_0,t_1)} < \infty\},$$

and $Z^m \equiv Z^m(0,T)$, where the norm $\|v\|_{Z^m(t_0,t_1)}$ is defined by

$$\|v\|_{Z^m(t_0,t_1)} \equiv \left\{ \sum_{j=0}^{m} \|v\|^2_{H^j(t_0,t_1;H^{m-j}(\Omega)^d)} \right\}^{1/2}.$$

The abbreviation LHS means the left-hand side.

2 A pressure-stabilized characteristics finite element scheme

In this section we present our pressure-stabilized characteristics finite element scheme for the Navier-Stokes equations [20, 22] with a natural stabilization parameter.

Let Ω be a bounded domain in $\mathbb{R}^d (d = 2,3)$, $\Gamma \equiv \partial\Omega$ be the boundary of Ω and T be a positive constant. We consider an initial boundary value problem; find $(u,p) : \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\frac{Du}{Dt} - \nabla\big(2\nu D(u)\big) + \nabla p = f \quad \text{in } \Omega \times (0,T), \tag{1a}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T), \tag{1b}$$

$$u = 0 \quad \text{on } \Gamma \times (0,T), \tag{1c}$$

$$u = u^0 \quad \text{in } \Omega,\ \text{at } t = 0, \tag{1d}$$

where u is the velocity, p is the pressure, $f : \Omega \times (0,T) \to \mathbb{R}^d$ is a given external force, $u^0 : \Omega \to \mathbb{R}^d$ is a given initial velocity, $\nu \in (0,\nu_0]$ is a viscosity for a fixed $\nu_0 > 0$, $D(u)$ is a strain-rate tensor defined by

$$D_{ij}(u) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (i,j = 1,\cdots,d),$$

D/Dt is a material derivation defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla.$$

Letting $V \equiv H_0^1(\Omega)^d$ and $Q \equiv L_0^2(\Omega)$, we define bilinear forms a on $V \times V$, b on $V \times Q$ and $\mathscr{A}$ on $(V \times Q) \times (V \times Q)$ by

$$\begin{aligned} a(u,v) &\equiv 2\big(D(u), D(v)\big), \quad b(v,q) \equiv -(\nabla \cdot v, q), \\ \mathscr{A}((u,p),(v,q)) &\equiv \nu a(u,v) + b(v,p) + b(u,q), \end{aligned}$$

respectively. Then, we can write the weak formulation of (1); find $(u,p): \ (0,T) \to V \times Q$ such that for $t \in (0,T)$

$$\Big(\frac{Du}{Dt}(t), v\Big) + \mathscr{A}((u,p)(t),(v,q)) = (f(t),v), \quad \forall (v,q) \in V \times Q, \tag{2}$$

with $u(0) = u^0$.

Let Δt be a time increment, $t^n \equiv n\Delta t$ for $n \in \mathbb{N} \cup \{0\}$ and $f^n \equiv f(\cdot, t^n)$ for a function f defined in $\Omega \times (0,T)$. Let $X : (0,T) \to \mathbb{R}^d$ be a solution of the ordinary differential equation,

$$\frac{dX}{dt} = u(X,t), \tag{3}$$

for a smooth function $u : \Omega \times (0,T) \to \mathbb{R}^d$. Then, it holds that

$$\frac{Du}{Dt}(X(t),t) = \frac{d}{dt} u\big(X(t),t\big).$$

Let $X(\cdot; x, t^n)$ be the solution of (3) subject to an initial condition $X(t^n) = x$. For a velocity $w: \ \Omega \to \mathbb{R}^d$ let $X_1(w,\Delta t) : \Omega \to \mathbb{R}^d$ be a function defined by

$$X_1(w,\Delta t)(x) \equiv x - w(x)\Delta t.$$

Since the position $X_1(u^{n-1},\Delta t)(x)$ is an approximation of $X(t^{n-1}; x, t^n)$, we can consider a first order approximation of the material derivative at $t = t^n (n \geq 1)$,

$$\frac{Du}{Dt}(x,t^n) = \frac{d}{dt} u\big(X(t;x,t^n),t\big)\Big|_{t=t^n} = \frac{u^n - u^{n-1} \circ X_1(u^{n-1},\Delta t)}{\Delta t}(x) + O(\Delta t),$$

where the symbol $\circ$ means the composition of functions,

$$v \circ X_1(w,\Delta t)(x) \equiv v(X_1(w,\Delta t)(x)),$$

for v and $w : \Omega \to \mathbb{R}^d$. $X_1(w,\Delta t)(x)$ is called an upwind point of x with respect to the velocity w. The next proposition proved in [28] gives a sufficient condition to guarantee all upwind points are in Ω.

Proposition 1 ([28], Proposition 1)**.** *Let $w \in W^{1,\infty}(\Omega)^d$ be a given function satisfying $w|_\Gamma = 0$, and assume*

$$\Delta t < \frac{1}{\|w\|_{1,\infty}}.$$

Then, it holds that

$$X_1(w,\Delta t)(\Omega) = \Omega.$$

For the sake of simplicity we assume that Ω is a polygonal $(d=2)$ or polyhedral $(d=3)$ domain. Let $\mathscr{T}_h = \{K\}$ be a triangulation of $\bar{\Omega}$ $(= \bigcup_{K\in\mathscr{T}_h} K)$, h_K be a diameter of $K \in \mathscr{T}_h$, and $h \equiv \max_{K\in\mathscr{T}_h} h_K$ be the maximum element size. Throughout this paper we consider a regular family of triangulations $\{\mathscr{T}_h\}_{h\downarrow 0}$ satisfying the inverse assumption [8], i.e., there exists a positive constant α_0 independent of h such that

$$\frac{h}{h_K} \le \alpha_0, \quad \forall K \in \mathscr{T}_h, \quad \forall h. \tag{4}$$

We define function spaces X_h, M_h, V_h and Q_h by

$$X_h \equiv \{v_h \in C^0(\bar{\Omega})^d;\ v_h|_K \in P_1(K)^d,\ \forall K \in \mathscr{T}_h\},$$
$$M_h \equiv \{q_h \in C^0(\bar{\Omega});\ q_h|_K \in P_1(K),\ \forall K \in \mathscr{T}_h\},$$

$V_h \equiv X_h \cap V$ and $Q_h \equiv M_h \cap Q$, respectively, where $P_1(K)$ is the polynomial space of linear functions on $K \in \mathscr{T}_h$. Let $N_T \equiv [T/\Delta t]$ be a total number of time steps, δ_0 be a positive constant and $(\cdot,\cdot)_K$ be the $L^2(K)^d$ inner product. We define bilinear forms $\mathscr{C}_h$ on $H^1(\Omega)\times H^1(\Omega)$ and $\mathscr{A}_h$ on $(V\times H^1(\Omega))\times(V\times H^1(\Omega))$ by

$$\mathscr{C}_h(p,q) \equiv -\delta_0 \sum_{K\in\mathscr{T}_h} h_K^2 (\nabla p, \nabla q)_K,$$
$$\mathscr{A}_h((u,p),(v,q)) \equiv \nu a(u,v) + b(v,p) + b(u,q) + \frac{1}{\nu}\mathscr{C}_h(p,q). \tag{5}$$

Suppose $f \in C^0([0,T];L^2(\Omega)^d)$ and $u^0 \in V$. Let an approximate function $u_h^0 \in V_h$ of u^0 be given. Our pressure-stabilized characteristics finite element scheme for (1) is to find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that for $n = 1,\cdots,N_T$

$$\Big(\frac{u_h^n - u_h^{n-1}\circ X_1(u_h^{n-1},\Delta t)}{\Delta t}, v_h\Big) + \mathscr{A}_h((u_h^n,p_h^n),(v_h,q_h)) = (f^n, v_h),$$
$$\forall (v_h,q_h) \in V_h \times Q_h. \tag{6}$$

Remark 1. *(i) The choice of the coefficient* $1/\nu$ *before* $\mathscr{C}_h(p,q)$ *in definition* (5) *of* $\mathscr{A}_h$ *is natural from the theoretical point of view as shown in Lemma 4 below.*

(ii) Scheme (6) *leads to a symmetric matrix of the form*

$$\begin{pmatrix} A & B^T \\ B & C \end{pmatrix},$$

where A, B and C correspond to $\frac{1}{\Delta t}(u_h^n, v_h) + \nu a(u_h^n, v_h)$, $b(u_h^n, q_h)$ *and* $\frac{1}{\nu}\mathscr{C}_h(p_h^n, q_h)$, *respectively.*

(iii) The matrix is independent of time step n and regular. The regularity is derived from the fact that $(u_h^n, p_h^n) = (0,0)$ *when* $u_h^{n-1} = f^n = 0$ *since we have*

$$\frac{1}{\Delta t}\|u_h^n\|_0^2 + 2\nu\|D(u_h^n)\|_0^2 + \frac{\delta_0}{\nu}\sum_{K\in\mathscr{T}_h} h_K^2 \|\nabla p_h^n\|_{L^2(K)^d}^2 = 0,$$

by substituting $(u_h^n, -p_h^n) \in V_h \times Q_h$ *into* (v_h, q_h) *in* (6).

(iv) There exists a unique solution (u_h^n, p_h^n) *if* $X_1(u_h^{n-1},\Delta t)$ *maps* Ω *into* Ω. *The condition is ensured if* $\Delta t\|u_h^{n-1}\|_{1,\infty} < 1$ *(cf. Proposition 1).*

3 Main results

In this section we present the main results of stability and error estimates, which are proved in section 4.

Let $\{u^n\}_{n=0}^{N_T}$ and $\{p^n\}_{n=0}^{N_T}$ be sequences of functions and $m \in \{1, \cdots, N_T\}$ be an integer. We use the following norms and seminorms, $\|\cdot\|_{V_h} \equiv \|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_{Q_h} \equiv \|\cdot\|_Q \equiv \|\cdot\|_0$,

$$\|u\|_{l^\infty_m(X)} \equiv \max_{n=0,\cdots,m} \|u^n\|_X, \qquad \|u\|_{l^\infty(X)} \equiv \|u\|_{l^\infty_{N_T}(X)},$$

$$\|u\|_{l^2_m(X)} \equiv \left\{\Delta t \sum_{n=1}^{m} \|u^n\|_X^2\right\}^{1/2}, \qquad \|u\|_{l^2(X)} \equiv \|u\|_{l^2_{N_T}(X)},$$

$$|q|_h \equiv \left\{\sum_{K\in\mathscr{T}_h} h_K^2 (\nabla q, \nabla q)_K\right\}^{1/2},$$

$$|p|_{l^\infty_m(|\cdot|_h)} \equiv \max_{n=0,\cdots,m} |p^n|_h, \qquad |p|_{l^\infty(|\cdot|_h)} \equiv |p|_{l^\infty_{N_T}(|\cdot|_h)},$$

for $X = L^\infty(\Omega)$, $W^{1,\infty}(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$. We additionally define norms $\|(v,q)\|_{X\times M,\nu} \equiv \{\nu\|v\|_X^2 + (1/\nu)\|q\|_M^2\}^{1/2}$ for $X\times M = V\times Q$ and $H^2(\Omega)^d \times H^1(\Omega)$. $\overline{D}_{\Delta t}$ is the backward difference operator defined by

$$\overline{D}_{\Delta t} a^n \equiv \frac{a^n - a^{n-1}}{\Delta t}.$$

After preparing a (pressure-stabilized) Stokes projection using P1/P1-element and two hypotheses, we give the main results.

Definition 1 (Stokes projection). *For $(u,p) \in V \times (Q\cap H^1(\Omega))$ we define the Stokes projection $(\hat{u}_h, \hat{p}_h) \in V_h \times Q_h$ of (u,p) by*

$$\mathscr{A}_h((\hat{u}_h,\hat{p}_h),(v_h,q_h)) = \mathscr{A}_h((u,p),(v_h,q_h)), \quad \forall (v_h,q_h) \in V_h \times Q_h. \tag{7}$$

Hypothesis 1. *The function $u^0 \in V$ satisfies the compatibility condition, $\nabla\cdot u^0 = 0$.*

Hypothesis 2. *The solution (u,p) of (2) satisfies $u \in C^0([0,T];W^{1,\infty}(\Omega)^d) \cap Z^2 \cap H^1(0,T;V\cap H^2(\Omega)^d)$ and $p \in H^1(0,T;Q\cap H^1(\Omega))$.*

Let Δt_* be any fixed positive constant.

Theorem 1. *Let (u,p) be the solution of (2). Suppose Hypotheses 1 and 2 hold. Then, there exist positive constants h_0 and c_0 independent of h and Δt such that the following hold for any h and Δt satisfying*

$$h \in (0,h_0], \quad \Delta t \le \min\{c_0 h^{d/4},\ \Delta t_*\}. \tag{8}$$

(i) Scheme (6) with the first component u_h^0 of the Stokes projection of $(u^0,0)$ by (7) has a unique solution (u_h,p_h).

(ii) There exists a positive constant

$$c_1(1/\nu,\ \|u\|_{C^0([0,T];W^{1,\infty}\cap H^2)},\ \|p\|_{C^0([0,T];H^1)}) \tag{9}$$

independent of h and Δt such that

$$\|u_h\|_{l^\infty(L^\infty)} \le c_1. \tag{10}$$

(iii) There exists a positive constant

$$c(1/\nu,\ T,\ \|u\|_{C^0([0,T];W^{1,\infty})\cap Z^2\cap H^1(0,T;H^2)},\ \|p\|_{H^1(0,T;H^1)}) \tag{11}$$

independent of h and Δt such that

$$\|u_h - u\|_{l^\infty(H^1)},\ \Big\|\overline{D}_{\Delta t} u_h - \frac{\partial u}{\partial t}\Big\|_{l^2(L^2)},\ \|p_h - p\|_{l^2(L^2)} \le c(\Delta t + h). \tag{12}$$

Remark 2. *Since the initial pressure p^0 is not given, we cannot practice the Stokes projection of (u^0, p^0). That is the reason why we employ the Stokes projection of $(u^0, 0)$ and set the first component as u_h^0. This choice is sufficient for the error estimates* (12).

4 Proof of Theorem 1

The section is devoted to the proof of Theorem 1.

We use c to represent the generic positive constant independent of the discretization parameters h and Δt. $c(A)$ means a positive constant depending on A, which monotonically increases as A increases. $c(\|u\|_{C^0([0,T];W^{1,\infty})})$ is simply denoted by $\tilde{c}$. The symbol "$\prime$ (prime)" is sometimes put in order to distinguish between two constants, e.g., c'.

4.1 Preparations

First we present some lemmas and a proposition directly used in the proof. They are fundamental and we omit the proofs.

Lemma 1 (discrete Gronwall's inequality, [17, 33])**.** *Let a_0 and a_1 be non-negative numbers, $\Delta t \in (0, 1/(2a_0)]$ be a real number, and $\{x_n\}_{n\ge 0}$, $\{y_n\}_{n\ge 1}$ and $\{b_n\}_{n\ge 1}$ be non-negative sequences. Suppose*

$$\overline{D}_{\Delta t} x_n + y_n \le a_0 x_n + a_1 x_{n-1} + b_n, \quad \forall n \ge 1.$$

Then, it holds that

$$x_n + \Delta t \sum_{i=1}^{n} y_i \le \exp\{(2a_0 + a_1)n\Delta t\}\Big(x_0 + \Delta t \sum_{i=1}^{n} b_i\Big), \quad \forall n \ge 1.$$

Lemma 2 (Korn's inequality, [11])**.** *Let Ω be a bounded domain with a Lipschitz-continuous boundary. Then, we have the following.*
(i) There exists a positive constant α_1 such that

$$(\|D(v)\|_0^2 + \|v\|_0^2)^{1/2} \ge \alpha_1 \|v\|_1, \quad \forall v \in H^1(\Omega)^d.$$

(ii) There exists a positive constant α_2 such that

$$\|v\|_0 \le \alpha_2 \|D(v)\|_0, \quad \forall v \in H_0^1(\Omega)^d,$$

and the norms $\|D(\cdot)\|_0$ and $\|\cdot\|_1$ are equivalent in $H_0^1(\Omega)^d$.

Lemma 3 ([8,9]). *(i) There exists a positive constant* α_3 *independent of* h *such that*

$$|q_h|_h \leq \alpha_3 \|q_h\|_0, \quad \forall q_h \in Q_h. \tag{13}$$

(ii) There exist an interpolation operator $\Pi_h : L^\infty(\Omega)^d \to X_h$ *and positive constants* α_{4k} $(k = 0, \cdots, 2)$ *independent of* h *such that*

$$\|\Pi_h v\|_{k,\infty} \leq \alpha_{4k} \|v\|_{k,\infty}, \quad \forall v \in W^{k,\infty}(\Omega)^d, \quad k = 0, 1, \tag{14a}$$

$$\|\Pi_h v - v\|_1 \leq \alpha_{42} h \|v\|_2, \quad \forall v \in H^2(\Omega)^d. \tag{14b}$$

(iii) There exist positive constants α_{50} *and* α_{51} *independent of* h *such that*

$$\|v_h\|_{0,\infty} \leq \alpha_{50} h^{-d/6} \|v_h\|_1, \quad \forall v_h \in V_h, \tag{15a}$$

$$\|v_h\|_{1,\infty} \leq \alpha_{51} h^{-d/2} \|v_h\|_1, \quad \forall v_h \in V_h. \tag{15b}$$

Remark 3. *(i) Although the inverse assumption* (4) *is supposed throughout the paper, it is not required for the estimates* (13) *and* (14). *The assumption that* $\{\mathscr{T}_h\}_{h\downarrow 0}$ *is regular is sufficient.*

(ii) The inverse inequality (15a) *is sufficient in this paper, while it is not optimal for* $d = 2$.

The next lemma shows a modified version of the stability inequality in [5, 15], and the lemma easily yields the following Proposition 2.

Lemma 4. *There exist positive constants* h_1 *and* α_6 *independent of* h *and* ν *such that for any* $h \in (0, h_1]$ *and* $\nu > 0$

$$\inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\mathscr{A}_h((u_h,p_h),(v_h,q_h))}{\|(u_h,p_h)\|_{V\times Q,\nu} \|(v_h,q_h)\|_{V\times Q,\nu}} \geq \alpha_6. \tag{16}$$

Proof. Introducing $(\tilde{u}_h, \tilde{p}_h) \equiv (\sqrt{\nu} u_h, (1/\sqrt{\nu}) p_h)$ and $(\tilde{v}_h, \tilde{q}_h) \equiv (\sqrt{\nu} v_h, (1/\sqrt{\nu}) q_h)$, we have

$$\begin{aligned}
&\text{LHS of (16)} \\
&= \inf_{(u_h,p_h)\in V_h\times Q_h} \sup_{(v_h,q_h)\in V_h\times Q_h} \frac{\nu a(u_h,v_h) + b(v_h,p_h) + b(u_h,q_h) + \frac{1}{\nu}\mathscr{C}_h(p_h,q_h)}{\|(u_h,p_h)\|_{V\times Q,\nu} \|(v_h,q_h)\|_{V\times Q,\nu}} \\
&= \inf_{(\tilde{u}_h,\tilde{p}_h)\in V_h\times Q_h} \sup_{(\tilde{v}_h,\tilde{q}_h)\in V_h\times Q_h} \frac{a(\tilde{u}_h,\tilde{v}_h) + b(\tilde{v}_h,\tilde{p}_h) + b(\tilde{u}_h,\tilde{q}_h) + \mathscr{C}_h(\tilde{p}_h,\tilde{q}_h)}{\|(\tilde{u}_h,\tilde{p}_h)\|_{V\times Q} \|(\tilde{v}_h,\tilde{q}_h)\|_{V\times Q}} \geq \alpha_6,
\end{aligned}$$

where the last inequality has been proved in [5, 15]. □

Remark 4. *Although the conventional inf-sup condition [16],*

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{b(v_h,q_h)}{\|v_h\|_1 \|q_h\|_0} \geq \beta^* > 0,$$

does not hold for the pair of V_h *and* Q_h*, the P1/P1 finite element spaces,* $\mathscr{A}_h$ *satisfies the stability inequality* (16) *for this pair.*

Proposition 2. *Suppose* $(u,p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$*. Then, there exist positive constants* h_1 *and* α_7 *independent of* h *and* ν *such that for any* $h \in (0,h_1]$ *and* $\nu > 0$ *the Stokes projection* $(\hat{u}_h, \hat{p}_h)$ *of* (u,p) *by* (7) *satisfies*

$$\sqrt{\nu}\|\hat{u}_h - u\|_1,\ \ \frac{1}{\sqrt{\nu}}\|\hat{p}_h - p\|_0,\ \ \frac{1}{\sqrt{\nu}}|\hat{p}_h - p|_h \le \alpha_7 h \|(u,p)\|_{H^2\times H^1,\nu}. \tag{17}$$

After preparing another lemma, we give the proof of Theorem 1 in the following subsections.

Lemma 5. *Let* $u \in W^{1,\infty}(\Omega)^d$ *be a velocity satisfying* $u|_\Gamma = 0$*. Then, there exists a constant* $\delta_1 \in (0,1)$ *independent of* Δt *such that the following hold for any* $\Delta t \in (0, \delta_1/\|u\|_{1,\infty}]$.
(i) The Jacobian $J \equiv \det(\partial X_1(u,\Delta t)/\partial x)$ *satisfies*

$$\frac{1}{2} \le J \le \frac{3}{2}.$$

(ii) There exists a positive constant α_8 *independent of* Δt *such that*

$$\|v - v\circ X_1(u,\Delta t)\|_0 \le \alpha_8 \|u\|_{0,\infty} \Delta t \|v\|_1, \quad \forall v \in V. \tag{18}$$

Proof. Let $y(x) \equiv X_1(u,\Delta t)(x)$. Since $(\partial y/\partial x)_{ij} = \delta_{ij} - \Delta t \partial u_i/\partial x_j$ for the Kronecker delta δ_{ij}, (i) is easily obtained. We prove (ii). For $s \in [0,1]$ we define $Y(x;s)$ by

$$Y(x;s) \equiv y(x) + s\{x - y(x)\} = x - (1-s)u(x)\Delta t.$$

From the identity

$$v(x) - v(y) = [v(y + s(x-y))]_{s=0}^1 = \Delta t \int_0^1 [\{u(x)\cdot\nabla\}v]\big(Y(x,s)\big)\,ds,$$

the Schwarz inequality and (i), we have

$$\begin{aligned}\big(\text{LHS of (18)}\big)^2 &= \Delta t^2 \int_\Omega \Big\{\int_0^1 [\{u(x)\cdot\nabla\}v]\big(Y(x;s)\big)\,ds\Big\}^2 dx \\ &\le \Delta t^2 \int_0^1 ds \int_\Omega [\{u(x)\cdot\nabla\}v]\big(Y(x;s)\big)^2 dx \\ &\le c\|u\|_{0,\infty}^2 \Delta t^2 \sum_{i,j=1}^d \int_\Omega \frac{\partial v_i}{\partial x_j}(x)^2\,dx,\end{aligned}$$

which implies (18). □

4.2 Estimates under an assumption

Let $\{(u,p)(t);\ t \in [0,T]\} \subset V \times Q$ be the solution of (2). Suppose that there exists a solution $\{(u_h^n, p_h^n)\}_{n=1}^m \subset V_h \times Q_h$ of scheme (6) with an initial value u_h^0 for an integer $m \in \{1, \cdots, N_T\}$. Let $(\hat{u}_h, \hat{p}_h)(t) \in V_h \times Q_h$ be the Stokes projection of $(u,p)(t) \in H^2(\Omega)^d \times H^1(\Omega)$ by (7) for $t \in [0,T]$ and set $\{e_h^n\}_{n=0}^m \subset V_h$, $\{\varepsilon_h^n\}_{n=1}^m \subset Q_h$ and $\{\eta_h(t);\ t \in [0,T]\} \subset V$ as

$$e_h^n \equiv u_h^n - \hat{u}_h^n, \quad \varepsilon_h^n \equiv p_h^n - \hat{p}_h^n, \quad \eta_h(t) \equiv (u - \hat{u}_h)(t).$$

From (2), (6), (7) and the identity $e_h^n = \eta_h^n - u^n + u_h^n$, it holds that for $n = 1, \cdots, m$

$$(\overline{D}_{\Delta t} e_h^n, v_h) + \mathscr{A}_h((e_h^n, \varepsilon_h^n), (v_h, q_h)) = \langle R_h^n, (v_h, q_h)\rangle, \quad \forall (v_h, q_h) \in V_h \times Q_h, \tag{19}$$

where

$$\begin{aligned}
\langle R_h^n, (v_h, q_h)\rangle &\equiv \sum_{i=1}^{4} \langle R_{hi}^n, v_h\rangle + \langle R_{h5}^n, q_h\rangle, \\
\langle R_{h1}^n, v_h\rangle &\equiv \Big(\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1(u^{n-1}, \Delta t)}{\Delta t}, v_h\Big), \\
\langle R_{h2}^n, v_h\rangle &\equiv \frac{1}{\Delta t}\Big(u^{n-1}\circ X_1(u_h^{n-1}, \Delta t) - u^{n-1}\circ X_1(u^{n-1}, \Delta t), v_h\Big), \\
\langle R_{h3}^n, v_h\rangle &\equiv \frac{1}{\Delta t}\Big(\eta_h^n - \eta_h^{n-1}\circ X_1(u_h^{n-1}, \Delta t), v_h\Big), \\
\langle R_{h4}^n, v_h\rangle &\equiv -\frac{1}{\Delta t}\Big(e_h^{n-1} - e_h^{n-1}\circ X_1(u_h^{n-1}, \Delta t), v_h\Big), \\
\langle R_{h5}^n, q_h\rangle &\equiv -\frac{1}{\nu}\mathscr{C}_h(p^n, q_h).
\end{aligned}$$

R_{h5}^n is derived from

$$\begin{aligned}
b(e_h^n, q_h) + \frac{1}{\nu}\mathscr{C}_h(\varepsilon_h^n, q_h) &= -b(\hat{u}_h^n, q_h) - \frac{1}{\nu}\mathscr{C}_h(\hat{p}_h^n, q_h) \\
&= -b(u^n, q_h) - \frac{1}{\nu}\mathscr{C}_h(p^n, q_h) = -\frac{1}{\nu}\mathscr{C}_h(p^n, q_h)
\end{aligned}$$

for $n = 1, \cdots, m$. Let L_m $(m = 0, \cdots, N_T)$ be a real number defined by

$$L_m \equiv \|u_h\|_{l_m^\infty(L^\infty)}.$$

In the next proposition we use Δt_*, h_1 and δ_1, the constants stated just before Theorem 1 and in Lemmas 4 and 5, respectively.

Proposition 3. *Let (u, p) be the solution of (2). Suppose Hypotheses 1 and 2 hold. Assume $h \in (0, h_1]$ and $\Delta t \in (0, \Delta t_*]$. Let u_h^0 be the first component of the Stokes projection of $(u^0, 0)$ by (7). Suppose that for an integer $m \in \{1, \cdots, N_T\}$ there exists a solution $\{(u_h^n, p_h^n)\}_{n=1}^{m-1}$ of scheme (6) satisfying*

$$\Delta t \|u_h\|_{l_{m-1}^\infty(W^{1,\infty})} \le \delta_1. \tag{20}$$

Then, the solution can be extended to (u_h^m, p_h^m) and there exists a positive constant

$$c_*(L_{m-1};\ 1/\nu,\ T,\ \|u\|_{C^0([0,T];W^{1,\infty})\cap Z^2\cap H^1(0,T;H^2)},\ \|p\|_{H^1(0,T;H^1)}) \tag{21}$$

independent of h and Δt such that

$$\|e_h\|_{l_m^\infty(H^1)},\ \ \|\overline{D}_{\Delta t} e_h\|_{l_m^2(L^2)},\ \ |\varepsilon_h|_{l_m^\infty(|\cdot|_h)} \le c_*(L_{m-1})(\Delta t + h). \tag{22}$$

For the proof we use the next lemma, which gives estimates of R_{hi} $(i = 1, \cdots, 5)$. It is proved in Appendix.

Lemma 6. *Under the assumptions in Proposition 3 it holds that for any $v_h \in V_h$, $q_h \in Q_h$ and $n = 1, \cdots, m$*

$$\langle R_{h1}^n, v_h\rangle \le \tilde{c}\sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)}\|v_h\|_0, \tag{23a}$$

$$\langle R_{h2}^n, v_h\rangle \le \tilde{c}\big\{\|e_h^{n-1}\|_0 + c(1/\nu)h\|(u,p)^{n-1}\|_{H^2\times H^1}\big\}\|v_h\|_0, \tag{23b}$$

$$\langle R_{h3}^n, v_h\rangle \le c(1/\nu)h\Big\{\frac{1}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + L_{n-1}\|(u,p)^{n-1}\|_{H^2\times H^1}\Big\}\|v_h\|_0, \tag{23c}$$

$$\langle R_{h4}^n, v_h\rangle \le cL_{n-1}\|e_h^{n-1}\|_1\|v_h\|_0. \tag{23d}$$

$$\langle R_{h5}^n, q_h\rangle \le \frac{\delta_0 h}{\nu}\|p^n\|_1|q_h|_h. \tag{23e}$$

Proof of Proposition 3. From (20) and Remark 1-(iv) the solution can be extended to (u_h^m, p_h^m). We prove (22). It holds that

$$\|\overline{D}_{\Delta t}e_h^n\|^2 + \overline{D}_{\Delta t}(\nu\|D(e_h^n)\|_0^2) + b(\overline{D}_{\Delta t}e_h^n, \varepsilon_h^n) \le \sum_{i=1}^4 \langle R_{hi}^n, \overline{D}_{\Delta t}e_h^n\rangle \tag{24}$$

for $n = 1, \cdots, m$ by (19) with $(v_h, q_h) = (\overline{D}_{\Delta t}e_h^n, 0) \in V_h \times Q_h$ and the inequality $(a^2 - b^2)/2 \le a(a-b)$. Recalling $u_h^0 \in V_h$ is the first component of the Stokes projection of $(u^0, 0)$ by (7), we denote by p_h^0 the second component. Then, it holds that

$$b(u_h^0, q_h) + \frac{1}{\nu}\mathscr{C}_h(p_h^0, q_h) = b(u^0, q_h) = 0, \quad \forall q_h \in Q_h. \tag{25}$$

Since $(\hat{u}_h^0, \hat{p}_h^0)$ is the Stokes projection of (u^n, p^n) for $n = 0$, we have

$$b(\hat{u}_h^0, q_h) + \frac{1}{\nu}\mathscr{C}_h(\hat{p}_h^0, q_h) = b(u^0, q_h) + \frac{1}{\nu}\mathscr{C}_h(p^0, q_h) = \frac{1}{\nu}\mathscr{C}_h(p^0, q_h). \tag{26}$$

Equations (25) and (26) imply

$$b(e_h^0, q_h) + \frac{1}{\nu}\mathscr{C}_h(\varepsilon_h^0, q_h) = -\frac{1}{\nu}\mathscr{C}_h(p^0, q_h),$$

where $\varepsilon_h^0 \equiv p_h^0 - \hat{p}_h^0$. Hence it holds that

$$b(\overline{D}_{\Delta t}e_h^n, q_h) + \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}\varepsilon_h^n, q_h) = -\frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}p^n, q_h), \quad \forall q_h \in Q_h,$$

for $n = 1, \cdots, m$. From the above equation with $q_h = \varepsilon_h^n$ we obtain

$$b(\overline{D}_{\Delta t}e_h^n, \varepsilon_h^n) + \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}\varepsilon_h^n, \varepsilon_h^n) = -\frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}p^n, \varepsilon_h^n) \tag{27}$$

for $n = 1, \cdots, m$. Subtracting (27) from (24) and using Lemma 6 with $v_h = \overline{D}_{\Delta t}e_h^n \in V_h$, the inequality $ab \le \beta a^2/2 + b^2/2\beta$ $(\beta > 0)$, Lemma 2 and the estimate of $|\overline{D}_{\Delta t}p^n|_h$,

$$|\overline{D}_{\Delta t}p^n|_h \le ch\|\overline{D}_{\Delta t}p^n\|_1 \le \frac{ch}{\sqrt{\Delta t}}\|p\|_{H^1(t^{n-1},t^n;H^1)},$$

we have

$$\|\overline{D}_{\Delta t}e_h^n\|_0^2 + \overline{D}_{\Delta t}\Big(\nu\|D(e_h^n)\|_0^2 + \frac{\delta_0}{2\nu}|\varepsilon_h^n|_h^2\Big) \le \sum_{i=1}^4 \langle R_{hi}^n, \overline{D}_{\Delta t}e_h^n\rangle + \frac{1}{\nu}\mathscr{C}_h(\overline{D}_{\Delta t}p^n, \varepsilon_h^n)$$

$$
\begin{aligned}
&\le \Big(\sum_{i=1}^{4}\beta_i\Big)\|\overline{D}_{\Delta t}e_h^n\|_0^2+\frac{\beta_5\delta_0}{\nu}|\varepsilon_h^n|_h^2+\frac{1}{\nu}\Big\{\frac{\tilde c}{\beta_2}+\frac{cL_{n-1}^2}{\beta_4}\Big\}\nu\|D(e_h^{n-1})\|_0^2\\
&\quad+\tilde c'(1/\nu)\Bigg[\frac{\Delta t}{\beta_1}\|u\|^2_{Z^2(t^{n-1},t^n)}+\frac{h^2}{\beta_2}\|(u,p)^{n-1}\|^2_{H^2\times H^1}+\frac{h^2}{\beta_5\Delta t}\|p\|^2_{H^1(t^{n-1},t^n;H^1)}\\
&\qquad+\frac{h^2}{\beta_3}\Big\{\frac{1}{\Delta t}\|(u,p)\|^2_{H^1(t^{n-1},t^n;H^2\times H^1)}+L_{n-1}^2\|(u,p)^{n-1}\|^2_{H^2\times H^1}\Big\}\Bigg]
\end{aligned}
\tag{28}
$$

for $n=1,\cdots,m$ and any positive numbers β_i $(i=1,\cdots,5)$. Thus, there exists a positive constant

$$
\tilde c(L_{m-1};\ 1/\nu,\ T,\ \|u\|_{C^0([0,T];W^{1,\infty})\cap Z^2\cap H^1(0,T;H^2)},\ \|p\|_{H^1(0,T;H^1)})
$$

independent of h and Δt such that

$$
\|e_h\|_{l^\infty_m(H^1)},\ \ \|\overline{D}_{\Delta t}e_h\|_{l^2_m(L^2)},\ \ |\varepsilon_h|_{l^\infty_m(|\cdot|_h)}\le \tilde c(L_{m-1})(\Delta t+h+\|e_h^0\|_1+|\varepsilon_h^0|_h) \tag{29}
$$

by applying Lemma 1 to (28) with proper β_i $(i=1,\cdots,5)$, e.g., $\beta_i=1/8$ for $i=1,\cdots,4$ and $\beta_5=1/(4\Delta t_*)$, and $L_{n-1}\le L_{m-1}$ for any $n=1,\cdots,m$.

Since (u_h^0,p_h^0) and $(\hat u_h^0,\hat p_h^0)$ are the Stokes projections of $(u^0,0)$ and (u^0,p^0) by (7), respectively, it holds that

$$
\begin{aligned}
\|e_h^0\|_1&=\|u_h^0-\hat u_h^0\|_1\le\|u_h^0-u^0\|_1+\|u^0-\hat u_h^0\|_1\le\frac{2\alpha_7 h}{\sqrt\nu}\|(u^0,p^0)\|_{H^2\times H^1,\nu}\\
&\le 2\alpha_7 h\max\{1,1/\nu\}\|(u^0,p^0)\|_{H^2\times H^1}\\
&\le c(1/\nu)h\|(u^0,p^0)\|_{H^2\times H^1},
\end{aligned}
\tag{30a}
$$

$$
\begin{aligned}
|\varepsilon_h^0|_h&=|p_h^0-\hat p_h^0|_h\le|p_h^0-0|_h+|\hat p_h^0-p^0|_h+|p^0|_h\\
&\le\alpha_3\{\|p_h^0-0\|_0+\|\hat p_h^0-p^0\|_0\}+h\|p^0\|_1\\
&\le h(2\alpha_3\alpha_7\sqrt\nu\|(u^0,p^0)\|_{H^2\times H^1,\nu}+\|p^0\|_1)\\
&\le h(2\alpha_3\alpha_7\max\{\nu,1\}\|(u^0,p^0)\|_{H^2\times H^1}+\|p^0\|_1)\\
&\le ch\|(u^0,p^0)\|_{H^2\times H^1}.
\end{aligned}
\tag{30b}
$$

Combining (30) with (29), we can take a positive constant $c_*(L_{m-1})$ independent of h and Δt of the form (21) such that (22) holds. □

4.3 Proof of Theorem 1

The proof is given by induction through three steps.

Step 1 (definitions of c_i, $i=0,\cdots,2$, and h_0): Let Δt_*, h_1, δ_1 and c_* be the constants stated just before Theorem 1, in Lemmas 4 and 5 and Proposition 3, respectively. We can take positive constants c_1 and c_2 such that the inequalities

$$
c_1\ge\max\big\{\|u_h^0\|_{0,\infty},\ 2\|\hat u_h\|_{C^0(L^\infty)}\big\}, \tag{31a}
$$

$$
c_2\ge\max\big\{\|u_h^0\|_{1,\infty}h^{d/4},\ 2\|\hat u_h\|_{C^0(W^{1,\infty})}h^{d/4}\big\}, \tag{31b}
$$

are valid for any $h\in(0,h_1]$ by the following estimates. From Proposition 2, (14) and (15) it holds that

$$
\|\hat u_h(t)\|_{0,\infty}\le\|\hat u_h(t)-\Pi_h u(t)\|_{0,\infty}+\|\Pi_h u(t)\|_{0,\infty}
$$

$$
\begin{aligned}
&\le \alpha_{50} h^{-d/6} \|\hat{u}_h(t) - \Pi_h u(t)\|_1 + \alpha_{40} \|u(t)\|_{0,\infty} \\
&\le \alpha_{50} h^{-d/6} \big(\|\hat{u}_h(t) - u(t)\|_1 + \|u(t) - \Pi_h u(t)\|_1 \big) + \alpha_{40} \|u(t)\|_{0,\infty} \\
&\le \alpha_{50} h^{1-d/6} \Big\{ \frac{\alpha_7}{\sqrt{\nu}} \|(u,p)(t)\|_{H^2 \times H^1, \nu} + \alpha_{42} \|u(t)\|_2 \Big\} + \alpha_{40} \|u(t)\|_{0,\infty} \\
&\le c(1/\nu) \big\{ h_1^{1-d/6} \|(u,p)\|_{C^0(H^2 \times H^1)} + \|u\|_{C^0(L^\infty)} \big\} < \infty, \qquad (32a)
\end{aligned}
$$

$$
\begin{aligned}
\|\hat{u}_h(t)\|_{1,\infty} h^{d/4} &\le \big(\|\hat{u}_h(t) - \Pi_h u(t)\|_{1,\infty} + \|\Pi_h u(t)\|_{1,\infty} \big) h^{d/4} \\
&\le \alpha_{51} h^{-d/4} \|\hat{u}_h(t) - \Pi_h u(t)\|_1 + \alpha_{41} h^{d/4} \|u(t)\|_{1,\infty} \\
&\le \alpha_{51} h^{-d/4} \big(\|\hat{u}_h(t) - u(t)\|_1 + \|u(t) - \Pi_h u(t)\|_1 \big) + \alpha_{41} h^{d/4} \|u(t)\|_{1,\infty} \\
&\le \alpha_{51} h^{1-d/4} \Big\{ \frac{\alpha_7}{\sqrt{\nu}} \|(u,p)(t)\|_{H^2 \times H^1, \nu} + \alpha_{42} \|u(t)\|_2 \Big\} + \alpha_{41} h^{d/4} \|u(t)\|_{1,\infty} \\
&\le c(1/\nu) \big\{ h_1^{1-d/4} \|(u,p)\|_{C^0(H^2 \times H^1)} + h_1^{d/4} \|u\|_{C^0(W^{1,\infty})} \big\} < \infty. \qquad (32b)
\end{aligned}
$$

Similar estimates are also obtained for $\|u_h^0\|_{0,\infty}$ and $\|u_h^0\|_{1,\infty} h^{d/4}$.

Then, we define a constant c_0 by

$$
c_0 \equiv \min \Big\{ \frac{\delta_1}{c_2}, \ \frac{c_2}{4\alpha_{51} c_*(c_1)} \Big\}. \qquad (33a)
$$

Let a positive constant h_2 be small enough to satisfy

$$
\begin{cases}
2\alpha_{50} c_*(c_1)(c_0 h_2^{d/12} + h_2^{1-d/6}) \le c_1, \\
4\alpha_{51} c_*(c_1) h_2^{1-d/4} \le c_2,
\end{cases} \qquad (33b)
$$

and we set $h_0 \equiv \min\{h_1, h_2\}$.

Step 2 (induction): Under the condition (8) we now consider the scheme (6) with the first component u_h^0 of the Stokes projection of $(u^0, 0)$ by (7). For $m \in \{0, \cdots, N_T\}$ we set property P(m),

$$
\mathrm{P}(m): \begin{cases}
\text{(a) The scheme (6) is solvable until } n = m, \\
\text{(b) } \|u_h\|_{l_m^\infty(L^\infty)} (= L_m) \le c_1, \\
\text{(c) } \|u_h\|_{l_m^\infty(W^{1,\infty})} h^{d/4} \le c_2.
\end{cases}
$$

It is trivial that P(0) holds true by the definitions of the constants c_1 and c_2. Supposing that P($m-1$) holds true for an integer $m \in \{1, \cdots, N_T\}$, we prove that P(m) also does. It holds that from (8), P($m-1$) and (33a)

$$
\Delta t \|u_h\|_{l_{m-1}^\infty(W^{1,\infty})} \le c_0 h^{d/4} \|u_h\|_{l_{m-1}^\infty(W^{1,\infty})} \le c_0 c_2 \le \delta_1,
$$

which implies that the assumptions of Proposition 3 are satisfied. We, therefore, obtain P(m)-(a) and the estimate (22). We show P(m)-(b) and (c). From (31) and (33) we have

$$
\begin{aligned}
\|u_h^m\|_{0,\infty} &\le \|u_h^m - \hat{u}_h^m\|_{0,\infty} + \|\hat{u}_h^m\|_{0,\infty} \le \alpha_{50} h^{-d/6} \|u_h^m - \hat{u}_h^m\|_1 + \frac{c_1}{2} \\
&\le \alpha_{50} c_*(c_1) h^{-d/6} (\Delta t + h) + \frac{c_1}{2} \\
&\le \alpha_{50} c_*(c_1)(c_0 h_2^{d/12} + h_2^{1-d/6}) + \frac{c_1}{2} \le c_1,
\end{aligned}
$$

$$\begin{aligned}
\|u_h^m\|_{1,\infty} h^{d/4} &\le \big(\|u_h^m - \hat{u}_h^m\|_{1,\infty} + \|\hat{u}_h^m\|_{1,\infty}\big) h^{d/4} \le \alpha_{51} h^{-d/4} \|u_h^m - \hat{u}_h^m\|_1 + \frac{c_2}{2} \\
&\le \alpha_{51} c_*(c_1) h^{-d/4} (\Delta t + h) + \frac{c_2}{2} \\
&\le \alpha_{51} c_*(c_1)(c_0 + h_2^{1-d/4}) + \frac{c_2}{2} \le c_2.
\end{aligned}$$

Thus, P(m) holds true, and the induction is completed.

Step 3 (existence (i), stability (ii) and error estimates (iii) of Theorem 1): The completed induction implies P(N_T) holds true. From (32) we can express c_1 in the form (9). Hence we have existence (i) and stability (ii). Since the inequalities

$$L_n \le L_{N_T} = \|u_h\|_{l^\infty(L^\infty)} \le c_1, \quad n = 0, \cdots, N_T, \tag{34}$$

are satisfied, the first and second inequalities of (12) in (iii) hold for $c \equiv c_*(c_1)$ by Proposition 3. The third inequality of (12) in (iii) is obtained as it holds that

$$\begin{aligned}
\|\varepsilon_h^n\|_0 &\le \sqrt{\nu} \|(e_h^n, \varepsilon_h^n)\|_{V \times Q, \nu} \le \frac{\sqrt{\nu}}{\alpha_6} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\mathscr{A}_h((e_h^n, \varepsilon_h^n), (v_h, q_h))}{\|(v_h, q_h)\|_{V \times Q, \nu}} \\
&= \frac{\sqrt{\nu}}{\alpha_6} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\langle R_h^n, (v_h, q_h) \rangle - (\overline{D}_{\Delta t} e_h^n, v_h)}{\|(v_h, q_h)\|_{V \times Q, \nu}} \\
&\le \tilde{c}(1/\nu, c_1) \Big[\|e_h^{n-1}\|_1 + \|\overline{D}_{\Delta t} e_h^n\|_0 + \sqrt{\Delta t} \|u\|_{Z^2(t^{n-1}, t^n)} \\
&\quad + h \Big(\frac{1}{\sqrt{\Delta t}} \|(u,p)\|_{H^1(t^{n-1}, t^n; H^2 \times H^1)} + \|(u,p)^{n-1}\|_{H^2 \times H^1} + \|p^n\|_1 \Big) \Big]
\end{aligned} \tag{35}$$

for $n = 1, \cdots, N_T$. Here we have used (13), (34) and Lemmas 4 and 6. Finally, $c_*(c_1)$, (9) and (35) derive the dependency (11) of the constant c in (12). □

5 Numerical results

In this section two and three dimensional test problems are computed by scheme (6) in order to observe the convergence order.

Quadrature formulae of degree five for $d = 2$ (seven points) and 3 (fifteen points) [30] are employed for computation of the integral

$$\int_K u_h^{n-1} \circ X_1(u_h^{n-1}, \Delta t)(x) v_h(x) \, dx$$

appearing in scheme (6). $\delta_0 = 0.05$ is chosen by some numerical experience. The system of linear equations is solved by MINRES.

Example 1. *In problem* (1) *we set* $\Omega = (0, \pi)^d$, $T = 1$ *and two values of* ν,

$$\nu = 1, \ 10^{-1}.$$

The functions f *and* u^0 *are given so that the exact solution is*

$$\begin{aligned}
&(u,p)(x,t) \\
&= \begin{cases} \big(-\phi(x_1,x_2,t), \phi(x_2,x_1,t), \rho(x_1,x_2,0,t)\big) & (d=2), \\ \big(\psi(x_1,x_2,x_3,t), \psi(x_2,x_3,x_1,t), \psi(x_3,x_1,x_2,t), \rho(x_1,x_2,x_3,t)\big) & (d=3), \end{cases}
\end{aligned}$$

where

$$\begin{aligned}\phi(a,b,t) &\equiv \sin^2 a \sin b \{\sin(a+t)+3\sin(a+2b+t)\},\\ \psi(a,b,c,t) &\equiv \sin^2 a \sin b \sin c \big[4\cos b \sin c \sin(c+a+t)\\ &\qquad - \sin b\{3\sin(b+2c+t)+\sin(b+t)\}\big],\\ \rho(a,b,c,t) &\equiv \sin(a+2b+c+t).\end{aligned}$$

Let N be the division number of each side of the domain. We set $N = 16, 32, 64, 128$ and 256 for $d = 2$ and $N = 8, 16, 32$ and 64 for $d = 3$, and (re)define $h \equiv \pi/N$. Sample meshes are shown in Fig. 1 for $d = 2$ (left, $N = 16$) and 3 (right, $N = 8$). The time increment Δt is set to be $\Delta t = 1/N = h/\pi$. Let (u_h, p_h) be the solution of scheme (6). The initial function u_h^0 is chosen as the first component of the Stokes projection of $(u^0, 0)$ by (7). We define Err by

$$Err \equiv \frac{\sqrt{\nu}\|u_h - \Pi_h u\|_{l^2(H^1)} + (1/\sqrt{\nu})\|p_h - \Pi_h p\|_{l^2(L^2)}}{\sqrt{\nu}\|\Pi_h u\|_{l^2(H^1)} + (1/\sqrt{\nu})\|\Pi_h p\|_{l^2(L^2)}}$$

as the relative error between (u, p) and (u_h, p_h). Fig. 2 shows graphs of Err versus h in logarithmic scale. We can see that Err is almost of first order in h for both $d = 2$ and 3, and the results are consistent with Theorem 1.

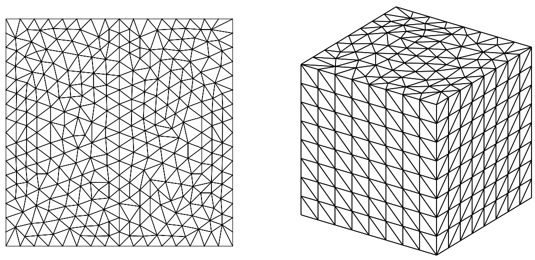

Figure 1: Sample meshes used for Example 1 in 2D (left, $N = 16$) and 3D (right, $N = 8$).

6 Conclusions

As for a pressure-stabilized characteristics finite element scheme for the Navier-Stokes equations, we have proved stability and convergence results with the optimal error estimates for the velocity and the pressure. The results hold under a condition on the time increment of the form $\Delta t \le ch^{d/4}$, which is the same as that of [31] by the characteristics finite element scheme using the conventional elements. The scheme is based on the method of characteristics, which works well for convection-dominated problems and leads to a symmetric coefficient matrix of the system of linear equations. Since a cheap P1/P1 finite element is employed, the degrees of freedom are smaller than that of conventional elements for the equations, e.g., P2/P1. These advantages, i.e., symmetry of the coefficient matrix and small degrees of freedom, reduces computation cost (time and memory). Two and three dimensional numerical results obtained by the linear solver MINRES have been shown and the numerical convergence orders have been recognized to be consistent with the theoretical results. Consequently,

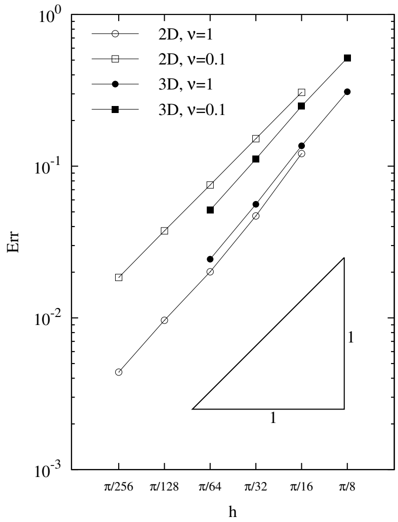

Figure 2: Err versus h for two and three dimensional test problems in Example 1.

the scheme leads to efficient computation especially in 3D as well as mathematical reliability with optimal error estimates, first order in both time and space.

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Appendix

Proof of Lemma 6 We firstly prove (23a). For $X(t) = X(t;x,t^n)$ and $Y^n(x;s) \equiv sX(t^{n-1}) + (1-s)X_1(u^{n-1},\Delta t)(x)$ $(s \in [0,1])$ we have

$$\begin{aligned}
&\frac{Du^n}{Dt} - \frac{u^n - u^{n-1}\circ X_1(u^{n-1},\Delta t)}{\Delta t} \\
&= \Big\{ \frac{Du^n}{Dt}(x) - \frac{u(X(t^n),t^n) - u(X(t^{n-1}),t^{n-1})}{\Delta t} \Big\} \\
&\quad - \frac{1}{\Delta t}\big\{u^{n-1}(X(t^{n-1})) - u^{n-1}\circ X_1(u^{n-1},\Delta t)(x)\big\} \\
&= \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} \Big\{ \frac{Du}{Dt}(X(t^n),t^n) - \frac{Du}{Dt}(X(t),t) \Big\} dt - \frac{1}{\Delta t}\big[u^{n-1}\big(Y^n(x;s)\big)\big]_{s=0}^1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} dt \int_t^{t^n} \frac{D^2 u}{Dt^2}(X(s),s)ds \\
&\quad - \frac{1}{\Delta t}\int_0^1 \big[\big\{\big(X(t^{n-1}) - X_1(u^{n-1},\Delta t)(x)\big)\cdot\nabla\big\}u^{n-1}\big]\big(Y^n(x;s)\big)ds \\
&\equiv R^n_{h11}(x) + R^n_{h12}(x),
\end{aligned}
$$

and

$$
\langle R^n_{h1}, v_h\rangle = (R^n_{h11}, v_h) + (R^n_{h12}, v_h) \le (\|R^n_{h11}\|_0 + \|R^n_{h12}\|_0)\|v_h\|_0. \tag{A.1}
$$

We evaluate $\|R^n_{h1i}\|_0$ $(i=1,2)$. From the Schwarz inequality it holds that

$$
\|R^n_{h11}\|_0 \le \sqrt{\frac{\Delta t}{3}}\Big\|\frac{D^2u}{Dt^2}\Big\|_{L^2(t^{n-1},t^n;L^2)} \le c\sqrt{\Delta t}\|u\|_{Z^2(t^{n-1},t^n)}. \tag{A.2}
$$

From the estimate,

$$
\begin{aligned}
&|X(t^{n-1};x,t^n) - X_1(u^{n-1},\Delta t)(x)| = \Big|u^{n-1}(x)\Delta t - \int_{t^{n-1}}^{t^n} u(X(t),t)\,dt\Big| \\
&= \Big|\Big\{u^n(x) - \int_{t^{n-1}}^{t^n}\frac{\partial u}{\partial t}(x,t)\,dt\Big\}\Delta t - \int_{t^{n-1}}^{t^n} u(X(t),t)\,dt\Big| \\
&= \Big|\int_{t^{n-1}}^{t^n}\{u^n(x) - u(X(t),t)\}\,dt - \Delta t\int_{t^{n-1}}^{t^n}\frac{\partial u}{\partial t}(x,t)\,dt\Big| \\
&= \Big|\int_{t^{n-1}}^{t^n}\big[u(X(s),s)\big]_{s=t}^{t^n}\,dt - \Delta t\int_{t^{n-1}}^{t^n}\frac{\partial u}{\partial t}(x,t)\,dt\Big| \\
&= \Big|\int_{t^{n-1}}^{t^n} dt\int_t^{t^n}\frac{Du}{Dt}(X(s),s)\,ds - \Delta t\int_{t^{n-1}}^{t^n}\frac{\partial u}{\partial t}(x,t)\,dt\Big| \\
&\le \Delta t\int_{t^{n-1}}^{t^n}\Big\{\Big|\frac{Du}{Dt}(X(t),t)\Big| + \Big|\frac{\partial u}{\partial t}(x,t)\Big|\Big\}dt,
\end{aligned}
$$

we have

$$
\|R^n_{h12}\|_0 \le \tilde c\sqrt{\Delta t}\Big(\Big\|\frac{Du}{Dt}\Big\|_{L^2(t^{n-1},t^n;L^2)} + \Big\|\frac{\partial u}{\partial t}\Big\|_{L^2(t^{n-1},t^n;L^2)}\Big) \le \tilde c\sqrt{\Delta t}\|u\|_{Z^1(t^{n-1},t^n)}. \tag{A.3}
$$

Combining (A.2) and (A.3) with (A.1), we obtain (23a).

For the estimate (23b) we have

$$
\begin{aligned}
&\Big|\frac{1}{\Delta t}\big\{u^{n-1}\circ X_1(u_h^{n-1},\Delta t) - u^{n-1}\circ X_1(u^{n-1},\Delta t)\big\}(x)\Big| \\
&= \Big|\frac{1}{\Delta t}\Big[u^{n-1}\Big(sX_1(u_h^{n-1},\Delta t)(x) + (1-s)X_1(u^{n-1},\Delta t)(x)\Big)\Big]_{s=0}^1\Big| \\
&= \Big|\int_0^1\{(u^{n-1} - u_h^{n-1})(x)\cdot\nabla\}u^{n-1}\Big(sX_1(u_h^{n-1},\Delta t) + (1-s)X_1(u^{n-1},\Delta t)\Big)\,ds\Big| \\
&= \Big|\int_0^1\{(\eta_h^{n-1} - e_h^{n-1})(x)\cdot\nabla\}u^{n-1}\Big(sX_1(u_h^{n-1},\Delta t) + (1-s)X_1(u^{n-1},\Delta t)\Big)\,ds\Big| \\
&\le \tilde c(|\eta_h^{n-1}(x)| + |e_h^{n-1}(x)|),
\end{aligned}
$$

which implies

$$
\langle R^n_{h2}, v_h\rangle = \frac{1}{\Delta t}\Big(u^{n-1}\circ X_1(u_h^{n-1},\Delta t) - u^{n-1}\circ X_1(u^{n-1},\Delta t), v_h\Big)
$$

$$\leq \tilde{c}(\|\eta_h^{n-1}\|_0 + \|e_h^{n-1}\|_0)\|v_h\|_0$$
$$\leq \tilde{c}\{c(1/\nu)h\|(u,p)^{n-1}\|_{H^2\times H^1} + \|e_h^{n-1}\|_0\}\|v_h\|_0.$$

The estimates (23c), (23d) and (23e) are obtained as

$$\begin{aligned}
\langle R_{h3}^n, v_h\rangle &= \frac{1}{\Delta t}\Big(\eta_h^n - \eta_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\Big)\\
&= (\overline{D}_{\Delta t}\eta_h^n, v_h) + \frac{1}{\Delta t}\Big(\eta_h^{n-1} - \eta_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\Big)\\
&\leq \|\overline{D}_{\Delta t}\eta_h^n\|_0\|v_h\|_0 + \alpha_8 L_{n-1}\|\eta_h^{n-1}\|_1\|v_h\|_0\\
&\leq \Big\{\frac{1}{\sqrt{\Delta t}}\Big\|\frac{\partial \eta_h}{\partial t}\Big\|_{L^2(t^{n-1},t^n;L^2)} + \alpha_8 L_{n-1}\|\eta_h^{n-1}\|_1\Big\}\|v_h\|_0\\
&\leq c(1/\nu)h\Big\{\frac{1}{\sqrt{\Delta t}}\|(u,p)\|_{H^1(t^{n-1},t^n;H^2\times H^1)} + L_{n-1}\|(u,p)^{n-1}\|_{H^2\times H^1}\Big\}\|v_h\|_0,\\
\langle R_{h4}^n, v_h\rangle &= -\frac{1}{\Delta t}\Big(e_h^{n-1} - e_h^{n-1}\circ X_1(u_h^{n-1},\Delta t), v_h\Big) \leq \alpha_8 L_{n-1}\|e_h^{n-1}\|_1\|v_h\|_0,\\
\langle R_{h5}^n, q_h\rangle &= -\frac{1}{\nu}\mathscr{C}_h(p^n, q_h) \leq \frac{\delta_0}{\nu}|p^n|_h|q_h|_h \leq \frac{\delta_0 h}{\nu}\|p^n\|_1|q_h|_h,
\end{aligned}$$

from (17) and (18). □

References

[1] Baba, K., Tabata, M.: On a conservative upwind finite element scheme for convective diffusion equations. RAIRO Analyse nuérique. **15**, 3–25 (1981)

[2] Barrett, R., Berry, M., Chan, T.F., Demmel, J., Donato, J., Dongarra, J., Eijkhout, V., Pozo, R., Romine, C., van der Vorst, H.: Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods. SIAM, Philadelphia (1994)

[3] Boukir, K., Maday, Y., Métivet, B., Razafindrakoto, E.: A high-order characteristics/finite element method for the incompressible Navier-Stokes equations. International Journal for Numerical Methods in Fluids. **25**, 1421–1454 (1997)

[4] Braack, M., Burman, E., John, V., Lube, G.: Stabilized finite element methods for the generalized Oseen problem. Computer Methods in Applied Mechanics and Engineering. **196**, 853–866 (2007)

[5] Brezzi, F., Douglas Jr., J.: Stabilized mixed methods for the Stokes problem. Numerische Mathematik. **53**, 225–235 (1988)

[6] Brezzi, F., Pitkäranta, J.: On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch editor, Efficient solutions of Elliptic Systems. Vieweg, Wiesbaden, 11–19 (1984)

[7] Brooks, A.N., Hughes, T.J.R.: Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. Computer Methods in Applied Mechanics and Engineering. **32**, 199–259 (1982)

[8] Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)

[9] Clément, P.: Approximation by finite element functions using local regularization. RAIRO Analyse nuérique. **2**, 77–84 (1975)

[10] Douglas Jr., J., Russell, T.F.: Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures. SIAM Journal on Numerical Analysis. **19**, 871–885 (1982)

[11] Duvaut, G., Lions, J.L.: Inequalities in mechanics and physics. Springer, Berlin (1976)

[12] Ewing, R.E., Russell, T.F.: Multistep Galerkin methods along characteristics for convection-diffusion problems. In Vichnevetsky, R. and Stepleman, R.S. editors, Advances in Computer Methods for Partial Differential Equations. **4**, 28–36, IMACS (1981)

[13] Ewing, R.E., Russell, T.F., Wheeler, M.F.: Simulation of miscible displacement using mixed method and modified method of characteristics. Proceedings of SPE Reservoir Simulation Symposium (15-18 November, 1983, San Francisco), Society of Petroleum Engineers. 71–81 (1983)

[14] Franca, L.P., Frey, S.L.: Stabilized finite element methods: II. The incompressible Navier-Stokes equations. Computer Methods in Applied Mechanics and Engineering. **99**, 209–233 (1992)

[15] Franca, L.P., Stenberg, R.: Error analysis of some Galerkin least squares methods for the elasticity equations. SIAM Journal on Numerical Analysis. **28**, 1680–1697 (1991)

[16] Girault, V., Raviart, P.-A.: Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms. Springer, Berlin (1986)

[17] Heywood, J.G., Rannacher, R.: Finite-element approximation of the nonstationary Navier-Stokes problem. Part IV: Error analysis for second-order time discretization. SIAM Journal on Numerical Analysis. **27**, 353–384 (1990)

[18] Hughes, T.J.R., Franca, L.P., Mallet, M.: A new finite element formulation for computational fluid dynamics: VI. Convergence analysis of the generalized SUPG formulation for linear time-dependent multidimensional advective-diffusive systems. Computer Methods in Applied Mechanics and Engineering. **63**, 97–112 (1987)

[19] Johnson, C.: Numerical Solution of Partial Differential Equations by the Finite Element Method. Cambridge Univ. Press. Cambridge (1987)

[20] Notsu, H.: Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme. Transactions of Japan Society for Computational Engineering and Science. ONLINE ISSN:1347-8826 (2008)

[21] Notsu, H., Rui, H., Tabata, M.: Development and L^2-analysis of a single-step characteristics finite difference scheme of second order in time for convection-diffusion problems. to appear in Journal of Algorithms & Computational Technology.

[22] Notsu, H., Tabata, M.: A combined finite element scheme with a pressure stabilization and a characteristic-curve method for the Navier-Stokes equations. Transactions of the Japan Society for Industrial and Applied Mathematics. **18**, 427–445 (2008) (in Japanese)

[23] Notsu, H., Tabata, M.: A single-step characteristic-curve finite element scheme of second order in time for the incompressible Navier-Stokes equations, Journal of Scientific Computing. **38**, 1–14 (2009)

[24] Notsu, H., Tabata, M.: Error estimates of a pressure-stabilized characteristics finite element scheme for the Oseen equations (submitted)

[25] Pironneau, O.: On the transport-diffusion algorithm and its applications to the Navier-Stokes equations. Numerische Mathematik. **38**, 309–332 (1982)

[26] Pironneau, O.: Finite Element Methods for Fluids. John Wiley & Sons, Chichester (1989)

[27] Pironneau, O., Tabata, M.: Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type. International Journal for Numerical Methods in Fluids. **64**, 1240–1253 (2010)

[28] Rui, H., Tabata, M.: A second order characteristic finite element scheme for convection-diffusion problems. Numerische Mathematik. **92**, 161–177 (2002)

[29] Saad, Y.: Iterative Methods for Sparse Linear Systems. SIAM, Philadelphia (2003)

[30] Stroud, A.H.: Approximate calculation of multiple integrals. Prentice-Hall, Englewood Cliffs, New Jersey (1971).

[31] Süli, E.: Convergence and nonlinear stability of the Lagrange-Galerkin method for the Navier-Stokes equations. Numerische Mathematik. **53**, 459–483 (1988)

[32] Tabata, M.: A finite element approximation corresponding to the upwind finite differencing. Memoirs of Numerical Mathematics. **4**, 47–63 (1977)

[33] Tabata, M., Tagami, D.: Error estimates for finite element approximations of drag and lift in nonstationary Navier-Stokes flows. Japan Journal of Industrial and Applied Mathematics. **17**, 371–389 (2000)

[34] Tezduyar, T.E., Mittal, S., Ray, S.E., Shih, R.: Incompressible flow computations with stabilized bilinear and linear equal-order-interpolation velocity-pressure elements. Computer Methods in Applied Mechanics and Engineering. **95**, 221–242 (1992)