WIAS Discussion Paper No.2010-008

Bidding behavior for a keyword auction in a sealed bid environment

March 31, 2011

Yoshio Kamijo
Waseda Institute for Advanced Study, Waseda University

早稲田大学高等研究所
Waseda Institute for Advanced Study

1-6-1 Nishiwaseda, Shinjuku-ku, Tokyo 169-8050, Japan
Tel: +81-3-5286-2460 ; Fax: +81-3-5286-2470

Bidding behavior for a keyword auction in a sealed bid environment

Yoshio Kamijo*

March, 2011

Abstract

In this paper, we explore bidding behavior for a repeatedly played keyword auction. In a keyword auction in practice, a bidder does not know the current bids submitted by the others, and thus, he cannot follow the greedy bidding strategy where he changes the bid to the one that produces the most favorable outcome for the bidder, taking other bidders' bids in the previous period as given. We propose a secure greedy bidding that can be executed under such sealed bid environment. We define a stable bid profile as the fixed point of the secure greedy bidding and show that even in the sealed bid situation, the stable bid profile exists and satisfies several good properties. Moreover, we also examine other versions of bidding behavior that needs neither the current bids of others nor the values of other bidders. We show that the bidding behavior that involves with the trial increase of the bid leads to the unique fixed point of the secure greedy bidding.
Keyword: keyword auction, sealed bid environment, bidding behavior, secure greedy bidding, trial and error bidding

*Corresponding author. Waseda Institute for Advanced Study, 1-6-1, Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan. E-mail: kami-jo@suou.waseda.jp. Tel: +81-3-3203-7391

1 Introduction

Internet advertisements that are shown along with search results for a keyword or a combination of keywords are sold through keyword auctions. Each time a user enters a search term into a search engine, an auction mechanism allocates the advertising slots within that user's search results. A keyword auction is done more than 1 million times in a day all over the world and the Internet advertisement revenue via the keyword auctions is a principal source of revenue of search engines.

The generalized second price auction (GSP) and the auction mechanisms based on it, are the most widely used for selling advertisements on Internet search engines. Based on the bids that advertisers submit for a keyword, the ad-slots are allocated according to the descending order of the bids, i.e., the top position is allocated to the bidder with the highest bid, the second position is allocated to the bidder with the second highest bid, and so on. Every time a search engine user clicks the advertisement, the advertiser pays the next highest bid. Thus, the advertiser in the highest position pays the bid of the advertiser in the second highest position, the advertiser in the second highest position pays the bid of the advertiser in the third highest position, and so on.

Since the payment of each advertiser does not depend on his bid but on the bid submitted by the advertiser in one lower position from him, the GSP auction has a similarity to the Vickrey auction for selling one object (Vickrey (1961)). In fact, when there is only one ad-slot, the GSP is equivalent to the Vickrey auction and thus, it has a nice property: for each advertiser, submitting his true expected revenue from the sponsored link is a dominant strategy and thus, advertisers do not need to distress themselves from determining their bids. However, when there are multiple ad-slots, the GSP does not have the truth-telling property (Edelman, Ostrovsky and Schwarz (2007)). This indicates that the actual bidding behavior in the GSP should exhibit the complicated figure. Edelman and Ostrovsky (2007) reported that bids observed in the GSP are largely fluctuated and this can be caused by the bidders' strategic behaviors.

In this paper, we explore bidding behavior for a keyword auction theoretically. As explained in the previous paragraph, the bids submitted by advertisers varies over period. This suggests that we should pay attention to the dynamic aspect of the bidding behavior. After describing the bidding behavior of the advertises in a keyword auction, we argue whether the stable bid profile against the bidding behavior exists or not, what property the stable bid profile possesses, and how long it takes until the stable bid profile is realized.

Our analysis considers a simplified model of keyword auctions. We assume that the values (expected revenue) per click of advertisers and the click through-rates (CTRs) of ad-slots are common knowledge. In each period, an advertiser can change his bid according to the result of the keyword auction played in a previous period. All the information that is available for the advertiser is his revenue, his payment to a search engine and how the ad-slots are assigned to advertisers, in the previous period. The advertiser does not know the actual bid profile of the other advertisers. This means that the advertisers cannot follow the greedy bidding strategy where in each period, the advertisers update their bids according to the best response dynamics. Since a keyword auction in practice adopt a *sealed bid* generalized second price auction, advertisers update their bids according to the limited information.

We first propose a greedy bidding strategy in a sealed bid keyword auction. The secure greedy bidding (SGB) is defined and the idea of SGB is partly from the balanced bidding proposed by Cary, Das, Edelman, Giotis, Heimerl, Karlin, Mathieu and Schwarz (2007) for the open bid environment. We show that the stable bid profile against the SGB (or the fixed point of their bidding behavior) exists, and in the bid profile, the ad-slots are assigned to advertisers in the same way as the one if all advertisers honestly announce their values, and the revenue of a search engine is the same as the one in the truth-telling equilibrium in the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey (1961); Clarke (1971); Groves (1973)).

Next, we examine whether their bids converge to the stable bid profile if they update their bids repeatedly according to the SGB. In a synchronous model where in each period, every bidder changes his bid according to the SGB strategy, we provide a counter example for the convergence. In this example of three ad-slots and three bidders, the bid cycle over periods arises. On the other hand, if we consider an asynchronous model where in each period, one bidder is randomly selected and this bidder changes his bid according to the SGB strategy, the convergence of the bidding behavior is guaranteed in the sealed bid repeated keyword auction. These are the same observation in a open bid environment reported by Cary et al. (2007). Non-convergence in a synchronous model and the convergence in an asynchronous model are found in a open bid environment.

A limitation of the results for the SGB is that it requires the information on the values of other advertisers instead of the current bid profile. Since the balanced bidding does not need such information, there is the trade-off between the SGB and the balanced bidding for the required information. Thus, we explore other two types of bidding behavior that need neither the current bids of others nor the values of other bidders.

The paper is organized as follows. In Section 2, we explain the model of a keyword auction. In Section 3, we introduce the secure greedy bidding for a sealed bid environment and show the basic properties of this bidding behavior. The convergence results are shown in Section 4. In Section 5, we discuss other versions of the bidding behavior for a sealed bid keyword auction.

2 Model

An auction on a keyword, simply a keyword-auction, is defined by the following components. There are N advertisers (bidders) participating in a keyword auction, each advertiser i having a value or expected revenue v_i for a click of the ad. We assume that $v_1 > v_2 > ... > v_N$. There are K ad-slots with click-through rates (CTRs) $\alpha_1 > \alpha_2 > ... > \alpha_K$, where α_k is the estimated probability of being clicked or the estimated number of clicks per given period, for an advertiser in the k-th ad-slot. We also set $\alpha_k = 0$ for all $k > K$ and assume $N \geqq K$. Each advertiser submits a bid to the auction. The bid submitted by i is denoted by b_i. We denote the bid profile of N advertisers by $\mathbf{b} = (b_1, \ldots, b_N)$.

In the generalized second price auction (GSP), advertisers are allocated with the ad-slots in the descending order of the bids $b_1, b_2, ..., b_N$. Let $d(k)$ denote the name of bidder who submits k-th highest bid among $\mathbf{b}$. Thus, bidder $d(k)$ acquires the ad-slot k. (Note that for $k > K$, bidder $d(k)$ does not acquire any ad-slot.) For each ad-slot k, advertiser $d(k)$ pays $b_{d(k+1)}$ for a click of its ad. Hence, the payment p_k is $\alpha_k b_{d(k+1)}$. (Note that for $k > K$, bidder $d(k)$ pays $\alpha_k b_{d(k+1)} = 0$ by the definition of α_ks.) Thus, the advertiser obtaining the ad-slot k pays the bid of the advertiser obtaining one lower ad-slot for each click. The payoff of the advertiser obtaining slot k is $\alpha_k v_{d(k)} - p_k = \alpha_k(v_{d(k)} - b_{d(k+1)})$.

3 Secure greedy bidding

In a keyword auction in use, each advertiser does not observe the actual bids submitted by the other advertisers. Each advertiser only observes the positions of the others and the current payment for each click, from which he can deduce the bid of the advertiser that is in the position immediately below from him.

In this section, we propose bidding behavior of an advertiser in a sealed bid environment. The part of the idea of the bidding behavior considered here is from the balanced bidding strategy by Cary et al. (2007), which is some type of the greedy bidding strategy in the open bid environment. Since the actual auction is in the sealed bid environment, the perfect greedy bidding strategy, where each bidder chooses the bid in the next period that is the best response to the current bids of the other bidders, cannot be achieved by the advertisers. They can only execute the incomplete version of greedy strategy. A greedy strategy in the sealed bid environment is that each bidder increases his bid so as to obtain the position immediately above as long as this increment in the bid does not lead to the decrease in the payoff even in the worst situation in the next period. This motivates the following definition of the greedy strategy in the sealed bid environment.

Definition 3.1. Let i be in the k-th slot. The i's secure greedy bidding for the slot immediately above is to choose the maximized bid b_i satisfying the following condition:

$$\alpha_k v_i - p_k \leq \alpha_{k-1} v_i - \alpha_{k-1} b_i. \tag{1}$$

The **secure greedy bidding for the slot immediately above (SGB for A)** is

$$\begin{aligned} b_i^A(k, p_k) &= (1 - \frac{\alpha_k}{\alpha_{k-1}})v_i + \frac{p_k}{\alpha_{k-1}} \\ &= (1 - r_k)v_i + r_k b_{d(k+1)} \end{aligned}$$

for each $k \leqq K$, where $r_k = \frac{\alpha_k}{\alpha_{k-1}}$ and to deal with the all slot uniformly, we define $\alpha_0 = 2\alpha_1$. For $k > K$, we set $r_k = 0$. Thus, $b_i^A(k, p_k) = v_i$ for $k > K$.

If b_i satisfies condition (1), the payoff of i does not decrease after i obtaining slot $k-1$ even if his payment is in the worst case. The greedy strategy in this setting is that among the bids satisfying (1), each bidder maximizes the possibility of obtaining the one higher slot. This is the definition of the secure greedy bidding for the slot immediately above. One remark on the definition of $b_i^A(k, p_k)$ is that it depends only on the identity of the bidder, his current position, and his current payment. The other information like the bids of the other advertisers is needless for each bidder to execute the secure greedy bidding for the slot immediately above.

Another interpretation of $b_i^A(k, p_k)$ is that it is in a sense a weakly dominant strategy of bidder i. Consider a situation that bidder $i = d(k)$ changes his bid so as to acquire one higher slot $k-1$ and ignore, for a moment, the bidders other than i and $d(k-1)$ and slots other than k and $k-1$. Let $b_{d(k-1)}$ be the current bid of bidder $d(k-1)$. Then, if $b_{d(k-1)} \leqq b_i^A(k, p_k)$, any new bid $\hat{b}_i$ of bidder i satisfying $\hat{b}_i > b_{d(k-1)}$ is his best response to $b_{d(k-1)}$. And if $b_{d(k-1)} \geqq b_i^A(k, p_k)$, any new bid $\hat{b}_i$ satisfying $\hat{b}_i < b_{d(k-1)}$ becomes his best response to $b_{d(k-1)}$. Combining these two observations, $b_i^A(k, p_k)$ is always best response to the bid of bidder in slot $k-1$. Thus, choosing $b_i^A(k, p_k)$ is interpreted as a weakly dominant strategy of bidder i, conditional that he tries to acquire one higher slot $k-1$.

The next is the secure greedy bidding that aims to obtain the one-lower slot. The idea is that in order to compare the payoffs in slot k and the payoff after obtaining slot $k+1$, advertiser i deduces the current payoffs of the bidder $d(k+1)$, who currently occupies the slot $k+1$. He deduces it from the information on his current payments p_k, from which he can know the bid of bidder $d(k+1)$ is p_k/α_k. Moreover, he assumes that other advertisers also follow the secure greedy bidding strategy and thus the current bid submitted by bidder $d(k+1)$ is $b^A_{d(k+1)}(k+1, p_{k+1})$. Thus, he can deduce the current payments of bidder $d(k+1)$ from the following equation:

$$\frac{p_k}{\alpha_k} = b^A_{d(k+1)}(k+1, \tilde{p}_{k+1}),$$

where $\tilde{p}_{k+1}$ is the payments of bidder $d(k+1)$ guessed by bidder i. From this, we have

$$\tilde{p}_{k+1} = p_k - (\alpha_k - \alpha_{k+1})v_{d(k+1)}.$$

Note that $\tilde{p}_{k+1}$ can be negative.

The secure greedy bidding for the position immediately below is defined as follows:

Definition 3.2. Let i be in the k-th slot. The i's **secure greedy bidding for the slot immediately below (SGB for B)** is as follows. If

$$\alpha_k v_i - p_k < \alpha_{k+1} v_i - \tilde{p}_{k+1}, \tag{2}$$

then, choose $b_i^B(k, p_k)$ defined by

$$\begin{aligned} b_i^B(k, p_k) &= \max\{(1 - \frac{\alpha_{k+1}}{\alpha_k})v_i + \frac{\tilde{p}_{k+1}}{\alpha_k}, 0\} \\ &= \max\{b_{d(k+1)} + (1 - r_{k+1})(v_i - v_{d(k+1)}), 0\} \end{aligned}$$

Our concern is what happens in the repeatedly played auction when each bidder follows the secure greedy bidding (for the position immediately above and below). To obtain the consequence from the dynamics generated by the secure greedy bidding, we first examine the stable state from the bidding behavior. The bid profile is stable under the secure greedy bidding if in the bid profile, no bidder changes the bid according to the secure greedy bidding. This motivates the following definition:

Definition 3.3. The bid profile **b** is **secure greedy bidding proof (SGBP)** if **b** is stable under the SGB of every bidder. Thus, at the secure greedy bidding proof profile b, each advertiser i with $d(k)$ submits the bid $b_i = b_i^A(k, p_k)$, and for each k, Inequality (2) does not hold, where p_k and $\tilde{p}_{k+1}$ are calculated from **b**.

The VCG mechanism has a more merit in the sealed bid environment than it in the open bid environment because truly submitting advertiser's own value is the best strategy irrespective of the other bidders' choices. The allocation of the ad-slots in the keyword auction is *truthful-output* if the allocation coincides with the one in the VCG. Since the VCG allocated with the ad-slots in the descending order of the bids and each advertiser submits his own value in the VCG, the resulting allocation of the ad-slots becomes an assortative allocation, i.e., the advertiser with k-th highest value acquires ad-slot k. In this context, it is known that an assortative allocation is efficient (i.e., maximizing the social surplus).

Theorem 3.1. *A SGBP allocation is truthful-output.*

Proof. Suppose that a SGBP allocation **b** is not truthful-output. Then, there must exist some k such that $v_{d(k)} < v_{d(k+1)}$. In this case, Inequality (2) holds for this k because

$$\begin{aligned} &\alpha_k v_{d(k)} - p_k - \big(\alpha_{k+1} v_{d(k)} - \tilde{p}_{k+1}\big) \\ &\quad = \alpha_k v_{d(k)} - p_k - \alpha_{k+1} v_{d(k)} \\ &\qquad + \big(p_k - (\alpha_k - \alpha_{k+1}) v_{d(k+1)}\big) \\ &\quad = (\alpha_k - \alpha_{k+1})(v_{d(k)} - v_{d(k+1)}) \\ &\quad < 0. \end{aligned}$$

This contradicts that **b** is SGBP. □

An important observation from the proof of this theorem is that Inequality (2) holds if and only if $v_i < v_{d(k+1)}$.

The theorem mentioned in the above indicates that the dynamics of the secure greedy bidding should stop at the efficient allocation.

The next result assures the existence of the the bid profile that can be a convergent point of the dynamics generated by the secure greedy bidding.

Theorem 3.2. *There exists a bidding profile* **b** *in a keyword auction that is a SGBP bid profile.*

Proof. Consider the bid profile $\mathbf{b}^*$ defined by the following manner:

$$b_k^* = \sum_{h=k}^{K+1} \frac{\alpha_{h-1} - \alpha_h}{\alpha_{k-1}} v_h \tag{3}$$

for each k with $1 \leqq k \leqq K$. For $k > K$, $b_k^* = v_k$.

We will show that $\mathbf{b}^*$ is SGBP bid profile. Note that the allocation of the ad-slots at $\mathbf{b}^*$ is assortative since by the definition of b_k^* and b_{k+1}^*, $1 \leqq k \leqq K$,

$$\begin{aligned} b_k^* - b_{k+1}^* &= \frac{\alpha_{k-1} - \alpha_k}{\alpha_{k-1}} v_k + (\frac{1}{\alpha_{k-1}} - \frac{1}{\alpha_k}) \sum_{h=k+1}^{K} (\alpha_{h-1} - \alpha_h) v_h \\ &= \frac{\alpha_{k-1} - \alpha_k}{\alpha_{k-1}\alpha_k} \left(\alpha_k v_k - \sum_{h=k+1}^{K} (\alpha_{h-1} - \alpha_h) v_h \right) \\ &> \frac{\alpha_{k-1} - \alpha_k}{\alpha_{k-1}\alpha_k} \left(\alpha_k v_k - \sum_{h=k+1}^{K} (\alpha_{h-1} - \alpha_h) v_k \right) \\ &= \frac{\alpha_{k-1} - \alpha_k}{\alpha_{k-1}\alpha_k} \alpha_K v_K > 0. \end{aligned}$$

We first show that for each $k \geqq 1$, $b_k^* = b_k^A(k, p_k^*)$, where p_k^* is a payment of bidder k who acquires slot k at bid profile $\mathbf{b}^*$. By the definition of $b_k^A(k, p_k^*)$,

$$\begin{aligned} b_k^A(k, p_k^*) &= (1 - \frac{\alpha_k}{\alpha_{k-1}}) v_k + \frac{p_k^*}{\alpha_{k-1}} \\ &= (1 - \frac{\alpha_k}{\alpha_{k-1}}) v_k + \frac{\alpha_k b_{k+1}^*}{\alpha_{k-1}} \\ &= \frac{\alpha_{k-1} - \alpha_k}{\alpha_{k-1}} v_k + \frac{\alpha_k}{\alpha_{k-1}} \sum_{h=k+1}^{K} \frac{\alpha_{h-1} - \alpha_h}{\alpha_k} v_h \\ &= \sum_{h=k}^{K} \frac{\alpha_{h-1} - \alpha_h}{\alpha_{k-1}} v_h = b_k^* \end{aligned}$$

We next show that Inequality (2) does not hold for any k at $\mathbf{b}^*$. Since at $\mathbf{b}^*$, the allocation is assortative, we have, for any k,

$$\begin{aligned} &\alpha_k v_k - p_k - (\alpha_{k+1} v_k - \tilde{p}_{k+1}) \\ &\quad = \alpha_k v_k - p_k - \alpha_{k+1} v_k + (p_k - (\alpha_k - \alpha_{k+1}) v_{k+1}) \\ &\quad = 0. \end{aligned}$$

□

The basic equilibrium concept adopted by Lahaie, Pennock, Saberi and Vohra (2007), Edelman et al. (2007), Varian (2007) and other researchers in this field to analyze the keyword auction is a locally envy-free equilibrium. A bid profile $\mathbf{b} = (b_1, \ldots, b_N)$ is a *locally envy-free equilibrium* if the following two conditions hold for any $k \geq 1$:

$$\underbrace{\alpha_k v_{d(k)} - \alpha_k b_{d(k+1)}}_{d(k)\text{'s current profit}} \quad \geq \quad \underbrace{\alpha_{k-1} v_{d(k)} - \alpha_{k-1} b_{d(k)}}_{\substack{d(k)\text{'s profit if he acquires ad-slot } k-1 \\ \text{with the current payment of } d(k-1)}}$$

and

$$\underbrace{\alpha_k v_{d(k)} - \alpha_k b_{d(k+1)}}_{d(k)\text{'s current profit}} \quad \geq \quad \underbrace{\alpha_{k+1} v_{d(k)} - \alpha_{k+1} b_{d(k+2)}}_{\substack{d(k)\text{'s profit if he acquires ad-slot } k-1 \\ \text{with the current payment of } d(k+1)}}$$

Thus, a bid profile is locally envy-free if each bidder is not better off by the exchange of his position with the position of the bidder immediately above or below. This means that an equilibrium concept when each bidder cares only about the neighboring bidders. This is very different from the Nash equilibrium where each bidder cares all of the other bidders. However, it is known that a locally envy-free equilibrium bid profile **b** is a Nash equilibrium of the normal form game with complete information (e.g., see Fukuda, Kamijo, Takeuchi, Masui and Funaki (2009)).

Next theorem states some of the pretty properties of SGBP bid profile.

Theorem 3.3. *Let **b** be a SGBP bid profile. Then,*

*(i) the revenue of the one shot complete information game of the GSP at **b** is the same as the revenue of the VCG at the dominant strategy equilibrium,*

*(ii) **b** is a locally envy-free equilibrium,*

*(iii) **b** is a Nash equilibrium of the one shot complete information game of the GSP, and*

*(iv) **b** is consistent in the sense that for each k, $p_k = \tilde{p}_k$ holds. In other words, the actual payment of the bidder in slot k equals the payment estimated by the bidder in slot $k-1$.*

Proof. (i). Suppose **b** is SGBP. Then, by Theorem 1, b is a truthful-output, and thus, $b_1 > b_2 > \cdots > b_N$. By the definition of the SGB, $b_k = v_k$ for each $k > K$. Since the payments of bidder K is v_{K+1}, it must hold that

$$b_K = b_K^A(K, \alpha_K v_{K+1}) = (1 - r_K)v_K + r_K v_{K+1}.$$

From this, the payments of the bidder k and his bid is determined as the following recursive manner: from $k = K - 1$ to $k = 2$

$$p_k = \alpha_k b_{k+1}$$

and

$$b_k = b_k^A(k, p_k) = (1 - r_k)v_k + r_k b_{k+1}.$$

It is easily checked that for each $k \geqq 1$, $b_k = b_k^*$ defined in (3). Varian (2007) shows that b^* is a bid profile that achieves the lower bound of the auctioneer's revenue among the set of all locally envy-free equilibrium. It is also known that this lower bound is the revenue of the dominant strategy equilibrium in VCG (Edelman et al. (2007)).

(ii) and (iii). From the proof of (i), **b** must be a locally envy-free equilibrium. It is a known result that a locally envy-free equilibrium is a Nash equilibrium (see, Varian (2007) and Fukuda et al. (2009)).

(iv). Because $\mathbf{b} = \mathbf{b}^*$ (except for bidder 1), we have, by definition of $\tilde{p}_k$, for each $k > 1$,

$$
\begin{aligned}
\tilde{p}_k &= p_{k-1} - (\alpha_{k-1} - \alpha_k)v_k \\
&= \alpha_{k-1} b_k^* - (\alpha_{k-1} - \alpha_k)v_k \\
&= \alpha_{k-1} \sum_{h=k}^{K} \frac{\alpha_{h-1} - \alpha_h}{\alpha_{k-1}} v_h - (\alpha_{k-1} - \alpha_k)v_k \\
&= \sum_{h=k}^{K} (\alpha_{h-1} - \alpha_h)v_h - (\alpha_{k-1} - \alpha_k)v_k \\
&= \sum_{h=k+1}^{K} (\alpha_{h-1} - \alpha_h)v_h \\
&= \alpha_k b_{k+1}^* = p_k.
\end{aligned}
$$

□

In the definition of the secure greedy bidding, the choice of the bid of each advertiser is based on the prediction on the payment of the one in the position immediately below from his. This means that in some situation, their behavior is caused by the wrong prediction on the others, and even in the SGBP bid profiles, such kinds of inconsistency of the prediction with the actual behavior may happen. However, Theorem 3.3 (iv) says that SGBP bid profile is consistent in the sense that at the SGBP bid profile, the prediction on the payment coincides with the actual payment.

One important remark is that the bid profile defined in Eq. (3) is a fixed point of the balanced bidding by Cary et al. (2007) for the open bid environment. Therefore, combining our results and the results of Cary et al. (2007), it is indicated that the stable bid profile in an open bid environment should be a unique stable bid profile in a sealed bid environment.

4 Convergence of the secure greedy bidding

In this section, we explore whether the convergence is attained in the repeatedly played keyword auction. We consider both a synchronous model and an asynchronous model.

The secure greedy bedding strategy in the repeatedly played GSP auction is as follows.

Definition 4.1. Given the current position of the slot and the current payment, the secure greedy bidding (SGB) strategy of bidder i with $d(k) = i$ and $k \leqq K$ is as follows.

- If the current profit of i is negative, i changes the bid to v_i in the next period,
- If the current profit of i is non-negative and Inequality (2) holds, i changes the bid to $b_i^B(k, p_k)$ in the next period, and
- if the current profit of i is non-negative and Inequality (2) does not hold, i changes the bid to $b_i^A(k, p_k)$ in the next period.

And if i does not have any slot (i.e., $k > K$), he changes the bid to $b_i^A(k, p_k) = b_i^A(k, 0) = v_i$.

We first consider a situation that in each period, every bidder changes his bid according to the SGB strategy (synchronous model). Similar to Cary et al. (2007), we show that there may be a cycle of bids in a keyword auction.

Example 4.1. Consider a repeated keyword auction where all bidders follow the SGB strategy and change their bids in each period. Then, for some initial bid profile, there exists a bid cycle in a repeated keyword auction.

Consider a situation where there are three bidders with values $v_1 = 100, v_2 = 80, v_3 = 60$ and three ad slots with $\alpha_1 = 25, \alpha_2 = 20, \alpha_3 = 4$. Thus $r_1 = 0.5, r_2 = 0.8, r_3 = 0.2$. Suppose their initial bid profile is $\mathbf{b}^1 = (74, 64, 0)$. Then, the next bid profile is $\mathbf{b}^2 = (82, 16, 48)$. Moreover, the third bid profile is $\mathbf{b}^3 = (74, 64, 0) = \mathbf{b}^1$. Thus, the bid cycle occurs.

The previous example means that the convergence of the bidding behavior under the sealed bid repeated keyword auction is not assured in a synchronous model. However, as discussed by Cary et al. (2007), an asynchronous model is more appropriate than a synchronous model as a approximation of a real keyword auction. In a asynchronous model, the convergence of the bidding behavior is guaranteed even in the sealed bid repeated keyword auction.

Theorem 4.1. *Consider a repeated keyword auction where all bidders follow the SGB strategy and in each period, one bidder is randomly chosen and change his bid according to the SGB strategy. Then, from any initial bid profile, the bid profile converges to the SGBP bid profile b^* defined in Equation (3).*

Proof. The proof is in the appendix. □

5 Discussion

In the previous sections, we propose and examine new bidding behavior for a keyword auction under a sealed bid environment. Our results (Theorems 3.1, 3.2, 3.3 and 4.1) indicate that even in the situation where each bidder cannot know the current bids of others, a market outcome is the same as the one suggested by Cary et al. (2007) for a keyword auction under the open bid environment. Because the auction in practice is played under the sealed bid environment, our results support the researches in this filed that use a locally envy free equilibrium outcome as their basic analysis. However, it should be noticed that instead of the information of the current bids of others, the SGB strategy requires another information, the values of other bidders, that it may be difficult for the bidders in practice to acquire. On the other hand, the balanced bidding proposed in Cary et al. (2007) needs only the information of the bids of others. Thus, there is a trade-off in the required information between the SGB and the balanced bidding.

In this section, we explore other versions of bidding behavior that can be executed by bidders in a real keyword auction where they know neither the current bids of others nor the values of other bidders.

5.1 Equilibrium bidding behavior

Let $\mathbf{b}$ be a locally envy-free equilibrium and p_k be the payments per click of slot k. Then, from the first inequality in the definition of a locally envy-free equilibrium, we have

$$\alpha_k(v_{d(k)} - p_k) \geqq \alpha_{k-1}(v_{d(k)} - p_{k-1}) \iff v_{d(k)} \leqq \frac{p_{k-1}\alpha_{k-1} - p_k\alpha_k}{\alpha_{k-1} - \alpha_k}$$

and from the second, we have

$$\alpha_k(v_{d(k)} - p_k) \geqq \alpha_{k+1}(v_{d(k)} - p_{k+1}) \iff v_{d(k)} \geqq \frac{p_k\alpha_k - p_{k+1}\alpha_{k+1}}{\alpha_k - \alpha_{k+1}.}$$

Combining these two inequalities, we obtain

$$\frac{p_k\alpha_k - p_{k+1}\alpha_{k+1}}{\alpha_k - \alpha_{k+1}} \leqq v_{d(k)} \leqq \frac{p_{k-1}\alpha_{k-1} - p_k\alpha_k}{\alpha_{k-1} - \alpha_k}. \tag{4}$$

Since the above inequality holds for any $k \geqq K$, we have

$$\frac{p_K\alpha_K}{\alpha_K} \leqq \frac{p_{K-1}\alpha_{K-1} - p_K\alpha_K}{\alpha_{K-1} - \alpha_K} \leqq \dots \leqq \frac{p_k\alpha_k - p_{k+1}\alpha_{k+1}}{\alpha_k - \alpha_{k+1}} \leqq \dots \leqq \frac{p_1\alpha_1 - p_2\alpha_2}{\alpha_1 - \alpha_2}. \tag{5}$$

From this, we can observe that at a locally envy-free equilibrium, $v_{d(k)} = v_k$ should hold for any $k \geqq K$, and thus, the allocation of the ad-slots at a locally envy-free equilibrium is efficient.

From these observations, we have the interesting properties of the equilibrium bid profiles. First, if we see α_k as the expected number of the clicks per given period, $p_{k-1}\alpha_{k-1} - p_k\alpha_k$ and $\alpha_{k-1} - \alpha_k$ are the increase of the cost and the number of the clicks, respectively, when $d(k)$ obtains one higher ad-slot $k-1$, and thus, $(p_{k-1}\alpha_{k-1} - p_k\alpha_k)/(\alpha_{k-1} - \alpha_k)$ can be seen as the marginal cost of clicks for $d(k)$. Second, from (5), the marginal cost of clicks is increasing. Third, from (4), the slots assigned to bidders are consistent with their profit maximization because they obtain their highest ad-slots among the ones where the marginal revenue (value) is greater than or equal to the marginal cost. (For more detail, see Varian (2007))

Based on this equilibrium predictions, Varian (2007) implicitly introduced the idea of the bidding behavior for keyword auctions. The idea is that if the marginal payments for obtaining the one higher ad-slot calculated from the current bid profile is less than the marginal revenue (the value of the advertiser), then the advertiser, say $d(k)$, should increase the bid to the one that he should choose if he is in slot $k-1$. This with the idea of the SGB motivates the following definition of new bidding behavior.

Definition 5.1. Given the current position of the slot and the current payment, the equilibrium bidding (EB) strategy of bidder i with $d(k) = i$ and $k \leqq K$ is as follows.

- If the current profit of i is negative, i changes the bid to v_i in the next period,
- If the current profit of i is non-negative and

$$v_i > \frac{p_{k-1}\alpha_{k-1} - p_k\alpha_k}{\alpha_{k-1} - \alpha_k} \tag{6}$$

  holds, i changes the bid to $b_i^A(k-1, p_{k-1})$ in the next period, and
- if the current profit of i is non-negative and Inequality (6) does not hold, i changes the bid to $b_i^A(k, p_k)$ in the next period.

And if i does not have any slot (i.e., $k > K$), he changes the bid to $b_i^A(k, p_k) = b_i^A(k, 0) = v_i$.

It should be emphasized that all the information that are needed for an advertiser to execute the EG strategy is his value, CTRs of ad-slots, his bid, and his payment. He needs neither the bids of others advertisers nor the values of others.

As is the SGBP, we define the stable bid according to the EB strategy.

Definition 5.2. The bid profile **b** is **equilibrium bidding proof (EBP)** if **b** is stable under the EB strategy of every bidder. Thus, at the EBP bid profile **b**, each advertiser i with $d(k)$ submits the bid $b_i = b_i^A(k, p_k)$, and for each k, Inequality (6) does not hold.

The existence of the EBP bid profiles is easily proved.

Theorem 5.1. *There exists a bidding profile* **b** *in a keyword auction that is a EBP bid profile.*

Proof. From (ii) of Theorem 3.3, the bid profile **b** defined in Eq. (3) is a locally envy-free equilibrium. In addition, from the proof of Theorem 3.2, in this bid profile, $b_{d(k)} = b^A_{d(k)}(k, p_k)$ holds for any k. Since Inequality (6) does not hold for any k when the current bid profile is a locally envy-free equilibrium, **b** is a EBP bid profile. □

In contrast to the SGBP bid profiles, as the following example will show, the EBP bid profile is not always a locally envy-free equilibrium. Moreover, it does not assure the truthful output (note that for any locally envy-free equilibrium bid profile, the allocation is truthful output, see Varian (2007)).

Example 5.1. Consider a situation where there exist three bidders with their values being $v_1 = 20, v_2 = 25$ and $v_3 = 10$ and two ad slots with CTRs being $\alpha_1 = 10$ and $\alpha_2 = 5$. Thus, $r_1 = r_2 = 1/2$. Consider a bid profile defined by

$$b_3 = 10,$$

$$b_2 = b^A_2(2, \alpha_2 b_3) = \frac{1}{2} \times 25 + (1 - \frac{1}{2}) \times 10 = 17.5,$$

and

$$b_1 = b^A_1(1, \alpha_1 b_2) = \frac{1}{2} \times 20 + (1 - \frac{1}{2}) \times 17.5 = 18.75.$$

For bidder 2,

$$v_2 = 25 \leqq \frac{17.5 \times 10 - 10 \times 5}{10 - 5} = \frac{p_1\alpha_1 - p_2\alpha_2}{\alpha_1 - \alpha_2}.$$

Thus, Inequality (6) does not hold and therefore, this bid profile is EBP. This bid profile is not truthful output and thus, is not a locally envy-free equilibrium.

5.2 Trial-and-error bidding behavior

In the actual bidding behavior in a real world, an advertiser often raises the bid as a trial and this may be a reason that the actual bidding behavior shows the complicated figure. To describe such a trial increase of a bid, we assume that in the beginning of each period, there is a very short period, called trial period, such that a bidder can change the bid and observe the resulting ad-slot assignment but this does not affect the profit of the advertisers unless the advertiser keeps this trial bid as his bid of this period. The combination of the trial-and-error and the equilibrium bidding mentioned in the previous subsection motivates the following bidding behavior.

Definition 5.3. Given the current position of the slot and the current payment, the trial-and error bidding (TEB) strategy of bidder i with $d(k) = i$ and $k \leqq K$ is as follows.

- If the current profit of i is negative, i changes the bid to v_i in the next period,
- If the current profit of i is non-negative, i changes the bid to $b^A_i(k-1, p_{k-1})$ in the trial period. If this change in his bid lead to the change in the allocation of the ad-slots, he keeps this bid as the bid of the next period. Otherwise, i changes the bid to $b^A_i(k, p_k)$ as the bid of the next period. If i has the top position, the trial bidding is removed and he chooses $b^A_i(k, p_k)$ as the bid of the next period.

And if i does not have any slot (i.e., $k > K$), he changes the bid to $b^A_i(k, p_k) = b^A_i(k, 0) = v_i$.

As are the SGBP and EBP, we define the stable bid according to the TEB strategy.

Definition 5.4. The bid profile **b** is **trial-and-error bidding proof (TEBP)** if **b** is stable under the TEB strategy of every bidder. Thus, at the TEBP bid profile **b**, each advertiser i with $d(k)$ submits the bid $b_i = b_i^A(k, p_k)$, and for each $k > 1$, $b^A_{d(k)}(k-1, p_{k-1}) < b_{d(k-1)}$.

The TEBP bid profile has the same good properties as the one the SGBP bid profiles satisfies.

Theorem 5.2. *The bid profile defined in Eq. (3) is the unique TEBP bid profile.*

Proof. From the definition of TEB strategy, if **b** is a TEBP bid profile, it must satisfy the following two conditions:

$$b_{d(k)} = v_{d(k)}$$

for any $k > K$, and

$$b_{d(k)} = b^A_{d(k)}(k, \alpha_k b_{d(k+1)})$$

for any $k \leqq K$. Thus, from the proof of Theorem 3.2, it suffices to show that **b** is assortative.

Assume that **b** is not assortative. Them, there exist $k \leqq K$ such that $v_{d(k)} > v_{d(k-1)}$. For bidder $d(k)$, his trial bid is greater than the bid of $d(k-1)$ because

$$\begin{aligned} b^A_{d(k)}(k-1, \alpha_k b_{d(k)}) &= (1 - r_{k-1})v_{d(k)} + r_{k-1}b_{d(k)} \\ &> (1 - r_{k-1})v_{d(k+1)} + r_{k-1}b_{d(k)} \\ &= b^A_{d(k-1)}(k-1, \alpha_k b_{d(k)}) \\ &= b_{d(k-1)}. \end{aligned}$$

This is a contradiction. □

Next theorem shows that the convergence result also holds for the TEB strategy.

Theorem 5.3. *Consider a repeated keyword auction where all bidders follow the TEB strategy and in each period, one bidder is randomly chosen and change his bid according to the TEB strategy. Then, from any initial bid profile, the bid profile converges to the TEBP bid profile b^* defined in Equation (3).*

Proof. The proof is in the appendix. □

References

Cary, M., Das, A., Edelman, B., Giotis, I., Heimerl, K., Karlin, A. R., Mathieu, C., Schwarz M., 2007. Greedy bidding strategies for keyword auctions. *Proceedings of the 8th ACM Conference on Electronic Commerce*, 262–271. San Diego, California, 2007.

Clarke, E. H., 2007. Multiple pricing of public goods. *Public Choice 11*, 17–33.

Edelman, B., Ostrovsky M., 2007. Strategic Bidder Behavior in Sponsored Search Auctions. *Decision Support Systems 43*, 192–198.

Edelman, B., Ostrovsky M., Schwarz M., 2007. Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords. *American Economic Review 91*, 242–259.

Fukuda, E., Kamijo, Y., Takeuchi A., Masui, M., Funaki, Y., 2009. Theoretical and experimental investigations of performances of the keyword auction mechanisms. mimeo.

Groves, T., 1973. Incentives in team. *Econometrica 41*, 617–631.

Lahaie, S., Pennock, D., Saberi A., Vohra R., 2007. Sponsored Search Auctions. In Nisan N., Roughgarden T., Tardos E., Vazirani V. (Eds.), Algorithmic Game Theory. Cambridge University Press, Cambridge, pp. 699–716.

Varian, H., 2007. Position Auctions. *International Journal Industrial Organization 25*, 1163–1178.

Vickrey, W., 1961. Counterspeculation, Auctions, and Competitive Sealed Tenders. *Journal of Finance 16*, 8–27.

Appendix

6 Proof of Theorem 4.2

We first introduce the following notation. For two bid profile **b** and **b'**, we say that **b'** can be realized from **b** by SGB and write **b** $\xrightarrow{sgb}$ **b'** if **b'** is realized from **b** by some one player's changing in his bid according to the SGB.

Lemma 6.1. *For any initial bid* $\boldsymbol{b}^0$*, there exists a finite sequence of the bid profiles,* $\boldsymbol{b}^1$*,* $\boldsymbol{b}^2$*, ...,* $\boldsymbol{b}^t$ *such that*

1. *for any* s *with* $0 \leqq s < t$*,* $\boldsymbol{b}^s \xrightarrow{sgb} \boldsymbol{b}^{s+1}$*, and*
2. $\boldsymbol{b}^t$ *satisfies the following two conditions:*
   - *(a) for any bidder* i *with* $i > K$*,* $b_i^t = v_i$*, and*
   - *(b) for any bidder* i*, he does not do overbid, i.e.,* $b_i^t \leqq v_i$*.*

Moreover, t *does not exceed* $2N + K^2 - K$*.*

Proof. To construct the sequence of bids in this lemma, we consider the several steps.

Step 1. There exists a sequence of bid profiles from $\mathbf{b}^0$ to $\mathbf{b}^{t^1}$ such that $\mathbf{b}^0 \xrightarrow{sgb} \mathbf{b}^1 \xrightarrow{sgb} \dots \xrightarrow{sgb} \mathbf{b}^{t^1}$, every bidder who does not obtain any slot submits his true value at $\mathbf{b}^{t^1}$, and t^1 is less than or equal to N.

To show Step 1, we consider the following procedure.

- Procedure (1).
- Initial conditions are bid profile $\mathbf{b}^0$, $S_0^1 = \{i :$ bidder i does not obtan an slot at $\mathbf{b}^0$ $\}$ and $S_0^2 = \{1, ..., N\}$.
- Repeat the following process from $t = 0$ until $S_t^1 \cap S_t^2$ becomes an empty set:
  - choose any i_t from $S_t^1 \cap S_t^2$ and i_t changes his bid to v_{i_t}.
  - $\mathbf{b}^{t+1}$ is the bid profile after i_t's change in his bid, S_{t+1}^1 is the set of bidders that do not obtain slots at $\mathbf{b}^{t+1}$, and $S_{t+1}^2 = S_t^2 \setminus \{i_t\}$.

Note that this procedure must be stopped in at most $t = N$ because in each time one element in S_t^2 is deleted and the cardinality of the initial S_0^2 is N. Let t^1 be the period where this procedure is stopped. It should be emphasized that for each t, bidder i_t follow the SGB strategy because i_t is chosen from the set of the bidders that do not obtain any slot at the current bid profile. This means that the sequence of bids from $\mathbf{b}^0$ to $\mathbf{b}^{t^1}$ is caused by the SGB. It is obvious from the definition of Procedure (1) at $\mathbf{b}^{t^1}$, every bidder who does not obtain any slot submits his true value.

Step 2. There exists a sequence of bid profiles from $\mathbf{b}^{t^1}$ to $\mathbf{b}^{t^1+t^2}$ such that $\mathbf{b}^{t^1} \xrightarrow{sgb} \mathbf{b}^{t^1+1} \xrightarrow{sgb} ... \xrightarrow{sgb} \mathbf{b}^{t^1+t^2}$, bidders $K+1, K+2, ..., N$ do not obtain ad-slots and submit their true values at $\mathbf{b}^{t^1+t^2}$, and t^2 is less than or equal to $N - K$.

Let S_0^3 be defined by $\{K+1, K+2, ..., N\} \cap \{i$: bidder i obtans an ad-slot at $\mathbf{b}^{t^1}$ $\}$. If S^3 is empty, skip this step and go to Step 3. If S^3 is not empty, put $t^2 = |S_0^3|$ and apply the following procedure (2).

- Procedure (2).
- Initial conditions are S_0^3 and $\mathbf{b}^{t^1}$.
- Repeat the following from $t = 0$ to t^2.
  - choose any i_t from S_t^3 and i_t changes his bid to v_{i_t}.
  - $\mathbf{b}^{t^1+t+1}$ is the bid profile after i_t's change in his bid and $S_{t+1}^3 = S_t^3 \setminus \{i_t\}$.

Note that for any $i \in S_0^3$, the profit of bidder i at bid profile $\mathbf{b}^{t^1}$ is negative because bid of $d(K+1)$ at $\mathbf{b}^{t^1}$ is $v_{d(K+1)}$ and $v_{d(K+1)}$ must be greater than v_i. The same situation holds for any i_t because the changes of bids of $i_1, i_2, ..., i_{t-1}$ to their true values only make i_t's position to a upper slot and i_t keeps the slot with paying more than his true value at $\mathbf{b}^{t^1+t}$. This means that the sequence of bids from $\mathbf{b}^{t^1}$ to $\mathbf{b}^{t^1+t^2}$ is caused by the SGB of bidders. It is easily checked that $t^2 \leqq N - K$. It is obvious from the definition of Procedure (2) that at $\mathbf{b}^{t^1+t^2}$, bidders $K+1, K+2, ..., N$ do not obtain ad-slots and submit their true values and other bidders submit a bid more than v_{K+1}.

Step 3. There exists a sequence of bids from $\mathbf{b}^{t^1+t^2}$ to $\mathbf{b}^{t^1+t^2+t^3}$ such that $\mathbf{b}^{t^1+t^2} \xrightarrow{sgb} \mathbf{b}^{t^1+t^2+1} \xrightarrow{sgb} ... \xrightarrow{sgb} \mathbf{b}^{t^1+t^2+t^3}$, bid profile $\mathbf{b}^{t^1+t^2+t^3}$ satisfies conditions (a) and (b) mentioned in this lemma, and t^3 is less than or equal to K^2.

Let S_0^4 be defined by $\{1, 2, ..., K\} \cap \{i$: bidder i does overbid at $\mathbf{b}^{t^1+t^2}$ $\}$. We put $r = |S_0^4|$ and $S_0^4 = \{i_1, ..., i_r\}$ where for $k < k' \leqq r$, i_k obtains the lower ad-slot than $i_{k'}$ at $\mathbf{b}^{t^1+t^2}$. Apply the following procedure sequentially from i_1 to i_r.

- Procedure (3) for i_k.
- i_k changes his bid according to the SGB until his bid is less than or equal to v_{i_k}. If i_k does not obtain any slot after this repetition of the SGB, i_k follows SGB one more time and submits his value v_i.

Consider Procedure (3) for i_k and suppose that currently i_k still does overbid and obtains slot ℓ. We separately consider the two cases: (1) $b_{d(\ell+1)} \leqq v_{i_k}$ and (2) $b_{d(\ell+1)} > v_{i_k}$. In case (1), by the definition of SGB, the bid in the next period of i_k is less than or equal to v_{i_k} in both SGB for A and SGB for B. On the other hand, in case (2), i_k changes the bid according to SGB for B because $v_{d(\ell+1)} \geqq b_{d(\ell+1)} > v_i$. (The first inequality follows from the fact that $d(\ell+1)$ does not do overbid. This fact is because we apply Procedure (3) from bidders in lower position to ones in higher position.) As the result of SGB for B, bidder i_k must obtain the lower slot than the current slot ℓ at the next bid profile. Thus, Procedure (3) for i_k should be stopped unless case (2) occurs in an infinite time. The infinite repetition of case (2) is impossible because in each time, bidder i_k obtains the lower slot than the current one and this has the limit of slot K (if i_k does not obtain any slot at some period, his bid must be less than or equal to v_{K+1}). Thus, Procedure (3) for i_k is stopped at most K times repetition of changes of i_k's bid.

From the discussion in the previous paragraph, the number of the repetition of the change in the bids from Procedure (3) for i_1 to Procedure (3) for i_r is not exceed $rK \leqq K^2$.

The final bid profile obtained after Step 3 satisfies all the properties mentioned in this lemma. This bid profile is realized by modifications of bids according to SGB less than or equal to $N+N-K+K^2 = 2N + K^2 - K$. Thus, the proof of this lemma is completed. $\square$

Next, we will show lemmas regarding on the properties of the SGB strategy.

Lemma 6.2. *Given some bid profile **b**, consider the SGB for B of i with $i = d(k), k < K$.*

(i) if $\tilde{p}_{k+1} = p_{k+1}$ and $b_{d(k+2)} \geqq v_{K+1}$ hold, then $b_i^B(k, \tilde{p}_{k+1}) > v_{K+1}$, and

(ii) if $\tilde{p}_{k+1} = p_{k+1}$ and $v_i \geqq b_{d(k+2)}$ hold, $b_i^B(k, \tilde{p}_{k+1}) > b_{d(k+2)}$.

Proof. (i). By the definition of SGB for B and the assumptions,

$$\begin{aligned} b_i^B(k, \tilde{p}_{k+1}) &= (1 - \frac{\alpha_{k+1}}{\alpha_k})v_{d(k+1)} + \frac{p_{k+1}}{\alpha_k} \\ &= (1 - r_{k+1})v_i + r_{k+1}b_{d(k+2)}. \end{aligned}$$

Since $v_i > v_{K+1}$ and $b_{d(k+2)} \geqq v_{K+1}$, $b_i^B(k, \tilde{p}_{k+1}) > v_{K+1}$.
(ii). From the calculation in the proof of (i), $v_i \geqq b_{d(k+2)}$ implies $b_i^B(k, \tilde{p}_{k+1}) > b_{d(k+2)}$. $\square$

Lemma 6.3. *Given some bid profile **b**, consider the SGB for A of i with $i = d(k), k < K$. If $b_{d(k+1)} \geqq v_{K+1}$, $b_i^A(k, p_k) > v_{K+1}$.*

Proof. Since $b_i^A(k, p_k) = (1 - r_k)v_i + r_k b_{d(k+1)}$, $b_{d(k+1)} \geqq v_{K+1}$ together with $v_i > v_{K+1}$ implies $b_i^A(k, p_k) > v_{K+1}$. $\square$

Lemma 6.4. *Given some bid profile **b**, take bidders i and j with $i < j (v_i > v_j)$. For any k with $k \leqq K$,*

1. *for any $p_k \geqq 0$, $b_i^A(k, p_k) > b_j^A(k, p_k)$*
2. *for any p_k and p'_k such that $p_k > p'_k > 0$, $b_i^A(k, p_k) > b_j^A(k, p'_k)$.*

Proof. The proof of this lemma is obvious from the definition of SGB for A. $\square$

Before moving to the next lemma, we prepare additional definitions. For any k with $1 \leqq k \leqq K$, bid profile **b** is k-consistent if for any h with $h > K$, $d(h) = h$, the values of bidders $d(K), d(K-1), ..., d(k)$ are increasing (i.e., $v_{d(K)} < v_{d(K-1)} < ... < v_{d(k)}$) and they submit the following bids:

$$b_{d(h)} = v_{d(h)} \text{ for any } h > K,$$

$$b_{d(K)} = b_{d(K)}^A(K, \alpha_K b_{d(K+1)})$$

$$\dots$$

$$b_{d(k)} = b^A_{d(k)}(k, \alpha_k b_{d(k+1)}).$$

Note that bid profile $\mathbf{b}^t$ described in Lemma 6.1 is $K+1$-consistent. One important remark is that if bid profile $\mathbf{b}$ is 1-consistent, this must be $\mathbf{b}^*$ defined in Equation (3). In other words, if we find the sequence of bid profiles that ends up with 1-consistent bid profile, we end the proof of Theorem 4.2.

Lemma 6.5. *Suppose* $\boldsymbol{b}$ *is* $k+1$*-consistent and there is no bidder who does overbid at this bid profile. Let bidder* i *currently occupy slot* k *with* $k \leqq K$ *and* $d(.)$ *denote the assignment of slots at* $\boldsymbol{b}$*.*

(i) The guess of i *to the payment of the bidder in slot* $k+1$*,* $\tilde{p}_{k+1}$*, is correct. In other words,* $\tilde{p}_{k+1} = p_{k+1}$*, where* p_k *is the actual payment of the bidder* $d(k+1)$ *at* $\boldsymbol{b}$*,*

(ii) If bidder i *changes his bid according to the SGB for B, then the following conditions must hold:*

$$b_{d(k+2)} < b^B_i(k, p_k) < b_{d(k+1)}.$$

Thus, i *obtains slot* $k+1$ *in the next period.*

(iii) If bidder i *changes his bid according to the SGB strategy, the new bid profile after* i*'s change in his bid is still* $k+1$*-consistent.*

Proof. (i). This holds because $\tilde{p}_{k+1}$ is defined by solution of $b_{d(k+1)} = b^A_{d(k+1)}(k+1, \tilde{p}_{k+1})$ and $\mathbf{b}$ is $k+1$-consistent.
(ii). From (i) of this lemma, the SGB for B of i is

$$b^B_i(k, p_k) = (1-r_{k+1})v_i + \frac{\tilde{p}_{k+1}}{\alpha_k} = (1-r_{k+1})v_i + \frac{p_{k+1}}{\alpha_k} = (1-r_{k+1})v_i + r_{k+1} b_{d(k+2)} = b^A_i(k+1, p_{k+1})$$

Since $v_i \geqq b_i > b_{d(k+2)}$, $b^B_i(k, p_k)$ is greater than $b_{d(k+2)}$.

Since i follows the SGB for B, Inequality (2) must hold and thus $v_i < v_{d(k+1)}$. Since $\mathbf{b}$ is $k+1$-consistent, $b_{d(k+1)} = b^A_{d(k+1)}(k+1, p_{k+1})$. Then, $b^B_i(k, p_k) < b_{d(k+1)}$ must hold because of Lemma 6.4.
(iii). If bidder i changes his bid according to SGB for A, his change of bid does not change the assignment of slots for slot $k+1, k+2, ..., K$. This means that after his changing a bid, the new bid profile is still $k+1$-consistent. On the other hand, we know from the proof of (ii) of this lemma that if bidder i changes his bid according to SGB for B, the resulting bid profile is still $k+1$-consistent. □

Lemma 6.6. *Suppose bid* $\boldsymbol{b}^0$ *is* $k+1$*-consistent, no bidder does overbid at this bid profile, and bidder* $d(k)$ *at the bid profile* $\boldsymbol{b}^0$ *satisfies Inequality (2). Then, there exists a finite sequence of bid profiles,* $\boldsymbol{b}^1$*,* $\boldsymbol{b}^2$*, ...,* $\boldsymbol{b}^t$ *such that*

1. *for any* s *with* $0 \leqq s < t$*,* $\boldsymbol{b}^s \xrightarrow{sgb} \boldsymbol{b}^{s+1}$*, and*
2. $\boldsymbol{b}^t$ *satisfies the following two conditions:*
   - *(a)* $\boldsymbol{b}^t$ *is* $k+1$*-consistent*
   - *(b) any bidder does not do overbid,*
   - *(c) for bidder* $d(k)$ *at* $\boldsymbol{b}^t$*, Inequality (2) does not hold (that is, if he is chosen as an active player, he changes his bid according to SGB for A).*

Moreover, t is less than or equal to $2(K-k)-1$.

Proof. For $h > k$, let i_h be bidder who occupies slot h at bid profile $\mathbf{b}^0$. Since Inequality (2) hold for bidder i in slot k at b^0 and $\mathbf{b}^0$ is $k+1$-consistent, there must exist $\ell > k$ such that

$$v_{i_K} < v_{i_{K-1}} < ... < v_{i_{\ell+1}} < v_i < v_{i_\ell} < ... < v_{i_{k+1}}$$

holds.

Step 1. There exists a sequence of bid profiles from $\mathbf{b}^0$ to $\mathbf{b}^{\ell-k}$ such that $\mathbf{b}^0 \xrightarrow{sgb} \mathbf{b}^1 \xrightarrow{sgb} ... \xrightarrow{sgb} \mathbf{b}^{\ell-k}$, for any h with $0 \leqq h \leqq \ell-k-1$, $\mathbf{b}^h \xrightarrow{sgb} \mathbf{b}^{h+1}$ is realized by bidder i's SGB for B, and bidder i is in slot $k+h$ at $\mathbf{b}^h$.

Given h with $0 \leqq h \leqq \ell-k-1$, suppose that $\mathbf{b}^h$ is the bid profile after i's h times change in his bid according to SGB for B. Moreover, we suppose that this bid profile is $k+h+1$-consistent and i occupies the slot $k+h$ at this bid profile. If bidder i follows the SGB strategy at $\mathbf{b}^h$, he must follow the SGB for B because $v_i < v_{i_\ell} \leqq v_{i_{k+h+1}}$. By Lemma 6.5, bidder i obtains slot $k+h+1$ after his change in a bid and the resulting bid profile, $\mathbf{b}^{h+1}$, is $k+h+1$-consistent, and thus, this must be also $k+h+2$-consistent. Applying this argument from $h=0$ to $h=\ell-k-1$, we obtain the desired sequence of bid profiles.

If $\ell = k+1$, bidder i_{k+1} is in slot k at the bid profile $b^{\ell-k}$ and $v_{i_{k+1}} > v_i$. This means that $\mathbf{b}^{\ell-k}$ is the desired bid profile of this lemma. Thus, we assume $\ell > k+1$. Let $t^1 = \ell - k$.

Step 2. There exists a sequence of bids from $\mathbf{b}^{t^1}$ obtained from Step 1 to $\mathbf{b}^{t^1+(\ell-k-1)}$ such that $\mathbf{b}^{t^1} \xrightarrow{sgb} \mathbf{b}^{t^1+1} \xrightarrow{sgb} ... \xrightarrow{sgb} \mathbf{b}^{t^1+(\ell-k-1)}$, for any h with $0 \leqq h \leqq \ell-k-2$, $\mathbf{b}^{t^1+h} \xrightarrow{sgb} \mathbf{b}^{t^1+h+1}$ is realized by bidder $i_{\ell-h}$'s SGB for A, and bidder $i_{h'}$ is in slot $h'-1$ at $\mathbf{b}^{t^1+h}$ for any h' with $\ell \geqq h' \geqq k+1$, and $\mathbf{b}^{t^1+h}$ is $\ell-h$-consistent.

Note that in the case of $h=0$, $\mathbf{b}^{\ell-k}$ is ℓ-consistent by Step 1. Consider that bidder $i_\ell, i_{\ell-1}, ..., i_{k+2}$ sequentially change their bids according to the SGB. Let $\mathbf{b}^{t^1} \xrightarrow{sgb} \mathbf{b}^{t^1+1} \xrightarrow{sgb} ... \xrightarrow{sgb} \mathbf{b}^{t^1+(\ell-k-1)}$ be the resulting sequence of bid profiles. We first show that $b_{i_\ell}^{t^1+1} < b_{i_{\ell-1}}^{t^1}$. By the construction of $\mathbf{b}^{t^1}$, $b_{i_{\ell-1}}^{t^1} = b_{i_{\ell-1}}^A(\ell-1, \alpha_{\ell-1} b_{i_\ell}^t)$. On the other hand, because $v_{i_\ell} > v_i$, i_ℓ follows the SGB for A and thus $b_{i_\ell}^{t^1+1} = b_{i_\ell}^A(\ell-1, \alpha_{\ell-1} b_i^t)$. Since $v_{i_{\ell-1}} > v_{i_\ell}$ and $b_{i_\ell}^t > b_i^t$, we have, by Lemma 6.5,

$$b_{i_\ell}^{t^1+1} = b_{i_\ell}^A(\ell-1, \alpha_{\ell-1} b_i^t) < b_{i_{\ell-1}}^A(\ell-1, \alpha_{\ell-1} b_{i_\ell}^t) = b_{i_{\ell-1}}^{t^1}.$$

This implies that i_ℓ's SGB for A does not change the allocation of slots.

Next, we show that given h with $1 \leqq h \leqq \ell-k-2$, if $b_{i_{\ell-h+1}}^{t^1+h} < b_{\ell-h}^{t^1}$, then $b_{i_{\ell-h}}^{t^1+h+1} < b_{i_{\ell-h-1}}^{t^1}$. By the construction of $\mathbf{b}^{t^1}$, $b_{i_{\ell-h-1}}^{t^1} = b_{i_{\ell-h-1}}^A(\ell-h-1, \alpha_{\ell-h-1} b_{i_{\ell-h}}^{t^1})$. On the other hand, because $v_{i_{\ell-h}} > v_{i_{\ell-h+1}}$, $i_{\ell-h}$ follows the SGB for A and thus $b_{i_{\ell-h}}^{t^1+h+1} = b_{i_z\ell-h}^A(\ell-h-1, \alpha_{\ell-h-1} b_{i_{\ell-h+1}}^{t^1+h})$. Since $v_{i_{\ell-1}} > v_{i_\ell}$ and $b_{i_{\ell-h+1}}^{t^1+h} < b_{\ell-h}^{t^1}$, we have, by Lemma 6.5,

$$b_{i_{\ell-h}}^{t^1+h+1} = b_{i_{\ell-h}}^A(\ell-h-1, \alpha_{\ell-h-1} b_{i_{\ell-h+1}}^{t^1+h}) < b_{i_{\ell-h-1}}^A(\ell-h-1, \alpha_{\ell-h-1} b_{i_{\ell-h}}^{t^1}) = b_{i_{\ell-h-1}}^{t^1}.$$

Applying the argument in the previous paragraph from $h=1$ to $h=\ell-k-2$, we know that for any h, bidder $i_{\ell-h}$ follows the SGB for A and the change of his bid does not affect the allocation of the ad-slots. Moreover, bid profile $\mathbf{b}^{t^1+h}$ is $\ell-h$-consistent. Therefore, at the final bid profile

$\mathbf{b}^{t^1+\ell-k-1}$, bidder i_{k+2} occupies the slot $k+1$, and thus bidder i_{k+1} occupies the slot k. This means that at this bid profile, Inequality (2) does not hold for the bidder in slot k. Moreover, this bid profile is $\ell-(\ell-k-1)=k+1$-consistent.

It is obvious that at the bid profile $\mathbf{b}^{t^1+\ell-k-1}$, no bidder does overbid. Finally, $t^1+\ell-k-1 = 2(\ell-k)-1 \leqq 2(K-k)-1$. □

Lemma 6.7. *Suppose bid $\boldsymbol{b}^0$ is $k+1$-consistent, no bidder does overbid at this bid profile, and bidder $d(k)$ at the bid profile $\boldsymbol{b}^0$ does not satisfy Inequality (2). Then, there exists a finite sequence of bid profiles, $\boldsymbol{b}^1$, $\boldsymbol{b}^2$, ..., $\boldsymbol{b}^t$ such that*

1. *for any s with $0 \leqq s < t$, $\boldsymbol{b}^s \xrightarrow{sgb} \boldsymbol{b}^{s+1}$, and*
2. *$\boldsymbol{b}^t$ satisfies the following two conditions:*
   (a) *$\boldsymbol{b}^t$ is k-consistent*
   (b) *any bidder does not do overbid,*

   Moreover, t is less than or equal to $2(K-k)(k-1)$.

Proof. Consider the SGB strategy of bidder $d(k)$ from $\mathbf{b}^1$ and let the new bid profile be $\mathbf{b}^1$. By assumption of this lemma, $d(k)$ must follow the SGB for A. If the change in his bid does not change the allocation of the slots, $\mathbf{b}^1$ is k-consistent and no bidder does not do overbid in this bid profile. Thus, we assume that the allocation of slots at $\mathbf{b}^1$ is not the same as one at $\mathbf{b}^0$. Let $i_k, i_{k-1}, ..., i_1$ be the bidders that occupy the slots $k, k-1, ..., 1$ at the bid profile $\mathbf{b}^0$, respectively. Then, there must exist $\ell < k$ such that at bid profile $\mathbf{b}^1$, bidder i_k obtains slot ℓ, bidder i_h with $k-1 \leqq h \leqq \ell$ obtain slot $h+1$, and bidder h with $1 \leqq h \leqq \ell-1$ keeps slot h.

Let $\mathbf{b}^{(1)} = \mathbf{b}^1$. Then, note that at the bid profile $\mathbf{b}^{(1)}$, the value of the bidder in slot k^* is greater than all the values of the bidders in slot $k+1, k+2, ..., K$. Consider the SGB strategy of bidder i_{k-1} at $\mathbf{b}^{(1)}$ who occupies the slot k at this bid profile. We separately consider the two cases: (1) i_{k-1} follows the SGB for A, and (2) i_{k-1} follow the SGB for B. In case (1), if the SGB for A of i_{k-1} does not change the allocation of $\mathbf{b}^{(1)}$, and thus this new bid profile is desired one. On the other hand, if the SGB for A of i_{k-1} changes the allocation of $\mathbf{b}^1$ and i_{k-1} obtains slot $\ell_1 < k$, let this new bid profile be $\mathbf{b}^{(2)}$. In case (2), by applying Lemma 6.6, after less than $2(K-k)$ times repetition of the changes in the bid by SGB strategy, there appears bid profile $\mathbf{b}^{(1')}$ such that this is $k+1$-consistent, any bidder do not overbid, and bidder j_1 in slot k at this bid profile does not satisfy Inequality (2). Then, we apply the argument in case (1) to bidder j_1 and bid profile $\mathbf{b}^{(1')}$ instead of i_{k-1} and $\mathbf{b}^{(1)}$. As the result, the new bid after the SGB for A of j_1 is the desired bid profile, or $\mathbf{b}^{(2)}$ are defined.

Next, we apply the same argument mentioned in the previous paragraph for bid profile $\mathbf{b}^{(2)}$ and bidder i_{k-2} who occupies slot k at this bid profile. The result is that either we obtain the desired bid profile or $\mathbf{b}^{(3)}$. Applying this argument sequentially unless the desired bid profile is obtained, there must exist bid profile $\mathbf{b}^{(t)}$ such that bidder in slot k at this bid profile had already changed his bid according to SGB for A in this procedure. Let the name of this bidder be i^* and consider the SGB of i^* at $\mathbf{b}^{(t)}$. Then, by construction of this process and the fact that this bidder previously change his bid according to SGB for A, at this time he must follow the SGB for A at this bid profile. Moreover, because payments of each slot h with $h \geqq k+1$ is non-increasing from $\mathbf{b}^{(1)}$ to $\mathbf{b}^{(t)}$, the SGB of i^* at $\mathbf{b}^{(t)}$ does not exceed $b_{i^*}^{(t)}$ that is the SGB for A at the past bid profile. Thus, the resulting bid profile is the desired one.

Therefore, we obtain the desired bid profile less than or equal to $(2(K-k)-1+1)*(k-\ell) \leqq 2(K-k)(k-1)$. □

Lemma 6.8. *Suppose bid b^0 is $K+1$-consistent, no bidder does overbid at this bid profile. Then, there exists a finite sequence of the bids, **b**^{1}, **b**^{2}, ..., **b**^{t} such that*

1. *for any s with $0 \leqq s < t$, **b**$^s \xrightarrow{sgb}$ **b**$^{s+1}$, and*
2. ***b**^{t} is 1-consistent.*

Moreover, t is less than or equal to $\frac{K(K^2+3K-4)}{6}$.

Proof. Apply Lemma 6.7 from $k = K$ to $k = 1$, and we obtain the desired bid profile no more than

$$\sum_{k=1}^{K} 2(K-k)(k-1) = \frac{K(K^2+3K-4)}{6}$$

□

Finally, we prove Theorem 4.2. From Lemma 6.1 and 6.8, we know that from any initial bid profile, there exist a sequence of bid profiles less than or equal to $2N + K^2 - K + \frac{K(K^2+3K-4)}{6}$ that realize **b*** defined in (3). This means that for any bid profile, there exists a small probability greater than $\eta > 0$ that **b*** is realized. This with the knowledge on the Markov process with infinite states guarantees that the convergence to **b*** occurs almost surely from any initial bid profile.

7 Proof of Theorem 5.3

For two bid profile **b** and **b'**, we say that **b'** can be realized from **b** by SGB and write **b** $\xrightarrow{teb}$ **b'** if **b'** is realized from **b** by some one plater's changing in his bid according to the TEB.

For the proof of this theorem, it suffices to show the following lemmas. Let **b*** be defined in Eq. (3).

Lemma 7.1. *For any initial bid **b**^{0}, there exists a finite sequence of bid profiles, **b**^{1}, **b**^{2}, ..., **b**^{t} such that*

1. *for any s with $0 \leqq s < t$, **b**$^s \xrightarrow{teb}$ **b**$^{s+1}$, and*
2. *no bidder does overbid at **b**^{t}*

Moreover, t is less than or equal to $2N$.

Proof. Let S be the set of bidders that do overbid at initial bid profile **b**^{0}. Take any $i \in S$ and let $i = d(k)$. Consider TEB strategy of bidder i. We separately consider the two cases: (1) i's profit at **b**^{0} is negative, and (2) it is non-negative. In case (1), bidder i changes his bid to v_i and thus, he does not do overbid after the change in his bid. In case (2), depending on the allocation after the change of i's bid according to the TEB strategy, we separately consider the two sub-cases: (2a) i obtains new ad-slot ℓ that is higher position than k, and (2b) i keeps the ad-slot k. In (2a), the profit of i becomes negative because i does overbid at **b**^{0} and now he obtain slot $\ell < k$. Thus, if he changes his bid according to the TEB strategy again, his new bid becomes v_i. In (2b), his bid is now $b^A(k, \alpha_k b_{d(k+1)}) = (1-r_k)v_i + r_k b_{d(k+1)}$. His bid is less than v_i because $b_{d(k+1)} < v_i$. Therefore, for any bidder $i \in S$, he stops the overbid after at most two times of change in his bid according to the TEB strategy.

We have the desired result because $|S| \leqq N$.

□

Lemma 7.2. *For any bid profile **b**^{0} satisfying the following two conditions: for some $k \leqq N$*

(a) *no bidder does overbid,*

(b) *for any bidder $i > k$, $b_i^0 = b_i^*$, and*

(c) *for any bidder $i > k$, $d(i) = i$ at the bid profile $\boldsymbol{b}^0$,*

there exists a finite sequence of bid profiles

$$\boldsymbol{b}^0 \xrightarrow{teb} \boldsymbol{b}^1 \xrightarrow{teb} \dots \xrightarrow{teb} \boldsymbol{b}^m$$

where $\boldsymbol{b}^m$ satisfying the following two conditions:

(a') *no bidder does overbid,*

(b') *for any bidder $i \geq k$, $b_i^m = b_i^*$, and*

(c') *for any bidder $i \geq k$, $d(i) = i$ at the bid profile $\boldsymbol{b}^m$,*

and m is less than or equal to $4k + 1$.

Proof. To construct the sequence of bids in this lemma, we consider the two steps.

Step 1. We construct a sequence of bids from $\mathbf{b}^0$ to $\mathbf{b}^{m^1}$ by the following procedure:

- Bidder k changes his bid according to TEB strategy repeatedly until the change of his bid does not change the bid profile.

Since the number of the bidder is finite, this procedure will stop at a finite time. Moreover, the number of the change in his bid is less than or equal to k and thus $m^1 \leqq k$.

Step 2. We construct a sequence of bids from $\mathbf{b}^{m^1}$ to $\mathbf{b}^{m^1+m^2}$ by the following procedure:

- Repeat the following three stages until the termination condition is satisfied at Stage 1. Let the current bid profile be $\mathbf{b}^{m^1+3t}$ and bidder k be in slot ℓ at this bid profile.

- Stage 1. The bidder in slot $\ell + 1$, say bidder j, changes his bid according to TEB. Let $\mathbf{b}^{m^1+3t+1}$ be the resulting bid profile. If $\mathbf{b}^{m^1+3t+1} = \mathbf{b}^{m^1+3t}$, stop this procedure and $\mathbf{b}^{m^1+3t+1}$ is the final bid profile $\mathbf{b}^{m^1+m^2}$. Otherwise go to Stage 2.

- Stage 2. Bidder j changes his bid according to TEB at bid profile $\mathbf{b}^{m^1+3t+1}$. Let $\mathbf{b}^{m^1+3t+2}$ be the resulting bid profile.

- Stage 3. Stage 3. Bidder k changes his bid according to TEB at bid profile $\mathbf{b}^{m^1+3t+2}$. Let $\mathbf{b}^{m^1+3t+3}$ be the resulting bid profile.

For the sequence of bid profiles constructed in Step 2, we will show the following claim.

Claim. Let $d(.)$ be defined and fixed at the bid profile $\mathbf{b}^{m^1+3t}$. Let bidder k be in slot ℓ at this bid profile and let $j = d(\ell + 1)$. Assume that $\mathbf{b}^{m^1+3t}$ satisfies the following conditions

1. $b_k^{m^1+3t} = b_k^A(\ell, \alpha_\ell b_j^{m^1+3t})$, and

2. for $i \in S := \{d(1), ..., d(\ell - 1)\}$,

$$b_i^{m^1+3t} > b_k^A(\ell - 1, \alpha_{\ell-1} b_k^{m^1+3t}).$$

If $v_j > v_k$, $\mathbf{b}^{m^1+3t+3}$ satisfies the following conditions:

1'. k is in slot $\ell+1$,

2'. $b_k^{m^1+3t+3} = b_k^A(\ell+1, \alpha_{\ell+1} b_{j'}^{m^1+3t+3})$ where $j' = d(\ell+2)$,

3'. for $i \in S \cup \{j\}$,

$$b_i^{m^1+3t+3} > b_k^A(\ell, \alpha_\ell b_k^{m^1+3t+3}).$$

For the simplicity, we use notations $\mathbf{b}$ instead of $\mathbf{b}^{m^1+3t}$ and $\mathbf{b}^i$ instead of $\mathbf{b}^{m^1+3t;i}$, $i = 1, 2, 3$.

We first consider Stage 1. The trial bid of bidder j is $b_j^A(\ell, \alpha_\ell b_j)$ and it satisfies $b_j^A(\ell, \alpha_\ell b_j) > b_k^A(\ell, \alpha_\ell b_j) = b_k$ since $v_j > v_k$. Thus, in the new bid profile $\mathbf{b}^1$, $b_j^1 = b_j^A(\ell, \alpha_\ell b_j)$, and j is in slot ℓ and k is in slot $\ell+1$.

In Stage 2, bidder j changes his bid again. His new bid b_j^2 must satisfy the following

$$b_j^2 \geqq b_j^A(\ell, \alpha_\ell b_k),$$

where note that $b_k = b_k^1$.

In Stage 3, bidder k change his bid according to the TEB strategy. His trial bid is $b_k^A(\ell, \alpha_\ell b_k)$ and it satisfies the following two conditions: for any $i \in S$,

$$b_k^A(\ell, \alpha_\ell b_k) < b_k^A(\ell, \alpha_{\ell-1} b_k) < b_i = b_i^2$$

where the third inequality is from the assumption of this claim, and

$$b_k^A(\ell, \alpha_\ell b_k) < b_j^A(\ell, \alpha_\ell b_k) \leqq b_j^2.$$

Therefore, j's trial bid does not change the allocation. Thus, $b_k^3 = b_k^A(\ell+1, \alpha_{\ell+1} b_{d(\ell+2)})$ and k is in slot $\ell+1$ at $\mathbf{b}^3$. Moreover, since $b_k^3 = b_k^A(\ell+1, \alpha_{\ell+1} b_{d(\ell+2)}) < b_k^A(\ell, \alpha_\ell b_{d(\ell+1)}) = b_k$, we have

$$b_k^A(\ell, \alpha_\ell b_k^3) \leqq b_k^A(\ell, \alpha_\ell b_k) < b_i^3$$

for any $i \in S \cup \{j\}$. The proof of the clam is finished.

We can apply this claim until $v_k < v_j$ is violated. Thus, this process is repeated less than or equal to k times. It is easily confirmed that the resulting bid profile $\mathbf{b}^{m^1+m^2}$ satisfying conditions (a') and (b) and m^2 is less than or equal to $3k+1$. Thus, $m^1 + m^2 \leqq 4k+1$. □

Applying this lemma from $k = N$ to $k = 1$, we obtain the finite sequence from any initial bid profile to $\mathbf{b}^*$ where each change in the bid is realized by TEB strategy of some bidder and the number of change in the bid is less than or equal to

$$\sum_{k=1}^{N} (4k+1) = 2N(N+1) + N.$$

Final argument is the same as the last paragraph in the proof of Theorem 4.2.