A study on the set of stationary solutions for the Gray-Scott model

Gray-Scott モデルにおける定常解集合の研究

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Introduction

In this thesis, we study the following reaction diffusion system known as Gray-Scott model;

$$\text{(GS)} \qquad \begin{cases} U_t = D_U \Delta U - k_1 U V^2 + k_f (U_0 - U) & \text{in} \quad \Omega \times (0, \infty), \\ V_t = D_V \Delta V + k_1 U V^2 - k_2 V & \text{in} \quad \Omega \times (0, \infty), \end{cases}$$

where $U(x,t)$ and $V(x,t)$ represent chemical substance concentrations at place $x \in \Omega$ and time $t > 0$. In (GS), D_U and D_V are diffusion coefficients of the chemical substances. And, k_1 and k_2 are chemical reaction speed constants in the following reactions

$$\begin{aligned} &U + 2V \to 3V, \\ &V \to P. \end{aligned}$$

In addition, k_f and U_0 are positive constants. The term $k_f(U_0 - U)$ represents that this chemical reaction is an open system. Furthermore, the region Ω is a bounded domain in $\mathbf{R^N}(N \geq 1)$ with smooth boundary $\partial\Omega$ or entire space $\mathbf{R^N}$.

The Gray-Scott model was first proposed as ordinary differential equations. (cf [12, 13, 14].) Using numerical simulation, Pearson [27] have found complex spatio-temporal patterns when he added the diffusion effect with the ordinary differential systems. For example, he found so-called self-replicating patterns, interesting pulse interaction phenomena and complicated stationary patterns. Since then, the Gray-Scott model with diffusion attracted a lot of researchers. Especially, stationary problems of (GS) have been studied by many mathematicians.

As in the same way of [17], we will describe (GS) as follows:

$$\text{(P)} \qquad \begin{cases} u_t = \Delta u - uv^2 + \lambda(1 - u) & \text{in} \quad \Omega \times (0, \infty), \\ \frac{\gamma}{d} v_t = \gamma \Delta v + uv^2 - v & \text{in} \quad \Omega \times (0, \infty), \end{cases}$$

where λ, γ and d satisfy

$$\lambda = \frac{k_1 k_f U_0^2}{k_2^2}, \quad \gamma = \frac{k_2 D_V}{k_1 U_0^2 D_U}, \quad d = \frac{D_V}{D_U}.$$

Stationary solutions of (P) satisfy the following elliptic equations;

$$\text{(SP)} \qquad \begin{cases} \Delta u - uv^2 + \lambda(1-u) = 0 & \text{in} \quad \Omega, \\ \gamma\Delta v + uv^2 - v = 0 & \text{in} \quad \Omega, \\ + \text{ boundary conditions} & \text{on} \quad \partial\Omega. \end{cases}$$

This thesis mainly treats with the stationary problem. Especially, we will discuss the following three problems;

A. Set of stationary solutions for (SP) in a bounded domain.

B. Existence and nonexistence of pulse solutions in entire domain.

C. Stability of front solutions for generalized stationary problem.

These problems will be treated in Chapters 2-4.

In **Chapter 1**, we will discuss non-diffusive case for (GS).

$$\begin{cases} U_t = -k_1UV^2 + k_f(U_0 - U) & \text{in} \quad (0,\infty), \\ V_t = k_1UV^2 - k_2V & \text{in} \quad (0,\infty). \end{cases} \tag{1}$$

We will study existence of time global solution, stability of equilibrium points, asymptotic behavior, and bifurcation of time periodic solution for this ordinary differential system. Put

$$\lambda = \frac{k_1k_fU_0^2}{k_2^2}.$$

If $\lambda < 4$, then there exists a unique global solution $(U(t), V(t))$ of (1) for any initial value (U_0, V_0) such that

$$\lim_{t\to\infty}(U(t), V(t)) = (1, 0).$$

However, behaviors of solutions for (1) are more complicated in case $\lambda > 4$. There are exactly three equilibrium points of (1). Among them, $(1, 0)$ is always stable and $\left(\frac{\lambda+\sqrt{\lambda^2-4\lambda}}{2\lambda}, \frac{\lambda-\sqrt{\lambda^2-4\lambda}}{2}\right)$ is always unstable, but $\left(\frac{\lambda-\sqrt{\lambda^2-4\lambda}}{2\lambda}, \frac{\lambda+\sqrt{\lambda^2-4\lambda}}{2}\right)$ changes its stability and Hopf bifurcation occurs at the turning point of the stability. Furthermore, we will show that $V(t)$ does not converge to zero if an initial value is sufficiently large.

Chapter 2 treats with the following stationary problem in a bounded domain $\Omega \subset R^N (N \geq 1)$ with smooth boundary $\partial\Omega$;

$$\text{(SP1)} \qquad \begin{cases} \Delta u - uv^2 + \lambda(1-u) = 0 & \text{in} \quad \Omega, \\ \gamma\Delta v + uv^2 - v = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\frac{\partial}{\partial n}$ is an outward normal derivative on $\partial\Omega$. Note that constant solutions of (SP1) are given by

(i)$(u, v) = (1, 0)$ if $\lambda < 4$,
(ii)$(u, v) = (1, 0)$, $(\frac{1}{2}, 2)$ if $\lambda = 4$,
(iii)$(u, v) = (1, 0)$, $\left(\frac{\lambda \pm \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda \mp \sqrt{\lambda^2 - 4\lambda}}{2}\right)$ if $\lambda > 4$.

Stationary problem (SP1) has been studied by many authors (see [20, 21, 22, 23, 26, 33, 36, 38].) Their results give us information about a priori bounds, bifurcation structure, and profiles of solutions for (SP1). However in order to obtain more information about the solution set of (SP1), we will analyze (SP1) through different approaches.

Our main purpose of this chapter is to show existence and non-existence results of non-constant solutions for (SP1). The following theorem is concerned with the non-existence of nontrivial solutions.

Theorem 0.1. *Let λ be fixed any positive parameter. Then there exists a positive constant $C(\lambda, \Omega)$ depending only λ and Ω such that* (SP1) *has no non-constant solutions provided $\gamma \geq C(\lambda, \Omega)$.*

Therefore a necessary condition of existence for non-trivial solutions for (SP1) is that γ is suitably small. Especially, when γ is near zero, many authors have constructed multi-peak solutions by using singular perturbation method. See [20, 21, 22, 23, 33, 36, 38]. However, if γ is not necessarily small, there are many open problems about the structure of non-constant solutions. Before stating our existence result, we will introduce the eigenvalues of the Laplacian with homogeneous Neumann boundary condition.

Notation 0.2. *Let $\{\mu_m\}_{m\geq 0}$ be the eigenvalues of*

$$\begin{cases} -\Delta u = \mu u & in \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & on \quad \partial\Omega, \end{cases}$$

satisfying $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$.

We put two positive constant solutions in case $\lambda > 4$ as follows;

$$(u_1, v_1) = \left(\frac{\lambda + \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda - \sqrt{\lambda^2 - 4\lambda}}{2}\right) \tag{2}$$

and

$$(u_2, v_2) = \left(\frac{\lambda - \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda + \sqrt{\lambda^2 - 4\lambda}}{2}\right). \tag{3}$$

In addition, we define the following function;

$$h(\mu, v_i) = \frac{\mu + \lambda - v_i^2}{\mu(\mu + \lambda + v_i^2)} \quad \text{for} \quad i = 1, 2, \tag{4}$$

where v_i for $i = 1, 2$ is the constant defined by (2) and (3).

Making use of the degree theory [2], we will obtain the following existence theorem.

Theorem 0.3. *Let* $\lambda \geq 4$ *and* $\gamma^{-1} \neq 3v_2^2 - \lambda + 2v_2\sqrt{2(v_2^2 - \lambda)}$. *Define* $h(\mu, v_i)$ *for* $i = 1, 2$ *by* (4). *Suppose that every eigenvalue* μ_m *is simple and that*

$$h(\mu_m, v_i) \neq h(\mu_n, v_j) \quad for \quad m \neq n, \; i, j = 1, 2,$$

provided $\lambda > 4$. *Then there exists a positive monotone decreasing sequence* $\{\gamma_k\}$ $(k = 1, 2, \cdots)$ *which converges to zero such that* (SP1) *has at least one non-constant positive solution if*

$$\gamma \in (\gamma_{2k}, \gamma_{2k-1}) \quad \text{for} \quad k = 1, 2, \cdots .$$

McGough-Riley have shown that non-trivial solutions bifurcate from the constant solution (u_i, v_i) for $i = 1, 2$ at $\gamma = h(\mu_m, v_i)$ defined by (4). In this chapter we will also treat with direction of the bifurcation and stability for the bifurcating solutions.

In **Chapter 3**, we study the following elliptic problem in entire space;

$$\text{(SP2)} \qquad \begin{cases} \Delta u - uv^2 + \lambda(1 - u) = 0 & x \in \mathbf{R^N}, \\ \gamma\Delta v + uv^2 - v = 0 & x \in \mathbf{R^N}, \\ \lim_{|x| \to +\infty}(u, v) = (1, 0). \end{cases}$$

Here λ and γ are positive parameters. A non-constant solution of (SP2) like Figure 1 is generally called standing pulse solution for one dimensional case, or spot solution for multi-dimensional case. Note that constant solution of (SP2) is uniquely determined by $(u, v) = (1, 0)$.

There are some known results related to (SP2). Doelman-Gardner-Kaper [9] and Doelman-Kaper-Zegeling [10] have discussed existence and stability for multi-pulse solutions in case $\lambda \ll 1$, $\gamma \ll 1$ and $\lambda\gamma \ll 1$. Hale-Peletier-Troy [17, 18] have constructed a unique one-pulse solution when $0 < \gamma < \frac{2}{9}$ and $|\lambda\gamma - 1|$ is sufficiently small. Wei [37] has obtained two one-spot solutions in case $\gamma \ll 1$. However the solution set of (SP2) for the other case remains largely open problem.

This chapter provides some sufficient conditions about the nonexistence of non-trivial solutions for (SP2).

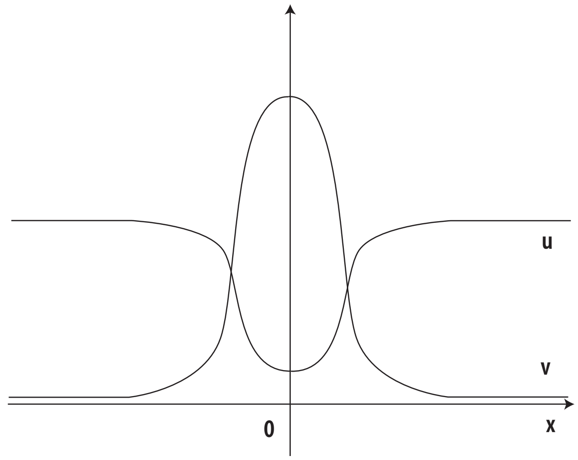

Figure 1: Pulse-like solution for (SP2).

Theorem 0.4. *Let* $\lambda\gamma > 1$. *Then there exists no nontrivial solution of* (SP2) *if* $\lambda \leq 4$.

Theorem 0.5. *Let* $X := \lambda\gamma < 1$. *Then* (SP2) *admits no nontrivial solution if one of the following conditions is satisfied*:
(i)$\gamma \geq \frac{1}{4}$,
(ii)$4 - 16\gamma \leq X \leq 4\gamma$ *with* $\frac{1}{5} \leq \gamma < \frac{1}{4}$,
(iii)$X \leq \frac{4}{5}$, $\gamma < \frac{1}{4}$ *with*

$$\begin{cases} X \geq \frac{4(1-4\gamma)}{(1+4\gamma-16\gamma^2)^2} & if \quad X \leq X^*(\gamma), \\ X \leq 4\gamma & if \quad X > X^*(\gamma), \end{cases}$$

where $X^*(\gamma)$ *is a monotone decreasing function on* γ *and satisfies*

$$X^*(\gamma) \to \frac{4}{5} \quad as \quad \gamma \to 0 \quad and \quad X^*(\gamma) \to \tilde{X} \quad as \quad \gamma \to \frac{1}{4}.$$

Here $\tilde{X} \in (0,1)$ *is a unique number satisfying* $\tilde{X} = (1-\tilde{X})\left(1+\sqrt{1-\tilde{X}}\right)^2$.

Figure 2 shows the existence and non-existence regions about non-trivial solutions for (SP2) in one dimensional case.

In order to prove Theorems 0.4 and 0.5, we will use the following strong maximum principle of second order linear elliptic equation. (See [29].)

Strong maximum principle. *Let* $w(\not\equiv 0)$ *be a classical solution of*

$$\Delta w + c(x)w = f(x), \quad x \in \mathbf{R^N},$$

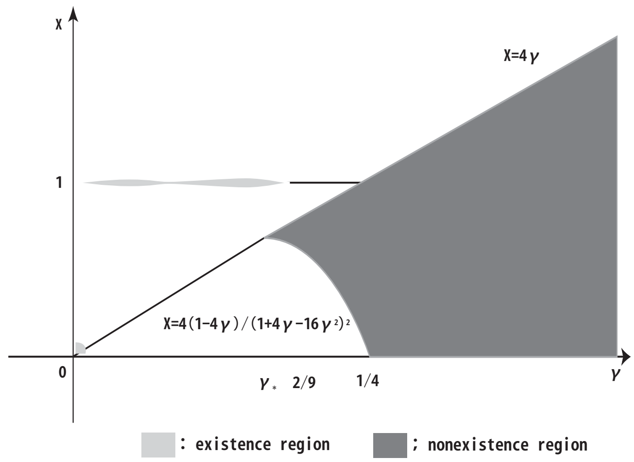

Figure 2: The existence and nonexistence regions.

where $c(x)$ *is a non-positive bounded continuous function on* $\mathbf{R^N}$. *Suppose that* $f(x) \leq 0$ *for* $x \in \mathbf{R^N}$. *If*

$$\liminf_{|x|\to\infty} w(x) \geq 0,$$

then it follows that

$$w(x) > 0 \quad for \quad x \in \mathbf{R^N}.$$

Here we mention an idea of the proof for Theorem 0.5. Making use of the strong maximum principle, we will derive a priori estimates for non-trivial solutions for (SP2). That is, for any non-trivial solution (u, v) for (SP2) there exists a positive constant C depending only on λ and γ such that

$$v(x) < C(1 - u(x)) \quad \text{for} \quad x \in \mathbf{R^N}.$$

If one can take $C \leq 4$, then

$$\max_{x\in\mathbf{R^N}} u(x)v(x) < 1.$$

Therefore it follows from (SP2) that

$$\gamma\Delta v = v(1 - uv) > 0.$$

Since $\lim_{|x|\to\infty} v(x) = 0$, v must be constant. Therefore u is also a constant. It takes much effort to choose the constant C properly. Especially it is difficult to prove (ii) and (iii) of Theorem 0.5.

In **Chapter 4**, we mainly study the following generalized stationary problem for one-dimensional entire space.

$$(\mathrm{SP3}) \qquad \begin{cases} u'' - uv^{\alpha} + \lambda(1-u) = 0, & x \in \mathbf{R}, \\ \gamma v'' + uv^{\alpha} - v = 0, & x \in \mathbf{R}, \\ (u,v)(-\infty) = (1,0), \quad (u,v)(+\infty) = (u_+, v_+), \end{cases}$$

where λ and γ are positive parameters, $\alpha > 1$ is a constant, u_+ and v_+ satisfies

$$u_+ v_+^{\alpha-1} = 1,$$

and v_+ is the largest solution of the following equation

$$v_+^{\alpha} - \lambda v_+^{\alpha-1} + \lambda = 0.$$

A stationary solution of (SP3) like Figure 3 is generally called front solution. There is no existing results except for [18]. Hale-Peletier-Troy [18] have constructed a unique monotone front solution in case

$$\lambda\gamma = 1 \quad \text{and} \quad \gamma = \gamma(\alpha), \tag{5}$$

where $\gamma(\alpha)$ is a constant depending only on α.

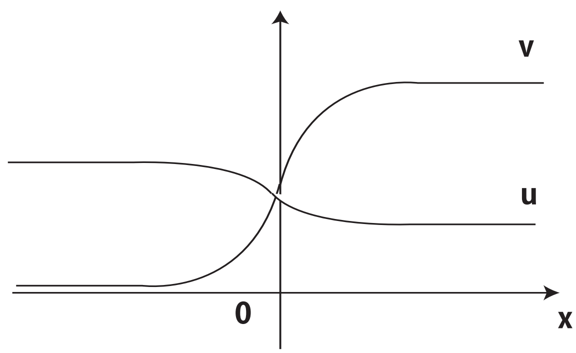

Figure 3: The profile of monotone front solution for (SP3).

In order to discuss stability of the front solution, we must consider the following non-stationary problem;

$$(\mathrm{NSP3}) \qquad \begin{cases} u_t = u'' - uv^{\alpha} + \lambda(1-u), & \text{in} \quad \mathbf{R} \times (0,\infty), \\ \frac{\gamma}{d} v_t = \gamma v'' + uv^{\alpha} - v, & \text{in} \quad \mathbf{R} \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{on} \quad \mathbf{R}. \end{cases}$$

Here $u_0(x)$ and $v_0(x)$ are non-negative continuous functions in $\mathbf{R}$. Moreover, λ, γ and d are positive parameters. Before stating our stability theorem, we must define the following function space;

Notation 0.6. $C_B(\mathbf{R})$ *is a function space of continuous bounded function on* $\mathbf{R}$.

Let $\zeta = (\varphi, \psi)$ be the monotone front solution given in [18, Theorem 4.1]. If the initial data $z_0 = (u_0(x), v_0(x)) \in C_B(\mathbf{R}) \times C_B(\mathbf{R})$, then we obtain our main result in Chapter 4.

Theorem 0.7. *Suppose that* λ *and* γ *satisfy* (5)*. If* $d = 1$ *and* $\alpha > 1$*, then* ζ *is asymptotically stable in the following sense: there exist constants* $\delta, M, \kappa > 0$*, and* $\xi \in \mathbf{R}$ *such that, if*

$$\|z_0 - \zeta(\cdot)\|_\infty < \delta,$$

then the solution $z(\cdot, t) = (u, v)$ *of* (NSP3) *corresponding to initial data* z_0 *satisfies*

$$\|z(\cdot, t) - \zeta(\cdot + \xi)\|_\infty \le M\|z(\cdot, 0) - \zeta(\cdot)\|_\infty e^{-\kappa t}.$$

This is an extension for the result of [17, Theorem 4.6]. The spectrum problem associated with linearization around $\zeta = (\varphi, \psi)$ is given by

$$\text{(LSP)} \qquad \begin{cases} -u'' + (\psi^\alpha + \frac{1}{\gamma})u + (\alpha\varphi\psi^{\alpha-1})v = \mu u, \\ -v'' - \frac{1}{\gamma}\psi^\alpha u + \frac{1}{\gamma}(1 - \alpha\varphi\psi^{\alpha-1})v = \mu v. \end{cases}$$

Spectrum consists of essential spectrum and isolated eigenvalues. Essential spectrum of (LSP) can be determined by the famous result of [19]. Therefore it suffices to study the isolated eigenvalue problem. When (5) is satisfied, the eigenvalue problem (LSP) can be reduced to the following single equation;

$$-v'' + f'(\psi)v = \mu v, \tag{6}$$

where

$$f'(\psi) = (1 + \alpha)\psi^\alpha - \frac{\alpha}{\gamma}\psi^{\alpha-1} + \frac{1}{\gamma}.$$

Observe that (φ, ψ) can be explicitly represented as

$$\varphi = \frac{1}{3}\left\{2 - \tan h\left(\frac{3x}{2\sqrt{2}}\right)\right\}, \quad \psi = \frac{3}{2}\left\{1 + \tan h\left(\frac{3x}{2\sqrt{2}}\right)\right\},$$

if $\alpha = 2$. Hale-Peletier-Troy [17] have used the result of Tistchmarsh [34] for determining the first and second eigenvalues for the eigenvalue problem (6) explicitly.

However, their method can not be applied for the eigenvalue problem in case $\alpha > 1$. To overcome this difficulty, we will focus on the zero point of eigenfunctions for (6).

Note that ψ' is the eigenfuction for zero eigenvalue. And ψ' has no zero point in $\mathbf{R}$. We will show by a contradiction argument that (6) has no negative eigenvalues. Therefore zero is the first eigenvalue for the problem (6) and the second eigenvalue is positive. Consequently, making use of the result for Henry [19], one can establish Theorem 0.7.

Chapter 1

Non-diffusion case

In this chapter, we will discuss non-diffusive case for (GS).

$$\begin{cases} U_t = -k_1 U V^2 + k_f(U_0 - U), \\ V_t = k_1 U V^2 - k_2 V. \end{cases} \tag{1.1}$$

Taking the following change of variables,

$$U = U_0 u, \quad V = \left(\frac{k_2}{k_1 U_0}\right) v, \quad \tilde{t} = \left(\frac{k_2^2}{k_1 U_0^2}\right) t, \quad \lambda = \frac{k_1 k_f U_0^2}{k_2^2}, \quad \eta = \frac{k_2}{k_1 U_0^2},$$

then we see from (1.1) that

$$u_t = -uv^2 + \lambda(1-u), \tag{1.2}$$

$$\eta v_t = uv^2 - v, \tag{1.3}$$

$$u(0) = u_0, \quad v(0) = v_0,$$

where u_0 and v_0 represent initial chemical concentration values. Throughout this chapter, this ordinary differential system is called (E).

1.1 Case $\lambda \leq 4$

First, we will deal with existence of global solution for (E) and asymptotic behavior of solutions in case $\lambda \leq 4$.

Theorem 1.1. *Assume $\lambda \leq 4$. For any nonnegative u_0 and v_0, there exists an unique global solution $(u(t), v(t))$ of* (E) *such that the following inequalities hold true;*

$$u(t) \leq \max\{u_0, 1\}, \quad v(t) \leq \max\{v_0, C_1(\eta, u_0, v_0)\}, \quad \text{for} \quad t > 0,$$

where $C_1(\eta, u_0, v_0)$ is a constant depending on η, u_0 and v_0. Furthermore, for any initial condition (u_0, v_0), the corresponding global solution of (E) *satisfies*

$$\lim_{t\to\infty}(u(t), v(t)) = (1, 0),$$

provided $\lambda < 4$.

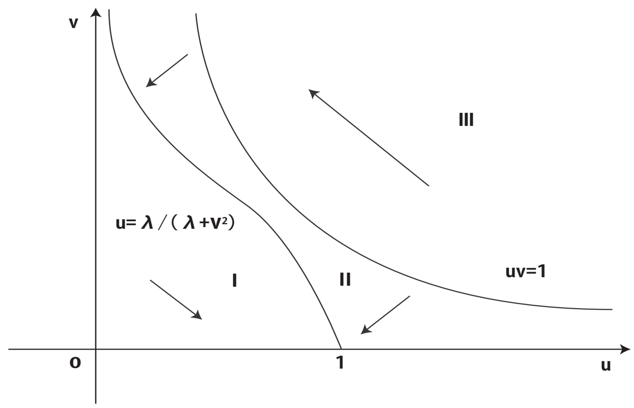

Figure 1.1: The phase plane in case $\lambda < 4$.

Proof. We will prove through phase plane method. Define the following three sets; (See Figure 1.1.)

$$\begin{aligned}
\mathrm{I} &= \left\{(u,v) \in \mathbf{R^2} \ : \ 0 \le u \le \frac{\lambda}{\lambda + v^2}, \ v \ge 0\right\}, \\
\mathrm{II} &= \left\{(u,v) \in \mathbf{R^2} \ : \ \frac{\lambda}{\lambda + v^2} < u \le \frac{1}{v}, \ v \ge 0\right\}, \\
\mathrm{III} &= \left\{(u,v) \in \mathbf{R^2} \ : \ u > \frac{1}{v}, \ v \ge 0\right\}.
\end{aligned}$$

Clearly, if $(u_0, v_0) \in$ I or II, then the corresponding solution $(u(t), v(t))$ of (E) satisfies

$$0 \le u(t) \le \max\{u_0, 1\}, \quad 0 \le v(t) \le v_0, \quad \text{for} \quad t > 0.$$

Therefore it is sufficient to consider the case $(u_0, v_0) \in$ III. If $(u_0, v_0) \in$ III is fixed, we claim that for the corresponding unique orbit $v = v(u)$ of (E) there exists a negative constant C not depending on u and v such that

$$\eta \frac{dv}{du} \ge C. \tag{1.4}$$

Recall that

$$\eta\frac{dv}{du} = \frac{dv}{dt}\Big/\frac{du}{dt}$$
$$= \frac{uv^2 - v}{-uv^2 + \lambda(1-u)}. \tag{1.5}$$

Choosing $C < -1$, one can derive the following estimate provided $(u, v) \in \text{III}$ and $\lambda \leq 4$;

$$\begin{aligned}
&(1+C)uv^2 + C\lambda(u-1) - v \\
&\leq (1+C)v + C\lambda\left(\frac{1}{v} - 1\right) - v \\
&= \frac{C(v^2 - \lambda v + \lambda)}{v} \\
&\leq 0.
\end{aligned} \tag{1.6}$$

Therefore the claim (1.4) holds true. Setting $C = -2$, then we have from Figure 1.1 that

$$0 \leq u(t) \leq \max\{u_0, 1\} \quad \text{and} \quad 0 \leq v(t) \leq C_1(\eta, u_0, v_0),$$

provided $(u_0, v_0) \in \text{III}$. Here $C_1(\eta, u_0, v_0)$ is the largest solution of the following equation;

$$\eta v^2 - (2u_0 + \eta v_0)v + 2 = 0.$$

Note that stationary point of (E) is uniquely determined by $(u, v) = (1, 0)$ in case $\lambda < 4$. Since the trajectory from any initial value (u_0, v_0) is bounded, we see from Poincare-Bendixson Theorem that the asymptotic behavior of solution $(u(t), v(t))$ is as follows;

$$\lim_{t\to\infty}(u(t), v(t)) = (1, 0).$$

Thus the proof is complete. □

The numerical computation presented in Figure 1.2 is made with Mathematica. The governing equation is

$$\begin{cases} u_t = -uv^2 + 1 - u, \\ v_t = uv^2 - v, \\ u_0 = 1, \quad v_0 = 2. \end{cases}$$

The vertical axis represents chemical substance concentration and the horizontal axis shows time. We see that the asymptotic behavior of the solution is

$$\lim_{t\to\infty}(u, v) = (1, 0).$$

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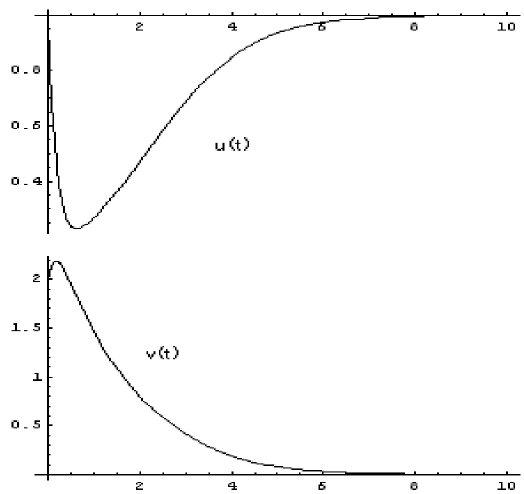


Figure 1.2: The typical numerical simulation in case $\lambda < 4$.

1.2 Case $\lambda > 4$

Next, we will treat (E) in case $\lambda > 4$. The following theorem is concerned with the global existence problem for (E).

Theorem 1.2. *Assume $\lambda > 4$. For any nonnegative u_0 and v_0, there exists a unique global solution $(u(t), v(t))$ of (E) which satisfies the following inequalities;*

$$u(t) \leq \max\{u_0, 1\} \quad \textit{and} \quad v(t) \leq \max\{v_0, C_2(\eta, u_0, v_0)\},$$

where $C_2(\eta, u_0, v_0)$ is a positive constant depending on η, u_0 and v_0.

Proof. The idea of the proof for Theorem 1.2 is the same as that of Theorem 1.1. The difference with the proof of Theorem 1.1 is the case when (u_0, v_0) is in III of Figure 1.3. Then we see from (1.5) and (1.6) that for the corresponding unique orbit $v = v(u)$ of (E) there exists a negative constant C not depending on t such that

$$\eta \frac{dv}{du} \geq C \quad \text{for} \quad (u, v) \in \mathrm{L}.$$

Here L is the two-dimensional region defined by

$$L = \left\{ (u, v) \in \mathbf{R^2} \ : \ v \geq \max\left\{ \frac{1}{u}, \frac{\lambda + \sqrt{\lambda^2 - 4\lambda}}{2} \right\} \right\}.$$

As in the proof of Theorem 1.1, this inequality completes the proof of Theorem 1.2. □

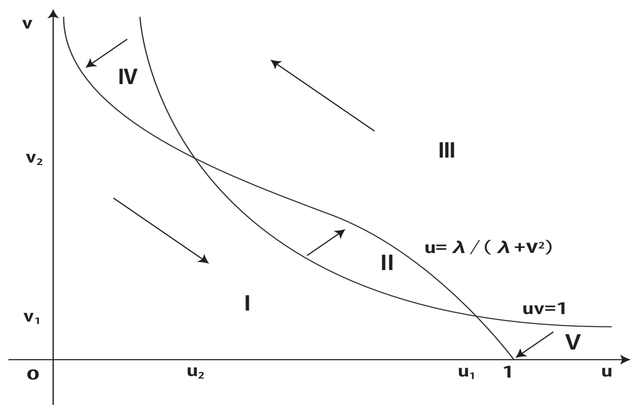


Figure 1.3: The phase plane in case $\lambda > 4$.

As for equilibrium points in case $\lambda > 4$, one can find that

$$(u, v) = (1, 0), \left(\frac{\lambda \pm \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda \mp \sqrt{\lambda^2 - 4\lambda}}{2} \right).$$

We will study stability of these equilibrium points. Suppose that $(\hat{u}, \hat{v})$ is the one of the equilibrium points. Linearizing (E) around $(\hat{u}, \hat{v})$, then we have

$$u_t = -(\hat{v}^2 + \lambda)u - 2\hat{u}\hat{v}v,$$
$$\eta v_t = \hat{v}^2 u + (2\hat{u}\hat{v} - 1)v.$$

The linearized eigenvalue problem of the equilibrium point is as follows;

$$-(\hat{v}^2 + \lambda)u - 2\hat{u}\hat{v}v = \mu u,$$
$$\frac{\hat{v}^2}{\eta}u + \left(\frac{2\hat{u}\hat{v} - 1}{\eta} \right) v = \mu v.$$

Then every eigenvalue μ satisfies the following equation;

$$\begin{vmatrix} \mu + \hat{v}^2 + \lambda & 2\hat{u}\hat{v} \\ -\frac{\hat{v}^2}{\eta} & \mu + \frac{1 - 2\hat{u}\hat{v}}{\eta} \end{vmatrix} = 0$$

Therefore, it follows that

$$(\mu + \hat{v}^2 + \lambda) \left(\mu + \frac{1 - 2\hat{u}\hat{v}}{\eta} \right) + \frac{2\hat{u}\hat{v}^3}{\eta} = 0. \tag{1.7}$$

If $(\hat{u}, \hat{v}) = (1, 0)$, then (1.7) is equivalent to

$$(\mu + \lambda) \left(\mu + \frac{1}{\eta} \right) = 0.$$

Hence we deduce that $\mu = -\lambda, -\frac{1}{\eta}$. Consequently, the following theorem holds true.

Theorem 1.3. *The equilibrium point* $(1,0)$ *is linearly stable for any* $\lambda > 4$ *and positive* η.

As for the stability for the equilibrium point $(u,v) = (1,0)$, we will show some stronger stability result. Define a set Σ as follows:

$$\Sigma = \left\{ (u,v) \in \mathbf{R^2} \ : \ 0 \le u \le 1, \ 0 \le v < 1 \right\}.$$

Then, we obtain the following theorem.

Theorem 1.4. *For any* $\lambda > 4$ *and positive* η*, the set* Σ *is an invariant region. Moreover, the corresponding solution* $(u(t), v(t))$ *of* (E) *for any* $(u_0, v_0) \in \Sigma$ *satisfies*

$$\lim_{t\to\infty} (u(t), v(t)) = (1,0).$$

Proof. If $u(t_0) \le 0$ for some $t_0 > 0$, then we have from (1.2) that

$$u'(t_0) = -u(t_0)v(t_0)^2 + \lambda(1 - u(t_0)) \ge \lambda.$$

This is impossible. Therefore $u(t) > 0$ for any $t > 0$.

According to (1.2), we see

$$u'(t) \le \lambda(1 - u(t)) \quad \text{for any} \quad t > 0.$$

Since $u(0) \le 1$, one can show that $u(t) \le 1$ for any $t > 0$.

As for $v(t)$, suppose that $v(t_1) < 0$ for some $t_1 > 0$. It follows from (1.3) that

$$\eta v'(t_1) = u(t_1)v(t_1)^2 - v(t_1) > 0.$$

This inequality implies that $v(0) < 0$, which is a contradiction. Hence $v(t) \ge 0$ for any $t > 0$.

Note that $u(t) \le 1$ for any $t > 0$. Owing to (1.3), we find

$$\eta v' \le v - v^2 \quad \text{for} \quad t > 0.$$

Observe that $v(0) < 1$. Then $v(t) < 1$ for any $t > 0$. Moreover,

$$\lim_{t\to+\infty} v(t) = 0.$$

Therefore, for any $\epsilon > 0$ there exists a large time T such that

$$\lambda - (\lambda + \epsilon)u \le u_t \le u + \lambda(1 - u) \quad \text{for} \quad t \ge T.$$

This inequality implies that

$$\frac{\lambda}{\lambda+\epsilon} \leq \liminf_{t\to\infty} u(x,t) \leq \limsup_{t\to\infty} u(x,t) \leq 1.$$

Taking $\epsilon \to 0$, then we conclude that

$$\lim_{t\to\infty} u(x,t) = 1.$$

$\square$

When $(\hat{u},\hat{v}) = \left(\frac{\lambda\pm\sqrt{\lambda^2-4\lambda}}{2\lambda}, \frac{\lambda\mp\sqrt{\lambda^2-4\lambda}}{2}\right)$, then (1.7) can be written as

$$\mu^2 + \left(\hat{v}^2 + \lambda - \frac{1}{\eta}\right)\mu + \frac{1}{\eta}(\hat{v}^2 - \lambda) = 0. \tag{1.8}$$

Note that $\hat{v}^2 - \lambda < 0$ if $(\hat{u},\hat{v}) = \left(\frac{\lambda+\sqrt{\lambda^2-4\lambda}}{2\lambda}, \frac{\lambda-\sqrt{\lambda^2-4\lambda}}{2}\right)$. Then (1.8) has one positive and one negative real solution. According to the result in [16, Theorem 9.29], we have the following theorem.

Theorem 1.5. *The equilibrium point* $(u_1, v_1) = \left(\frac{\lambda+\sqrt{\lambda^2-4\lambda}}{2\lambda}, \frac{\lambda-\sqrt{\lambda^2-4\lambda}}{2}\right)$ *is a saddle point.*

Figure 1.4 is described as a neighborhood of the equilibrium point (u_1, v_1).

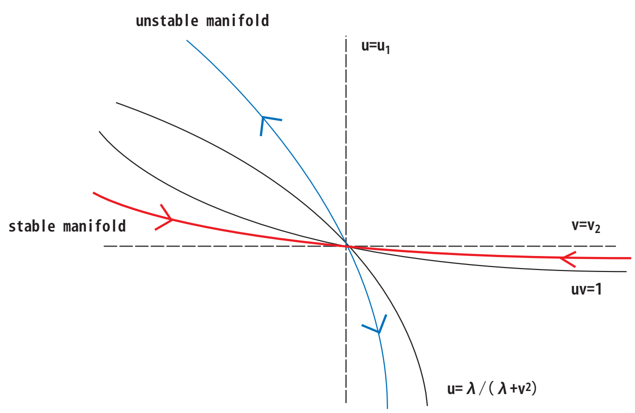


Figure 1.4: The flow in a neighborhood of the saddle point (u_1, v_1).

Finally, we will discuss the stability of

$$(u_2, v_2) = \left(\frac{\lambda-\sqrt{\lambda^2-4\lambda}}{2\lambda}, \frac{\lambda+\sqrt{\lambda^2-4\lambda}}{2}\right). \tag{1.9}$$

Note that $v_2^2 - \lambda > 0$ and the discriminant D of (1.8) is

$$D = \hat{\eta}^2 - 2(3v_2^2 - \lambda)\hat{\eta} + (v_2^2 + \lambda)^2,$$

where

$$\hat{\eta} = \frac{1}{\eta}. \tag{1.10}$$

After some calculation, one can show the following theorem.

Theorem 1.6. *Suppose that* (u_2, v_2) *and* $\hat{\eta}$ *are defined by* (1.9) *and* (1.10) *respectively. Then the equilibrium point* (u_2, v_2) *is* (i) *a stable point if* $0 < \hat{\eta} < A_-$, (ii) *a stable spiral point if* $A_- < \hat{\eta} < B$, (iii) *an unstable spiral point if* $B < \hat{\eta} < A_+$ (iv) *an unstable point if* $A_+ < \hat{\eta}$. *Here*

$$A_{\pm} = 3v_2^2 - \lambda \pm 2v_2\sqrt{2(v_2^2 - \lambda)}, \quad B = v_2^2 + \lambda.$$

Figure 1.5 is depicted as a neighborhood of the equilibrium point (u_2, v_2).

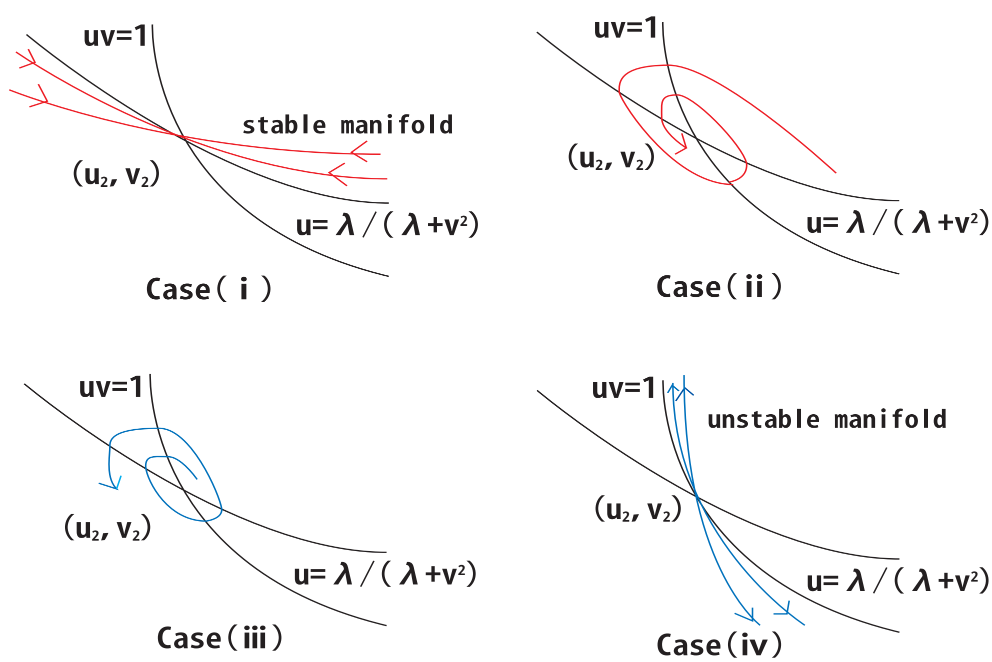


Figure 1.5: The flow in a neighborhood of (u_2, v_2).

We will use the following Poincare-Andronov-Hopf theorem to show existence of periodic solutions encircling the equilibrium point (u_2, v_2).

Theorem 1.7 ([16],Theorem 11.12). *Let* $\dot{x} = A(\lambda)x + F(\lambda, x)$ *be a* C^k*, with* $k \geq 3$*, planar vector field depending on a scalar parameter* λ *such that* $F(\lambda, 0) = 0$ *and* $D_x F(\lambda, 0) = 0$ *for all sufficiently small* $|\lambda|$*. Assume that the linear part* $A(\lambda)$ *at the origin has the eigenvalues* $\alpha(\lambda) \pm i\beta(\lambda)$ *with* $\alpha(0) = 0$ *and* $\beta(0) \neq 0$*. Furthermore, suppose that the eigenvalues cross the imaginary axis with nonzero speed, that is,*

$$\frac{d\alpha}{d\lambda}(0) \neq 0.$$

Then, in any neighborhood U *of the origin in* $\mathbf{R^2}$ *and any given* $\lambda_0 > 0$ *there is a* $\bar{\lambda}$ *with* $|\bar{\lambda}| < \lambda_0$ *such that the differential equation* $\dot{x} = A(\bar{\lambda})x + F(\bar{\lambda}, x)$ *has a nontrivial periodic orbit in* U*.*

Denote

$$\mu = \alpha(\hat{\eta}) \pm i\beta(\hat{\eta}),$$

with

$$\alpha(\hat{\eta}) = \frac{\hat{\eta} - v_2^2 - \lambda}{2} \quad \text{and} \quad \beta(\hat{\eta}) = \pm\frac{1}{2}\sqrt{-\hat{\eta}^2 + 2(3v_2^2 - \lambda)\hat{\eta} - (v_2^2 + \lambda)^2}.$$

If $\hat{\eta} = v_2^2 + \lambda$, one can see that

$$\alpha(\hat{\eta}) = 0, \quad \beta(\hat{\eta}) = \pm\sqrt{v_2^4 - \lambda^4}.$$

Because $\frac{d\alpha}{d\hat{\eta}} \neq 0$ at $\hat{\eta} = v_2^2 + \lambda$, one can derive the following theorem.

Theorem 1.8. *Let*

$$\eta_h = \frac{1}{v_2^2 + \lambda}.$$

For any neighborhood U *of* (u_2, v_2) *in* $\mathbf{R^2}$ *and any given* $\epsilon_0 > 0$ *there is a* $\bar{\eta}$ *with* $|\bar{\eta} - \eta_h| < \epsilon_0$ *such that* (E) *has a nontrivial periodic solution in* U *if* $\eta = \bar{\eta}$*.*

Even if $\lambda > 4$, we can see from Theorem 1.4 that the solution of (E) satisfies

$$\lim_{t\to\infty}(u(t), v(t)) = (1, 0)$$

if the initial value (u_0, v_0) is sufficiently small. However, if (u_0, v_0) is sufficiently large, then asymptotic behavior of the solution for (E) is different.

Theorem 1.9. *Let $\lambda\eta \geq 1$ and $\eta < \frac{1}{4}$. Assume that an initial value (u_0, v_0) satisfies the following inequalities*;

$$u_0 + \eta v_0 - 1 \geq 0 \quad \textit{and} \quad v_0 \geq \frac{1 - \sqrt{1 - 4\eta}}{2\eta}.$$

Then the corresponding solution $(u(t), v(t))$ of (E) *satisfies the following estimate*;

$$\liminf_{t \to +\infty} v(t) \geq \frac{1 + \sqrt{1 - 4\eta}}{2\eta}.$$

Proof. Define a new function $p = u + \eta v - 1$. If $\lambda\eta \geq 1$, then (1.2) + (1.3) implies that

$$p_t = -\lambda p + (\lambda\eta - 1)v \geq -\lambda p.$$

Observe that $p(0) = u_0 + \eta v_0 - 1 \geq 0$. Then we have

$$p(t) \geq 0 \quad \text{for} \quad t > 0.$$

Since $u(t) \geq 1 - \eta v(t)$ for any $t > 0$, one can see from (1.3) that

$$\eta v_t \geq -v(\eta v^2 - v + 1). \tag{1.11}$$

In view of

$$v_0 \geq \frac{1 - \sqrt{1 - 4\eta}}{2\eta},$$

we find according to (1.11) that

$$\liminf_{t \to +\infty} v(t) \geq \frac{1 + \sqrt{1 - 4\eta}}{2\eta}.$$

Thus the proof is complete. □

Theorem 1.10. *Suppose that $\lambda\eta \leq 1$ and $\lambda^2\eta > 4$. Let an initial value (u_0, v_0) satisfy*

$$u_0 + \eta v_0 - \lambda\eta \geq 0 \quad \textit{and} \quad v_0 > \frac{\lambda\eta - \sqrt{\lambda^2\eta^2 - 4\eta}}{2\eta}. \tag{1.12}$$

Then the corresponding solution $(u(t), v(t))$ of (E) *satisfies the following estimate*;

$$\liminf_{t \to +\infty} v(t) \geq \frac{\lambda\eta + \sqrt{\lambda^2\eta^2 - 4\eta}}{2\eta}.$$

Proof. Denote $q = u + \eta v - \lambda\eta$. Adding (1.2) with (1.3), we have

$$q_t = -\frac{1}{\eta}q + \left(\frac{1}{\eta} - \lambda\right) v \geq -\frac{1}{\eta}q,$$

provided $\lambda\eta \leq 1$. Since $q(0) = u_0 + \eta v_0 - \lambda\eta \geq 0$, then we find that

$$q(t) \geq 0 \quad \text{for} \quad t > 0.$$

It follows from (1.3) that

$$\begin{aligned} \eta v_t &\geq (\lambda\eta - \eta v)v^2 - v \\ &= -v(\eta v^2 - \lambda\eta v + 1). \end{aligned}$$

In view of (1.12), we deduce that

$$\liminf_{t\to+\infty} v(t) \geq \frac{\lambda\eta + \sqrt{\lambda^2\eta^2 - 4\eta}}{2\eta}.$$

Consequently, the proof is complete. □

The numerical simulation in Figure 1.6 is also made with Mathematica. The governing equation is

$$\begin{cases} u_t = -uv^2 + 5 - 5u, \\ v_t = uv^2 - v, \\ u(0) = u_0, \quad v(0) = v_0. \end{cases}$$

For case (A), the initial values are given by

$$u_0 = 0.8 \quad \text{and} \quad v_0 = 1.35.$$

On the other hand, the initial values are determined by

$$u_0 = 0.8 \quad \text{and} \quad v_0 = 1.37,$$

for case (B). The vertical axis represents chemical substance concentration and the horizontal axis shows time. We see from Figure 1.6 that the solution (u, v) in case (A) satisfies

$$\lim_{t\to\infty} (u, v) = (1, 0). \tag{1.13}$$

However the solution in case (B) does not satisfy (1.13).

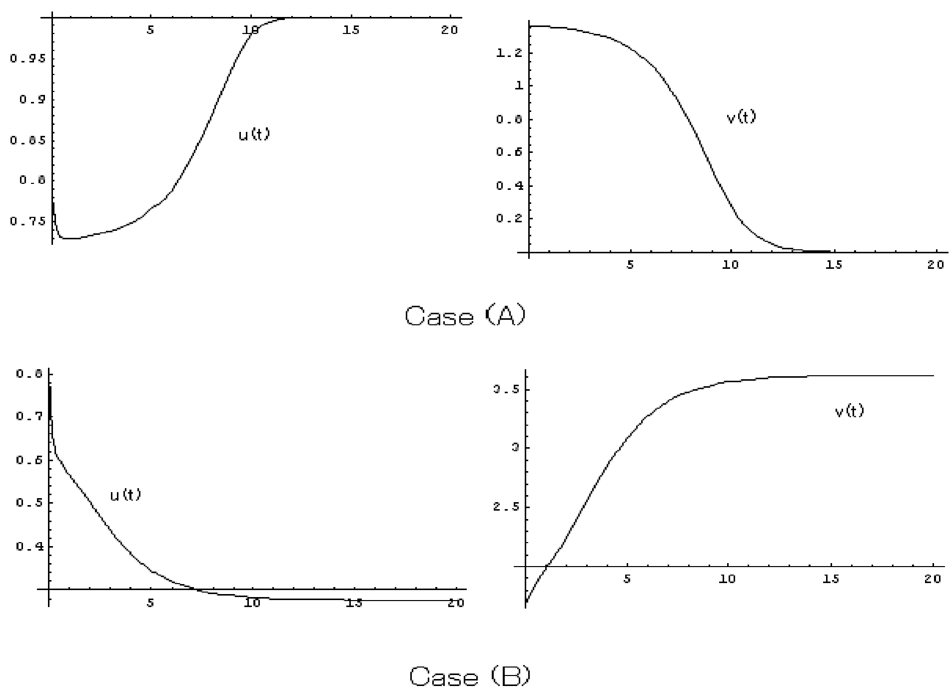


Figure 1.6: The numerical simulation in case $\lambda > 4$.

Chapter 2

Solution set of stationary problem in a bounded domain

In this chapter, we will discuss the following stationary problem in a bounded domain with homogeneous Neumann boundary condition.

$$\Delta u - uv^2 + \lambda(1-u) = 0 \qquad \text{in} \quad \Omega, \tag{2.1}$$

$$\gamma\Delta v + uv^2 - v = 0 \qquad \text{in} \quad \Omega, \tag{2.2}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega. \tag{2.3}$$

where λ and γ are positive parameters, $\frac{\partial}{\partial n}$ is an outward normal derivative on $\partial\Omega$, and $\Omega \subset \mathbf{R}^{\mathbf{N}}$ is a bounded domain with smooth boundary $\partial\Omega$. This stationary problem is called (SP1) throughout this chapter. In [27], the numerical simulation has shown that (SP1) has rich solution structure. Therefore it is important to study non-constant solutions of (SP1). Note that constant solutions of (SP1) are given by

(i) $(u, v) = (1, 0)$ if $\lambda < 4$,
(ii) $(u, v) = (1, 0)$, $(\frac{1}{2}, 2)$ if $\lambda = 4$,
(iii) $(u, v) = (1, 0)$, $\left(\frac{\lambda \pm \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda \mp \sqrt{\lambda^2 - 4\lambda}}{2}\right)$ if $\lambda > 4$.

There are many known results concerned with (SP1) (see [20, 21, 22, 23, 26, 30, 33, 36, 38].) Their results give us information about profiles of solutions, a priori bounds, and bifurcation structure. However, in order to understand more information about the solution set of (SP1), further study is needed.

This chapter mainly study the following three problems related to (SP1).

(a) A priori estimates of non-trivial solutions for (SP1).

(b) Sufficient conditions about existence and nonexistence of non-trivial solutions.

(c) Bifurcation structure for (SP1).

In Section 2.1, we will treat with the problem (a). The following a priori estimates is our main result of Section 2.1.

Theorem 2.1. *Let* (u, v) *be any solution for* (SP1) *except for* $(1, 0)$. *Then there exists a positive constant* C *depending on* λ, γ, Ω *and* N *such that*

$$\frac{1}{C} \leq u(x), v(x) \leq C \quad for \quad x \in \bar{\Omega}.$$

As for the problem (b), we first consider sufficient conditions about the non-existence of nontrivial solutions for (SP1). In Section 2.2, we can show that (SP1) admits no nontrivial solutions if γ is sufficiently large.

Theorem 2.2. *Let* $\lambda \leq 4$. *Then* (SP1) *has no nontrivial solutions if* $\gamma \geq \frac{1}{4}$.

Before stating our second non-existence result, we will introduce the eigenvalues of the Laplacian as follows:

Notation 2.3. *Let* $\{\mu_m\}_{m \geq 0}$ *be the eigenvalues of*

$$\begin{cases} -\Delta u = \mu u & in \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & on \quad \partial\Omega, \end{cases}$$

satisfying $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$.

Then we obtain the following theorem.

Theorem 2.4. *Set* $\lambda > 4$ *and* $\lambda\gamma \geq 1$. *Then, there is a positive constant* $C(\lambda, \mu_1)$ *which depends on* λ *and* μ_1 *such that* (SP1) *admits only trivial solutions if* $\gamma \geq C(\lambda, \mu_1)$. *Here* μ_1 *is the first positive eigenvalue defined by* Notation 2.3.

Figure 2.1 shows the non-existence regions given by Theorems 2.2 and 2.4.

These a priori estimates and non-existence results are useful in studying the existence of non-constant solutions for (SP1). In order to mention our main result in Section 2.3, we put two positive constants in case $\lambda > 4$ as follows:

Definition 2.5. *Let* $\lambda > 4$. *Then we denote*

$$(u_1, v_1) = \left(\frac{\lambda + \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda - \sqrt{\lambda^2 - 4\lambda}}{2} \right) \tag{2.4}$$

and

$$(u_2, v_2) = \left(\frac{\lambda - \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda + \sqrt{\lambda^2 - 4\lambda}}{2} \right). \tag{2.5}$$

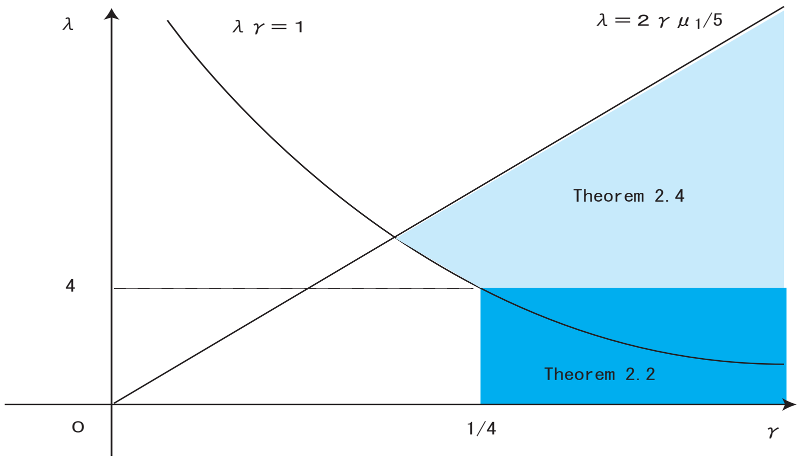


Figure 2.1: The non-existence regions.

In addition, we will define the following functions; (see Figure 2.2.)

Definition 2.6. *Let* $\lambda > 4$*. Then we define*

$$h(\mu, v_i) = \frac{\mu + \lambda - v_i^2}{\mu(\mu + \lambda + v_i^2)} \quad \text{for} \quad i = 1, 2,$$

where v_i *for* $i = 1, 2$ *is the constant given by* Definition 2.5*.*

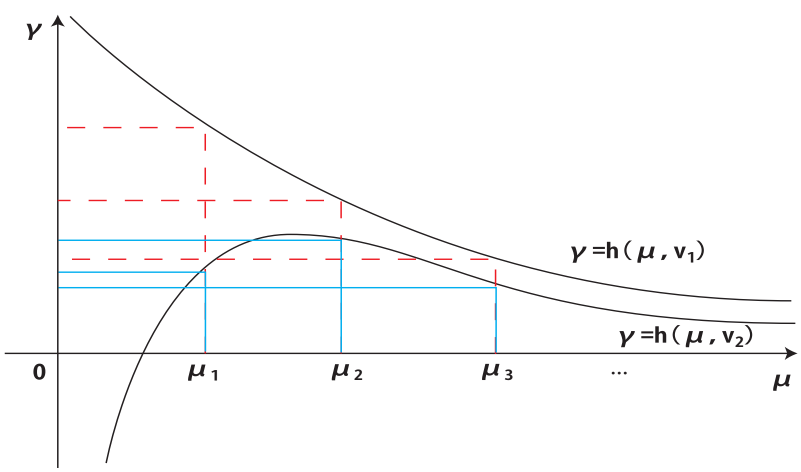


Figure 2.2: The graph of $h(\mu, v_i)$ for $i = 1, 2$.

Making use of the degree theory [2], we have the following existence theorem in Section 2.3.

Theorem 2.7. *Suppose that* $\lambda \geq 4$ *and define the function* $h(\mu, v_i)$ *for* $i = 1, 2$ *as* Definition 2.6. *Assume that every eigenvalue* μ_m *is simple, and that*

$$h(\mu_m, v_i) \neq h(\mu_n, v_j) \quad for \quad m \neq n,\ i, j = 1, 2,$$

provided $\lambda > 4$. *Then there exists a positive monotone decreasing sequence* $\{\gamma_k\}$ $(k = 1, 2, \cdots)$ *which converges to zero such that* (SP1) *has at least one non-constant positive solution if*

$$\gamma \in (\gamma_{2k}, \gamma_{2k-1}) \quad \text{for} \quad k = 1, 2, \cdots .$$

Finally, we deal with the problem (c) in Section 2.4. McGough-Riley have shown that non-trivial solutions bifurcate from the constant solution (u_i, v_i) for $i = 1, 2$ at $\gamma = h(\mu_m, v_i)$. In Section 2.4, we will mainly treat with direction of the bifurcation and stability for the bifurcating solutions.

2.1 A priori estimates

In this section, we will mainly focus on upper and lower bound of stationary solutions for (SP1). Upper bound estimates were derived by McGough-Riley [26] for two dimensional case. Here we can obtain upper bound estimates for any dimensional case.

In order to get upper bound estimates, we will use the following maximum principle derived by Lou and Ni [25] and strong maximum principle [31].

Maximum principle ([25]) *Suppose that* $g \in C(\bar{\Omega} \times R^1)$.
(i) *If* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad in \quad \Omega, \quad \frac{\partial w}{\partial \mu} \leq 0 \quad on \quad \partial\Omega,$$

and $w(x_0) = \max_{\bar{\Omega}} w$, *then* $g(x_0, w(x_0)) \geq 0$.
(ii) *If* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \quad in \quad \Omega, \quad \frac{\partial w}{\partial \mu} \geq 0 \quad on \quad \partial\Omega,$$

and $w(x_0) = \min_{\bar{\Omega}} w$, *then* $g(x_0, w(x_0)) \leq 0$.

Strong maximum principle ([31, p64]) *Let a nonnegative function* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfy the following differential inequality and the homogeneous Neumann boundary condition*;

$$\Delta w + c(x)w \leq 0 \quad in \quad \Omega, \quad \frac{\partial w}{\partial n} = 0 \quad on \quad \partial\Omega,$$

where $c(x) \in C(\Omega)$ is a nonnegative function. Then,

$$v(x) > 0 \quad or \quad v(x) \equiv 0 \quad in \quad \bar{\Omega}.$$

Applying the maximum principle and the strong maximum principle to (SP1), we can show the following three lemmas.

Lemma 2.8. *Let (u, v) be any nontrivial solution of* (SP1)*. Then*

$$0 < u(x) \leq 1 \quad and \quad v(x) \geq 0 \quad for \quad x \in \bar{\Omega}.$$

Furthermore, if $(u, v) \neq (1, 0)$, then it follows that

$$v(x) > 0 \quad for \quad x \in \bar{\Omega}.$$

Proof. Suppose $u(x_0) = \min_{x \in \bar{\Omega}} u(x)$. Using the maximum principle, we see from (2.1) that

$$-u(x_0)v(x_0)^2 + \lambda(1 - u(x_0)) \leq 0,$$

which shows

$$u(x_0) \geq \frac{\lambda}{\lambda + v(x_0)^2} > 0.$$

Therefore, $\min_{x \in \bar{\Omega}} u(x) > 0$.

Next, put $u(y_0) = \max_{x \in \bar{\Omega}} u(x)$. Then one can apply the maximum principle to (2.1) that

$$u(y_0) \leq \frac{\lambda}{\lambda + v(y_0)^2} \leq 1.$$

Finally, set $v(x_1) = \min_{x \in \bar{\Omega}} v(x)$. Then we have from (2.2) that

$$v(x_1)(u(x_1)v(x_1) - 1) \leq 0.$$

Hence,

$$v(x_1) \geq 0.$$

Observe that

$$\gamma \Delta v - v = -uv^2 \leq 0 \quad \text{in} \quad \Omega. \tag{2.6}$$

Applying the strong maximum principle to (2.6), we deduce that

$$v(x) > 0 \quad \text{in} \quad \bar{\Omega},$$

provided $v(x) \not\equiv 0$. Thus the proof is complete. □

Lemma 2.9. *Assume that* (u, v) *is any solution of* (SP1). *If* $\lambda\gamma \le 1$, *then*

$$u(x) + \gamma v(x) \le 1 \quad for \quad x \in \bar{\Omega}.$$

Proof. Define $p = u + \gamma v - 1$. By combining (2.1) and (2.2), then it follows that

$$\Delta p - \frac{1}{\gamma} p = \left(\lambda - \frac{1}{\gamma} \right) v \ge 0.$$

Since $\frac{\partial p}{\partial n} = 0$ on $\partial\Omega$ by (2.3), we see from the maximum principle that $\max_{x \in \bar{\Omega}} p(x) \le 0$, as required. □

Lemma 2.10. *Let* (u, v) *be any solution for* (SP1). *If* $\lambda\gamma \ge 1$, *then*

$$u(x) + \gamma v(x) \le \lambda\gamma \quad for \quad x \in \bar{\Omega}.$$

Proof. Put $q = u + \gamma v - \lambda\gamma$. Then (2.1)+(2.2) implies that

$$\Delta q - \frac{1}{\gamma} q = \left(\lambda - \frac{1}{\gamma} \right) u \ge 0.$$

Since $\frac{\partial q}{\partial n} = 0$ on $\partial\Omega$, the maximum principle enables us to show $\max_{x \in \bar{\Omega}} q(x) \le 0$. Thus the proof is complete. □

Next, we intend to derive lower bound estimates for any solutions of (SP1) except for $(1, 0)$. The following lemma is concerned with lower bound for u.

Lemma 2.11. *Suppose that* (u, v) *is any solution for* (SP1). *Then there exists a positive constant* $C_1(\lambda, \gamma)$ *depending only on* λ *and* γ *such that*

$$u(x) \ge C_1(\lambda, \gamma) \quad for \quad x \in \bar{\Omega}.$$

Remark 2.12. *As for the positive constant* $C_1(\lambda, \gamma)$ *in* Lemma 2.11, *one can take it as follows*;

$$C_1(\lambda, \gamma) = \frac{\lambda}{\lambda + C(\lambda, \gamma)^2} \quad \text{with} \quad C(\lambda, \gamma) = \max\left\{ \lambda, \frac{1}{\gamma} \right\}.$$

Proof. Assume that $\min_{x\in\bar{\Omega}} u(x) = u(x_m)$. Using the maximum principle, we see that

$$u(x_m) \leq \frac{\lambda}{\lambda + v(x_m)^2}. \tag{2.7}$$

It follows from Lemmas 2.9 and 2.10 that

$$v(x) \leq \max\left\{\lambda, \frac{1}{\gamma}\right\} \quad \text{in} \quad \bar{\Omega}. \tag{2.8}$$

Thus the conclusion follows from (2.7) and (2.8).

□

In order to obtain lower bound estimates for v, we will need the following Harnack inequality given by Lin-Ni-Takagi [24].

Harnack inequality([24]) *Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive classical solution of the following second order elliptic equation;*

$$\Delta w(x) + c(x)w(x) = 0 \quad in \quad \Omega, \quad \frac{\partial w}{\partial n} = 0 \quad on \quad \partial\Omega.$$

Here $c(x) \in C(\bar{\Omega})$. Then there exists a positive constant $C_* = C_*(N, \Omega, \mu)$ such that

$$\max_{\bar{\Omega}} w \leq C_* \min_{\bar{\Omega}} w,$$

where μ is a positive constant satisfying $\mu \geq ||c||_\infty$.

Then one can establish the following lemma.

Lemma 2.13. *Assume that (u, v) is any solution of* (SP1) *except for* $(1, 0)$*. Then, there exists some positive constant $C_2(\lambda, \gamma, \Omega, N)$ depending on λ, γ, Ω and N such that*

$$v(x) \geq C_2(\lambda, \gamma, \Omega, N) \quad for \quad x \in \bar{\Omega}.$$

Proof. We can describe (2.2) as

$$\Delta v + c(x)v = 0 \quad \text{in} \quad \Omega,$$

where

$$c(x) = \frac{1}{\gamma}(1 - u(x)v(x)).$$

Note that $u(x) \le 1$ and $v(x) \le \max\left\{\lambda, \frac{1}{\gamma}\right\}$ for any $x \in \bar{\Omega}$ from Lemmas 2.8-2.10. Then one can derive the following estimates;

$$||c(x)||_\infty \le \frac{1}{\gamma}\left\{1 + \max\left(\lambda, \frac{1}{\gamma}\right)\right\}.$$

Since $v(x) > 0$ in $\bar{\Omega}$, the Harnack inequality enables us to show that

$$v_{min} \ge C_*(N, \Omega, \mu) \max_{\bar{\Omega}} v, \quad \text{with} \quad \mu = \frac{1}{\gamma}\left\{1 + \max\left(\lambda, \frac{1}{\gamma}\right)\right\}. \tag{2.9}$$

Setting $\max_{\bar{\Omega}} v = v(x_M)$, then we see from the maximum principle that

$$v(x_M)(u(x_M)v(x_M) - 1) \ge 0.$$

In view of $v(x_M) > 0$, we find

$$\max_{\bar{\Omega}} v \ge \frac{1}{u(x_M)} \ge 1. \tag{2.10}$$

Hence the conclusion follows from (2.9) and (2.10). □

Consequently, one can prove Theorem 2.1.

Proof of Theorem 2.1. For any solutions of (SP1) except for $(u, v) = (1, 0)$, it follows from Lemmas 2.8-2.13 that the following inequalities hold true;

$$C_1(\lambda, \gamma) \le u(x) \le 1 \quad \text{for} \quad x \in \bar{\Omega} \tag{2.11}$$

and

$$C_2(\lambda, \gamma, \Omega, N) \le v(x) \le \max\left\{\lambda, \frac{1}{\gamma}\right\} \quad \text{for} \quad x \in \bar{\Omega}. \tag{2.12}$$

Here the constants C_1 and C_2 are uniformly bounded for $\gamma \to +\infty$. Combining (2.11) and (2.12), we conclude that the theorem holds true. □

2.2 Nonexistence of nontrivial solutions

In this section, we deal with some sufficient conditions about the non-existence of non-trivial solutions for (SP1). We first establish the following nonexistence results by using the maximum principle.

Theorem 2.14. *Let $\lambda\gamma \le 1$. Then* (SP1) *admits only constant solutions if $\gamma \ge \frac{1}{4}$.*

Proof. If $\lambda\gamma \le 1$, then it follows from Lemma 2.9 that

$$u(x) + \gamma v(x) - 1 \le 0 \quad \text{for} \quad x \in \bar{\Omega}.$$

Hence we see from (2.2) that

$$\begin{aligned} \gamma \Delta v &= v(1 - uv) \\ &\ge v(\gamma v^2 - v + 1) \\ &\ge 0, \end{aligned} \tag{2.13}$$

provided $\gamma \ge \frac{1}{4}$. Multiplying (2.13) by v and integrating on Ω, then

$$\gamma \int_\Omega |\nabla v|^2 dx \le 0.$$

This implies that $v(x)$ is a constant. Therefore we have from (2.2) that $u(x)$ is also a constant. Thus the proof is complete. □

Theorem 2.15. *Let $\lambda \le 4$ and $\lambda\gamma > 1$. Then* (SP1) *has no non-constant solutions.*

Proof. Define $s = \lambda(u-1) + v$. Then (2.1) and (2.2) lead to

$$\Delta s - \left\{ \left(1 - \frac{1}{\lambda\gamma}\right) v^2 + \lambda \right\} s = \left(\frac{1}{\lambda\gamma} - 1\right) v(v^2 - \lambda v + \lambda).$$

If $\lambda\gamma > 1$ and $\lambda \le 4$, then

$$\left(1 - \frac{1}{\lambda\gamma}\right) v^2 + \lambda > 0 \quad \text{and} \quad \left(\frac{1}{\lambda\gamma} - 1\right) v(v^2 - \lambda v + \lambda) \le 0.$$

Since $\frac{\partial s}{\partial n} = 0$ on $\partial\Omega$, we can use the maximum principle to show that

$$s(x) \ge 0 \quad \text{for} \quad x \in \bar{\Omega}. \tag{2.14}$$

Adding (2.1) and (2.2), then

$$\Delta u + \gamma \Delta v = \lambda(u-1) + v \quad \text{in} \quad \Omega. \tag{2.15}$$

Integrate (2.15) in Ω, then

$$\int_\Omega \lambda(u-1) + v dx = 0. \tag{2.16}$$

Combing (2.14) and (2.16), we deduce that

$$\lambda(u(x)-1)+v(x)=0 \quad \text{for} \quad x\in\bar{\Omega}. \tag{2.17}$$

Insert (2.17) into (2.2), then

$$\lambda\gamma\Delta v = v(v^2-\lambda v+\lambda)\geq 0,$$

according to $\lambda \leq 4$. As in the proof of Theorem 2.14, we conclude that $u(x)$ and $v(x)$ are constant functions. □

Then we can prove Theorem 2.2.

Proof of Theorem 2.2. The conclusion follows from Theorems 2.14 and 2.15. □

Finally, making use of Lemmas 2.8 and 2.10, one can prove Theorem 2.4.

Proof of Theorem 2.4. Denote

$$\bar{u}=\frac{1}{|\Omega|}\int_\Omega u dx \quad \text{and} \quad \bar{v}=\frac{1}{|\Omega|}\int_\Omega v dx.$$

If we multiply (2.1) by $u-\bar{u}$ and integrate on Ω, then

$$\begin{aligned}
&\int_\Omega |\nabla u|^2 dx \\
&= \int_\Omega \left[-uv^2+\lambda(1-u)-\left\{-\bar{u}\bar{v}^2+\lambda(1-\bar{u})\right\}\right](u-\bar{u})dx \\
&= -\int_\Omega v^2(u-\bar{u})^2dx - \bar{u}\int_\Omega (v+\bar{v})(u-\bar{u})(v-\bar{v})dx - \lambda\int_\Omega (u-\bar{u})^2 dx \\
&\leq -\lambda\int_\Omega (u-\bar{u})^2dx - \bar{u}\int_\Omega (v+\bar{v})(u-\bar{u})(v-\bar{v})dx.
\end{aligned}$$

Note that the following inequality holds true from Lemmas 2.8 and 2.10;

$$\begin{aligned}
&\left|\bar{u}\int_\Omega (v+\bar{v})(u-\bar{u})(v-\bar{v})dx\right| \\
&\leq 2\lambda\int_\Omega |u-\bar{u}||v-\bar{v}|dx \\
&\leq \epsilon_1\int_\Omega (u-\bar{u})^2dx + \frac{\lambda}{2\epsilon_1}\int_\Omega (v-\bar{v})^2dx,
\end{aligned}$$

where ϵ_1 is any positive constant. Hence, we see that

$$\int_\Omega |\nabla u|^2dx \leq (\epsilon_1-\lambda)\int_\Omega (u-\bar{u})^2dx + \frac{\lambda}{2\epsilon_1}\int_\Omega (v-\bar{v})^2dx. \tag{2.18}$$

Next, if we multiply (2.2) by $(v-\bar{v})$ and integrate on Ω, then we have according to Lemmas 2.8 and 2.10 that

$$
\begin{aligned}
\gamma\int_\Omega |\nabla v|^2 dx &= \int_\Omega \left\{uv^2 - v - (\bar{u}\bar{v}^2 - \bar{v})\right\}(v-\bar{v})dx \\
&= \int_\Omega u(v+\bar{v})(v-\bar{v})^2 dx + \int_\Omega \bar{v}^2(u-\bar{u})(v-\bar{v})dx - \int_\Omega (v-\bar{v})^2 dx \\
&\leq 2\lambda\int_\Omega (v-\bar{v})^2 dx + \lambda^2\int_\Omega (u-\bar{u})(v-\bar{v})dx - \int_\Omega (v-\bar{v})^2 dx \\
&\leq (2\lambda-1)\int_\Omega (v-\bar{v})^2 dx + \epsilon_2\int_\Omega (u-\bar{u})^2 dx + \frac{\lambda^2}{4\epsilon_2}\int_\Omega (v-\bar{v})^2 dx \\
&\leq \epsilon_2\int_\Omega (u-\bar{u})^2 dx + \left(\frac{\lambda^2}{4\epsilon_2} + 2\lambda - 1\right)\int_\Omega (v-\bar{v})^2 dx, \qquad (2.19)
\end{aligned}
$$

where ϵ_2 is any positive constant.

Adding (2.18) and (2.19), then

$$
\begin{aligned}
&\gamma\int_\Omega |\nabla v|^2 dx + \int_\Omega |\nabla u|^2 dx \\
&\leq (\epsilon_1 + \epsilon_2 - \lambda)\int_\Omega (u-\bar{u})^2 dx + \left(\frac{\lambda}{2\epsilon_1} + \frac{\lambda^2}{4\epsilon_2} + 2\lambda - 1\right)\int_\Omega (v-\bar{v})^2 dx. \qquad (2.20)
\end{aligned}
$$

If we take

$$\epsilon_1 = \epsilon_2 = \frac{\lambda}{2},$$

then it follows from (2.20) that

$$\gamma\int_\Omega |\nabla v|^2 dx \leq D(\lambda)\int_\Omega (v-\bar{v})^2 dx,$$

where

$$D(\lambda) = \frac{5}{2}\lambda.$$

Therefore we can use Poincare's inequality [32, Theorem 11.11] to show that

$$\gamma\mu_1\int_\Omega (v-\bar{v})^2 dx \leq \gamma\int_\Omega |\nabla v|^2 dx \leq D(\lambda)\int_\Omega (v-\bar{v})^2 dx. \qquad (2.21)$$

If γ satisfies the following inequality

$$\gamma \geq \frac{D(\lambda)}{\mu_1},$$

then it follows from (2.21) that

$$\int_\Omega |\nabla v|^2 dx = 0.$$

Hence, $v(x)$ is a constant. On the other hand, owing to (2.20), we see

$$\int_\Omega |\nabla u|^2 dx = 0.$$

Thus $u(x)$ is also a constant, as required. □

2.3 Existence of nontrivial positive solutions

In this section, we will discuss existence of non-trivial positive stationary solutions for (SP1) through the degree theory [2].

Set an auxiliary parameter

$$\gamma_s = s\gamma + (1-s)M \quad \text{for} \quad s \in [0,1],$$

where M is a large constant determined later. And define an operator

$$T_s(w) = \begin{pmatrix} (-\Delta + I)^{-1}(f(u,v)+u) \\ (-\Delta + I)^{-1}(\gamma_s^{-1} g(u,v)+v) \end{pmatrix} \quad \text{with} \quad w = (u,v).$$

Here $f(u,v) = -uv^2 + \lambda(1-u)$ and $g(u,v) = uv^2 - v$. If we put a functional space X as $X = C(\Omega) \times C(\Omega)$, the operator $T_s : X \to X$ is a compact operator. Then every solution of (SP1) in case $\gamma = \gamma_s$ becomes a fixed point of T_s in X.

Define a function space W as follows;

$$W = \left\{ w = (u,v) \in X \mid \frac{1}{C} \le u, v \le C \right\}, \tag{2.22}$$

where C is a constant to be determined as follows. Let any γ_* be fixed and consider γ satisfying $\gamma > \gamma_*$. Then we see from Theorem 2.1 that for any solutions for (SP1) except for $(u,v) = (1,0)$ the following inequalities hold true;

$$C_1(\lambda, \gamma_*) \le u(x) \le 1 \quad \text{for} \quad x \in \bar{\Omega}$$

and

$$C_2(\lambda, \gamma_*, \Omega, N) \le v(x) \le \max\left\{\lambda, \frac{1}{\gamma_*}\right\} \quad \text{for} \quad x \in \bar{\Omega}.$$

Here C_1 and C_2 are constants defined by (2.11) and (2.12). So, if we take

$$C = 2\max\left\{\frac{1}{C_1}, \frac{1}{C_2}, 1, \max\left\{\lambda, \frac{1}{\gamma_*}\right\}\right\},$$

then the operator T_s $(0 \le s \le 1)$ has no fixed point on the boundary ∂W for any positive γ_s.

We will define an integer $j_0(\gamma)$ as follows;

Definition 2.16. *An integer $j_0(\gamma)$ is the number of eigenvalues μ_m* (counting algebraic multiplicity) *which satisfy*

$$\gamma < \frac{1}{\mu_m + 8}.$$

Then we will establish the following existence result.

Theorem 2.17. *Assume $\lambda = 4$ and $j_0(\gamma)$ is the integer given by* Definition 2.16*. If $j_0(\gamma)$ is odd, then* (SP1) *admits at least one positive nonconstant solution.*

For any fixed $\lambda > 4$, we define the following integer $j_i(\gamma)$ for $i = 1, 2$;

Definition 2.18. *Let $\lambda > 4$ and $\gamma^{-1} \neq 3v_2^2 - \lambda + 2v_2\sqrt{2(v_2^2 - \lambda)}$. An integer $j_i(\gamma)$ for $i = 1, 2$ is the number of positive eigenvalues μ_n* (counting algebraic multiplicity) *satisfying*

$$\gamma < h(\mu_n, v_i),$$

where $h(\mu, v_i)$ for $i = 1, 2$ is the function given by Definition 2.6.

Then the following theorem holds true.

Theorem 2.19. *Assume $\lambda > 4$ and $\gamma^{-1} \neq 3v_2^2 - \lambda + 2v_2\sqrt{2(v_2^2 - \lambda)}$. Furthermore, $j_i(\gamma)$ $(i = 1, 2)$ is the integer given by* Definition 2.18*. If $j_1(\gamma) + j_2(\gamma)$ is odd, then* (SP1) *has at least one positive nontrivial solution.*

These two existence results will be proved in subsections 2.3.1 and 2.3.2. Then one can prove Theorem 2.7.

Proof of Theorem 2.7. Assume $\lambda = 4$ and define

$$\gamma_m = \frac{1}{\mu_{m-1} + 8} \quad \text{for} \quad m \ge 1.$$

Since every eigenvalue μ_m is simple, we see that $j_0(\gamma)$ given by Definition 2.16 is odd if and only if

$$\gamma \in (\gamma_{2m}, \gamma_{2m-1}) \quad \text{for} \quad m = 1, 2, \cdots .$$

Making use of Theorem 2.17, one can show Theorem 2.7 in case $\lambda = 4$.

Next, we consider the case $\lambda > 4$. We see from Figure 2.2 that $h(\mu_m, v_1)$ given by Definition 2.6 is always positive for any $m \geq 1$. On the other hand, there exists a natural number m_* such that

$$h(\mu_m, v_2) \leq 0 \quad \text{for} \quad m < m_*$$

and

$$h(\mu_m, v_2) > 0 \quad \text{for} \quad m \geq m_*.$$

Denote $\gamma_{m,1} = h(\mu_m, v_1)$ for $m \geq 1$ and

$$\gamma_{m,2} = h(\mu_m, v_2) \quad \text{with} \quad m \geq m_*.$$

If we rearrange the sequences $\{\gamma_{m,1}\}_{m\geq 0}$ and $\{\gamma_{m,2}\}_{m\geq m_*}$, we can construct a monotone decreasing sequence $\{\gamma_n\}_{n\geq 1}$ which converges to zero as $n \to \infty$.

Because every eigenvalue μ_m is simple, we find that $j_1(\gamma) + j_2(\gamma)$ given by Definition 2.18 is odd if and only if

$$\gamma \in (\gamma_{2n}, \gamma_{2n-1}) \quad \text{for} \quad n = 1, 2, \cdots .$$

Thus the conclusion follows from Theorem 2.19. □

2.3.1 Proof of Theorem 2.17

We will prove theorem 2.17 in this subsection. Observe that constant fixed point of T_s in W given by (2.22) is uniquely determined by

$$w_0 := \left(\frac{1}{2}, 2\right), \tag{2.23}$$

provided $\lambda = 4$. The Leray-Schauder index property [2] implies that

$$\text{Index}(T_s, w_0) = (-1)^{\sigma_0(s)},$$

where $\sigma_0(s)$ is the number of real negative eigenvalues (counting algebraic multiplicity) of $I - DT_s(w_0)$. Then the following lemma holds true.

Lemma 2.20. *Let* $j_0(\gamma)$ *be the integer given by* Definition 2.16. *Suppose that* w_0 *is the constant defined by* (2.23). *Then,*

$$\mathrm{Index}(T_s, w_0) = (-1)^{j_0(\gamma_s)}.$$

Remark 2.21. *If* $\gamma_0 = M$ *satisfies* $M > \frac{1}{8}$, *then*

$$\mathrm{Index}(T_0, w_0) = 1.$$

Proof. Set $w_0 = (\hat{u}, \hat{v})$. Then we see that

$$\begin{aligned} DT_s(w_0) &= \begin{pmatrix} (-\Delta + I)^{-1} \{ (f_u(\hat{u}, \hat{v}) + 1)\, u + f_v(\hat{u}, \hat{v}) v \} \\ (-\Delta + I)^{-1} \{ \gamma_s^{-1} g_u(\hat{u}, \hat{v}) u + (\gamma_s^{-1} g_v(\hat{u}, \hat{v}) + 1)\, v \} \end{pmatrix} \\ &= \begin{pmatrix} (-\Delta + I)^{-1} \{ (1 - \lambda - \hat{v}^2) u - 2\hat{u}\hat{v} v \} \\ (-\Delta + I)^{-1} [\gamma_s^{-1} \hat{v}^2 u + \{ \gamma_s^{-1} (2\hat{u}\hat{v} - 1) + 1 \}\, v] \end{pmatrix} \\ &= \begin{pmatrix} (-\Delta + I)^{-1} (-7u - 2v) \\ (-\Delta + I)^{-1} \{ 4\gamma_s^{-1} u + (\gamma_s^{-1} + 1) v \} \end{pmatrix}. \end{aligned} \tag{2.24}$$

Every eigenvalue μ of $I - DT_s(w_0)$ satisfies

$$(I - DT_s(w_0))\, z = \mu z, \quad \text{with} \quad z = (u, v). \tag{2.25}$$

According to (2.24), one can describe (2.25) as follows;

$$\begin{cases} (\mu - 1)\Delta u + (8 - \mu) u + 2v = 0, \\ (\mu - 1)\Delta v - 4\gamma_s^{-1} u - (\gamma_s^{-1} + \mu) v = 0. \end{cases}$$

Therefore it is sufficient to study the following infinitely many equations;

$$\begin{vmatrix} (1 - \mu)\mu_m + 8 - \mu & 2 \\ -4\gamma_s^{-1} & (1 - \mu)\mu_m - \gamma_s^{-1} - \mu \end{vmatrix} = 0 \quad \text{for} \quad m = 0, 1, 2, \cdots. \tag{2.26}$$

After some calculation, one can show that (2.26) becomes

$$(\mu_m + 1)^2 \mu^2 + A_s \mu + B_s = 0, \tag{2.27}$$

where

$$\begin{aligned} A_s &= (\mu_m + 1)(\gamma_s^{-1} - 2\mu_m - 8), \\ B_s &= \mu_m(\mu_m + 8 - \gamma_s^{-1}). \end{aligned}$$

Let D_s be the discriminant of (2.27). Then we see after some computation that

$$\operatorname{sign} D_s = (\gamma_s^{-1} - 8)^2 \geq 0,$$

where

$$\operatorname{sign} D = \begin{cases} 1 & \text{if} \quad D > 0 \\ 0 & \text{if} \quad D = 0 \\ -1 & \text{if} \quad D < 0 \end{cases} . \tag{2.28}$$

(a)If $\mu_m = 0$, then $A_s = \gamma_s^{-1} - 8$ and $B_s = 0$. Thus the number of negative solution for (2.27) is exactly one provided $\gamma_s < \frac{1}{8}$, and is zero provided $\gamma_s \geq \frac{1}{8}$.

(b)Assume $\mu_m > 0$. Note that B_s is negative if and only if $\gamma_s < \frac{1}{\mu_m + 8}$. After some computation, we see that (2.27) has exactly one negative real solution if $\gamma_s < \frac{1}{\mu_m+8}$, and has no negative real solutions if $\gamma_s \geq \frac{1}{\mu_m+8}$.

From the above argument, we conclude that $\sigma_0(s)$ is equal to $j_0(\gamma_s)$. □

Consequently, one can prove Theorem 2.17 by using the homotopy invariance property for the degree.

Proof of Theorem 2.17. We use a contradiction argument. Suppose that (SP1) has no nontrivial solutions if $\gamma_s = \gamma$.

Let W is the set given by (2.22). Then we see that that T_s has no fixed point on ∂W. Making use of Theorem 2.2 and Remark 2.21, we have

$$\deg(I - T_0, W, 0) = \operatorname{index}(T_0, w_0) = 1,$$

provided $\gamma_0 = M$ is sufficiently large. On the other hand, according to Lemma 2.20, one can see

$$\deg(I - T_1, W, 0) = \operatorname{index}(T_1, w_0) = (-1)^{j_0(\gamma)} = -1.$$

It is a contradiction owing to the homotopy invariance property for $\deg(I - T_s, W, 0)$ $(0 \leq s \leq 1)$. Thus the proof is complete. □

2.3.2 Proof of Theorem 2.19

Next, we will fix $\lambda > 4$ and regard γ as a parameter. Then constant fixed points of T_s on the set W given in (2.22) are determined by

$$w_1 := (u_1, v_1), \quad w_2 := (u_2, v_2), \tag{2.29}$$

where (u_1, v_1) (resp. (u_2, v_2)) is defined by (2.4) (resp.(2.5)).

The Lerey-Schauder index property asserts that

$$\mathrm{Index}(T_s, w_1) = (-1)^{\sigma_1(s)}, \quad \mathrm{Index}(T_s, w_2) = (-1)^{\sigma_2(s)}.$$

Here $\sigma_1(s)$ (resp. $\sigma_2(s)$) is the number of real negative eigenvalues (counting algebraic multiplicity) of $I - DT_s(w_1)$ (resp. $I - DT_s(w_2)$). Then we have the following lemma.

Lemma 2.22. *Let $j_i(\gamma)$ $(i = 1, 2)$ be the integer given by* Definition 2.18. *Suppose that w_1 and w_2 are the constants given by* (2.29). *Moreover, for any fixed $s \in [0, 1]$, assume that $\gamma_s^{-1} \neq 3v_2^2 - \lambda + 2v_2\sqrt{2(v_2^2 - \lambda)}$. Then,*

$$\mathrm{Index}(T_s, w_1) = (-1)^{j_1(\gamma_s)+1}, \quad \mathrm{Index}(T_s, w_2) = (-1)^{j_2(\gamma_s)}.$$

Remark 2.23. *Let $\gamma_0 = M$ satisfy*

$$M > \frac{\mu_1 + \lambda - v_1^2}{\mu_1(\mu_1 + \lambda + v_1^2)},$$

where v_1 is defined by (2.4). *Then it follows that*

$$\mathrm{Index}(T_0, w_1) = -1, \quad \mathrm{Index}(T_0, w_2) = 1.$$

Proof. For each $i = 1, 2$, we will study the following eigenvalue problem;

$$(I - DT_s(w_i))\, z = \mu z, \quad \text{with} \quad z = (u, v),$$

where $s \in [0, 1]$, and $w_i (i = 1, 2)$ is defined as (2.29).

As in the same way of proof for Lemma 2.20, it suffices to discuss the following infinitely many equations,

$$(\mu_m + 1)^2\mu^2 + A_s\mu + B_s = 0, \quad \text{for} \quad m = 0, 1, 2, \cdots. \tag{2.30}$$

Here

$$\begin{aligned} A_s &= (\mu_m + 1)(\gamma_s^{-1} - 2\mu_m - \lambda - v_i), \\ B_s &= \mu_m^2 + (\lambda + v_i^2 - \gamma_s^{-1})\mu_m + \gamma_s^{-1}(v_i^2 - \lambda), \end{aligned}$$

where v_1 and v_2 are defined as (2.4) and (2.5) for each $i = 1, 2$.

(i)Assume $v_i = v_1$ and denote the discriminant of (2.30) by D_1. Then

$$\begin{aligned} \mathrm{sign}\, D_1 &= \mathrm{sign}\left\{(\gamma_s^{-1} - 2\mu_m - \lambda - v_1^2)^2 - 4B_s\right\} \\ &= \mathrm{sign}\left\{(\lambda + v_1^2)^2\gamma_s^2 + 2(\lambda - 3v_1^2)\gamma_s + 1\right\} \\ &= +1, \end{aligned} \tag{2.31}$$

for each positive γ_s. Here sign D is defined by (2.28). Therefore, according to (2.31), the number of real solutions for (2.30) is exactly two for any number $m \geq 0$.

(a)If $\mu_m = 0$, then $B_s = \gamma_s^{-1}(v_1^2 - \lambda) < 0$. Hence the number of negative real solution for (2.30) remains exactly one for any positive γ_s.

(b)For any fixed $\mu_m > 0$, the sign of B_s changes from negative to positive if γ_s becomes larger. Note that B_s=0 is equivalent to

$$\gamma_s = h(\mu_m, v_1),$$

where $h(\mu, v_1)$ is the function given by Definition 2.6. Therefore, (2.30) has exactly one negative solution provided $\gamma < h(\mu_m, v_1)$ and has no negative real solutions provided $\gamma \geq h(\mu_m, v_1)$.

Consequently,

$$\text{Index}(T_s, w_1) = (-1)^{\sigma_1(s)} = (-1)^{j_1(\gamma_s)+1},$$

as required.

(ii)Next, we will deal with case $v_i = v_2$. Let D_2 be the discriminant for (2.30). After some computation, we see

$$\text{sign}\, D_2 = \text{sign}\left\{(\lambda + v_2^2)^2\gamma_s^2 + 2(\lambda - 3v_2^2)\gamma_s + 1\right\}. \tag{2.32}$$

Put

$$\alpha(\lambda) = \frac{3v_2^2 - \lambda - 2v_2\sqrt{2(v_2^2 - \lambda)}}{(\lambda + v_2^2)^2}, \quad \beta(\lambda) = \frac{3v_2^2 - \lambda + 2v_2\sqrt{2(v_2^2 - \lambda)}}{(\lambda + v_2^2)^2}.$$

Observe that the discriminant D_2 is positive provided $\gamma_s \in (0, \alpha(\lambda)) \cup (\beta(\lambda), +\infty)$, and is negative provided $\gamma_s \in (\alpha(\lambda), \beta(\lambda))$ according to (2.32). This is the different point with the previous case (i).

If $\gamma_s = \beta(\lambda)$, then we see after some calculation that A_s is negative. However, if $\gamma_s = \alpha(\lambda)$, then $A_s = 2(\mu_m + 1)\left\{v_2^2 - \lambda + v_2\sqrt{2(v_2^2 - \lambda)} - \mu_m\right\}$ can be positive and negative. If $\gamma_s^{-1} \neq 3v_2^2 - \lambda + 2v_2\sqrt{2(v_2^2 - \lambda)}$, we see that (2.30) has no negative real solution or exactly two negative real solution provided $\gamma_s > h(\mu_m, v_2)$, and has unique negative real solution provided $\gamma_s < h(\mu_m, v_2)$. Therefore

$$\text{Index}(T_s, w_2) = (-1)^{\sigma_2(s)} = (-1)^{j_2(\gamma_s)+2\Sigma} = (-1)^{j_2(\gamma_s)},$$

where Σ is the finite number of eigenvalues μ_m when (2.30) has exactly two negative real solutions. Thus the proof is complete. □

As in the proof of Theorem 2.17, one can prove Theorem 2.19 by using the homotopy invariance principle for the degree.

Proof of Theorem 2.19. We see that $\deg(I - T_s, W, 0)(0 \le s \le 1)$ is well defined because T_s has no fixed point on ∂W. Here the set W is defined by (2.22). Then the homotopy invariance property shows that

$$\deg(I - T_0, W, 0) = \deg(I - T_1, W, 0). \tag{2.33}$$

By setting $\gamma_0 = M$ as sufficiently large constant, stationary problem (SP1) has no nontrivial solution in view of Theorem 2.4. Hence it follows from Remark 2.23 that

$$\deg(I - T_0, W, 0) = \mathrm{Index}(T_0, w_1) + \mathrm{Index}(T_0, w_2) = 0. \tag{2.34}$$

Assume that (SP1) has no nontrivial solution if $\gamma_s = \gamma$. Then, according to Lemma 2.22, we find

$$\begin{aligned} \deg(I - T_1, W, 0) &= \mathrm{Index}(T_1, w_1) + \mathrm{Index}(T_1, w_2) \\ &= (-1)^{j_1(\gamma)+1} + (-1)^{j_2(\gamma)} \\ &= \pm 2, \end{aligned}$$

which is a contradiction with (2.33) and (2.34). Thus the proof is complete. □

2.4 Bifurcation analysis

2.4.1 Stability of constant stationary solutions

In this subsection, we study stability of constant stationary solutions for (SP1). We will treat with the following non-stationary problem;

$$\begin{cases} u_t = \Delta u - uv^2 + \lambda(1-u) & \text{in } \Omega \times (0,\infty), \\ \frac{\gamma}{d} v_t = \gamma \Delta v + uv^2 - v & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{2.35}$$

Here λ, γ and d are parameters, and $(u_0(x), v_0(x))$ is a pair of nonnegative initial functions.

As for constant stationary solutions of (SP1), we see that
(i) $(u,v) = (1,0)$ if $\lambda < 4$,
(ii) $(u,v) = (1,0)$, $(\frac{1}{2}, 2)$ if $\lambda = 4$,
(iii) $(u,v) = (1,0)$, $\left(\frac{\lambda \pm \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda \mp \sqrt{\lambda^2 - 4\lambda}}{2}\right)$ if $\lambda > 4$.

First, we discuss stability of $(u,v)=(1,0)$. Linearizing (2.35) around $(u,v)=(1,0)$, then one can easily show that (1,0) is linearly stable. Here we will show some stability result for $(1,0)$. Define a set Σ as follows;

$$\Sigma=\{(u,v)\in \mathrm{C}(\Omega)\times \mathrm{C}(\Omega);\, 0\le u\le a,\ 0\le v<b\}, \tag{2.36}$$

where a and b satisfy the following inequalities

$$a\ge 1 \quad \text{and} \quad ab\le 1. \tag{2.37}$$

Before stating our stability result, we define an invariant region as follows;

Definition 2.24. *A function space Σ is called an invariant region if for any initial data $(u_0(x),v_0(x))\in\Sigma$, there exists a unique global solution $(u(x,t),v(x,t))$ of* (2.35) *satisfying*

$$(u(x,t),v(x,t))\in\Sigma \quad \text{for any} \quad (x,t)\in\bar{\Omega}\times(0,\infty).$$

Then we have the following theorem.

Theorem 2.25. *Let Σ be given by* (2.36) *and* (2.37)*. Then Σ is an invariant region. Moreover, every unique global solution $(u(x,t),v(x,t))$ of* (2.35) *for $(u_0,v_0)\in\Sigma$ satisfies*

$$\lim_{t\to\infty}(u(x,t),v(x,t))=(1,0).$$

Proof. We define

$$V=\left\{-uv^2+\lambda(1-u),\frac{d}{\gamma}(uv^2-v)\right\}.$$

(i)If we denote $G=-u$, then

$$\nabla G\cdot V|_{u=0}=(-1,0)\cdot\left(\lambda,-\frac{d}{\gamma}v\right)=-\lambda<0.$$

(ii)Let $G=-v$. Then it follows that

$$\nabla G\cdot V|_{v=0}(0,-1)\cdot(\lambda(u-1))=0.$$

(iii)Setting $G=u-a$, then one can see from (2.37) that

$$\begin{aligned}\nabla G\cdot V|_{u=a}&=(1,0)\cdot\left(-av^2+\lambda(1-a),\frac{d}{\gamma}(av^2-v)\right)\\&=-av^2+\lambda(1-a)\\&\le 0.\end{aligned}$$

(iv)Put $G = v - b$. Then we have from (2.37) that

$$\begin{aligned}\nabla G \cdot V|_{v=b} &= (0,1) \cdot \left(-b^2 u + \lambda(1-u), \frac{d}{\gamma}(b^2 u - b)\right) \\ &= \frac{d}{\gamma}(b^2 u - b) \\ &\le \frac{d}{\gamma}(ab^2 - b) \\ &\le 0.\end{aligned}$$

Making use of the result of Smoller [32, Corollary 14.8], we find that Σ is an invariant region.

Now, since $u(x,t) \le a$ for any $x \in \Omega$ and $t \ge 0$, it follows from (2.35) that

$$\frac{d}{\gamma} v_t \le \gamma \Delta v + av^2 - v.$$

Observe that $v(x,0) < b \le \frac{1}{a}$. According to the comparison principle [32, Theorem 10.1], we deduce that

$$\lim_{t\to\infty} v(x,t) = 0.$$

Therefore for any $\epsilon > 0$, there exists a large time T such that

$$\Delta + \lambda - (\lambda + \epsilon)u \le u_t \le \Delta u + \lambda(1-u) \quad \text{for} \quad t \ge T.$$

This inequality implies that

$$\frac{\lambda}{\lambda + \epsilon} \le \liminf_{t\to\infty} u(x,t) \le \limsup_{t\to\infty} u(x,t) \le 1.$$

Taking $\epsilon \to 0$, then we conclude

$$\lim_{t\to\infty} u(x,t) = 1.$$

□

If $\lambda = 4$, then McGough-Riley [26] have obtained the following stability result.

Theorem 2.26 ([26]). *Let $\lambda = 4$. Suppose that w_0 is the constant solution given by* (2.23). *Then w_0 is linearly stable if $\lambda > K$, but is linearly unstable if $\gamma < K$. Here*

$$K = \frac{1}{8} \max\{1, d\}.$$

Next, we consider case $\lambda > 4$. Then the following theorems have also shown by McGough-Riley.

Theorem 2.27 ([26]). *Assume $\lambda > 4$. Let w_1 be the constant solution given by* (2.29)*. Then w_1 is linearly unstable for any positive γ and d.*

Theorem 2.28 ([26]). *Let $\lambda > 4$. Suppose that w_2 is the constant defined by* (2.29)*. Denote*

$$L = \max\left\{\frac{d}{\lambda + v_2^2}, M\right\},$$

where

$$M = \max_{n \in N}\left\{\frac{\mu_n + \lambda - v_2^2}{\mu_n(\mu_n + \lambda + v_2^2)}\right\}.$$

Then w_2 is linearly stable provided $\gamma > L$. On the other hand, w_2 is linearly unstable provided $\gamma < L$.

2.4.2 Existence and stability for bifurcating solutions

In this subsection, we intend to study existence and stability of nontrivial small amplitude solutions for (SP1) through local bifurcation theory. The following existence result has been given by McGough and Riley.

Theorem 2.29 ([26]). *Suppose that $\lambda \geq 4$ and w_i for $i = 1, 2$ is the constant solution defined by* (2.29)*. Assume that every eigenvalue $\mu_n\,(n = 1, 2, 3 \cdots)$ given by* Notation 2.3 *is simple. Denote*

$$\gamma_{n,i} := h(\mu_n, v_i). \tag{2.38}$$

Then nontrivial solutions bifurcate from w_i at $\gamma = \gamma_{n,i}$. Here $h(\mu, v_i)$ for $i = 1, 2$ is the function given by Definition 2.6.

Remark 2.30. (A) *When $\lambda = 4$, it follows that*

$$w_1 = w_2 = \left(\frac{1}{2}, 2\right)$$

and (2.38) *is equivalent to*

$$\gamma_{n,i} = \frac{1}{\mu_n + 8}.$$

Therefore the sequence $\{\gamma_{n,i}\}$ is strictly decreasing.

(B) *Assume $\lambda > 4$. Then the sequence $\{\gamma_{n,1}\}$ strictly decreases and converges to zero. If $\mu_1 \geq D(\lambda)$, then the sequence $\{\gamma_{n,2}\}$ is decreasing and tends to 0 as $n \to \infty$. However, if $\mu_1 < D(\lambda)$, then the sequence $\{\gamma_{n,2}\}$ increases at first, but decreases later and converges to 0. Here $D(\lambda) = \lambda - v_2^2 + v_2\sqrt{2\,(v_2^2 - \lambda)}$.*

When γ is in a neighborhood of $\gamma_{n,i}$ for $i = 1, 2$, it follows from local bifurcation theory [6] that nontrivial solutions $(u, v, \gamma) = (\Phi_{n,i}(\epsilon), \Psi_{n,i}(\epsilon), \gamma_{n,i}(\epsilon))$ given by Theorem 2.29 can be represented as follows;

$$\left\{ \begin{array}{l} \Phi_{n,i}(\epsilon) = u_i + \epsilon \left\{(p_i \phi_n) + \tilde{u}_i\right\}, \\ \Psi_{n,i}(\epsilon) = v_i + \epsilon \left\{(q_i \phi_n) + \tilde{v}_i\right\}, \\ \gamma_{n,i}(\epsilon) = \gamma_{n,i} + \epsilon \gamma'_{n,i}(0) + o(\epsilon), \end{array} \right. \tag{2.39}$$

where p_i and q_i are constants which satisfy

$$p_i^2 + q_i^2 = 1, \quad v_i^2 p_i + (1 - \gamma_{n,i}\mu_n) q_i = 0, \quad p_i < 0 < q_i, \tag{2.40}$$

and ϕ_n is the eigenfunction corresponding to μ_n. Furthermore, $\tilde{u}_i$ and $\tilde{v}_i$ are $o(1)$ functions.

The direction of the bifurcation has still been unknown. One can show the following theorem.

Theorem 2.31. *Let $\lambda \geq 4$. Suppose that $(\Phi_{n,i}(\epsilon), \Psi_{n,i}(\epsilon), \gamma_{n,i}(\epsilon))$ for $i = 1, 2$ are bifurcating solutions given in (2.39) and (2.40). Then the direction of bifurcation $\gamma'_{n,i}(0)$ satisfies*

$$\gamma'_{n,i}(0) = \frac{(s_i - r_i)(2v_i p_i + u_i q_i)\int_\Omega \phi_n^3 dx}{s_i \mu_n \int_\Omega \phi_n^2 dx}. \tag{2.41}$$

where r_i and s_i are positive constants satisfying

$$r_i^2 + s_i^2 = 1, \quad 2r_i + (\gamma_{n,i}\mu_n - 1)s_i = 0. \tag{2.42}$$

Remark 2.32. *For fixed $i = 1, 2$, the coefficient $2v_i p_i + u_i q_i$ of (2.41) is positive provided $\mu_n > 3v_i^2 - \lambda$, but is negative provided $\mu_n < 3v_i^2 - \lambda$. After some computation, one can see that $r_i < s_i$.*

Proof. We only prove the case $i = 1$. The other case can be treated in the same way. Define the linearlized operator $L : H^2(\Omega) \times H^2(\Omega) \times \mathbf{R}_+ \to L^2(\Omega) \times L^2(\Omega)$ as follows;

$$L(u, v, \gamma) = (\Delta u - (v_1^2 + \lambda)u - 2v, \gamma\Delta v + v_1^2 u + v).$$

Note that $(u, v, \gamma) = (p_1\phi_n, q_1\phi_n, \gamma_{n,1})$ satisfies

$$L(u, v, \gamma) = 0,$$

and

$$\mathrm{Ker}(L(u, v, \gamma_{n,1})) = \{(p_1\phi_n, q_1\phi_n)\}.$$

Here r_1 and s_1 are the constants given by (2.40). On the other hand, one can see from the result of [6] that

$$(\tilde{u_1}, \tilde{v_1}) \perp \mathrm{Ker}(L^*(u, v, \gamma_{n,1})),$$

and

$$\mathrm{Ker}(L^*(u, v, \gamma_{n,1})) = \{(r_1\phi_n, s_1\phi_n)\},$$

where $\tilde{u_1}$ and $\tilde{v_1}$ are the functions defined by (2.39), r_1 and s_1 are the constants given by (2.42), and L^* is the adjoint operator of L.

Substituting (2.39) into (2.1) and (2.2), then

$$L(\tilde{u_1}, \tilde{v_1}, \gamma_{n,1}) + (0, (\gamma_{n,1}(\epsilon) - \gamma_{n,1})(\Delta(q_1\phi_n) + \Delta\tilde{v_1})) + \epsilon(-A, A) + O(\epsilon^2) = 0,$$

where

$$A = u_1(q_1\phi_n + \tilde{v_1})^2 + 2v_1(p_1\phi_n + \tilde{u_1})(q_1\phi_n + \tilde{v_1}).$$

Taking $L^2(\Omega) \times L^2(\Omega)$ inner product with $(r_1\phi_n, s_1\phi_n)$, then we see that

$$s_1\mu_n(\gamma_{n,1}(\epsilon) - \gamma_{n,1})\left(q_1\int_\Omega \phi_n^2 dx + \int_\Omega \phi_n\tilde{v_1}dx\right) + \epsilon(r_1 - s_1)\int_\Omega A\phi_n dx + O(\epsilon^2) = 0.$$

If we differentiate this equation with respect to ϵ and take $\epsilon \to 0$, then

$$s_1\mu_n\gamma'_{n,1}(0)\int_\Omega \phi_n^2 dx + (r_1 - s_1)(2v_1p_1 + u_1q_1)\int_\Omega \phi_n^3 dx = 0.$$

This completes the proof of Theorem 2.31. □

Finally, we discuss stability for the bifurcating solutions given by (2.39). By linearizing (2.35) around a stationary solution $(u, v) = (\varphi, \psi)$ for (SP1), then the corresponding linearlized eigenvalue problem is as follows;

$$\begin{cases} \Delta u - (\psi^2 + \lambda)u - (2\varphi\psi)v = \mu u \\ d\Delta v + \left(\frac{d\psi^2}{\gamma}\right)u + \frac{d}{\gamma}(2\varphi\psi - 1)v = \mu v \end{cases} \tag{2.43}$$

Here we define the stability index known as Morse Index.

Definition 2.33. *We define Morse Index of* (φ, ψ) *as the number of unstable eigenvalues of* (2.43)*, and call it* $\text{Morse Index}\,(\varphi, \psi)$.

Then the following theorem can be shown by using the result of [7].

Theorem 2.34. *Let* $\gamma'_{n,i}(0)$ $(i = 1, 2)$ *be defined by* (2.39)*. Assume that*

$$(1 - \gamma_{n,i}\mu_m)d \neq \gamma_{n,i}(\mu_m + \lambda + v_i^2) \quad \text{for any} \quad m \in \mathbf{N}. \tag{2.44}$$

If $\gamma'_{n,i}(0) \neq 0$*, then there exists some small positive constant* $\delta_{n,i}$ *such that*

$$\begin{aligned} &\text{Morse Index}\,(\Phi_{n,i}(\epsilon), \Psi_{n,i}(\epsilon)) \\ &= \begin{cases} \text{Morse Index}\,(u_i, v_i) + 1 & \text{if} \quad \gamma_{n,i}(\epsilon) \in (\gamma_{n,i}, \gamma_{n,i} + \delta_{n,i}), \\ \text{Morse Index}\,(u_i, v_i) - 1 & \text{if} \quad \gamma_{n,i}(\epsilon) \in (\gamma_{n,i} - \delta_{n,i}, \gamma_{n,i}), \end{cases} \end{aligned}$$

for $i = 1, 2$.

Proof. For fixed $i = 1, 2$, we write by $\mu_i(\epsilon)$ (resp. $\tilde{\mu}_i(\epsilon)$) the eigenvalues of (2.43) for $(\varphi, \psi) = (u_i, v_i)$ (resp. $(\varphi, \psi) = (\Phi_{n,i}(\epsilon), \Psi_{n,i}(\epsilon))$) in case $\gamma = \gamma_{n,i}(\epsilon)$. Note that every eigenvalue $\mu_i(0)$ except for zero satisfies

$$\text{Re}\,\mu_i(0) \neq 0,$$

in view of (2.44). By using the perturbation result of [7, Lemma 1.3], it suffices to study the the eigvenvalue $\tilde{\mu}_i(\epsilon)$ with $\tilde{\mu}_i(0) = 0$. Then it follows from the result of [7, Theorem 1.16] that

$$\lim_{\epsilon \to 0,\, \tilde{\mu}_i(\epsilon) \neq 0} \frac{-\epsilon \gamma'_{n,i}(\epsilon) \mu'_i(\epsilon)}{\tilde{\mu}_i(\epsilon)} = 1.$$

This implies that the theorem holds true. □

Remark. After writing this thesis, the author has become aware of the paper R. Peng and M. X. Wang [30], in which similar results discussed in this chapter are investigated.

Chapter 3

Standing pulse solution

In this chapter, we mainly discuss the following boundary value problem in entire space;

$$\Delta u - uv^2 + \lambda(1-u) = 0, \qquad x \in \mathbf{R^N}, \tag{3.1}$$

$$\gamma\Delta v + uv^2 - v = 0, \qquad x \in \mathbf{R^N}, \tag{3.2}$$

$$\lim_{|x|\to+\infty}(u,v) = (1,0), \tag{3.3}$$

where λ and γ are positive parameters, and $N \geq 1$ is an integer. Throughout this chapter, the stationary problem (3.1)-(3.3) is called (SP2). Note that constant solution of (SP2) is uniquely determined by $(u,v) = (1,0)$. A non-constant solution of (SP2) like Figure 3.1 is generally called standing pulse solution for one dimensional case, or spot solution for multi-dimensional case, and it is regarded chemically as stationary chemical wave.

There are many known results concerned with (SP2). For one dimensional case, Doelman-Gardner-Kaper [9] and Doelman-Kaper-Zegeling [10] have constructed multi-pulse solutions in case $\lambda \ll 1$, $\gamma \ll 1$ and $\lambda\gamma \ll 1$ through geometric singular perturbation theory, and have studied their stability properties. Hale-Peletier-Troy [17, 18] have treated (SP2) in case $\lambda\gamma \approx 1$ for one dimensional case. They have established a unique one-pulse solution making use of phase plane method and implicit function theorem, and shown its stability result. Ai [1] has also discussed (SP2) when $\lambda\gamma$ is near one and obtained the unique one-pulse solution using contraction mapping theorem. One the other hand, there are few results concerned with (SP2) for multi-dimensional case. For two dimensional case, Wei [37] has studied existence and stability for two radially symmetric one-spot solutions in case $\gamma \ll 1$ by singular perturbation method.

In this chapter, we will mainly deal with the following three problems;

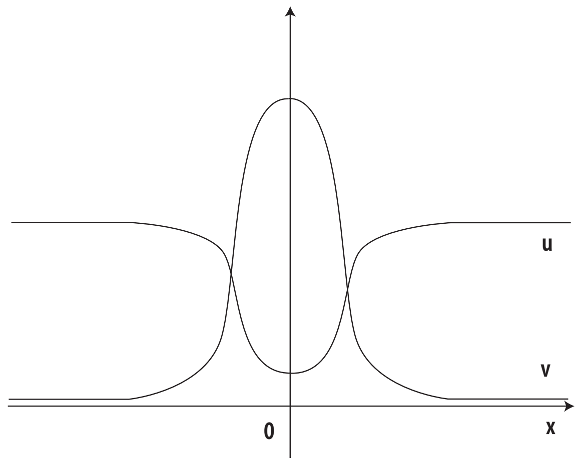


Figure 3.1: Pulse-like solution for (SP2).

(a) Existence and stability of non-trivial solutions for multi-dimensional case.

(b) Sufficient conditions about the nonexistence of nontrivial solutions for (SP2).

(c) If we replace uv^2 in (SP2) by $uv^\alpha(\alpha > 1)$, then how about the solution structure of (SP2) changes?

In Section 3.1, we will study the problem (a) in case λ and γ satisfy

$$\lambda\gamma = 1. \tag{3.4}$$

Then we have the following theorem.

Theorem 3.1. *Let λ and γ satisfy* (3.4) *and $N \geq 2$. If $0 < \gamma < \frac{2}{9}$, then* (SP2) *has a solution $(u(x), v(x))$ with the following properties*

(i)$u(x) = u(|x|), \quad v(x) = v(|x|), \quad u(x) = 1 - \gamma v(x) \quad for \quad x \in \mathbf{R^N}$;

(ii)$u'(r) > 0 \quad and \quad v'(r) < 0 \quad for \quad r = |x| > 0$;

(iii)$\lim_{r\to+\infty}(u(x), u'(x), v(x), v'(x)) = (1, 0, 0, 0)$.

On the other hand, if $\gamma \geq \frac{2}{9}$, then (SP2) *has no nontrivial solutions.*

This existence result for multi-dimensional case is an extension of the existence theorem for one-dimensional case given by [18]. Section 3.1 also deals with profile of the solutions given by Theorem 3.1. Especially, we will focus on maximum value and decay rate of the non-trivial solutions.

By setting $s = u - 1$, then the original problem (P) can be described as

$$\text{(NSP2)} \qquad \begin{cases} s_t = \Delta s - (1+s)v^2 - \lambda s, & (x,t) \in \mathbf{R^N} \times (0,\infty), \\ \frac{\gamma}{d} v_t = \gamma \Delta v + (1+s)v^2 - v, & (x,t) \in \mathbf{R^N} \times (0,\infty), \\ s(x,0) = s_0(x), \quad v(x,0) = v_0(x) & x \in \mathbf{R^N}, \end{cases}$$

where $s_0(x)$ and $v_0(x)$ are nonnegative smooth functions in $\mathbf{R}^{\mathbf{N}}$. Furthermore, λ, γ and d are positive parameters.

Set functional space W as follows

$$W = L^2(\mathbf{R}^{\mathbf{N}}) \times L^2(\mathbf{R}^{\mathbf{N}}).$$

Then we will also derive the following stability result in Section 3.1.

Theorem 3.2. *Assume $d = 1$, then the radial symmetric solutions given by* Theorem 3.1 *are linearly unstable in W.*

Concerned with the problem (b), the following nonexistence results will be shown in Section 3.2.

Theorem 3.3. *Let $\lambda\gamma > 1$. Then there exists no nontrivial solution of* (SP2) *if* $\lambda \leq 4$.

Theorem 3.4. *Let $X := \lambda\gamma < 1$. Then* (SP2) *admits no nontrivial solution if one of the following conditions is satisfied*:
(i)$\gamma \geq \frac{1}{4}$,
(ii)$4 - 16\gamma \leq X \leq 4\gamma$ *with* $\frac{1}{5} \leq \gamma < \frac{1}{4}$,
(iii)$X \leq \frac{4}{5}$, $\gamma < \frac{1}{4}$ *with*

$$\begin{cases} X \geq \frac{4(1-4\gamma)}{(1+4\gamma-16\gamma^2)^2} & if \quad X \leq X^*(\gamma), \\ X \leq 4\gamma & if \quad X > X^*(\gamma), \end{cases}$$

where $X^*(\gamma)$ is a monotone decreasing function on γ and satisfies

$$X^*(\gamma) \to \frac{4}{5} \quad as \quad \gamma \to 0 \quad and \quad X^*(\gamma) \to \tilde{X} \quad as \quad \gamma \to \frac{1}{4}.$$

Here $\tilde{X} \in (0,1)$ is a unique number satisfying $\tilde{X} = (1 - \tilde{X})\left(1 + \sqrt{1 - \tilde{X}}\right)^2$.

Remark 3.5. *These two nonexistence theorems hold true for any dimensional case.*

Therefore we can draw the picture about the existence and nonexistence regions of nontrivial solutions for (SP2) in one-dimensional case. See Figure 3.2.

Section 3.2 is devoted to prove the non-existence theorems. To prove the theorems, we need a priori estimates derived by Strong Maximum Principle.

In Section 3.3, we study the following generalized stationary problem related to the problem (c);

$$\text{(GSP)} \qquad \begin{cases} \Delta u - uv^\alpha + \lambda(1-u) = 0, & x \in \mathbf{R}^{\mathbf{N}}, \\ \gamma\Delta v + uv^\alpha - v = 0, & x \in \mathbf{R}^{\mathbf{N}}, \\ \lim_{x\to+\infty}(u,v) = (1,0), \end{cases}$$

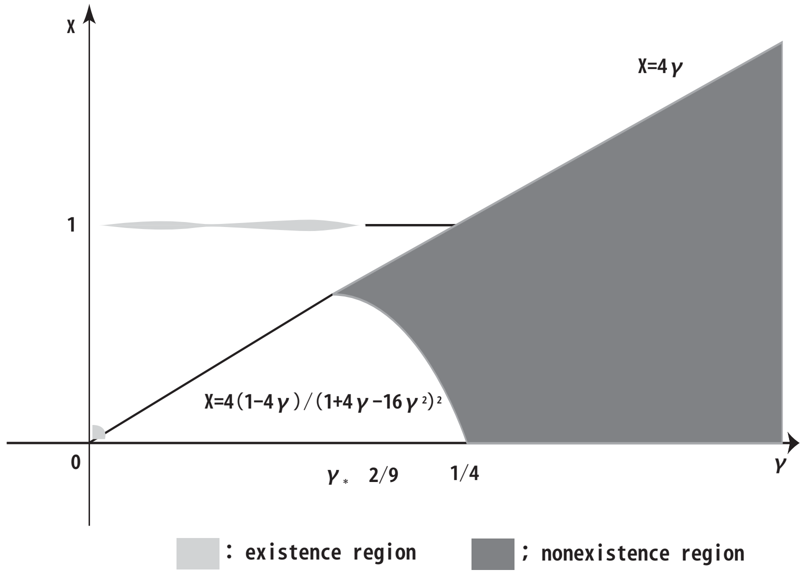


Figure 3.2: The existence and nonexistence regions.

where λ and γ are parameters, and α is a constant satisfying $\alpha > 1$. Here the corresponding chemical reaction is as follows;

$$\begin{cases} U + \alpha V \to (1+\alpha)V, \\ V \to P. \end{cases}$$

This generalized stationary problem has been first discussed by [18] in one-dimensional case. Hale-Peletier-Troy [18] have obtained a unique one-pulse solution of (GSP) in case $\lambda\gamma \approx 1$. In this section, using the method developed in Section 3.2, we will derive some non-existence theorems of (GSP) for multi-dimensional space. Moreover, we will show that unique constant solution $(1,0)$ of (GSP) is globally stable for some special parameter case.

3.1 Non-trivial solutions of (SP2) in case $\lambda\gamma = 1$

In this section, we first prove Theorem 3.1.

Proof of Theorem 3.1. Define a function $p = u + \gamma v - 1$. According to (3.1)-(3.3), we have

$$\Delta p = \frac{1}{\gamma} p, \qquad \lim_{|x|\to\infty} p = 0.$$

Therefore, strong maximum principle enables us to show that p must identically zero. Setting $u = 1 - \gamma v$ in (3.2), then

$$\Delta v + f(v, \gamma) = 0, \tag{3.5}$$

where

$$f(v, \gamma) = -\frac{1}{\gamma} v(\gamma v^2 - v + 1). \tag{3.6}$$

Since $v(x) \to 0$ as $r = |x| \to +\infty$, then it follows from the result of [15] that every positive ground state solution of (3.5) must be radially symmetric. Hence (3.5) can be reduced to the following ordinary differential problem;

$$v'' + \frac{N-1}{r} v' + f(v, \gamma) = 0, \quad v(r) > 0, \quad \lim_{r \to +\infty} v'(r) = 0. \tag{3.7}$$

Note that the reaction term $f(v, \gamma)$ satisfies the following condition (J) if $0 < \gamma < \frac{2}{9}$.

(J) There exists a number $\xi > 0$ such that $F(\xi) > 0$, where

$$F(\xi) := \int_0^\xi f(s, \gamma) ds.$$

Using the result of [5], we find that (3.7) has a solution $v = v(r)$ which satisfies $v'(r) < 0$.

On the other hand, $f(v, \gamma)$ does not satisfy the condition (J) if $\gamma > \frac{2}{9}$. Therefore (3.7) has no nontrivial solution. □

Proposition 3.6. *Assume that* (u, v) *is the radially symmetric solution given by* Theorem 3.1. *Then the convergence rate of the solution is exponential.*

Proof. We begin to show that $v(r)$ and $v'(r)$ decay to zero exponentially. Recall that

$$\lim_{v \to 0} \frac{f(v, \gamma)}{v} = -\frac{1}{\gamma}.$$

Then it follows from the result of [28] that

$$\lim_{r \to \infty} v(r) e^{\left(\sqrt{\frac{1}{\gamma} - \epsilon}\right) r} < \infty,$$

for any ϵ in $\left(0, \frac{1}{\gamma}\right)$. Moreover, owing to the result of [28], we find

$$\lim_{r \to \infty} \frac{v'(r)}{v(r)} = -\sqrt{\frac{1}{\gamma}}.$$

Hence v and v' decay to exponentially zero as $r \to \infty$.

Note that $u(r) - \gamma v(r) + 1 \equiv 0$ for $r \in [0, +\infty]$. Then one can see

$$\lim_{r\to\infty} \frac{u'(r)}{1-u(r)} = \lim_{r\to\infty} \frac{\gamma v'(r)}{\gamma v(r)} = -\sqrt{\frac{1}{\gamma}}, \tag{3.8}$$

and

$$\lim_{r\to\infty} (1-u) e^{\left(\sqrt{\frac{1}{\gamma}}-\epsilon\right) r} = \lim_{r\to\infty} \gamma v(r) e^{\left(\sqrt{\frac{1}{\gamma}}-\epsilon\right) r} < \infty. \tag{3.9}$$

Combining (3.8) and (3.9), we deduce that $u-1$ and u' also decay to zero exponentially as $r \to \infty$. □

Proposition 3.7. *As for the maximum values $u(0)$ and $v(0)$ for the solution (u, v) given by* Theorem 3.1*, the following inequalities hold true.*

$$\frac{1-\sqrt{1-4\gamma}}{2} < u(0) < \frac{1+\sqrt{4-18\gamma}}{3}, \quad \frac{2-\sqrt{4-18\gamma}}{3\gamma} < v(0) < \frac{1+\sqrt{1-4\gamma}}{2\gamma}.$$

Proof. Define

$$\xi_0 = \inf\left\{\xi > 0 : \int_0^\xi f(s,\gamma)ds > 0\right\}, \quad \beta = \inf\left\{\xi > \xi_0 : f(\xi,\gamma) > 0\right\},$$

where $f(v, \gamma)$ is defined by (3.6). After some computation, we find

$$\xi_0 = \frac{2-\sqrt{4-18\gamma}}{3\gamma}, \quad \beta = \frac{1+\sqrt{1-4\gamma}}{2\gamma}.$$

Using the result of [5], one can see

$$v(0) \in (\xi_0, \beta).$$

In view of $u(0) - \gamma v(0) + 1 = 0$, it follows that $u(0) \in (1-\gamma\beta, 1-\gamma\xi_0)$. This means that

$$\frac{1-\sqrt{1-4\gamma}}{2} < u(0) < \frac{1+\sqrt{4-18\gamma}}{3}.$$

Thus the proof is complete. □

Concerned with the uniqueness of the solutions given by Theorem 3.1, we obtain the following proposition.

Proposition 3.8. *Let (u, v) be the radially symmetric solution given by* Theorem 3.1*. Then it is unique (up to translation) solution for* (SP2).

Proof. See the result of [35]. □

Making use of Proposition 3.8, we will show the following proposition.

Proposition 3.9. *Let $v_N(r)$ denote the solution given by* Theorem 3.1 *for each dimension N. Then it follows that*

$$v_{N+1}(r) < v_N(r) \quad \text{for} \quad r \in [0, +\infty) \quad \text{and} \quad N \geq 1.$$

Proof. Note that $v_N(r)$ and $v_{N+1}(r)$ satisfy

$$\gamma v_N'' + \frac{N-1}{r} v_N' + f(v_N) = 0, \tag{3.10}$$

and

$$\gamma v_{N+1}'' + \frac{N-1}{r} v_{N+1}' + f(v_{N+1}) = -\frac{1}{r} v_{N+1}' > 0.$$

Therefore $v_{N+1}(r)$ is a lower solution of (3.10).

Moreover, $v_{N-1}(r)$ satisfies the following inequality,

$$\gamma v_{N-1}'' + \frac{N-1}{r} v_{N-1}' + f(v_{N-1}) = \frac{1}{r} v_{N-1}' < 0.$$

Hence $v_{N-1}(r)$ is an upper solution of (3.10). Using the comparison principle and uniqueness result given by Theorem 3.8, one can see that

$$v_{N+1}(r) < v_N(r) < v_{N-1}(r) \quad \text{for} \quad r \in [0, +\infty).$$

Consequently, the proof is complete. □

Finally, we will prove Theorem 3.2. In order to construct an unstable eigenvalue, we must use the result of [5].

Proof of Theorem 3.2. Let (φ, ψ) be the radially symmetric solution given by Theorem 3.1. Then

$$\varphi(x) + \gamma\psi(x) - 1 = 0 \quad \text{for} \quad x \in \mathbf{R^N}. \tag{3.11}$$

By linearizing (NSP2) around $(\varphi - 1, \psi)$, the corresponding linearlized eigenvalue problem in case $d = 1$ and $\lambda\gamma = 1$ is as follows:

$$\begin{cases} \Delta s - (\psi^2 + \frac{1}{\gamma})s - 2\varphi v = \mu u, \\ \Delta v + \frac{\psi^2}{\gamma} s + \frac{1}{\gamma}(2\varphi - 1)v = \mu v. \end{cases} \tag{3.12}$$

Set some functional space W_0 as

$$W_0 = \left\{(s,v) \in W : s(x) + \gamma v(x) = 0 \text{ for any } x \in \mathbf{R^N}\right\},$$

where $W = L^2(\mathbf{R^N}) \times L^2(\mathbf{R^N})$. If $(s,v) \in W_0$, then it follows from (3.11) and (3.12) that, $s + \gamma v = 0$ and

$$\Delta v + \frac{1}{\gamma}\{-3\gamma\psi^2 + 2\psi - 1\}v = \mu v. \tag{3.13}$$

According to the result of [5], the eigenvalue problem (3.13) has a positive real eigenvalue with the corresponding eigenfunction $v(x) \in L^2(\mathbf{R^N})$. Thus (3.12) has also at least one unstable eigenvalue. □

3.2 A priori estimates and proofs of non-existence results

In section 3.1, we have used the fact that (SP2) can be reduced to a single equation when $\lambda\gamma = 1$. However, if $\lambda\gamma$ is not 1, the property breaks down. Therefore, we must consider essentially system problem in case $\lambda\gamma \neq 1$. It becomes generally a difficult problem.

To prove Theorems 3.3 and 3.4, we need some a priori estimates for solutions of (SP2). Our analysis is based on the strong maximum principle in the following form (see, e.g. Peletier-Troy [29]).

Strong maximum principle. *Let $w(\not\equiv 0)$ be a classical solution of*

$$\Delta w + c(x)w = f(x), \quad x \in \mathbf{R^N},$$

where $c(x)$ is a non-positive bounded continuous function on $\mathbf{R^N}$. Suppose that $f(x) \leq 0$ for $x \in \mathbf{R^N}$. If

$$\liminf_{|x|\to\infty} w(x) \geq 0,$$

then it follows that

$$w(x) > 0 \quad for \quad x \in \mathbf{R^N}.$$

Using the strong maximum principle, we can show the following lemmas.

Lemma 3.10. *Assume $\lambda\gamma > 1$ and $\lambda \leq 4$. Then*

$$\lambda(u(x) - 1) + v(x) > 0 \quad for \quad x \in \mathbf{R^N}.$$

Proof. Define $s = \lambda(u-1)+v$. Then (3.1) and (3.2) lead to

$$\Delta s - \left\{ \left(1 - \frac{1}{\lambda\gamma}\right) v^2 + \lambda \right\} s = \left(\frac{1}{\lambda\gamma} - 1\right) v(v^2 - \lambda v + \lambda), \qquad \lim_{|x|\to\infty} s = 0.$$

If $\lambda\gamma > 1$ and $\lambda \le 4$, then

$$\left(1 - \frac{1}{\lambda\gamma}\right) v^2 + \lambda > 0 \quad \text{and} \quad \left(\frac{1}{\lambda\gamma} - 1\right) v(v^2 - \lambda v + \lambda) \le 0.$$

Therefore we can use the strong maximum principle to show that $s(x) > 0$ for $x \in \mathbf{R^N}$. □

Making use of Lemma 3.10, we can prove Theorem 3.3.

Proof of Theorem 3.3. Addition of (3.1) and (3.2) gives that

$$\Delta u + \gamma \Delta v = \lambda u + v - \lambda. \tag{3.14}$$

Integrating (3.14) on $\mathbf{R^N}$, then we have

$$0 = \int_{\mathbf{R^N}} (\lambda(u-1)+v)dx. \tag{3.15}$$

But Lemma 3.10 means that the right side of (3.15) is positive. This is a contradiction. Thus the proof is complete. □

Next, we will mainly discuss a priori estimate of nontrivial solution for (SP2) in case $\lambda\gamma < 1$.

Lemma 3.11. *Let (u,v) be any nontrivial solution of* (SP2). *Then*

$$0 < u(x) < 1, \quad \textit{and} \quad v(x) > 0 \quad \textit{for} \quad x \in \mathbf{R^N}.$$

Proof. We first show $u(x) > 0$ for $x \in \mathbf{R^N}$ by contradiction. Assume $\inf_{x\in\mathbf{R^N}} u(x) \le 0$. Since $\lim_{|x|\to\infty} u(x) = 1$, u has its minimum at $x = x_0$;

$$\inf_{x\in\mathbf{R^N}} u(x) = u(x_0) \le 0.$$

Note $\Delta u(x_0) \ge 0$. So it follows from (3.1) that

$$0 \le \Delta u(x_0) = u(x_0)v(x_0)^2 + \lambda(u(x_0)-1) \le -\lambda < 0.$$

This is a contradiction. Therefore, $\inf_{x\in\mathbf{R^N}} u(x) = \min_{x\in\mathbf{R^N}} u(x) > 0$.

We next put $u - 1 = w$. Then (3.1) and (3.3) imply that

$$\Delta w - \lambda w = uv^2 \geq 0 \quad \text{and} \quad \lim_{|x| \to \infty} w(x) = 0.$$

We can use the strong maximum principle to show $w(x) < 0$ for $x \in \mathbf{R^N}$, which shows

$$u(x) < 1 \quad \text{for} \quad x \in \mathbf{R^N}.$$

Finally (3.2) and (3.3) give

$$\gamma \Delta v - v = -uv^2 \leq 0, \quad \lim_{|x| \to \infty} v(x) = 0.$$

Using the strong maximum principle again, we get

$$v(x) > 0 \quad \text{for} \quad x \in \mathbf{R^N}.$$

Thus the proof is complete. □

Lemma 3.12. *Let* (u, v) *be any nontrivial solution of* (SP2). *Then the following inequalities hold true.*
(i)*If* $\lambda\gamma < 1$, *then* $u(x) + \gamma v(x) - 1 < 0$ *for* $x \in \mathbf{R^N}$.
(ii)*If* $\lambda\gamma > 1$, *then* $u(x) + \gamma v(x) - 1 > 0$ *for* $x \in \mathbf{R^N}$.

Proof. We first consider the case $\lambda\gamma < 1$. Define $p = u + \gamma v - 1$; then (3.1)-(3.3) leads to

$$\Delta p - \lambda p = (1 - \lambda\gamma)v \geq 0 \quad \text{and} \quad \lim_{|x| \to \infty} p(x) = 0.$$

Then the strong maximum principle yields

$$p(x) < 0 \quad \text{for} \quad x \in \mathbf{R^N}.$$

Thus the proof is complete in case $\lambda\gamma < 1$. The proof for $\lambda\gamma > 1$ is almost the same; so we omit it. □

Lemma 3.13. *Let* (u, v) *be any nontrivial solution of* (SP2). *Then the following inequalities hold true.*
(i)*If* $\lambda\gamma < 1$, *then* $u(x) + \gamma v(x) > \lambda\gamma$ *for* $x \in \mathbf{R^N}$.
(ii)*If* $\lambda\gamma > 1$, *then* $u(x) + \gamma v(x) < \lambda\gamma$ *for* $x \in \mathbf{R^N}$.

Proof. We give the proof only in the case $\lambda\gamma < 1$. The other case can be treated almost in the same way. Define $q = u + \gamma v - \lambda\gamma$, then

$$\Delta q - \frac{1}{\gamma} q = \left(\lambda - \frac{1}{\gamma}\right) u \leq 0.$$

Since $\lim_{|x|\to\infty} q(x) = 1 - \lambda\gamma > 0$, we see from the strong maximum principle that

$$q(x) = u(x) + \gamma v(x) - \lambda\gamma > 0 \quad \text{for} \quad x \in \mathbf{R^N}.$$

□

For any solution (u, v) of (SP2) define

$$u_{min} = \min_{x\in\mathbf{R^N}} u(x) \quad \text{and} \quad v_{max} = \max_{x\in\mathbf{R^N}} v(x).$$

Then we can show the following lemma.

Lemma 3.14. *Assume* $\lambda\gamma < 1$.
(i)*If* $\lambda\gamma^2 \geq \frac{1}{4}$, *then* $u_{min} = 1$ *and* $v_{max} = 0$.
(ii)*If* $\lambda\gamma^2 < \frac{1}{4}$, *then*

$$u_{min} \geq \frac{1 - \sqrt{1 - 4\lambda\gamma^2}}{2} \quad \textit{and} \quad v_{max} \leq \frac{1 + \sqrt{1 - 4\lambda\gamma^2}}{2\gamma}.$$

Proof. Assume $u \not\equiv 1$ and $v \not\equiv 0$. Let $u_{min} = u(x_m)$ for some $x_m \in \mathbf{R^N}$. In view of (3.1), observe that

$$0 \leq \Delta u(x_m) = u(x_m)(\lambda + v(x_m)^2) - \lambda.$$

Hence

$$u_{min} = u(x_m) \geq \frac{\lambda}{\lambda + v(x_m)^2} \geq \frac{\lambda}{\lambda + v_{max}^2}. \tag{3.16}$$

Set $v_{max} = v(x_M)$ for some $x_M \in \mathbf{R^N}$. Then one can see from Lemma 3.12 that

$$\gamma v_{max} = \gamma v(x_M) < 1 - u(x_M) \leq 1 - u_{min}. \tag{3.17}$$

Combing (3.16) with (3.17), we get

$$\frac{\lambda}{\lambda + v_{max}^2} \leq u_{min} < 1 - \gamma v_{max}. \tag{3.18}$$

If $4\lambda\gamma^2 \geq 1$, then it is impossible to find (u_{min}, v_{max}) satisfying (3.18). So (SP2) has no nontrivial solution in case $4\lambda\gamma^2 \geq 1$. If $4\lambda\gamma^2 < 1$, it is easy to see from (3.18) that

$$\frac{1-\sqrt{1-4\lambda\gamma^2}}{2} < u_{min} < \frac{1+\sqrt{1-4\lambda\gamma^2}}{2},$$
$$\frac{1-\sqrt{1-4\lambda\gamma^2}}{2\gamma} < v_{max} < \frac{1+\sqrt{1-4\lambda\gamma^2}}{2\gamma}.$$

Thus the proof is complete. □

Lemma 3.15. *Let* $\lambda\gamma < 1$ *and* $4\lambda\gamma^2 \leq 1$. *Assume that there exists a positive number* a *satisfying*

$$1 > a\gamma \geq \frac{1+\sqrt{1-4\lambda\gamma^2}}{2}, \tag{3.19}$$
$$\gamma a^2 - a + 4(1-\lambda\gamma) \geq 0. \tag{3.20}$$

Then any solution (u, v) *of* (SP2) *satisfies*

$$a(u(x)-1) + v(x) < 0 \quad for \quad x \in \mathbf{R^N}.$$

Proof. Multiply (3.1) by a and multiply (3.2) by $\frac{1}{\gamma}$. Adding the resulting expressions leads to

$$a\Delta u + \Delta v = \left(a - \frac{1}{\gamma}\right) uv^2 + a\lambda(u-1) + \frac{1}{\gamma}v$$

If we set $r = a(u-1) + v$, then r satisfies $\lim_{|x|\to\infty} r(x) = 0$ and

$$\Delta r - \left\{\lambda + \left(1 - \frac{1}{a\gamma}\right) v^2\right\} r = \frac{1-a\gamma}{a\gamma} v \left\{v^2 - av + \frac{a(1-\lambda\gamma)}{1-a\gamma}\right\}. \tag{3.21}$$

Here

$$\begin{aligned} v^2 - av + \frac{a(1-\lambda\gamma)}{1-a\gamma} &= \left(v - \frac{a}{2}\right)^2 - \frac{a^2}{4} + \frac{a(1-\lambda\gamma)}{1-a\gamma} \\ &\geq \frac{a}{4(1-a\gamma)} \left\{\gamma a^2 - a + 4(1-\lambda\gamma)\right\}; \end{aligned}$$

so that the right hand side of (3.21) is nonnegative by (3.20). As to the coefficient of r in (3.21), it follows from Lemma 3.14 that

$$\begin{aligned} \lambda + \left(1 - \frac{1}{a\gamma}\right) v^2 &\geq \lambda - \frac{1-a\gamma}{a\gamma} \left(\frac{1+\sqrt{1-4\lambda\gamma^2}}{2\gamma}\right)^2 \\ &= \frac{1+\sqrt{1-4\lambda\gamma^2}}{2\gamma^2} \left(1 - \frac{1+\sqrt{1-4\lambda\gamma^2}}{2a\gamma}\right); \end{aligned}$$

which is nonnegative by (3.19). Then the strong maximum principle enables us to show $r(x) < 0$ for $x \in \mathbf{R^N}$. ☐

Remark 3.16. *In* Lemma 3.15, (3.20) *holds true whenever* $D := 16\lambda\gamma^2 - 16\gamma + 1 \leq 0$. *On the other hand, if* $D > 0$, *then* (3.20) *is equivalent to*

$$a \geq \frac{1+\sqrt{D}}{2\gamma} \quad \text{or} \quad a \leq \frac{1-\sqrt{D}}{2\gamma}.$$

Therefore, in order to choose a satisfying (3.19) *and* (3.20), *it is possible to take*

$$a \geq \frac{1+\sqrt{1-4\lambda\gamma^2}}{2\gamma}$$

if

$$\lambda \leq \max\left\{\frac{16\gamma-1}{16\gamma^2}, \frac{4}{5\gamma}\right\},$$

and

$$a \geq \frac{1+\sqrt{16\lambda\gamma^2-16\gamma+1}}{2\gamma}$$

if

$$\lambda > \max\left\{\frac{16\gamma-1}{16\gamma^2}, \frac{4}{5\gamma}\right\}.$$

Lemma 3.17. *Assume* $\lambda\gamma < 1$ *and* $\gamma < \frac{1}{4}$. *Let*

$$a = \begin{cases} \frac{1+\sqrt{1-4\lambda\gamma^2}}{2\gamma} & if \quad \lambda \leq \frac{4}{5\gamma}, \\ \frac{1+\sqrt{16\lambda\gamma^2-16\gamma+1}}{2\gamma} & if \quad \lambda > \frac{4}{5\gamma}. \end{cases} \tag{3.22}$$

Then the following inequalities hold true;

$$\begin{cases} u_{min} = 1, \ v_{max} = 0 & if \quad \frac{4}{25\gamma^2} \leq \lambda \leq 4, \\ u_{min} \geq \frac{a-\sqrt{a^2-4\lambda}}{2a}, \ v_{max} \leq \frac{a+\sqrt{a^2-4\lambda}}{2} & otherwise. \end{cases}$$

Proof. Since $4\lambda\gamma^2 < 1$, Lemma 3.14 means $u_{min} \geq \frac{1-\sqrt{1-4\lambda\gamma^2}}{2}$. Moreover,

$$\max\left\{\frac{16\gamma-1}{16\gamma^2}, \frac{4}{5\gamma}\right\} = \frac{4}{5\gamma},$$

in case $\gamma < \frac{1}{4}$. It is seen from Remark 3.16 that, if a is defined by (3.22), then a satisfies (3.19) and (3.20). Hence Lemma 3.15 implies

$$v_{max} < a(1-u_{min}). \tag{3.23}$$

So, one can repeat the proof of Lemma 3.14 and use (3.23) in place of (3.17). Making use of (3.22), one should note that

$$a^2 \leq 4\lambda \quad \text{if and only if} \quad \frac{1}{5} \leq \gamma < \frac{1}{4} \text{ and } \frac{4}{25\gamma^2} \leq \lambda \leq 4.$$

Hence the conclusion follows from (3.16) and (3.23) as in the proof of Lemma 3.14. □

Lemma 3.18. *Let $\lambda\gamma < 1$, $\gamma < \frac{1}{4}$ and define a by* (3.22). *Assume that there exists a positive number b satisfying*

$$a > b \geq \frac{a+\sqrt{a^2-4\lambda}}{2\gamma a}, \tag{3.24}$$

$$\gamma b^2 - b + 4(1-\lambda\gamma) \geq 0. \tag{3.25}$$

Then any solution (u, v) of (SP2) *satisfies*

$$b(u(x)-1) + v(x) < 0 \quad for \quad x \in \mathbf{R^N}.$$

Proof. The idea of the proof is similar to that of Lemma 3.15. □

Remark 3.19. *If $\lambda \leq \frac{4}{5\gamma}$, then we can show conditions* (3.24) *and* (3.25) *are equivalent to*

$$\begin{cases} b \geq \frac{1+\sqrt{1-\frac{4\lambda}{a^2}}}{2\gamma} & \text{if} \quad X \leq X^*(\gamma), \\ b \geq \frac{1+\sqrt{16\lambda\gamma^2-16\gamma+1}}{2\gamma} & \text{if} \quad X > X^*(\gamma). \end{cases} \tag{3.26}$$

where $a = \frac{1+\sqrt{1-4\lambda\gamma^2}}{2\gamma}$, $X = \lambda\gamma$ and $X^*(\gamma) \in (0,1)$ is a unique number satisfying $X^* = (1-X^*)(1+\sqrt{1-4\gamma X^*})^2$. It is not difficult to verify the condition of X^* given in (iii) *of* Theorem 3.4.

Using Lemmas 3.11-3.18, one can prove Theorem 3.4.

Proof of (i) *for Theorem 3.4.* Assume that (SP2) has a nontrivial solution (u, v). Define the following set

$$A = \left\{ (u, v) \in \mathbf{R}^2 : 0 < u < 1,\ 0 < v < \frac{1}{\gamma}(1 - u) \right\}.$$

Lemmas 3.11-3.12 imply $(u(x), v(x)) \in A$ for every $x \in \mathbf{R}^{\mathbf{N}}$. Here observe that every $(u, v) \in A$ satisfies $uv < 1$ if $\gamma \geq \frac{1}{4}$. Therefore, $\gamma \Delta v = v(1 - uv) > 0$ everywhere in $\mathbf{R}^{\mathbf{N}}$. Since $\lim_{|x| \to \infty} v(x) = 0$, then the strong maximum principle shows that $v(x) < 0$. This is impossible. □

Proof of (ii), (iii) *for Theorem 3.4.* For $\gamma < \frac{1}{4}$ and $\lambda < \frac{1}{\gamma}$, define

$$B = \left\{ (u, v) \in \mathbf{R}^2 : \frac{1 - \sqrt{1 - 4\lambda\gamma^2}}{2} \leq u < 1,\ 0 < v < a(1 - u) \right\},$$

where a is a positive number satisfying (3.19) and (3.20). Then by Lemmas 3.11-3.14, $(u(x), v(x)) \in B$ for every $x \in \mathbf{R}^{\mathbf{N}}$. Here we should note that every $(u, v) \in B$ satisfies $uv < 1$ if a satisfies $a \leq 4$. As in the proof of (i), (SP2) has no nontrivial solutions if one can choose a satisfying (3.19), (3.20), and $a \leq 4$. Since a is given by (3.22), this condition is possible when λ and γ satisfy

$$\frac{4}{\gamma} - 16 \leq \lambda \leq 4 \quad \text{with} \quad \frac{1}{5} \leq \gamma < \frac{1}{4}. \tag{3.27}$$

Thus we see the nonexistence result in case (ii).

On the other hand, for the case where λ and γ do not satisfy (3.27), define the following set

$$C = \left\{ (u, v) \in \mathbf{R}^2 : \frac{a - \sqrt{a^2 - 4\lambda}}{2} \leq u < 1,\ 0 < v < b(1 - u) \right\},$$

where b is a positive number which satisfies (3.26). Similarly to the proof of (ii), it is sufficient to show $b \leq 4$ in order to prove the nonexistence of nontrivial solutions. Recall that we can take $b = \frac{1 + \sqrt{1 - \frac{4\lambda}{a^2}}}{2\gamma}$ with $a = \frac{1 + \sqrt{1 - 4\gamma X}}{2\gamma}$ if $X \leq X^*(\gamma)$. Then $b \leq 4$ is equivalent to

$$\sqrt{1 - \frac{4\lambda}{a^2}} \leq 8\gamma - 1.$$

After some calculations, it follows that

$$X \geq \frac{4(1 - 4\gamma)}{(1 + 4\gamma - 16\gamma^2)^2}.$$

For $X > X^*(\gamma)$, we choose $b = \frac{1+\sqrt{16\lambda\gamma^2-16\gamma+1}}{2\gamma}$. Then $b \leq 4$ is equivalent to $X \leq 4\gamma$. Thus the proof is complete. □

Figure 3.3 is represented as the regions A, B and C in the proof of Theorem 3.4.

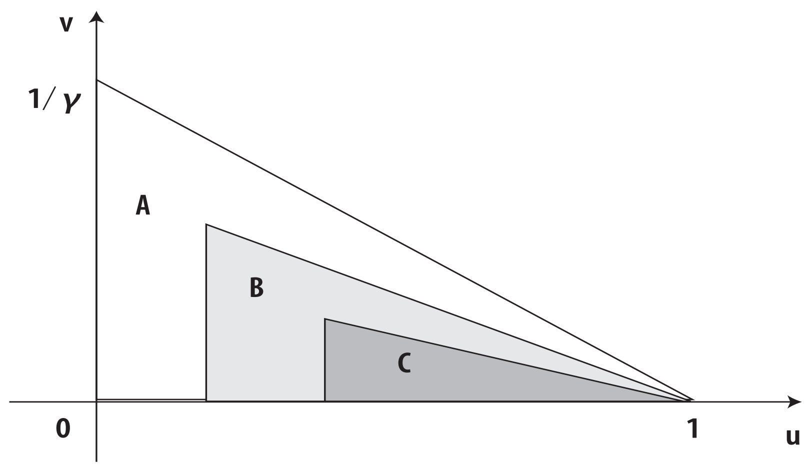


Figure 3.3: The regions A, B and C.

3.3 Generalized stationary problem

In this section, we will discuss stationary solutions for the following generalized problem;

$$\text{(GP)} \qquad \begin{cases} u_t = \Delta u - uv^\alpha + \lambda(1-u) & \text{in} \quad \mathbf{R^N} \times (0,\infty), \\ \frac{\gamma}{d} v_t = \gamma \Delta v + uv^\alpha - v & \text{in} \quad \mathbf{R^N} \times (0,\infty), \end{cases}$$

where λ, γ and d are positive parameters, and α is a constant satisfying $\alpha > 1$. Here the corresponding chemical reaction is as follows;

$$\begin{cases} U + \alpha V \to (1+\alpha)V, \\ V \to P. \end{cases}$$

We mainly treat with the following stationary problem for (GP) in this section.

$$\text{(GSP)} \qquad \begin{cases} \Delta u - uv^\alpha + \lambda(1-u) = 0, & x \in \mathbf{R^N}, \\ \gamma \Delta v + uv^\alpha - v = 0, & x \in \mathbf{R^N}, \\ \lim_{x \to \pm\infty}(u,v) = (1,0), \end{cases}$$

Note that constant solution of (GSP) is uniquely determined by $(u,v) = (1,0)$. There exists almost no results of (GSP) except for [18]. We will discuss existence, non-existence, and stability of non-trivial solutions for (GSP). Moreover, we will deal with global stability for the constant solution.

3.3.1 One dimensional case

For one dimensional case, (GSP) can be described as

$$u'' - uv^\alpha + \lambda(1-u) = 0, \qquad x \in \mathbf{R}, \tag{3.28}$$
$$\gamma v'' + uv^\alpha - v = 0, \qquad x \in \mathbf{R}, \tag{3.29}$$
$$\lim_{x\to\pm\infty}(u,v) = (1,0). \tag{3.30}$$

Throughout this subsection we call this stationary problem (GSP1). Here we define

$$\gamma_1(\alpha) = \frac{2^{1/(\alpha-1)}(\alpha+2)(\alpha-1)}{\alpha(\alpha+1)^{\alpha/(\alpha-1)}}. \tag{3.31}$$

Then it is well known that the following theorem holds true.

Theorem 3.20 ([18]). *Let $\lambda\gamma = 1$ and $\gamma_1(\alpha)$ be the constant defined by* (3.31)
(a)*If $0 < \gamma < \gamma_1(\alpha)$, then problem* (GSP1) *has a unique solution $(u(x), v(x))$ with the following properties*
(i)$u(-x) = u(x), \quad v(-x) = v(x), \quad u(x) = 1 - \gamma v(x) \quad for \quad x \in \mathbf{R}$;
(ii)$u'(x) > 0 \quad and \quad v'(x) < 0 \quad for \quad x > 0$;
(iii)$\lim_{x\to\pm\infty}(u(x), u'(x), v(x), v'(x)) = (1,0,0,0)$.
(ii)*If $\gamma \geq \gamma_1(\alpha)$, then problem* (GSP1) *has no non-trivial solution.*

Remark 3.21. *In* [18]*, Hale-Peletier-Troy have used the implicit function theorem to show that* (GSP1) *has a unique one-pulse solution when $\lambda\gamma = 1 + \epsilon$ and $0 < \gamma < \gamma_1(\alpha)$ if ϵ is sufficiently small constant.*

We will consider maximum value and convergence rate of the solutions given by Theorem 3.20. Then the following proposition holds true in [18].

Proposition 3.22 ([18]). *Let v_{max} denote the maximum of v given in* Theorem 3.20*. Then v_{max} is the smallest positive solution for the following equation*;

$$2\gamma(\alpha+1)v^\alpha - 2(\alpha+2)v^{\alpha-1} + (\alpha+1)(\alpha+2) = 0.$$

Proof. For the reader's convenience, we will give the outline of the proof. Observe that v satisfies the following differential equation in case $\lambda\gamma = 1$;

$$\gamma v'' = v - v^\alpha + \gamma v^{\alpha+1}. \tag{3.32}$$

Multiplying (3.32) by v' and integrate in $\mathbf{R}$, then

$$\frac{\gamma}{2}(v')^2 = \frac{1}{2}v^2 - \frac{1}{\alpha+1}v^{\alpha+1} + \frac{\gamma}{\alpha+2}v^{\alpha+2}. \tag{3.33}$$

Denoting $v_{max} = v(x_M)$, we see that $v'(x_M) = 0$. Because $v(x_M) > 0$, then

$$2\gamma(\alpha+1)v_{max}^{\alpha} - 2(\alpha+2)v_{max}^{\alpha-1} + (\alpha+1)(\alpha+2) = 0. \tag{3.34}$$

In view of (3.33), v_{max} is the smallest positive solution for (3.34). Thus the proof is complete. □

Proposition 3.23 ([18]). *Let* (u, v) *be the one-pulse solution given by* Theorem 3.20*. Then it follows that*

$$\lim_{x\to\pm\infty} (1-u(x))\, e^{\sqrt{\frac{1}{\gamma}}|x|} < \infty \quad \textit{and} \quad \lim_{x\to\pm\infty} v(x)e^{\sqrt{\frac{1}{\gamma}}|x|} < \infty.$$

Proof. The convergence result for v can be immediately derived by (3.32). Note that

$$u(x) + \gamma v(x) - 1 = 0 \quad \text{for} \quad x \in \mathbf{R}.$$

Hence one can derive the estimate for u. Thus the proof is complete. □

Next, we will study stability for the one-pulse solutions given by Theorem 3.20. By setting

$$s(x,t) = u(x,t) - 1 \quad \text{for} \quad (x,t) \in \mathbf{R}\times(0,\infty),$$

then (GP) can be written as

$$\begin{cases} s_t = s'' - (1+s)v^{\alpha} - \lambda s, & \text{in} \quad \mathbf{R}\times(0,\infty), \\ \frac{\gamma}{d}v_t = \gamma v'' + (1+s)v^{\alpha} - v, & \text{in} \quad \mathbf{R}\times(0,\infty), \\ s(x,0) = s_0(x), \quad v(x,0) = v_0(x) & \text{on} \quad \mathbf{R}, \end{cases}$$

where $s_0(x)$ and $v_0(x)$ denote non-negative initial values.

Then one can show the following stability result as in the proof of Theorem 3.2.

Theorem 3.24. *Let* $d = 1$ *and denote* $W = L^2(\mathbf{R}) \times L^2(\mathbf{R})$*. Then the one-pulse solutions given by* Theorem 3.20 *are linearly unstable in* W*.*

Next, we will discuss the case

$$\lambda\gamma \neq 1. \tag{3.35}$$

The following propositions are concerned with upper bound estimates of non-trivial solutions for (GSP1) in case λ and γ satisfy (3.35).

Proposition 3.25. *Let* (u, v) *be any nontrivial solution of* (GSP1)*. Then*

$$0 < u(x) < 1 \quad \textit{and} \quad v(x) > 0 \quad \textit{for} \quad x \in \mathbf{R^N}.$$

Proposition 3.26. *Let* (u, v) *be any nontrivial solution of* (GSP1)*. Then the following inequalities hold true.*
(i)*If* $\lambda\gamma < 1$*, then* $u(x) + \gamma v(x) - 1 < 0$ *for* $x \in \mathbf{R^N}$*.*
(ii)*If* $\lambda\gamma > 1$*, then* $u(x) + \gamma v(x) - 1 > 0$ *for* $x \in \mathbf{R^N}$*.*

Proposition 3.27. *Suppose that* (u, v) *is any nontrivial solution of* (GSP1)*. Then the following inequalities hold true.*
(i)*If* $\lambda\gamma < 1$*, then* $u(x) + \gamma v(x) > \lambda\gamma$ *for* $x \in \mathbf{R^N}$*.*
(ii)*If* $\lambda\gamma > 1$*, then* $u(x) + \gamma v(x) < \lambda\gamma$ *for* $x \in \mathbf{R^N}$*.*

The proof of the propositions are similar as that of Lemmas 3.11-3.13. So we omit it here.

As for the lower bound for solutions of (GSP1), we can obtain the following property.

Proposition 3.28. *Assume that* (u, v) *is any solutions of* (GSP1)*. Then*

$$u(x) \geq \frac{\lambda}{\lambda + C^\alpha} \quad for \quad x \in \mathbf{R},$$

where

$$C = \max\left\{\lambda, \frac{1}{\gamma}\right\}.$$

The proposition can be shown by using the same method of Lemma 3.14. We omit its proof here.

Now, we will study the convergence rate of non-trivial solutions for (GSP1) in case $|x|$ is sufficiently large. By setting

$$u' = p, \quad v' = q,$$

then (GSP1) can be expressed as the following ODE system;

$$\begin{aligned} u' &= p, \\ p' &= uv^\alpha + \lambda(u - 1), \\ v' &= q, \\ q' &= -\frac{1}{\gamma}uv^\alpha + \frac{1}{\gamma}v. \end{aligned}$$

Linearizing these differential equations around $(u, p, v, q) = (1, 0, 0, 0)$, then we have

$$u' = p, \quad p' = \lambda u, \quad v' = q, \quad q' = \frac{1}{\gamma}v. \tag{3.36}$$

Denote

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{\gamma} & 0 \end{bmatrix}.$$

Then (3.36) can be written as

$$z' = Az \quad \text{for} \quad z = (u, p, v, q).$$

Every eigenvalue μ of A satisfies the following equation;

$$\begin{vmatrix} \mu & -1 & 0 & 0 \\ -\lambda & \mu & 0 & 0 \\ 0 & 0 & \mu & -1 \\ 0 & 0 & -\frac{1}{\gamma} & \mu \end{vmatrix} = (\mu^2 - \lambda)\left(\mu^2 - \frac{1}{\gamma}\right) = 0.$$

Thus, it follows that

$$\mu = \pm\sqrt{\lambda},\ \pm\sqrt{\frac{1}{\gamma}}.$$

Then the corresponding eigenvector (u, p, v, q) of μ satisfies

$$p = \mu u, \quad \lambda u = \mu p, \quad q = \mu v, \quad \frac{1}{\gamma} = \mu q.$$

Therefore we see that the normalized eigenvector is
(i) $(u, p, v, q) = \frac{1}{\sqrt{1+\lambda}}(1, \sqrt{\lambda}, 0, 0)$ if $\mu = \sqrt{\lambda}$,
(ii) $(u, p, v, q) = \frac{1}{\sqrt{1+\lambda}}(1, -\sqrt{\lambda}, 0, 0)$ if $\mu = -\sqrt{\lambda}$,
(iii) $(u, p, v, q) = \sqrt{\frac{\gamma}{1+\gamma}}\left(0, 0, 1, \sqrt{\frac{1}{\gamma}}\right)$ if $\mu = \sqrt{\frac{1}{\gamma}}$,
(iv) $(u, p, v, q) = \sqrt{\frac{\gamma}{1+\gamma}}\left(0, 0, 1, -\sqrt{\frac{1}{\gamma}}\right)$ if $\mu = -\sqrt{\frac{1}{\gamma}}$.

Hence if $x \to -\infty$, then

$$(u - 1, p, v, q) \approx Ae^{\sqrt{\lambda}x}\vec{p_1} + Be^{\sqrt{\frac{1}{\gamma}}x}\vec{p_2},$$

where $A < 0$ and $B > 0$ are some constants, and

$$\vec{p_1} = \left(e^{\sqrt{\lambda}x}, \sqrt{\lambda}e^{\sqrt{\lambda}x}, 0, 0\right) \quad \text{and} \quad \vec{p_2} = \left(0, 0, e^{\sqrt{\frac{1}{\gamma}}x}, \sqrt{\frac{1}{\gamma}}e^{\sqrt{\frac{1}{\gamma}}x}\right).$$

On the other hand, if $x \to +\infty$, then we see

$$(u-1, p, v, q) \approx Ce^{-\sqrt{\lambda}x}\vec{q_1} + De^{-\sqrt{\frac{1}{\gamma}}x}\vec{q_2},$$

where $C < 0$ and $D > 0$ are some constants. Furthermore,

$$\vec{q_1} = \left(e^{-\sqrt{\lambda}x}, \sqrt{\lambda}e^{-\sqrt{\lambda}x}, 0, 0\right) \quad \text{and} \quad \vec{q_2} = \left(0, 0, e^{-\sqrt{\frac{1}{\gamma}}x}, -\sqrt{\frac{1}{\gamma}}e^{-\sqrt{\frac{1}{\gamma}}x}\right).$$

Thus the convergence rate of non-trivial solutions for (GSP1) do not depend on α. Moreover, if λ (resp. $\frac{1}{\gamma}$) is sufficiently large, a nontrivial solution u (resp. v) can be like a needle shape.

Finally, we will derive important non-existence results of (GSP1). Put

$$\gamma_2(\alpha) = \frac{\alpha - 1}{\alpha^{\frac{\alpha}{\alpha-1}}}. \tag{3.37}$$

Then one can show the following nonexistence theorem in case $\lambda\gamma > 1$ through the same method of the proof for Theorem 3.3.

Theorem 3.29. *Let $\lambda\gamma > 1$ and $\gamma_2(\alpha)$ be the constant defined by* (3.37). *If $\lambda \le \frac{1}{\gamma_2(\alpha)}$, then* (GSP1) *has no nontrivial solution.*

Proof. Define a function $s := \lambda(u-1) + v$. Then s satisfies

$$s'' - \left\{\left(1 - \frac{1}{\lambda\gamma}\right)v^\alpha + \lambda\right\}s = \left(\frac{1}{\lambda\gamma} - 1\right)v(v^\alpha - \lambda v^{\alpha-1} + \lambda), \qquad \lim_{x\to\pm\infty} s = 0.$$

If $\lambda\gamma > 1$ and $\lambda \le \frac{1}{\gamma_2(\alpha)}$, then we have

$$\left(1 - \frac{1}{\lambda\gamma}\right)v^\alpha + \lambda > 0 \quad \text{and} \quad \left(\frac{1}{\lambda\gamma} - 1\right)v(v^\alpha - \lambda v^{\alpha-1} + \lambda) \le 0.$$

Therefore we can use the strong maximum principle to show that $s(x) > 0$ for $x \in \mathbf{R}$.

Addition of (3.28) and (3.29) implies that

$$u'' + \gamma v'' = \lambda u + v - \lambda. \tag{3.38}$$

Integrating (3.38) on $\mathbf{R^N}$ leads to

$$\int_{\mathbf{R}} (\lambda(u-1) + v)dx = 0. \tag{3.39}$$

But the left side of (3.39) is positive. This is a contradiction. Thus the proof is complete. □

Using the similar method as the proof for (i) of Theorem 3.4, one can show the following nonexistence theorem.

Theorem 3.30. *Let $\gamma_2(\alpha)$ be the constant defined* (3.37)*. If $\lambda\gamma < 1$ and $\gamma \geq \gamma_2(\alpha)$, then* (GSP1) *has no nontrivial solution.*

Remark 3.31. (i) *If $\alpha = 1$, then* (GSP1) *has only trivial solution.*
(ii) *When α is sufficiently large, we can derive the following estimate*;

$$\lim_{\alpha\to\infty} \gamma_2(\alpha) = 1.$$

Proof. Making use of (i) for Proposition 3.26, then

$$\begin{aligned} \gamma v'' &= v(1 - uv^{\alpha-1}) \\ &\geq v(\gamma v^{\alpha} - v^{\alpha-1} + 1) \\ &\geq 0, \end{aligned}$$

provided $\gamma \geq \gamma_2(\alpha)$. Since $\lim_{|x|\to\infty} v(x) = 0$, $v(x)$ is a constant. Therefore $u(x)$ is also a constant. Thus the proof is complete. □

Theorems 3.29 and 3.30 imply that nonexistence region of non-trivial solutions for (GSP1) can be larger if α is near one, but can be smaller if α is sufficiently large. Figure 3.4 is depicted as the nonexistence regions given by Theorems 3.29 and 3.30.

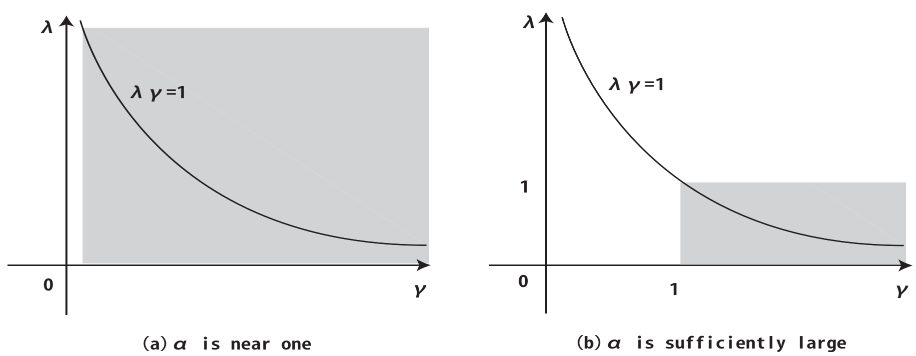


Figure 3.4: The non-existence regions in case $\alpha \approx 1$ and $\alpha \to \infty$.

3.3.2 Higher dimensional case

For higher dimensional case, one can derive the following existence and stability results when $\lambda\gamma = 1$ as in Section 3.1.

Theorem 3.32. *Let $\lambda\gamma = 1$ and $N \geq 2$. Suppose that $\gamma_1(\alpha)$ is the constant defined by* (3.31).
(a)*If $0 < \gamma < \gamma_1(\alpha)$, then problem* (GSP) *has a solution $(u(x), v(x))$ with the following properties*
(i)$u(x) = u(|x|), \quad v(x) = v(|x|), \quad u(x) = 1 - \gamma v(x) \quad for \quad x \in \mathbf{R^N}$;
(ii)$u'(r) > 0 \quad and \quad v'(r) < 0 \quad for \quad r = |x| > 0$;
(iii)$\lim_{r\to+\infty}(u(x), u'(x), v(x), v'(x)) = (1, 0, 0, 0)$.
(b) *If $\gamma \geq \gamma_1(\alpha)$, then problem* (GSP) *has no nontrivial solution.*

Remark 3.33. *As for the convergence rate of the radially symmetric solution $(u(r), v(r))$ given by* Theorem 3.32, *one can show that for any $\epsilon \in \left(0, \frac{1}{\gamma}\right)$*

$$\lim_{r\to\infty} u(r)e^{r\sqrt{\frac{1}{\gamma}-\epsilon}} < \infty \quad \text{and} \quad \lim_{r\to\infty} v(r)e^{r\sqrt{\frac{1}{\gamma}-\epsilon}} < \infty.$$

In order to study stability for the solutions given in Theorem 3.32, we will consider the following parabolic problem as in Theorem 3.24;

$$\begin{cases} s_t = \Delta s - (1+s)v^\alpha - \lambda s & \text{in} \quad \mathbf{R^N} \times (0, \infty), \\ \frac{\gamma}{d} v_t = \gamma \Delta v + (1+s)v^\alpha - v & \text{in} \quad \mathbf{R^N} \times (0, \infty), \\ s(x, 0) = s_0(x), \quad v(x, 0) = v_0(x) & \text{on} \quad \mathbf{R^N}, \end{cases} \tag{3.40}$$

where $s_0(x)$ and $v_0(x)$ represent the initial functions. Then the following theorem can be shown by the same method of the proof for Theorem 3.24.

Theorem 3.34. *Let $\lambda\gamma = 1$ and define functional space $W = L^2(\mathbf{R^N}) \times L^2(\mathbf{R^N})$. If $d = 1$, then the radially symmetric solutions given by* Theorem 3.32 *are linearly unstable in W.*

Proof. The proof is almost same as that of Theorem 3.24. Therefore we omit it. □

Finally, we establish the following non-existence result of (GSP) for higher dimensional case.

Theorem 3.35. *Let $N \geq 2$ and $\gamma_2(\alpha)$ be the constant defined in* (3.31)*. If*

$$\lambda \leq \frac{1}{\gamma_2(\alpha)} \quad \textit{and} \quad \gamma \geq \gamma_2(\alpha),$$

then (GSP) *has no nontrivial solutions.*

Proof. The proof of the theorem is almost same as that of Theorems 3.29 and 3.30. So we omit it here. □

3.3.3 Global stability for constant solution

In this subsection, we will show that the constant solution $(1,0)$ of (GSP) is globally stable for some special parameter case. Hence we treat with the following Cauchy problem;

$$\begin{cases} u_t = \Delta u - uv^\alpha + \lambda(1-u) & \text{in} \quad \mathbf{R}^\mathbf{N} \times (0,\infty), \\ \frac{\gamma}{d} v_t = \gamma \Delta v + uv^\alpha - v & \text{in} \quad \mathbf{R}^\mathbf{N} \times (0,\infty), \\ \lim_{|x|\to\infty}(u,v) = (1,0) & \text{for} \quad t > 0, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{on} \quad \mathbf{R}^\mathbf{N}. \end{cases} \tag{3.41}$$

In order to establish global stability results, we will need the following maximum principle [4, 11], comparison principle, and a solution property for the linear heat equation.

Theorem 3.36 ([4, 11]). *Let $z(x,t)$ be a continuous bounded function of*

$$\begin{cases} z_t - \Delta z + cz \le 0 & in \quad \mathbf{R}^\mathbf{N} \times (0,\infty), \\ \lim_{|x|\to\infty} z(x,t) \le 0 & for \quad t > 0, \\ z(x,0) \le 0 & on \quad \mathbf{R}^\mathbf{N}. \end{cases}$$

Here $c = c(x,t)$ is a non-negative smooth function in $\mathbf{R}^\mathbf{N} \times (0,\infty)$. Then it follows that

$$z(x,t) \le 0 \quad in \quad \mathbf{R}^\mathbf{N} \times (0,\infty).$$

Theorem 3.37. *Let w and z be smooth functions on $\mathbf{R}^\mathbf{N} \times (0,\infty)$. Suppose that w and z satisfy*

$$\begin{aligned} & w_t - d\Delta w - f(w) \ge z_t - d\Delta z - f(z), && (x,t) \in \mathbf{R}^\mathbf{N} \times (0,\infty), \\ & w(x,0) \ge z(x,0), && x \in \mathbf{R}^\mathbf{N}, \\ & \lim_{|x|\to+\infty} w(x,t) \ge \lim_{|x|\to+\infty} z(x,t), && t \in (0,\infty). \end{aligned}$$

Then the following inequality holds true;

$$w(x,t) \ge z(x,t) \quad for \quad (x,t) \in \mathbf{R}^\mathbf{N} \times (0,\infty).$$

Theorem 3.38. *Assume that $z(x,t)$ is a solution for the following Cauchy problem*;

$$z_t = \Delta z - Az \quad in \quad \mathbf{R}^\mathbf{N}, \quad z(x,0) = z_0,$$

where A is a non-negative constant, and an initial function satisfies $z_0 \in L^1(\mathbf{R^N})$. Then it holds that

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R^N}} |z(x,t)| = 0.$$

Here we define the following functional space.

Notation 3.39. *We denote $Y = C_B(\mathbf{R^N}) \times L^2(\mathbf{R^N})$. Here $f \in C_B(\mathbf{R^N})$ is a continuous bounded function on $\mathbf{R^N}$.*

Using these theorems, we can obtain the following lemma concerned with a priori estimates and asymptotic behavior of solutions for (3.41).

Lemma 3.40. *Let (u, v) be any solution for (3.41). Suppose that $(u_0(x), v_0(x)) \in Y \times Y$. Then there exists a positive constant M such that*

$$0 \le u(x,t) \le M, \quad v(x,t) \ge 0 \quad for \quad (x,t) \in \mathbf{R^N} \times (0,\infty).$$

Moreover, it follows that

$$\limsup_{t\to+\infty} u(x,t) \le 1 \quad in \quad x \in \mathbf{R^N}.$$

Proof. Assume that $u(x_0, t_0) < 0$ for some $x_0 \in \mathbf{R^N}$ and $t_0 > 0$. Since $\Delta u(x_0, t_0) \ge 0$, then we find that

$$u_t(x_0,t_0) = \Delta u(x_0,t_0) - u(x_0,t_0)v(x_0,t_0)^\alpha + \lambda(1 - u(x_0,t_0)) > \lambda.$$

This is a contradiction. Hence $u(x,t) \ge 0$ for any $x \in \mathbf{R^N}$ and $t > 0$.

Recall that

$$u_t = \Delta u - uv^2 + \lambda(1-u) \le \Delta u + \lambda(1-u) \quad \text{in} \quad \mathbf{R^N} \times (0,\infty).$$

Using Theorems 3.37 and 3.38, then

$$\limsup_{t\to+\infty} u(x,t) \le 1 \quad \text{for} \quad x \in \mathbf{R^N}.$$

Finally, note that

$$v_t - d\Delta v + \frac{d}{\gamma}v = \frac{d}{\gamma}uv^\alpha \ge 0 \quad \text{in} \quad \mathbf{R^N} \times (0,\infty), \qquad \lim_{|x|\to\infty} v = 0 \quad \text{for} \quad t > 0.$$

Therefore we see from Theorem 3.36 that

$$v(x,t) \ge 0 \quad \text{in} \quad \mathbf{R^N} \times (0,\infty).$$

Thus the proof is complete. $\square$

Next, we will derive some information about the upper bound and asymptotic behavior for v in case $d = 1$.

Lemma 3.41. *Assume $d = 1$ and (u, v) is any solution for (3.41). Define a function $p(x, t) := u(x, t) + \gamma v(x, t) - 1$. If $\lambda\gamma \leq 1$ and $p(x, 0) \in Y$, there exists a positive constant M such that $p(x, t) \leq M$ for any $x \in \mathbf{R^N}$ and $t > 0$. Moreover,*

$$\limsup_{t\to+\infty} p(x, t) \leq 0 \quad for \quad x \in \mathbf{R^N}.$$

Proof. Observe that $p(x, t)$ satisfies the following equations;

$$\begin{cases} p_t - \Delta p + \lambda p = (\lambda\gamma - 1)v \leq 0 & \text{in} \quad \mathbf{R^N} \times (0, \infty), \\ \lim_{|x|\to\infty} p = 0 & \text{for} \quad t > 0. \end{cases}$$

It follows from Theorems 3.37 and 3.38 that

$$\limsup_{t\to+\infty} p(x, t) \leq 0 \quad \text{in} \quad x \in \mathbf{R^N}.$$

Thus the proof is complete. □

Lemma 3.42. *Suppose that $d = 1$ and (u, v) is any solution for (3.41). Define $q(x) = u(x) + \gamma v(x) - \lambda\gamma$. If $\lambda\gamma \geq 1$ and $q(x, 0) \in Y$ then there exists some positive constant M such that*

$$q(x, t) \leq M \quad for \quad (x, t) \in \mathbf{R^N} \times (0, \infty),$$

and

$$\limsup_{t\to+\infty} q(x, t) \leq 0 \quad for \quad x \in \mathbf{R^N}.$$

Proof. Notice that $q(x, t)$ satisfies

$$\begin{cases} q_t - \Delta q + \frac{1}{\gamma} q = (1 - \lambda\gamma)u \leq 0 & \text{in} \quad \mathbf{R^N} \times (0, \infty), \\ \lim_{|x|\to\infty} q = 1 - \lambda\gamma \leq 0 & \text{for} \quad t > 0. \end{cases}$$

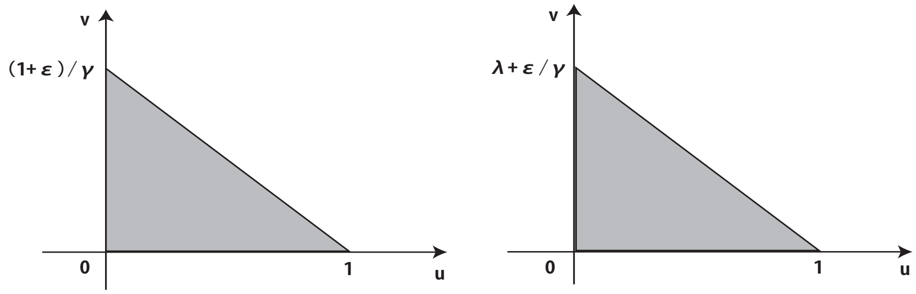
As in the proof of Lemma 3.41, these inequalities complete the proof. □

Now, set two-dimensional region D_ϵ as follows;

$$D_\epsilon = \begin{cases} \{(u, v) \in \mathbf{R^2} : 0 < u \leq 1, v \geq 0, u + \gamma v \leq 1 + \epsilon\} & \text{if} \quad \lambda\gamma \leq 1, \\ \{(u, v) \in \mathbf{R^2} : 0 < u \leq 1, v \geq 0, u + \gamma v \leq \lambda\gamma + \epsilon\} & \text{if} \quad \lambda\gamma \geq 1, \end{cases} \tag{3.42}$$

for any fixed positive constant ϵ. See Figure 3.5.

Making use of Lemmas 3.40-3.42, we can obtain the following theorem.



(a) λγ < 1　　(b) λγ > 1

Figure 3.5: The region D_ϵ defined by (3.42).

Theorem 3.43. *Let $d = 1$. Then for any $(u_0(x), v_0(x)) \in Y \times Y$, there exists a unique time global solution $(u(x,t), v(x,t))$ of* (3.41) *such that*

$$(u(x,t), v(x,t)) \in D_\epsilon \quad for \quad (x,t) \in \mathbf{R^N} \times (T, \infty),$$

if T is sufficiently large constant.

Finally, we will show that the constant solution $(1, 0)$ is globally stable if we impose some suitable conditions with the parameters for (3.41). Our first stability result is in case $\lambda\gamma \leq 1$ and $d = 1$.

Theorem 3.44. *Suppose that $\lambda\gamma \leq 1$ and $d = 1$. If $\gamma > \gamma_2(\alpha)$, then the corresponding solution (u, v) of* Theorem 3.43 *satisfies*

$$\lim_{t\to\infty} (u, v) = (1, 0) \quad uniformly \ in \ \mathbf{R^N},$$

for any initial data $(u_0(x), v_0(x)) \in Y \times Y$. Here $\gamma_2(\alpha)$ is defined by (3.37).

Proof. It follows from Theorem 3.43 that

$$\begin{aligned} \max_{x\in R, t>T} uv^{\alpha-1} &\leq \max_{x\in R, t>T} (1 - \gamma v + \epsilon) v^{\alpha-1} \\ &\leq \max_{x\in R, t>T} (1 - \gamma v) v^{\alpha-1} + \max_{x\in R, t>T} \epsilon v^{\alpha-1} \\ &\leq \frac{\gamma_2(\alpha)}{\gamma} + \epsilon \left(\frac{1}{\gamma}\right)^{\alpha-1} \\ &\leq 1, \end{aligned} \tag{3.43}$$

provided $\gamma > \gamma_2(\alpha)$ and T is sufficiently large. Then we see

$$v_t = \Delta v + \frac{1}{\gamma} v(uv^{\alpha-1} - 1) \leq \Delta v \quad \text{for} \quad (x,t) \in \mathbf{R^N} \times (T, \infty).$$

According to Theorems 3.37 and 3.38, we find

$$\lim_{t\to\infty} v(x,t) = 0 \quad \text{uniformly in } x \in \mathbf{R^N}.$$

Thus for any constant $\delta > 0$ there exists a large constant T' not depending on x such that

$$|v(x,t)| \le \delta \quad \text{in} \quad \mathbf{R^N} \times (T', \infty).$$

It follows from (3.41) that

$$u_t \ge \Delta u - \delta^\alpha u + \lambda(1-u) \quad \text{in} \quad \mathbf{R^N} \times (T', \infty).$$

Hence

$$\liminf_{t\to\infty} u(x,t) \ge \frac{\lambda}{\lambda + \delta^\alpha}.$$

Taking $\delta \to 0$, then we deduce that

$$\lim_{t\to\infty} u(x,t) = 1 \quad \text{uniformly in } x \in \mathbf{R^N}.$$

Thus the proof is complete. □

Finally, we establish the following global stability result concerned with case $\lambda\gamma \ge 1$ and $d = 1$.

Theorem 3.45. *Assume that $d = 1$ and $\lambda\gamma \ge 1$. Let $\gamma_2(\alpha)$ be defined in* (3.37). *If $\lambda < \frac{1}{\gamma_2(\alpha)}$, then the solution (u,v) given by* Theorem 3.43 *satisfies*

$$\lim_{t\to\infty} (u,v) = (1,0) \quad uniformly \ in \ \mathbf{R^N},$$

for any initial condition $(u_0(x), v_0(x)) \in Y \times Y$.

Proof. Define a new function $s = \lambda(u-1) + v$. Then $s(x,t)$ satisfies

$$s_t - \Delta s + \left\{\lambda + \left(1 - \frac{1}{\lambda\gamma}\right) v\right\} s = \left(1 - \frac{1}{\lambda\gamma}\right) v(v^\alpha - \lambda v^{\alpha-1} + \lambda), \tag{3.44}$$

for $(x,t) \in \mathbf{R^N} \times (0,\infty)$. Since $\lambda < \frac{1}{\gamma_2(\alpha)}$, then the right side of (3.25) is non-negative. Because

$$\lim_{|x|\to+\infty} s(x,t) = 0 \quad \text{for} \quad t > 0,$$

it follows from Theorems 3.37 and 3.38 that

$$\limsup_{t\to+\infty} s(x,t) \le 0 \quad \text{for} \quad x \in \mathbf{R^N}.$$

Choosing T as sufficiently large, then we deduce

$$v_t \le \Delta v - \frac{1}{\lambda\gamma} v(v^{\alpha} - \lambda v^{\alpha-1} + \lambda) \le \Delta v \quad \text{for} \quad (x,t) \in \mathbf{R^N} \times (T, \infty).$$

As in the proof of Theorem 3.44, this inequality completes the proof of Theorem 3.45. □

Chapter 4

Monotone front solution for generalized stationary problem

In this chapter, we will primarily study the following generalized boundary value problem;

$$\text{(SP3)} \qquad \begin{cases} u'' - uv^\alpha + \lambda(1-u) = 0, & \text{in} \quad \mathbf{R}, \\ \gamma v'' + uv^\alpha - v = 0, & \text{in} \quad \mathbf{R}, \\ (u,v)(-\infty) = (1,0), \quad (u,v)(+\infty) = (u_+, v_+), \end{cases}$$

where α is a constant satisfying $\alpha > 1$. Furthermore, (u_+, v_+) is a pair of constants which satisfies

$$u_+ = 1 - \frac{1}{\lambda} v_+, \tag{4.1}$$

and v_+ is the smallest positive solution of the following equation

$$v^\alpha - \lambda v^{\alpha-1} + \lambda = 0. \tag{4.2}$$

A solution of (SP3) is generally called front solution. And it is regarded as stationary chemical front wave and plays an important role for pattern formation of the Gray-Scott model.

This chapter mainly treats with the following two problems;

(a) Existence and non-existence of monotone front solutions in case $\alpha = 2$.

(b) Existence and stability of monotone front solutions in case $\lambda\gamma = 1$ and $\alpha > 1$.

Here a monotone front solution of (SP3) is defined as follows;

Definition 4.1. *A monotone front solution is a solution for* (SP3) *which satisfies*

$$u'(x) \le 0, \quad v'(x) \ge 0 \quad for \quad x \in \mathbf{R}.$$

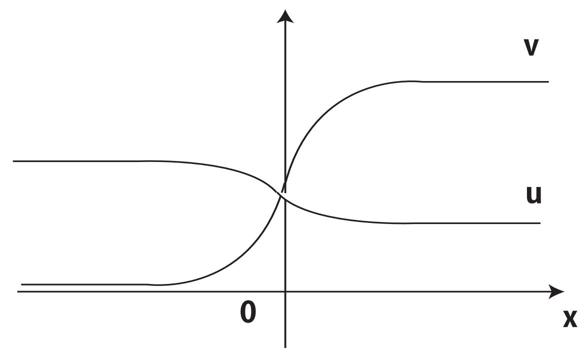


Figure 4.1: The profile of monotone front solution for (SP3).

Figure 4.1 shows the profile of monotone front solution.

In Section 4.1, we mainly deal with the problem (a). Hale-Peletier-Troy have studied (SP3) in case

$$\lambda\gamma \approx 1 \quad \text{and} \quad \gamma \approx \frac{2}{9}, \tag{4.3}$$

and obtained a unique monotone front solution. However, if λ and γ do not satisfy (4.3), the solution set of (SP3) has been largely open problem.

Let

$$(u_2, v_2) = \left(\frac{\lambda - \sqrt{\lambda^2 - 4\lambda}}{2\lambda}, \frac{\lambda + \sqrt{\lambda^2 - 4\lambda}}{2} \right). \tag{4.4}$$

Then Section 4.1 gives the following non-existence results of monotone front solution.

Theorem 4.2. *Let $\lambda\gamma \leq 1$ and $\alpha = 2$. Then* (SP3) *has no monotone front solution provided*

$$\gamma > \min\left\{ \frac{4v_2 - 6}{3v_2^2}, \frac{6}{\lambda(v_2 + 3)} \right\},$$

where v_2 is defined by (4.4).

Theorem 4.3. *Let $\lambda\gamma \geq 1$ and $\alpha = 2$. Then* (SP3) *admits no monotone front solution provided*

$$\gamma < \max\left\{ \frac{4v_2 - 6}{3v_2^2}, \frac{6}{\lambda(v_2 + 3)} \right\}.$$

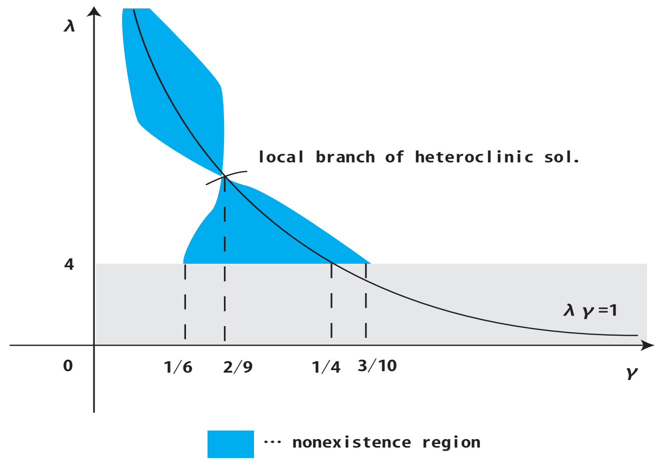


Figure 4.2: The nonexistence regions of monotone front solution for (SP3).

Figure 4.2 represents the non-existence regions given in these theorems.

Concerned with the problem (b), Hale-Peletier-Troy have discussed (SP3) in case $\lambda\gamma = 1$ and $\gamma = \gamma_1$. Here

$$\gamma_1 = \frac{2^{1/(\alpha-1)}(\alpha-1)(\alpha+2)}{\{\alpha(\alpha+1)\}^{\alpha/(\alpha-1)}}. \tag{4.5}$$

Then they have constructed the following monotone front solution.

Theorem 4.4 ([18]). *Let γ_1 be the constant defined by* (4.5)*. Suppose that $\lambda\gamma = 1$ and $\gamma = \gamma_1$. Then* (SP3) *has a unique solution (φ, ψ) which satisfies*

$$\varphi + \gamma\psi - 1 = 0, \quad \varphi' < 0, \quad \psi' > 0 \quad \text{for} \quad x \in \mathbf{R}.$$

In order to discuss stability of the monotone front solution, we must consider the following non-stationary problem;

$$\text{(NSP3)} \qquad \begin{cases} u_t = u'' - uv^\alpha + \lambda(1-u), & \text{in} \quad \mathbf{R} \times (0,\infty), \\ \frac{\gamma}{d} v_t = \gamma v'' + uv^\alpha - v, & \text{in} \quad \mathbf{R} \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{on} \quad \mathbf{R}. \end{cases}$$

Here $u_0(x)$ and $v_0(x)$ are non-negative continuous functions in $\mathbf{R}$. Furthermore, λ, γ and d are positive parameters.

Notation 4.5. *$C_B(\mathbf{R})$ is a function space of continuous bounded functions on $\mathbf{R}$.*

Let $\zeta = (\varphi, \psi)$ be the monotone front solution given by Theorem 4.4. If the initial data $z_0 = (u_0(x), v_0(x)) \in C_B(\mathbf{R}) \times C_B(\mathbf{R})$, then we obtain our main stability result in Section 4.2.

Theorem 4.6. *Suppose that $\lambda\gamma = 1$ and γ satisfies* (4.5)*. If $d = 1$ and $\alpha > 1$, then ζ is asymptotically stable in the following sense: there exist constants $\delta, M, \kappa > 0$, and $\xi \in \mathbf{R}$ such that, if*

$$\|z_0 - \zeta(\cdot)\|_\infty < \delta,$$

then the solution $z(\cdot, t) = (u, v)$ of (NSP3) *corresponding to initial data z_0 satisfies*

$$\|z(\cdot, t) - \zeta(\cdot + \xi)\|_\infty \leq M\|z(\cdot, 0) - \zeta(\cdot)\|_\infty e^{-\kappa t}.$$

This theorem is an extension of the result of [18]. Hale et al. have used the result of Tistchmarsh [34] for proving Theorem in case $\alpha = 2$. However, their method can not be applied for the stability problem in case $\alpha > 1$. To overcome this difficulty, we will focus on zero point of eigenfunctions for linearized eigenvalue problem.

Finally, traveling front solution related to (SP3) is discussed in Section 4.3. We primarily study the following equations;

$$\begin{cases} U'' + cU' - UV^\alpha + \lambda(1 - U) = 0, & \xi \in \mathbf{R}, \\ \gamma V'' + \frac{c\gamma}{d} V' + UV^\alpha - V = 0, & \xi \in \mathbf{R}. \end{cases} \tag{4.6}$$

Here the prime is the derivative with respect to $\xi := x - ct$. Moreover, λ, γ and d are positive parameters, and c is a constant called traveling wave speed. Here we impose the following boundary condition;

$$(U(-\infty), V(-\infty)) = (u_+, v_+), \quad (U(+\infty), V(+\infty)) = (1, 0), \tag{4.7}$$

where (u_+, v_+) is defined by (4.1) and (4.2).

In Section 4.3, we treat with existence and stability for traveling front solutions satisfying (4.6) and (4.7). Especially, we will show that traveling front solutions are linearly stable for the special parameter case.

4.1 Proofs of non-existence results

If $\alpha = 2$, then (SP3) can be written as

$$u'' - uv^2 + \lambda(1 - u) = 0, \qquad \text{in } \mathbf{R}, \tag{4.8}$$

$$\gamma v'' + uv^2 - v = 0, \qquad \text{in } \mathbf{R}, \tag{4.9}$$

$$(u, v)(-\infty) = (1, 0), \quad (u, v)(+\infty) = (u_2, v_2),$$

where (u_2, v_2) is defined by (4.4). This problem is called (SP3-1). In this section, we will deal with existence and non-existence of monotone front solutions defined by Definition 4.1.

Then the following theorem is well known in case $\lambda\gamma = 1$.

Theorem 4.7 ([18]). *Let* $\lambda\gamma = 1$. *If* $\gamma = \frac{2}{9}$ *then* (SP3-1) *has a unique solution* (φ, ψ) *which satisfies the following properties*;

$$\varphi + \gamma\psi - 1 = 0, \quad \varphi' < 0, \quad \psi' > 0 \quad \text{for} \quad x \in \mathbf{R}.$$

If we purturbe $\gamma = \frac{2}{9}$, then the following existence theorem has also shown by Hale-Peletier-Troy.

Theorem 4.8 ([18]). *Let* (φ, ψ) *denote the monotone front solution of Theorem* 4.7 *and define* $\gamma_* = \frac{2}{9}$. *Then there is an* $\epsilon_0 > 0$ *such that there exist smooth branches of front solutions* $\{(u(\epsilon), v(\epsilon), \gamma(\epsilon)) : |\epsilon| < \epsilon_0\}$ *of* (SP3-1) *such that*

$$u(\epsilon) \to \varphi, \quad v(\epsilon) \to \psi \quad \text{and} \quad \text{as} \quad \epsilon \to 0$$

uniformly on $\mathbf{R}$. *Moreover,*

$$\gamma'(0) = \frac{\pi^2 - 7}{3}\gamma_*.$$

Making use of Auto 97 [8], Hale-Peletier-Troy [18] have also made numerical simulation about continuation for the branch of the front solutions. The curve named C in Figure 4.3 is depicted as the global branch for front solutions.

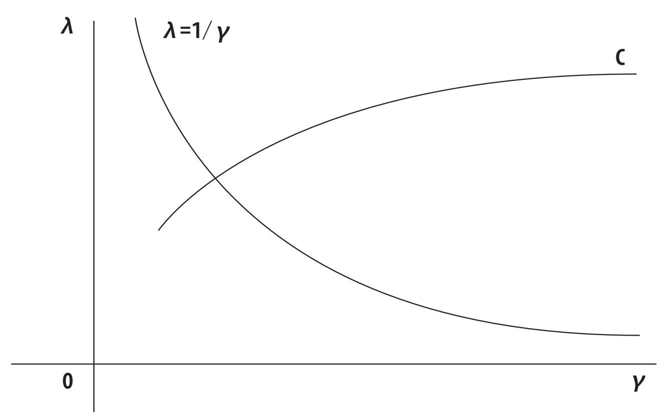


Figure 4.3: The global branch of front solutions (cf [18]).

Next, we will discuss the non-existence results given by Theorems 4.2 and 4.3. To prove these theorems, we will need the following a priori estimates derived by the strong maximum principle in Section 3.2.

Lemma 4.9. *Let* (u, v) *be any solution for* (SP3-1). *Define* $p(x) = u(x)+\gamma v(x)-1$, *then the following inequalities hold true.*
(i) *If* $\lambda\gamma \leq 1$, *then* $p(x) < 0$ *for any* $x \in \mathbf{R}$,
(ii) *If* $\lambda\gamma \geq 1$, *then* $p(x) > 0$ *for any* $x \in \mathbf{R}$.

Proof. We only prove the case $\lambda\gamma \leq 1$. The other case can be treated in almost the same way. Adding (4.8) with (4.9), we see

$$p'' - \lambda p = (1 - \lambda\gamma)v \geq 0.$$

By taking $x \to \pm\infty$, then it follows that

$$\lim_{x\to-\infty} p(x) = 0,$$

and

$$\begin{aligned}\lim_{x\to+\infty} p(x) &= u_2 + \gamma v_2 - 1,\\ &\leq u_2 + \frac{1}{\lambda} v_2 - 1\\ &= 0.\end{aligned}$$

Therefore one can see from the strong maximum principle to show that

$$p(x) < 0 \quad \text{for} \quad x \in \mathbf{R}.$$

This completes the proof of Lemma 4.9. □

Lemma 4.10. *Suppose that* (u, v) *is any solution for* (SP3-1). *Let* $q(x) = u(x) + \gamma v(x) - \lambda\gamma$. *Then the following estimates hold true.*
(i) *If* $\lambda\gamma \leq 1$, *then* $q(x) > 0$ *for any* $x \in \mathbf{R}$,
(ii) *If* $\lambda\gamma \geq 1$, *then* $q(x) < 0$ *for any* $x \in \mathbf{R}$.

Proof. We give the proof only for the case $\lambda\gamma \leq 1$. The proof of the other case is almost the same. If we add (4.8) with (4.9), then

$$q'' - \lambda q = \left(\lambda - \frac{1}{\gamma}\right) u \geq 0.$$

Taking $x \to \pm\infty$, then we have

$$\lim_{x\to-\infty} q(x) = 1 - \lambda\gamma \geq 0,$$

and

$$\begin{aligned}\lim_{x\to+\infty} q(x) &= u_2 + \gamma v_2 - \lambda\gamma, \\ &\geq u_2 + \frac{1}{\lambda} v_2 - 1 \\ &= 0.\end{aligned}$$

Making use of the strong maximum principle, we deduce that

$$q(x) > 0 \quad \text{for} \quad x \in \mathbf{R}.$$

□

Using Lemmas 4.9-4.10, one can prove Theorems 4.2 and 4.3.

Proof of Theorems 4.2 and 4.3. We first prove Theorem 4.2 in case $\lambda\gamma \leq 1$. According to Lemma 4.9, we have

$$\begin{aligned}\gamma v'' &= v - uv^2 \\ &> v + (\gamma v - 1)v^2 \\ &= \gamma v^3 - v^2 + v.\end{aligned} \tag{4.10}$$

Note that $v'(x) \geq 0$ for any $x \in \mathbf{R}$. If we multiply (4.10) by v', then

$$\gamma v'' v' \geq (\gamma v^3 - v^2 + v)v'. \tag{4.11}$$

Integrating (4.11) from $-\infty$ to ∞, then we find

$$3\gamma v_2^2 - 4v_2 + 6 \leq 0. \tag{4.12}$$

Therefore if

$$\gamma > \frac{4v_2 - 6}{3v_2^2},$$

then the left hand side of (4.12) is positive. It is a contradiction.

On the other hand, it follows from Lemma 4.10 that

$$\gamma v'' \leq \gamma v^3 - \lambda\gamma v^2 + v. \tag{4.13}$$

Multiplying (4.13) by v' and integrating on $\mathbf{R}$, then

$$3\gamma v_2^2 - 4\lambda\gamma v_2 + 6 \geq 0. \tag{4.14}$$

Note that v_2 satisfies

$$v_2^2 - \lambda v_2 + \lambda = 0.$$

Since

$$\gamma > \frac{6}{\lambda(v_2 + 3)},$$

the left hand side of (4.14) is negative. This is a contradiction. Thus the proof is complete for $\lambda\gamma \leq 1$. The proof of $\lambda\gamma \geq 1$ is almost the same, so we omit it. □

4.2 Generalized stationary problem

This section is devoted to prove Theorem 4.6. We always assume that

$$\lambda\gamma = d = 1, \quad \gamma = \gamma_1,$$

where γ_1 is defined by (4.5).

Let L be the linearized operator about ζ and denote its spectrum by $\sigma(L)$: we define for $\eta = (u, v)^t$

$$L\eta = -\begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + \begin{pmatrix} \psi^\alpha + \frac{1}{\gamma} & \alpha\varphi\psi^{\alpha-1} \\ -\frac{1}{\gamma}\psi^\alpha & -\frac{\alpha}{\gamma}\varphi\psi^{\alpha-1} + \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is well known that the following stability result holds true.

Theorem 4.11 ([19]). *Assume $\sigma(L) = \{0\} \cup \sigma^*$, where 0 is a simple eigenvalue and $\operatorname{Re}\sigma^* \geq \nu$ with some $\nu > 0$. Then ζ is stable in the following sense: for any small neighborhood U in $C_B(\mathbf{R}) \times C_B(\mathbf{R})$ of ζ, there are constants $\kappa > 0, M > 0$ and $\xi \in \mathbf{R}$ such that, for any initial data in U, the corresponding solution $z(x,t)$ of* (NSP3) *satisfies:*

$$\|z(\cdot, t) - \zeta(\cdot + \xi)\|_\infty \leq M\|z(\cdot, 0) - \zeta(\cdot)\|_\infty e^{-\kappa t}.$$

First we study the essential spectrum $\sigma_e(L)$ of L.

Lemma 4.12. *It holds that*

$$\sigma_e(L) \subset \{\mu \in \mathbf{R} \,|\, \mu \geq c\},$$

where

$$c = \begin{cases} \frac{1}{\gamma} & \alpha \geq 2, \\ \frac{\alpha(\alpha-1)}{2\gamma} & 1 < \alpha < 2. \end{cases}$$

Proof. In order to show Lemma 4.12, we will use Theorem A.2 in the monograph of Henry [19, p140]. His result asserts

$$\sigma_e(L) \subset S_+ \cup S_- \quad \text{for} \quad L\eta = -\eta_{xx} + N(x)\eta,$$

where $S_\pm = \{\mu \,|\, \det(\tau^2 I + N_\pm - \mu I) = 0 \text{ for some real } \tau, -\infty < \tau < \infty\}$ with $N(x) \to N_\pm$ as $x \to \pm\infty$.

We begin to determine S_+. Since L is written as

$$L\eta = -\eta_{xx} + \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \eta,$$

we see that

$$a(x) \to a_+, \quad b(x) \to b_+, \quad c(x) \to c_+, \quad d(x) \to d_+ \quad \text{as} \quad x \to +\infty$$

with

$$a_+ = v_+^\alpha + \frac{1}{\gamma}, \quad b_+ = \alpha u_+ v_+^{\alpha-1}, \quad c_+ = -\frac{1}{\gamma} v_+^\alpha, \quad d_+ = -\frac{\alpha}{\gamma} u_+ v_+ + \frac{1}{\gamma}.$$

Recall that $\mu \in S_+$ satisfies

$$\det \left\{ \tau^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix} - \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = 0.$$

Hence

$$\mu = \tau^2 + \frac{A \pm B}{2},$$

with

$$\begin{aligned} A &= a_+ + d_+ = \frac{2}{\gamma} - \frac{\alpha}{\gamma} u_+ v_+^{\alpha-1} + v_+^\alpha, \\ B &= \sqrt{(a_+ - d_+)^2 + 4 b_+ c_+} = \left| v_+^\alpha - \frac{\alpha}{\gamma} u_+ v_+^{\alpha-1} \right|, \\ u_+ &= 1 - \gamma v_+ = \frac{2}{\alpha(\alpha-1)}, \\ v_+ &= \left(\frac{\alpha(\alpha+1)}{2} \right)^{1/\alpha-1}. \end{aligned}$$

After some calculations, one can show

$$S_+ = \begin{cases} \{\mu \in \mathbf{R} \,|\, \mu \geq \frac{1}{\gamma}\}, & \alpha \geq 2, \\ \{\mu \in \mathbf{R} \,|\, \mu \geq \frac{\alpha(\alpha-1)}{2\gamma}\}, & 1 < \alpha < 2. \end{cases}$$

Similarly, we have

$$S_- = \left\{ \mu \in \mathbf{R} \,|\, \mu \geq \frac{1}{\gamma} \right\}.$$

Therefore, the proof of Lemma 4.12 is complete. □

Next, we consider isolated eigenvalues with finite multiplicity. We introduce a new function $p = u + \gamma v$, then $L\eta = \mu\eta$ is equivalent to

$$\begin{cases} -p'' + \frac{1}{\gamma} p = \mu p, \\ -v'' - \frac{\varphi^\alpha}{\gamma} p + \{(1+\alpha)\varphi^\alpha - \frac{\alpha}{\gamma}\varphi^{\alpha-1} + \frac{1}{\gamma}\} v = \mu v. \end{cases} \tag{4.15}$$

Denote (4.15) by $\hat{L}\theta = \mu\theta$ for $\theta = (p, v)^t$. Clearly μ is an eigenvlaue of L if and only if μ is an eigenvalue of $\hat{L}$. By the property of $\hat{L}$, the eigenvalue μ must be real. Furthermore, the following lemma holds true.

Lemma 4.13. *$\mu = 0$ is a simple eigenvalue.*

Proof. The proof is almost the same as [17]. □

Lemma 4.14. *There is no negative eigenvalue of L.*

Proof. Suppose that μ is a negative eigenvalue of L. By (4.15), $-p'' + (\frac{1}{\gamma} - \mu)p = 0$. Since p is uniformly bounded, p must be identically zero. Therefore, there exists $v \neq 0$ such that

$$Tv \stackrel{\text{def}}{=} -v'' + f'(\psi)v = \mu v, \tag{4.16}$$

where

$$f'(\psi) = (1+\alpha)\psi^\alpha - \frac{\alpha}{\gamma}\psi^{\alpha-1} + \frac{1}{\gamma}.$$

Note that ψ' is the eigenfunction corresponding to zero eigenvalue, then

$$T\psi' = 0. \tag{4.17}$$

Now making use of (4.16) and (4.17), and integrating by parts, we see

$$\mu \int_{-r}^{r} v\psi' dx = g(r),$$

where

$$g(r) = -v'(r)\psi'(r) + v'(-r)\psi'(-r) + v(r)\psi''(r) - v(-r)\psi''(-r).$$

Since $f'(\psi(r)) \to \frac{\alpha(\alpha-1)}{2\gamma}$ as $r \to \infty$ and $f'(\psi(r)) \to \frac{1}{\gamma}$ as $r \to -\infty$, v and ψ' decays exponentially to zero as $r \to \pm\infty$. Therefore, it holds that

$$g(r) \to 0 \quad \text{as} \quad r \to +\infty.$$

Because μ is negative, we see

$$\int_{-\infty}^{\infty} v\psi' dx = 0.$$

Since ψ' is positive, v has at least one zero point. Recall that v decays exponentially to zero as $r \to +\infty$; then we can assume that there exists $x_* \in \mathbf{R}$ such that

$$v(x_*) = 0 \quad \text{and} \quad v > 0 \quad \text{in} \quad (x_*, \infty).$$

Now v and ψ' satisfy the following equations:

$$-v'' + f'(\psi)v = \mu v, \tag{4.18}$$
$$-(\psi')'' + f'(\psi)\psi' = 0. \tag{4.19}$$

Multiply (4.18)(resp. (4.19)) by ψ'(resp. v); then we get

$$-v''\psi' + (\psi')''v = \mu v\psi'. \tag{4.20}$$

Integrate (4.20) from x_* to $+\infty$. Integrating by parts and making use of $v(x_*) = 0$, one can see

$$v'(x_*)\psi'(x_*) = \mu \int_{x_*}^{+\infty} v\psi' dx.$$

Recall that $v'(x_*) > 0, \psi'(x_*) > 0$ and $\int_{x_*}^{+\infty} v\psi' dx > 0$; then $\mu > 0$. This is a contradiction. □

Making use of the above Lemmas 4.12-4.14, we can prove Theorem 4.6.

Proof of Theorem 4.6. To complete the proof of Theorem 4.6, it is sufficient to verify the assumption of Theorem 4.11. Lemmas 4.13 and 4.14 show that there exists the least positive isolated eigenvalue β with finite multiplicity. Spectrum of L consists of essential spectrum and isolated eigenvalues with finite multiplicity. Therefore combining with Lemma 4.12, we see

$$\sigma(L) = \{0\} \cup \sigma^* \quad \text{with} \quad \operatorname{Re}\sigma^* \geq \nu,$$

where

$$\nu = \min\{c, \beta\} > 0.$$

Thus the proof is complete. □

4.3 Traveling front Solution

In this section, we will discuss existence and stability of traveling wave solutions for the original problem (P). First, we define the traveling wave as follows;

Definition 4.15. *A traveling wave solution is a non-negative solution of* (P) *which satisfies the following equation;*

$$(u(x,t), v(x,t)) = (U(\xi), V(\xi)) \quad \text{with} \quad \xi = x - ct\,(c > 0),$$

where c is called a traveling wave speed.

By definition, every traveling wave solution $(U(\xi), V(\xi))$ satisfies the following equations;

$$\begin{cases} U'' + cU' - UV^\alpha + \lambda(1-U) = 0, & \xi \in \mathbf{R}, \\ \gamma V'' + \frac{c\gamma}{d}V' + UV^\alpha - V = 0, & \xi \in \mathbf{R}. \end{cases} \tag{4.21}$$

Here the prime is the derivative with respect to ξ. In this section, we always impose the following boundary condition;

$$(U(-\infty), V(-\infty)) = (u_+, v_+)\,, \quad (U(+\infty), V(+\infty)) = (1, 0), \tag{4.22}$$

where (u_+, v_+) is defined by (4.1) and (4.2).

Let λ, γ, and d satisfy the following relation;

$$\lambda\gamma = 1, \quad d = 1, \quad 0 < \gamma < \gamma_1, \tag{4.23}$$

where γ_1 is given by (4.5). Then we can obtain the following existence theorem.

Theorem 4.16. *Suppose that λ, γ and d satisfy* (4.23). *Then there exists a unique solution $(U, V, c) = (\varphi, \psi, c_*)$ of* (4.21) *with* (4.22) *such that*

$$\varphi' > 0, \quad \psi' < 0, \quad \varphi + \gamma\psi - 1 = 0.$$

Proof. For the case of (4.23), one can describe (4.21) as

$$\begin{cases} U'' + cU' - UV^\alpha + \frac{1}{\gamma}(1-U) = 0, & \xi \in \mathbf{R}, \\ \gamma V'' + c\gamma V' + UV^\alpha - V = 0, & \xi \in \mathbf{R}. \end{cases} \tag{4.24}$$

Define a new function $P = U + \gamma V - 1$. Then we see

$$P'' + cP' - \frac{1}{\gamma}P = 0.$$

Because P is bounded in $\xi \in \mathbf{R}$, we have

$$P(\xi) = 0 \quad \text{for} \quad \xi \in \mathbf{R}.$$

Substituting $U + \gamma V - 1 = 0$ into second equation of (4.24), then

$$V'' + cV' - \frac{1}{\gamma}V(1 - V^{\alpha-1} + \gamma V^{\alpha}) = 0. \tag{4.25}$$

Therefore it follows from the result of [3] that (4.25) has a unique pair of solution and speed $(V, c) = (\psi, c_*)$ which satisfies

$$(\psi(-\infty), \psi(+\infty)) = (v_+, 0),$$

and

$$\psi' < 0 \quad \text{in} \quad \mathbf{R}.$$

Denote $\varphi = 1 - \gamma\psi$, then

$$(\varphi(-\infty), \varphi(+\infty)) = (u_+, 1),$$

and

$$\varphi'(\xi) > 0 \quad \text{for} \quad \xi \in \mathbf{R}.$$

Thus the proof is complete. □

Next, we will treat with stability for the traveling front solution given by Theorem 4.16. Then we have the following theorem.

Theorem 4.17. *The traveling wave solution $\zeta := (\varphi, \psi)$ given by* Theorem 4.16 *is linearly stable. That is, for any small neighborhood $U \subset C_B(\mathbf{R}) \times C_B(\mathbf{R})$ of ζ, there exist positive constants κ and M such that the global solution $z(x,t) = (u(x,t), v(x,t))$ of* (NSP3) *for any initial data in U satisfies the following inequality;*

$$\|z(\cdot,t) - \zeta(\cdot + \xi)\|_\infty \leq M\|z(\cdot,0) - \zeta(\cdot)\|_\infty e^{-\kappa t}.$$

As in the proof of Theorem 4.6, it suffices to study the following linearized spectrum problem;

$$\begin{cases} -u'' - cu' + (\psi^{\alpha} + \frac{1}{\gamma})u + (\alpha\varphi\psi^{\alpha-1})v = \mu u, \\ -v'' - cv' - \frac{1}{\gamma}\psi^{\alpha}u + \frac{1}{\gamma}(1 - \alpha\varphi\psi^{\alpha-1})v = \mu v. \end{cases} \tag{4.26}$$

Here φ and ψ is the traveling wave solution given in Theorem 4.16. And we denote the speed by $c := c_*$.

Spectrum consists of essential spectrum and isolated eigenvalue with finite multiplicity. First, we will determine the essential spectrum for (4.26).

Lemma 4.18. *Let σ_e be the essential spectrum of* (4.26). *Then the following relation holds true*;

$$\sigma_e \subset \left\{ a + bi \in \mathbf{C} : a \geq \frac{b^2}{c^2} + v_+^\alpha + \frac{1-\alpha}{\gamma} \right\}.$$

Proof. Define

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}$$

and

$$N(\xi) = \begin{pmatrix} \psi^\alpha + \frac{1}{\gamma} & \alpha\varphi\psi^{\alpha-1} \\ -\frac{1}{\gamma}\psi^\alpha & -\frac{\alpha}{\gamma}\varphi\psi^{\alpha-1} + \frac{1}{\gamma} \end{pmatrix}, \quad w = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then (4.26) can be rewritten as

$$-Dw'' + Mw' + N(\xi)w = \mu w.$$

Observe that

$$N_+ := \lim_{\xi \to +\infty} N(\xi) = \begin{pmatrix} v_+^\alpha + \frac{1}{\gamma} & \alpha \\ -\frac{1}{\gamma}v_+^\alpha & \frac{1}{\gamma}(1-\alpha) \end{pmatrix}, \quad N_- := \lim_{\xi \to -\infty} N(\xi) = \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix}.$$

Then we define $S_\pm$ as follows;

$$S_\pm = \{\mu \,|\, \det(\tau^2 D + i\tau M + N_\pm - \mu I) = 0 \text{ for some real } \tau,\ -\infty < \tau < \infty\}.$$

After some computation, we see that

$$S_+ = \left\{ a + bi \in \mathbf{C} \,|\, a = \frac{b^2}{c^2} + \frac{1}{\gamma} \right\}, \tag{4.27}$$

and

$$S_- = \left\{ a + bi \in \mathbf{C} \,|\, a = \frac{b^2}{c^2} + \frac{1}{\gamma} \text{ or } a = \frac{b^2}{c^2} + v_+^\alpha + \frac{1-\alpha}{\gamma} \right\}.$$

Since

$$\frac{1}{\gamma} > v_+^\alpha + \frac{1-\alpha}{\gamma},$$

then

$$S_- \subset \left\{ a + bi \in \mathbf{C} : a \geq \frac{b^2}{c^2} + v_+^\alpha + \frac{1-\alpha}{\gamma} \right\}. \tag{4.28}$$

It follows from the result of [19] that

$$\sigma_e(L) \subset S_+ \cup S_-. \tag{4.29}$$

Thus the conclusion follows from (4.27)-(4.29). □

Next, we will study the isolated eigenvalues with finite multiplicities.

Lemma 4.19. *If μ satisfies*

$$\operatorname{Re}\mu < v_+^\alpha + \frac{1-\alpha}{\gamma},$$

then the eigenvalues of (4.26) *must be real.*

Proof. Put $q = u + \gamma v$. According to (4.26), we have

$$q'' + cq' + \left(\mu - \frac{1}{\gamma}\right) q = 0. \tag{4.30}$$

The characterized equation is equivalent to

$$\rho^2 + c\rho + \mu - \frac{1}{\gamma} = 0. \tag{4.31}$$

Define $\rho = a + bi$ and $\mu = r + si$ $\left(r < \frac{1}{\gamma}, s \neq 0\right)$. Then (4.31) can be expressed as

$$a^2 + ac + r - b^2 - \frac{1}{\gamma} + (2ab + bc + s)i = 0.$$

Hence it follows that

$$a^2 + ac + r - b^2 - \frac{1}{\gamma} = 0, \tag{4.32}$$

$$2ab + bc + s = 0. \tag{4.33}$$

We see from (4.33) that $(2a + c)b = -s$. Assume that $2a + c = 0$. Then, owing to (4.32), we find

$$a^2 + b^2 + \frac{1}{\gamma} - r = 0.$$

This is a contradiction because $r < \frac{1}{\gamma}$.

Substitute $b = -\frac{s}{2a+c}$ into (4.32), then

$$a^2 + ac + r - \frac{1}{\gamma} = \frac{s^2}{(2a + c)^2}. \tag{4.34}$$

After some computation, we see that (4.34) has exactly one negative and one positive solution. Let a_+ and a_- $(a_- < 0 < a_+)$ be the solutions of (4.34). If we denote

$$b_\pm = -\frac{s}{2a_\pm + c},$$

then the solution of (4.30) can be represented as

$$q = Ae^{(a_+ + b_+ i)\xi} + Be^{(a_- + b_- i)\xi} \qquad (A, B : \text{constant}).$$

Since q is bounded for $\xi \to \pm\infty$, we find $A = B = 0$. Therefore,

$$q(\xi) = 0 \quad \text{for} \quad \xi \in \mathbf{R}.$$

Inserting $u = -\gamma v$ into the second equation of (4.26), then

$$-v'' - cv' + f(\psi)v = \mu v,$$

where

$$f(\psi) = (1+\alpha)\psi^\alpha - \frac{\alpha}{\gamma}\psi^{\alpha-1} + \frac{1}{\gamma}.$$

If we introduce a new function $y = e^{\frac{c}{2}\xi}v$, then

$$-y'' + \left(\frac{c^2}{4} + f(\psi)\right) y = \mu y. \tag{4.35}$$

Note that $f(\psi(\xi)) \to v_+^\alpha + \frac{1-\alpha}{\gamma}$ provided $\xi \to +\infty$, and $f(\psi(\xi)) \to \frac{1}{\gamma}$ provided $\xi \to -\infty$. Since $\operatorname{Re}\mu < v_+^\alpha + \frac{1-\alpha}{\gamma} = \min\left\{\frac{1}{\gamma}, v_+^\alpha + \frac{1-\alpha}{\gamma}\right\}$, then $|y|$ decay exponentially to zero as $\xi \to \pm\infty$. If we multiply (4.35) by $\overline{y}$ and integrate from $-\infty$ to $+\infty$, then

$$\int_{-\infty}^{\infty} |y'|^2 d\xi + \left(\frac{c^2}{4} + f(\psi)\right) \int_{-\infty}^{+\infty} |y|^2 d\xi = \mu \int_{-\infty}^{+\infty} |y|^2 d\xi.$$

Consequently, μ must be real number, as required. □

Lemma 4.20. 0 *is a simple eigenvalue for* L.

Proof. We use a Wronskian argument for the proof. Recall that $(u, v) = (\varphi', \psi')$ is the eigenfunction of the eigenvalue $\mu = 0$. If we denote $\psi' = \hat{v}$, then

$$-\hat{v}'' - c\hat{v}' + f(\psi)\hat{v} = 0.$$

Define $h_1 := e^{\frac{c}{2}\xi}\hat{v}$. Then we see

$$-h_1'' + \left(\frac{c^2}{4} + f(\psi)\right) h_1 = 0. \tag{4.36}$$

Let $(\tilde{u}, \tilde{v})$ be another eigenfunction for $\mu = 0$ and define $h_2 = e^{\frac{c}{2}\xi}\tilde{v}$. Then it follows that

$$-h_2'' + \left(\frac{c^2}{4} + f(\psi)\right) h_2 = 0. \tag{4.37}$$

Here we consider the Wronskian $W(h_1, h_2) = h_1' h_2 - h_1 h_2'$. According to (4.36) and (4.37), we have

$$W' = h_1'' h_2 - h_1 h_2'' = 0.$$

Observe that h_1 and h_2 exponentially decay to zero in case $\xi \to +\infty$. Then we deduce that

$$W(\xi) = 0 \quad \text{for} \quad \xi \in \mathbf{R}.$$

Thus the proof is complete. $\square$

Lemma 4.21. *There is no negative real eigenvalue of L.*

Proof. We use a contradiction argument. Assume that μ is a negative real eigenvalue of L. Then there exists $v \neq 0$ such that $-v'' - cv' + f(\psi)v = \mu v$. If we denote $j = e^{\frac{c}{2}\xi} v$, then

$$Tj \stackrel{\text{def}}{=} -j'' + \left(\frac{c^2}{4} + f(\psi)\right) j = \mu j. \tag{4.38}$$

Recall that ψ' is an eigenfunction to zero-eigenvalue. Putting $k := e^{\frac{c}{2}\xi}\psi'$, then we have

$$Tk = 0. \tag{4.39}$$

It follows from (4.38) and (4.39) that

$$\mu \int_{-l}^{l} jk dx = g(l),$$

where

$$g(l) = -j'(l)k(l) + j'(-l)k(-l) + j(l)k'(l) - j(-l)k'(-l).$$

Note that j and k exponentially decay to zero when $l \to \pm\infty$. Hence we have

$$g(l) \to 0 \quad \text{as} \quad l \to +\infty.$$

Since μ is negative, we see

$$\int_{-\infty}^{\infty} jk dx = 0.$$

Because k is a positive function, j has at least one zero point. Observe that j exponentially decay to zero if $l \to +\infty$. Then there exists $x_* \in \mathbf{R}$ such that

$$j(x_*) = 0 \quad \text{and} \quad j > 0 \quad \text{in} \quad (x_*, \infty).$$

Now, j and k satisfy the following equations;

$$\begin{aligned} -j'' + f(\psi)j &= \mu j, \quad (4.40) \\ -k'' + f(\psi)k &= 0. \quad (4.41) \end{aligned}$$

If we multiply (4.40)(resp.(4.41)) by j(resp.k), then we see

$$-j''k + k''j = \mu jk. \quad (4.42)$$

Integrating (4.42) from x_* to $+\infty$, it follows that

$$j'(x_*)k(x_*) = \mu \int_{x_*}^{+\infty} jk dx.$$

Since $j'(x_*) > 0, k(x_*) > 0$ and $\int_{x_*}^{+\infty} jk dx > 0$, then $\mu > 0$. This is a contradiction. Thus the proof of Lemma 4.21 is complete. □

Using Lemmas 4.18-4.21, we can prove Theorem 4.17

Proof of Theorem 4.17. We see from Lemmas 4.19-4.21 that the eigenvalue with the least real part must be zero eigenvalue and the second eigenvalue (if exists) must have positive real part. If we define the second eigenvalue by β, then it follows from Lemma 4.18 that

$$\sigma(L) = \{0\} \cup \sigma^* \quad \text{with} \quad \mathrm{Re}\, \sigma^* \geq \nu,$$

where

$$\nu = \min \left\{ v_+^\alpha + \frac{1-\alpha}{\gamma}, \mathrm{Re}\, \beta \right\} > 0.$$

Because zero is a simple eigenvalue for L, then one can verify the assumption of Theorem 4.11. Thus the proof is complete. □

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