Chapter 2

Theoretical Concepts

2.1 Treating the Fuzziness and Randomness

In term of fuzzy random variable concept, the study is focused on considering the fuzziness and randomness simultaneously. Fuzziness is characterised as the absence of sharp boundaries of human perception while randomness is caused by mechanism of some chance. Thus, to address these uncertainties, fuzzy variable has become an important tool as standalone fuzzy theory or probability theory cannot be directly applied to the so-called hybrid uncertainties circumstances. Fuzzy random variables are defined as a measurable function linking a probability space to a collection of fuzzy numbers (Kwakernaak, 1978; 1979). Fuzzy arithmetic and fuzzy operations for fuzzy numbers have also been studied through the use of the extension principle that involves the concept of possibility (Nguyen, 1978; Zadeh, 1975a; Zadeh, 1975b; Zadeh, 1975c). In possibility theory, an impression is expressed by using a possibility distribution. Thus, the fuzzy parameters are associated with possibility distributions as opposed to the random variables that are associated with probability distributions. Then, the possibilistic concept is used with fuzzy random variables explanation.

Let us assume that Pos is a possibility measure defined on the power set $P(\Gamma)$ of Γ in a universe Γ. Given $\Re$ as the set of real numbers, a function $Y:\Gamma \to \Re$ is said to be a fuzzy variable defined on Γ (see Nahmias, 1978). The possibility distribution μ_Y of Y is defined by $\mu_Y(t) = Pos\{Y = t\}, t \in \Re$, which is the possibility of event $\{Y = t\}$. For fuzzy variable Y with possibility distribution μ_Y, the possibility and necessity of event $\{Y \le r\}$ are given, respectively, in the following forms:

$$\begin{aligned} Pos\{Y \le r\} &= \sup_{t \le r} \mu_Y(t), \\ Nec\{Y \le r\} &= 1 - \sup_{t > r} \mu_Y(t). \end{aligned} \tag{2.1}$$

The expectation based on an average of possibility and necessity is defined based on Liu and Liu (2002). The possibility expresses a level of overlapping and the necessity articulates a degree of inclusion. The expected value of a fuzzy variable is presented as follows:

Definition 2.1: Let Y be a fuzzy variable. Under the assumption that the two integrals are finite, the expected value of Y is defined as follows:

$$E[Y] = \int_0^{\infty} \left(\frac{1}{2} \left[1 + \sup_{t \ge r} \mu_Y(t) - \sup_{t < r} \mu_Y(t) \right] \right) dr - \int_{-\infty}^{0} \left(\frac{1}{2} \left[1 + \sup_{t \le r} \mu_Y(t) - \sup_{t > r} \mu_Y(t) \right] \right) dr \tag{2.2}$$

From Equation (2.2), the expected value of Y is defined as $E[Y] = \frac{a^l + 2c + a^r}{4}$.

Definition 2.2: Suppose that (Ω, Σ, Pr) is a probability space and F_v is a collection of fuzzy variables defined on possibility space $(\Gamma, P(\Gamma), \mathrm{Pos})$. A fuzzy random variable is a map $X : \Omega \to F_v$ such that for any Borel subset B of $\Re$, $Pos\{X(\omega) \in B\}$ is a measureable function of ω.

Let X be a fuzzy random variable on Ω. From the above definition, $X(\omega)$ is a fuzzy variable for each $\omega \in \Omega$. Furthermore, a fuzzy random variable X is said to be positive if for every ω, X is almost surely positive.

Let V be a random variable on probability space $(\Omega, \Sigma, \mathrm{Pr})$. $X(\omega) = (V(\omega) - 2, V(\omega) + 2, V(\omega) + 6)_{\Delta}$ is a triangular fuzzy variable for every $\omega \in \Omega$ on some possibility space $(\Gamma, P(\Gamma), \mathrm{Pos})$. As a result, X is a triangular fuzzy random variable.

The expected value of the fuzzy variable $X(\omega)$ is denoted by $E[X(\omega)]$ for any fuzzy random variable X on Ω, which has been proved to be a measurable function of ω (Liu and Liu, 2003). Given this, the expected value of the fuzzy random variable X is defined as the mathematical expectation of the random variable $E[X(\omega)]$.

Definition 2.3: Let X be a fuzzy random variable defined on a probability space $(\Omega, \Sigma, \Pr)$. The expected value of X is defined as

$$E[X] = \int_{\Omega}\left[\int_{0}^{\infty}\left(\frac{1}{2}\left[1+\sup_{t\geq r}\mu_{Z(\omega)}(t)-\sup_{t<r}\mu_{Z(\omega)}(t)\right]\right)dr-\int_{-\infty}^{0}\left(\frac{1}{2}\left[1+\sup_{t\leq r}\mu_{Z(\omega)}(t)-\sup_{t>r}\mu_{Z(\omega)}(t)\right]\right)dr\right]\Pr(d\omega) \quad (2.3)$$

Definition 2.4: Let X be a fuzzy random variable defined on a probability space $(\Omega, \Sigma, \Pr)$ with expected value e. The variance of X is defined as

$$\operatorname{var}[X] = E\left[(X\text{-}e)^2\right] \quad (2.4)$$

where $e = E[X]$ given by Definition 3.3.

In this section, fuzzy random variables are introduced as an integral component of regression models with the presence of random and fuzzy information, which is the main backbone of developed model throughout this study. The developed regression models based on fuzzy random variables are provided in Chapter 3.

2.2 Fuzzy Goal Programming: An Additive Model

Classical goal programming is constructed with objective functions, constraint, and target values, which are all deterministic values. When the knowledge of experts is imprecise or unavailable, it is difficult to get the exact value for developing a model. In such uncertain and imprecise situations, fuzzy values are used in the goal programming description. The inexact values in the goal programming model reflect the vagueness or tolerance of the decision maker and also the imprecision of the knowledge of experts.

Tiwari *et al.* (1987) have created and implemented an additive model in the fuzzy goal programming context that aggregates the collective fuzzy goals. In their model, the aspiration levels for goals are assumed to be fuzzy. Despite the deviation variables used in the goal programming, a generalised fuzzy goal programming model is defined by a membership function as follows:

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{to satisfy} && G_i(\mathbf{X}) \tilde{\geq} g_i, \quad i = 1,2, \cdots, m, \\ &\text{subject to} && \mathbf{AX} \leq \mathbf{b}, \\ & && \mathbf{X} \geq 0, \end{aligned} \tag{2.5}$$

where $\mathbf{X}$ is a vector with components $x_1, x_2, \cdots, x_n$, and $\mathbf{AX} \leq \mathbf{b}$ are the system constraints in vector notation. The fuzzification of the aspiration level is denoted by $\tilde{\geq}$. The i^{th} fuzzy goal $G_i(\mathbf{X}) \tilde{\geq} g_i$ in (3) indicates that the decision maker is satisfied even if the value of the i^{th} fuzzy goal $G_i(\mathbf{X})$ is less than g_i up to a certain tolerance limit. A membership function yields a degree of closeness of each goal to its desired attainment level using the interval $[0,1]$ to represent the degree of membership of each goal. The worst possible value for an objective function makes a grade of membership zero. A linear membership function u_i for the i^{th} fuzzy goal $G_i(\mathbf{X}) \tilde{\geq} g_i$ can be expressed according to Zimmerman (1987) as

$$u_i = \begin{cases} 1 & \text{if } G_i(\mathbf{X}) \geq g_i, \\ \dfrac{G_i(\mathbf{X}) - L_i}{g_i - L_i} & \text{if } L_i \leq G_i(\mathbf{X}) \leq g_i, \\ 0 & \text{if } G_i(\mathbf{X}) \leq L_i, \end{cases} \tag{2.6}$$

where L_i is the lower tolerance limit for the fuzzy goal $G_i(\mathbf{X})$.

In the case of the goal $G_i(\mathbf{X}) \tilde{\leq} g_i$, the membership function is defined as

$$u_i = \begin{cases} 1 & \text{if } G_i(\mathbf{X}) \leq g_i, \\ \dfrac{U_i - G_i(\mathbf{X})}{U_i - g_i} & \text{if } g_i \leq G_i(\mathbf{X}) \leq U_i, \\ 0 & \text{if } G_i(\mathbf{X}) \geq U_i, \end{cases} \tag{2.7}$$

where U_i is the upper tolerance limit for the fuzzy goal $G_i(\mathrm{X})$.

The additive model of the fuzzy goal programming (Tiwari *et al.*, 1987) problem (2.8) is formulated by substituting all membership functions in the model (2.5) as follows:

$$\begin{aligned} &max && V(\mu) = \sum_{i=1}^{m} \mu_i \\ &\text{subject to} && \mu_i = \frac{G_i(\mathbf{X}) - L_i}{g_i - L_i} \\ &&& \mathbf{AX} \leq \mathbf{b}, \\ &&& \mu_i \leq 1, \\ &&& X, \mu_i \geq 0, \\ &&& i = 1,2,\cdots,m, \end{aligned} \tag{2.8}$$

where $V(\mu)$ is called the fuzzy achievement function or fuzzy decision function. Note that $AX \leq b$ is the crisp system constraints in vector. This is the single objective optimisation problem, which can be solved by employing an appropriate classical technique. Unlike the conventional goal programming function (minimising the deviations), it is easy to maximise the fuzzy decision function consisting of μ_i. This use of an additive model allows us to obtain the maximum sum of the achievement degree for the goals.

2.3 Possibilistic Programming

Possibility distributions are assumed to be obtained subjectively from the knowledge of experts, whereas probability distributions are estimated from observations. From the perspective of possibility concept (Zadeh, 1978), impression can be expressed in terms of a possibility distribution. For instance, an expression '*about one million dollars*' contains a fuzzy number. Furthermore, given a proposition '*it is possible to invest about one million dollars*,' it can be understood as the possibility of the investment.

Let us interpret the possibility concept and specify possibility distribution $\Pi_F(x)$ as $\Pi_F(x) \triangleq \mu_F(x)$, with the knowledge $F \triangleq$ '*about*' as specified in the following:

Definition 2.5: Given a possibility distribution $\Pi_F(x)$, the possibility measure of a fuzzy set A specified by $\mu_A(x)$ is defined as $\Pi_F(A) = \sup_x \mu_A(x) \wedge \Pi_F(x)$.

In the possibilistic programming approach, a vague aspiration is represented by a fuzzy goal G_i. A fuzzy goal G_i is fuzzy set whose membership function μ_{G_i} expresses a degree of satisfaction to a soft constraint such as '*considerably larger than* g_i' and '*considerably smaller than* g_i'.

The membership function of linear fuzzy goal G_i is as follows:

$$\mu_{G_i}(r) = \max\left\{\min\left(1 - \frac{r - g_i}{d_i}, 1\right), 0\right\} \tag{2.9}$$

or

$$\mu_{G_i}(r) = \max\left\{\min\left(1 - \frac{g_i - r}{d_i}, 1\right), 0\right\} \tag{2.10}$$

where g_i is the center value of the target goal and d_i is the width. The linear fuzzy goals G_i defined by (2.9) or (2.10) are written as $G_i = \,]g_i, d_i)$ and $G_i = (g_i, d_i[$, respectively, to show the relationship of the decision maker's goal.

Example 2.1: The linear fuzzy goal that corresponds to the linguistic expression provided by decision maker as "*significantly smaller than 5 million dollars*" are defined by $G_1 = \ ]5,0.002)$, with center value 5, and width 0.02.

Meanwhile, the ambiguous data is represented by a possibility distribution π_{ij}. A possibility distribution is regarded as a fuzzy restriction that performs as a flexible constraint on the value that may be assigned to a variable. Thus, a possibility distribution π_{ij} is defined in terms of a fuzzy set A_{ij} presenting the linguistic expression such as '*about* a_{ij}' as $\pi_{ij} = \mu_{A_{ij}}$, where $\mu_{A_{ij}}$ is a membership function of '*about* a_{ij}' A_{ij}.

A symmetric triangular fuzzy number $A_{ij} = \langle a_{ij}, d_{ij} \rangle$ is used to define a possibility distribution π_{ij} with the following membership function:

$$\mu_{A_{ij}}(r) = max\left\{1 - \frac{|r - a_{ij}|}{d_{ij}}, 0\right\} \tag{2.11}$$

Thus, under probabilistic programming perspective, the expressions are useful and meaningful to formulate the real-world problem that contains such uncertainty.

Example 2.2: Assume that machine capacity is expressed as a fuzzy number. The machine capacity of Product A a_1 at Machine 1 is described with linguistic expression 'about 4'. A fuzzy number is illustrated with the membership function $\mu_{A_1}(r) = max\left\{1 - \frac{|r-4|}{0.7}, 0\right\}$. $\mu_{A_1}(r)$ shows the possibility degree of the event of the machine capacity, for Product A at Machine 1 is r. So, μ_{A_1} can be regarded as a possibility distribution of the processing time of Product A at machine 1 and a_1 can be considered as a possibilistic variable restricted by the possibility distribution μ_{A_1}.

A possibilistic programming (Inuiguchi *et al.*, 1994) is written as follows:

$$\begin{aligned} &Y_i \underline{\underline{\Delta}} \sum_{j=1}^{n} \alpha_{ij} x_j \gtrsim g_i, i = 1, \ldots, m, \\ &x_j \geq 0, j = 1, \ldots, n, \end{aligned} \tag{2.12}$$

where α_{ij} is a possibilistic variable restricted by a possibility distribution and defined by a triangular fuzzy number $A_{ij} = \langle a_{ij}, d_{ij} \rangle$ with centre a_{ij} and width d_{ij}, and $\tilde{\geq}$ is fuzzy inequalities that express the 'considerably larger than'. Thus, $\tilde{\geq} g$ has a linguistic expression 'considerably larger than g_i' that corresponds to a fuzzy goal G_i, defined by a fuzzy set with linear membership function.