Determination of an Optimal Control Limit and the Value of Optimal Control Scheme in Cost Variances Investigation

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I. Introduction

The principle of management by exception, which was stated many years ago by Frederick Taylor, is considered an essential requirement for effective management contol systems. It states that management should give primary attention to significant exceptions and, therefore, little or no attention to the relatively large number of situations in which performance is considered satisfactory. Since an investigation involves an expenditure of organizational resources, it should only be undertaken when the benefits expected from it exceed its costs.

The principle of management by exception is followed in a standard cost control system. When actual cost, measured in dollar terms, deviates from a predetermined standard, a cost accountant produces a "variance report" which focuses management's attention on exceptional or significant variances. A certain amount of variance from a standard may be considered random fluctuations beyond management control. Alternatively, its size may not be large enough to warrant an investigation because the costs of investigation and correction are greater than the benefits of correction. The control system thus needs a criterion for determining exceptional or significant variances.

Little attention, however, has been given to the problem of determining control limits on an economic basis and the formal expression of optimal control limits is not central to the accounting literature. Consequently, in practice management may establish control limits by judgment and use them as criteria to determine whether a difference between actual and standard costs is worth investigation.

The purpose of this paper is to present a mathematical model for determining optimal control limits in a cost variance investigation problem. It provides a method to calculate these criteria under sound practical assumptions so that the relevant cost-benefit function is optimized. The value of the control system with optimal control limits will be defined and it will be shown that the value is non-negative.

II. Background and Existing Research

Standard costs are basically estimates of what costs are expected to be under particular operating conditions. Since we cannot specify exactly what will happen in the future, we cannot expect actual costs to be the same as standard costs. A cost variance, therefore, is calculated as the difference between the actual and standard costs. Such a variance is computed periodically (e. g. one week, two weeks or one month). A cost accountant then uses this deviation to aid management who determines whether or not they should investigate the process to see if it should be changed. For this purpose a cost accountant needs some criteria for distinguishing significant from insignificant variances.

The principle of management by exception regards the standard as a mean about which cost deviations will occur and sets control limits or action limits about this mean. Only those variances which fall outside these control limits are considered exceptional and significant and consequently investigated. In practice, this principle can be implemented by using one of several models suggested in the literature. They may be classified into the following three categories[1]:

(A) Traditional approaches

(B) Statistical quality control approaches

(C) Statistical decision theory approaches.

A. Traditional Approaches

The easiest and crudest method is to set control limits using the absolute size of the cost variance or some fixed percentage of the standard, for example, ± 10% of the standard. As Bierman and Dyckman (1971) point out, these measures rely upon the intuition of management in deciding whether the variance should be investigated and whether the investigation should then lead to corrective action.[2] This type of approach, which is illustrated in Figure 1, is similar to that used by many firms for statistical sampling for quality control of output. However, the formal expression of control limits are not given in this approach.[3] Furthermore, the traditional method implicitly assumes that any variance which is less than the cutoff is due to random factors and hence, an investigation of the process in such a case would not yield any benefits. In other words, it is implied that as far as a variance stays less than the cutoff, the process remains in the state of control.

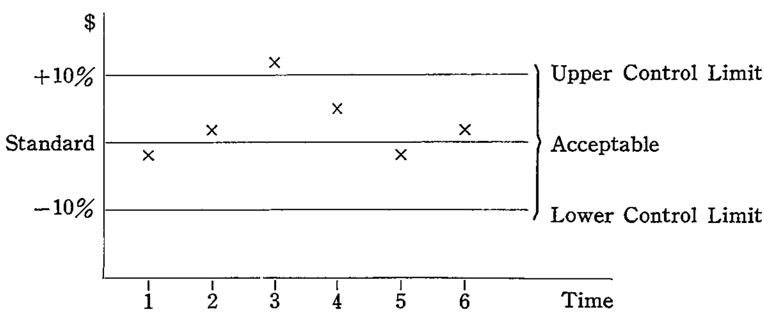

Figure 1. Traditional Approach to Control Limits

B. Statistical Quality Control Approaches

In recent years more scientific approaches have been adopted. They are designed to account for random fluctuations while maintaining a high probability that the significant changes in operating structure which cause a larger deviation will be identified.

A statistical quality control rule determines confidence limits for random variation around a standard cost based upon the mean value of observed actual costs and some estimate of the standard deviation of this mean. The classic example of such a rule is the $\bar{X}$ chart which was suggested first by Shewhart (1931). On the $\bar{X}$ chart, as shown in Figure 2, the central line is set at $\bar{X}''$ and the control limits are taken as $\bar{X}'' \pm$ m $(\sigma''/\sqrt{n})$ where $\bar{X}''$ and σ'' are the mean value and the standard deviation provided from past experience, respectively. Samples of size n are taken from the process periodically and the sample $\bar{X}$ is plotted on the chart. If a sample $\bar{X}$ falls outside of the control limits, it is assumed that some change in the average X has occurred and an investigation is undertaken for an "assignable cause." The probability of observing a sample average $\bar{X}$ outside the control limits varies with parameter m. Usually m is specified so that the probability of producing an erroneous signal when the process is in control is very small.(4)

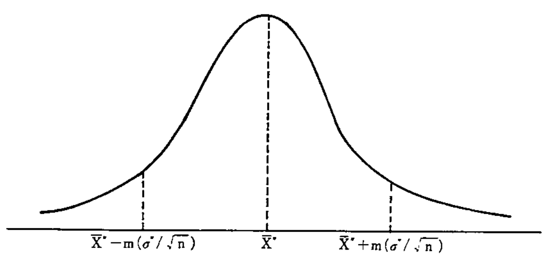

Figure 2. The X Charts and Control Limits

In the quality control literature there are studies which attempt to design control charts economically in which the parameters of the $\bar{X}$ chart are determined so that the relevant function is optimized.

Duncan (1956) showed how to determin the sample size, the interval between samples, and the control limits that yield approximately maximum average net income. Later, Goel, Jain and Wu (1968) proposed an algorithm for determining the economic design of $\overline{X}$ charts based on Duncan's model. However, the basic equation of their algorithm cannot be solved explicitly so that we must resort to a numerical method in order to find optimal parameters which satisfy the equation. Gibra (1971) proposed an $\overline{X}$ chart model which was used not only as a device for detecting out-of-control conditions but also as a basis for maintaining a prescribed quality level of the product. Like Goel, et al.'s procedure, optimal parameters in his model can be found by trial and error, which seems cumbersome and may decrease its applicability.

A cumulative sum procedure, introduced by Page (1954), is distinguished from the $\overline{X}$ chart approach by the characteristic that it uses previous observations for detecting a non-random change inthe mean of a process. A series of partial sums is calculated as follows:

$$S_t=\sum_{k=1}^{t}(\overline{X}_k-\mu_0)$$

where $\overline{X}_k$ is the current observation and μ_0 represents the target mean. For a two sided control scheme, a symmetric V mask is constructed and placed on the chart with the vertex of the V pointing horizontally forwards and at a distance d ahead of the current point (t, S_t). The total angle of the V is 2θ. Notice that a particular procedure in this method is specified by two parameters, here denoted by d and θ. A lack of process control is assumed to be indicated when a previously plotted point falls outside the limbs of the V-mask. Taylor (1968) developed an approximate formula for the long run average cost per unit time as a function of the parameters of the cumulative sum chart. In a manner analogous to that used by Taylor, Goel and Wu (1973) proposed a different procedure to design cumulative sum charts economically.

C. Statistical Decision Theory Approaches

The earliest statistical decision theory approach is best exemplified by the Bierman, Fouraker and Jaedicke (1961) model. It is a simple decision model which consists of two actions (investigate and do not investigate) and two states (the unfavorable variances resulted from noncontrollable causes and controllable causes). After comparing the expected cost of investigating with that of not investigating, the following decision criterion was obtained: Investigation is warranted if, given an unfavorable variance, the probability of that variance resulting from noncontrollable causes is less than $(L-C)/L$, where C represents the cost of investigation and L is the cost incurred due to not investigating a variance which can be controlled.

Dyckman (1969) produced the same probability criterion as that proposed by Bierman, et al. although L was interpreted as a constant savings from investigating an out-of-control situation. Let the revised state probability for in-control state θ_1 after n cost observations be denoted by $f_n(\theta_1)$. Then, the expected costs of investigation was given by

$$Cf_n(\theta_1)+[C-L][1-f_n(\theta_1)].$$

If this expectation is less than zero, investigation should be undertaken. Alternatively, if $f_n(\theta_1)<(L-C)/L$, an investigation is signaled. Besides the case in which a process once corrected remains in a state of control, which was initially considered by Bierman et al., Dyckman discussed two other multi-period situations: (i) a change from the in-control state to the out-of-control state may occur during any time period but not vice versa, and (ii) changes in both directions are possible. Although the state probabilities, $f_n(\theta_1)$ and $f_n(\theta_2)$, were calculated differently, the above criterion was applied to all three situations.[5]

Based on the Girshick and Rubin (1952) procedure[6] for production process control, Kaplan proposed a probabilistic multi-period model for the accounting variance investigation decision. The system was represented by a simple two state Markov process with transition matrix:

$$P=\begin{pmatrix} g & 1-g \\ 0 & 1 \end{pmatrix}$$

where g is the probability that the system would remain in the in-control state during the reporting period and, accordingly, $1-g$ is the probability that the system would go to the out-of-control state sometime within the reporting period. Dynamic programming was used to compute optimal solutions which minimized discouted future costs. At each stage the expected costs incurred when investigating were compared with the expected costs of not taking any action. q_i was defined to be the posterior (after observing the i-th output) probability that the process would be in control during period i+1. q_i^* was calculated as the value of q_i which made both cost terms in the minimization equal. Obviously, the form of optimal policy depends upon q_i. If $q_i<q_i^*$, investigation is undertaken. If $q_i>q_i^*$, no action is taken and another output from the process is observed.

Unlike Bierman et al.'s model, Kaplan (1969) and Dyckman (1969) formulated investigation decision models based upon multiple observations. Prior information about in or out-of-control probabilities was combined with the most recent observation by using Bayes' theorem. To the extent that the outputs from the process are believed to be dependent random variables from a stochastic process, then a reasonable decision should make use of prior observations as well as the most recent one.[7]

D. Limitations in the Previous Models

All the three models discussed above dealt with a two state system, i. e., in-control and out-of-control. As Kaplan (1975) admitted, this assumption seems to be so simple and unrealistic as a description of reality that it may limit the applicability of those models. The system may drift away from standards through an evolutionary process of mismanagement and could be situated in several states within the in-control and the out-of-control situations and between the two situations. It may be possible to incorporate a finer classification in which discrete amounts of "controlness" are allowed and a process can be in a countable number of states i, ($i=0, 1, 2, \ldots\ldots$), with a cost variance as a function of the state. However, a more realistic control situation may be to assume a continuous state of the world rather than that of a countable number of states. For example, we may develop a model which allows the state of the system to be the level of cost variances which are described by a continuous variable.(8)

The other unrealistic simplification in the previous models is that the cost of investigation and correction is assumed to be a known constant and the benefit from investigation and correction, especially in Bierman, et al. and Dyckman, is assumed to be fixed. In reality, they may depend on the causes of cost variances and environments in which the system operates. Although the final judgment on the appropriateness of the cost and benefit functions must be made on the basis of empirical studies, a more reasonable assumption is to express them as a function of the degree of out-of-controlness of a system. They may be approximated as a function of the size of the cost variances.

The other limitation in the previous studies is the assumption on probability distribution of actual costs and cost variances. Bierman, et al. and Dyckman assumed that the actual cost was normally distributed with the mean as standard and known standard deviations. Similarly, in Duvall's (1967) model the distributions of controllable and non-controllable cost variances were assumed to be normal. In the $\bar{X}$ chart approach, it was assumed that a normally distributed measurable characteristic was produced.

In a study conducted by Louis Tuzi (1964), 36 historical monthly variances from standard for each overhead account were tested for normality by means of the "chi-square" test. Having found that a number of these monthly variances formed a skewed distribution, he concluded that the use of control charts based on the normal distribution was inappropriate.

Recognizing these critical limitations in the preceding developments, the next section will develop a model which allows the state of the system to be a continuum and can be applied to any type of continuous distribution functions of cost variances.

III. A Proposed Model for an Optimal Control Limit

In this section we will seek to find an optimal control limit for the case where there is a continuum of states. To do this, the following symbols must be introduced.

Let x be a variance and k a control limit. An investigation is undertaken if

$$x>k, \qquad 0<k<\infty.$$

Alternatively, if x is less than or equal to k, an investigation is not signaled. The variance described in this model is measured in physical units such as labor hours and quantities of material. Therefore, the cost variance is calculated as the difference between the actual and the standard units multiplied by the standard cost per unit. Obviously, x must be either non-positive (favorable variances or zero) or positive (unfavorable variances). If the deviation is negative, management enjoys cost savings, Ax, which are proportional to the size of the deviation. If the deviation is positive but not greater than the critical value k, the costs due to unfavorable variances, Bx, are incurred, which are also proportional to the size of the deviation. If it is larger than k, an investigation is undertaken to find causes of that deviation. The cost of an investigation is assumed to be composed of two parts: fixed cost F and variable cost Cx.

What are the possible outcomes of an investigation? There may be two possibilities. First, a cost accountant may discover that a deviation arose from the specific decisions of operating management and was caused by factors within the jurisdiction and control of management. A corrective action will be taken in this case. While a company will incur a cost of correction, $H+Dx$, it will gain the benefits from correction, denoted by Lx, which are proportional to the size of the deviation. Benefits of a corrective action will be estimated on the assumption that the correction would eliminate subsequent variances by bringing about adjustment to future performance. In other words, L is defined as the opportunity costs which will be incurred in the future if a corrective action is not taken.

A second possibility is that the investigation may disclose that the variance was caused by a change in the operating conditions of a company or any other factors beyond the control of management. In this case, no corrective action will be taken but the standard may be adjusted at the fixed amount of the expense, E, to take these changes into consideration so that future cost variances will be eliminated.

In order to deal with those two possibilities, it is assumed that given an unfavorable variance, a cost accountant can assign a subjective probability, p, that the variance resulted from controllable causes. Hence, $1-p$ is the probability that the variance resulted from noncontrollable causes. Given the above definitions and assumptions, the costs and benefits for this investigation decision problem are summarized in Table 1.

Table 1 Payoffs for Corresponding Size of Variance

Size of Variance	Costs and/or Benefits
$-\infty<x\leq 0$	Ax (Cost Savings)
$0<x\leq k$	Bx (Unfavorable Cost Variances)
$k<x<+\infty$	$Bx+(F+Cx)+\begin{cases}(H+Dx)-Lx & \text{(First Possibility)} \\ E & \text{(Second Possibility)}\end{cases}$

By assuming the existence of positive density function of a continuous distribution of cost variance, x, we may express the expected total costs, $U(k)$, as follows:

$$U(k)=\int_{-\infty}^{0} Axf(x)dx+\int_{0}^{k} Bxf(x)dx+\int_{k}^{\infty}(Bx+F+Cx)f(x)dx$$
$$+p\int_{k}^{\infty}(H+Dx-Lx)f(x)dx+(1-p)\int_{k}^{\infty}Ef(x)dx. \quad (1)$$

Since each term in (1) is expressed by a definite integral, variable x will drop in the process of calculation and the expected total costs are expressed as a function of k.

For notational simplicity we define:

$$W_1\equiv\int_{-\infty}^{0} xf(x)dx$$
$$W_2(k)\equiv\int_{0}^{k} xf(x)dx$$
$$W_3(k)\equiv\int_{k}^{\infty} xf(x)dx=\int_{-\infty}^{+\infty} xf(x)dx-\int_{-\infty}^{0} xf(x)dx-\int_{0}^{k} xf(x)dx \quad (2)$$
$$\pi(k)\equiv\int_{k}^{\infty} f(x)dx=1-\int_{-\infty}^{0} f(x)dx-\int_{0}^{k} f(x)dx.$$

Using the above simplified notation, we can rewrite (1) in the following way:

$$U(k)=AW_1+BW_2(k)+F\pi(k)+(B+C)W_3(k)$$
$$+pH\pi(k)+p(D-L)W_3(k)+(1-p)E\pi(k). \quad (3)$$

Recall that our problem is to find an optimal k which minimizes $U(k)$.[9] The critical level k° which satisfies the equation

$$U'(k^\circ)=0 \quad (4)$$

with respect to k will be equal to the optimal k^*, provided that $U(k)$ is convex at k°, i, e.,

$$U''(k^\circ)>0 \quad (5)$$

and that k° is unique and positive

$$k^\circ>0. \quad (6)$$

Differentiating each integral with respect to its lower or upper limit in (2), we have, for $k=k^\circ$,[10]

$$\frac{dW_1}{dk^\circ}=0; \qquad \frac{d\pi(k^\circ)}{dk^\circ}=-f(k^\circ) \quad (7)$$

$$\frac{dW_2(k^\circ)}{dk^\circ}=k^\circ f(k^\circ)\,; \qquad \frac{dW_3(k^\circ)}{dk^\circ}=-k^\circ f(k^\circ).$$

Using (7), the first derivative of $U(k^\circ)$ for $k=k^\circ$ becomes

$$U'(k^\circ)=f(k^\circ)\{-F-Ck^\circ-pH+p(L-D)k^\circ-(1-p)E\}. \qquad (8)$$

Since by assumption $f(k^\circ)>0$, we obtain from (4)

$$k^\circ=\frac{F+pH+(1-p)E}{p(L-D)-C}>0 \qquad \text{if } p(L-D)-C>0. \qquad (9)$$

Without loss of generality we can assume $p(L-D)-C$ to be positive since, a priori, an investigation will be undertaken only if the benefits from it exceed its costs. In other words, if this quantity is not positive, then there is no economic incentive to undertake an investigation and take a corrective action. Calculating the second derivative of (3) for $k=k^\circ$, we have

$$U''(k^\circ)=f'(k^\circ)\{-F-Ck^\circ-pH+p(L-D)k^\circ-(1-p)E\} + f(k^\circ)\{p(L-D)-C\}>0. \qquad (10)$$

From the preceding analysis, we find k° given in (9) an optimal control limit, denoted by k^*, in cost variance investigation, i. e.,

$$k^*=\frac{F+pH+(1-p)E}{p(L-D)-C}\ . \qquad (11)$$

It is interesting to note that the numerator in (11) represents the expected fixed costs of investigation and correction while the denominator the expected marginal benefit yielded through investigation and correction. The optimal value k^* is simply given by this ratio. This formula is similar to that used in cost-volume-profit analysis. It represents that critical level of cost variance which equates the expected benefits to expected costs of investigation and correction —that is, a break-even point. The optimal value k^* depends upon all the cost and benefit terms except for B and probability p. As was intuitively expected, non-investigation range $(-\infty, k^*)$ will broaden when the cost term increases and the benefit term decreases. Since we have made no restrictive assumptions on the probability distribution of x, k^* in (11) can be applied to any kind of continuous distributions with a positive density function of x for $-\infty<x<\infty$.

For the specific example, suppose that C=\$12; D=\$20; L=\$60; E=\$500; F=\$12,900; and H=\$18,750. The variance is caused by factors within the control of management 80% of the time. Then, the optimal control limit, k^*, is calculated as follows:

$$k^*=\frac{\$12{,}900+(.80)(\$18{,}750)+(1-.80)(\$500)}{(.80)(\$60-\$20)-\$12}=\frac{\$28{,}000}{\$20}=1{,}400.$$

The optimal action in this case is to investigate the process if the variance is greater than 1,400 units.

IV. An Exploratory Investigation and Its Effects on a Full Investigation

In this section we will incorporate an exploratory investigation in our framework and obtain two optimal control limits. Suppose that a company has the option of conducting two levels of investigation as first introduced by Dyckman (1969). The first level of investigation which is essentially exploratory in nature costs less than a second level, full investigation. But there is a risk that it will not be able to disclose the causes of the cost variances. It is assumed that the costs of such an investigation are variable costs, $F'+C'x$, ($F'<F$ and $C'<C$) and the probability that the cause of deviation will be detected when it exists is h, $0<h<1$.

An exploratory investigation is supposed to be signaled when x falls outside $k_1(0<k_1<k_2)$ but inside k_2. There are three possible outcomes from such an investigation:

(1) The cause of deviation will be discovered and it resulted from controllable factors;

(2) The cause of deviation will be discovered but it resulted from non-controllable factors; and

(3) The cause will not be discovered whether controllable or not.

The joint probabilities and costs for these outcomes are represented in the following table:[11]

Table 2. Probabilities and Costs for Four Cases

Outcome	Probability	Cost
Discovered and Controllable	q	$H'+D'x-Lx$ $(H'<H,\ D'<D)$
Discovered but Uncontrollable	$h-q$	E' $(E'<E)$
Not discovered and Controllable	$p-q$	$M'x$
Not discovered and Uncontrollable	$1-p-h+q$	0

The avoidable costs, $M'x$, will be charged for an exploratory investigation if it fails to discover the cause of deviation resulted from controllable factors. But, the costs will be zero when it is resulted from uncontrollable factors since no corrective action could be taken even if the cause was discovered.

The expected total costs in this problem becomes

$$\begin{aligned} U(k_1, k_2) = & \int_{-\infty}^{0} Axf(x)dx + \int_{0}^{k_1} Bxf(x)dx \\ & + \int_{k_1}^{k_2} (Bx+F'+C'x)f(x)dx \\ & + q\int_{k_1}^{k_2} (H'+D'x-Lx)f(x)dx \\ & + (h-q)\int_{k_1}^{k_2} E'f(x)dx + (p-q)\int_{k_1}^{k_2} M'xf(x)dx \\ & + \int_{k_2}^{\infty} (Bx+F+Cx)f(x)dx \\ & + p\int_{k_2}^{\infty} (H+Dx-Lx)f(x)\,dx + (1-\mathrm{p})\int_{k_2}^{\infty} Ef(x)dx \\ = & AW_1 + BW_2(k_1) + F'\pi(k_1, k_2) \\ & + (B+C')W(k_1, k_2) + qH'\pi(k_1, k_2) \\ & + q(D'-L)W(k_1, k_2) + (h-q)E'\pi(k_1, k_2) \\ & + (p-q)M'W(k_1, k_2) + F\pi(k_2) + (B+C)W_3(k_2) \\ & + pH\pi(k_2) + p(D-L)W_3(k_2) + (1-p)E\pi(k_2) \end{aligned} \tag{12}$$

where

$$W_2(k_1) \equiv \int_0^{k_1} xf(x)dx$$

$$W(k_1, k_2) \equiv \int_{k_1}^{k_2} xf(x)dx = \int_0^{k_2} xf(x)dx - \int_0^{k_1} xf(x)dx \tag{13}$$

$$W_3(k_2)\equiv\int_{k_2}^{\infty}xf(x)dx=\int_{-\infty}^{\infty}xf(x)dx-\int_{-\infty}^{0}xf(x)dx$$
$$-\int_{0}^{k_2}xf(x)dx$$
$$\pi(k_2)\equiv\int_{k_2}^{\infty}f(x)dx=1-\int_{-\infty}^{0}f(x)dx-\int_{0}^{k_2}f(x)dx$$
$$\pi(k_1, k_2)\equiv\int_{k_1}^{k_2}f(x)dx.$$

Partially differentiating $U(k_1, k_2)$ for $k_1=k_1^\circ$ and $k_2=k_2^\circ$, we find

$$\begin{aligned}\frac{\partial U(k_1^\circ, k_2)}{\partial k_1^\circ}&=f(k_1^\circ)\{-F'-C'k_1^\circ-qH'+q(L-D')k_1^\circ\\&\quad-(h-q)E'-(p-q)M'k_1^\circ\}\\\frac{\partial U(k_1, k_2^\circ)}{\partial k_2^\circ}&=f(k_2^\circ)\{F'-F+(C'-C)k_2^\circ+qH'-pH\\&\quad+[q(D'-L)-p(D-L)]k_2^\circ\\&\quad+(h-q)E'-(1-p)E+(p-q)M'k_2^\circ\}.\end{aligned}\tag{14}$$

By setting the equations equal to zero, we get

$$\begin{aligned}k_1^\circ&=\frac{Y'}{X'}\\k_2^\circ&=\frac{Y-Y'}{X-X'}\end{aligned}\tag{15}$$

where

$$X=p(L-D)-C$$
$$X'=q(L-D')-C'-(p-q)M'$$
$$Y=F+pH+(1-p)E$$
$$Y'=F'+qH'+(h-q)E'.$$

X' is considered positive for the same reason as explained for k in the previous section. The numerator and denominator in the second equation are also positive. The expected fixed costs of a full investigation, Y, must be larger than those of an exploratory investigation, Y', while the expected marginal benefit yielded from the former, X, must be greater than those from the latter, X'.

The condition $k_1\leq k_2$ is satisfied if

$$\frac{X}{X'}\leq\frac{Y}{Y'}\ .\tag{16}$$

The above inequality implies that the relative fixed costs of a full investigation compared to an exploratory investigation must not be less than the relative expected marginal benefit of the former compared to the latter. This is a necessary condition for keeping an exploratory investigation useful. If the inequality does not hold, i. e., the relative marginal benefits exceed the relative fixed costs, the exploratory investigation becomes totally unattractive. For the full investigation brings larger net benefits regardless of size of cost variances.

Examining the second-order or sufficient conditions for a minimum, we find

$$\frac{\partial^2 U(k_1^\circ, k_2)}{\partial k_1^{\circ 2}} > 0 \qquad \frac{\partial^2 U(k_1, k_2^\circ)}{\partial k_2^{\circ 2}} > 0$$

$$\begin{vmatrix} \dfrac{\partial^2 U(k_1^\circ, k_2)}{\partial k_1^{\circ 2}} & \dfrac{\partial^2 U(k_1^\circ, k_2^\circ)}{\partial k_1^\circ \, \partial k_2^\circ} \\ \dfrac{\partial^2 U(k_1^\circ, k_2^\circ)}{\partial k_2^\circ \, \partial k_1^\circ} & \dfrac{\partial^2 U(k_1, k_2^\circ)}{\partial k_2^{\circ 2}} \end{vmatrix} > 0 \qquad (17)$$

Hence, we conclude that k_1° and k_2° given in (15) are optimal control limits which produce the minimum expected costs. It is important to note that the optimal control limit for an exploratory investigation, k_1^*, is independent of the decision of a full investigation while the optimal control limit for the latter, k_2^*, is affected by the cost and benefit parameters and probabilities associated with the former. As shown in (15), k_2^* depends upon the differences in the expected fixed costs and marginal benefits between both investigations. As the exploratory investigation becomes more successful, i. e., the probability that it will discover the cause of a variance becomes larger, and the difference in the fixed costs relative to the marginal benefit increases, k_2^* becomes bigger and the range of the exploratory investigation will be broaden.

To measure how an exploratory investigation affects the optimal control limit for a full investigation derived in the previous section, we take the difference between k_2^* and $\bar{k}^*$,

$$k_2^* - k^* = \frac{X'Y - XY'}{(X - X')X} \geq 0 \tag{18}$$

It follows from (16) that the difference is nonnegative and hence introducing an exploratory investigation shifts the control limit for a full investigation away from the origin (zero variance).

V. The Value of an Optimal Control Scheme

As we described in the second section, the traditional cost variance investigation scheme does not have a formal expression for control limits. The famous "materiality" rule of accounting, which is most often used as 10% of the standard cost, does not have any underlying basis. When statistical quality control charts are applied to accounting variance investigation decision, control limits are virtually always set at two or three standard deviations. But, their justification are given on a statistical basis rather than on an economic basis.

This section defines the value of an optimal control scheme with the control limit derived in the third section and shows that its expected cost is less than that of a traditional rule of thumb. In so doing, we assume that a cost variance is normally distributed with a mean zero and standard deviation σ, that is,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \tag{19}$$

By means of the method of substitution, we get

$$\int x f(x)dx = \int x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} dx = -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} + C$$
$$= -\sigma^2 f(x) + C \tag{20}$$

Using this result, we calculate

$$\int_{-\infty}^{0} x f(x)dx = -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\Big|_{-\infty}^{0} = -\frac{\sigma}{\sqrt{2\pi}} = -\sigma^2 f(0)$$
$$\int_{0}^{k} x f(x)dx = -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\Big|_{0}^{k} = -\sigma^2 f(k) - (-\sigma^2) f(0)$$
$$= \sigma^2 f(0) - \sigma^2 f(k) \tag{21}$$
$$\int_{k}^{\infty} x f(x)dx = -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\Big|_{k}^{\infty} = 0 - (-\sigma^2) f(k) = \sigma^2 f(k).$$

If the optlmal control limit is used, the expected total costs become

$$
\begin{aligned}
U(k^*) &= \int_{-\infty}^{0} Axf(x)dx + \int_{0}^{k^*} Bxf(x)dx \\
&\quad + \int_{k^*}^{\infty} (Bx+F+Cx)f(x)dx \\
&\quad + p\int_{k^*}^{\infty} (H+Dx-Lx)f(x)dx + (1-p)\int_{k^*}^{\infty} Ef(x)dx \\
&= -A\sigma^2 f(0) + B\{\sigma^2 f(0) - \sigma^2 f(k^*)\} + F\{1-F(k^*)\} \qquad (22) \\
&\quad + (B+C)\sigma^2 f(k^*) + pH\{1-F(k^*)\} \\
&\quad - p(L-D)\sigma^2 f(k^*) + (1-p)E\{1-F(k^*)\} \\
&= -A\sigma^2 f(0) + B\sigma^2 f(0) + F\{1-F(k^*)\} + C\sigma^2 f(k^*) \\
&\quad + pH\{1-F(k^*)\} - p(L-D)\sigma^2 f(k^*) \\
&\quad + (1-p)E\{1-F(k^*)\}
\end{aligned}
$$

where

$$
F(k^*) = \int_{-\infty}^{k^*} f(x)dx = \frac{1}{2} + \int_{0}^{k^*} f(x)dx.
$$

Suppose that we adopt the rule of thumb which states that if cost variances fall outside $m\sigma$, an investigation is undertaken. The expected total costs under this policy are

$$
\begin{aligned}
U(m) &= \int_{-\infty}^{0} Axf(x)dx + \int_{0}^{m\sigma} Bxf(x)dx \\
&\quad + \int_{m\sigma}^{\infty} (Bx+F+Cx)f(x)dx \\
&\quad + p\int_{m\sigma}^{\infty} (H+Dx-Lx)f(x)dx + (1-p)\int_{m\sigma}^{\infty} Ef(x)dx \\
&= -A\sigma^2 f(0) + B\{\sigma^2 f(0) - \sigma^2 f(m\sigma)\} + F\{1-F(m\sigma)\} \qquad (23) \\
&\quad + (B+C)\sigma^2 f(m\sigma) \\
&\quad + pH\{1-F(m\sigma)\} - p(L-D)\sigma^2 f(m\sigma) \\
&\quad + (1-p)E\{1-F(m\sigma)\} \\
&= -A\sigma^2 f(0) + B\sigma^2 f(0) + F\{1-F(m\sigma)\} + C\sigma^2 f(m\sigma) \\
&\quad + pH\{1-F(m\sigma)\} - p(L-D)\sigma^2 f(m\sigma) \\
&\quad + (1-p)E\{1-F(m\sigma)\}.
\end{aligned}
$$

The value of an optimal control scheme with k^*, denoted by $V(k^*, m)$, is defined as the difference between $U(m)$ and $U(k^*)$,

$$
\begin{aligned}
V(k^*, m) &= U(m) - U(k^*) \\
&= \{F+pH+(1-p)E\}\{F(k^*)-F(m\sigma)\} \\
&\quad + \{p(L-D)-C\}\{f(k^*)-f(m\sigma)\}\sigma^2 \\
&= \{F+pH+(1-p)E\}\{F_N(z^*)-F_N(m)\} \\
&\quad + \{p(L-D)-C\}\{f_N(z^*)-f_N(m)\}\sigma
\end{aligned} \tag{24}
$$

where $f_N(\cdot)$ and $F_N(\cdot)$ are the density function and the cumulative probability of the standardized normal distribution, respectively, and $z^*=k^*/\sigma$.

Since k^* is the optimal control limit which brings minimum expected costs, $V(k^*, m)$ must be nonnegative and it takes minimum value, zero, when $m\sigma$ is set equal to the optimal level, k^*. To prove this, we take the first derivative of $V(k^*, m)$ for $m=m^\circ$ and set it equal to zero,

$$
\begin{aligned}
\frac{dV(k^*, m^\circ)}{dm^\circ} &= \{F+pH+(1-p)E\}\left(-\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{1}{2}m^2} \\
&\quad + m\{p(L-D)-C\}\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}m^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}m^2}[\{p(L-D)-C\}m\sigma - \{F \\
&\quad +pH+(1-p)E\}]=0.
\end{aligned} \tag{25}
$$

Since the second derivative for $m=m^\circ$ is positive, i. e.,

$$
\frac{d^2V(k^*, m^\circ)}{dm^{\circ 2}} = \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}m^2}\right)[\sigma\{p(L-D-C\}]>0 \tag{26}
$$

$V(k^*, m)$ takes its minimum value if

$$
m\sigma = \frac{F+pH+(1-p)E}{p(L-D)-C} \tag{27}
$$

which is derived from (25).

As stated, $m\sigma$ in (27) is equal to k^* given in (11). Hence, the expected total costs are minimized by using the proposed rule as compared to the rule of thumb.

A simple numerical example may be useful to illustrate this point. For the specific problem introduced earlier, suppose that a cost accountant has used 2σ as a control limit and $\sigma=500$. Referring to a normal distribution table, we obtain

$$F_N(z^*) = .99744; \qquad F_N(2) = .9772$$
$$f_N(z^*) = .00792; \qquad f_N(2) = .05399.$$

Then, $V(k^*, m)$ is calculated as follows:

$$\begin{aligned} V(k^*, m) &= \$28{,}000(.99744 - .9772) + \$20(500)(.00792 - .05399) \\ &= \$566.72 - \$460.70 \\ &= \$106.02. \end{aligned}$$

Thus \$106.02 will be saved by using the optimal control limit k^*.

VI. Summary

Many management control systems operate on the principle of management by exception. Standard cost control systems are one such example. For its operation the principle presupposes some criterion which makes a distinction between the situations that warrant management attention and those that do not. A control limit is the criterion which has been heavily used in standard cost control. By setting control limits about the standard, and investigating only those variances outside the control limits, management can concentrate on exceptional items.

In spite of its popularity, little attention has been given to the problem of determining control limits on an economic basis and the formal expression of optimal control limits has never been sought in the accounting literature. Consequently, in practice, managements establish control limits by judgment and determine without any analyses whether a cost variance was worth investigation.

This papar has proposed a model for setting optimal control limits in the case of a single period and a continuum of states. It was obtained as an explicit function of the cost and benefit parameters in the investigation decision problem. The model was extended to include an exploratory investigation. The option of conducting two levels of investigation was examined. The first control limit would then serve as a warning signal for an exploratory investigation while the second one would become a criterion to trigger a full investigation. Also, the proposed model is applicable regardless of the form of the continuous distribution of a cost variance. This is of great practical value since a cost variance may not necessarily be normally distributed as assumed in most of the previous studies. Finally, comparison of the proposed control scheme with the traditional rule of thumb was made and it was shown that it leads to lower expected costs as compared to other models.

Footnotes

(1) In a comprehensive survey of models and techniques, Kaplan (1975) classified existing models for determining when to investigate cost variances along two dimensions. The first was whether the investigation decision is made on the basis of a single observation or whether some past sequence of observation is considered in the decision. Models were thus classified as single-period or multi-period. The second dimension was whether or not the model explicitly includes the expected costs and benefits of investigation in determining when to investigate a variance. His classification is summarized in a 2×2 table as shown below.

A Taxonomy of Deviation Investigation Models

	Costs and Benefits of Investigation not Considered	Costs and Benefits of Investigation Considered
Sngle-Period	Zannetos (1964) Juers (1967), Koehler (1968) Luh (1968), Probst (1971) Buzby (1974)	Duncan (1956) Bierman, Fouraker, and Jaedicke (1961)
Multi-Period	Cumulative-Sum Chart as in Page (1954). Also Barnard (1959) Chernoff and Zacks (1964)	Duvall (1967) Kaplan (1969) Dyckman (1969) Bather (1963)

(2) Bierman and Dyckman (1971) : 33.

(3) Amey and Egginton (1973) : 488–9.

(4) In practice, the $\overline{X}$ charts are modified on an ad hoc basis to detect a run of observations in excess of 2σ or 3σ. Sometime they are used with action limits at 3σ and warning limits at 2σ. But, as Kaplan (1975) claims, there is no generally accepted modification to this classic control chart.

(5) In the case where budgetary and manpower constraints are imposed on period's investigation activities, the investigation decision problem may be formulated as a mathematical program. Dyckman (1969) expressed one means of selecting the optimal set of processes to be investigated in any period by an integer programming method which would maximize the expected return based on the state probabilities for each process at the end of the period.

Ozan and Dyckman (1971) viewed the investigation decision as a capital investment decision under uncertainty and incorporated the three objective functions which allowed a company to specify its attitude toward risk involved in the investigation decision under uncertainty.

(6) Girshick and Rubin (1952) studied the statistical control model in a production process which consisted of a machine with four possible states: two performance levels and two states occurring during overhauls. They gave the minimum cost control procedure for the case in which the time to breakdown had a geometric distribution. This optimal procedure is complex and severely restricted because of the introduction of equilibrium distributions. As Bather (1963) suggested and Kaplan (1969) adopted, the difficulties associated with statistical equilibrium can be avoided by using the methods of dynamic programming.

(7) See Kaplan (1975): 320.

(8) Duvall (1966) developed a model in which the state of the system was described by a continuous variable. Each observed deviation, x, was composed of two parts: (i) a part due to non-controllable causes, w, and (ii) a part due to controllable causes, y. His benefit function from an investigation was assumed to be a direct function of the continuous variable, y. On the other hand, he also assumed that the cost of an investigation was some constant, C. Hence, the profit resulting from an investigation was written as

$$P(y)=\begin{cases} ky-C & \text{if } y>0 \\ -(Ly+C) & \text{if } y\leq 0. \end{cases}$$

(9) I am indebted to Professor Jacob Marschak for suggesting the technique used in this paper. For the analysis of management-by-exception information structures, see Marschak and Radner (1972): 206–217.

(10) The derivative of a definite integral with respect to the upper limit of integration is equal to the value of the integrand at this upper limit:

$$\frac{d}{dk}\int_a^k f(x)\,dx=f(k).$$

(11) If the probability that the cause of the deviation is discovered is independent of the probability that the cause resulted from controllable factors, q will be equal to ph. It is assumed that a full investigation will discover the cause of deviation resulting from controllable factors.

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