INVESTMENT AND OUTPUT

——In Case of the I-O Analyses——

Yuichi Shinzawa

Introduction

Input-output analysis in the modern economics originated with Dr. Wassily W. Leontief's celebrated work entitled "The Structure of American Economy."(1) Since its publication in 1941, Leontief himself, needless to say, has elaborated his theory in many subsequent papers and books. Likewise, numerous authors have devoted great effort to the theoretical and mathematical development of the analysis, to its actual applications and to the theoretical attempts to combine it with Walrasian, Keynesian or even Marxian economic theories.

Recently, however, there seems to be a tendency to presume that those interested in economics, from experts to laymen, hold a common understanding of this popular input-output theory. Technicians in the field of mathematics, especially, have been attempting to analyze rather pseudo input-output systems without paying any attention to their significance in economics. So as to be helpful to such people this paper presents a rather fundamental and theoretical study of the structure of so-called "input-output analysis" in order to show one of the proper ways to work with the theory.

In the first section, comparing the original input-output table of Leontief with my own "Generalized Reproduction Scheme,"(2) the relation between the net input-output table and the gross input-output

(1) Wassily W. Leontief, "The Structure of American Economy, 1919—1939, An empirical application of equilibrium analysis," (Oxford University Press, Second Edition, Enlarged, 3rd. Printing, 1960).

(2) Yuichi Shinzawa, "An Essay on Solving the So-called Transformation Problem," Waseda Business and Economic Studies, No. 14, 1978.

table is described.

Section 2 deals with Leontief's concept of saving coefficients.(3)

In section 3, Oskar Lange's reproduction scheme(4) will be analyzed in detail.

In section 4, using my scheme, the meaning of investment in the input-output table will be clarified, and one of new ideas on investment will be presented.

1. Leontief's Original Input-Output Table and the Gross Input-Output Table: —A Comparison Based on Consepts of the Generalized Reproduction Scheme—

The original input-output table presented by Leontief in 1941 consisted of the following equations system as basic equations in a stationary equilibrium (a hypothetical state of simple reproduction with neither saving nor investment).(5)

At the output side

$$
\begin{array}{l}
-X_1+x_{12}+x_{13}+\cdots\cdots\cdots\cdots\cdots\cdots\cdots+x_{1j}+\cdots\cdots+x_{1n}=0\\
x_{21}-X_2+x_{23}+\cdots\cdots\cdots\cdots\cdots\cdots\cdots+x_{2j}+\cdots\cdots+x_{2n}=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{i1}+x_{i2}+x_{i3}+\cdots\cdots-X_i+\cdots\cdots+x_{ij}+\cdots\cdots+x_{in}=0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{n1}+x_{n2}+x_{n3}+\cdots\cdots\cdots\cdots\cdots\cdots\cdots+x_{nj}+\cdots\cdots-X_n=0
\end{array}
\qquad (1)
$$

where X_i indicates the net output of the ith industry which is equal to the total output minus the amounts of its products consumed within the same industry, and where x_{i1}, x_{i2}, x_{i3}, ……, x_{ij}, ……, x_{in} stand for the amounts of its products absorbed by the jth industry respectively. For the sake of uniformity in developing this discussion, notice that the subscripts representing a transfer of output from the ith industry

(3) Leontief, W. W., *ibid.*, Part II, The theoretical scheme, pp. 33—45.

(4) Oskar Lange, "Theory of Reproduction and Accumulation," Pergamon Press, PWN-Polish Scientific Publishers, Warzawa, 1st English edition, 1969.

(5) Leontief, W. W., *ibid.*, p. 35.

to the jth industry is denoted by ij, not ji as seen in the Leontief's table, *i.e.*, ij does not denote $i \longleftarrow j$, but $i \longrightarrow j$.

Leontief emphasized the exclusion of all intertransaction (x_{ii}) in the same industry from his original table in order to avoid double counting output. Mathematically speaking, diagonal elements x_{ii} ($i=1, 2, \cdots\cdots, n$) are eliminated from the matrix of X.

$$X \equiv \begin{pmatrix} x_{11}\, x_{12} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \, x_{1n} \\ x_{21}\, x_{22} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \, x_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n1}\, x_{n2} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \, x_{nn} \end{pmatrix} \qquad (2)$$

This type of input-output table excluding diagonal elements is called a net input-output table. Since "The Input-Output Table of the United States in 1947"(6) was published, however, the gross input-output tables including the diagonal elements, *i.e.*, intertransactions in the same industry, have been more widely used than the net input-output table. As to this tendency, Dr. Michio Hatanaka said that "although the superiority of the former (the gross input-output table) has not been established, ..."(7)

Even Leontief himself preferring the gross input-output concepts to his original table, developed accordingly his works such as in 1965.(8) The problem of choosing between either a net input-output table or a gross input-output table will be discussed at the end of this section.

Returning to the original input-output table, some concepts of Leontief's scheme will now be compared with those of mine.

Sectors:

Leontief devided the entire economy into n sectors and regarded the household, the nth sector as a pseudo industry. In my model, whole

(6) Wassily W. Leontief and others, "Studies in the Structure of the American Economy," New York, Oxford University Press, 1953.

(7) Michio Hatanaka, "The Workability of Input-Output Analysis," Fachverlag für Wirtschaftstheorie und Ökonometrie, 1960, p. 13.

(8) Wassily Leontief, "Input-Output Economics," New York, Oxford University Press, 1966.

industries consist of n sectors and the household is regarded as a separate sector. In the household sector, people, 1) offer their labour power and services, 2) hold rights to capital assets invested in the form of stock shares, securities, savings accounts or money, 3) receive income as wages, salaries, yields of capital assets invested in industries and spend money to buy goods and services produced in the industries. If the household is regarded as an industry, it is the $n+1$th pseudo industry. Therefore there is no essential difference between the two tables other than the question of the number of industries. In this section, it is assumed that the number of industries is n, including the household as the nth pseudo industry, in order that both tables may stand on the equal ground for discussion.

Variables:

Leontief denoted net output in the ith sector by X_i and the amount of products of the ith industry, absorbed by jth industry by x_{ij}. In my model, Q_i represents total output, including intertransactions in the same industry, and Q_{ij} is the same as that of Leontief's x_{ij}. Therefore $X_i<Q_i$ and $x_{ij}=Q_{ij}$. Regarding prices, P_i is defined as the price of products measured by money value of the ith industry in Leontief's table. However in my scheme, P_i stands for the absolute price[9] of products measured by the value of labour power in the ith industry. If it is assumed that both unit values of money and labour power are constant or in a constant proportional relationship with each other, both standards for prices may be exchangeable. If this assumption is deemed acceptable, both tables may be considered analogous to each other. Comparison of Leontief's table with that of mine, however, is not so simple.

The relationship between the two schemes

$$X_i=Q_i-Q_{ii}. \tag{3}$$

Therefore the technological coefficients of production are different

(9) As to the relative price, refer to "An Essay on Solving the So-called Transformation Problem," Shinzawa, Yuichi, WBES (No. 14), 1978.

from each other.

$$a'_{ij}=\frac{X_{ij}}{X_i}=\frac{Q_{ij}}{Q_i-Q_{ii}} \tag{4}$$

in the Leontief's table, and

$$a_{ij}=\frac{Q_{ij}}{Q_i} \tag{5}$$

in my scheme.

Then

$$a'_{ij}>a_{ij}.$$

If $Q_{ii}=0$ then $\frac{Q_{ij}}{Q_i}\longrightarrow\frac{X_{ij}}{X_i}=a'_{ij}.$ (6)

If x_{ii} is added to X_i, then

$$\frac{x_{ij}}{X_i+x_{ii}}\longrightarrow\frac{Q_{ij}}{Q_i}=a_{ij}. \tag{7}$$

Then the following question arises as "what is the true character or meaning of total net output?" Before discussing this problem, it is best to rearrange both schemes as follows.

Output phase

Leontief's scheme

$$(\mathrm{I})\quad \begin{array}{l} -X_1+x_{12}+x_{13}+\cdots\cdots+x_{1n}=0 \\ x_{21}-X_2+x_{23}+\cdots\cdots+x_{2n}=0 \\ \vdots \\ x_{n1}+x_{n2}+x_{n3}+\cdots\cdots-X_n=0 \end{array}$$

my scheme

$$(\mathrm{II})\quad \begin{array}{l} Q_{11}+Q_{12}+Q_{13}+\cdots\cdots+Q_{1n-1}+Q_{1n}=Q_1 \\ Q_{21}+Q_{22}+Q_{23}+\cdots\cdots+Q_{2n-1}+Q_{2n}=Q_2 \\ \vdots \\ Q_{n-11}+Q_{n-12}+Q_{n-13}+\cdots\cdots+Q_{n-1\,n-1}+Q_{n-1\,n}=Q_{n-1} \\ \quad -X_i=Q_{ii}-Q_i \end{array} \tag{3$'$}$$

Input phase

Leontief's scheme

$$\text{(III)}\quad\begin{aligned}&-X_1P_1+x_{21}P_2+x_{31}P_3+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+x_{n1}P_n=0\\&x_{12}P_1-X_2P_2+x_{32}P_3+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+x_{n2}P_n=0\\&\vdots\\&x_{1n}P_1+x_{2n}P_2+x_{3n}P_3+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad-X_nP_n=0\end{aligned}$$

my scheme

$$\text{(IV)}\quad\begin{aligned}&Q_{11}P_1+Q_{21}P_2+Q_{31}P_3+\cdots\cdots+Q_{n-11}P_{n-1}+(1+\gamma_1\beta_1)V_1=Q_1\\&Q_{12}P_1+Q_{22}P_2+Q_{32}P_3+\cdots\cdots+Q_{n-12}P_{n-1}+(1+\gamma_2\beta_2)V_2=Q_2\\&\vdots\\&Q_{1n-1}P_1+Q_{2n-1}P_2+Q_{3n-1}P_3+\cdots\cdots\\&\qquad+Q_{n-1\,n-1}P_{n-1}+(1+\gamma_{n-1}\beta_{n-1})V_{n-1}=Q_{n-1}\end{aligned}$$

Dividing both sides of the set of equations (IV) by Q_i $(i=1, 2, \cdots\cdots, n)$

$$\text{(V)}\quad\begin{aligned}&a_{11}P_1+a_{21}P_2+a_{31}P_3+\cdots\cdots+a_{n-11}P_{n-1}+(1+\gamma_1\beta_1)L_1=P_1\\&a_{12}P_1+a_{22}P_2+a_{32}P_3+\cdots\cdots+a_{n-12}P_{n-1}+(1+\gamma_2\beta_2)L_2=P_2\\&\vdots\\&a_{1n-1}P_1+a_{2n-1}P_2+a_{3n-1}P_3+\cdots\cdots\\&\qquad+a_{n-1n-1}P_{n-1}+(1+\gamma_{n-1}\beta_{n-1})L_{n-1}=P_{n-1}.\end{aligned}$$

As $\quad a_{ij}=\dfrac{Q_{ij}}{Q_i}\quad$ then $\quad a_{ij}Q_i=Q_{ij}.$ (8)

If one substitutes the relations (8) into (II), then

$$\text{(VI)}\quad\begin{aligned}&a_{11}Q_1+a_{12}Q_2+a_{13}Q_3+\cdots\cdots+a_{1n-1}Q_{n-1}+(1+\gamma_1\beta_1)Q_n=Q_1\\&a_{21}Q_1+a_{22}Q_2+a_{23}Q_3+\cdots\cdots+a_{2n-1}Q_{n-1}+(1+\gamma_2\beta_2)Q_n=Q_2\\&\vdots\\&a_{n-11}Q_1+a_{n-12}Q_2+a_{n-13}Q_3+\cdots\cdots\\&\qquad+a_{n-1\,n-1}Q_{n-1}+(1+\gamma_{n-1}\beta_{n-1})Q_n=Q_{n-1}.\end{aligned}$$

Dividing both sides of the set of equations (III) by X_i $(i=1, 2, \cdots, n)$

$$\text{(VII)}\quad\begin{aligned}&-P_1\quad+a'_{21}P_2+a'_{31}P_3+\cdots\cdots+a'_{n1}P_n=0\\&a'_{12}P_1-\ P_2\ +a'_{32}P_3+\cdots\cdots+a'_{n2}P_n=0\\&\vdots\\&a'_{1n}P_1+a'_{2n}P_2+a'_{3n}P_3+\ \cdots\cdots\cdots\ -P_n=0\end{aligned}$$

subtracting P_i $(i=1, 2, \cdots\cdots, n)$ from both sides of each equation of

(V), the set of equations of (V) is

$$
\text{(VIII)}\quad
\begin{aligned}
&-(1-a_{11})P_1+a_{21}P_2+a_{31}P_3+\cdots\cdots+a_{n-11}P_{n-1}+(1+\gamma_1\beta_1)L_1=0\\
&a_{12}P_1-(1-a_{22})P_2+a_{32}P_3+\cdots\cdots+a_{n-12}P_{n-1}+(1+\gamma_2\beta_2)L_2=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{1n-1}P_1+a_{2n-1}P_2+a_{3n-1}P_3+\cdots\cdots\\
&\qquad-(1-a_{n-1\,n-1})P_{n-1}+(1+\gamma_{n-1}\beta_{n-1})L_{n-1}=0.
\end{aligned}
$$

As each element in (VIII) corresponds to that in (VII) except elements of the nth sector, *i. e.*, those of the household, subtracting the ith equation in (VII) from the ith equation in (VIII)

$$
\begin{aligned}
&(a'_{1i}-a_{1i})P_1+(a'_{2i}-a_{2i})P_2+(a'_{3i}-a_{3i})P_3+\cdots\cdots-a_{ii}P_i+\cdots\cdots\\
&+(a'_{n-1i}-a_{n-1i})P_{n-1}+\{a'_{ni}P_n-(1+\gamma_i\beta_i)L_i\}=0. \qquad (9)
\end{aligned}
$$

Dividing the ith equation by the ith diagonal element a_{ii}, each element of the equation (9) has the following relation:

$$
\text{if}\quad -a_{ii}P_i+\{a'_{ni}P_n-(1+\gamma_i\beta_i)L_i\}=0,\quad \frac{a'_{ji}-a_{ji}}{a_{ii}}=a'_{ji}. \qquad (10)
$$

Therefore

$$
a'_{ji}(1-a_{ii})=a_{ji}. \qquad (11)
$$

At the first stage of the original input-output table, Leontief assumes that a_{ii} is equal to zero and that there is no investment. Namely $\gamma_i=1$, and $a'_{ni}P_n=(1+\beta_i)L_i$. Therefore as long as the relation of the equation (11) is kept, both systems are reversible to each other and no substantial difference between the net input-output table and the gross input-output table exists at this stage of simple reproduction.

2. Saving Coefficients in the Leontief's Input-Output Table:

In the preceeding section, relationships between the net input-output table and the gross input-output table were described for the case of simple reproduction scheme. In this section beginning by quoting some passages from "The Structure of American Economy" the concept of investment in the original input-output table of Leontief will be analyzed.

In Part I, Leontief described the fundamental principles of the input-output table as follows:

"The economic activity of the whole country is visualized as if covered by one huge accounting system. ...... Each business enterprise as well as each individual household is treated as a separate accounting unit. ......For our particular purpose, only one is important: the expenditure and revenue account. It registers on its credit side the outflow of goods and services from the enterprise or household (which corresponds to total receipts or sales) and on the debit side acquisition of goods or services by the particular enterprise or household (p. 11). ......The structure of the expenditure and revenue account is very similar to that of the balance of trade of a country; it covers explicitly all the commodity and service transactions, but not the so-called capital items (p. 12)."

And in Part II "The Theoretical Scheme," he wrote,

"the difference between the aggregate expenditures of a household or an enterprise (or a groupe of households or enterprises) and its aggregate revenue is defined as investment when it is positive and as saving when it is negative (p. 42). ...... Introduction of savings and investments obviously requires modification of all the cost equations. The value product of any industry (or household), instead of being simply equal to its aggregate outlays, can now be either larger or smaller. In other words, total cost must now be equated to the total revenue divided by a certain saving coefficient, B_i:

$$-\frac{X_iP_i}{B_i}+(x_{1i}P_1+x_{2i}P_2+x_{3i}P_3+\cdots+x_{ki}P_k+\cdots+x_{ni}P_n)=0 \quad (12)$$

Whenever B_i is greater than 1, the particular industry shows positive savings; it equals 1 if the total revenue of the enterprise or household exactly covers its outlays; and it becomes smaller than 1 in the case of negative savings *i. e.,* positive investment (p. 43)."

Substituting the relational equation (11) into the ith equation of VII,

$$a_{1i}P_1+a_{2i}P_2+a_{3i}P_3+\cdots\cdots-\frac{(1-a_{ii})}{B_i}P_i+\cdots\cdots+a_{ni}P_n=0. \quad (13)$$

By subtracting the ith equation of (V) from the equation (13) and rearranging the remaining terms, the following equation is obtained.

$$\left(\frac{1}{B_i}-1\right)(1-a_{ii})P_i=a_{ni}P_n-(1+\gamma_i\beta_i)L_i \quad (14)$$

In the equation (12) $a_{ni}\ P_n$ is the input of revenue or labour power of the household to a unit of products in the ith industry in the Leontief's system in other words, income per a unit of products paid for the household from ith industry. But $(1+\gamma_i\beta_i)\ L_i$ is the input of labour power from which investment is excluded, in other words, the part of income which can be spent for consumers' goods and services.

As quoted above Leontief's conditions on B_i are

if $B_i>1$, savings is positive,

if $B_i<1$, investment is positive

and if $B_i=1$, savings is equal to investment and zero.

From the equation (14), if $B_i \gtreqless 1$, then

$$0 \gtreqless a_{ni}P_n-(1+\gamma_i\beta_i)L_i. \quad (15)$$

As $Y_{li}=(1+\beta_i)V_i$ and $I_{li}=(1-\gamma_i)\ \beta_i V_i$, the equation (15) can be re-written

$$Y_{li}-I_{li} \gtreqless a_{ni}P_nQ_i \quad (16)$$

and then,

$$S_{li} \gtreqless a_{ni}P_nQ_i. \quad (17)$$

Whenever one accepts the concept of the saving coefficient B_i, and the assumption, that $a_{ni}P_nQ_i$ is neither income from the ith industry to the household, nor the consumption part of the household, then $a_{ni}P_nQ_i$ has to be equal to investment.

And, as long as $a_{ni}P_nQ_i$ is defined as investment,
if $B_i \gtreqless 1$ then the Leontief's conclusion that $S_{li} \gtreqless I_{li}$ is seemingly valid. However, since $a_{ni}P_nQ_i$ is, as previously defined the input of service

or labour power of the household to a unit of products in the ith industry, there is nothing guaranteeing that $a_{ni}P_nQ_i$ is investment. In my system $a_{ni}P_nQ_i$ is only corresponding concept to that of $(1+\gamma_i\beta_i)V_i$, *i. e.*, investment has been made.

It is quite obvious that the principle of saving coefficients was based on the concept of equilibrium between savings and investment, which was derived from the theoretical developments of economics in the earlier years of 1930s, before J. M. Keynes published "The General Theory of Employment, Interest and Money."(10) Keynes, defining the concepts of income, investment, saving, aggregate sales and user cost, proved the identity of investment and saving. He described that "whilst, therefore, the amount of saving is an outcome of the collective behaviour of individual consumers and the amount of investment of the collective behaviour of individual entrepreneurs, these two amounts are necessarily equal, since each of them is equal to the excess of income over consumption. Moreover, this conclusion in no way depends on any subtleties or peculiarities in the definition of income given above. Provided it is agreed that income is equal to the value of current output, that current investment is equal to the value of that part of current output which is not consumed, and that saving is equal to the excess of income over consumption——all of which is conformable both to common sense and to the traditional usage of the great majority of economists——the equality of saving and investment necessarily follows. In short——

Income = value of output = consumption + investment.

Saving = income − consumption.

Therefore saving = investment."(11)

Disregarding the general notion of the saving coefficients from the original input-output table of Leontief, $B_i=1$, accordingly,

(10) John Maynard Keynes, "The General Theory of Employment, Interest and Money," MacMillan and Co., 1st edition, 1936.

(11) Keynes, J. M., "The General Theory," Ch. 6, p. 63.

$$a_{ni}P_n=(1+\gamma_i\beta_i)L_i \tag{18}$$

and there are no contradictions among Leontief's system, Keynes' system, and my system. In the Leontief's system, the concept of the value of net output (A_i^L) is equal to that of the value of aggregate sales (or the value of finished output which will have been sold) of the ith industry (A_i^K) in Keynes' definition, and the value of total output (A_i) in my system, in spite of the gross concept, is equal to ($A_i^K+a_{ii}$)

$$Y_i=A_i-U_i=(A_i-a_{ii})-(U_i-a_{ii})=A_i^K-U_i^K=A_i^L-U_i^L. \tag{19}$$

There is no essential difference among the three systems nor between net output and gross output for income analysis, except for whether or not selfconsumption is included in each system.

3. The Reproduction Scheme Presented by Oskar Lange:

In order to make the meaning of investment in the input-output table more clear, the reproduction scheme constructed by Dr. Oskar Lange will now be analyzed.

In the "Theory of Reproduction and Accumulation,"(12) Lange presented the fundamental explanation for the traditional theory of reproduction and showed relevant data on the rates of surplus values from countries such as Poland, Great Britain, the Soviet Union and U. S. A. Lange stated that the high rate of surplus value in the United States, "is due to the fact that the surplus products contains the capitalists' income, part of which is invested, and part is used to cover non-productive expenditure, primarily for all kinds of services. In the United States services of this kind (*e. g.* advertising expenditure) are more developed than in the European countries."(13) This statement reveals that Lange recognized the fact that a part of surplus value (products) is reinvested into the reproduction process; however, he did not try to

(12) Lange, Oskar, *ibid.*
(13) *op. cit.,* p. 15.

insert this idea with an iteration process into his reproduction model. Therefore in developing his scheme in "Chapter 2: Equilibrium Conditions,"[14] in spite of many similarities, Lange's model is quite different from that of mine. His scheme consists of the following equation.

$$c_i+v_i+m_{ic}+m_{iv}+m_{ik}=P_i \qquad (20)$$

This equation does not include iteration process in it. But our scheme[15] with iteration process of surplus value reinvested is defined as follows

$$\begin{aligned} &c_{it}+v_{it}+M_{ict}+M_{ivt}+M_{ikt}=P_t \\ &=(c_{it}+M_{ict})+(v_{it}+M_{ivt})+M_{ikt} \\ &=C_{it}+V_{it}+M_{ikt}. \end{aligned} \qquad (21)^{[16]}$$

If Lange's reproduction scheme is rewritten[17] and developed according to my way of reasoning it is transformed as follows,

$$\begin{aligned} &A_{11}+A_{21}+\cdots\cdots+A_{n1}+v_1+m_1=A_1 \\ &A_{12}+A_{22}+\cdots\cdots+A_{n2}+v_2+m_2=A_2 \\ &A_{13}+A_{23}+\cdots\cdots+A_{n3}+v_3+m_3=A_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &A_{1n}+A_{2n}+\cdots\cdots+A_{nn}+v_n+m_n=A_n \end{aligned} \qquad (22)$$

$$\begin{aligned} &a_{11}P_1+a_{21}P_2+\cdots\cdots+a_{n1}P_n+(1+\beta_1)L_1=P_1 \\ &a_{12}P_1+a_{22}P_2+\cdots\cdots+a_{n2}P_n+(1+\beta_2)L_2=P_2 \\ &a_{13}P_1+a_{23}P_2+\cdots\cdots+a_{n3}P_n+(1+\beta_3)L_3=P_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{1n}P_1+a_{2n}P_2+\cdots\cdots+a_{nn}P_n+(1+\beta_n)L_n=P_n \end{aligned} \qquad (23)$$

(14) *op. cit.*, pp. 22—43.

(15) Yuichi Shinzawa, "An Interpretation on the Reproduction Process," Waseda Business & Economic Studies (No. 13), 1977.

(16) Lange, *ibid.*, p. 41.

(17) Lange, *ibid.*, p. 46.

$$A'P + Dt = P \tag{24}$$

where

$$A' \equiv \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}, \quad P \equiv \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}, \quad t \equiv \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$$

and

$$D \equiv \begin{bmatrix} \frac{1+\beta_1}{\alpha_1+1+\beta_1} & & \\ & \ddots & \\ & & \frac{1+\beta_n}{\alpha_n+1+\beta_n} \end{bmatrix} \tag{25}$$

Then if relative value μ_n^L=relative price ν_n^L where,

$$\mu_n^L = \left[\mathrm{I} - A'_{nn} - D_{nn} \right]^{-1} g_n \tag{26}$$

$$\mu_n^L \equiv \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{n-1} \end{bmatrix} \qquad A'_{nn} \equiv \begin{bmatrix} a_{11} & \cdots & a_{n-1\,1} \\ \vdots & & \vdots \\ a_{n\,n-1} & \cdots & a_{n-1\;n-1} \end{bmatrix}$$

$$D \equiv \begin{bmatrix} \frac{1+\beta_1}{\alpha_1+1+\beta_1} & & \\ & \ddots & \\ & & \frac{1+\beta_{n-1}}{\alpha_{n-1}+1+\beta_{n-1}} \end{bmatrix} \quad \text{and}$$

$$g_n \equiv \begin{bmatrix} a_{n1} \\ \vdots \\ a_{n\;n-1} \end{bmatrix} \tag{27}$$

The ith diagonal elements of this inverse matrix is

$$1-a_{ii}-\frac{1+\beta_i}{\alpha_i+1+\beta_i}>0. \tag{28}$$

Comparing this elememt with the equivalent factor of my scheme, the element of Lange is just the same as that of mine when propensity to consume of capitalists (γ_i) is assumed to be zero.

In chapters 3 and 4, Lange explained an expanded input-output balance-sheet table including both reproduction and investment flows, and presented the following balance equations

$$\sum_{j=1}^{n} x_{ji}+x_{0i}+m_i=X_i \tag{29}$$

$$X_i=\sum_{j=1}^{n} x_{ij}+\sum_{j=1}^{n} I_{ij}+x_i^{(0)} \quad (i=1,\ 2,\ \cdots\cdots n) \tag{30}$$

or

$$X_i=\sum_{j=1}^{n}(x_{ij}+I_{ij})+x_i^{(0)} \tag{31}$$

where

X_i is the aggregate product of the ith branch

$x_i^{(0)}$ is the part consumed

I_i is the invested part of the final product

and

x_{0i} is the amount of labour employed.

m_i in the equation (29) is

$$m_i=\sum_{j=1}^{n} m_{ji}+m_{x_{0i}}+m_{ki}. \tag{32}$$

Substituting the equation (32) into (29)

$$\sum_{j=1}^{n}(x_{ji}+m_{ji})+x_{0i}+m_{x_{0i}}+m_{ki}=X_i. \tag{33}$$

From the equation (29) and (30) we get

$$x_{0i}+m_i=x_i^{(0)}+I_i. \tag{34}$$

If both sides of the equation (33) as to i are calculated,

$$\sum_{i=1}^{n} x_{0i} + \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ji} + \sum_{i=1}^{n} m_{vi} + \sum_{i=1}^{n} m_{ki} = \sum_{i=1}^{n} x_i^{(0)} + \sum_{i=1}^{n} I_i. \tag{35}$$

As $\sum_{i=1}^{n} x_i^{(0)}$ is consumption, the equation (35) denotes income. And if consumption is defined as follows:

$$\sum_{i=1}^{n} x_{0i} + \sum_{i=1}^{n} m_{ki} = \sum_{i=1}^{n} x_i^{(0)} \tag{36}$$

Investment is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} m_{ji} + \sum_{i=1}^{n} m_{vi} = \sum_{i=1}^{n} I_i. \tag{37}$$

However in my scheme including iteration process of surplus value if neglecting inventory problem for simplicity's sake, consumption (C_{lt}) is

$$C_{lt} = v_t + M_{vt} + M_{kt} \tag{38}$$

investment (I_{lt}) $\quad I_{lt} = M_t - M_{kt} = M_{ct} + M_{vt}$ (39)

and income (Y_{lt}) is

$$\begin{aligned} Y_{lt} &= C_{lt} + I_{lt} \\ &= v_t + M_{vt} + M_{ct} + M_{vt} + M_{kt} \\ &= V_t + M_t. \end{aligned} \tag{40}$$

In spite of rather the iteration process is included or not, investment factor of Lange's system is corresponding to that of my system. As for consumption, C_{lt}^L, however, as $\sum_{i=1}^{n} x_{0i} = v_t$ and $\sum_{i=1}^{n} m_{ki} = M_{kt}$, the equation (36) is not equal to (38).

$$v_t + M_{kt} = \sum_{i=1}^{n} x_i^{(0)} \neq C_{lt} = v_t + M_{vt} + M_{kt} \tag{41}$$

Investment is

$$I_{lt}^L = M_{ct} + M_{vt} \tag{42}$$

and income is

$$Y_{lt}^L = C_{lt}^L + I_{lt}^L = v_t + M_{kt} + M_{ct} + M_{vt} = v_t + M_t. \tag{43}$$

Consequently we can conclude that in Lange's system as long as M_{vt}

is included in I^{L}_{lt}, M_{vt} can create additional demand for consumption goods of workers so that total consumption has to be equal to C_{lt}. Superficially glancing at this conclusion, Lange seems to have made a mistake in building his scheme; but if we regard M_{vt} as an advance to workers, his scheme is entirely valid. In the subsequent chapters, Lange treats M_{ct} as investment and consumption is regarded as consisting of $V_t+M_{vt}+M_{kt}$. In this case, if investment is equal to saving as a definition and saving is the residual part of income from which consumption is subtracted, as Keynes proved in "The General Theory,"(18) then investment has to be equal to M_t-M_{kt} $(=M_{ct}+M_{vt})$. There is no contradiction in his theory as long as he assumes as Marx did that m_v is advanced.

By introducing the concept of the advance to workers into his scheme, Lange can be said to stand on Marxian ground more than Keynesian ground in spite of his opinion of Marx's "Capital."

In "Marxian Economics and Modern Economic Theory"(19) Lange said that "In a capitalist economy it requires, as Marx has shown himself in the third volume of Das Kapital, certain modifications due to differences in the organic composition of capital (*i. e.*, the ratio of the capital invested in capital goods to the capital invested in payment of wages) in different industries. Thus the labour theory of value has no qualities which would make it, from the Marxist point of view, superior to the modern more elaborate theory of economic equilibrium. It is only a more primitive form of the latter, restricted to the narrow field of pure competition and even not without its limitations in this field. Further, its most relevant statement (*i. e.*, the quality of price to average cost plus 'normal' profit) is included in the modern theory of economic equilibrium. Thus the labour theory of value can not possibly be the source of the superiority of Marxian over bourgeois'

(18) Keynes, J. M., *ibid.*, p. 63.

(19) Oskar Lange, "Marxian Economics and Modern Economic Theory," The Review of Economic Studies, June, 1935.

economics in explaining the phenomena of economic evaluation. In fact, the adherence to an antiquated form of the theory of economic equilibrium is the cause of the inferiority of Marxian economics in many field."[20]

In this paper I have no intention to be "*cymini sectores*" to the works of distinguished authors. My work is devoted to finding a workable way to synthesize economic theories by analyzing their similarities, analogies and differences. I have no prejudices against any theories as accepting the labour theory of value, I proved the possibilities of combining both theories of Marx and Keynes. Briefly speaking, whereas my work evolves from the Keynesian school of thought, I accept the Marxian theory; Lange, however, a follower of the Marxian economics, disregarded the important concept of the labour theory of value.

4. The Meaning of Investment in the Input-Output Table:

This section introduces my fundamental conception of the input-output table and discuss an applicable way of measuring investment.

The input-output table is divided into two phases——the input phase and the output phase. As to selecting between a net or a gross input-output table, the gross input-output table is chosen without hesitation because the products produced and used up within the same sector can be thought to constitute that part of output which I call a total output. Gimmick concepts such as the saving coefficient, the advanced money or pseudo industries will not be employed.

There are three kinds of input relations as to value, price and quantity respectively. Whole industries are divided into n sectors, and the household is the $n+1$ sector. Concepts used in the income analysis in this paper are assumed to be measured by the value of labour power. And a standard of such value in the ith sector is the total

(20) Lange, Oskar, *ibid.*, "Marx and Modern Economics" edited by David Horowitz, MacGibbon & Kee, 1968, pp. 77—79.

value paid for workers who offer their labour power to production in the ith sector, that is, V_i ($i=1, 2, \cdots\cdots, n$). V_i consists of the number of workers (N_i), the average working hour per worker (T_i) and the unit value of labour power (l) in common with each sector.

$$V_i = N_i \cdot T_i \cdot l \tag{44}$$

Whenever more realistic schemes evaluated by money are desired, the relational equation (44) can be easily transformed by utilizing concepts of wage unit, unit value of labour power l and unit utility of money as illustrated in my previous paper.[21] This problem will not be discussed here. Variables and parameters are defined as follows:

A_i······Value of total output in the ith sector

A_{ij}······Value of products from the ith sector to the jth sector. (In case of A_{ii}, it is products consumed within the same sector i.)

P_i······the absolute price of products in the ith sector which can be easily transformed into a relative price or production price as I proved in my previous paper.

Q_i······the volume of total output in the ith sector

β_i······the rate of surplus value in the ith sector

γ_i······the rate of capitalists' propensity to consume

L_i······the value paid for workers per unit of products.

$$L_i = \frac{V_i}{Q_i}$$

$(1+\gamma_i\beta_i)V_i$······a part of income from the ith sector, which people can spend for consumption and if neglecting the problem of inventory, it is used for consumption.

$(1+\beta_i)V_i$······income paid for people from the ith sector.

Formulating various input-output tables by from the variables and parameters defined above, results in the following six sets of equation systems:

(21) Yuichi Shinzawa, *ibid.*, WBES No. 13.

Input-output relations in an input phase:

(I-1) as to values

$$A_{11}+A_{21}+A_{31}+\cdots\cdots+A_{n1}+(1+\gamma_1\beta_1)V_1=A_1$$
$$A_{12}+A_{22}+A_{32}+\cdots\cdots+A_{n2}+(1+\gamma_2\beta_2)V_2=A_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{1n}+A_{2n}+A_{3n}+\cdots\cdots+A_{nn}+(1+\gamma_n\beta_n)V_n=A_n$$

(I-2) as to prices

$$a_{11}P_1+a_{21}P_2+a_{31}P_3+\cdots\cdots+a_{n1}P_n+(1+\gamma_1\beta_1)L_1=P_1$$
$$a_{12}P_1+a_{22}P_2+a_{32}P_3+\cdots\cdots+a_{n2}P_n+(1+\gamma_2\beta_2)L_2=P_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{1n}P_1+a_{2n}P_2+a_{3n}P_3+\cdots\cdots+a_{nn}P_n+(1+\gamma_n\beta_n)L_n=P_n \qquad \text{and,}$$

(I-3) as to quantities

$$Q_{11}+Q_{21}\frac{P_2}{P_1}+Q_{31}\frac{P_3}{P_1}+\cdots\cdots+Q_{n1}\frac{P_n}{P_1}+(1+\gamma_1\beta_1)\frac{V_1}{P_1}=Q_1$$
$$Q_{12}\frac{P_1}{P_2}+Q_{22}+Q_{32}\frac{P_3}{P_2}+\cdots\cdots+Q_{n2}\frac{P_n}{P_2}+(1+\gamma_2\beta_2)\frac{V_2}{P_2}=Q_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Q_{1n}\frac{P_1}{P_n}+Q_{2n}\frac{P_2}{P_n}+Q_{32}\frac{P_3}{P_n}+\cdots+Q_{nn}\ (1+\gamma_n\beta_n)\frac{V_n}{P_n}=Q_n.$$

In an output phase:

(II-1) as to values

$$A_{11}+A_{12}+A_{13}+\cdots\cdots+A_{1n}+A_{1n+1}=A_1$$
$$A_{21}+A_{22}+A_{23}+\cdots\cdots+A_{2n}+A_{2n+1}=A_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_{n1}+A_{n2}+A_{n3}+\cdots\cdots+A_{nn}+A_{n\ n+1}=A_n$$

(II-2) as to prices

$$a_{11}P_1+a_{12}P_1+a_{13}P_1+\cdots\cdots+a_{1n}P_1+\frac{Q_{1n+1}}{Q_1}P_1=P_1$$
$$a_{21}P_2+a_{22}P_2+a_{23}P_2+\cdots\cdots+a_{2n}P_2+\frac{Q_{2n+1}}{Q_2}P_2=P_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}P_n+a_{n2}P_n+a_{n3}P_n+\cdots\cdots+a_{nn}P_n+\frac{Q_{nn+1}}{Q_n}P_n=P_n \qquad \text{and,}$$

(II-3) as to quantities

$$a_{11}Q_1+a_{12}Q_2+a_{13}Q_3+\cdots\cdots+a_{1n}Q_n+Q_{1n+1}=Q_1$$
$$a_{21}Q_1+a_{22}Q_2+a_{23}Q_3+\cdots\cdots+a_{2n}Q_n+Q_{2n+1}=Q_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}Q_1+a_{n2}Q_2+a_{n3}Q_3+\cdots\cdots+a_{nn}Q_n+Q_{nn+1}=Q_n.$$

As A_{ji} includes a part of new investment to the jth industry, which denotes I_{ji}, if no investment occurs in the jth industry the following equation can be used:

$$\bar{A}_{ji}=A_{ji}-I_{ji}. \tag{45}$$

Therefore the following equation

$$A_{1i}+A_{2i}+A_{3i}+\cdots\cdots+A_{ni}+(1+\gamma_i\beta_i)V_i=A_i \tag{46}$$

can be rewritten as

$$\bar{A}_{1i}+\bar{A}_{2i}+\bar{A}_{3i}+\cdots\cdots+\bar{A}_{ni}+\sum_{j=1}^{n}I_{ji}+(1+\gamma_i\beta_i)V_i=A_i \tag{47}$$

and as

$$\sum_{j=1}^{n}I_{ji}=I_i=(1-\gamma_i)\beta_iV_i. \tag{48}$$

The equation (47) is

$$\bar{A}_{1i}+\bar{A}_{2i}+\bar{A}_{3i}+\cdots\cdots+\bar{A}_{ni}+(1+\beta_i)V_i=A_i.$$

The input-output table expressed by this relationship is exactly the same as the table from which Dr. Michio Hatanaka analyzed as the input-output table of 1947. In this table investment is excluded from A_{ji} in the ith sector and is assumed to be independent in a pseudo sector such as $(n-1)$. I will call this input-output table the investment separated type of input-output table.

Investment separated type:

in an input phase:

(I′-1) as to values

$$\bar{A}_{11}+\bar{A}_{21}+\bar{A}_{31}+\cdots\cdots+\bar{A}_{n1}+(1+\beta_1)V_1=A_1$$
$$\bar{A}_{12}+\bar{A}_{22}+\bar{A}_{32}+\cdots\cdots+\bar{A}_{n2}+(1+\beta_2)V_2=A_2$$

....

$$\bar{A}_{1n}+\bar{A}_{2n}+\bar{A}_{3n}+\cdots\cdots+\bar{A}_{nn}+(1+\beta_n)V_n=A_n$$

(I ′-2) as to prices

$$\bar{a}_{11}P_1+\bar{a}_{21}P_2+\bar{a}_{31}P_3+\cdots\cdots+\bar{a}_{n1}P_n+(1+\beta_1)L_1=P_1$$
$$\bar{a}_{12}P_1+\bar{a}_{22}P_2+\bar{a}_{32}P_3+\cdots\cdots+\bar{a}_{n2}P_n+(1+\beta_2)L_2=P_2$$

....

$$\bar{a}_{1n}P_1+\bar{a}_{2n}P_2+\bar{a}_{3n}P_3+\cdots\cdots+\bar{a}_{nn}P_n+(1+\beta_n)L_n=P_n,$$ and

in an output phase:

(II′-3) as to quantities

$$\bar{a}_{11}Q_1+\bar{a}_{12}Q_2+\bar{a}_{13}Q_3+\cdots\cdots+\bar{a}_{1n}Q_n+Q_{1n+1}+\frac{I_1}{P_1}=Q_1$$
$$\bar{a}_{21}Q_1+\bar{a}_{22}Q_2+\bar{a}_{23}Q_3+\cdots\cdots+\bar{a}_{2n}Q_n+Q_{2n+1}+\frac{I_2}{P_2}=Q_2$$

....

$$\bar{a}_{n1}Q_1+\bar{a}_{n2}Q_2+\bar{a}_{n3}Q_3+\cdots\cdots+\bar{a}_{nn}Q_n+Q_{nn+1}+\frac{I_n}{P_n}=Q_n$$

If one can subtract (I ′-2) from (I -2) in the input phase, and (II′-3) from (II-3) in the output phase the following sets of equations are formed:

(III-1)

$$(a_{11}-\bar{a}_{11})P_1+(a_{21}-\bar{a}_{21})P_2+(a_{31}-\bar{a}_{31})P_3+\cdots\cdots+(a_{n1}-\bar{a}_{n1})P_n=\beta_1(1-\gamma_1)$$
$$(a_{12}-\bar{a}_{12})P_1+(a_{22}-\bar{a}_{22})P_2+(a_{32}-\bar{a}_{32})P_3+\cdots\cdots+(a_{n2}-\bar{a}_{n2})P_n=\beta_2(1-\gamma_2)$$

....

$$(a_{1n}-\bar{a}_{1n})P_1+(a_{2n}-\bar{a}_{2n})P_2+(a_{3n}-\bar{a}_{3n})P_3+\cdots\cdots+(a_{nn}-\bar{a}_{nn})P_n=\beta_n(1-\gamma_n)$$

and

(III-2)

$$(a_{11}-\bar{a}_{11})Q_1+(a_{12}-\bar{a}_{12})Q_2+(a_{13}-\bar{a}_{13})Q_3+\cdots\cdots+(a_{1n}-\bar{a}_{1n})Q_n=\frac{I_1}{P_1}$$

$$(a_{21}-\bar{a}_{21})Q_1+(a_{22}-\bar{a}_{22})Q_2+(a_{23}-\bar{a}_{23})Q_3+\cdots\cdots+(a_{2n}-\bar{a}_{2n})Q_n=\frac{I_2}{P_2}$$

$$\cdots\cdots$$

$$(a_{n1}-\bar{a}_{n1})Q_1+(a_{n2}-\bar{a}_{n2})Q_2+(a_{n3}-\bar{a}_{n3})Q_3+\cdots\cdots+(a_{nn}-\bar{a}_{nn})Q_n=\frac{I_n}{P_n}$$

Defining that

$$A\equiv\begin{pmatrix} a_{11}\cdots\cdots a_{1n} \\ \vdots \qquad \vdots \\ a_{n1}\cdots\cdots a_{nn}\end{pmatrix},\quad \bar{A}\equiv\begin{pmatrix} \bar{a}_{11}\cdots\cdots \bar{a}_{1n} \\ \vdots \qquad \vdots \\ \bar{a}_{n1}\cdots\cdots \bar{a}_{nn}\end{pmatrix}$$

$$B\equiv\begin{pmatrix} (1+\gamma_1\beta_1) & & \\ & \ddots & \\ & & (1+\gamma_n\beta_n)\end{pmatrix},\quad \bar{B}\equiv\begin{pmatrix} (1+\beta_1) & & \\ & \ddots & \\ & & (1+\beta_n)\end{pmatrix}$$

$$L\equiv\begin{pmatrix} L_1 \\ \vdots \\ L_n\end{pmatrix},\quad P\equiv\begin{pmatrix} P_1 \\ \vdots \\ P_n\end{pmatrix},\quad Q\equiv\begin{pmatrix} Q_1 \\ \vdots \\ Q_n\end{pmatrix},\quad C\equiv\begin{pmatrix} Q_{1\ n+1} \\ \vdots \\ Q_{n\ n+1}\end{pmatrix} \qquad (49)$$

and

$$D\equiv\begin{pmatrix} (a_{11}-\bar{a}_{11}) & \cdots\cdots & (a_{1n}-\bar{a}_{1n}) \\ \vdots & & \vdots \\ (a_{n1}-\bar{a}_{n1}) & \cdots\cdots & (a_{nn}-\bar{a}_{nn})\end{pmatrix}$$

the equation systems above are rewritten as

(Ⅰ-2) $A'P+BL=P$

(Ⅰ′-2) $\bar{A}'P+\bar{B}L=P$

(Ⅱ-3) $AQ+C=Q$

(Ⅱ′-3) $\bar{A}Q+C+\begin{pmatrix} \frac{I_1}{P_1} \\ \vdots \\ \frac{I_n}{P_n}\end{pmatrix}=Q$

(Ⅲ-1) $(\boldsymbol{A}'-\bar{\boldsymbol{A}}')\boldsymbol{P}=(\boldsymbol{B}'-\bar{\boldsymbol{B}}')\boldsymbol{L}=\begin{pmatrix}\frac{I_1}{Q_1}\\ \vdots \\ \frac{I_n}{Q_n}\end{pmatrix}=\boldsymbol{D}'\boldsymbol{P}$

(Ⅲ-2) $(\boldsymbol{A}-\bar{\boldsymbol{A}})\boldsymbol{Q}=\begin{pmatrix}\frac{I_1}{P_1}\\ \vdots \\ \frac{I_n}{P_n}\end{pmatrix}=\boldsymbol{DQ}.$

(Ⅲ-1) and (Ⅲ-2) have a character of duality with each other as follows:

From (Ⅲ-1)

$$D'P=\begin{pmatrix}\frac{I_1}{Q_1}\\ \vdots \\ \frac{I_n}{Q_n}\end{pmatrix} \longrightarrow \begin{pmatrix}Q_1 & & \\ & \ddots & \\ & & Q_n\end{pmatrix}D'P=\begin{pmatrix}I_1\\ \vdots \\ I_n\end{pmatrix}$$

$$\longrightarrow P'D\begin{pmatrix}Q_1 & & \\ & \ddots & \\ & & Q_n\end{pmatrix}=[I_1\cdots\cdots I_n] \tag{50}$$

From (Ⅲ-2)

$$DQ=\begin{pmatrix}\frac{I_1}{P_1}\\ \vdots \\ \frac{I_n}{P_n}\end{pmatrix}=\begin{pmatrix}\frac{1}{P_1} & & \\ & \ddots & \\ & & \frac{1}{P_n}\end{pmatrix}\begin{pmatrix}I_1\\ \vdots \\ I_n\end{pmatrix} \longrightarrow$$

$$\begin{pmatrix}P_1 & & \\ & \ddots & \\ & & P_n\end{pmatrix}DQ=\begin{pmatrix}I_1\\ \vdots \\ I_n\end{pmatrix} \longrightarrow Q'D'\begin{pmatrix}P_1 & & \\ & \ddots & \\ & & P_n\end{pmatrix}=[I_1\cdots\cdots I_n] \tag{51}$$

$$\therefore \quad P'DQ=Q'D'P. \tag{52}$$

I will redefine the variables and parameters in the tth period as follows:

$$a_{jit}=\frac{Q_{jit}}{Q_{it}}=\frac{A_{jit}}{P_{jt}Q_{it}} \tag{53}$$

$$\bar{a}_{jit}=\frac{\bar{Q}_{jit}}{Q_{it}}=\frac{\bar{A}_{jit}}{P_{jt}Q_{it}} \tag{54}$$

$$\Delta a_{jit}=a_{jit}-\bar{a}_{jit} \tag{55}$$

$$A_{jit}-\bar{A}_{jit}=I_{jit} \tag{56}$$

or $$(a_{jit}-\bar{a}_{jit})Q_{it}P_{jt}=I_{jit} \tag{57}$$

then $$\Delta a_{jit}Q_{it}P_{jt}=I_{jit}. \tag{58}$$

Summing up the equation (56) with respect to j

$$\sum_{j=1}^{n}(A_{jit}-\bar{A}_{jit})=\sum_{j=1}^{n}I_{jit}=I_{it}. \tag{59}$$

Total capital assets (K_{it}) in the ith industry in the tth period require goods and services from the jth industry (A_{ji}) to produce its products. f_{jit} is defined as the ratio of purchase from the jth industry and capital assets of the ith industry. Then

$$A_{jit}=f_{jit}K_{it} \tag{60}$$

and $$A_{jit-1}=f_{jit-1}K_{it-1}. \tag{61}$$

If both sides of equations (60) and (61) are calculated with respect to j, then we get

$$\sum_{j=1}^{n}A_{jit}=K_{it}\sum_{j=1}^{n}f_{jit}=K_{it}f_{it} \tag{62}$$

and

$$\sum_{j=1}^{n}A_{jit-1}=K_{it-1}\sum_{j=1}^{n}f_{jit-1}=K_{it}f_{it-1} \tag{63}$$

where

$$f_{it}=\sum_{j=1}^{n}f_{jit} \qquad \text{and} \qquad f_{it-1}=\sum_{j=1}^{n}f_{jt-1}. \tag{64}$$

As investment of the ith industry in the tth period is equal to the increment of capital assets,

$$I_{it}=K_{it}-K_{it-1}=\frac{\sum_{j=1}^{n}A_{jit}}{f_{it}}-\frac{\sum_{j=1}^{n}A_{jit-1}}{f_{it-1}}. \tag{65}$$

From the equations (59) and (65)

$$\left(1-\frac{1}{f_{it}}\right)\sum_{j=1}^{n}A_{jit}=\sum_{j=1}^{n}\bar{A}_{jit}-\frac{\sum_{j=1}^{n}A_{jit-1}}{f_{it-1}}. \tag{66}$$

Therefore

$$\sum_{j=1}^{n}\bar{A}_{jit}=\left(1-\frac{1}{f_{it}}\right)\sum_{j=1}^{n}A_{jit}+\frac{\sum_{j=1}^{n}A_{jit-1}}{f_{it-1}}. \tag{67}$$

Taking a certain element $\bar{a}_{jit}Q_{it}P_{jt}$ and its corresponding element from both sides of the equation (67)

$$\bar{a}_{jit}Q_{it}P_{jt}=\left(1-\frac{1}{f_{it}}\right)a_{jit}Q_{it}P_{jt}+\frac{1}{f_{it-1}}a_{jit-1}Q_{it-1}P_{jt-1}. \tag{68}$$

Dividing both sides of the equation (68) by $Q_{it}P_{jt}$

$$\bar{a}_{jit}=\left(1-\frac{1}{f_{it}}\right)a_{jit}+\frac{1}{f_{it-1}}a_{jit-1}\frac{Q_{it-1}P_{jt-1}}{Q_{it}P_{jt}}. \tag{69}$$

From the equation (56) and (57)

$$\frac{I_{jit}}{A_{jit}}=\frac{I_{jit}}{Q_{jt}P_{jt}a_{jit}}=1-\frac{\bar{a}_{jit}}{a_{jit}}. \tag{70}$$

Substituting the equation (69) into (70)

$$\begin{aligned}\frac{I_{jit}}{A_{jit}}&=\frac{1}{f_{it}}-\frac{1}{f_{it-1}}\cdot\frac{a_{jit-1}}{a_{jit}}\cdot\frac{Q_{it-1}P_{it-1}}{Q_{it}P_{jt}}\\&=\frac{1}{f_{it}}-\frac{1}{f_{it-1}}\left\{\frac{1}{(1+\dot{a}_{jit})(1+\dot{Q}_{it})(1+\dot{P}_{jt})}\right\}.\end{aligned} \tag{71}$$

In the equations (69) and (71) if purchases per capital assets, technology coefficients and price in the ith industry are assumed to be constant,

$$\bar{a}_{jit}=\left\{1-\frac{1}{f_{it}}\left(1-\frac{Q_{it-1}}{Q_{it}}\right)\right\}=a_{jit} \tag{72}$$

and

$$\frac{I_{jit}}{A_{jit}}=\frac{1}{f_{it}}\left\{1-\frac{1}{(1+\dot{Q}_{it})}\right\}. \tag{73}$$

And in addition to this assumption if volumes of products in the ith industry can not change,

$$\bar{a}_{jit}=\left(1-\frac{1}{f_{it}}\right)a_{jit} \tag{74}$$

and

$$\frac{I_{jit}}{A_{jit}}=0, \tag{75}$$

therefore no investment at all occurs in this simple reproduction scheme.

As a plausible assumption, if $f_{it}=f_{it-1}$ and $a_{jit}=a_{jit-1}$,

$$\bar{a}_{jit}=\left\{1-\frac{1}{f_{it}}\left(1-\frac{Q_{it-1}P_{jt-1}}{Q_{it}P_{jt}}\right)\right\} \tag{76}$$

and

$$\begin{aligned}\frac{I_{jit}}{A_{jit}}&=\frac{1}{f_{it}}\left\{1-\frac{Q_{it-1}}{Q_{it}}\cdot\frac{P_{jt-1}}{P_{jt}}\right\}\\&=\frac{K_{it}}{\sum_{j=1}^{n}A_{jit}}\left\{1-\frac{Q_{it-1}}{Q_{it}}\cdot\frac{P_{jt-1}}{P_{jt}}\right\}.\end{aligned} \tag{77}$$

Therefore,

$$I_{it}=\sum_{j=1}^{n}I_{jit}=K_{it}\left\{\frac{Q_{it}P_{jt}-Q_{it-1}P_{jt-1}}{Q_{it}P_{jt}}\right\} \tag{78}$$

and then,

$$\frac{I_{it}}{K_{it}}=\frac{\varDelta QP_{jt-1}+\varDelta PQ_{it-1}+\varDelta P\varDelta Q}{Q_{it}P_{jt}} \tag{79}$$

In the equation (79) if absolute prices do not change

$$\frac{I_{it}}{K_{it}}=\frac{\varDelta Q}{Q_{it}}. \tag{80}$$

Actually, parameters and variables are never constant with the lapse of time as seen in so many published input-output tables.

Regarding this point, I would agree with Hatanaka's view: he said in his preface of "The workability of input-output analysis" that "Many economists have criticized input-output analysis by pointing out plausible reasons for changes in input coefficients from the theoretical point of view. However, the real problem concerning the appraisal of input-output analysis is not whether the input coefficients are constant, or they are not. The real problem is concerned with the degree of changes as well as the pattern of these changes which actually take place in the input coefficients. At the present stage of economics theoretical investigation has not led us to a fruitful analysis of the variations of the input coefficients."(22)

(22) Michio Hatanaka, *ibid.*, Author's preface, p. v.