Renormalized Solutions to Stochastic Conservation Laws

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1 Introduction

In this paper we study the first order stochastic conservation law of the following type

$$du + \operatorname{div}(A(u))dt = \Phi(u)dW(t) \quad \text{in } \Omega \times Q, \tag{1.1}$$

with the initial condition

$$u(0, \cdot) = u_0(\cdot) \quad \text{in } \Omega \times D, \tag{1.2}$$

and the formal boundary condition

$$\text{“}u = u_b\text{”} \quad \text{on } \Omega \times \Sigma. \tag{1.3}$$

Here $D \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary ∂D, $T > 0$, $Q = (0, T) \times D$, $\Sigma = (0, T) \times \partial D$ and W is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. More precisely, $(\mathscr{F}_t)$ is a complete right-continuous filtration and $W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k$ with $(\beta_k)_{k\geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)$ and $(e_k)_{k\geq 1}$ a complete orthonormal system in a separable Hilbert space H (cf. [4] for example).

In the deterministic case of $\Phi = 0$, the problem has been studied by many authors, e.g. see [2], [11], [13], [17], [18].

It is natural for applications in the wide variety of fields as physics, finance, biology, medicine and others to add a stochastic forcing $\Phi(u)dW(t)$. These stochastic cases have been investigated by Kim [12], Feng and Naualart [7], Debussche and Vovelle [5], Bauzet et al. [1]. Also see [3], [6], [15], [20]. In particular, by using a notion of kinetic solution the authors [14] proved the uniqueness and the existence of kinetic solutions to the initial-boundary problem for stochastic conservation laws. In the preceding paper [14] the boundary defect measures $\bar{m}^{\pm}$ were cut off or renormalized on each finite interval $(-N, N)$ of $\mathbb{R}_\xi$, but the defect measure m was not. On the other hand, Noboriguchi

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[19] proved the equivalence between renormalized kinetic solutions and renormalized entropy solutions. To prove this equivalence we have to cut off the defect measure m and introduce renormalized kinetic defect measures $m_N^{\pm}$.

Our purpose of this paper is to present a definition of kinetic solutions with renormalized defect measures $\bar{m}_N^{\pm}$ and to prove a result of the uniqueness of such solutions. The idea of the proof is almost the same as in [14], but a difficulty occurs in the course of the proof of the L^1-contraction property. In [14] this property was proved by using the decay condition on the defect measure m. However, we now have to proceed with the weaker decay condition on the renormalized defect measures $m_N^{\pm}$ (see (2.1)) than that on the defect measure m in [14]. This difficulty will be overcome by showing a convergence of the derivative of $\mu_N(\xi) = \mathbb{E}m_N^{\pm}([0,T)\times D\times(-N,\xi))$ instead of $\mathbb{E}m([0,T)\times D\times(\xi,\infty))$ (see [14, Lemma 3.3]).

We now give the precise assumptions in this paper:

(H_1) The flux function A: $\mathbb{R} \to \mathbb{R}^d$ is of class C^2 and its derivatives have at most polynomial growth.

(H_2) For each $z \in L^2(D)$, $\Phi(z) : H \to L^2(D)$ is defined by $\Phi(z)e_k = g_k(\cdot, z(\cdot))$, where $g_k \in C(D \times \mathbb{R})$ satisfies the following conditions:

$$G^2(x,\xi) = \sum_{k=1}^{\infty} |g_k(x,\xi)|^2 \le L(1+|\xi|^2), \tag{1.4}$$

$$\sum_{k=1}^{\infty} |g_k(x,\xi) - g_k(y,\zeta)|^2 \le L\left(|x-y|^2 + |\xi-\zeta|\, r(|\xi-\zeta|)\right) \tag{1.5}$$

for every $x, y \in D$, $\xi, \zeta \in \mathbb{R}$. Here, L is a constant and r is a continuous nondecreasing function on $\mathbb{R}_+$ with $r(0) = 0$.

(H_3) $u_0 \in L^\infty(\Omega \times D)$ and is $\mathscr{F}_0 \otimes \mathscr{B}(D)$-measurable. $u_b \in L^\infty(\Omega \times \Sigma)$ and $\{u_b(t)\}$ is predictable, in the following sense: For every $p \in [1,\infty)$, the $L^p(\partial D)$-valued process $\{u_b(t)\}$ is predictable with respect to the filtration $(\mathscr{F}_t)$.

Note that by (1.4) one has

$$\Phi : L^2(D) \to L_2(H; L^2(D)), \tag{1.6}$$

where $L_2(H; L^2(D))$ denotes the set of Hilbert-Schmidt operators from H to $L^2(D)$.

2 Kinetic solution and generalized kinetic solution

We give the definition of solution in this section. We mainly follows the notations of [5] and [11]. We choose a finite open cover $\{U_{\lambda_i}\}_{i=0,\dots,M}$ of $\overline{D}$ and a partition of unity $\{\lambda_i\}_{i=0,\dots,M}$ on $\overline{D}$ subordinated to $\{U_{\lambda_i}\}$ such that $U_{\lambda_0} \cap \partial D = \emptyset$, for $i = 1, \dots, M$,

$$D^{\lambda_i} := D \cap U_{\lambda_i} = \{x \in U_{\lambda_i}; (\mathcal{A}_i x)_d > h_{\lambda_i}(\overline{\mathcal{A}_i x})\} \text{ and}$$
$$\partial D^{\lambda_i} := \partial D \cap U_{\lambda_i} = \{x \in U_{\lambda_i}; (\mathcal{A}_i x)_d = h_{\lambda_i}(\overline{\mathcal{A}_i x})\},$$

with a Lipschitz function $h_{\lambda_i} : \mathbb{R}^{d-1} \to \mathbb{R}$, where $\mathcal{A}_i$ is an orthogonal matrix corresponding to a change of coordinates of $\mathbb{R}^d$ and $\bar{y}$ stands for $(y_1, \ldots, y_{d-1})$ if $y \in \mathbb{R}^d$. For the sake of clarity, we will drop the index i of λ_i and we will suppose that the matrix $\mathcal{A}_i$ equals to the identity. We also set $Q^\lambda = (0,T) \times D^\lambda$, $\Sigma^\lambda = (0,T) \times \partial D^\lambda$ and $\Pi^\lambda = \{\bar{x}; x \in B^\lambda\}$.

To regularize functions that are defined on D^λ and $\mathbb{R}$, let us consider a standard mollifier ρ on $\mathbb{R}$, that is, ρ is a nonnegative and even function in $C_c^\infty((-1,1))$ such that $\int_{\mathbb{R}} \rho = 1$. We set $\rho^\lambda(x) = \Pi_{i=1}^{d-1} \rho(x_i) \rho(x_d - (L_\lambda + 1))$ for $x = (x_1, \ldots, x_d)$ with the Lipschitz constant L_λ of h_λ on Π^λ. Moreover we denote by ψ a standard mollifier on $\mathbb{R}_\xi$. For $\varepsilon, \delta > 0$ we set $\rho_\varepsilon^\lambda(x) = \frac{1}{\varepsilon^d} \rho^\lambda(\frac{x}{\varepsilon})$ and $\psi_\delta(\xi) = \frac{1}{\delta} \psi(\frac{\xi}{\delta})$.

Definition 2.1 (Kinetic measure)**.** A set $\{m_N; N > 0\}$ of maps m_N from Ω to $\mathcal{M}_b^+([0,T) \times D \times (-N,N))$, the set of non-negative finite measures over $[0,T) \times D \times (-N,N)$, is said to be a kinetic measure if

(i) for each $N > 0$, m_N is weak measurable,

(ii) if $A_N = [0,T) \times D \times \{\xi \in \mathbb{R}; N-1 \leq |\xi| \leq N\}$ then

$$\lim_{N \to \infty} \mathbb{E} m_N (A_N) = 0, \tag{2.1}$$

(iii) for all $\phi \in C_b(D \times (-N,N))$, the process

$$t \mapsto \int_{[0,t] \times D \times (-N,N)} \phi(x,\xi) \, dm_N(s,x,\xi) \tag{2.2}$$

is predictable.

Definition 2.2 (Kinetic solution)**.** Let u_0 and u_b satisfy (H_3). A measurable function $u : \Omega \times Q \to \mathbb{R}$ is said to be a kinetic solution of (1.1)-(1.3) if $\{u(t)\}$ is predictable, for all $p \geq 1$ there exists a constant $C_p \geq 0$ such that for a.e. $t \in [0,T]$,

$$\|u(t)\|_{L^p(\Omega \times D)} \leq C_p, \tag{2.3}$$

there exist kinetic measures $\{m_N^\pm\}$ and, for any $N > 0$, there exist increasing $\bar{m}_N^+ \in L^1(\Omega \times \Sigma \times (-N,N))$ and decreasing $\bar{m}_N^- \in L^1(\Omega \times \Sigma \times (-N,N))$ such that $\{\bar{m}_N^\pm(t)\}$ is predictable, $\bar{m}_N^+(N-1) = \bar{m}_N^-(-N+1) = 0$ for sufficiently large $N > 0$ and $f_+ := \mathbf{1}_{u > \xi}$, $f_- := f_+ - 1 = -\mathbf{1}_{u \leq \xi}$ satisfy: for all $\varphi \in C_c^\infty([0,T) \times \overline{D} \times (-N,N))$,

$$\begin{aligned}
&\int_Q \int_{-N}^N f_\pm (\partial_t + a(\xi) \cdot \nabla) \varphi \, d\xi dx dt + \int_D \int_{-N}^N f_\pm^0 \varphi(0) \, d\xi dx + M_N \int_\Sigma \int_{-N}^N f_\pm^b \varphi \, d\xi d\sigma dt \\
&= -\sum_{k=1}^\infty \int_0^T \int_D g_k(x,u) \, \varphi(x,t,u) \, dx d\beta_k(t) - \frac{1}{2} \int_Q G^2(x,u) \, \partial_\xi \varphi(x,t,u) \, dx dt \\
&\quad + \int_{[0,T) \times D \times (-N,N)} \partial_\xi \varphi \, dm_N^\pm + \int_\Sigma \int_{-N}^N \partial_\xi \varphi \, \bar{m}_N^\pm \, d\xi d\sigma dt \qquad \text{a.s.,}
\end{aligned} \tag{2.4}$$

where $a(\xi) = A'(\xi)$, $M_N = \max_{-N \leq \xi \leq N} |a(\xi)|$. In (2.4), $f_+^0 = \mathbf{1}_{u_0 > \xi}$, $f_+^b = \mathbf{1}_{u_b > \xi}$, $f_-^0 = f_+^0 - 1$ and $f_-^b = f_+^b - 1$.

For the sake of the proof of the existence of a kinetic solution, it is useful to introduce the notion of generalized kinetic solution. We start with the definition of kinetic function.

Definition 2.3 (Kinetic function)**.** Let (X, μ) be a finite measure space. We say that a measurable function $f_+ : X \times \mathbb{R} \to [0,1]$ is a kinetic function if there exists a Young measure ν on X such that for every $p \geq 1$,

$$\int_X \int_{\mathbb{R}} |\xi|^p \, d\nu_z(\xi) d\mu(z) < +\infty \tag{2.5}$$

and for μ-a.e. $z \in X$, for all $\xi \in \mathbb{R}$,

$$f_+(z, \xi) = \nu_z(\xi, +\infty).$$

Here we recall that a Young measure ν on X is a weak measurable mapping $z \mapsto \nu_z$ from X into the space of probability measures on $\mathbb{R}$. For a kinetic function $f_+ : X \times \mathbb{R} \to [0,1]$ we denote the conjugate function by $f_- = f_+ - 1$. Observe that if $f_+ = \mathbf{1}_{u>\xi}$, then it is a kinetic function with the corresponding Young measure $\nu = \delta_{u=\xi}$, the Dirac measure centered at u, and its conjugate $f_- = -\mathbf{1}_{u \leq \xi}$.

We introduce the definition of generalized kinetic solution.

Definition 2.4 (Generalized kinetic solution)**.** Let u_0 and u_b satisfy (H$_3$). A measurable function $f_+ : \Omega \times Q \times \mathbb{R} \to [0,1]$ is said to be a generalized kinetic solution of (1.1)-(1.3) if the following conditions (i)-(iii) hold:

(i) $\{f_+(t)\}$ is predictable.

(ii) f_+ is a kinetic function with the associated Young measure ν on $\Omega \times Q$ such that for all $p \geq 1$, there exists $C_p \geq 0$ satisfying that for a.e. $t \in [0,T]$,

$$\mathbb{E} \int_D \int_{\mathbb{R}} |\xi|^p \, d\nu_{t,x}(\xi) dx \leq C_p. \tag{2.6}$$

(iii) There exist kinetic measures $\{m_N^\pm\}$ and, for any $N > 0$, there exist increasing $\bar{m}_N^+ \in L^1(\Omega \times \Sigma \times (-N, N))$ and decreasing $\bar{m}_N^- \in L^1(\Omega \times \Sigma \times (-N, N))$ such that $\{\bar{m}_N^\pm(t)\}$ is predictable, $\bar{m}_N^+(N-1) = \bar{m}_N^-(-N+1) = 0$ for sufficiently large $N > 0$ and for all $\varphi \in C_c^\infty([0,T) \times \overline{D} \times (-N, N))$,

$$\begin{aligned}
&\int_Q \int_{-N}^N f_\pm (\partial_t + a(\xi) \cdot \nabla) \varphi \, d\xi dx dt + \int_D \int_{-N}^N f_\pm^0 \varphi(0) \, d\xi dx + M_N \int_\Sigma \int_{-N}^N f_\pm^b \varphi \, d\xi d\sigma dt \\
&= -\sum_{k=1}^\infty \int_0^T \int_D \int_{-N}^N g_k \, \varphi \, d\nu_{t,x}(\xi) dx d\beta_k(t) - \frac{1}{2} \int_Q \int_{-N}^N G^2 \, \partial_\xi \varphi \, d\nu_{t,x}(\xi) dx dt \\
&\quad + \int_{[0,T) \times D \times (-N,N)} \partial_\xi \varphi \, dm_N^\pm + \int_\Sigma \int_{-N}^N \partial_\xi \varphi \, \bar{m}_N^\pm \, d\xi d\sigma dt \qquad \text{a.s.}
\end{aligned} \tag{2.7}$$

The following proposition due to [5, Proposition 8] shows that any generalized kinetic solution admits left and right limits at every $t \in [0,T]$.

Lemma 2.5. *Let f_+ be a generalized kinetic solution of* (1.1)-(1.3). *Then f_+ admits almost surely left and right limits at all points $t^* \in [0,T]$ in the following sense: For all $t^* \in [0,T]$ there exist some kinetic functions $f_+^{*,\pm}$ on $\Omega \times D \times \mathbb{R}$ such that $\mathbb{P}$-a.s.,*

$$\int_{D\times\mathbb{R}} f_+(t^* \pm \varepsilon)\varphi\, d\xi dx \to \int_{D\times\mathbb{R}} f_+^{*,\pm}\varphi\, d\xi dx$$

as $\varepsilon \to +0$ for all $\varphi \in C_c^1(D\times\mathbb{R})$. Moreover, almost surely, $f_+^{*,+} = f_+^{*,-}$ for all $t^* \in [0,T]$ except some countable set.

In what follows, for a generalized kinetic solution f_+, we will define $f_+^{\pm}$ by $f_+^{\pm}(t^*) = f_+^{*,\pm}$ for $t^* \in [0,T]$.

In order to prove uniqueness we need to extend test functions in (2.7) to the class of $C_c^\infty([0,T)\times\mathbb{R}^d\times\mathbb{R})$. To this end we introduce the cutoff functions as follows.

$$\Psi_\eta(\xi) = \int_{-\infty}^{\xi} \{\psi_\eta(\zeta + N - \eta) - \psi_\eta(\zeta - N + \eta)\}\, d\zeta, \quad \eta > 0.$$

Proposition 2.6. *Let f_+ be a generalized kinetic solution of* (1.1)-(1.3). *Let $\bar{f}_\pm^\lambda$ be any weak* limit of $\{f_\pm^{\lambda,\varepsilon}\}$ as $\varepsilon \to +0$ in $L^\infty(\Sigma^\lambda \times \mathbb{R})$ for any element λ of the partition of unity $\{\lambda_i\}$ on $\overline{D}$, where $f_\pm^{\lambda,\varepsilon}$ is denoted by*

$$f_\pm^{\lambda,\varepsilon}(t,x,\xi) = \int_{D^\lambda} f_\pm(t,x,\xi)\rho_\varepsilon^\lambda(y-x)dy,$$

and let $\bar{f}_\pm = \sum_{i=0}^M \lambda_i \bar{f}_\pm^{\lambda_i}$.

(i) *For a.s. there exists a full set $\mathbb{L}$ of Σ such that $\bar{f}_\pm(t,x,\xi)$ is non-increasing in ξ for all $(t,x) \in \mathbb{L}$.*

(ii) *For any $\varphi \in C_c^\infty(\mathbb{R}^d\times\mathbb{R})$, for any $t \in [0,T)$ and for any $\eta > 0$,*

$$\begin{aligned}
&-\int_D \int_{-N}^{N} \Psi_\eta f_\pm^+(t)\varphi d\xi dx + \int_0^t \int_D \int_{-N}^{N} \Psi_\eta f_\pm a(\xi)\cdot\nabla\varphi d\xi dx ds \\
&\quad + \int_D \int_{-N}^{N} \Psi_\eta f_\pm^0 \varphi d\xi dx + \int_0^t \int_{\partial D} \int_{-N}^{N} \Psi_\eta(-a(\xi)\cdot\mathbf{n})\bar{f}_\pm \varphi d\xi d\sigma ds \\
&= -\sum_{k\ge 1} \int_0^t \int_D \int_{-N}^{N} \Psi_\eta g_k\, \varphi\, d\nu_{s,x}(\xi) dx d\beta_k(s) \\
&\quad -\frac{1}{2}\int_0^t \int_D \int_{-N}^{N} \Psi_\eta \partial_\xi \varphi\, G^2\, d\nu_{s,x}(\xi) dx ds + \int_{[0,t]\times D\times(-N,N)} \Psi_\eta \partial_\xi \varphi\, dm_N^\pm \\
&\quad -\frac{1}{2}\int_0^t \int_D \int_{-N}^{N} \Big(\psi_\eta(\xi+N-\eta) - \psi_\eta(\xi-N+\eta)\Big) G^2\, \varphi\, d\nu_{s,x}(\xi) dx ds \\
&\quad + \int_{[0,t]\times D\times(-N,N)} \Big(\psi_\eta(\xi+N-\eta) - \psi_\eta(\xi-N+\eta)\Big)\varphi\, dm_N^\pm \quad a.s. \qquad (2.8)
\end{aligned}$$

(iii) *P-a.s., for a.e. $(t,x) \in \Sigma$, the weak* limits $-a(\xi) \cdot \mathbf{n}(\bar{x})\bar{f}_{\pm}(t,x,\xi)$ coincide with $M_N f^b_{\pm}(t,x,\xi) + \partial_\xi \bar{m}^{\pm}_N(t,x,\xi)$ for a.e. $\xi \in (-N,N)$.*

Proof. The result can be proved by a minor change of the proof of [14, Proposition 2.7]. □

3 Uniqueness

In this section we prove the main result of the paper.

Theorem 3.1 (L^1-contraction property). *Let $f_{i,+}$, $i = 1,2$, be generalized kinetic solutions to* (1.1)-(1.3) *with data $(f^0_{i,+}, f^b_{i,+}) = (\mathbf{1}_{u_{i,0}>\xi}, \mathbf{1}_{u_{i,b}>\xi})$, respectively. Under the assumptions* (H$_1$)-(H$_3$) *we have for a.e. $t \in [0,T)$*

$$-\mathbb{E}\int_D\int_{\mathbb{R}} f_{1,+}(t)f_{2,-}(t) \le -\mathbb{E}\int_D\int_{\mathbb{R}} f^0_{1,+}f^0_{2,-} - M\mathbb{E}\int_0^t\int_{\partial D}\int_{\mathbb{R}} f^b_{1,+}(s)f^b_{2,-}(s), \quad (3.1)$$

where $M = \max\{|a(\xi)| : |\xi| \le \|u_{1,b}\|_{L^\infty(\Omega\times\Sigma)} \vee \|u_{2,b}\|_{L^\infty(\Omega\times\Sigma)}\}$.

Corollary 3.2 (Uniqueness, Reduction). *Under the same assumptions as in the above theorem, if f_+ is a generalized solution to* (1.1)-(1.3) *with initial datum $\mathbf{1}_{u_0>\xi}$ and boundary datum $\mathbf{1}_{u_b>\xi}$, then there exists a kinetic solution u to* (1.1)-(1.3) *with initial datum u_0 and boundary datum u_b such that $f_+(t,x,\xi) = \mathbf{1}_{u(t,x)>\xi}$ a.s. for a.e. (t,x,ξ). Moreover, for a.e. $t \in [0,T)$,*

$$\mathbb{E}\,\|u_1(t) - u_2(t)\|_{L^1(D)} \le \mathbb{E}\,\|u_{1,0} - u_{2,0}\|_{L^1(D)} + M\mathbb{E}\int_0^t \|u_{1,b}(s) - u_{2,b}(s)\|_{L^1(\partial D)}\,ds, \quad (3.2)$$

where u_i, $i = 1,2$, are the corresponding kinetic solutions to (1.1)-(1.3) *with data $(u_{i,0}, u_{i,b})$.*

To prove the uniqueness theorem we define the non-decreasing functions $\mu_N(\xi)$ and $\mu_\nu(\xi)$ on $\mathbb{R}$ by

$$\mu_N(\xi) = \mathbb{E}m_N([0,T) \times D \times (-N,\xi)), \quad (3.3)$$

$$\mu_\nu(\xi) = \mathbb{E}\int_{Q\times(-\infty,\xi)} d\nu_{t,x}(\xi)dxdt, \quad (3.4)$$

where $\{m_N\}$ and ν are a kinetic measure and a Young measure satisfying (2.5), respectively. Let $\mathbb{D}_N$ be the sets of $\xi \in (N-1,N)$ such that both of μ_N and μ_ν are differentiable at $-\xi$ and ξ. We also set $\mathbb{D} = \cup_{N=1}^\infty \mathbb{D}_N$. It is easy to see that $\mathbb{D}_N$ and $\mathbb{D}$ are full sets of $(N-1,N)$ and $(0,\infty)$, respectively.

Lemma 3.3. *It holds true:*

(i) *Let $N_0 \in \mathbb{N}$. If $a \in \mathbb{D}_{N_0}$, then for all $N \in \mathbb{N}$ with $N \ge N_0$, as $\delta \downarrow 0$*

$$\int_{-N}^N \psi_\delta(\xi \pm a)\,d\mu_N(\xi) \to \mu'_N(\mp a)$$

$$\int_{-N}^N (1+|\xi|^2)\psi_\delta(\xi \pm a)\,d\mu_\nu(\xi) \to (1+a^2)\mu'_\nu(\mp a).$$

(ii) *There exists a sequence* $\{a_N\}$ *with* $a_N \in \mathbb{D}_N$ *such that*

$$\liminf_{N\to\infty} \mu_N'(\pm a_N) = 0 \quad \text{and} \quad \liminf_{N\to\infty} a_N^p \mu_\nu'(\pm a_N) = 0 \textit{ for } p \geq 0. \tag{3.5}$$

Proof. We prove the lemma only in the case of μ_N. The case of μ_ν will be done in a similar fashion. Let $a \in \mathbb{D}_{N_0}$. Since $\mu_N(\xi \mp a) = \mu_N(\mp a) + \mu_N'(\mp a)\xi + o(\xi)$ for each $N \in \mathbb{N}$ with $N \geq N_0$, it follows that

$$\int_{-N}^{N} \psi_\delta(\xi \pm a)\ d\mu_N(\xi) = -\int_{-\delta}^{\delta} \mu_N(\xi \mp a)\ d\psi_\delta(\xi) = \mu_N'(\mp a) - \int_{-\delta}^{\delta} o(\xi)\psi_\delta'(\xi)\ d\xi.$$

Besides, the last term of the right hand on the above equality tends to 0 as $\delta \to +0$. To see this take an arbitrary $\varepsilon > 0$. There exists $\delta_0 > 0$ such that if $|\xi| < \delta_0$ then $|o(\xi)| \leq \varepsilon\,|\xi|$. If $0 < \delta < \delta_0$, then

$$\left| \int_{-\delta}^{\delta} o(\xi)\psi_\delta'(\xi)\ d\xi \right| \leq \varepsilon \int_{-\delta}^{\delta} |\xi\psi_\delta'(\xi)|\ \ d\xi \leq \varepsilon.$$

Thus we obtain the claim of (i).

Next, let us assume that there exists a number $k \in \mathbb{N}$ such that for any $N \geq k$,

$$\mu_N'(\xi) > \frac{1}{k}, \quad \xi \in \mathbb{D}_N.$$

Since the function $\xi \mapsto \mu_N(\xi)$ is non-decreasing, for all $N \in \mathbb{N}$ with $N \geq k$

$$\mu_N(N) - \mu_N(N-1) \geq \int_{N-1}^{N} \mu_N'(\xi)\ d\xi \geq \frac{1}{k} > 0.$$

This contradictions the limit (2.1). Thus for each $k \in \mathbb{N}$, there exist a number $N_k \geq k$ and $a_k \in \mathbb{D}_{N_k}$ such that $\mu_{N_k}'(a_k) \leq \frac{1}{k}$. □

Proposition 3.4 (Doubling variable)**.** *Let* $f_{i,+}$, $i = 1, 2$, *be generalized kinetic solutions to* (1.1)-(1.3)*with data* $(f_{i,+}^0, f_{i,+}^b)$. *Then, for* $t \in [0, T)$, *for* $\varepsilon, \delta > 0$, *for* $N \in \mathbb{N}$ *and for any element* λ *of the partition of unity* $\{\lambda_i\}$ *on* $\overline{D}$, *we have*

$$\begin{aligned}
&-\mathbb{E}\int_{D_x^\lambda \times D_y \times (-a_N, a_N)^2} \lambda(x)\rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta) f_{1,+}^{+}(t,x,\xi) f_{2,-}^{+}(t,y,\zeta) d\xi d\zeta dx dy \\
&\leq -\mathbb{E}\int_{D_x^\lambda \times D_y \times (-a_N, a_N)^2} \lambda(x)\rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta) f_{1,+}^{0}(x,\xi) f_{2,-}^{0}(y,\zeta) d\xi d\zeta dx dy \\
&\quad -\mathbb{E}\int_{(0,t)\times\partial D_x^\lambda \times D_y \times (-a_N, a_N)^2} \lambda(x)\rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta)(-a(\xi)\cdot \mathbf{n}(x)) \\
&\qquad\qquad \times \bar{f}_{1,+}^{\lambda}(s,x,\xi) f_{2,-}(s,y,\zeta) d\xi d\zeta d\sigma(x) dy ds \\
&\quad + I_1 + I_2 + I_3 + I_N,
\end{aligned} \tag{3.6}$$

where $\{a_N\}$ is a sequence of $\mathbb{D}_N$ satisfying (3.5),

$$I_1 = -\mathbb{E}\int_{(0,t)\times D_x^\lambda\times D_y\times(-a_N,a_N)^2} f_{1,+}(s,x,\xi)f_{2,-}(s,y,\zeta)(a(\xi)-a(\zeta)) \cdot\nabla_x\rho_\varepsilon^\lambda(y-x)\lambda(x)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds,$$

$$I_2 = -\mathbb{E}\int_{(0,t)\times D_x^\lambda\times D_y\times(-a_N,a_N)^2} f_{1,+}(s,x,\xi)f_{2,-}(s,y,\zeta)a(\xi) \cdot\nabla_x\lambda(x)\rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta)d\xi d\zeta dxdyds,$$

$$I_3 = \frac{1}{2}\mathbb{E}\int_{(0,t)\times D_x^\lambda\times D_y\times(-a_N,a_N)^2} \lambda(x)\rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta) \times\sum_{k=1}^\infty |g_k(x,\xi)-g_k(y,\zeta)|^2\, d\nu_{s,x}^1(\xi)\otimes d\nu_{s,y}^2(\zeta)dxdyds,$$

$\limsup_{N\to\infty} I_N = 0$ with I_N defined by (3.8) below.

Here $m_N^{i,\pm}$, ν^i, $i=1,2$, are the kinetic measures and the Young measures associated with the generalized kinetic solutions $f_{i,+}$, $\bar{f}_{i,\pm}^\lambda$ any weak* limits of $\{f_{i,\pm}^{\lambda,\varepsilon'}\}$ as $\varepsilon'\to 0$ in $L^\infty(\Sigma^\lambda\times\mathbb{R})$, and C a constant which is independent of ε, δ, N.

Proof. We will follow the proof of [5, Proposition 9]. Let $\varphi_1\in C_c^\infty(\mathbb{R}_x^d\times\mathbb{R}_\xi)$ and $\varphi_2\in C_c^\infty(\mathbb{R}_y^d\times\mathbb{R}_\zeta)$. Define the cutoff function as

$$\Psi_\eta(\xi) = \int_{-\infty}^\xi \Big(\psi_\eta(r+a_N)-\psi_\eta(r-a_N)\Big)dr.$$

Set

$$\begin{aligned}
F_{1,+}(t) &= \sum_{k=1}^\infty \int_0^t\int_{D_x^\lambda}\int_{-N}^N \Psi_\eta(\xi)g_{k,1}\varphi_1^\lambda d\nu_{s,x}^1(\xi)dxd\beta_k(s),\\
G_{1,+}(t) &= \int_0^t\int_{D_x^\lambda}\int_{-N}^N \Psi_\eta(\xi)f_{1,+}(s,x,\xi)a(\xi)\cdot\nabla_x\varphi_1^\lambda d\xi dxds\\
&+\frac{1}{2}\int_0^t\int_{D_x^\lambda}\int_{-N}^N \Psi_\eta(\xi)\partial_\xi\varphi_1^\lambda G_1^2 d\nu_{s,x}^1(\xi)dxds\\
&+\int_0^t\int_{\partial D_x^\lambda}\int_{-N}^N \Psi_\eta(\xi)(-a(\xi)\cdot\mathbf{n}(x))\bar{f}_{1,+}^\lambda(s,x,\xi)\varphi_1^\lambda d\xi d\sigma(x)ds\\
&-\int_{[0,t]\times D_x^\lambda\times(-N,N)} \Psi_\eta(\xi)\partial_\xi\varphi_1^\lambda dm_N^{1,+}(s,x,\xi)\\
&+\frac{1}{2}\int_0^t\int_{D_x^\lambda}\int_{-N}^N \Big(\psi_\eta(\xi+a_N)-\psi_\eta(\xi-a_N)\Big)\varphi_1^\lambda G_1^2 d\nu_{s,x}^1(\xi)dxds\\
&-\int_{[0,t]\times D_x^\lambda\times(-N,N)} \Big(\psi_\eta(\xi+a_N)-\psi_\eta(\xi-a_N)\Big)\varphi_1^\lambda dm_N^{1,+}(s,x,\xi).
\end{aligned}$$

On the other hand we set

$$F_{2,-}(t) = \sum_{k=1}^{\infty} \int_0^t \int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) g_{k,2} \varphi_2 d\nu_{s,y}^2(\zeta) dy d\beta_k(s),$$

$$\begin{aligned} G_{2,-}(t) = & \int_0^t \int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) f_{2,-}(s,x,\zeta) a(\zeta) \cdot \nabla_y \varphi_2 d\zeta dy ds \\ & + \frac{1}{2} \int_0^t \int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) \partial_\zeta \varphi_2 G_2^2 d\nu_{s,y}^2(\zeta) dy ds \\ & + \int_0^t \int_{\partial D_y} \int_{-N}^{N} \Psi_\eta(\zeta)(-a(\zeta) \cdot \mathbf{n}(y)) \bar{f}_{2,-}(s,y,\zeta) \varphi_2 d\zeta d\sigma(y) ds \\ & - \int_{[0,t] \times D_y \times (-N,N)} \Psi_\eta(\zeta) \partial_\zeta \varphi_2 dm_N^{2,-}(s,y,\zeta) \\ & + \frac{1}{2} \int_0^t \int_{D_y} \int_{-N}^{N} \Big(\psi_\eta(\zeta + a_N) - \psi_\eta(\zeta - a_N) \Big) \varphi_2 G_2^2 d\nu_{s,y}^2(\zeta) dy ds \\ & - \int_{[0,t] \times D_y \times (-N,N)} \Big(\psi_\eta(\zeta + a_N) - \psi_\eta(\zeta - a_N) \Big) \varphi_2 dm_N^{2,-}(s,y,\zeta). \end{aligned}$$

By (2.8) we have

$$\int_{D_x^\lambda} \int_{-N}^{N} \Psi_\eta(\xi) f_{1,+}^{+}(t) \varphi_1^\lambda d\xi dx = F_{1,+}(t) + G_{1,+}(t) + \int_{D_x^\lambda} \int_{-N}^{N} \Psi_\eta(\xi) f_{1,+}^{0} \varphi_1^\lambda d\xi dx$$

and

$$\int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) f_{2,-}^{+}(t) \varphi_2 d\zeta dy = F_{2,-}(t) + G_{2,-}(t) + \int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) f_{2,-}^{0} \varphi_2 d\zeta dy$$

Set $\alpha(x,\xi,y,\zeta) = \varphi_1(x,\xi)\varphi_2(y,\zeta)$ and $\Psi_\eta(\xi,\zeta) = \Psi_\eta(\xi)\Psi_\eta(\zeta)$. Using Itô's formula for $F_{1,+}(t)F_{2,-}(t)$, integration by parts for functions of finite variation (see [21, p.6]) for

$$\left(G_{1,+}(t) + \int_{D_x^\lambda} \int_{-N}^{N} \Psi_\eta(\xi) f_{1,+}^{0} \varphi_1^\lambda d\xi dx \right) \left(G_{2,-}(t) + \int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) f_{2,-}^{0} \varphi_2 d\zeta dy \right),$$

and integration by parts for functions of finite variation and continuous martingales (see [21, p.152]) for

$$F_{1,+}(t) \left\{ G_{2,-}(t) + \int_{D_y} \int_{-N}^{N} \Psi_\eta(\zeta) f_{2,-}^{0} \varphi_2 d\zeta dy \right\},$$

we obtain

$$
\begin{aligned}
&-\mathbb{E}\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{1,+}^{+}(t)f_{2,-}^{+}(t)\alpha^\lambda d\xi d\zeta dxdy \\
&= -\mathbb{E}\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{1,+}^{0}f_{2,-}^{0}\alpha^\lambda d\xi d\zeta dxdy \\
&\quad -\sum_{k=1}^{\infty}\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)g_{k,1}g_{k,2}\alpha^\lambda d\nu_{s,x}^1(\xi)\otimes d\nu_{s,y}^2(\zeta)dxdyds \\
&\quad -\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{1,+}(s)f_{2,-}(s)(a(\xi)\cdot\nabla_x+a(\zeta)\cdot\nabla_y) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\alpha^\lambda d\xi d\zeta dxdyds \\
&\quad -\frac{1}{2}\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{1,+}(s)\partial_\zeta\alpha^\lambda G_2^2 d\nu_{s,y}^2(\zeta)d\xi dxdyds \\
&\quad -\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{\partial D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{1,+}(s)\bar{f}_{2,-}^{(\lambda)}(s)(-a(\zeta)\cdot\mathbf{n}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\alpha^\lambda d\xi d\zeta dxd\sigma(y)ds \\
&\quad +\mathbb{E}\int_{[0,t]\times D_y\times(-N,N)}\int_{D_x^\lambda}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{1,+}^{-}(s)\partial_\zeta\alpha^\lambda d\xi dxdm_N^{2,-}(s,y,\zeta) \\
&\quad -\frac{1}{2}\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi)f_{1,+}(s)\Big[\psi_\eta(\zeta+a_N) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\psi_\eta(\zeta-a_N)\Big]G_2^2\alpha^\lambda d\nu_{s,y}^2(\zeta)d\xi dxdyds \\
&\quad +\mathbb{E}\int_{[0,t]\times D_y\times(-N,N)}\int_{D_x^\lambda}\int_{-N}^{N}\Psi_\eta(\xi)f_{1,+}^{+}(s)\Big[\psi_\eta(\zeta+a_N) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\psi_\eta(\zeta-a_N)\Big]\alpha^\lambda d\xi dxdm_N^{2,-}(s,y,\zeta) \\
&\quad -\frac{1}{2}\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{2,-}(s)\partial_\xi\alpha^\lambda G_1^2 d\nu_{s,x}^1(\xi)d\zeta dxdyds \\
&\quad -\mathbb{E}\int_0^t\int_{\partial D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)\bar{f}_{1,+}^{(\lambda)}(s)f_{2,-}(s)(-a(\xi)\cdot\mathbf{n}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\alpha^\lambda d\xi d\zeta d\sigma(x)dyds \\
&\quad +\mathbb{E}\int_{[0,t]\times D_x^\lambda\times(-N,N)}\int_{D_y}\int_{-N}^{N}\Psi_\eta(\xi,\zeta)f_{2,-}^{+}(s)\partial_\xi\alpha^\lambda d\zeta dydm_N^{1,+}(s,x,\xi) \\
&\quad -\frac{1}{2}\mathbb{E}\int_0^t\int_{D_x^\lambda}\int_{D_y}\int_{-N}^{N}\int_{-N}^{N}\Psi_\eta(\zeta)f_{2,-}(s)\Big[\psi_\eta(\xi+a_N) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\psi_\eta(\xi-a_N)\Big]G_1^2\alpha^\lambda d\nu_{s,x}^1(\xi)d\zeta dxdyds \\
&\quad +\mathbb{E}\int_{[0,t]\times D_x^\lambda\times(-N,N)}\int_{D_y}\int_{-N}^{N}\Psi_\eta(\zeta)f_{2,-}^{-}(s)\Big[\psi_\eta(\xi+a_N)
\end{aligned}
$$

$$-\psi_\eta(\xi - a_N)\Big]\alpha^\lambda d\zeta dy dm_N^{1,+}(s,x,\xi), \tag{3.7}$$

where $\alpha^\lambda = \alpha(x,\xi,y,\zeta)\lambda(x)$. Noting that $C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi) \otimes C_c^\infty(\mathbb{R}_y^d \times \mathbb{R}_\zeta)$ is dense in $C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi \times \mathbb{R}_y^d \times \mathbb{R}_\zeta)$ and that m^i and ν^i, $i = 1, 2$, vanish for large ξ thanks to (2.1) and (2.5), by an approximation argument we can take $\alpha(x,\xi,y,\zeta) = \rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta)$ in (3.7). In this case note that $\alpha^\lambda = \lambda(x)\rho_\varepsilon^\lambda(y-x)\psi_\delta(\xi-\zeta)$ and $\rho_\varepsilon^\lambda(y-x) = 0$ on $D_x^\lambda \times \partial D_y$. Using the identity $(\partial_\xi + \partial_\zeta)\psi_\delta = 0$, we compute the fourth and sixth terms on the right hand of (3.7) as follows.

$$\begin{aligned}
&-\frac{1}{2}\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \Psi_\eta(\xi,\zeta) f_{1,+}(s)\partial_\zeta \alpha^\lambda G_2^2 d\nu_{s,y}^2(\zeta) d\xi dx dy ds \\
&= \frac{1}{2}\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \Psi_\eta(\xi,\zeta) f_{1,+}(s)\partial_\xi \alpha^\lambda G_2^2 d\nu_{s,y}^2(\zeta) d\xi dx dy ds \\
&= -\frac{1}{2}\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \Psi_\eta(\zeta)\Big[\psi_\eta(\xi + a_N) - \psi_\eta(\xi - a_N)\Big] \\
&\qquad\qquad \times f_{1,+}(s)\alpha^\lambda G_2^2 d\nu_{s,y}^2(\zeta) d\xi dx dy ds \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \Psi(\xi,\zeta)\alpha^\lambda G_2^2 d\nu_{s,x}^1(\xi) d\nu_{s,y}^2(\zeta) dx dy ds
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \psi_\eta(\xi,\zeta) f_{1,+}^-(s)\partial_\zeta \alpha^\lambda d\xi dx dm_N^{2,-}(s,y,\zeta) \\
&= \mathbb{E}\int_{[0,t]\times D_y \times(-N,N)}\int_{D_x^\lambda}\int_{-N}^N \Psi_\eta(\zeta)\Big[\psi_\eta(\xi + a_N) \\
&\qquad\qquad -\psi_\eta(\xi - a_N)\Big] f_{1,+}^-(s)\alpha^\lambda d\xi dx dm_N^{2,-}(s,y,\zeta) \\
&\quad -\mathbb{E}\int_{[0,t]\times D_y \times(-N,N)}\int_{D_x^\lambda}\int_{-N}^N \Psi_\eta(\xi,\zeta)\alpha^\lambda d\nu_{s,x}^{1,-}(\xi) dx dm_N^{2,-}(s,y,\zeta) \\
&\le -\mathbb{E}\int_{[0,t]\times D_y \times(-N,N)}\int_{D_x^\lambda}\int_{-N}^N \Psi_\eta(\zeta)\Big[\psi_\eta(\xi + a_N) \\
&\qquad\qquad -\psi_\eta(\xi - a_N)\Big] f_{1,+}^-(s)\alpha^\lambda d\xi dx dm_N^{2,-}(s,y,\zeta).
\end{aligned}$$

Similarly, the ninth and eleventh terms can be computed. We then calculate the terms produced by the truncation function Ψ_η, namely, the terms containing the functions $\psi_\eta(\xi \pm a_N)$ or $\psi_\eta(\zeta \pm a_N)$.

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \Psi_\eta(\xi) f_{1,+}(s)\psi_\eta(\zeta \pm a_N) G_2^2 \alpha^\lambda d\nu_{s,y}^2(\zeta) d\xi dx dy ds \\
&\le C\,\mathbb{E}\int_0^t \int_{D_x^\lambda}\int_{D_y}\int_{-N}^N\int_{-N}^N \psi_\eta(\zeta \pm a_N)\left(1 + |\zeta|^2\right)\alpha^\lambda d\nu_{s,y}^2(\zeta) d\xi dx dy ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C \int_{\mathbb{R}} \psi_\eta(\zeta \pm a_N) \left(1 + |\zeta|^2\right) \mathbb{E} \int_0^T \int_D d\nu^2_{s,y}(\zeta) dy ds \\
&= C \int_{\mathbb{R}} \psi_\eta(\zeta \pm a_N) \left(1 + |\zeta|^2\right) d\mu_{\nu^2}(\zeta) \to C(1 + a_N^2)\mu'_{\nu^2}(\pm a_N)
\end{aligned}
$$

as $\eta \to +0$ by virtue of Lemma 3.3, where μ_{ν^2} is defined by (3.4). A similar argument yields that all the other terms containing the function ψ_η on the right hand of (3.7) are estimated from above as $\eta \to +0$ by

$$
I_N = C \left(\mu'_{m_N^{1,+}}(a_N) + \mu'_{m_N^{2,-}}(a_N) + (1 + a_N^2)(\mu'_{\nu^1}(a_N) + \mu'_{\nu^2}(a_N)) \right), \tag{3.8}
$$

which is convergent to 0 as $N \to \infty$ by Lemma 3.3. Consequently, letting $\eta \to +0$ in (3.7) and then using the identity $(\nabla_x + \nabla_y)\rho^\lambda_\varepsilon = 0$ in the third term on the right hand we obtain (3.6) with I_N defined by (3.8). □

Proof of Theorem 3.1. Set for $t \geq 0$ and $N > 0$,

$$
\begin{aligned}
\eta^t_N(\varepsilon, \delta) = &-\mathbb{E} \int_{D^\lambda_x} \int_{D_y} \int_{-a_N}^{a_N} \int_{-a_N}^{a_N} \lambda(x) \rho^\lambda_\varepsilon(y - x) \psi_\delta(\xi - \zeta) \\
&\qquad \times f_{1,+}(t, x, \xi) f_{2,-}(t, y, \zeta) d\xi d\zeta dx dy \\
&+\mathbb{E} \int_{D^\lambda} \int_{-a_N}^{a_N} \lambda(x) f_{1,+}(t, x, \xi) f_{2,-}(t, x, \xi) d\xi dx.
\end{aligned}
$$

It is easy to see that $\lim_{\varepsilon,\delta \to 0} \eta^t_N(\varepsilon, \delta) = 0$ uniformly in N. Also set

$$
\begin{aligned}
r_N(\varepsilon, \delta) = &-\mathbb{E} \int_0^t \int_{\partial D^\lambda_x} \int_{D_y} \int_{-a_N}^{a_N} \int_{-a_N}^{a_N} \lambda(x) \rho^\lambda_\varepsilon(y - x) \psi_\delta(\xi - \zeta) \\
&\times (-a(\xi) \cdot \mathbf{n}(x)) \bar{f}^\lambda_{1,+}(s, x, \xi) f_{2,-}(s, y, \zeta) d\xi d\zeta d\sigma(x) dy ds \\
&+\mathbb{E} \int_0^t \int_{\partial D^\lambda_x} \int_{-a_N}^{a_N} \lambda(x)(-a(\xi) \cdot \mathbf{n}(x)) \bar{f}^\lambda_{1,+}(s, x, \xi) \bar{f}^\lambda_{2,-}(s, x, \xi) d\xi d\sigma(x) ds.
\end{aligned}
$$

Since there exists a sequence $\{\varepsilon_n\} \downarrow 0$ such that $f_{2,-} * \rho^\lambda_{\varepsilon_n}$ converges as $n \to \infty$ to $\bar{f}^\lambda_{2,-}$ in $L^\infty(\Sigma^\lambda \times \mathbb{R})$-weak*, we see that $\lim_{\varepsilon_n,\delta \to 0} r_N(\varepsilon_n, \delta) = 0$ for each $N > 0$. Therefore, it follows from Proposition 3.4 that

$$
\begin{aligned}
&-\mathbb{E} \int_{D^\lambda} \int_{-a_N}^{a_N} \lambda(x) f^+_{1,+}(t, x, \xi) f^+_{2,-}(t, x, \xi) d\xi dx \\
&\leq -\mathbb{E} \int_{D^\lambda} \int_{-a_N}^{a_N} \lambda(x) f^0_{1,+}(x, \xi) f^0_{2,-}(x, \xi) d\xi dx \\
&\quad -\mathbb{E} \int_{\partial D^\lambda} \int_{-a_N}^{a_N} \lambda(x)(-a(\xi) \cdot \mathbf{n}(x)) \bar{f}^\lambda_{1,+}(s, x, \xi) \bar{f}^\lambda_{2,-}(s, x, \xi) d\xi d\sigma(x) ds \\
&\quad +I_1 + I_2 + I_3 + I_N + \eta^t_N(\varepsilon_n, \delta) + \eta^0_N(\varepsilon_n, \delta) + r_N(\varepsilon_n, \delta).
\end{aligned}
$$

On the domain U_{λ_0} a similar argument also deduces the same inequality as above, but the term on the boundary ∂D^{λ_0} vanishes. By virtue of Lemma 2.6 (iii) it holds that

$a(\xi)\cdot\mathbf{n}(x)\bar{f}^{\lambda}_{2,-}=a(\xi)\cdot\mathbf{n}(x)\bar{f}_{2,-}$ a.e. on $[0,T)\times\partial D^{\lambda}\times(-N,N)$, and hence

$$\begin{aligned}&\sum_{i=0}^{M}\mathbb{E}\int_0^t\int_{\partial D^{\lambda_i}}\int_{-a_N}^{a_N}\lambda_i(x)(-a(\xi)\cdot\mathbf{n}(x))\bar{f}^{\lambda_i}_{1,+}(s,x,\xi)\bar{f}^{\lambda_i}_{2,-}(s,x,\xi)d\xi d\sigma(x)ds\\&=\mathbb{E}\int_0^t\int_{\partial D^{\lambda_i}}\int_{-a_N}^{a_N}(-a(\xi)\cdot\mathbf{n}(x))\bar{f}_{2,-}\sum_{i=1}^{M}\lambda_i\bar{f}^{\lambda_i}_{1,+}d\xi d\sigma(x)ds\\&=\mathbb{E}\int_0^t\int_{\partial D^{\lambda_i}}\int_{-a_N}^{a_N}(-a(\xi)\cdot\mathbf{n}(x))\bar{f}_{1,+}\bar{f}_{2,-}d\xi d\sigma(x)ds.\end{aligned}$$

Here recall that $\bar{f}_{1,+}=\sum_{i=0}^{M}\lambda_i\bar{f}^{\lambda_i}_{1,+}$. Thus, summing over $i=0,\ldots,M$ yields

$$\begin{aligned}&-\mathbb{E}\int_D\int_{-a_N}^{a_N}f^+_{1,+}(t,x,\xi)f^+_{2,-}(t,x,\xi)d\xi dx\\&\le-\mathbb{E}\int_D\int_{-a_N}^{a_N}f^0_{1,+}(x,\xi)f^0_{2,-}(x,\xi)d\xi dx\\&\quad-\mathbb{E}\int_{\partial D}\int_{-a_N}^{a_N}(-a(\xi)\cdot\mathbf{n}(x))\bar{f}_{1,+}(s,x,\xi)\bar{f}_{2,-}(s,x,\xi)d\xi d\sigma(x)ds\\&\quad+\sum_{i=0}^{M}(I_1+I_2+I_3+I_N+\eta^t_N(\varepsilon,\delta)+\eta^0_N(\varepsilon,\delta)+r_N(\varepsilon,\delta)).\end{aligned}\tag{3.9}$$

Now note that

$$\lim_{\varepsilon,\delta\to0}\sum_{i=0}^{M}I_2=-\mathbb{E}\int_0^t\int_D\int_{-a_N}^{a_N}\nabla(\sum_{i=0}^{M}\lambda_i)a(\xi)f_{1,+}f_{2,-}d\xi dxds=0.\tag{3.10}$$

In a similar way as in the proof of [5, Theorem 11] we obtain

$$|I_1|\le C\delta\varepsilon^{-1},\quad|I_2|\le C\left(\varepsilon^2\delta^{-1}+r(\delta)\right).\tag{3.11}$$

Finally, we compute the boundary term on the right hand side of (3.9) as follows:

$$\begin{aligned}&-\int_{-a_N}^{a_N}(-a\cdot\mathbf{n})\bar{f}_{1,+}\bar{f}_{2,-}d\xi\\&=-\int_{-a_N}^{u_{2,b}}(-a\cdot\mathbf{n})\bar{f}_{1,+}\bar{f}_{2,-}d\xi-\int_{u_{2,b}}^{u_{1,b}\vee u_{2,b}}(-a\cdot\mathbf{n})\bar{f}_{1,+}\bar{f}_{2,-}d\xi\\&\quad-\int_{u_{1,b}\vee u_{2,b}}^{a_N}(-a\cdot\mathbf{n})\bar{f}_{1,+}\bar{f}_{2,-}d\xi\\&\le-\int_{-a_N}^{u_{2,b}}\bar{f}_{1,+}\partial_\xi\bar{m}^{2,-}_N d\xi+M_b\int_{u_{2,b}}^{u_{1,b}\vee u_{2,b}}d\xi-\int_{u_{1,b}\vee u_{2,b}}^{a_N}\partial_\xi\bar{m}^{1,+}_N\bar{f}_{2,-}d\xi\\&\le-M_b\int_{\mathbb{R}}f^b_{1,+}f^b_{2,-}d\xi.\end{aligned}\tag{3.12}$$

Now we take $\delta=\varepsilon_n^{4/3}$. Letting $\varepsilon_n\to0$ and then letting $N\to\infty$, we immediately deduce (3.1) from (3.9), (3.10), (3.11) and (3.12). □

Proof of Corollary 3.2. Let f_+ be a generalized solution to (1.1)-(1.3) with the initial datum $\mathbf{1}_{u_0>\xi}$ and the boundary datum $\mathbf{1}_{u_b>\xi}$. It follows from Theorem 3.1 and Lemma 2.5 that for $t \in [0,T)$,

$$\mathbb{E}\int_D\int_{\mathbb{R}} f_+^{\pm}(t,x,\xi)(1-f_+^{\pm}(t,x,\xi))d\xi dx = 0.$$

By Fubini's theorem, for $t \in [0,T)$ there is a set E_t of full measure in $\Omega \times D$ such that, for $(\omega,x) \in E_t$, $f_+^{\pm}(\omega,t,x,\xi) \in \{0,1\}$ for a.e. $\xi \in \mathbb{R}$. Since $f_+^{\pm}(t,x,\xi) = \nu_{t,x}(\xi,\infty)$ with a Young measure ν on $\Omega \times Q$, there exists $u(\omega,t,x) \in \mathbb{R}$ such that $f_+^{\pm}(\omega,t,x,\xi) = \mathbf{1}_{u(\omega,t,x)>\xi}$ for a.e. (ω,x,ξ). This gives that $u(\omega,t,x) = \int_{\mathbb{R}}(f_+^{\pm}(\omega,t,x,\xi) - \mathbf{1}_{\xi<0})d\xi$ and hence u is predictable. Moreover, (2.3) is a direct consequence of (2.6). Consequently, we see that u is a kinetic solution to (1.1)-(1.3). □

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