ON PROJECTIONS OF KNOTS, LINKS AND SPATIAL GRAPHS

BY
RYO HANAKI

A dissertation submitted for the degree of
Doctor of Science at Waseda University
November 2009

Acknowledgements

I am deeply grateful to Professors Shin'ichi Suzuki and Kouki Taniyama for their valuable suggestions, comments and encouragements. I thank Professors Makoto Ozawa and Ryo Nikkuni for their valuable suggestions, comments and encouragements.

Ryo Hanaki
Content Studies of Mathematics,
Curriculum Area Sciences Major,
Graduate School of Education,
Waseda University,
Shinjuku, Tokyo,
Japan.

Contents

Introduction

The knots, links and spatial graphs are one circle, disjoint circles and graphs which are embedded in 3-dimensional space $\mathbf{R}^3$ or $\mathbf{S}^3$, respectively. The projections are the images of them obtained by the natural projection. The projection has some information of the original knot, link or spatial graph. We investigate some information from the projection.

In Chapter 1, we introduce the notion of *pseudo diagram*. A pseudo diagram of a spatial graph is a spatial graph projection on the 2-sphere with over/under information at some of the double points. We introduce the trivializing (resp. knotting) number of a spatial graph projection by using its pseudo diagrams as the minimum number of the crossings whose over/under information lead the triviality (resp. nontriviality) of the spatial graph. We determine the set of non-negative integers which can be realized by the trivializing (resp. knotting) numbers of knot and link projections, and characterize the projections which have a specific value of the trivializing (resp. knotting) number.

In Chapter 2, we show that a finite set of specific knotted double-handcuff graphs is shown to be minimal among those which produce all projections of knotted double-handcuff graphs. In addition, we show that a double-handcuff graph has no strongly almost trivial spatial embeddings.

In Chapter 3, we present new classes of graphs which have a strongly almost trivial embedding and that of graphs which have no strongly almost trivial embeddings. We show that both a property that a graph has a strongly almost trivial embedding and a property that a graph has no strongly almost trivial embeddings are not inherited by minors.

Throughout this paper we work in the piecewise linear category. Let G be a finite graph. We consider G as a topological space in the usual way. We give other definitions in each chapter.

Chapter 1

Pseudo diagrams of knots, links and spatial graphs

1.1 Introduction

Let G be a finite graph which does not have degree zero or one vertices. Let f be an embedding of G into the 3-sphere $\mathbf{S}^3$. Then f is called a *spatial embedding* of G and the image $\mathcal{G} = f(G)$ is called a *spatial graph*. In particular, $f(G)$ is called a *knot* if G is homeomorphic to a circle and an *r-component link* if G is homeomorphic to disjoint r circles. In this paper, we say that two spatial graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *ambient isotopic* if there exists an orientation-preserving self-homeomorphism h on $\mathbf{S}^3$ such that $h(\mathcal{G}_1) = \mathcal{G}_2$. A graph G is said to be *planar* if there exists an embedding of G into the 2-sphere $\mathbf{S}^2$. A spatial graph $\mathcal{G}$ is said to be *trivial* (or *unknotted*) if $\mathcal{G}$ is ambient isotopic to a graph in $\mathbf{S}^2$ where we consider $\mathbf{S}^2$ as a subspace of $\mathbf{S}^3$. Thus only planar graphs have trivial spatial graphs. We consider only planar graphs from now on. It is known in [23] that a trivial spatial graph of G is unique up to ambient isotopy in $\mathbf{S}^3$.

A continuous map $\varphi : G \to \mathbf{S}^2$ is called a *regular projection*, or simply a *projection*, of G if the multiple points of φ are only finitely many transversal double points away from the vertices. Then $P = \varphi(G)$ is also called a *projection*. A *diagram* D is a projection P with over/under information at the every double point. Then we say that D *is obtained from* P and P *is a projection of* D. A diagram D uniquely represents a spatial graph up to ambient isotopy. Let $\mathcal{G}$ be a spatial graph represented by D and $\mathcal{G}'$ a spatial graph ambient isotopic to $\mathcal{G}$. Then we also say that P *is a projection of* $\mathcal{G}'$. A double point with over/under information and a double point without over/under information are called a *crossing* and a *pre-crossing*, respectively. Thus a diagram has crossings and has no pre-crossings, and a projection has pre-crossings and has no crossings.

A projection P is said to be *trivial* if any diagram obtained from P represents a trivial spatial graph. On the other hand, a projection P is said to be *knotted* [38] if any diagram obtained from P represents a nontrivial spatial graph. Moreover, the following definitions for a projection P are known. A projection P is said to be *identifiable* [16] if every diagram obtained from P yields a unique labeled spatial graph, and *completely distinguishable* [26] if any two different diagrams obtained from P represent different labeled spatial graphs. Nikkuni showed in [25, Theorem 1.2] that a projection P is identifiable if and only if P is trivial.

Let $\mathcal{G}$ be a spatial graph and P a projection of $\mathcal{G}$. Then we ask the following question.

Question 1.1.1. *Can we determine from P whether the original spatial graph $\mathcal{G}$ is trivial or knotted?*

If P is neither trivial nor knotted, then the (non)triviality of $\mathcal{G}$ cannot be determined from P. For example, let P be a projection of a circle with 3 pre-crossings as illustrated in Fig. 1.1. Then we have 2^3 diagrams obtained from P. Two diagrams represent a nontrivial knot and six diagrams represent a trivial knot.

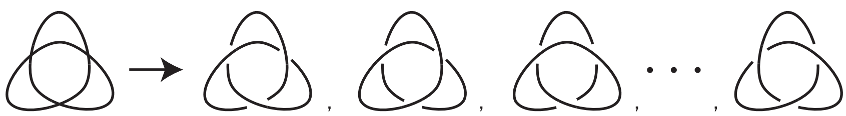


Figure 1.1: Projection and diagrams obtained from it.

It is well known in knot theory that for any projection P of disjoint circles there exists a diagram D obtained from P such that D represents a trivial link. Namely P never admits a knotted projection. However it is known in [38] that there exists a knotted projection of a planar graph. For example, let $\mathcal{G}$ be a spatial graph of the octahedron graph and P a projection of $\mathcal{G}$ as illustrated in Fig. 1.2. Then we can see that any diagram obtained from

P contains a diagram of a Hopf link. Namely P is knotted. However there exists a projection of $\mathcal{G}$ which is neither trivial nor knotted. In general, we have the following proposition.

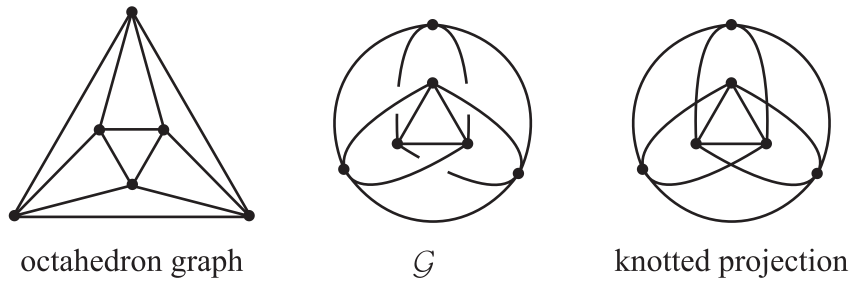


Figure 1.2: Octahedron graph and a knotted projection of it.

Proposition 1.1.2. *For any spatial graph $\mathcal{G}$ of a graph G, there exists a projection P of $\mathcal{G}$ such that P is neither trivial nor knotted.*

We give a proof of Proposition 1.1.2 in section 1.2.

Then it is natural to ask the following question.

Question 1.1.3. *Let $\mathcal{G}$ be a spatial graph and P a projection of $\mathcal{G}$. Which pre-crossings of P and the over/under information lead the (non)triviality of $\mathcal{G}$?*

Now we introduce the notion of a pseudo diagram as a generalization of a projection and a diagram. Let P be a projection of a graph G. A *pseudo diagram* Q of G is a projection P with over/under information at some of the pre-crossings. Then we say that Q *is obtained from* P and P *is a projection of* Q. Thus a pseudo diagram Q has crossings and pre-crossings. Here we allow the possibility that a pseudo diagram has no crossings or has no pre-crossings, that is, a pseudo diagram is possibly a projection or a diagram. We denote the number of crossings and pre-crossings of Q by $c(Q)$ and $p(Q)$, respectively. For a pseudo diagram Q, by giving over/under information to some of the pre-crossings, we can get another (possibly same) pseudo diagram Q'. Then we say that Q' is *obtained from* Q.

We say that a pseudo diagram Q is *trivial* if for any diagram obtained from Q represents a trivial spatial graph. On the other hand, we say that Q is *knotted* if any diagram obtained from Q represents a nontrivial spatial graph. For example, in Fig. 1.3, a pseudo diagram (a) is trivial, (b) is knotted, and (c) is neither trivial nor knotted.

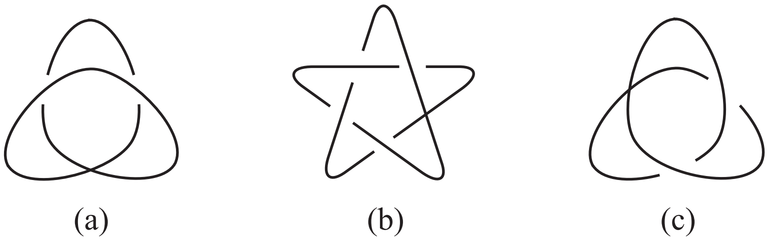


Figure 1.3: Pseudo diagrams.

Let P be a projection of a graph G. Then we define the *trivializing number* (resp. *knotting number*) of P by the minimum of $c(Q)$, where Q varies over all trivial (resp. knotted) pseudo diagrams obtained from P, and denote it by $tr(P)$ (resp. $kn(P)$). Note that there does not exist a knotted (resp. trivial) pseudo diagram obtained from P if and only if $tr(P) = 0$ (resp. $kn(P) = 0$), namely P is trivial (resp. knotted). In this case we define that $kn(P) = \infty$ (resp. $tr(P) = \infty$). Note that for any graph G there exists a projection P of G with $kn(P) = \infty$. For example, P is an image of a planar embedding of G. We also note that for a certain graph G there exists a projection P of G with $tr(P) = \infty$ as in Fig. 1.2.

We remark here that the observation of DNA knots was an opportunity of this research, namely we cannot determine over/under information at some of the crossings in some photos of DNA knots. DNA knots barely become visual objects by examining the protein-coated one by electromicroscope. However there are still cases in which it is hard to confirm the over/under information of some of the crossings. If we can know the (non-)triviality of a knot without checking every over/under information of crossings, then it may give a reasonable way to detect the (non-)triviality of a DNA knot. In addition, it is known that there exists an enzyme, called topoisomerase, which plays a role of crossing change. The research of pseudo diagrams may provide an effective method to change a given DNA knot to a trivial (nontrivial) one. See [8, 4, 24] on DNA knots.

We start from two questions on the trivializing number and the knotting number of projections of a circle.

Question 1.1.4. *For any non-negative integer n, does there exist a projection P of a circle with $tr(P) = n$?*

Question 1.1.5. *For any non-negative integer n, does there exist a projection P of a circle with $kn(P) = n$?*

We have the following theorem and propositions as answers to Questions 1.1.4 and 1.1.5.

Theorem 1.1.6. *For any projection P of a circle, the trivializing number of P is even.*

Proposition 1.1.7. *For any non-negative even number n, there exists a projection P of a circle with $tr(P) = n$.*

Proposition 1.1.8. *There does not exist a projection of a circle whose knotting number is less than three. For any positive integer $n \geq 3$, there exists a projection P of a circle with $kn(P) = n$.*

We give proofs of Theorem 1.1.6 and Proposition 1.1.7 in section 1.3 and a proof of Proposition 1.1.8 in section 1.4. Moreover we see from the following proposition that there are no relations between trivializing number and knotting number in general.

Proposition 1.1.9. *For any non-negative even number n and any positive integer $l \geq 3$, there exists a projection P of a circle with $tr(P) = n$ and $kn(P) = l$.*

We give a proof of Proposition 1.1.9 in section 1.5. In addition, we have the following theorems.

Theorem 1.1.10. *Let P be a projection of disjoint circles. Then $tr(P) = 2$ if and only if P is obtained from one of the projections as illustrated in Fig. 1.4 (a) and (b) where m is a positive integer by possibly adding trivial circles and by a series of replacing a sub-arc of P as illustrated in Fig. 1.4 (c) where a trivial circle means an embedding of a circle into $\mathbf{S}^2$ which does not intersect any other component of the projection.*

We see that for any projection P of disjoint circles, $tr(P) \leq p(P)$ by the definitions. We also see that for any projection P with $kn(P) \neq \infty$, $kn(P) \leq p(P)$ by the definitions. Then we have the following theorems.

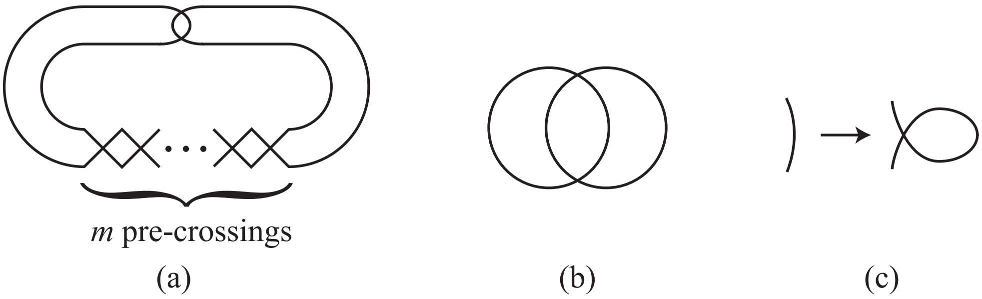


Figure 1.4:

Theorem 1.1.11. *Let P be a projection of a circle with at least one pre-crossing. Then it holds that $tr(P) \leq p(P) - 1$. The equality holds if and only if P is one of the projections as illustrated in Fig. 1.5 where m is a positive odd integer.*

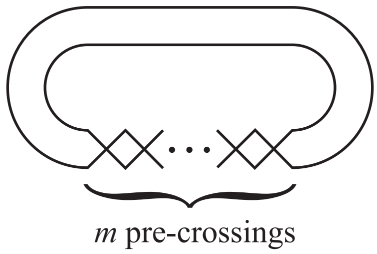


Figure 1.5:

Theorem 1.1.12. *Let P be a projection of n disjoint circles. Let $C_1, C_2, \ldots, C_n$ be the image of the circles of P. Then $tr(P) = p(P)$ if and only if each of $C_1, C_2, \ldots, C_n$ has no self-pre-crossings where a self-pre-crossing is a pre-crossing whose preimage is contained in a circle.*

Theorem 1.1.13. *Let P be a projection of disjoint circles. Then $kn(P) = p(P)$ if and only if P is obtained from one of the projections as illustrated in Fig. 1.6 by possibly adding trivial circles.*

We give proofs of Theorems 1.1.10, 1.1.11 and 1.1.12 in section 1.3 and a proof of Theorem 1.1.13 in section 1.4.

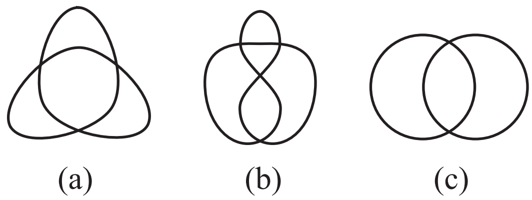


Figure 1.6: Projections P of a circle with $kn(P) = p(P)$.

Let Q be a pseudo diagram of a circle. By giving an orientation to the circle, we can regard Q as a *singular knot*, namely an immersion of a circle into $\mathbf{S}^3$ whose multiple points are only finitely many transversal double points of arcs spanning a sufficiently small flat plane. We consider a singular knot up to ambient isotopy preserving the flatness at each double point. A singular knot K is said to be *trivial* if K is deformed by ambient isotopy preserving the flatness at each double point to a singular knot in $\mathbf{S}^2$. See [31] for details. We can also regard a singular knot as a spatial 4-valent graph up to *rigid vertex isotopy*, see [18, 46]. Then we have the following.

Theorem 1.1.14. *Let Q be a trivial pseudo diagram of a circle. Let K_Q be a singular knot obtained from Q by giving an orientation to the circle. Then K_Q is trivial.*

We give a proof of Theorem 1.1.14 in section 1.3. In section 1.6 we give an application of the trivializing number and the knotting number.

1.2 Fundamental property

First of all, we prove Proposition 1.1.2.

Proof of Proposition 1.1.2. First we show that $\mathcal{G}$ has a projection which is not knotted. For any spatial graph $\mathcal{G}$ we can transform $\mathcal{G}$ into a trivial spatial graph by crossing changes and ambient isotopies. Thus any spatial graph can be expressed as a band sum of a trivial spatial graph and Hopf links, see Fig. 1.7. See [34, 47, 41] for details. Then we can get a diagram D of $\mathcal{G}$ which is identical with a planar embedding of G except the Hopf bands. Let P be the projection of D. Then P is also a projection of a band sum of

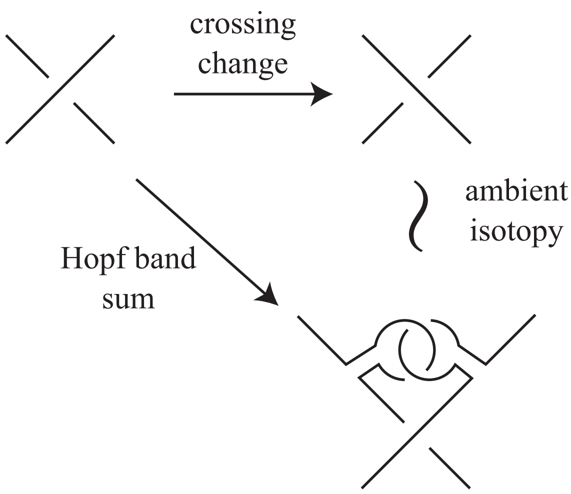


Figure 1.7:

a trivial spatial graph and trivial 2-component links which is itself a trivial spatial graph. Therefore P is not knotted.

If P is not trivial then P is neither trivial nor knotted. Suppose that P is trivial. Let l be a simple arc in P which belongs to the image of a cycle of P. Let P' be a projection obtained from P by applying the local deformation to l as illustrated in Fig. 1.8. Then P' is also a projection of $\mathcal{G}$ which is neither trivial nor knotted. $\square$

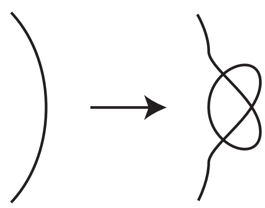


Figure 1.8:

In the rest of this section, we show fundamental properties of the trivializing number and the knotting number which are needed later. Let P be a projection of a circle. We say that a simple closed curve S in $\mathbf{S}^2$ is a *decomposing circle* of P if the intersection of P and S is the set of just two transversal double points. See Fig. 1.9.

Proposition 1.2.1. *Let P be a projection of a circle and S a decomposing circle of P. Let $\{q_1, q_2\} = P \cap S$. Let B_1 and B_2 be the disks such that $B_1 \cup B_2 = \mathbf{S}^2$ and $B_1 \cap B_2 = S$. Let l be one of the two arcs on S joining q_1 and q_2. Let $P_1 = (P \cap B_1) \cup l$ and $P_2 = (P \cap B_2) \cup l$. Then $tr(P) = tr(P_1) + tr(P_2)$ and $kn(P) = \min\{kn(P_1), kn(P_2)\}$.*

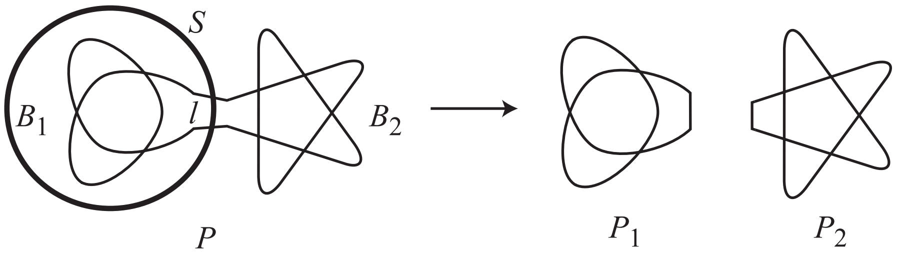


Figure 1.9: Decomposing circle.

Proof. Let Q be a pseudo diagram obtained from P. Let Q_1 (resp. Q_2) be the pseudo diagram obtained from P_1 (resp. P_2) corresponding to Q. Then Q is trivial if and only if both Q_1 and Q_2 are trivial. This implies that $tr(P) = tr(P_1) + tr(P_2)$. We also see that Q is knotted if and only if either Q_1 or Q_2 is knotted. This implies that $kn(P) = \min\{kn(P_1), kn(P_2)\}$. $\square$

The following proposition is shown in [6, 29, 36, 37] as a characterization of trivializing number zero projections of disjoint circles.

Proposition 1.2.2. *[6, 29, 36, 37] Let P be a projection of disjoint circles. Then $tr(P) = 0$ if and only if P is obtained from the projection in Fig. 1.10 (a) by possibly adding trivial circles and by a series of replacing a sub-arc of P as illustrated in Fig. 1.4 (c).*

As an example we illustrate a projection of two circles whose trivializing number equals to zero in Fig. 1.10 (b).

Let P be a projection of disjoint circles. A pre-crossing p of a projection P is said to be *nugatory* if the number of connected components of $P - p$

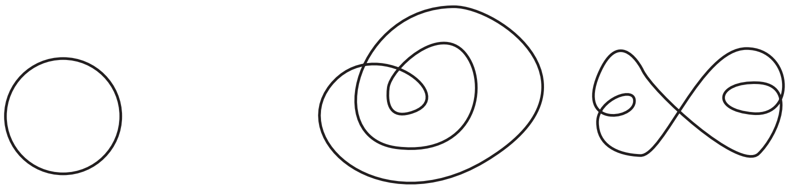


Figure 1.10: Projections P of a circle with $tr(P) = 0$.

is greater than that of P. A crossing c of a diagram D obtained from a projection P is also said to be *nugatory* if the pre-crossing corresponding to c is nugatory in P. Then we can rephrase that P is a projection of disjoint circles with $tr(P) = 0$ if and only if all pre-crossings of P are nugatory. A projection P (resp. a diagram D) is said to be *reduced* if P (resp. D) has no nugatory pre-crossings (resp. no nugatory crossings). Then the following propositions hold.

Proposition 1.2.3. *Let P be a projection of disjoint circles with nugatory pre-crossings and $tr(P) = k$. Let p be a nugatory pre-crossing of P. Let Q be a trivial pseudo diagram obtained from P with k crossings. Then p is a pre-crossing of Q.*

Proof. Suppose that p is a crossing in Q. By forgetting the over/under information of p, we can get another trivial pseudo diagram. Then we have $tr(P) < k$. This is a contradiction. □

Similarly we have the following proposition.

Proposition 1.2.4. *Let P be a projection of disjoint circles with nugatory pre-crossings and $kn(P) = k$. Let p be a nugatory pre-crossing of P. Let Q be a knotted pseudo diagram obtained from P with k crossings. Then p is a pre-crossing of Q.* □

1.3 Trivializing number

In this section, we study trivializing number. First we prove Theorem 1.1.6 and Proposition 1.1.7.

For a pseudo diagram of a circle, we recall a chord diagram of pre-crossings to prove Theorem 1.1.6. Let Q be a pseudo diagram of a circle with n pre-crossings. A *chord diagram* of Q is a circle with n chords marked on it by dashed line segment, where the preimage of each pre-crossing is connected by a chord. We denote it by CD_Q. For example, let Q be a pseudo diagram (a) in Fig. 1.11. Then a chord diagram (b) in Fig. 1.11 is CD_Q. Note that for each chord of a chord diagram of a projection, each of the two arcs in the circle bounded by the end points of the chord contains even number of end points of the other chords. Moreover, a realization problem of a chord diagram by a projection is known in [11].

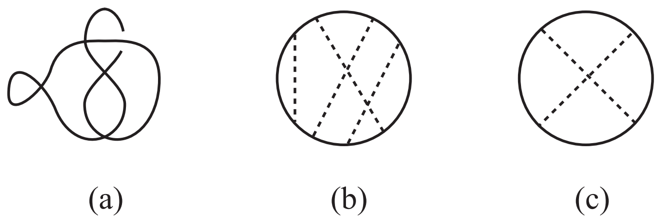


Figure 1.11: Chord diagram.

To prove Theorem 1.1.6, we regard a pseudo diagram of a circle as a singular knot by giving an orientation to the circle and consider the Vassiliev invariant. Let v be a knot invariant which takes values in an additive group. We can extend v to singular knots by the Vassiliev skein relation:

$$v(K_\times) = v(K_+) - v(K_-)$$

where $K_\times$, K_+ and K_- are singular knots which are identical except inside the depicted regions as illustrated in Fig. 1.12. Then v is called a *Vassiliev invariant of order k* if $v(K) = 0$ for any singular knot K with more than k double points and there exists a singular knot J with exactly k double points such that $v(J) \neq 0$. See [44, 2, 3, 31] for Vassiliev invariants. Then the following lemmas hold.

Lemma 1.3.1. *Let Q be a trivial pseudo diagram of a circle with $p(Q) > 0$. Let K_Q be a singular knot obtained from Q by giving an orientation to the circle. Then $v(K_Q) = 0$ where v is a Vassiliev invariant of oriented knots.*

Proof. It is clear from the definitions of Vassiliev invariants. □

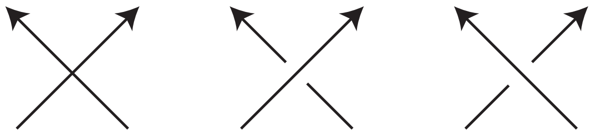


Figure 1.12:

Lemma 1.3.2. *Let Q be a pseudo diagram of a circle with two pre-crossings such that CD_Q is (c) in Fig. 1.11. Then Q is not trivial.*

Proof. Let K_Q be a singular knot obtained from Q. Let a_2 be the second coefficient of the Conway polynomial which is extended to singular knots as above. It is well known that $a_2(K_Q) = 1$. Thus Q is not trivial by Lemma 1.3.1. □

We have the following lemma by applying Lemma 1.3.2.

Lemma 1.3.3. *Let Q be a trivial pseudo diagram of a circle. Then CD_Q contains no sub-chord diagrams as in Fig. 1.11 (c).*

Proof. Suppose that Q contains sub-chord diagrams as in Fig. 1.11 (c). Let Q' be a pseudo diagram obtained from Q such that $CD_{Q'}$ is (c) in Fig. 1.11. By Lemma 1.3.2, a diagram representing nontrivial knot is obtained from Q', hence from Q. This implies that Q is not trivial. This completes the proof. □

Proof of Theorem 1.1.6. Let CD be a sub-chord diagram of CD_P with the maximum number of chords over all sub-chord diagrams of CD_P which do not contain (c) in Fig. 1.11. We show that a trivial pseudo diagram whose chord diagram is CD is obtained from P. Let p_1 be a pre-crossing of P which corresponds to an outer most chord c_1 in CD and l_1 the sub-arc on P which corresponds to the outer most arc. By giving over/under information to each pre-crossing on l_1 so that l_1 goes over the others as in Fig. 1.13, we obtain a pseudo diagram Q_1 from P. Next, let p_2 be a pre-crossing of Q_1 which corresponds to an outer most chord c_2 under forgetting c_1 in CD, and l_2 the sub-arc on Q_1 which corresponds to the outer most arc. By giving over/under information to each pre-crossing on l_2 so that l_2 goes over the others except

l_1, we obtain a pseudo diagram Q_2 from Q_1. By repeating this procedure until all of the chords are forgotten, we obtain a pseudo diagram Q from P. For any diagram D obtained from Q, first we can vanish the crossings on l_1 and the crossing corresponding to p_1, next we can vanish the crossings on l_2 and the crossing corresponding to p_2, similarly we can vanish all crossings of D. Therefore, we see that Q is trivial. Moreover $c(Q)$ is even because each l_i has no self-crossings by the maximality of chords in CD. Since $tr(P) = c(Q)$ by Lemma 1.3.3, $tr(P)$ is even. □

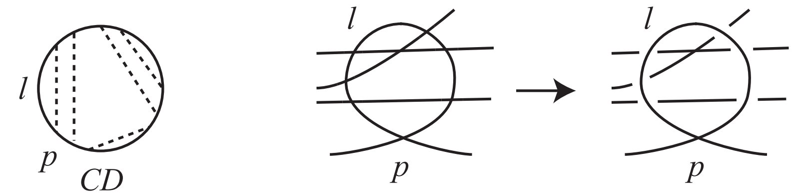


Figure 1.13:

Proof of Proposition 1.1.7. The projection of Fig. 1.5 where $m = n + 1$ has trivializing number n. □

Then we have the following corollary of Theorem 1.1.6 for projections of n disjoint circles.

Corollary 1.3.4. *Let P be a projection of n disjoint circles. Let $C_1, C_2, \ldots, C_n$ be the images of the circles of P. Then the following formula holds.*

$$tr(P) = \sum_{1 \le i < j \le n} \sharp(C_i \cap C_j) + \sum_{k=1}^{n} tr(C_k)$$

where $\sharp A$ is the cardinality of a set A. Therefore, $tr(P)$ is even.

Proof. First we show that

$$tr(P) \ge \sum_{1 \le i < j \le n} \sharp(C_i \cap C_j) + \sum_{k=1}^{n} tr(C_k).$$

Let Q be a trivial pseudo diagram obtained from P. Suppose that there exists a pre-crossing in $C_i \cap C_j (i \neq j)$ such that it is also a pre-crossing of Q. Then a diagram whose sub-diagram represents a 2-component link with nonzero linking number is obtained from Q, namely Q is not trivial. Thus each of the pre-crossings in $C_i \cap C_j$ is a crossing of Q. Note that $\sharp(C_i \cap C_j)$ is even. Moreover each $C_k (1 \leq k \leq n)$ has to be a trivial pseudo diagram in Q. This implies that the above inequality holds.

Next we construct a trivial pseudo diagram obtained from P with $\sum_{1 \leq i < j \leq n} \sharp(C_i \cap C_j) + \sum_{k=1}^{n} tr(C_k)$ crossings. We give over/under information to the pre-crossings in $C_i \cap C_j$ so that C_i goes over C_j for $i > j$ and some pre-crossings of C_k so that a pseudo diagram obtained from C_k is trivial and has $tr(C_k)$ crossings. Then it is easy to see that the pseudo diagram obtained from P by the above way is trivial. This completes the proof. □

In general, we have the following proposition.

Proposition 1.3.5. *Let P a projection of a graph. Then $tr(P) \neq 1$.*

Proof. Suppose that there exists a projection P with $tr(P) = 1$. Let Q be a trivial pseudo diagram obtained from P with only one crossing c. Let Q' be the pseudo diagram obtained from Q by changing the over/under information of c. We show that Q' is trivial. Let D be a diagram obtained from Q'. The mirror image diagram of D is obtained from Q. Since the mirror image of a trivial spatial graph is also trivial, D represents a trivial spatial graph. Hence Q' is trivial. Thus this implies that $tr(P) = 0$. This is a contradiction. □

However, for a certain graph G there exists a projection P of G with $tr(P) = 3$. For example, let G be a graph which is homeomorphic to the disjoint union of a circle and a θ-curve as illustrated in the left side of Fig. 1.14. Then there exists a projection P of G with $tr(P) = 3$, see the right side of Fig. 1.14. Moreover for each $n \geq 2$ there exists a projection P_n of G with $tr(P_n) = n$, see Fig. 1.15.

Next we prove Theorem 1.1.10 that characterizes trivializing number two projections of disjoint circles.

Proof of Theorem 1.1.10. The 'if' part is obvious. Let P be a projection of n disjoint circles with $tr(P) = 2$. Let $C_1, C_2, \ldots, C_n$ be the image of the circles in P. Suppose that there exist pre-crossings in $C_i \cap C_j (i \neq j)$. In this case,

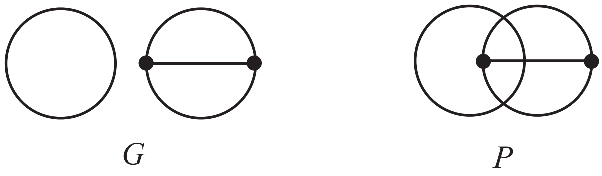


Figure 1.14:

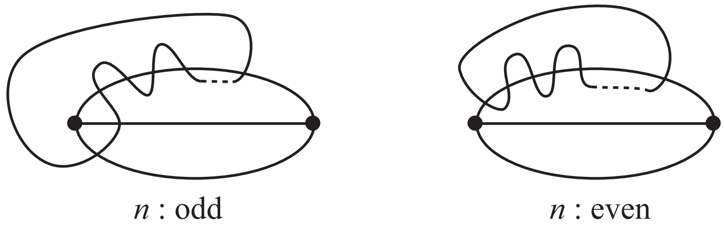


Figure 1.15:

such pre-crossings must be crossings in a trivial pseudo diagram by the same reason as we said in the proof of Corollary 1.3.4. Since $tr(P) = 2$, such pre-crossings belong to the intersection of only one pair of C_i and C_j and each C_i is a trivial projection by Corollary 1.3.4. Thus P is a projection obtained from (b) in Fig. 1.4 by adding trivial circles and by a series of replacing a sub-arc of P as illustrated in Fig. 1.4 (c).

Suppose that $C_i \cap C_j = \emptyset (i \neq j)$. Since $tr(P) = 2$, by Theorem 1.1.6 and Corollary 1.3.4, only one of $C_1, C_2, \ldots, C_n$ is not a trivial projection. Then by the proof of Theorem 1.1.6 we see that CD_P is obtained from one of the chord diagrams (a) or (b) in Fig. 1.16 by adding chords which do not cross the other chords. These chord diagrams (a) or (b) in Fig. 1.16 are realized by the projections (a) in Fig. 1.4. It follows from [11, Theorem 1] that the realizations of these chord diagrams are unique up to mirror image and ambient isotopy. Adding chords which do not cross the other chords corresponds to a series of replacing a sub-arc as illustrated in Fig. 1.4 (c). This completes the proof. □

We use the following procedure which is called a *descending procedure* to prove Theorem 1.1.11 and Proposition 1.1.8. Let P be a projection of n disjoint circles. Let $C_1, C_2, \ldots, C_n$ be the image of the circles in P. We give an

Figure 1.16:

arbitrary orientation and an arbitrary base point which is not a pre-crossing to each C_i. We trace $C_1, C_2, \ldots, C_n$ in order and from their base points along their orientation. We give the over/under information to each pre-crossing of P so that every crossing may be first traced as an over-crossing as illustrated in Fig. 1.17. Then the diagram obtained from P by the procedure as above represents a trivial link.

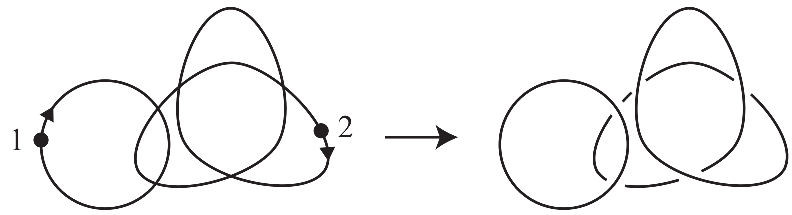


Figure 1.17: A descending procedure.

Proof of Theorem 1.1.11. First we show that $tr(P) \leq p(P) - 1$. Let P be a projection of a circle. We give an orientation to the circle. Let b_1 be a base point on P which is not a pre-crossing. Let p be the pre-crossing of P which first appears when we trace P from b_1 along the orientation. Let b_2 be a base point which is slightly before it than p with respect to the orientation.

Let D_1 (resp. D_2) be the diagram obtained from P by the descending procedure from a base point b_1 (resp. b_2) along the orientation. Here each of D_1 and D_2 represents a trivial knot. The difference of D_1 and D_2 is only the over/under information of p. Let Q be the pseudo diagram obtained from D_1 (or D_2) by forgetting the over/under information of p. Then Q is trivial. This implies that $tr(P) \leq p(P) - 1$.

Next we show that the equality holds if and only if P is one of the projections as illustrated in Fig. 1.5. The 'if' part is obvious. Let P be a projection of a circle with $tr(P) = p(P) - 1$. Then CD_P is a chord diagram in Fig. 1.18 since there exists no pair of parallel chords by the proof of Theorem 1.1.6. Note that CD_P has odd chords. These chord diagrams are realized by the projections as illustrated in Fig. 1.5 where m is a positive odd integer. It follows from [11, Theorem 1] that the realizations of these chord diagrams are unique up to mirror image and ambient isotopy. This completes the proof. □

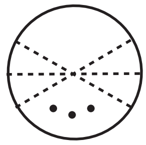


Figure 1.18:

Proof of Theorem 1.1.12. This is an immediate consequence of Theorem 1.1.11 and Corollary 1.3.4. □

Note that similar results on the *unknotting number* for knot diagrams and link diagrams as Theorem 1.1.11 and Theorem 1.1.12 are known in [42, Theorem 1.4, Theorem 1.5].

In the rest of this section, we prove Theorem 1.1.14. To accomplish this, we use the following Theorem 1.3.6. Let D be a diagram of a circle and K a knot represented by D. Then a disk E in $\mathbf{S}^3$ is called a *crossing disk* for a crossing of D if E intersects K only in its interior exactly twice with zero algebraic intersection number and these two intersections correspond the crossing.

Theorem 1.3.6. *[1] Let K be a trivial knot and D a diagram of K. Let $c_1, c_2, \ldots, c_n$ be crossings of D and $E_1, E_2, \ldots, E_n$ crossing disks corresponding to $c_1, c_2, \ldots, c_n$ respectively. Suppose that for any nonempty subset $C \subset \{c_1, c_2, \ldots, c_n\}$ the diagram obtained from D by crossing changes at C represents a trivial knot. Then K bounds an embedded disk in the complement of $\partial E_1 \cup \partial E_2, \cup \cdots \cup \partial E_n$.*

Proof of Theorem 1.1.14. Let $p_1, p_2, \ldots, p_n$ be all of the pre-crossings of Q. Let D be a diagram representing a trivial knot K obtained from Q. Let $c_1, c_2, \ldots, c_n$ be the crossings of D corresponding to $p_1, p_2, \ldots, p_n$ respectively. Let $E_1, E_2, \ldots, E_n$ be crossing disks corresponding to $c_1, c_2, \ldots, c_n$ respectively. For any nonempty subset C of $\{c_1, c_2, \ldots, c_n\}$, a diagram obtained from D by crossing changes at C represents a trivial knot by the definition of a trivial pseudo diagram. By Theorem 1.3.6, there exists an embedded disk H whose boundary is K in the complement of $\partial E_1 \cup \partial E_2, \cup \cdots \cup \partial E_n$. By taking sufficiently small sub-disk of E_i if necessary, we may assume that each $H \cap E_i (i = 1, 2, \ldots, n)$ is a simple arc. By contracting each simple arc to a point, we obtain a singular disk bounding K_Q. Here, we stick two disks at each double point of K_Q as illustrated in Fig. 1.19. Then we have a disk containing K_Q. Therefore, K_Q is trivial. □

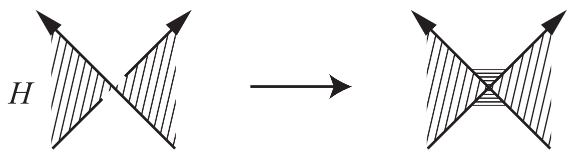


Figure 1.19:

1.4 Knotting number

In this section, we study knotting number and give proofs of Proposition 1.1.8 and Theorem 1.1.13.

Proof of Proposition 1.1.8. First we show that there does not exist a projection of a circle whose knotting number is less than three. Suppose that there exists a projection P of a circle with $kn(P) = 2$. Let Q be a knotted pseudo diagram obtained from P with two crossings c_1 and c_2. Let p_1 and p_2 be the pre-crossings of P which correspond to c_1 and c_2 respectively.

Without loss of generality, we may assume that the position of p_1 and p_2 (resp. c_1 and c_2) on P (resp. Q) is (a) or (b) (resp. (c) or (d)) as in Fig. 1.20. We give an orientation and a base point to the image of the circle as illustrated in Fig. 1.20. In case (a) (resp. (b)), let D_1 (resp. D_2) be the diagram obtained from P by the descending procedure from a base point b.

Here under any of the over/under information of c_1 and c_2, each of D_1 and D_2 represents a trivial knot. This is a contradiction. In case (c) (resp. (d)), let D_3 (resp. D_4) be the diagram obtained from Q by the descending procedure from a base point b_1 (resp. b_2). Then each of D_3 and D_4 represents a trivial knot. This is a contradiction.

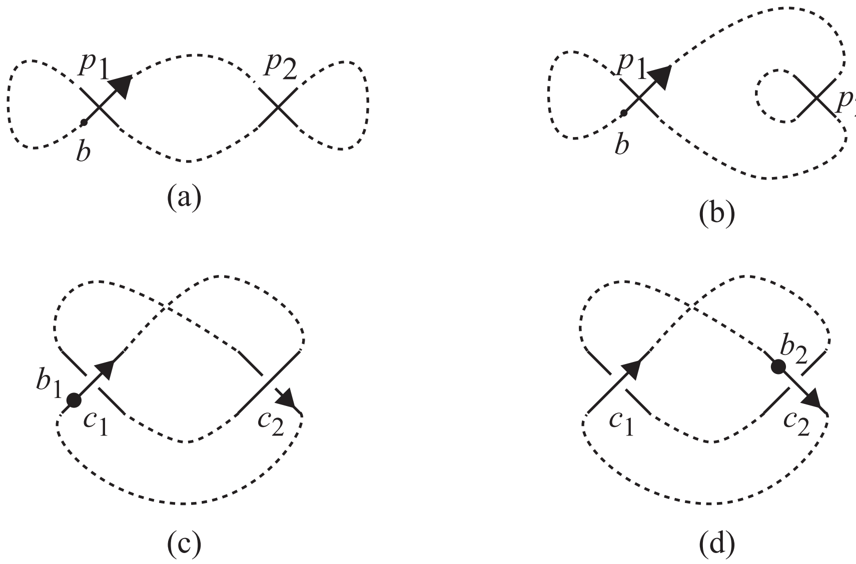


Figure 1.20:

Similarly we can show that there do not exist projections of a circle whose knotting number is less than two.

For $n \geq 3$, the projection of Fig. 1.5 where $m = 2n - 3$ has knotting number n. This completes the proof. □

Note that there exists a projection P of two circles with $kn(P) = 2$ as (c) in Fig. 1.6. In general, we have the following proposition which is similar to Proposition 1.3.5.

Proposition 1.4.1. *Let P be a projection of a graph G. Then $kn(P) \neq 1$.*

Proof. Since the mirror image of a nontrivial spatial graph is also nontrivial, we can prove it in the same way as the proof of Proposition 1.3.5. □

We prepare some known theorems to prove Theorem 1.1.13. Let D be a diagram of disjoint circles. We give an orientation to the image of each circle in D. Then each crossing has a sign as illustrated in Fig. 1.21. A diagram D is said to be *positive* if all crossings of D are positive. Then the following is known.

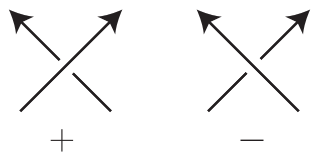


Figure 1.21:

Theorem 1.4.2. *[6, 43, 29, 7] Let D be a positive diagram of disjoint circles with a crossing which is not nugatory. Then D represents a nontrivial link.*

A diagram D is said to be *almost positive* if all crossings except one crossing of D are positive. The following theorem is shown in [33, 30] for knots and in [30] for links.

Theorem 1.4.3. *[33, 30] Let D be an almost positive diagram representing a trivial link. Then D can be obtained from one of the diagrams (a), (b), (c) in Fig. 1.22 by possibly adding trivial circles and by a series of replacing a sub-arc by a part as illustrated in Fig. 1.22 (d).*

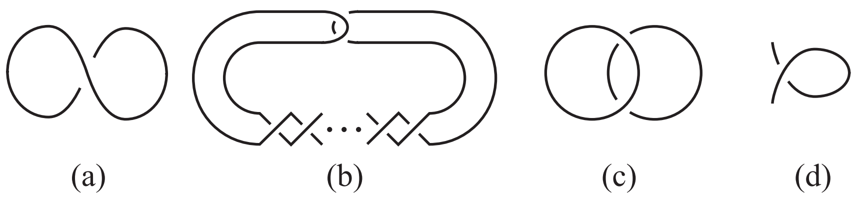


Figure 1.22:

Proof of Theorem 1.1.13. The 'if' part is obvious. Let P be a projection with $tr(P) \neq 0$ which is not obtained from any of the projections as illustrated in Fig. 1.6 by possibly adding trivial circles. We show that there exists a knotted pseudo diagram with at least one pre-crossing obtained from P, that is, $kn(P) < p(P)$.

First we suppose that P has a nugatory pre-crossing p_1. By Proposition 1.2.4 there exists a knotted pseudo diagram obtained from P with a pre-crossing p_1. This implies that $kn(P) < p(P)$.

Next we suppose that P has no nugatory pre-crossings. Suppose that P is not a projection as (a) or (b) in Fig. 1.4. Let p_2 be a pre-crossing of P and Q_2 the pseudo diagram obtained from P by giving over/under information to all pre-crossings except p_2 to be positive. We show that Q_2 is knotted. Let D_{2+} be the diagram obtained from Q_2 by giving the over/under information to p_2 to be positive. Since D_{2+} is a positive diagram, D_{2+} represents a nontrivial link by Theorem 1.4.2. Let D_{2-} be the diagram obtained from Q by giving the over/under information to p_2 to be negative. Since D_{2-} is an almost positive diagram, D_{2-} represents a nontrivial link by Theorem 1.4.3. Thus Q_2 is knotted.

Suppose that P is a projection (a) in Fig. 1.4. Note that $m > 2$ since P is not obtained from one of the projections as illustrated in Fig. 1.6. Let p_3 be one of m pre-crossings in a row. Let Q_3 be the pseudo diagram obtained from P by giving over/under information to all crossings except p_3 to be positive. We show that Q_3 is knotted. Let D_{3+} be the diagram obtained from Q_3 by giving the over/under information to p_3 to be positive. Since D_{3+} is a positive diagram, D_{3+} represents a nontrivial link by Theorem 1.4.2. Let D_{3-} be the diagram obtained from Q_3 by giving the over/under information to p_3 to be negative. We deform D_{3-} into D'_{3-} as illustrated in Fig. 1.23. Since D'_{3-} is a positive diagram with crossings which are not nugatory, D'_{3-} represents a nontrivial link by Theorem 1.4.2. Thus Q_3 is knotted. $\square$

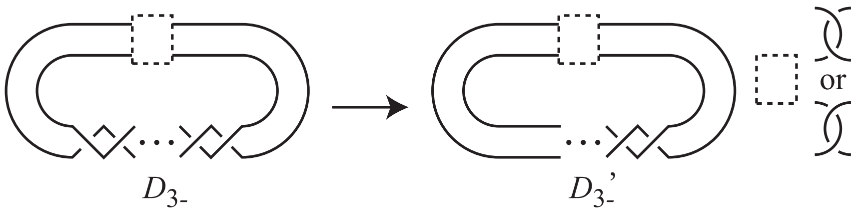


Figure 1.23:

Note that for a certain graph G there exist infinitely many projections P of G with $kn(P) = p(P)$. For example, let G be a handcuff graph and $\{P_i\}_{i=1,2,\ldots}$ is the family of the projections as illustrated in Fig. 1.24. It is known in [39] that a diagram representing a nontrivial spatial graph is obtained from P_i $(i = 1, 2, 3, \ldots)$. Then it is easy to check $kn(P_i) = p(P_i)$.

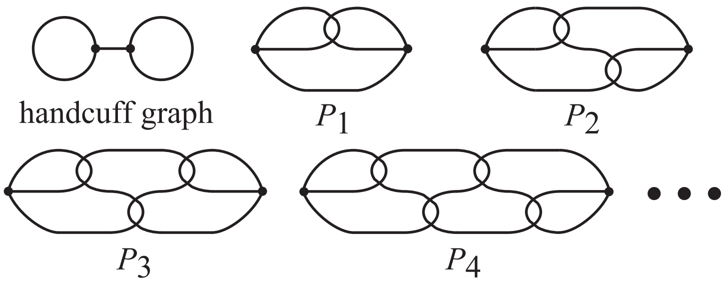


Figure 1.24:

1.5 Relations between trivializing number and knotting number

In this section, we study relations between the trivializing number and the knotting number. We give a proof of Proposition 1.1.9.

Proof of Proposition 1.1.9. Let P_1 be a projection of a circle as illustrated in Fig. 1.4 where $l = 2m - 5$. Then we have $tr(P_1) = 2$ and $kn(P_1) = l$. Let P be the projection which is the composition of $n/2$ copies of P_1 as illustrated in Fig. 1.25. Thus $tr(P) = n$ and $kn(P_1) = l$ by Proposition 1.2.1. $\square$

1.6 An application of trivializing number and knotting number

We ask the following question. For a projection P of a graph, how many diagrams obtained from P which represent trivial spatial graphs (resp. nontrivial spatial graphs)? We denote the number of diagrams obtained from P which represent trivial spatial graphs (resp. nontrivial spatial graphs) by $n_{\mathrm{tri}}(P)$ (resp. $n_{\mathrm{nontri}}(P)$). Then we have the following inequality between $n_{\mathrm{tri}}(P)$ (resp. $n_{\mathrm{nontri}}(P)$) and $tr(P)$ (resp. $kn(P)$) for any graphs.

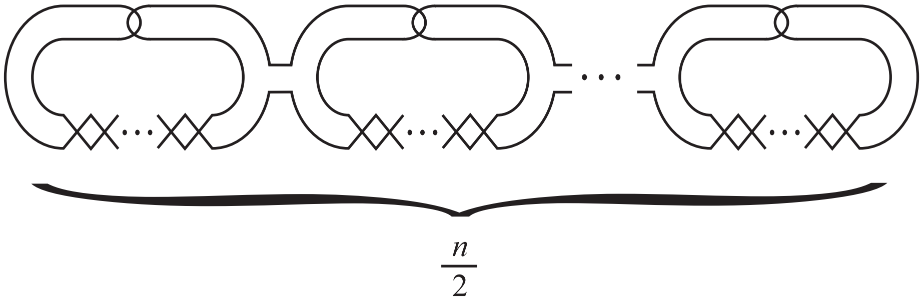


Figure 1.25:

Proposition 1.6.1. *Let P be a projection of a graph. If P is neither trivial nor knotted, then $n_{\mathrm{tri}}(P) \geq 2^{p(P)-tr(P)+1}$ and $n_{\mathrm{nontri}}(P) \geq 2^{p(P)-kn(P)+1}$.*

Proof. We show that $n_{\mathrm{tri}}(P) \geq 2^{p(P)-tr(P)+1}$. Let Q be a trivial pseudo diagram obtained from P with $tr(P)$ crossings. Then $2^{p(P)-tr(P)}$ diagrams which represent trivial spatial graphs are obtained from Q. Let Q' be the pseudo diagram obtained from Q by changing over/under information at all crossings of Q. Then Q' is trivial in the same way as the proof of Proposition 1.3.5. Then $2^{p(P)-tr(P)}$ diagrams which represent spatial graphs are obtained from Q'. Thus $n_{\mathrm{tri}}(P) \geq 2^{p(P)-tr(P)+1}$. Similarly we can show that $n_{\mathrm{nontri}}(P) \geq 2^{p(P)-kn(P)+1}$. $\square$

Chapter 2

Regular projections of knotted double-handcuff graphs

2.1 Introduction

An embedding of G into $\mathbf{R}^3$ is called a *spatial embedding* of G, and its image is called a *spatial graph*. Two spatial embeddings f and f' of G are *equivalent* if there exists a (possibly orientation reversing) self-homeomorphism $h : \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(f(G)) = f'(G)$. We consider spatial embeddings of a graph up to this equivalence. A spatial embedding f of G is *trivial* (or *unknotted*) if there exists a spatial embedding f' of G which is equivalent to f such that $f'(G) \subset \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$.

A continuous map $\varphi : G \to \mathbf{R}^2$ is called a *regular projection*, or simply a *projection*, of G if its multiple points are only finitely many transversal double points away from vertices. Then its image is also called a (regular) projection and we denote it by $\widehat{G} = \varphi(G)$. Similarly we denote the image of a subspace A of G by $\widehat{A} = \varphi(A)$. A double point of a projection is called a *crossing*. In particular, a crossing whose preimage is contained in an edge is called a *self-crossing*. For a spatial embedding f of G, we say that φ is a *projection of f* if there exists a spatial embedding f' of G which is equivalent to f such that $\varphi = \pi \circ f'$ where $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ is the natural projection. Then we also say that $\widehat{G}$ is a *projection of* f, and f *is obtained from* φ *(or* $\widehat{G}$*).*

A projection with over/under information of crossings is called a *regular diagram*. A regular diagram uniquely represents a spatial graph up to equivalence. Thus, for a spatial embedding f of G, $\widehat{G}$ is a projection of f if and only if there exists a regular diagram produced by $\widehat{G}$ which represents $f(G)$. A projection φ of G is said to be *trivial* if only trivial spatial embeddings of G are obtained from φ. A set $\mathcal{E}$ of nontrivial spatial embeddings of G is called *elementary* if every nontrivial projection of G is a projection of at least one element of $\mathcal{E}$ and no proper subset of $\mathcal{E}$ satisfies this property. In general an elementary set of G is not unique. We denote by $elm(G)$ the minimal cardinality of all elementary sets of G, and we call it the elementary number of G.

It is shown in [36] that if a graph G is homeomorphic to a circle, then $\mathcal{E}$ consists of a trefoil knot, therefore $elm(G) = 1$. It is shown in [37] that if a graph G is homeomorphic to the disjoint union of two circles, then $\mathcal{E}$ consists of the Hopf link and the split union of the trefoil knot and the trivial knot, therefore $elm(G) = 2$. In these cases, $\mathcal{E}$ is uniquely determined. It is shown in [20] and [14] that if a graph G is a θ-curve, then the set that consists of three spatial embeddings illustrated in Fig. 2.1 is an elementary set, and $elm(G) = 3$. In general for each $n \geq 3$, an elementary set of $G = \theta_n$ is shown in [14] and we have $elm(\theta_n) = n$.

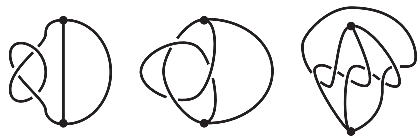


Figure 2.1: An elementary set of θ-curve

It is shown in [39] that if a graph G is the handcuff graph illustrated in Fig. 2.2, then the set $\mathcal{E}$ that consists of infinite spatial embeddings illustrated in Fig. 2.3 is an elementary set, and there exist no finite elementary sets of G, therefore $elm(G) = \infty$. In general it is shown in [39] that if a graph G is a connected planar graph with a cut edge e such that both components of $G-$ int$e$ contain cycles, then $elm(G) = \infty$.

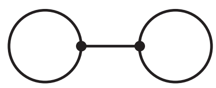


Figure 2.2: A handcuff graph

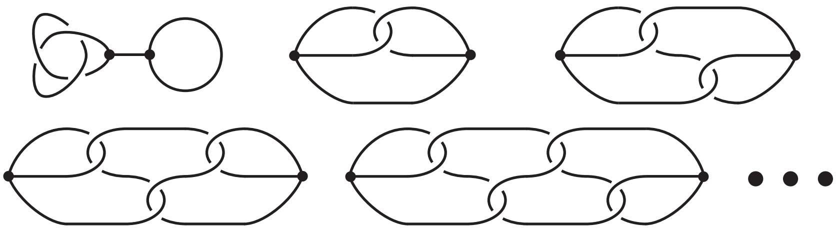


Figure 2.3: An elementary set of a handcuff graph

Our purpose in this paper is to find an elementary set and determine the elementary number of the *double-handcuff graph*. Here the *double-handcuff graph* H is a graph illustrated in Fig. 2.4 with four vertices $u_1,\ u_2,\ v_1,\ v_2$ and six edges $e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6$, where each of e_1 and e_2 joins u_1 and v_1, each of e_3 and e_4 joins u_2 and v_2, e_5 joins u_1 and u_2, and e_6 joins v_1 and v_2. Note that a double-handcuff graph is a graph that is obtained from a subdivision of a handcuff graph or from a subdivision of a θ-curve by adding an edge which is not a loop.

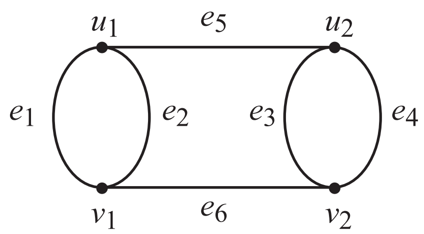


Figure 2.4: A double-handcuff graph

We shall show that $elm(H) = 7$. We note that the elementary number of H is finite even if H contains a handcuff graph whose elementary is infinite.

Theorem 2.1.1. *Let $\mathcal{E}$ be the set of nontrivial spatial embeddings of a double-handcuff graph illustrated in* Fig. 2.5*. Then $\mathcal{E}$ is an elementary set of H, and $elm(H) = 7$.*

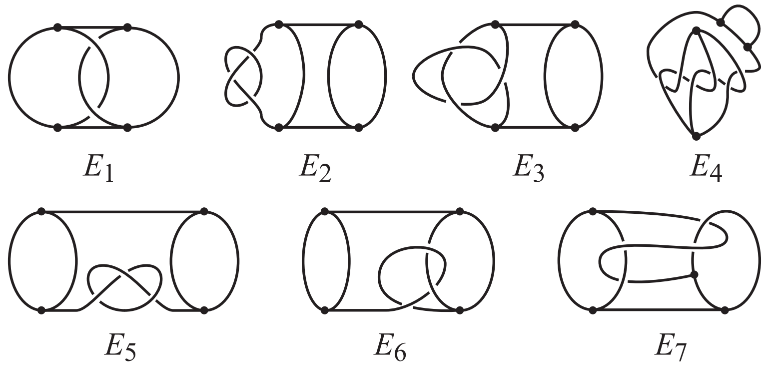


Figure 2.5: An elementary set of H

2.2 Proof of Theorem 2.1.1

In this section we shall prove Theorem 2.1.1 by some lemmas. We show that for any nontrivial projection φ of H it is a projection of at least one element of $\mathcal{E}$, that is, $elm(H) \leq 7$ by Lemmas 2.2.3, 2.2.4, 2.2.5, and 2.2.6, and $elm(H) \geq 7$ by Lemma 2.2.7. Before proving these lemmas, we introduce Propositions 2.2.1 and 2.2.2.

Proposition 2.2.1. *Let $\widehat{P}$ be a projection that has a self-crossing s. Let l be the sub-arc of $\widehat{P}$ that is from s to s. Let $\widehat{P}'$ be a projection obtained from $\widehat{P}$ by eliminating l. If a spatial graph G is obtained from $\widehat{P}'$, then G is also obtained from $\widehat{P}$.*

Proof. See Fig. 2.6. □

A crossing c of a projection $\widehat{P}$ of G is *nugatory* if the number of connected components of $\widehat{P} - c$ is more than that of $\widehat{P}$. A projection $\widehat{P}$ is *reduced* if $\widehat{P}$ has no nugatory crossings. We denote the set of all spatial embeddings of G obtained from $\widehat{P}$ by $\mathrm{EMB}(\widehat{P})$. Then we get the following proposition.

Proposition 2.2.2. *For any projection $\widehat{P}$ of G that has nugatory crossings, there exists a reduced projection $\widehat{P}'$ of G such that* $\mathrm{EMB}(\widehat{P}') = \mathrm{EMB}(\widehat{P})$.

Proof. Let $\widehat{P_1}$ be a connected component of a projection $\widehat{P}$ of G with a nugatory crossing c. Let $\widehat{T_1}$ and $\widehat{T_2}$ be the parts of $\widehat{P_1}$ such that $\widehat{P_1} = \widehat{T_1} \cup \widehat{T_2}$ and $\widehat{T_1} \cap \widehat{T_2} = \{c\}$. Let $\widehat{P_2}$ be a projection illustrated in Fig. 2.7. Then

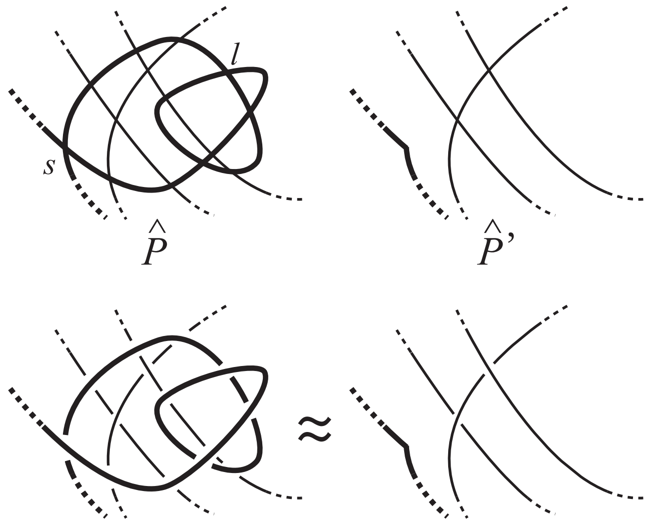


Figure 2.6: Proof of Proposition 2.2.1

it is clear that EMB($\widehat{P}_1$) =EMB($\widehat{P}_2$). Therefore we can eliminate nugatory crossings. $\square$

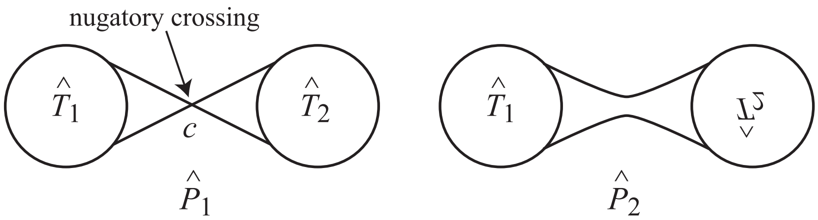


Figure 2.7: Eliminating a nugatory crossing

By Proposition 2.2.2, we may assume that φ in Lemmas 2.2.3, 2.2.4, 2.2.5, and 2.2.6 is a reduced projection. For points x and y on an edge e, we denote by $[x, y; e] = [x, y]$ the simple arc in e bounded by x and y.

Lemma 2.2.3. *If* $(\hat{e}_1 \cup \hat{e}_2) \cap (\hat{e}_3 \cup \hat{e}_4)$ *is not empty, then* φ *is a projection of* E_1.

Proof. Without loss of generality, we may assume $\hat{e}_1 \cap \hat{e}_3$ is not empty. Let p be the crossing on $\hat{e}_1$ that is the nearest to u_1 in $\hat{e}_1 \cap \hat{e}_3$. Let $p_1 = \varphi^{-1}(p) \cap e_1$

and $p_2 = \varphi^{-1}(p) \cap e_3$. Let p'_2 be a point on e_3 that is slightly nearer to v_2 than p_2, p''_2 a point on e_3 that is slightly farther from v_2 than p_2. We suppose that there are no self-crossings in $[u_1, p_1; e_1]$ and $[u_2, p_2; e_3]$ by Proposition 2.2.1.

Let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = 0, h(v_1) = 1, h(u_2) = -1, h(v_2) = 0$.
2. $h_1([u_1, p_1]) = \{0\}$. $h_1|_{[p_1, v_1]}$ is injective.
3. h_i is injective $(i = 2, 4, 5, 6)$.
4. $h_3([v_2, p'_2]) = \{0\}$. $h_3|_{[p'_2, p_2]} \to [0, \varepsilon]$ is homeomorphism. $h_3|_{[p_2, p''_2]} \to [0, \varepsilon]$ is homeomorphism. $h_3|_{[p''_2, u_2]} \to [-1, 0]$ is homeomorphism.
5. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times h$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.8. Here Fig. 2.8 is a regular diagram on the xz-plane and boxes B_1 and B_2 represent pure 3-braids. Then by the repeated application of deformation of Fig. 2.9, we see that H' is ambient isotopic to E_1. $\square$

Let $\mathbf{S}^2 \subset \mathbf{S}^3$ be the one point compactification of the pair $\mathbf{R}^2 \subset \mathbf{R}^3$. We consider projections on $\mathbf{S}^2$ and spatial double-handcuff graphs in $\mathbf{S}^3$ for the convenience. A circle S on $\mathbf{S}^2$ is called a *separating circle* if S meets $\hat{e}_5$ transversally at one point and $\hat{e}_6$ transversally at one point and does not meet any other edges. Note that S bounds two 2-balls A_1 and A_2 in $\mathbf{S}^2$. We may assume without loss of generality that $A_1 \supset \hat{e}_1 \cup \hat{e}_2$ and $A_2 \supset \hat{e}_3 \cup \hat{e}_4$.

Lemma 2.2.4. *If there exists a separating circle S, then φ is a projection of $E_2, E_3, E_4, E_5,$ or E_6.*

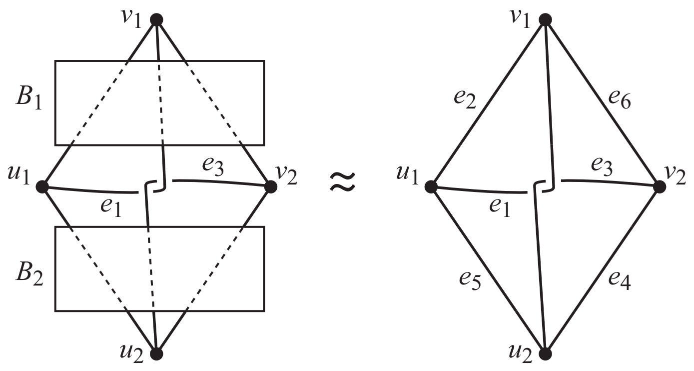


Figure 2.8:

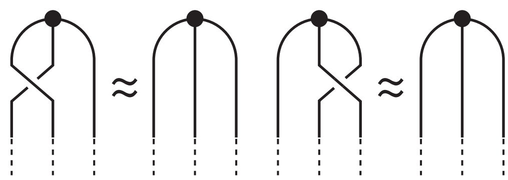


Figure 2.9: Eliminating a crossing

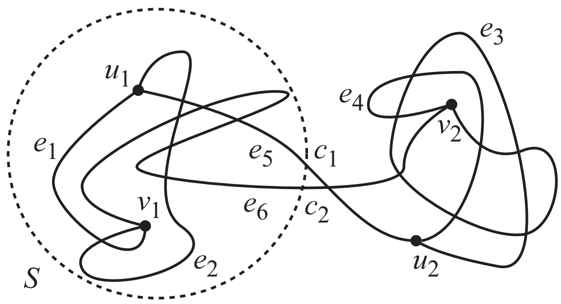


Figure 2.10: A separating circle

Proof. By contracting A_1 (resp. A_2) to a point in $\mathbf{S}^2$, we may consider $A_1 \cup \hat{e}_5 \cup \hat{e}_6$ (resp. $A_2 \cup \hat{e}_5 \cup \hat{e}_6$) to be a projection of an edge, say e_0 (resp. e_0'). Therefore we consider φ to be a projection, say ψ (resp. ψ'), of a θ-curve which consists of two vertices u_2 and v_2 (resp. u_1 and v_1) and three edges e_3, e_4 and e_0 (resp. e_1, e_2 and e_0'). Here ψ or ψ' is nontrivial because φ is nontrivial. Without loss of generally, we may suppose that ψ is nontrivial.

There exists a height function h such that $\psi \times h$ is a spatial embedding of an elementary set illustrated in Fig. 2.1 ([14]). Let $c_1 = S \cap \hat{e}_5, c_2 = S \cap \hat{e}_6$. We can extend h to a height function h' of a double-handcuff graph with the following properties, where $[u_1, c_1]$ (resp. $[v_1, c_2]$) is the simple arc in e_5 (resp. e_6) bounded by u_1 and c_1 (resp. v_1 and c_2), ε is a sufficiently small positive real number.

1. $h'(c_2) - h'(c_1) = \varepsilon$.

2. $h'(c_1) < h'(u_1) < h'(v_1) < h'(c_2)$.

3. $h'|_{e_i}$, $h'|_{[u_1,c_1]}$, and $h'|_{[v_1,c_2]}$ is injective ($i = 1, 2$).

4. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h'(x) \neq h'(y)$ for all $x, y \in H$.

Let H' be the image of $\varphi \times h'$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph. Here Fig. 2.11 is a part of a regular diagram on the xz-plane.

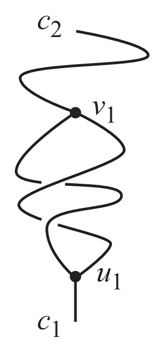


Figure 2.11:

We eliminate crossings in Fig. 2.11. Hence $\varphi \times h'$ is E_2, E_3, E_4, E_5, or E_6. $\square$

Lemma 2.2.5. *If $\hat{e}_5$ or $\hat{e}_6$ has self-crossings, then φ is a projection of E_5 or E_6.*

Proof. Without loss of generality, we may assume $\hat{e}_5$ has self-crossings. We can choose a self-crossing p of $\hat{e}_5$ such that the sub-arc, say l, of $\hat{e}_5$ from p to p has no other self-crossings. Let R_1 be the region as illustrated in Fig. 2.12. We can suppose that a shape of R_1 is a teardrop in the right side of Fig. 2.12, since we consider that a projection is on $\mathbf{S}^2$.

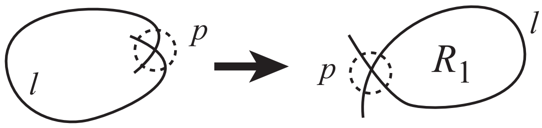


Figure 2.12:

Case 1. $l \cap \hat{e}_5$ is not empty.

Let $p' \in l \cap \hat{e}_5$. Let $\{p_1, p_3\} = \varphi^{-1}(p), \{p_2, p_4\} = \varphi^{-1}(p')$ and we may suppose without loss of generality that u_1, p_1, p_2, p_3, p_4 and u_2 are arranged in this order on e_5. Let p'_i be a point on e_5 that is slightly nearer to u_1 than p_i, and p''_i a point on e_5 that is slightly farther from u_1 than p_i $(i = 1, 3, 4)$.

Let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = -1, h(v_1) = h(u_2) = -2, h(v_2) = -3$.
2. h_i is injective $(i = 1, 2, 3, 4, 6)$.
3. All of the following maps are homeomorphisms and $h_5([p''_1, p''_3]) = \{0\}$.
   $h_5|_{[u_1,p'_1]} \to [-1, 0]$, $h_5|_{[p'_1,p_1]} \to [0, \varepsilon]$, $h_5|_{[p_1,p''_1]} \to [0, \varepsilon]$, $h_5|_{[p''_3,p'_4]} \to [0, 1]$, $h_5|_{[p'_4,p_4]} \to [\varepsilon, 1]$, $h_5|_{[p_4,p''_4]} \to [0, \varepsilon]$, and $h_5|_{[p''_4,u_2]} \to [-2, 0]$.
4. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times h$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.13. Here Fig. 2.13 is a regular diagram on the xz-plane and boxes B_1 and B_2 represent pure 2-braids, boxes B_3 and B_4

represent pure 3-braids. First we eliminate crossings in B_1, B_2 and B_4. By deforming, we obtain H' in lower left of Fig. 2.13 and we eliminate crossings in B_3. Thus we see that H' is ambient isotopic to E_5.

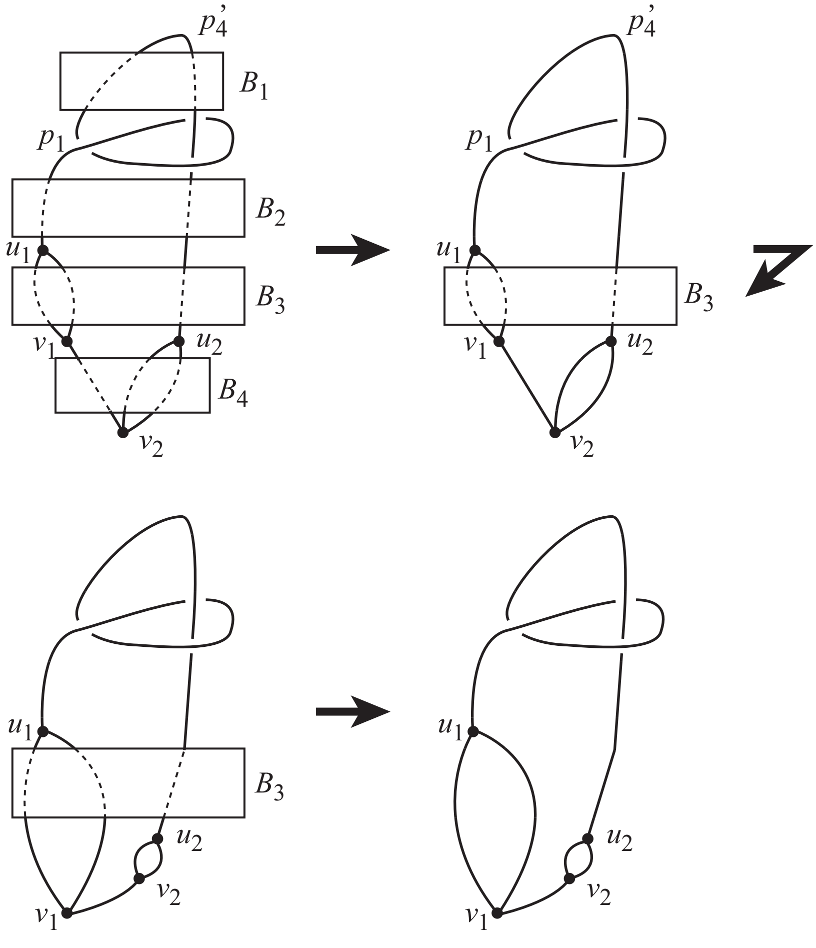


Figure 2.13: Case 1

Case 2. $\hat{v}_1 \in R_1$ or $\hat{v}_2 \in R_1$.

Without loss of generality, we may assume $\hat{v}_1 \in R_1$. Let $\{p_1, p_2\} = \varphi^{-1}(p)$ and we may suppose without of loss generality that u_1, p_1, p_2 and u_2 are arranged in this order on e_5. Let p_1' be a point on e_5 that is slightly nearer to u_1 than p_1, and p_1'' a point on e_5 that is slightly farther from u_1 than p_1.

Let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = 1, h(v_1) = 0, h(u_2) = -1, h(v_2) = -2$.

2. h_i is injective $(i = 1, 2, 3, 4, 6)$.

3. All of the following maps are homeomorphism and $h_5|_{[p_1'', p_2]} \to \{0\}$.

   $h_5|_{[u_1, p_1']} \to [0, 1]$, $h_5|_{[p_1', p_1]} \to [-\varepsilon, 0]$, $h_5|_{[p_1, p_1'']} \to [-\varepsilon, 0]$, and $h_5|_{[p_2, u_2]} \to [-1, 0]$.

4. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times f$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.14. Here Fig. 2.14 is a regular diagram on the xz-plane and boxes B_1 and B_3 represent pure 3-braids, a box B_2 represents a pure 2-braid. Then we eliminate crossings in B_1, B_2 and B_3. Thus we see that H' is ambient isotopic to E_5.

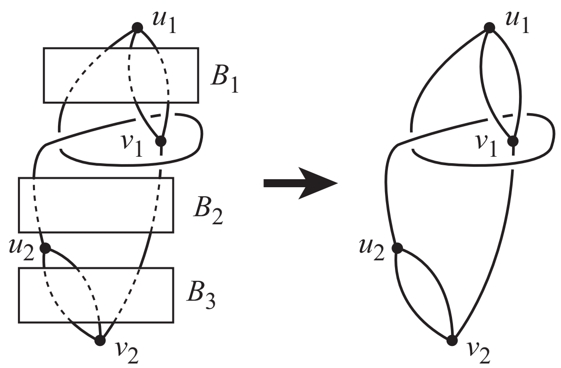


Figure 2.14: Case 2

Case 3. $\hat{e}_i \cap l (i = 1, 2, 3, 4)$ is not empty, $\hat{v}_1 \notin R_1$ and $\hat{v}_2 \notin R_1$.

Without loss of generality, we may assume $\hat{e}_1 \cap l$ is not empty. Let $\{p_1, p_2\} = \varphi^{-1}(p)$ and we may suppose without loss of generality that u_1, p_1, p_2 and u_2 are arranged in this order on e_5. Let p_1' be a point on e_5 that is slightly nearer to u_1 than p_1, and p_1'' a point on e_5 that is slightly farther from u_1 than p_1. Let p_3 be a point on e_1 such that $\varphi(p_3) \in R_1$ and p_4 a point on e_2 that is a sufficiently near to v_1.

Let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = 1, h(v_1) = h(u_2) = -1, h(v_2) = -2$.
2. h_1 is injective and $h_1(p_3) = 0$.
3. h_2 is injective and $h_2(p_4) = 0$.
4. h_i is injective $(i = 3, 4, 6)$.
5. All of the following maps are homeomorphisms and $h_5([p_1'', p_2]) = \{0\}$.
   $h_5|_{[u_1,p_1']} \to [0, 1]$, $h_5|_{[p_1',p_1]} \to [-\varepsilon, 0]$, $h_5|_{[p_1,p_1'']} \to [-\varepsilon, 0]$, and $h_5|_{[p_2,u_2]} \to [-1, 0]$.
6. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times h$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.15. Here Fig. 2.15 is a regular diagram on the xz-plane and boxes B_1 and B_3 represent pure 3-braids, a box B_2 represents a pure 2-braid. First we eliminate crossings in B_1 and B_3. Then we eliminate crossings in B_2. Thus we see that H' is ambient isotopic to E_6.
Case 4. $\hat{e}_6 \cap l$ is not empty.

Let $\{p_1, p_2\} = \varphi^{-1}(p)$ and we may suppose that u_1, p_1, p_2 and u_2 are arranged in this order on e_5. Let p_1' be a point on e_5 that is slightly nearer to u_1 than p_1, and p_1'' a point on e_5 that is slightly farther from u_1 than p_1. Let p_3 be a point on e_6 in R_1.

Let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = 2, h(v_1) = 1, h(u_2) = -1, h(v_2) = -2$.
2. h_i is injective $(i = 1, 2, 3, 4)$.

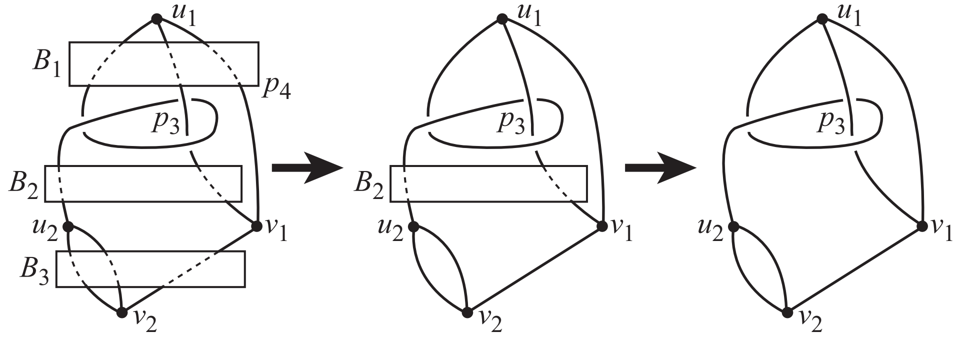


Figure 2.15: Case 3

3. All of the following maps are homeomorphisms and $h_5([p_1'', p_2]) = \{0\}$.
   $h_5|_{[u_1,p_1']} \to [0,2]$, $h_5|_{[p_1',p_1]} \to [-\varepsilon, 0]$, $h_5|_{[p_1,p_1'']} \to [-\varepsilon, 0]$, and $h_5|_{[p_2,u_2]} \to [-1, 0]$.
4. h_6 is injective and $h_6(p_3) = 0$.
5. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times h$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.16. Here Fig. 2.16 is a regular diagram on the xz-plane and boxes B_1 and B_4 represent pure 3-braids, boxes B_2 and B_3 represent pure 2-braids. Then we eliminate crossings in B_1, B_2, B_3 and B_4. Thus we see that H' is ambient isotopic to E_5. $\square$

We suppose that $(\hat{e}_1 \cup \hat{e}_2) \cap (\hat{e}_3 \cup \hat{e}_4)$ is empty. Let R_1 be the closure of the union of the connected components of $S^2 - (\hat{e}_1 \cup \hat{e}_2)$ that does not contain $\hat{e}_3 \cup \hat{e}_4$. Let $\bar{R}_1$ be the closure of $S^2 - R_1$. Let M_1 be a sufficiently small regular neighbourhood of R_1 in S^2. Then M_1 is homeomorphic to a closed disk.

Similarly let R_2 be the closure of the union of the connected components of $S^2 - (\hat{e}_3 \cup \hat{e}_4)$ that does not contain M_1. Let $\bar{R}_2$ be the closure of $S^2 - R_2$. Let M_2 be a sufficiently small regular neighbourhood of R_2 in S^2. Then M_2 is also homeomorphic to a closed disk.

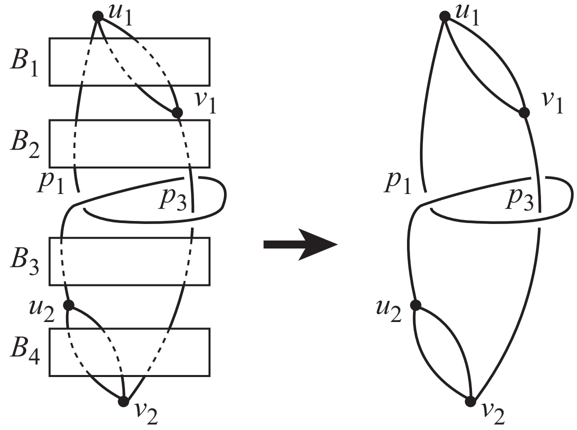


Figure 2.16: Case 4

Let $[0,1]$ be the unit closed interval. Let $\bar{e}_5 : [0,1] \to e_5$ be a parametrization with $\bar{e}_5(0) = u_1$ and $\bar{e}_5(1) = u_2$. Let $\bar{e}_6 : [0,1] \to e_6$ be a parametrization with $\bar{e}_6(0) = v_1$ and $\bar{e}_6(1) = v_2$.

Lemma 2.2.6. *If $(\hat{e}_1 \cup \hat{e}_2) \cap (\hat{e}_3 \cup \hat{e}_4)$ is empty, there exist no separating circles, and $\hat{e}_5$ and $\hat{e}_6$ has no self-crossings, then φ is a projection of E_6 or E_7.*

Proof. First we shall show that if there exists a pair of points x, y in $[0,1]$ such that $x < y, \varphi \circ \bar{e}_5(x) \in R_2$ and $\varphi \circ \bar{e}_5(y) \in R_1$, then it is a projection of E_7. In this case we note that there exist no separating circles.

Let $U = \{(x,y) | x, y \in [0,1], x < y, \varphi \circ \bar{e}_5(x) \in R_2, \varphi \circ \bar{e}_5(y) \in R_1\}$, $X = \rho_1(U)$, $Y = \rho_2(U)$ where $\rho_1(x,y) = x$, $\rho_2(x,y) = y$. Let $x_1 \in X$ such that $\varphi \circ \bar{e}_5(x_1)$ is the nearest to $\hat{v}_2$ on $\hat{e}_3$. Let $y_1 = \min\{y | x_1 < y, y \in Y\}$.

Without loss of generality, we may assume that $\hat{e}_1 \cap \varphi \circ \bar{e}_5(y_1)$ is not empty. Let $t_5 = \bar{e}_5(x_1), t_3 = \varphi^{-1}(\varphi(t_5)) \cap e_3$ and $s_5 = \bar{e}_5(y_1), s_1 = \varphi^{-1}(\varphi(s_5)) \cap e_1$. Let $\{p_1, p_2, \ldots, p_n\} = \hat{e}_1 \cap \varphi \circ \bar{e}_5((0, x_1])$ (possibly $\hat{e}_1 \cap \varphi \circ \bar{e}_5((0, x_1]) = \emptyset$), and $p_{5,1} = \varphi^{-1}(p_1) \cap e_5, p_{5,2} = \varphi^{-1}(p_2) \cap e_5, \ldots, p_{5,n} = \varphi^{-1}(p_n) \cap e_5$. Here we can suppose that $\bar{e}_5^{-1}(p_{5,1}) < \bar{e}_5^{-1}(p_{5,2}) < \cdots < \bar{e}_5^{-1}(p_{5,n})$. Let $p'_{5,i}$ be a point on e_5 that is slightly nearer to u_1 than $p_{5,i}$, and $p''_{5,i}$ a point on e_5 that is slightly farther from u_1 than $p_{5,i}$ $(i = 1, 2, \ldots, n)$. Let t'_5 be a point on e_5 that is slightly nearer to u_1 than t_5, and t''_5 a point on e_5 that is slightly farther from u_1 than t_5. Let s'_5 be a point on e_5 that is slightly nearer to u_1

than s_5, and s_5'' a point on e_5 that is slightly farther from u_1 than s_5. We can suppose that there are no self-crossings in $[u_1, s_1; e_1]$ and $[u_2, t_3; e_3]$ by Proposition 2.2.1.

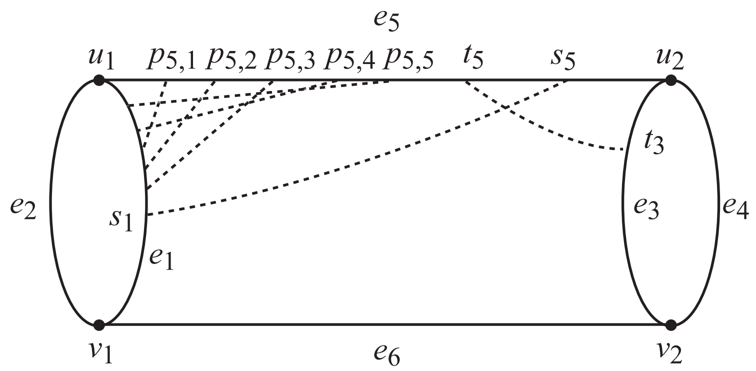


Figure 2.17:

Let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = 0, h(v_1) = 1, h(u_2) = -1, h(v_2) = 0$.

2. $h_1([u_1, s_1]) = \{0\}$. $h_1|_{[s_1,v_1]}$ is injective.

3. h_i is injective $(i = 2, 4, 6)$.

4. $h_3([v_2, t_3]) = \{0\}$. $h_3|_{[t_3,u_2]}$ is injective.

5. All of the following maps are homeomorphisms and $h_5([u_1, p'_{5,1}]) = h_5([p''_{5,1}, p'_{5,2}]) = h_5([p''_{5,2}, p'_{5,3}]) = \cdots = h_5([p''_{5,n-1}, p'_{5,n}]) = h_5([p''_{5,n}, t'_5]) = h_5([t''_5, s'_5]) = \{0\}$.

   $h_5|_{[p'_{5,1},p_{5,1}]} \to [-\varepsilon, 0]$, $h_5|_{[p_{5,1},p''_{5,1}]} \to [-\varepsilon, 0]$, $h_5|_{[p'_{5,2},p_{5,2}]} \to [-\varepsilon, 0]$, $h_5|_{[p_{5,2},p''_{5,2}]} \to [-\varepsilon, 0]$, $\cdots$ $h_5|_{[p'_{5,n},p_{5,n}]} \to [-\varepsilon, 0]$, $h_5|_{[p_{5,n},p''_{5,n}]} \to [-\varepsilon, 0]$, $h_5|_{[t'_5,t_5]} \to [-\varepsilon, 0]$, $h_5|_{[t_5,t''_5]} \to [-\varepsilon, 0]$, $h_5|_{[s'_5,s_5]} \to [0, \varepsilon]$, and $h_5|_{[s_5,u_2]} \to [-1, \varepsilon]$.

6. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times h$.

We eliminate crossings $p_1, p_2, \ldots, p_n$ as illustrated in Fig. 2.18. Then there are two cases (I), (II) as illustrated in Fig. 2.19. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.20. Here Fig. 2.20 is a regular diagram on the xz-plane and boxes B_1 and B_2 represent pure 3-braids. Then we eliminate crossings in B_1 and B_2. Thus we see that H' is ambient isotopic to E_7.

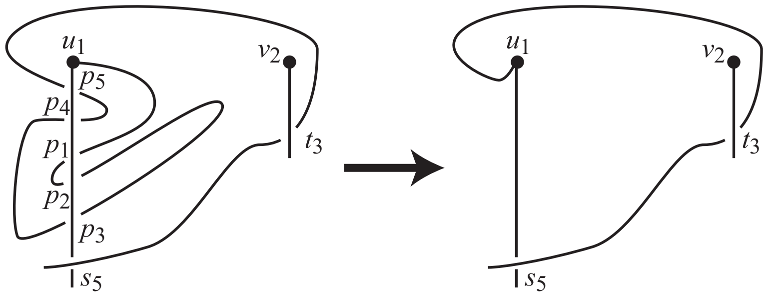
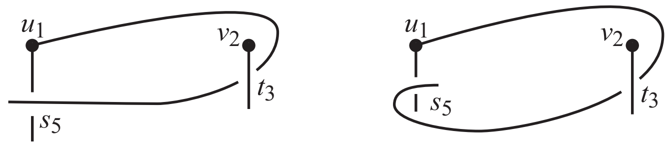


Figure 2.18: Eliminating crossings $p_1, p_2, \ldots, p_n$



(I) (II)

Figure 2.19:

Similarly if there exists a pair of points x, y in $[0, 1]$ such that $x < y, \varphi \circ \bar{e}_6(x) \in R_2$ and $\varphi \circ \bar{e}_6(y) \in R_1$, then φ is a projection of E_7.

Next we shall show that if there exist no pairs as above, then φ is a projection of E_6.

Let $x_{11} = \min \{x \in [0, 1] | \varphi \circ \bar{e}_5(x) \in \bar{R}_1\}$. Here if $x_{11} = 0$ and $\varphi \circ \bar{e}_5(\varepsilon) \notin \bar{R}_1$, let $x_{11} = \min \{x \in (0, 1] | \varphi \circ \bar{e}_5(x) \in \bar{R}_1\}$, where ε is a sufficiently small positive real number. Let $x_{12} = \min \{x \in [x_{11}, 1] | \varphi \circ \bar{e}_5(x) \in \bar{R}_1\}$. There may

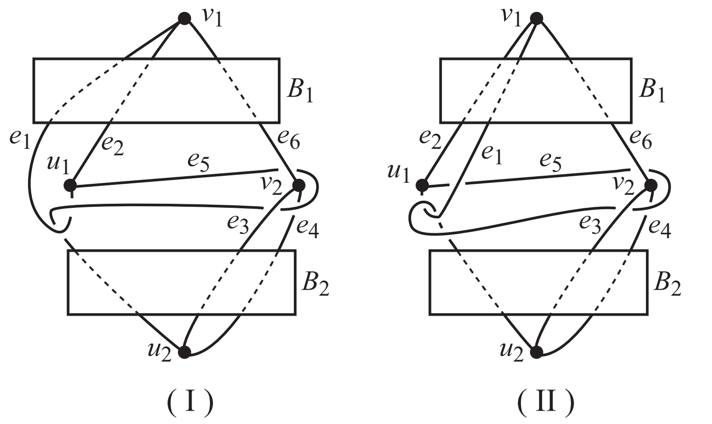


Figure 2.20:

not exist x_{12}. Let $x_{21} = \min\ \{x \in [x_{12}, 1] | \varphi \circ \bar{e}_5(x) \in \bar{R}_1\}$. Let $x_{22} = \min\ \{x \in [x_{21}, 1] | \varphi \circ \bar{e}_5(x) \in \bar{R}_1\}$. Then we repeat the process above. Similarly let $y_{11} = \min\ \{y \in [0, 1] | \varphi \circ \bar{e}_6(y) \in \bar{R}_1\}$. Here if $y_{11} = 0$ and $\varphi \circ \bar{e}_6(\varepsilon) \notin \bar{R}_1$, let $y_{11} = \min\ \{y \in (0, 1] | \varphi \circ \bar{e}_6(y) \in \bar{R}_1\}$, where ε is a sufficiently small positive real number. Let $y_{12} = \min\ \{y \in [y_{11}, 1] | \varphi \circ \bar{e}_6(y) \in \bar{R}_1\}$. There may not exist y_{12}. Let $y_{21} = \min\ \{y \in [y_{12}, 1] | \varphi \circ \bar{e}_6(y) \in \bar{R}_1\}$. Let $y_{22} = \min\ \{y \in [y_{21}, 1] | \varphi \circ \bar{e}_6(y) \in \bar{R}_1\}$. Then we repeat the process above.

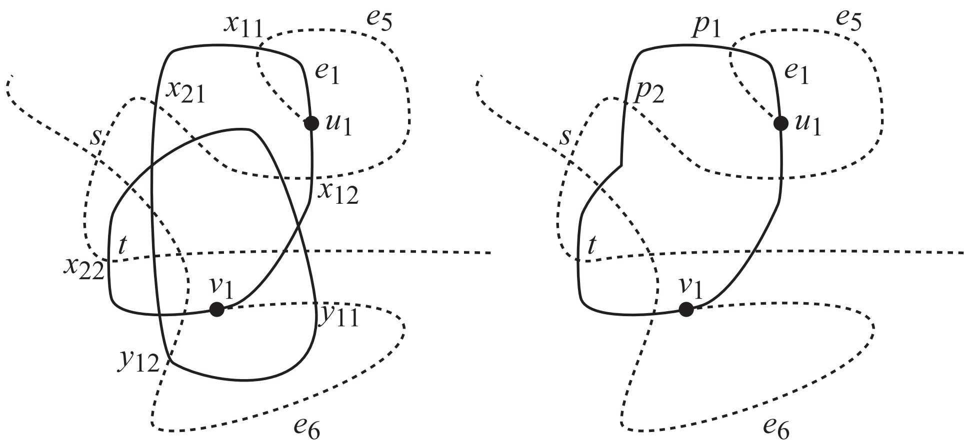


Figure 2.21:

Here $\varphi \circ \bar{e}_5([x_{11}, x_{12}] \cup [x_{21}, x_{22}] \cup \cdots \cup [x_{m1}, x_{m2}]) \cap \hat{e}_6$ is not empty or $\varphi \circ \bar{e}_6([y_{11}, y_{12}] \cup [y_{21}, y_{22}] \cup \cdots \cup [y_{n1}, y_{n2}]) \cap \hat{e}_5$ is not empty because there exist no separating circles. Without loss of generality, we may assume $\varphi \circ \bar{e}_5([x_{11}, x_{12}] \cup [x_{21}, x_{22}] \cup \cdots \cup [x_{m1}, x_{m2}]) \cap \hat{e}_6$ is not empty. Let $\hat{s} \in \varphi \circ \bar{e}_5([x_{11}, x_{12}] \cup [x_{21}, x_{22}] \cup \cdots \cup [x_{m1}, x_{m2}]) \cap \hat{e}_6$ such that $\hat{s}$ is the nearest to $\hat{v}_1$ on $\hat{e}_6$. Let $s = \varphi^{-1}(\hat{s}) \cap e_6$. Let s' be a point on e_6 that is slightly nearer to v_1 than s, and s'' a point on e_6 that is slightly farther from v_1 than s. Suppose that $\hat{s} \in \varphi \circ \bar{e}_5([x_{i1}, x_{i2}])$. Let $\hat{t}$ be $\varphi \circ \bar{e}_5(x_{i2})$. We can assume that $\varphi^{-1}(t) \in e_1$. Let $t_1 = e_1 \cap \varphi^{-1}(t)$, $t_5 = e_5 \cap \varphi^{-1}(t)$. We can suppose that there are no self-crossings in $[u_1, t_1; e_1]$ by Proposition 2.1 (Fig. 2.21). Let $\{p_1, p_2, \ldots, p_n\} = \varphi([u_1, t_1; e_1]) \cap \varphi \circ \bar{e}_5((0, x_1])$ (possibly $\hat{e}_1 \cap \varphi \circ \bar{e}_5((0, x_1]) = \emptyset$), and $p_{5,1} = \varphi^{-1}(p_1) \cap e_5, p_{5,2} = \varphi^{-1}(p_2) \cap e_5, \ldots, p_{5,n} = \varphi^{-1}(p_n) \cap e_5$. Here we can suppose that $\bar{e}_5^{-1}(p_{5,1}) < \bar{e}_5^{-1}(p_{5,2}) < \cdots < \bar{e}_5^{-1}(p_{5,n})$. Let $p'_{5,i}$ be a point on e_5 that is slightly nearer to u_1 than $p_{5,i}$, and $p''_{5,i}$ a point on e_5 that is slightly farther from u_1 than $p_{5,i}$ $(i = 1, 2, \ldots, n)$. Let t'_5 be a point on e_5 that is slightly nearer to u_1 than t_5, and t''_5 a point on e_5 that is slightly farther from u_1 than t_5.

There are four cases about the position of $\hat{v}_1$ and $\varphi([u_1, t_1; e_1]) \cup \varphi([u_1, t_5; e_5])$ in $\mathbf{R}^2$ as illustrated in Fig. 2.22.

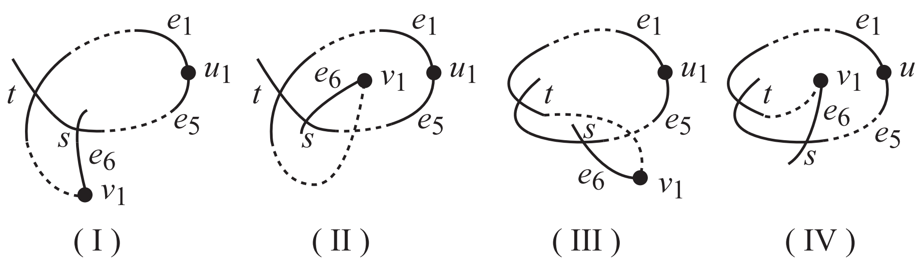


Figure 2.22:

If the projection satisfies case (I) or (IV) (resp. (II) or (III)), let $h : H \to \mathbf{R}$ be a continuous function with the following properties, where $h_i : e_i \to \mathbf{R}$ $(i = 1, 2, \ldots, 6)$ is a restriction map of h and ε is a sufficiently small positive real number.

1. $h(u_1) = 0, h(v_1) = 1, h(u_2) = -2, h(v_2) = -1.$

2. $h_1([u_1, t_1]) = \{0\}$. $h_1|_{[t_1, v_1]}$ is injective.

3. h_i is injective ($i = 2, 3, 4$).

4. All of the following maps are homeomorphisms and $h_5([u_1, p'_{5,1}]) = h_5([p''_{5,1}, p'_{5,2}]) = h_5([p''_{5,2}, p'_{5,3}]) = \cdots = h_5([p''_{5,n-1}, p'_{5,n}]) = h_5([p''_{5,n}, s'_5]) = h_5([s''_5, t'_5]) = \{0\}$.

   $h_5|_{[p'_{5,1}, p_{5,1}]} \to [-\varepsilon, 0]$, $h_5|_{[p_{5,1}, p''_{5,1}]} \to [-\varepsilon, 0]$, $h_5|_{[p'_{5,2}, p_{5,2}]} \to [-\varepsilon, 0]$, $h_5|_{[p_{5,2}, p''_{5,2}]} \to [-\varepsilon, 0]$, $\cdots$, $h_5|_{[p'_{5,n}, p_{5,n}]} \to [-\varepsilon, 0]$, $h_5|_{[p_{5,n}, p''_{5,n}]} \to [-\varepsilon, 0]$, $h_5|_{[s'_5, s_5]} \to [-\varepsilon, 0]$, $h_5|_{[s_5, s''_5]} \to [-\varepsilon, 0]$, $h_5|_{[t'_5, t_5]} \to [0, \varepsilon]$, and $h_5|_{[t_5, u_2]} \to [-2, \varepsilon]$.

5. h_6 is injective and $h_6(s'') = 0$ (resp. $h_6(s') = 0$).

6. If $x \neq y$ and $\varphi(x) = \varphi(y)$ then $h(x) \neq h(y)$ for all $x, y \in H$.

Let $(\varphi \times h) : H \to \mathbf{R}^2 \times \mathbf{R}^1 = \mathbf{R}^3$ be a spatial embedding of H defined by $(\varphi \times h)(x) = (\varphi(x), h(x))$. Let H' be the image of $\varphi \times h$. By a deformation that is similar to the deformation illustrated in Fig. 2.18, we eliminate crossings $p_1, p_2, \ldots, p_n$. By an ambient isotopy preserving the third coordinate H' is deformed into a spatial double-handcuff graph illustrated in Fig. 2.23. Here Fig. 2.23 is a regular diagram on the xz-plane and boxes B_1 and B_3 represent pure 3-braids, a box B_2 represents a pure 2-braid. Then we eliminate crossings in B_1, B_2 and B_3. Thus we see that H' is ambient isotopic to E_6. □

Lemma 2.2.7. *$elm(H) \geq 7$.*

Proof. Let $\widehat{P}_1, \widehat{P}_2, \ldots, \widehat{P}_6$ and $\widehat{P}_7$ be projections of H illustrated in Fig. 2.24. Let $E(\widehat{P}_i)$ be the set of all nontrivial spatial embeddings obtained from $\widehat{P}_i$ ($i = 1, 2, \ldots, 7$).

We shall show that $E(\widehat{P}_i) \cap E(\widehat{P}_j) (i \neq j)$ is empty. The set $E(\widehat{P}_1)$ consists of only one element and it contains a Hopf link. The set $E(\widehat{P}_2)$ consists of only one element and three trefoil knots are obtained from it as subgraphs. The set $E(\widehat{P}_3)$ consists of only one element and it contains exactly one trefoil. Any element of $E(\widehat{P}_4)$ contains neither a trefoil knot nor a Hopf link. The set $E(\widehat{P}_5)$ consists of only one element and two trefoil knots are obtained from it as subgraphs. The set $E(\widehat{P}_6)$ consists of only one element and four trefoil knots are obtained from it as subgraphs. The set $E(\widehat{P}_7)$ consists of two

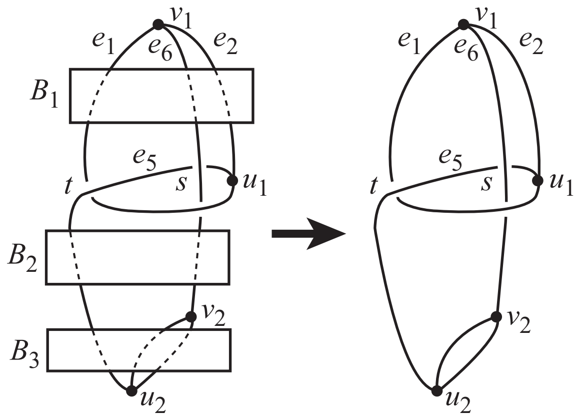


Figure 2.23:

elements. We call one of $E(\widehat{P}_7)$ E_7 as illustrated in Fig. 2.5 and another E_7' as illustrated in Fig. 2.25. Then E_7 contains exactly one trefoil, E_7' contains exactly one figure-eight knot. Thus it remains that we show E_3 is not ambient isotopic to E_7. The trefoil knot obtained from E_3 is constructed by two edges, the trefoil knot obtained from E_7 is constructed by four edges. Therefore E_3 is not ambient isotopic to E_7. $\square$

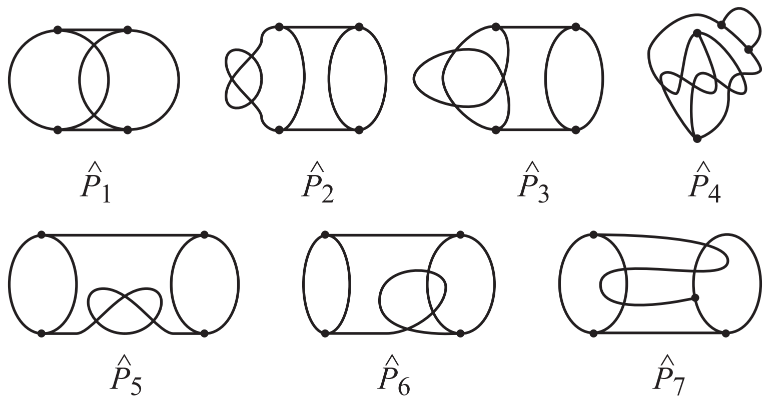


Figure 2.24:

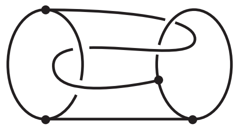


Figure 2.25: E_7'

It is known in [20] that the set that consists of three spatial embeddings illustrated in Fig. 2.26 is also an elementary set of a θ-curve. Then the proof of Theorem 2.1.1 implies that the set $\mathcal{E}'$ that is obtained from $\mathcal{E}$ by replacing E_4 with the spatial embedding E_4' illustrated in Fig. 2.27 is also an elementary set of a double-handcuff graph. However we have not characterized an elementary set of a double-handcuff graph yet.

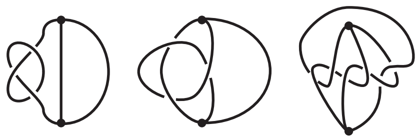


Figure 2.26: An elementary set of a θ-curve

Figure 2.27: The spatial embedding E_4' of a double-handcuff graph

2.3 Corollary

A nontrivial spatial embedding f of a planar graph G is said to be *strongly almost trivial* if there exists a projection $\hat{f}$ of f such that $\hat{f}|_H$ is trivial for any proper subgraph H of G. For example, a handcuff graph and a θ-curve have strongly almost trivial spatial embeddings. See Fig. 2.28(1) and (2). Huh and Oh [15] showed certain sufficient conditions for planar graphs to have no strongly almost trivial spatial embeddings.

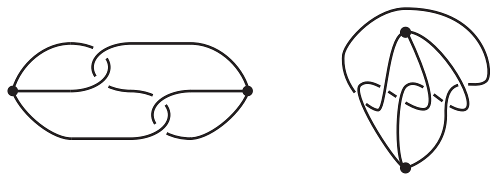


Figure 2.28: Strongly almost trivial spatial embeddings

Theorem 2.3.1. [15, Theorem 1.1] *Let G be a connected planar graph whose vertices are not separating and have valency at least 3. Suppose that G satisfies the following three assumptions;*

1. *G does not contain a cycle consisting exactly a pair of edges.*
2. *For any pair of disjoint edges of G, there exist two disjoint cycles of edges each of which contains an edge of the pair.*
3. *For any path consisting of three consecutive edges of G, there exists a cycle of edges containing the path.*

Then G has no strongly almost trivial spatial embeddings.

In addition, Huh and Oh [15] showed that although the complete graph on four vertices K_4 does not satisfy (ii) in Theorem 2.3.1, K_4 has no strongly almost trivial spatial embeddings [15]. A double-handcuff graph does not satisfy (i) and (ii) in Theorem 2.3.1. However the following corollary holds.

Corollary 2.3.2. *A double-handcuff graph H has no strongly almost trivial spatial embeddings.*

Lemma 2.3.3. *Let H be a double-handcuff graph, $\mathcal{E}$ the elementary set in Theorem 2.1.1. For each element f of $\mathcal{E}$, there exists a proper subgraph H' of H such that $f|_{H'}$ is a nontrivial spatial embedding.*

Proof. See Fig. 2.5. □

Proof of Corollary 2.3.2. Let f be a nontrivial spatial embedding of H, and $\hat{f}$ a projection of f. Since $\hat{f}$ is nontrivial, there exists an element g of $\mathcal{E}$ such that g is obtained from $\hat{f}$ by Theorem 2.1.1. By Lemma 2.3.3 there exists a proper subgraph H' of H such that $f|_{H'}$ is a nontrivial spatial embedding. Hence $\hat{f}|_{H'}$ is nontrivial. □

2.4 Question

A handcuff graph is a 1-connected graph, and the elementary number of it is infinite. A double-handcuff graph is a 2-connected graph, and the elementary number of it is finite. In general, we suggest the following question.

Question 2.4.1. *Is the elementary number of a 2-connected planar graph always finite?*

Chapter 3

On strongly almost trivial embeddings of graphs

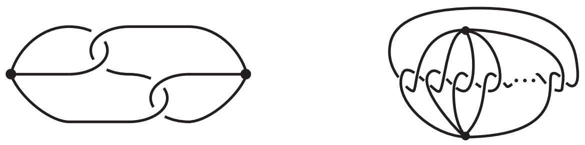
3.1 Introduction

We allow multiple edges and loops and suppose that G does not have a degree two vertex. We refer the reader to [5] and [10] for fundamental terminology of graph theory. An embedding of G into $\mathbf{R}^3$ is called a *spatial embedding* of G, and its image is called a *spatial graph*. In particular, $f(G)$ is called a *knot* (resp. a *link*) if G is homeomorphic to a circle (resp. disjoint circles). Two spatial embeddings f and f' of G are *equivalent* if there exists a (possibly orientation reversing) self-homeomorphism h on $\mathbf{R}^3$ such that $h(f(G)) = f'(G)$. We consider spatial embeddings of a graph up to this equivalence. A spatial embedding f of G is said to be *trivial* (or *unknotted*) if there exists a spatial embedding f' such that f' is equivalent to f and $f'(G) \subset \mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$. Note that only planar graphs have a trivial spatial embedding. A spatial embedding f of G is said to be *minimally knotted* if f is nontrivial and for any proper subgraph H of G, $f|_H$ is trivial.

A continuous map $\varphi : G \to \mathbf{R}^2$ is called a *projection* of G if its multiple points are only finitely many transversal double points away from the vertices. Then its image is also called a projection. A double point of a projection is called a *crossing*. For a spatial embedding f of G, we say that φ is a *projection of f* if there exists a spatial embedding f' of G which is equivalent to f such that $\varphi = \pi \circ f'$ where $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ is the natural projection. Then we also say that f *is obtained from φ*. A projection φ of G is said to be *trivial* if only trivial spatial embeddings of G are obtained from φ.

A projection with over/under information of the crossings is called a *diagram*. Then a crossing with over/under information is also called a *crossing*. A diagram D uniquely represents a spatial embedding f up to the equivalence. Then we say that D *is a diagram of* f.

A spatial embedding f of a planar graph G is said to be *strongly almost trivial*, or simply *SAT*, ([20] and [15]) if f is nontrivial and there exists a projection $\hat{f}$ of f such that $\hat{f}|_H$ is trivial for any proper subgraph H of G. Then we call such a projection a *strongly almost trivial projection*. For example, a handcuff graph which consists of two loops and an edge joining these loops has a SAT embedding. Also a θ_n-curve which consists of two vertices and n multiple edges joining these vertices and $n \geq 3$ has a SAT embedding. See Fig. 3.1(a) and (b). Note that the embedding (a) is appeared in [39] and (b) is known in [35]. We see that these diagrams without over/under information are SAT projections.



(a) (b)

Figure 3.1: SAT embeddings

We introduce known results on SAT embeddings. From the definitions, we see that if f is SAT then f is minimally knotted. It is known in [19] and [45] that any planar graph without vertices of degree less than two has a minimally knotted embedding. However, Huh and Oh showed in [15] that there exists a planar graph which has no SAT embeddings. For example, in Fig. 3.2, the graph P_5 and the complete graph with four vertices have no SAT embeddings [15], the double-handcuff graph also has no SAT embeddings [12]. However, it is open which graphs have a SAT embedding and which graphs have no SAT embeddings.

The following is a question on SAT embeddings in [21, Problem 5.16] due to Kinoshita and Mikasa.

Question 3.1.1. *Does there exist an embedding of a θ-curve which is minimally knotted but not strongly almost trivial?*

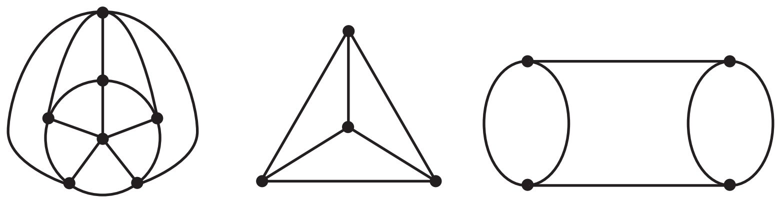


Figure 3.2: Graphs which have no SAT embeddings

This question is investigated in [13] and is conjectured that the answer is yes. In this article, we have new classes of graphs which have a SAT embedding. Therefore, we have the following.

Question 3.1.2. *Does there exist a planar graph G which has a strongly almost trivial embedding such that G has a minimally knotted but not strongly almost trivial embedding?*

First, we present fundamental propositions.

Proposition 3.1.3. *Let φ be a projection of a planar graph G. Then φ is trivial if and only if for any subgraph H of G which has a strongly almost trivial embedding, $\varphi|_H$ is trivial.*

Recently, the following is defined and studied in [9]. A projection φ of G is said to be *knottable* (resp. *linkable*) [9] if there exists a nontrivial spatial embedding of G obtained from φ whose image contains nontrivial knot (resp. nonsplittable link). Moreover, Kobayashi generalizes the definitions. A projection φ of G is said to be *twistable* [22] if there exists a planar subgraph H of G such that $\varphi|_H$ is not trivial. Note that if φ of G is knottable or linkable then φ is twistable, and if φ is twistable then φ is not trivial. A graph G is said to be *intrinsically knottable* (resp. *intrinsically linkable*) [9] if every projection of G is knottable (resp. linkable). A graph G is said to be *intrinsically twistable* [22] if every projection of G is twistable. We have the following.

Proposition 3.1.4. *Let G be an intrinsically twistable graph. For any projection φ of G, there exists a planar subgraph H of G which has a strongly almost trivial embedding such that $\varphi|_H$ is not trivial.*

We present new classes of graphs which have a SAT embedding and that of graphs which have no SAT embeddings.

Theorem 3.1.5. *Let G be an n-bouquet which consists of one vertex with n loops. Then G has a strongly almost trivial embedding.*

As an example, we give a SAT embedding of an n-bouquet as illustrated in Fig. 3.3.

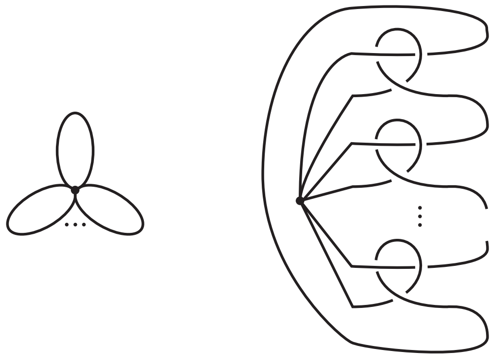


Figure 3.3: n-bouquet and a SAT embedding of it

Proposition 3.1.6. *Let G be a disconnected graph without cut edges such that G is not homeomorphic to two disjoint circles. Then G has no strongly almost trivial embeddings.*

Let F be a forest, namely a graph which does not contain a cycle. We define G_F to be the graph obtained from F by adding a loop to the vertices v with $d_F(v) \leq 1$ where $d_F(v)$ denotes the degree of v in F. For example, see Fig. 3.4.

Theorem 3.1.7. *Let F be a forest with at least one edge. Then G_F has a strongly almost trivial embedding.*

As an example, we give a SAT embedding of G_F in Fig. 3.4 as illustrated in Fig. 3.5.

Theorem 3.1.8. *Let G be a connected graph with exactly one cut edge e such that G is not homeomorphic to a handcuff graph and each connected component of $G - e$ has at least one cycle. Then G has no strongly almost trivial embeddings.*

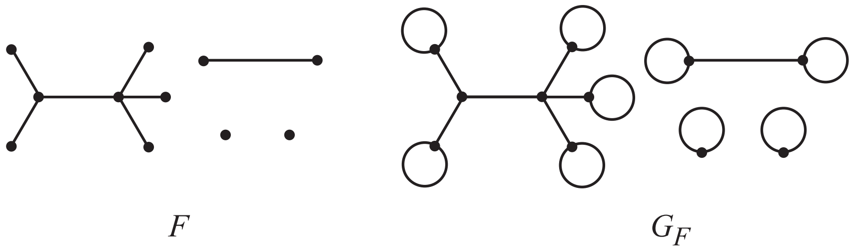


Figure 3.4: Forest F and G_F

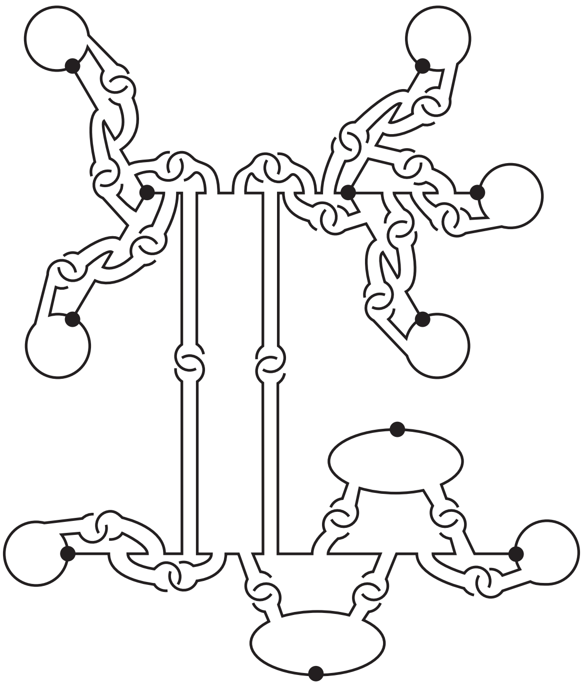


Figure 3.5: SAT embedding of G_F

As an example, in Fig. 3.6, we give graphs which satisfy the conditions in Theorem 3.1.8.

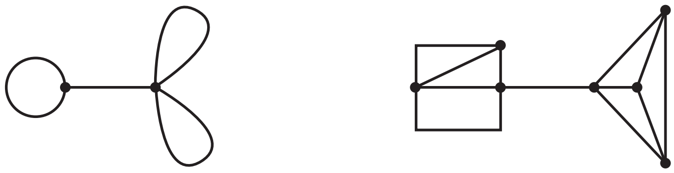


Figure 3.6: Graphs which have no SAT embeddings

From Theorems 3.1.5, 3.1.7 and 3.1.8, we get the following corollary on graph minors with respect to SAT.

Corollary 3.1.9. *Both a property that a graph has a strongly almost trivial embedding and a property that a graph has no strongly almost trivial embeddings are not inherited by minors.* □

As an example, we give the following graphs as illustrated in Fig. 3.7 where $G_1 \prec_m G_2$ denotes that G_1 is a minor of G_2. We see that the graphs (a) and (c) have a SAT embedding from Theorems 3.1.5 and 3.1.7, and (b) has no SAT embeddings from Theorem 3.1.8.

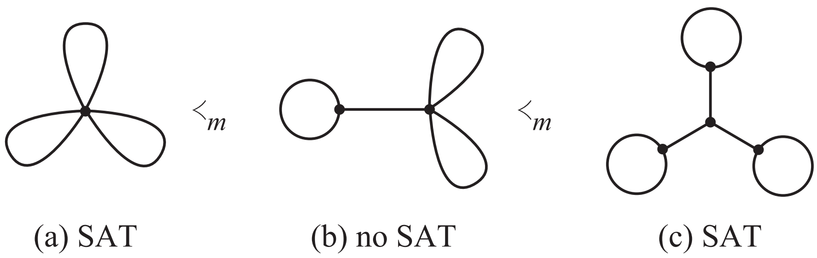


Figure 3.7: On graph minors

We give proofs in sections 3.2, 3.3, 3.4 and 3.5. In section 3.6, we introduce some topics related to SAT.

3.2 Proofs of Propositions 3.1.3, 3.1.4 and 3.1.6

We give proofs of Propositions 3.1.3, 3.1.4 and 3.1.6.

Proof of Proposition 3.1.3. The 'only if' part is obvious. We show that if φ is not trivial then there exists a subgraph H of G which has a SAT embedding such that $\varphi|_H$ is not trivial. If G has a SAT embedding then we put $H = G$. If G has no SAT embeddings then there exists a proper subgraph H_1 of G such that $\varphi|_{H_1}$ is not trivial from the definition of SAT. If H_1 has a SAT embedding then we put $H = H_1$. Similarly, if H_1 has no SAT embeddings then there exists a proper subgraph H_2 of H_1 such that $\varphi|_{H_2}$ is not trivial. We continue in this manner. Since G is finite, H_i has a SAT embedding for some i. Thus we put $H = H_i$. □

Proof of Proposition 3.1.4. Let φ be a projection of G. Since G is intrinsically twistable, there exists a planar graph H' such that $\varphi|_{H'}$ is not trivial. By Proposition 3.1.3, there exists a subgraph H of H' which has a SAT embedding such that $\varphi|_H$ is not trivial. □

We recall the following lemma in [15, Lemma 2.1] to show Proposition 3.1.6.

Lemma 3.2.1. *Let C_1 and C_2 be disjoint cycles of a graph G and $G \neq C_1 \cup C_2$. Let φ be a projection of G. If $\varphi(C_1) \cap \varphi(C_2) \neq \emptyset$, then $\varphi|_{C_1 \cup C_2}$ is not trivial, namely φ is not strongly almost trivial.*

Proof of Proposition 3.1.6. Let φ be a SAT projection. If $\varphi(G)$ is disconnected, then there exists a proper subgraph H such that $\varphi|_H$ is not trivial and $\varphi(H)$ is connected. Let v be a vertex not in H. Then $\varphi|_{G-v}$ is not trivial, this is a contradiction. Suppose that $\varphi(G)$ is connected. Let H_1 and H_2 be connected components such that $\varphi(H_1) \cap \varphi(H_2) \neq \emptyset$. Let $c \in \varphi(H_1) \cap \varphi(H_2)$ be a crossing. Let e_1 (resp. e_2) be the edge in H_1 (resp. H_2) such that $c \in \varphi(e_1)$ (resp. $c \in \varphi(e_2)$). Since e_1 (resp. e_2) is not a cut edge, there exists a cycle C_1 (resp. C_2) such that $e_1 \in E(C_1)$ (resp. $e_2 \in E(C_2)$). Then, $\varphi(C_1) \cap \varphi(C_2) \neq \emptyset$. By Lemma 3.2.1, φ is not SAT. □

3.3 Proof of Theorem 3.1.5

First we recall a color invariant defined in [17] to prove nontriviality of spatial embeddings. In a diagram, edges are divided at each crossing. Here, each part of edge is called a *segment*. A spatial graph diagram is said to be *3-colorable* if the diagram has the following four properties: (1) each segment is drawn by one of three colors, (2) at least two colors are used, (3) either all three colors or only one color are appeared at each crossing, (4) all segments with the same end vertex are assigned by the same color. It is known in [17] that for any connected graph G without odd degree vertices, if a spatial embedding f of G has a 3-colorable diagram then f is nontrivial.

Proof of Theorem 3.1.5. Let f be a spatial embedding which is represented by the diagram in Fig. 3.3. Since the diagram is 3-colorable as in Fig. 3.8, f is nontrivial. Let $\hat{f}$ be the projection which is the diagram without over/under information in Fig. 3.3. It is obvious that for any proper subgraph H of G, $\hat{f}|_H$ is trivial. $\square$

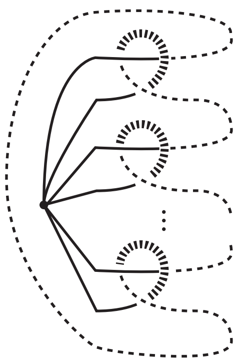


Figure 3.8: 3-colorable diagram

3.4 Proof of Theorem 3.1.7

First we recall the theorem in [40] to prove nontriviality of spatial embeddings. Let $\mathbf{S}^2 \subset \mathbf{S}^3$ be the one point compactification of the pair $\mathbf{R}^2 \subset \mathbf{R}^3$. In this section, we consider projections on $\mathbf{S}^2$ and spatial graphs in $\mathbf{S}^3$ for the convenience. A spatial graph $\mathcal{G}$ is said to be *irreducible* if for any 2-sphere S in $\mathbf{S}^3$ which intersects with $\mathcal{G}$ at most one point, $\mathcal{G}$ is contained in one of the two 3-balls which are bounded by S. A 2-disk D embedded in $\mathbf{S}^3$ is said to be *good* for $\mathcal{G}$ if ∂D is contained in $\mathcal{G}$, $\mathrm{int}D \cap \mathcal{G}$ contains at most finitely many points and for any $x \in \mathrm{int}D \cap \mathcal{G}$ a neighborhood of x looks like Fig. 3.9 where p and q are some positive integers. Here, there is a possibility that x is an interior point of an edge or a vertex. Then a 2-disk D embedded in $\mathbf{S}^3$ is said to be *contractible* for $\mathcal{G}$ if D is good for $\mathcal{G}$, and $\mathrm{int}D \cap \mathcal{G}$ is not empty or $\partial D \cap \mathrm{cl}(G - \partial D)$ is not just one point where cl denotes the closure. For example, see Fig. 3.10.

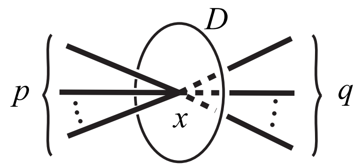


Figure 3.9:

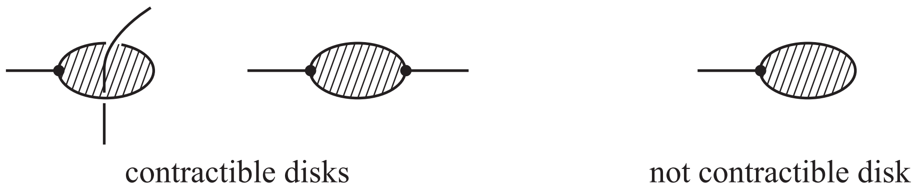


Figure 3.10: Contractible disks and not contractible disk

Theorem 3.4.1. *[40] Let $\mathcal{G}$ be a spatial graph in $\mathbf{S}^3$ and D a contractible disk for $\mathcal{G}$. Let $\mathcal{G}'$ be a spatial graph obtained from $\mathcal{G}$ by contracting D to one point. If $\mathcal{G}'$ is irreducible then $\mathcal{G}$ is irreducible.*

This theorem is useful to show that a spatial embedding f of G is non-trivial for a graph G which has a cut edge or a cut vertex or a disconnected graph G. In fact, if a spatial embedding f of such a graph is trivial then a spatial graph $f(G)$ is not irreducible. Therefore, if $f(G)$ is irreducible then f is not trivial. We give a proof of Theorem 3.1.7.

Proof of Theorem 3.1.7. Let $G_1, G_2, \ldots, G_n$ be the connected components of G_F such that each of $G_1, \ldots, G_m$ has an edge which is not a loop and each of $G_{m+1}, \ldots, G_n$ has only one edge as a loop. We construct a SAT embedding f of G_F in the following way. We consider on a diagram. Here, we denote a Hopf band by a broken line as in Fig. 3.11.

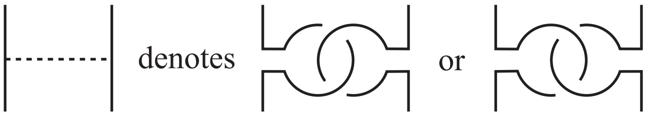
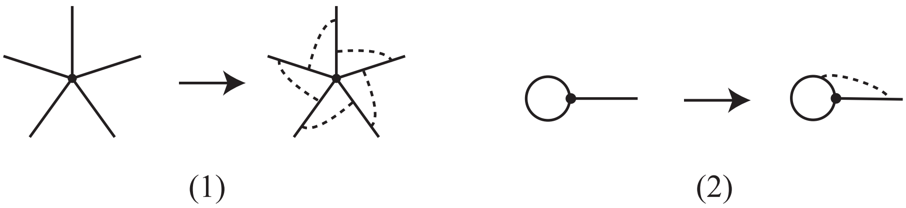


Figure 3.11: Hopf band

1. For each set of the edges which are incident with a vertex v with $d_F(v) \geq 3$, we attach a Hopf band to the edges as in Fig. 3.12.
2. For each edge which is incident with a vertex v with $d_F(v) = 1$, we attach a Hopf band to the edges as in Fig. 3.12.
3. For each edge which is not a loop, we replace Hopf bands as in Fig. 3.13(3).
4. We choose an edge e_1 of G_1 which is not a loop. We add Hopf bands as in Fig. 3.13(4) so that e_1 has Hopf bands on oneself $\lfloor n/2 \rfloor$ times.
5. We choose an edge e_i of G_i which is not a loop for each $i = 2, 3, \ldots, m$. We add Hopf bands as in Fig. 3.13(4) so that e_i has Hopf bands on oneself. For each pair of edges e_1 and e_i and that of edges e_1 and a loop of G_k $(k = m+1, \ldots, n)$, we replace Hopf bands as in Fig. 3.14(4').



(1) (2)

Figure 3.12: Constructions (1) and (2)

Figure 3.13: Construction (3) and (4)

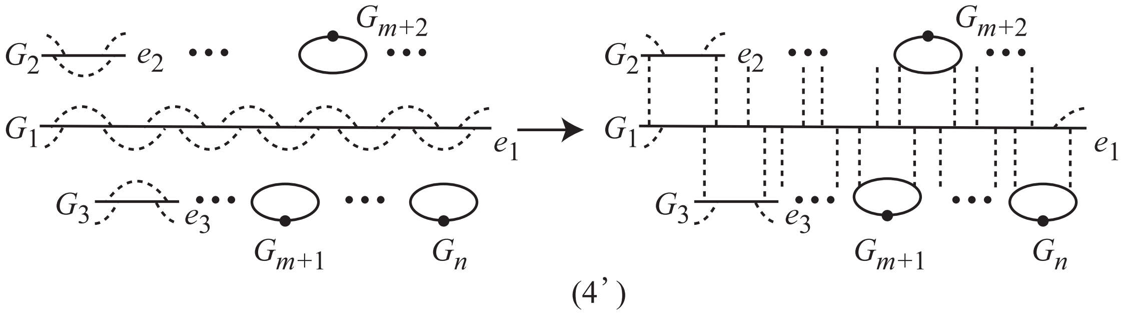


Figure 3.14: Construction (4)

We will see that each connected component is a SAT embedding by constructions (1), (2) and (3). The diagram in Fig. 3.5 is obtained in the above way.

First we show that f is nontrivial by applying Theorem 3.4.1. Assume that G_F is connected. We recall that there exists a vertex v in a tree such that every vertex adjacent to v, except possibly for one, has degree one. Let v be such a vertex in F. Let $v_1, v_2, \ldots, v_l$ be the vertices adjacent to v with $d_F(v_i) = 1$ $(i = 1, 2, \ldots, l)$. Let D_i be the disk in $\mathbf{S}^3$ whose boundary is the loop incident with v_i $(i = 1, 2, \ldots, l)$. Note that the disks are contractible. We contract the disks as in Fig. 3.15. Then we contract the resultant disks as in Fig. 3.15. If F is a star graph (resp. $E(F)$ is the set of one edge), then we get a wheel graph (resp. a θ-curve) as in Fig. 3.16, and hence f is nontrivial by Theorem 3.4.1. If F is the others, then we repeat this process and get a wheel graph or a θ-curve, and hence f is nontrivial by Theorem 3.4.1.

Assume that G_F is disconnected. First, we contract each G_i except e_i as above $(i = 1, 2, \ldots, m)$. Then we get a diagram as in Fig. 3.17, and further contract. The graph obtained in Fig. 3.17 is 2-connected. Hence f is irreducible by Theorem 3.4.1, and therefore f is nontrivial.

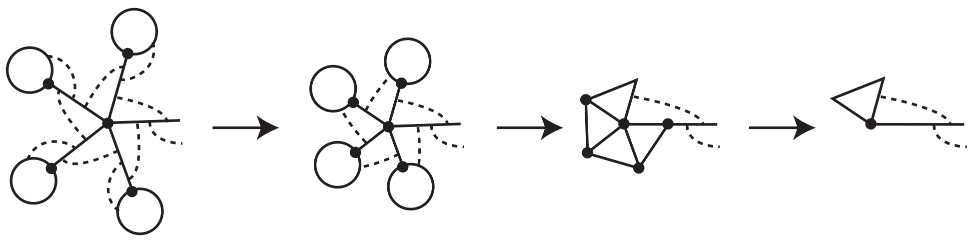


Figure 3.15: Contracting contractible disks

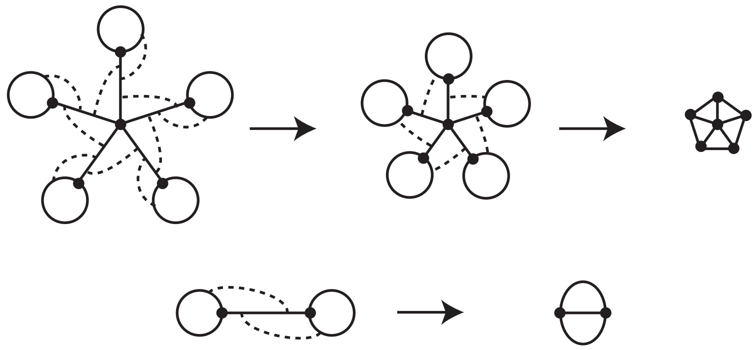


Figure 3.16: Contracting contractible disks

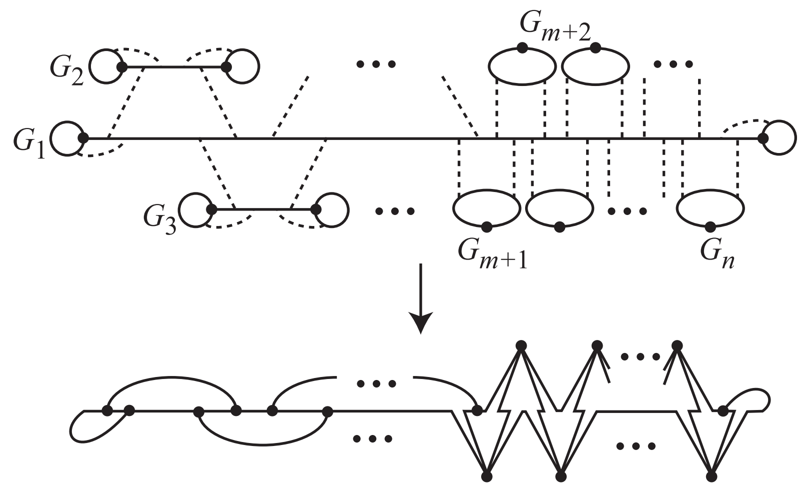


Figure 3.17: Contracting contractible disks

Let $\hat{f}$ be the projection which is a diagram obtained above without over/under information. We show that for any proper subgraph H of G_F, $\hat{f}|_H$ is trivial. For the purpose we construct a digraph $D_{\hat{f}}$ from $\hat{f}$ in the following way. Let $V(D_{\hat{f}})$ be the set of the vertices corresponding to the edges of G_F and $A(D_{\hat{f}})$ the set of the arcs such that an arc joins a vertex corresponding to e to a vertex corresponding e' if and only if $f(e')$ has a cutting circle in $\hat{f}|_{G-e}$ or e' is a loop adjacent to e where a cutting circle intersects with $\hat{f}(e')$ transversally at exactly one point on $\mathbf{S}^2$. Note here that for any diagram D obtained from $\hat{f}$, we can vanish all crossings on e in D as in Fig. 3.18 if $\hat{f}(e)$ has a cutting circle.

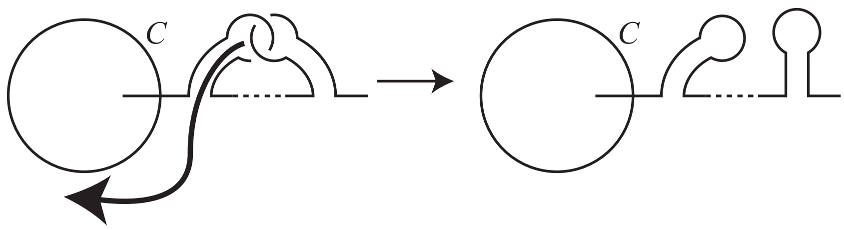


Figure 3.18: Vanishing crossings

Hence, it is sufficient to show that $D_{\hat{f}}$ is strongly connected, namely for any pair of vertices w and w' there exists a directed path from w to w'. For vertices in $D_{\hat{f}}$ corresponding to the edges which are incident with a vertex v with $d_F(v) \geq 3$, the induced subdigraph of $D_{\hat{f}}$ by these vertices is as in Fig. 3.19(a) from construction (1) and (3). For a pair of vertices in $D_{\hat{f}}$ corresponding to a loop and the edge incident with the loop, the induced subdigraph of $D_{\hat{f}}$ by these vertices is as in Fig. 3.19(b) from construction (2) and (3). For a pair of vertices in $D_{\hat{f}}$ corresponding to edges e_1 and e_j and edges e_1 and a loop of G_k, the induced subdigraph of $D_{\hat{f}}$ by these vertices is as in Fig. 3.19(c) from construction (4). For example, the digraph in Fig. 3.20 is constructed from a projection as in Fig. 3.5. Therefore we see that $D_{\hat{f}}$ is strongly connected. □

3.5 Proof of Theorem 3.1.8

In this section, we give a proof of Theorem 3.1.8.

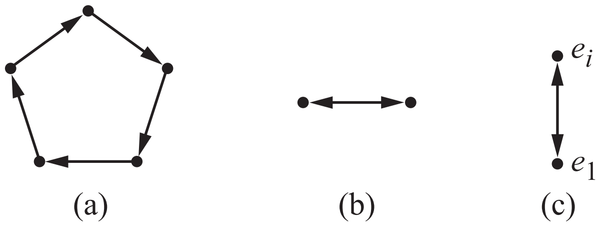


Figure 3.19: Subdigraphs

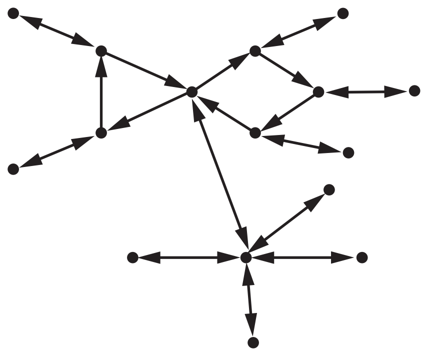


Figure 3.20: Digraph of a projection in Fig. 3.5

Proof of Theorem 3.1.8. We show that for any projection φ of G which is not trivial, there exists a proper subgraph H of G such that $\varphi|_H$ is not trivial. Let H_1 and H_2 be the connected subgraphs of G such that $H_1 \cup H_2 = G - e$. If $\varphi|_{H_1}$ (resp. $\varphi|_{H_2}$) is not trivial then we put $H = H_1$ (resp. $H = H_2$). We may assume that both $\varphi|_{H_1}$ and $\varphi|_{H_2}$ are trivial.

Assume that $\varphi(H_1) \cap \varphi(H_2) \neq \emptyset$. Let $c \in \varphi(H_1) \cap \varphi(H_2)$ be a crossing. Let e_1 (resp. e_2) be the edge in H_1 (resp. H_2) such that $c \in \varphi(e_1)$ (resp. $c \in \varphi(e_2)$). Since e_1 (resp. e_2) is not a cut edge, there exists a cycle C_1 (resp. C_2) such that $e_1 \in E(C_1)$ (resp. $e_2 \in E(C_2)$). Let $H = C_1 \cup C_2$. Then $\varphi|_H$ is not trivial by Lemma 3.2.1.

Assume that $\varphi(H_1) \cap \varphi(H_2) = \emptyset$. We show that there exists a subgraph H of G such that H is a handcuff graph and $\varphi|_H$ is not trivial. Now, there exist no cutting circles on $\varphi(e)$. Because, if there exists a cutting circle on $\varphi(e)$, then φ is trivial since both $\varphi|_{H_1}$ and $\varphi|_{H_2}$ are trivial. Let $v_1 \in V(H_1)$ ($v_2 \in V(H_2)$) be the end vertex of e. Let $\bar{e} : [0,1] \to e$ be a parameter with $\bar{e}(v_1) = 0$ and $\bar{e}(v_2) = 1$. Let $t_1 = \max\{t \in [0,1] | \varphi \circ \bar{e}(t) \in \varphi(H_1)\}$. If $t_1 = 0$ then this contradicts to the fact that there exist no cutting circles on $\varphi(e)$. We have $t_1 > 0$. Let $p_1 = \varphi \circ \bar{e}(t_1)$. Let C_1' be a cycle of H_1 such that $p_1 \in \varphi(C_1')$. Similarly let $t_2 = \min\{t \in [0,1] | \varphi \circ \bar{e}(t) \in \varphi(H_2)\}$. If $t_2 = 1$ then this contradicts to the fact that there exist no cutting circles on $\varphi(e)$. We have $t_2 < 1$. Let $p_2 = \varphi \circ \bar{e}(t_2)$. Let C_2' be a cycle of H_2 such that $p_m \in \varphi(C_2')$. Let P be a shortest path of G such that P contains both a vertex of C_1' and that of C_2'. Let $H = C_1' \cup C_2' \cup P$. Note that H is a handcuff graph. Since there exist no cutting circles on $\varphi(e)$, $\varphi(H)$ has no cutting circles. It is known in [39, Lemma 2] that one of the spatial embeddings in Fig. 3.21 is obtained from $\varphi|_H$ if $\varphi(H)$ has no cutting circles. Note that each spatial embedding in Fig. 3.21 is nontrivial since we have a θ-curve by contracting contractible disks. Therefore $\varphi|_H$ is not trivial. $\square$

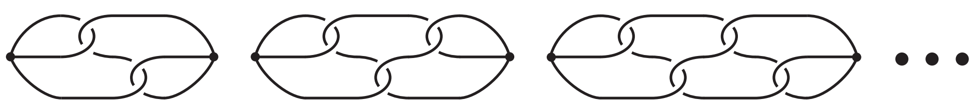


Figure 3.21: Nontrivial spatial handcuff graphs

3.6 Related Topics

We introduce some topics related to SAT embeddings. The following definition is known for knots. A knot diagram D is said to be *everywhere n-trivial* [32] if for any subset C with n crossings of the set of the crossings of D, the diagram obtained from D by switching over/under information at the crossings of C represents the trivial knot. Then it is known that the trivial knot, the trefoil knot and the figure-eight-knot have everywhere 1-trivial diagrams as in Fig. 3.22. Moreover, Stoimenow and Askitas conjecture that the only knots which have an every-where 1-trivial diagram are the trivial knot, the trefoil knot and the figure-eight-knot [32, Conjecture 5.2].

Figure 3.22: Everywhere 1-trivial diagrams of the trefoil knot and the figure-eight-knot

Now, we generalize this definition for diagrams of a spatial embedding. Namely, a diagram D is said to be *everywhere n-trivial* if for any subset C with n crossings of the set of the crossings of D, the diagram obtained from D by switching over/under information at the crossings of C represents the trivial spatial graph. Then we see that the diagram in Fig. 3.3 is everywhere 1-trivial and so are diagrams constructed in the proof of Theorem 3.1.7. Also, the diagrams in Fig. 3.1 and Fig. 3.21 are everywhere 1-trivial. Finally we remark that these spatial graphs have n-trivial diagrams for certain n in the sense of [28]. Therefore they have trivial finite type invariants of order less than n in the sense of [31].

Bibliography

[1] N. Askitas, E. Kalfagianni, *On knot adjacency*, Topology Appl. **126** (2002), 63–81.

[2] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[3] J. S. Birman, X.-S. Lin, *Knot polynomials and Vassiliev's invariants.* Invent. Math. **111** (1993), 225–270.

[4] A. D. Bates and A. Maxwell, "DNA Topology" (2nd ed.), Oxford university press, 2005.

[5] J. Clark and D. A. Holton, "A first look at graph theory", World Scientific Publishing Co., 1991.

[6] T. D. Cochran and R. E. Gompf, *Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and property P*, Topology **27** (1988) 495–512.

[7] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. (2) **39** (1989), 535–552.

[8] F. B. Dean, A. Stasiak, Th. Koller, N. R. Cozzarelli, *Duplex DNA knots produced by Escherichia coli topoisomerase I, structure and requirements for formation*, J. Biol. Chem. **260** (1985), 4975–4983.

[9] A. DeCelles, J. Foisy, C. Versace and A. Wilson, *On graphs for which every planar immersion lifts to a knotted spatial embedding*, Involve **1** (2008), 145–158.

[10] R. Diestel, "Graph theory", Springer-Verlag, 2005.

[11] C. H. Dowker, B. Thistlethwaite, *Classification of knot projections*, Topology Appl. **16** (1983), 19–31.

[12] R. Hanaki, *Regular projections of knotted double-handcuff graphs*, to appear in J. Knot Theory Ramifications.

[13] Y. Huh, G. T. Jin and S. Oh, *Strongly almost trivial θ-curves*, J. Knot Theory Ramifications **11** (2002), 153–164.

[14] Y. Huh, G. T. Jin and S. Oh, *An elementary set for θ_n-curve projections*, J. Knot Theory Ramifications **11** (2002), 1243–1250.

[15] Y. Huh and S. Oh, *Planar graphs producing no strongly almost trivial embedding*, J. Graph Theory **43** (2003), 319–326.

[16] Y. Huh, K. Taniyama, *Identifiable projections of spatial graphs*, J. Knot Theory Ramifications **13** (2004), 991–998.

[17] Y. Ishii and A. Yasuhara, *Color invariant for spatial graphs*, J. Knot Theory Ramifications **6** (1997), 319–325.

[18] L. H. Kauffman, *Invariants of graphs in three-space*, Trans. Amer. Math. Soc. **311** (1989), 697–710.

[19] A. Kawauchi, *Almost identical imitations of (3, 1)-dimensional manifold pairs*, Osaka J. Math. **26** (1989), 743–758.

[20] S. Kinoshita and J. Mikasa, *On projections of spatial theta-curves*, Kwansei Gakuin University (1993) In Japanese.

[21] R. Kirby, *Problems in low-dimensional topology*, Geometric topology, Amer. Math. Soc., Providence, RI, (1997), 35–473.

[22] K. Kobayashi, private communication.

[23] W. Mason, *Homeomorphic continuous curves in 2-space are isotopic in 3-space*, Trans. Amer. Math. Soc. **142** (1969), 269–290.

[24] K. Murasugi, "Knot theory & its applications", Modern Birkhäuser Classics, 2008.

[25] R. Nikkuni, *A remark on the identifiable projections of planar graphs*, Kobe J. Math. **22** (2005), 65–70.

[26] R. Nikkuni, *Completely distinguishable projections of spatial graphs*, J. Knot Theory Ramifications **15** (2006), 11–19.

[27] R. Nikkuni, *Regular projections of spatial graphs*, *Knot Theory for Scientific Objects, Osaka City University Advanced Mathematical Institute Studies* **1** no. 1, pp. 111–128, Osaka Municipal Universities Press, 2007.

[28] Y. Ohyama, *Vassiliev invariants and similarity of knots*, Proc. Amer. Math. Soc. **123** (1995), 287–291.

[29] J.H. Przytycki, *Positive knots have negative signature*, Bull. Polish Acad. Sci. Math. **37** (1989), 559–562 (1990).

[30] J.H. Przytycki and K. Taniyama, *Almost positive links have negative signature*, preprint.

[31] T. Stanford, *Finite-type invariants of knots, links, and graphs*, Topology **35** (1996), 1027–1050.

[32] A. Stoimenow, *On unknotting numbers and knot trivadjacency*, Math. Scand. **94** (2004), 227–248.

[33] A. Stoimenow, *Gauss diagram sums on almost positive knots*, Compos. Math. **140** (2004), 228–254.

[34] S. Suzuki, *Local knots of 2-spheres in 4-manifolds*, Proc. Japan Acad. **45** (1969), 34–38.

[35] S. Suzuki, *Almost unknotted θ_n-curves in 3-sphere*, Kobe J. Math. **1** (1984), 19–22.

[36] K. Taniyama, *A partial order of knots*, Tokyo J. Math. **12** (1989), 205–229.

[37] K. Taniyama, *A partial order of links*, Tokyo J. Math. **12** (1989), 475–484.

[38] K. Taniyama, *Knotted projections of planar graphs*, Proc. Amer. Math. Soc. **123** (1995), 3357–3579.

[39] K. Taniyama and C. Yoshioka, *Regular projections of knotted handcuff graphs*, J. Knot Theory Ramifications **7** (1998), 509–517.

[40] K. Taniyama, *Irreducibility of spatial graphs*, J. Knot Theory Ramifications **11** (2002), 121–124.

[41] K. Taniyama and A. Yasuhara, *Clasp-pass moves on knots, links and spatial graphs*, Topology Appl. **122** (2002), 501–529.

[42] K. Taniyama, *Unknotting numbers of diagrams of a given nontrivial knot are unbounded*, to appear in J. Knot Theory Ramifications.

[43] P. Traczyk, *Nontrivial negative links have positive signature*, Manuscripta Math. **61** (1988), 279–284.

[44] V.A. Vassiliev, *Cohomology of knot spaces*, In Theory of singularities and its applications (ed. V. Arnold) (Advances in Soviet Mathematics, 1, AMS Providence, RI, 1990).

[45] Y. Q. Wu, *Minimally knotted embeddings of planar graphs*, Math. Z. **214** (1993), 653–658.

[46] S. Yamada, *An invariant of spatial graphs*, J. Graph Theory **13** (1989), 537–551.

[47] M. Yamamoto, *Knots in spatial embeddings of the complete graph on four vertices*, Topology Appl. **36** (1990), 291–298.

A list of papers by Ryo Hanaki

1. Regular projections of knotted double-handcuff graphs, to appear in J. Knot Theory Ramifications.
2. On an inequality between unknotting number and crossing number of links, to appear in J. Knot Theory Ramifications. (with Junsuke Kanadome)
3. Pseudo diagrams of knots, links and spatial graphs, to appear in Osaka J. Math.
4. On strongly almost trivial embeddings of graphs, submitted.
5. グラフ理論の教材化の試み〜中学・高校の離散数学教材〜，日本数学教育学会，第 38 回数学教育論文発表会論文集 pp.649-654，2005 年 10 月 29 日.
6. 一筆がき問題に関する教材研究ー中学・高等学校向け離散グラフ教材ー（生野 隆氏との共同研究），日本数学教育学会，第 40 回数学教育論文発表会論文集，pp.277-282，2007 年 11 月 3 日.
7. 最短経路問題に関する教材研究ー中学・高等学校向け離散グラフ教材ー，日本数学教育学会，第 40 回数学教育論文発表会論文集，pp.853-858，2007 年 11 月 3 日.
8. 一筆がき問題に関する教材について，日本数学教育学会，第 41 回数学教育論文発表会論文集，pp.225-230，2008 年 11 月 1 日.
9. 数学的帰納法の証明をよむ活動について，日本数学教育学会，第 42 回数学教育論文発表会論文集，掲載予定.