Treatise on Incomplete Asset Markets

Indeterminacy and Inefficiency of Equilibria with Incomplete Asset Markets

不完備資産市場の研究

不完備資産市場均衡の非決定性と非効率性

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November, 2005

Contents

Chapter 1

Introduction

In this chapter,[1] I first present a basic model which will be used throughout this dissertation. This model is familiar in this field and is known to cause various theoretical problems. I explicate what are the problems and make some survey on how these problems have been delt with. In this process, I articulate the theme of this dissertation, giving an explanation to each subject treated in each chapter. Finally I make a brief survey on further developments of the theory of incomplete asset markets.

[1]This chapter is partly based on R.Nagata, *Theory of Regular Economies*, chap.11, World Scientific, 2004.

1.1 Background of Incomplete Asset Markets

There are many kinds of uncertainty in our real economic world among which I consider a specific one; that is, the uncertainty concerning future events. Since no one knows what will happen in the future, this uncertainty is quite universal.

It may be safely said that a typical agent facing this uncertainty will try to prepare somehow or other for the future. A rational agent would never ignore the future in the present. This propensity to prepare for the future yields new goods called assets. Note that the term 'asset' is a little different than the familiar one in our everyday life. Indeed, an insurance that promises to deliver a certain amount of money contingent on some event in the future is included in the assets and is referred to as a nominal asset. In this connection, there exists another kind of asset called a real asset that promises to deliver a bundle of goods contingent on some future event.

Let's proceed to the model building, taking account of these new factors. The model I work with throughout this dessertation is very simple but known as a basic model in this field, main features of which are summarized as follows. First, the model is based on the pure exchange economy. Second, it has only two periods, i.e. the present ($t = 0$) and the future ($t = 1$). Third, all the relevant variables are confined to some finite dimensional space.

Now let's spell out the details of the model. An economic environment which may happen in the future is called a state of nature. The conceivable set of states of nature is assumed to be finite, numbered by $s = 1, \dots, S$. I call date $t = 0$, state $s = 0$ so that there are $S + 1$ states in all including the present and the future. There are L goods and I consumers that are common in each state. Each consumer i ($i = 1, \dots, I$) is characterized by its consumption set C^i, initial endowments ω^i and utility function u^i for which the following conditions are assumed. Since a rational consumer would take account of all future possibilities, C^i can be properly assumed to be a subset of $\boldsymbol{R}^{L(S+1)}$. Accordingly, u^i turns out to be a map of C^i to $\boldsymbol{R}$ that is possibly interpreted as a von Neumann-Morgenstern expected utility function given by

$$\sum_{s=1}^{S} \rho_s U^i(\boldsymbol{x}_0^i, \boldsymbol{x}_s^i)$$

in which $\boldsymbol{x}_s^i$ is obviously a consumption vector at state s ($s = 1, \dots, S$) and $\rho_s > 0$ denotes the subjective probability of state s and $\sum_{s=1}^{S} \rho_s = 1$. However, the argument below does not depend on this particular type of function. Finally, ω^i is obviously an element of $\boldsymbol{R}^{L(S+1)}$.

For simplicity of notation, I set $\omega^i = (\boldsymbol{\omega}_0^i, \boldsymbol{\omega}_1^i, \dots, \boldsymbol{\omega}_S^i)$, $\omega_{\mathbf{1}}^i = (\boldsymbol{\omega}_1^i, \dots, \boldsymbol{\omega}_S^i)$ and $x^i = (\boldsymbol{x}_0^i, \boldsymbol{x}_1^i, \dots, \boldsymbol{x}_S^i)$, $x_{\mathbf{1}}^i = (\boldsymbol{x}_1^i, \dots, \boldsymbol{x}_S^i)$.

Then I turn to the assets that are, as I have mentioned, separated into two groups: real assets and nominal assets. A real asset is a contract that promises to deliver a bundle of goods at each state in the future. Let $\boldsymbol{a}_j(s)$ be the bundle of goods ($\in \boldsymbol{R}^L$) that a real asset j delivers at state s ($s = 1, \dots, S$). Then the whole returns of j are represented by a LS-vector $(\boldsymbol{a}_j(1), \dots, \boldsymbol{a}_j(S))$.

On the other hand, a nominal asset is a contract that promises to deliver an exogenously given stream of units of account across the states at date 1. Thus, if I denote a given amount of units of account a nominal asset j delivers at state s by $a_j(s)$, the whole returns of the asset are expressed by a S-vector $(a_j(1), \dots, a_j(S))$.

Let's first consider a real asset model, emphasizing its equilibrium. Suppose that there exist J real assets. If one sees the returns vector $(\boldsymbol{a}_j(1), \dots, \boldsymbol{a}_j(S))$ of each asset j as a column vector, then the real asset structure is obtained as follows.

Definition 1.1 *The real asset structure of J real assets is a $LS \times J$ matrix given by*

$$A = \begin{pmatrix} \boldsymbol{a}_1(1) & \boldsymbol{a}_2(1) & \dots & \boldsymbol{a}_J(1) \\ \boldsymbol{a}_1(2) & \boldsymbol{a}_2(2) & \dots & \boldsymbol{a}_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_1(S) & \boldsymbol{a}_2(S) & \dots & \boldsymbol{a}_J(S) \end{pmatrix}$$

In order to complete the model, prices should be considered. Note that there are spot market prices at each state at date 1 as well as the present market prices. In addition, the prices of the real assets should be taken into account. Needless to say, the asset markets are only open at date 0 (the present) since there exist only two periods. Let $\boldsymbol{p}_0$ be the price vector of goods at date 0 and $\boldsymbol{p}_s$ be the (spot market) price vector of goods at state s ($s = 1, \dots, S$) at date 1. The price vector of J assets is denoted by $\boldsymbol{q}$ $(= (q_1, \dots, q_J))$. I assume that all prices are strictly positive. Similarly to consumption vectors, I set $p = (\boldsymbol{p}_0, \boldsymbol{p}_1, \dots, \boldsymbol{p}_S)$ and $p_{\mathbf{1}} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_S)$ in the following.

Given a real asset structure A, one obtains the date 1 matrix of revenues brought in by A which is called a dividend matrix.

Definition 1.2 *A dividend matrix of a real asset structure A is the matrix given by*

$$D(p_{\mathbf{1}}, A) = \begin{pmatrix} \boldsymbol{p}_1 \cdot \boldsymbol{a}_1(1) & \dots & \boldsymbol{p}_1 \cdot \boldsymbol{a}_J(1) \\ \vdots & \ddots & \vdots \\ \boldsymbol{p}_S \cdot \boldsymbol{a}_1(S) & \dots & \boldsymbol{p}_S \cdot \boldsymbol{a}_J(S) \end{pmatrix}$$

Note that this matrix is obviously obtained by premultiplying A by the following $S \times LS$ matrix P consisting of $\boldsymbol{p}_s$ ($s = 1, \dots, S$).

$$P = \begin{pmatrix} \boldsymbol{p}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{p}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \boldsymbol{p}_S \end{pmatrix}$$

where each $\boldsymbol{p}_s$ $(s = 1, \dots, S)$ is interpreted as a row vector. Thus, D can be seen as a smooth map with $p_{\mathbf{1}}$ and A as its independent variables.

It follows from the above argument that the economy with assets is specified by the three kinds of parameters; that is, each consumer's utility function and initial endowments plus an asset structure. Thus, a tuple $(u, \omega; A)$ is called an economy with real assets where $u = (u^1, \dots, u^I)$ and $\omega = (\omega^1, \dots, \omega^I)$.

Let's consider the behavior of each consumer given an economy. Although a consumer faces the uncertainty concerning future events, he/she is able to adjust his/her income among the present and the future through the assets. Thus, he/she will seek to obtain the optimal intertemporal consumption allocation by buying and selling those assets. The demand vector of consumer i for J assets is called a portfolio of i and denoted by $\boldsymbol{z}^i$ $(= (z_1^i, \dots, z_J^i))$. Note that the positive (negative) element of the portfolio implies the demand (supply) of the corresponding asset. For simplicity, I set $z = (\boldsymbol{z}^1, \dots, \boldsymbol{z}^I)$ in the following.

Considering this course of the behavior of each consumer, the definition of equilibria of an economy with real assets is straightforward. Before describing the definition, however, a specific operation called the box product is needed to simplify a notation.

Let a and b be two n sets of k-vector; that is, $a = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n),\ b = (\boldsymbol{b}_1, \dots, \boldsymbol{b}_n)$ where $\boldsymbol{a}_i, \boldsymbol{b}_i \in \boldsymbol{R}^n,\ i = 1, \dots, n$. Then the box product $a \,\square\, b$ of a and b is defined as

$$a \,\square\, b = (\boldsymbol{a}_1\boldsymbol{b}_1, \dots, \boldsymbol{a}_n\boldsymbol{b}_n).$$

I am now in a position to state the definition of an equilibrium for an economy with real assets.

Definition 1.3 *For an economy with real assets $(u, \omega; A)$, an equilibrium is a pair of prices and actions $(p, \boldsymbol{q}; x, z)$ satisfying that*

(1) $(x^i, \boldsymbol{z}^i)$ is a solution for the following optimization problem $(i = 1, \dots, I)$.

$$\begin{aligned} \max_{x^i} \quad & u^i(x^i) \\ s.t. \quad & \boldsymbol{p}_0(\boldsymbol{x}_0^i - \boldsymbol{\omega}_0^i) + \boldsymbol{q} \cdot \boldsymbol{z}^i \leq 0 \\ & p_{\mathbf{1}} \,\square\, (x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i) \leq D(p_{\mathbf{1}}, A)\boldsymbol{z}^i. \end{aligned}$$

(2) $\sum_{i=1}^I (x^i - \omega^i) = 0$.

(3) $\sum_{i=1}^I \boldsymbol{z}^i = 0$.

Note in the above definition that (1) implies the subjective equilibrium for each consumer and that (2), (3) are the market clearance conditions respectively for goods and assets.

Now that I have obtained a real asset model, I turn to a nominal asset model. It is, however, easy to characterize the latter model because the only difference between a real asset and a nominal asset is the asset structure. One may assume that there exist J nominal assets in the economy. Since nominal assets promise returns denominated in the unit of account, its asset structure is described as follows.

Definition 1.4 *The nominal asset structure of J nominal assets is a $S \times J$ matrix given by*

$$A = \begin{pmatrix} a_1(1) & a_2(1) & \dots & a_J(1) \\ a_1(2) & a_2(2) & \dots & a_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(S) & a_2(S) & \dots & a_J(S) \end{pmatrix}$$

It is worth noting that the nominal asset structure coincides with its dividend matrix since a given returns of each nominal asset is independent of spot prices in the future. Consequently, an equilibrium of an economy with nominal assets is defined as follows.

Definition 1.5 *For an economy with nominal assets $(u, \omega; A)$, an equilibrium is a pair of prices and actions $(p, \boldsymbol{q}; x, z)$ satisfying that*

(1) $(x^i, \boldsymbol{z}^i)$ is a solution for the following optimization problem $(i = 1, \dots, I)$.

$$\begin{array}{ll} \max\limits_{x^i} & u^i(x^i) \\ s.t. & \boldsymbol{p}_0(\boldsymbol{x}_0^i - \boldsymbol{\omega}_0^i) + \boldsymbol{q} \cdot \boldsymbol{z}^i \leq 0 \\ & p_{\mathbf{1}} \,\square\, (x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i) \leq A\boldsymbol{z}^i. \end{array}$$

(2) $\sum_{i=1}^{I}(x^i - \omega^i) = 0$.

(3) $\sum_{i=1}^{I} \boldsymbol{z}^i = 0$.

A comment on the treatment of the nominal assets seems to be in order. As is easily seen from the foregoing arguments, the model is based on the so called Arrow-Debreu competitive equilibrium model, so that money is left out of account in the model (or even if money is introduced, it only performs a very restricted function, namely, to act as a unit of account). Hence, what each nominal asset delivers at each state in the future should be interpreted as a specified amount written on the credit side of an agent's account which is only available at a corresponding state. Since the nominal asset's return is specified independently of the spot prices, its purchasing power is inversely proportional to the price level.

It is worth noting that the above definitions of an equilibrium do not necessarily make sense. In order for these definitions to be acceptable, the possibility of the so called arbitrage should be ruled out . First, consider this issue in a real asset model. In the following, I postulate that the utility function of each consumer satisfies the monotonocity.

Since the income for consumer i obtained by trading the assets consists of $-\boldsymbol{q} \cdot \boldsymbol{z}^i$ for the present and $D(p_{\mathbf{1}}, A)\boldsymbol{z}^i$ for the states of the future, if there exists a portfolio $\boldsymbol{z}^i$ satisfying that

$$\begin{pmatrix} -\boldsymbol{q} \\ D(p_{\mathbf{1}}, A) \end{pmatrix} \boldsymbol{z}^i > 0$$

then not only i but all consumers desire such a portfolio as much as possible, which prevents the clearance in each asset market. Thus, in order for an equilibrium to exist, one needs the condition that there exists no portfolio $\boldsymbol{z}$ such that

$$\begin{pmatrix} -\boldsymbol{q} \\ D(p_{\mathbf{1}}, A) \end{pmatrix} \boldsymbol{z} > 0$$

To this condition the following proposition, called Stiemke's theorem, is applicable.

Theorem 1.1 *For each given $n \times m$ matrix A, either*
(I) $A \cdot \boldsymbol{x} \leq 0$ has a solution $\boldsymbol{x} \in \boldsymbol{R}^m$
or
(II) $\boldsymbol{y} \cdot A = 0,\ \boldsymbol{y} > 0$ has a solution $\boldsymbol{y} \in \boldsymbol{R}^n$
but never both.

For proof of the theorem, see Mangasarian (1969,1994), chap.2. §4.

Thus, there must exist a $S+1$-vector $\boldsymbol{\lambda}$ $(= (\lambda_0, \lambda_1, \ldots, \lambda_S)) > 0$ such that

$$\boldsymbol{\lambda} \begin{pmatrix} -\boldsymbol{q} \\ D(p_{\mathbf{1}}, A) \end{pmatrix} = 0$$

It can be easily seen that the normalization of the vector $\boldsymbol{\lambda}$ is permissible, thus I set $\lambda_0 = 1$ for the present and also set $\lambda_{\mathbf{1}} = (\lambda_1, \ldots, \lambda_S)$. Then I have that $\boldsymbol{q} = \lambda_{\mathbf{1}} D(p_{\mathbf{1}}, A)$ which is written in component form as follows.

$$q_j = \sum_{s=1}^{S} \lambda_s D_s^j(p_{\mathbf{1}}, A),\ j = 1, \ldots, J$$

where $D_s^j(p_{\mathbf{1}}, A)$ denotes the dividend brought by asset j at state s. Hence $\lambda_{\mathbf{1}}$ is a coefficient vector that associates the revenue of each asset at each state with its present price; that is, $\lambda_{\mathbf{1}}$ is interpreted as the vector of the discount rates. Accordingly, the absence of arbitrage opportunities implies the existence of the vector of the discount rates common to all assets. Such a vector is often called a present value vector. In addition, the presence of such a present value vector is often called the no-arbitrage condition.

Under the no-arbitrage condition, for any portfolio satisfying the budget constraints for consumer i, the following condition is immediately obtained.

$$\sum_{s=0}^{S} \lambda_s \boldsymbol{p}_s(\boldsymbol{x}_s^i - \boldsymbol{\omega}_s^i) = 0.$$

Set $\boldsymbol{p}_s^* = \lambda_s \boldsymbol{p}_s$ $(s = 0, 1, \ldots, S)$, which can be interpreted as the present value price vector of spot prices at each state.

Then, noting that the set $\{\ \boldsymbol{z} \in \boldsymbol{R}^J \mid p_{\mathbf{1}} \Box (x_{\mathbf{1}} - \omega_{\mathbf{1}}) = D(p_{\mathbf{1}}, A) \cdot \boldsymbol{z}\}$ is equal to the set $\{\ \boldsymbol{z} \in \boldsymbol{R}^J \mid p^*_{\mathbf{1}} \Box (x_{\mathbf{1}} - \omega_{\mathbf{1}}) = D(p^*_{\mathbf{1}}, A) \cdot \boldsymbol{z}\}$ since $D(p^*_{\mathbf{1}}, A) \cdot \boldsymbol{z} = \lambda_{\mathbf{1}} \Box D(p_{\mathbf{1}}, A) \cdot \boldsymbol{z}$, the optimization problem for i can be rewritten as follows.

$$\begin{array}{ll} \max\limits_{x^i} & u^i(x^i) \\ s.t. & p^*(x^i - \omega^i) = 0 \\ & p^*_{\mathbf{1}} \Box (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \in sp[D(p^*_{\mathbf{1}}, A)] \end{array}$$

where $sp[D(p^*_{\mathbf{1}}, A)]$ denotes the linear subspace of $\boldsymbol{R}^S$ spanned by the column vectors of $D(p^*_{\mathbf{1}}, A)$.

It is worth noting that in the above formulation $\boldsymbol{z}^i$ and $\boldsymbol{q}$ are excluded; that is, the portfolio selecting behavior of i is put aside.

Moreover, it is possible to dispense with the market clearance condition of the assets. Indeed, if the subjective equilibrium demand of each consumer for goods, i.e. the solution for the above problem, satisfies the market clearance condition, then there always exists an optimal portfolio allocation $(\bar{\boldsymbol{z}}^1, \ldots, \bar{\boldsymbol{z}}^I)$ such that $\sum_{i=1}^I \bar{\boldsymbol{z}}^i = 0$. In fact, noting that for the solution x^i of the above problem there exists a portfolio $\boldsymbol{z}^i$ $(= (z^i_1, \ldots, z^i_J))$ such that $\boldsymbol{p}^*_s \cdot (\boldsymbol{x}^i_s - \boldsymbol{\omega}^i_s) = \sum_{j=1}^J D^j_s(p^*_{\mathbf{1}}, A) z^i_j$ for all s, I may set $\bar{z}^i_j = z^i_j - \sum_{i=1}^I z^i_j / I, j = 1, \ldots, J$ to obtain that

$$\begin{aligned} \boldsymbol{p}^*_s \cdot (\boldsymbol{x}^i_s - \boldsymbol{\omega}^i_s) = \sum_{j=1}^J D^j_s(p^*_{\mathbf{1}}, A) \bar{z}^i_j,\ i = 1, \ldots I \\ \sum_{i=1}^I \bar{z}^i_j = 0,\ j = 1, \ldots, J. \end{aligned}$$

Hence, under the no-arbitrage condition, a simplified definition of an equilibrium for the economy with real assets is obtainable.

Definition 1.6 *For an economy with real assets $(u, \omega; A)$, an equilibrium under the no-arbitrage condition is a pair of prices and actions (p, x) satisfying that*

(1) x^i is a solution for the following optimization problem $(i = 1, \ldots, I)$.

$$\begin{array}{ll} \max\limits_{x^i} & u^i(x^i) \\ s.t. & p \cdot (x^i - \omega^i) = 0 \\ & p_{\mathbf{1}} \Box (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \in sp[D(p_{\mathbf{1}}, A)]. \end{array}$$

(2) $\sum_{i=1}^I (x^i - \omega^i) = 0.$

It is worth noing that the equilibrium price q_j of asset j is equal to $\sum_{s=1}^S D^j_s(p_{\mathbf{1}}, A),\ j = 1, \ldots, J$ where $p_{\mathbf{1}}$ is the equilibrium prices of goods at date 1. This is because the prices for the goods in the above definition are interpreted as the present value prices. Since there exists no equilibrium under the presence of arbitrage opportunities, I may confine myself to the no-arbitrage equilibrium in the following.

As for the case of a nominal asset model, a similar discussion is applicable except for one concideration. In a nominal asset model, even if the present value price vector of spot prices is considered, it is not allowed to dispense with the vector of the discount rates in rephrasing the budget constraints in date 1 for each agent. As can be easily checked, these constraints are described as follows.

$$p^*_{\mathbf{1}} \,\square\, (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) = \begin{pmatrix} \lambda_1 a_1(1) & \lambda_1 a_2(1) & \dots & \lambda_1 a_J(1) \\ \lambda_2 a_1(2) & \lambda_2 a_2(2) & \dots & \lambda_2 a_J(2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_S a_1(S) & \lambda_S a_2(S) & \dots & \lambda_S a_J(S) \end{pmatrix} \boldsymbol{z}^i.$$

Thus, the reduced form of conditions concerning an equilibrium with nominal assets is derived as follows.

Definition 1.7 *For an economy with nominal assets $(u, \omega; A)$, a pair of prices and actions (p, x) is an equilibrium under the no-arbitrage condition if and only if there exists a strictly positive S-vector $(\lambda_1, \dots, \lambda_S)$ such that the (p, x) satisfies that (1) x^i is a solution for the following optimization problem $(i = 1, \dots, I)$.*

$$\begin{array}{ll} \max\limits_{x^i} & u^i(x^i) \\ s.t. & p \cdot (x^i - \omega^i) = 0 \\ & p_{\mathbf{1}} \,\square\, (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \in sp[\Lambda A]. \end{array}$$

(2) $\sum_{i=1}^{I} (x^i - \omega^i) = 0$
where Λ designates a diagonal matrix with $(\lambda_1, \dots, \lambda_S)$ as its diagonal.

In light of the above definitions of an equilibrium regarding an economy with assets (real or nominal), it is easily seen that the term $sp[D(p_{\mathbf{1}}, A)]$ (resp. $sp[\Lambda A]$) plays a crucial role. Indeed, if $sp[D(p_{\mathbf{1}}, A)](resp.\ sp[\Lambda A]) = \boldsymbol{R}^S$, then the second budget constraint for consumer i is always met and is dispensable. Hence, in this case, the equilibrium condition turns out to be the one familiar in the standard Arrow-Debreu model without assets. This observation leads to the following dichotomy.

Definition 1.8 *If $sp[D(p_{\mathbf{1}}, A)](resp.\ sp[\Lambda A]) = \boldsymbol{R}^S$, then the real (resp. nominal) asset markets are said to be complete. Otherwise, they are incomplete.*

Since a complete asset model is not novel at all in that it can be reduced to the usual Arrow-Debreu model, it is quite legitimate to concentrate on the case of incomplete markets.

It is worth noting that if the number of assets (J) is strictly less than the number of states (S), then the asset market is necessarily incomplete. Thus, it is very often postulated in the literature concerning incomplete asset markets that $J < S$. In this dissertation, I also assume mostly that $J < S$. Needless to say, however, the condition that $J > S$ does not always assure the completeness of the asset market. For the possibility of the incompleteness in the case where $J > S$, see Magill and Shafer (1990).

1.2 Theory of General Equilibrium with Incomplete Asset Markets (GEI)

Once one decides to focus on an economy with incomplete asset markets, he/she has to prepare for various theoretical difficulties. These difficulties have proved to be so fundamental and serious as to necessitate a separete research field concerning this issue. This requirement developed the so called *theory of general equilibrium with incomplete asset markets* (GEI). The theory is characterized as follows. "The GEI model studies the character of economic activity when there may be more than one missing market, and more than one budget constraint."(Geanakoplos (1990), p.3). Note that the incomplete asset market model provided in the previous section possesses these characters. The archetype GEI model was initially formulated by Radner (1972), though he more emphasized on the expectation for the future and the role of firms than the model described above. Thus, an equilibrium previously defined is often called a Radner equilibrium (see, e.g., Mas-Colell, Whinston and Green (1995),19 E). Although in the initial stage of the development of GEI many practical problems were treated, such as finicing of firms, option pricing and macroeconomics (see, e.g., Stiglitz (1974), Drèze (1974), Grossman and Hart (1979), Ross (1976), Lucas (1978,1980), Prescott and Mehra (1980) etc.), arising theoretical difficulties were left untouched. Then, what were the theoretical difficulties ?

Let's consider the following numerical example due to Geanakoplos (1990).

example : There are two consumers (a, b), two goods (x, y), two states of nature $(s = 1, 2)$ at date 1 and two assets $(j = 1, 2)$. At state 1 two goods are both tradable, but only x is tradable at state 2 as well as at state 0 (the present). Let (x_1, y_1) denote the quantity vector of the two goods at state 1 and x_2 denote the one at state 2. The prices for those goods are respectively designated by $p_{x_1},\ p_{y_1}$ and p_{x_2}. On the other hand, let $q_1,\ q_2$ denote the prices respectively for assets 1 and 2. The asset structure of these assets is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

where the 1st and the 2nd row respectively designate the quantity of goods x and y delivered at state 1, while the 3rd row indicates the quantity of good x delivered at state 2. Hence, the dividend matrix of these assets is as follows.

$$\begin{pmatrix} p_{x_1} & p_{y_1} \\ p_{x_2} & p_{x_2} \end{pmatrix}$$

Utilities of consumers a and b are respectively given by the following utility functions.

$$\begin{aligned} u^a &= \ln x_1^a + 2\ln y_1^a + \ln x_2^a \\ u^b &= \ln x_1^b + \ln y_1^b + 2\ln x_2^b. \end{aligned}$$

Note that neither consumer cares for x_0. Endowments for them are as follows.

$$\begin{aligned} \omega^a &= (\bar{x}_0^a, \bar{x}_1^a, \bar{y}_1^a, \bar{x}_2^a) \\ &= (0,\ 1,\ 1,\ 2) \\ \omega^b &= (\bar{x}_0^b, \bar{x}_1^b, \bar{y}_1^b, \bar{x}_2^b) \\ &= (0,\ 1,\ 2,\ 1). \end{aligned}$$

In this framework, the behavior of each consumer is summarized as the following optimization program. That is, for consumer a

$$\begin{aligned} \max \quad & \ln x_1^a + 2\ln y_1^a + \ln x_2^a \\ s.t. \quad & q_1\theta_1^a + q_2\theta_2^a = 0 \\ & p_{x_1}(x_1^a - 1) + p_{y_1}(y_1^a - 1) = p_{x_1}\theta_1^a + p_{y_1}\theta_2^a \\ & p_{x_2}(x_2^a - 2) = p_{x_2}\theta_1^a + p_{x_2}\theta_2^a \end{aligned}$$

where θ_j^a $(j = 1,\ 2)$ denotes the quantity demanded by a for asset j whereas for b I have that

$$\begin{aligned} \max \quad & \ln x_1^b + \ln y_1^b + 2\ln x_2^b \\ s.t. \quad & q_1\theta_1^b + q_2\theta_2^b = 0 \\ & p_{x_1}(x_1^b - 1) + p_{y_1}(y_1^b - 2) = p_{x_1}\theta_1^b + p_{y_1}\theta_2^b \\ & p_{x_2}(x_2^b - 1) = p_{x_2}\theta_1^b + p_{x_2}\theta_2^b \end{aligned}$$

where θ_j^b $(j = 1,\ 2)$ is alike θ_j^a.

On the other hand, the market clearance conditions are as follows.

$$\begin{aligned} \theta_j^a + \theta_j^b &= 0, \quad j = 1,\ 2 \\ x_1^a + x_1^b &= 2 \\ y_1^a + y_1^b &= 3 \\ x_2^a + x_2^b &= 3. \end{aligned}$$

It can be easily obtained by computation that the solutions for p_{y_1} and q_2 are both 1 but those for the other variables are all 0/0, which implies that this numerical example is unsolvable, though the example possesses no abnormal features.

This example shows that there may exist no equilibrium in an economy with incomplete asset markets. This difficulty was early noticed by Hart (1974,1975). In addition to the existence puzzle, Hart also pointed out that an equilibrium of an incomplete asset market model, if any, may not be Pareto optimal. He suggested this point by giving an example where one equilibrium Pareto dominates another equilibrium (Hart (1975)). These negative observations concerning such fundamental properties as the existence and the efficency of an equilibrium threatened a logical basis for the theory.

In the special case of numeraire assets, it turned out that an existence theorem could be derived by means of a standard fixed point technique (Geanakoplos and Polemarchakis (1986), Chae (1988)), where a numeraire asset is an asset which promises to deliver a given amount of the numeraire good alone in each state. For the general case, however, this technique proved to be useless. It is the theory of regular economies that had a share in the breakthrough of this deadlock (for the theory of regular economies, see Nagata (2001b, 2004)). Once the method of this theory proved to be useful (Repullo (1986)), GEI achieved a remarkable improvement in the pure theory. That is to say, for the case of real assets, Duffie and Shafer (1985) solved the problem of the non-existence of equilibrium, showing that GEI equilibrium generically exists in initial endowments and asset structures (for their procedure, see Nagata (2001a)). Their result was, thereafter, refined through elaborate mathematical techniques such as algebraic topology (Hussein, Lasry and Magill (1990)) and vector bundle (Hirsh, Magill and Mas-Colell (1990)). In addition, another way of proof through the investigation of pseudo-equilibria, which is a key concept for the existence proof, has been recently developed (Chichilnisky and Heal (1996), Zhou (1997a, b)). On the other hand, for the nominal asset case, after a suggestive prospect given by Cass (1984), Werner (1985) proved the existence of equilibrium without recourse to the method of regular economies. Then, the structure of the equilibrium set was studied at length through the method of regular economies (Balasko and Cass (1989), Geanakoplos and Mas-Colell (1989), Werner (1986,1990)).

In the process of the pure theoretic development of GEI, it turned out that there is a crucial distinction concerning the structure of equilibria between a real asset model and a nominal asset model. It is true that there always exists in both models a possibility of a multiplicity of equilibria. But, in the case of real assets, a local uniqueness of an equilibrium is generically guaranteed, which implies that one can have finite equilibria in some cases. By contrast, in the case of nominal assets, equilibria, if any, could form a continuum, resulting in a possibility of indeterminacy as Cass (1985) first exemplified. Indeed, it has been generically shown that the degree of indeterminacy is $S - 1$ when the nominal asset prices are taken to be endogenous variables (Geanakoplos and Mas-Colell(1989)), whereas the degree amounts to be $S - J$ if the asset prices are exogenously fixed (Balasko and

Cass(1989)), where the degree indicates the maximum dimension of a manifold contained in the set of equilibria.

The first two chapters (i.e., chap. 2 and 3) of this dissertation are concerned with this characteristic of the equilibrium set. In chapter 2, I investigate from the generic viewpoit the structure of the equilibrium set when there coexists real assets and nominal assets. As was stated above, one type of asset yields a different structure of the equilibrium set from the other type does, which naturally leads to the question: what happens to the equilibrium set with both types? Since only a very limited case has been discussed with regard to this issue, I try to consider the problem within a general framework. As a result, it is demonstrated that regardless of the presence of real assets there is generically still real indeterminacy of equilibria whose degree is the same as without the real assets. Otherwise stated, the indeterminacy caused by nominal assets is robust in that it is not alleviated through introduction of real assets. Then, in chapter 3, I show that the indeterminacy of equilibria with nominal assets is also robust with respect to agents' preference. In the literature on incomplete asset markets, a preference of an agent is mostly characterized by some kind of convexity (e.g., quasi-concave utility function), which facilitates the successive analytical arguments in that such a characterization possibly leads to a consistent demand function. I try to free the preference from any convexity; namely I require a utility function to be only monotone, which does not assure any well-defined demand function. Then, I demonstrate that even in these circumstances the indeterminacy character of the equilibrium set exactly carries over; that is, the degree of indeterminacy is preserved even without any convexity of utility functions.

Another problem involved in the GEI is concerned with efficiency of equilibria. As I have stated before, the equilibrium allocations with incomplete asset markets need not be Pareto optimal, as Hart (1975) first suggested. This observation led to investigation of a less demanding criterion concerning the welfare of those equilibria, resulting in a new concept of optimality called the constrained optimality (or efficiency). This was initially presented by Diamond (1967), then through the work of Stiglitz (1982), elaborated and extended to the general model of the incomplete asset markets by Geanakoplos and Polemarchakis (1986). This specific notion of optimality is descrived as follows. That is to say, a constrained Pareto optimal allocation (or constrained efficient allocation) is an allocation which is Pareto dominant among feasible allocations under the condition that a fictional planner can control the portfolio allocation among the agents, leaving the allocation of goods to the price mechanism. However, even with this weak form of optimality, it has been shown that except for special cases all equilibria with incomplete asset markets are generically constrained suboptimal, which substantially means that they are typically not constrained optimal (Geanakoplos and Polemarchakis (1986), Magill and Shafer (1991)). This result offers some grounds for the intervention of the government in the private economy with incomplete asset markets in order to improve an allocation in terms of Pareto efficiency. Thus, thereafter, some authors argued effectiveness of gevernment policies in an economy with incomplete asset markets (Kajii (1994), Citanna, Kajii and Villanacci

(1998), Villanacci et al. (2002)). In this connection, the possibility of welfare improvement via financial innovation has been argued (Elul (1995, 1999), Cass and Citanna (1998)). From the above arguments, one should conclude that incomplete market equilibria are generally irrelevant to Pareto optimality, whether in a pure or constrained sense. Then, is there any other optimality notion than Pareto efficiency that supports incomplete market equilibria? To this inquiry, Grossman (1977) produced the notion called social Nash optimality. Loosely speaking, an allocation $\bar{x}$ is a social Nash optimum if it is impossible to improve every consumer's utility by reallocating only $(\bar{x}_s^i)_i$ at state s with $(\bar{x}_k^i)_i,\ k \neq s$ unchanged, $s = 0,\ 1, \ldots,\ S$. Grossman demonstrated that an incomplete market equilibrium allocation is virtually equivalent to a social Nash optimal allocation.

As long as the efficiency problem with incomplete asset markets is concerned, it seems that the constrained (in)efficiency has been a central issue. In contrast, (in)efficiency itself, which is obviously more fundamental, has not been fully argued. Indeed, there exists very few analytical literature dealing with this topic (e.g., Magill and Quinzii (1996a), Villanacci et al. (2002)). The latter half of this dissertation (i.e., chap. 4 and 5) focuses on this basic but uncultivated phase of incomplete asset markets. In view of the arguments mentioned above, it may be hardly surprising that one can establish generic inefficiency of an equilibrium with incomplete asset markets. My interest is not in this phenomenon itself but in the reason why this occurs. In chapter 3, I demonstrate for both a real and a nominal asset model that generic inefficiency of an equilibrium stems from the trade system peculiar to incomplete asset markets, independent of a subjective optimization behavior of each agent. Specifically, given an assumption of current monotonicity, it is shown on the basis of a numerical conditions regarding agents, goods and assets that the basic structure of incomplete asset markets, i.e., a multiplicity of missing markets and budget constraints, gives rise to generic inefficiency of an equilibrium, no matter how an agent behaves, where current monotonicity means monotonicity of a utility function only with respect to the consumption at date 0. Thus, roughly speaking, in incomplete asset markets agents are kept away from Pareto optimal allocations before they declare their demand, whether the assets are real or nominal. Chapter 4 is a sophistication of chapter 3. It is worth noting that the claim of chapter 3 is based on a specific topology for the function space, namely the compact open topology, resulting in necessitating a fairly severe condition on the number of agents, goods and assets. Thus, in chapter 4, I adopt the more appropriate topology for the function space, namely the Whitney topology, and use an elaborate mathematical technique similar to the one developed in chapter 2, succeeding in an improvement on the numerical conditions. In fact, as long as a real asset case is concerned, it turns out that one only need a very trivial condition on the numbers of agents, goods and assets to have the same outcome. In addition, for a nominal asset case, it is shown that the result gained there is independent of the asset pricing, that is, it holds whether the asset price is endogenous or exogenous.

As can be easily seen from the above arguments, the theme of this dissertation is solely concerned with fundamental aspects of incomplete asset markets. However, the analytical tools to be used are not fundamental. Thus, in the last chapter, I give a full explanation of specific mathematical techniques originally developed for the purpose of facilitation of analysis of this sort of issue.

Incidentally, I will try to make each chapter self-contained, thus one will see some passages appear repeatedly from chapter 2 to chapter 5.

1.3 Developments of GEI

In the previous sections, I made arguments and surveys which are all concerned with the basic model of GEI, namely, two-period pure exchange economy model. Though all the discussions of this dissertation are solely based on this basic model, a brief survey on further developments of GEI is in order.

It seems that there exist two marked features characterizing the basic model; namely, the pure exchange and the finiteness of all the elements forming the framework of the basic model. More specifically for the latter, the number of agents, goods, states of nature and the periods is implied. From the viewpoint of generalization, it is desirable to weaken these conditions. As a matter of fact, such generalizations have been consciously or unconsciously practiced in the literature. It is, however, worth noting that any extension of the basic model brings some structural complexities and technical difficulties to the madel. Let's first consider the departure from the pure exchange, i.e., the introduction of firms into the basic model.

As has been described in the previous section, the present value vector, i.e., the vector of discount rates, for the future revenues is not uniquely determined in an incomplete market model (see the preceding passages of definition 1.6 and 1.7), which causes a serious difficulty to the model particularly with multi-owner firms. First suppose that a firm is owned by one proprietor. Then, under usual technical assumptions, the owner would be able to choose an optimal production plan in accordance with his/her utility maximization. This is equivalent to say that there exists a specific present value vector for the owner such that the maximization of the value of a production plan evaluated by the present value vector is consistent with the maximization of his/her utility maximization. Now think of a multi-owner firm like a partnership or a corporation. In this case, a specific present value vector for each owner is different with each other, which means that the optimal production plan of the firm is different among the owners. In fact, Duffie and Shafer (1986b) have shown in a simple numercal example with 2 agents, one good, 2 states and 1 firm that a slight difference between the agents' utility function hinders the occurrence of the optimal production plan common to both of them, which can be attributed to the differnce of the present value vector evaluated by each agent. Thus, this problem, namely, the production indeterminacy in an incomplete market model with multi-owner firms, formed an issue for many authors to deal with. As a result, various ways were proposed for unifying the objective function of a firm by coordinating the different present value vectors among owners (Drèze (1974), Grossman and Hart (1979), Kreps (1979), Marimon (1987), Kelsey and Milne (1996), Bonnisseau and Lachiri (2004) see also Harris and Raviv (1988)).

In addition to this conceptual problem, one has to be involved in fundamental theoretical problems, namely, the existence and the efficiency of equilibrium of a production economy with incomplete asset markets. In fact, Momi (2001) has given an example in which generic existence of equilibrium no longer holds insofar as the Drèze criterion to determine a firm's objective function and short sales of the stocks are allowed, though this is not the case for other cases (Magill and Quinzii (1988), Geanakoplos, Magill, Quinzii and Drèze (1990)). Regarding the efficiency problem of equilibrium, it has been shown that the introduction of production to an incomplete asset market model does not save the situation (Geanakoplos, Magill, Quinzii and Drèze (1990)).

Finally for the production economy, one should refer to the financing of a firm, which naturally leads to the investigation of the relevancy of the so called Modigliani-Miller theorem in an incomplete asset market model. Regarding this issue, it has been shown that the M-M theorem can fail to hold in some cases (Stiglitz (1969,1974), Hellwig (1981), DeMarzo (1988)).

Now let's proceed to the other issue, namely, the finiteness of all the elements of the basic model. In the literature concerning the general competitive equilibrium, we have had a long history of the departure from the finiteness since the pioneering works of Aumann (1964) and Bewley (1972). From the nature of the matter, the arguments during the history have centered on the number of agents and goods, having yielded a variety of meaningful results. Thus, it is easily seen that novel findings to be expected would be limited even in the GEI model if one is only concerned with the number of agents and goods. In other words, what one should do with the GEI model is to relax the numeral restriction on periods and states of nature that are specific to GEI. In fact, most researches on the departure from the finiteness have been concerned with the periods and/or the states of nature in the development of GEI.

First, an extension of the periods will be refered to. If one speaks of incomplete markets with more than two periods, one should notice that Radner (1972) has already considered the issue in a form of a sequence of markets. However, it is Werner (1986,1990) who first extended the standard two-period model described in the previous section to the multi-period. The extention of periods makes the model complicated in that one needs to incorporate retrading markets for long-lived assets. Werner has considered a three-period model with nominal assets in which both short-lived (one-period) and long-lived (two-period) assets exist, where the latter are supposed to be originally traded at date 0 and then retrated at date 1. For T-period economies, see Florenzano and Gourdel (1994).

However, once one excludes the constraint of two periods, one easily proceeds over finite (multi-) periods to infinite periods, though one must cope with additional difficulties in the latter case. The crucial difficulty among others consists in the nature of debt constraints. In the finite horizon case, the condition of no debt beyond the terminal date, together with the budget constraint, puts limits on debt at earlier stage, while in the infinite case this kind of constraint is not imposed. In addition, in incomplete markets the current value of future endowments is not equal among traders because marginal rates of substitution may not be equal among them, which implies that the so called solvency requirement is no longer valid, preventing an equilibrium from existing. Thus, in the late 90's many authors worked with this challenging issue of the infinite horizon incomplete markets (Hernández and Santos (1993,1996), Magill and Quinzii (1994,1996b), Levine and Zame (1996), see also Araujo,Monteiro and Páscoa (1996)). In this connection, one should also notice researches in the OLG model with incomplete markets (Cass, Green and Spear (1992), Gottardi (1996), see also Florenzano, Gourdel and Páscoa (2001)) and the recursive equilibria with infinitely lived agents and incomplete markets (Duffie, Geanakoplos, Mas-Colell and MacLennan (1994), Kehoe and Levine (2001), Kubler and Schmedders (2002, 2003), Krebs (2004)).

Finally, I proceed to the other element, namely states of nature. Since the basic model has multiple (finite) states of nature in the future, numeral relaxation on this element necessarily indicates the consideration of an infinite number of states. The more problematic is the case of a continuum of states (i.e., uncountably many states), where one must suffer the following difficulties which stem from the fact that a typical setting in this case is based on some probability space. First, in order to prove an equilibrium, one is required to use some elaborate mathematical technique which one would dispense with in the finite case. Second, even if one can circumvent this technical matter, a resulting existence-proof is, as all the known results have shown, likely to depend on a very strong assumption from the economic viewpoint. Third, the structural property of equilibria is not necessarily desirable; in fact Mas-Colell (1991,1992) has already shown that there exists an open set of economies of incomplete real asset markets with a continuum of states such that the set of equilibria for every member of the open set has at least the cardinality of the continuum, which implies that the property of the finiteness of the equilibria is not generic for the case in which there are uncountably many states of nature even if only real assets are considered.

As for the technical treatment concerning the existence-proof, one may mention Hellwig (1996), Mas-Colell and Monteiro (1996) and Monteiro (1996) as representative approaches. However, all of these approaches need a strong assumption that portfolio returns can be covered by the initial endowments for almost all states and all admissible portfolios. This assumption, moreover, has been shown to be crucial to those approaches (Mas-Colell and Zame (1996)), which is, of course, controversial. In order to circumvent this trap, Araujo, Monteiro and Páscoa (1998) make resort to a new concept of bankruptcy, which is a clever, but, in a sense, ad hoc means. About the structural property of equilibria, it has been shown that if one pays attention to a topological structure instead of a cardinal one, one can gain promising consequences concerning local uniqueness of the equilibria (Dávila (1998), Monteiro and Páscoa (2000)).

There are many other researches concerning the GEI model that have been achieved from different angles than those provided above. It is impossible to survy all of them here since those researches are enormous in quantity. I can only refer to a limited amount of them. In order to efficiently sum up those works, let me classify them into two groups; namely, one is reconsideration of various classical issues concerning the usual Arrow-Debreu model in the framework of GEI and the other is an application of the so-called new wave of economic theory to the GEI model.

First group: ∗ An extension of a characterization of the excess demand functions initiated by Sonnenschein, Mantel and Debreu to incomplete markets has been performed by Bottazzi and Hens (1996), Detemple and Gottardi (1998), Gottardi and Hens (1999), Chiappori and Ekeland (1999) and Hens (2001). ∗ The problem of transaction costs has been argued in the framework of incomplete markets by Préchac (1996), Arrow and Hahn (1999). ∗ Computaion of equilibria a là Scarf has been investigated through various algorithm in the GEI model (see Brown et.al.,(1996), DeMarzo and Eaves (1996), Schmedders (1998, 1999), Kulber and Schmedders (2000) and Kulber (2001)). ∗ The fact that the issue of uniqueness of an equilibrium has been extensively argued for a long time in the standard general competitive equilibrium theory naturally led to an investigation of the same issue in the framework of incomplete markets (Bettzüge (1998)). ∗ Non-Walrasian equilibrium, opposed to Walrasian equilibrium, has been widely argued since the pioneering works by Benassy (1975) and Drèze (1975). This issue also has been investigated in the framework of incomplete markets (Nagata (1992), Herings and Polemerchakis (2002)). ∗ Money and monetary equilibrium with incomplete markets have been discussed (Magill and Quizii (1988, 1992, 1996a), Dubey and Geanakoplos (1992, 2003)).

Second group: In this group the works centered on (asymmetric) information and game theory in incomplete markets are included. ∗ Rational expectations equilibrium initiated by Radner (1979) has been extensively exploited by many authors (Polemarchakis and Siconolfi (1993), Rahi (1995), Pietra and Siconolfi(1998), Stahn (2000), Citanna and Villanacci (2000), Donati and Momi (2003). see also Pietra and Siconolfi(1996, 1997)). ∗ Recently, attention has begun to focus on the application of game theoretic approaches to various aspects of the GEI model (Giraud and Stahn (2003), Giraud and Weyers (2004), Kühn (2004)).

Before closing this chapter, let me mention CAPM very briefly. CAPM is short for the Capital Asset Pricing Model, thus it is related to the GEI model in the sense that it concentrates on determination of asset prices. However, the basic concept for the CAPM is the risk attitude of an agent, which is not fully considered in the GEI model. Moreover, incomplete markets are not necessarily underlying the CAPM. Hence, instead of going far into the literature of the CAPM with incomplete markets, I limit myself to refer to Duffie (1992) and Magill and Quinzii (1996a, cap.3) as relevant references.

Chapter 2

Mixture of Real and Nominal Assets and Indeterminacy of Equilibria

This chapter[1] investigates the real indeterminacy of equilibria in an incomplete market model in which there are two periods, with uncertainty in the second, and both real and nominal assets exist. As is well known, the equilibria of a model with real assets behave very differently from the equilibria of a model with nominal assets. Then, what happens if real and nominal assets coexist ? This is the question I will try to answer in this chapter. As a result, the robustness of real indeterminacy of equilibria is demonstrated within a general framework. Specifically, it is shown that regardless of the presence of real assets there is generically still real indeterminacy of equilibria whose dimension is the same as without the real assets.

[1]This chapter is based on R.Nagata, "Real Indeterminacy of Equilibria with Real and Nominal Assets", *Advances in Mathematical Economics*, vol. 7 , 95-111, 2005.

2.1 Introduction

It is well known that the consequences of a real asset model and a nominal asset model are very different when markets are incomplete, where a real asset is a contract which promises to deliver a bundle of goods at each state in the future, whereas a nominal asset promises to deliver a given stream of units of account across the states. In a real asset model with two periods the equilibrium set is shown to be generically finite (Duffie and Shafer (1985)), whereas in a nominal asset model there generically exists real indeterminacy of equilibria (Cass (1984, 1985), Werner (1985)). The real indeterminacy of equilibria indicates that the set of equilibrium allocation of goods among the agents constitutes a continuum, which has been shown to generically contain a definite dimensional smooth manifold. This observation leads to measurement of the degree of indeterminacy by the dimension of the relevant manifold. Then, it has been shown that the degree of indeterminacy is $S-1$ when the nominal asset prices are taken to be endogenous variables (Geanakoplos and Mas-Colell (1989)), whereas the degree amounts to be $S-J$ if the asset prices are exogenously fixed (Balasko and Cass (1989)), where S indicates the number of states of nature in future and J the number of nominal assets.

In reality it is very hard to imagine that there exists only one type of asset. There are usually both nominal and real assets in actuality, which naturally leads us to the investigation of incomplete markets with both assets. however, there is very little literature considering this issue. Geanakoplos and Mas-Colell argue in the chapter cited above that the dimension of real indeterminacy is robust to the addition of real assets to nominal assets. But they have only considered a special kind of real assets, that is, real numeraire assets which promise to pay only in commodity 1 (the numeraire) at each state. The reason why they particularly chose real numeraire assets is that any nominal asset is transformed into a real numeraire asset, which obviously makes the matter simple. Considering the peculiarity of real numeraire assets, however, the introduction of those assets alone is far from a satisfactory generalization. Indeed, they say in this respect, "I will not make here an effort to get the best possible result" (*ibid.*, p. 36). It is intuitively expected in general that the larger is the proportion of real assets, the smaller is the indeterminacy associated with nominal assets. Magill and Shafer say " ... if the returns matrix consists of a mixture of real and nominal assets ...the equilibrium set contains the image of ... an open set which is typically of dimension less than $S-1$" (Magill and Shafer (1991), p. 1572).

In this chapter I investigate the real indeterminacy of equilibria when there exist *ordinary* real assets as well as nominal assets. I generically analyze the properties of the equilibrium set with both types of assets with respect to two classes of parameters: one is the asset structures of both assets and the other is initial endowments among agents. Asset prices are considered to be endogenous variables. It is shown within a very general framework that real indeterminacy of equilibria occurs generically and its dimension is still $S-1$ as long as the numbers of states and agents are both larger than the total number of both assets. This result does not depend on a component ratio of both assets. On the other hand, it turns out that once the total number of assets exceeds the number of states, the equilibrium set generically fails to be a continuum, so that real indeterminacy disappears. In section 2, I present a two-period model with two types of assets based on a pure exchange economy which is shown to be transformed into a model having only a particular kind of real assets. In section 3, first the existence of equilibria for the model is established from the generic viewpoint, then it is shown under some usual and moderate assumptions regarding a utility function and the numbers of states, assets and agents that there is generically real indeterminacy of equilibria and the dimension of the indeterminacy is $S-1$ in incomplete markets, whereas in potentially complete markets the equilibrium set generically amounts to be at most countable so that the real indeterminacy of equilibria does not matter. Finally, in section 4, the relation between the result of this chapter and other literature, especially that of Geanakoplos and Mas-Colell, is addressed.

2.2 The Model

I consider a pure exchange economy under uncertainty. To keep matters simple, the model has only two periods ($t = 0, 1$) with uncertainty in the second. At date 1 one of S states ($s = 1, \ldots, S$) occurs. For simplicity I call date $t = 0$, state $s = 0$, so that in total there are $S + 1$ states. The economy consists of I consumers ($i = 1, \ldots, I$) and L goods ($l = 1, \ldots, L$).

In each state there are L goods, so that the basic real commodity space is $\boldsymbol{R}^{L(S+1)}$. Each consumer i ($i = 1, \ldots, I$) has an initial endowment of goods $\omega^i = (\boldsymbol{\omega}^i_0, \boldsymbol{\omega}^i_1, \ldots, \boldsymbol{\omega}^i_S) \in \boldsymbol{R}^{L(S+1)}_{++}$, where $\boldsymbol{\omega}^i_s \in \boldsymbol{R}^L_{++}$ is the vector of goods in state s. The preference of agent i is represented by a utility function $u^i : \boldsymbol{R}^{L(S+1)}_{+} \to \boldsymbol{R}$ defined over consumption bundles $x^i = (\boldsymbol{x}^i_0, \boldsymbol{x}^i_1, \ldots, \boldsymbol{x}^i_S)$ in the consumption set $X^i = \boldsymbol{R}^{L(S+1)}_{+} (i = 1, \ldots, I)$. I make a usual assumption on each agent's utility function.

Assumption 2.1 Each utility function u^i ($i = 1, \ldots, I$) satisfies the following conditions:

1. $u^i \in C\big(\boldsymbol{R}^{L(S+1)}_{+}, \boldsymbol{R}\big)$, $u^i \in C^\infty\big(\boldsymbol{R}^{L(S+1)}_{++}, \boldsymbol{R}\big)$.
2. $Du^i_{\boldsymbol{x}} \in \boldsymbol{R}^{L(S+1)}_{++}$ for each $\boldsymbol{x} \in \boldsymbol{R}^{L(S+1)}_{++}$.
3. for each $\boldsymbol{x} \in \boldsymbol{R}^{L(S+1)}_{++}$, $\boldsymbol{v}^t D^2 u^i_{\boldsymbol{x}} \boldsymbol{v} < 0$ for all $\boldsymbol{v} \neq 0$ such that $Du^i_{\boldsymbol{x}} \boldsymbol{v} = 0$, where the superscript 't' indicates the transpose.
4. if $U^i(\bar{\boldsymbol{x}}) = \{\boldsymbol{x} \in \boldsymbol{R}^{L(S+1)}_{+} \mid u^i(\boldsymbol{x}) \geq u^i(\bar{\boldsymbol{x}})$, then $U^i(\bar{\boldsymbol{x}}) \subset \boldsymbol{R}^{L(S+1)}_{++}\}$ for each $\bar{\boldsymbol{x}} \in \boldsymbol{R}^{L(S+1)}_{++}$.

I investigate the case where real and nominal assets coexist. A real asset is a contract which promises to deliver a bundle of the L goods in each state s at date 1. On the other hand, a nominal asset promises to pay an exogenously given amount of units of account in each state at date 1. I assume that there are J real assets ($j = 1, \ldots, J$) and K nominal assets ($k = 1, \ldots, K$) in the economy. Our main interest is in the case of incomplete asset markets, so that $J + K < S$. A real asset j is clearly characterized by commodity vector configuration $A^j = (\boldsymbol{a}_1^j, \ldots, \boldsymbol{a}_S^j) \in \boldsymbol{R}^{LS}$ at date 1, where $\boldsymbol{a}_s^j$ $(= (a_{s1}^j, \ldots, a_{sL}^j))$ is a vector of the L goods delivered in state s ($s = 1, \ldots, S$). If I see A^j ($j = 1, \ldots, J$) as a column vector and put them altogether, I obtain a $LS \times J$ matrix A as follows:

$$A = [A^1, \ldots, A^J] = \begin{pmatrix} a_{11}^1 & \ldots & a_{11}^J \\ \vdots & \ddots & \vdots \\ a_{SL}^1 & \ldots & a_{SL}^J \end{pmatrix}$$

which is called the real asset structure. The nominal asset structure is, however, represented by the following $S \times K$ matrix B:

$$B = [B^1, \ldots, B^K] = \begin{pmatrix} b_1^1 & \ldots & b_1^K \\ \vdots & \ddots & \vdots \\ b_S^1 & \ldots & b_S^K \end{pmatrix}$$

where $B^k (= (b_1^k, \ldots, b_S^k))$ is a vector of returns denominated in the unit of account in each state that a nominal asset k promises to pay ($k = 1, \ldots, K$).

Let $p = (\boldsymbol{p}^0, \boldsymbol{p}^1, \ldots, \boldsymbol{p}^S) \in \boldsymbol{R}_{++}^{L(S+1)}$ be an array of vectors of spot prices in each state, $\boldsymbol{q} = (q_1, \ldots, q_J) \in \boldsymbol{R}_{++}^J$ be the vector of real asset prices, and $\boldsymbol{r} = (r_1, \ldots, r_K) \in \boldsymbol{R}_{++}^K$ be the vector of nominal asset prices. Then the optimization problem an agent i faces is expressed as follows ($i = 1, \ldots, I$):

$$\begin{aligned} \max \quad & u^i(\boldsymbol{x}^i) \\ s.t. \quad & \boldsymbol{p}^0 \boldsymbol{x}_0^i - \boldsymbol{p}^0 \boldsymbol{\omega}_0^i = -\boldsymbol{q}\boldsymbol{y}^i - \boldsymbol{r}\boldsymbol{z}^i \\ & \boldsymbol{p}^s \boldsymbol{x}_s^i - \boldsymbol{p}^s \boldsymbol{\omega}_s^i = (\boldsymbol{p}^s \boldsymbol{a}_s^1, \boldsymbol{p}^s \boldsymbol{a}_s^2, \ldots, \boldsymbol{p}^s \boldsymbol{a}_s^J)\boldsymbol{y}^i + (b_s^1, b_s^2, \ldots, b_s^K)\boldsymbol{z}^i, \quad s = 1, \ldots, S \end{aligned}$$

where $\boldsymbol{y}^i \in \boldsymbol{R}^J$ (resp. $\boldsymbol{z}^i \in \boldsymbol{R}^K$) is a vector of quantities purchased of real assets (resp. nominal assets). Let $\mathcal{E}((u^i, \omega^i)_i, A, B)$ denote the resulting economy in which each agent i has a utility function-endowment pair $(u^i, \omega^i)(i = 1, \ldots, I)$ and there coexist real and nominal asset structures A, B.

Definition 2.1 *An equilibrium for the economy* $\mathcal{E}((u^i, \omega^i)_i, A, B)$ *is a pair* $((x^i, \boldsymbol{y}^i, \boldsymbol{z}^i)_i$, $(p, \boldsymbol{q}, \boldsymbol{r}))$ *such that*

(i) $(x^i, \boldsymbol{y}^i, \boldsymbol{z}^i)$ *is a solution of the optimization problem provided above with* $(p, \boldsymbol{q}, \boldsymbol{r})$, $i = 1, \ldots, I$

(ii) $\sum_{i=1}^I (x^i - \omega^i) = 0$

(iii) $\sum_{i=1}^{I} \boldsymbol{y}^i = 0$

(iv) $\sum_{i=1}^{I} \boldsymbol{z}^i = 0$

In order to consider non-arbitrage, I specify the matrix of returns of real assets at date 1 which is given by

$$V(p_{\mathbf{1}}) = \begin{pmatrix} \boldsymbol{p}^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \boldsymbol{p}^S \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1^1 & \dots & \boldsymbol{a}_1^J \\ \vdots & \ddots & \vdots \\ \boldsymbol{a}_S^1 & \dots & \boldsymbol{a}_S^J \end{pmatrix} = \begin{pmatrix} \boldsymbol{p}^1\boldsymbol{a}_1^1 & \dots & \boldsymbol{p}^1\boldsymbol{a}_1^J \\ \vdots & \ddots & \vdots \\ \boldsymbol{p}^S\boldsymbol{a}_S^1 & \dots & \boldsymbol{p}^S\boldsymbol{a}_S^J \end{pmatrix}$$

Then the budget set of agent i is described as follows, $i = 1, \dots, I$:

$$\{x^i \in \boldsymbol{R}_+^{L(S+1)} \mid \boldsymbol{p}^0(\boldsymbol{x}_0^i - \boldsymbol{\omega}_0^i) = -\boldsymbol{q}\boldsymbol{y}^i - \boldsymbol{r}\boldsymbol{z}^i,\ p_{\mathbf{1}} \square (x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i) = V(p_{\mathbf{1}})\boldsymbol{y}^i + B\boldsymbol{z}^i\} \cdots (*)$$

where $p_{\mathbf{1}} = (\boldsymbol{p}^1, \dots, \boldsymbol{p}^S)$, $x_{\mathbf{1}}^i = (\boldsymbol{x}_1^i, \dots, \boldsymbol{x}_S^i)$, $\omega_{\mathbf{1}}^i = (\omega_1^i, \dots, \omega_S^i)$ and $\square$ denotes box product. To ensure the absence of arbitrage opportunities it is required that there exists no $(\boldsymbol{y}^i, \boldsymbol{z}^i)$ such that

$$\begin{pmatrix} -\boldsymbol{q} & -\boldsymbol{r} \\ V(p_{\mathbf{1}}) & B \end{pmatrix} \begin{pmatrix} \boldsymbol{y}^i \\ \boldsymbol{z}^i \end{pmatrix} > 0$$

where $(\boldsymbol{y}^i, \boldsymbol{z}^i)$ is, of course, interpreted as a column vector. It follows from Stiemke's lemma (see Mangasarian (1969, 1994)) that there must exist an $\alpha \in \boldsymbol{R}_{++}^{S+1}$ such that

$$\alpha \begin{pmatrix} -\boldsymbol{q} & -\boldsymbol{r} \\ V(p_{\mathbf{1}}) & B \end{pmatrix} = 0.$$

where α is interpreted as a vector of present value coefficients, thus $\alpha_s \boldsymbol{p}^s,\ s = 0, 1, \dots, S$ can be viewed as present value prices. I shall show that these present value prices give rise to the new set of budget constraints without $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{y}^i, \boldsymbol{z}^i$, which is equivalent to $(*)$. Let p^* denote an array of vectors $(\alpha_0\boldsymbol{p}^0, \alpha_1\boldsymbol{p}^1, \dots, \alpha_S\boldsymbol{p}^S)$. Then it is easy to see that the budget equations of $(*)$ lead to

$$p^*(x^i - \omega^i) = 0,$$

which forms a part of the new set of constraints. In order to obtain the rest of them, I need to consider the transformation of nominal assets into real assets.

It is known that if there are only nominal assets in the economy, then a nominal asset equilibrium can be viewed as an equilibrium in which all assets are real numeraire assets (Geanakoplos and Mas-Colell (1989)). That is to say, $((x^i, \boldsymbol{z}^i)_i, (p, \boldsymbol{r}))$ is an equilibrium with the nominal asset structure B if and only if there exists a strictly positive diagonal matrix Λ such that $((x^i, \boldsymbol{z}^i)_i, (p, \boldsymbol{r}))$ is an equilibrium with the real numeraire asset structure

ΛB, where a real numeraire asset is a contract which pays only a certain units of good 1 (the numeraire) in each state, so that a real numeraire asset structure is a $S \times K$ returns matrix with only good 1 if there exist K real numeraire assets in the economy.

To embed a real numeraire asset into the general real asset space, consider a $LS \times S$ matrix Ω given by

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

which I call a real asset operator. For any matrix C with S rows, let C^* denote the product ΩC. Then it can be shown that the system of mixed assets $(A,\ B)$ is equivalent to a system of real assets $(A,\ (\Lambda B)^*)$, where Λ is any strictly positive $S \times S$ diagonal matrix.

Proposition 2.1 *$(x^i, \boldsymbol{y}^i, \boldsymbol{z}^i)_i$ is an equilibrium allocation for the economy $\mathcal{E}((u^i, \omega^i)_i, A, B)$ if and only if there exists a strictly positive diagonal matrix Λ such that $(x^i, \boldsymbol{y}^i, \boldsymbol{z}^i)_i$ is an equilibrium allocation for the economy $\mathcal{E}((u^i, \omega^i)_i, A, (\Lambda B)^*)$, where $(A, (\Lambda B)^*)$ is interpreted as a real asset structure.*

Proof: *First note that if both of A and $(\Lambda B)^*$ are a real asset structure, budget equations of each agent i $(i = 1, \dots, I)$ amount to the following;*

$$\begin{aligned} \boldsymbol{p}^0(\boldsymbol{x}_0^i - \boldsymbol{\omega}_0^i) &= -\boldsymbol{q}\boldsymbol{y}^i - \boldsymbol{r}\boldsymbol{z}^i \\ p_{\mathbf{1}}\Box(x_{\mathbf{1}}^i - \omega_{\mathbf{1}}^i) &= \begin{pmatrix} \boldsymbol{p}^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \boldsymbol{p}^S \end{pmatrix} A\boldsymbol{y}^i + \begin{pmatrix} \boldsymbol{p}^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \boldsymbol{p}^S \end{pmatrix} (\Lambda B)^*\boldsymbol{z}^i. \end{aligned}$$

$\rightarrow$ *: Let $\boldsymbol{p}^s = (p_1^s, \dots, p_L^s)$ $(s = 1, \dots, S)$ be an equilibrium price system in each state. Then I can choose $(1/p_1^s)$ as each diagonal element of Λ $(s = 1, \dots, S)$.*
$\leftarrow$ *: Given a $(\Lambda B)^*$, the budget equation in each state at date 1 is degree 0 homogeneous*

with respect to the spot prices. Thus I may convert $\boldsymbol{p}^s$ *into* $\boldsymbol{p}^{s\prime} = \boldsymbol{p}^s/(\lambda_s p_1^s), s = 1, \ldots, S$, *where* λ_s (> 0) *is a diagonal element of* Λ $(s = 1, \ldots, S)$, *obtaining the budget equations at date* 1 *for the economy with mixed asset structures* $(A,\ B)$.

Consequently, for any strictly positive diagonal matrix Λ, an equilibrium for the economy $\mathcal{E}((u^i, \omega^i)_i, A, (\Lambda B)^*)$, if any, constitutes an equilibrium for the economy $\mathcal{E}((u^i, \omega^i)_i, A, B)$. Otherwise stated, the set of equilibria for the mixed assets economy is parameterized by Λ.

Now that the mixed asset structure substantially turns into a real asset structure, all budget equations at date 1 are homogeneous functions of the spot prices. Hence, by the non-arbitrage condition I can transform the date 1 budget equations to the following:

$$p^*_{\mathbf{1}} \square (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \in \langle (P^*_{\mathbf{1}} A,\ P^*_{\mathbf{1}} (\Lambda B)^*) \rangle$$

where $p^*_{\mathbf{1}}$ is a configuration of present value vectors of date 1 prices (i.e. $p^*_{\mathbf{1}} = (\boldsymbol{p}^{*1}, \ldots, \boldsymbol{p}^{*S}) = (\alpha_1 \boldsymbol{p}^1, \ldots, \alpha_S \boldsymbol{p}^S)$) and $P^*_{\mathbf{1}}$ is a $S \times LS$ matrix given by

$$P^*_{\mathbf{1}} = \begin{pmatrix} \boldsymbol{p}^{*1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \boldsymbol{p}^{*S} \end{pmatrix}$$

and $\langle M \rangle$ is the subspace spanned by the columns of a matrix M.

It is known that according to the non-arbitrage condition, the definition of an equilibrium is simplified in the case of real assets, that is, it reduces to the one without portfolios and asset prices (see eg. Magill and Shafer (1991)). In the case of real asset structure $(A,\ (\Lambda B)^*)$, such simplification leads to the following definition for a non-arbitrage equilibrium.

Definition 2.2 *A non-arbitrage equilibrium for the economy* $\mathcal{E}((u^i, \omega^i)_i, A, (\Lambda B)^*)$ *is a pair* $((x^i)_i, p^*)$ *such that*

(i) $x^i = argmax\{u^i(x^i) \mid x^i \in D(p^*, \omega^i; A,\ (\Lambda B)^*)\} \quad i = 1, \ldots, I$

(ii) $\sum_{i=1}^I (x^i - \omega^i) = 0$

(iii) $p^* \in \mathbb{S}^{L(S+1)-1}_{++}$

where $D(p^*, \omega^i; A,\ (\Lambda B)^*)$ *designates the following budget constraint set for consumer i*

$$\{x^i \in \boldsymbol{R}^{L(S+1)}_+ \mid p^*(x^i - \omega^i) = 0, p^*_{\mathbf{1}} \square (x^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}) \in \langle (P^*_{\mathbf{1}} A,\ P^*_{\mathbf{1}} (\Lambda B)^*) \rangle \}$$

and $\mathbb{S}^{L(S+1)-1}_{++}$ *is the* $L(S+1) - 1$ *dimensional strictly positive unit sphere.*

The reason why p^* is confined in $\mathbb{S}_{++}^{L(S+1)-1}$ is, of course, because of the degree 0 homogeneity of the demand function of each agent in this case.

In the sequel, I use p (resp. $p_{\mathbf{1}}, P_{\mathbf{1}}$) for p^* (resp. $p^*_{\mathbf{1}}, P^*_{\mathbf{1}}$) for simplicity of notation.

Let the set of non-arbitrage equilibria for a given economy be $E((u^i, \omega^i)_i, A, (\Lambda B)^*)$. As long as I consider the real indeterminacy of equilibria for the mixed asset economy $\mathcal{E}((u^i, \omega^i)_i, A, B)$ on the basis of non-arbitrage, I should work with the following set:

$$\Gamma = \bigcup_{\Lambda \in N} E((u^i, \omega^i)_i, A, (\Lambda B)^*)$$

where N is the set of all strictly positive $S \times S$ diagonal matrices (note that, as I have remarked above, the set of equilibria for the mixed assets economy is parameterized by Λ).

2.3 Main Results

I first establish the generic existence of equilibria for the economy with a specific real asset structure defined by the following:

$$\begin{pmatrix}
a_1^1(1) & \dots & a_1^J(1) & b_1^1 & \dots & b_1^K \\
a_2^1(1) & \dots & a_2^J(1) & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_L^1(1) & \dots & a_L^J(1) & 0 & \dots & 0 \\
a_1^1(2) & \dots & a_1^J(2) & b_2^1 & \dots & b_2^K \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_L^1(2) & \dots & a_L^J(2) & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_1^1(S) & \dots & a_1^J(S) & b_S^1 & \dots & b_S^K \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_L^1(S) & \dots & a_L^J(S) & 0 & \dots & 0
\end{pmatrix}$$

I may identify the set of those real assets with $\boldsymbol{R}^{LSJ+SK}$. Let (a, b) be an element of $\boldsymbol{R}^{LSJ+SK}$, i.e. $a \in \boldsymbol{R}^{LSJ}$ and $b \in \boldsymbol{R}^{SK}$.

Proposition 2.2 *Under assumption 2.1, $E((u^i, \omega^i)_i, a, b^*) \neq \emptyset$ for almost all $(\omega, a, b) \in \boldsymbol{R}_{++}^{L(S+1)I} \times \boldsymbol{R}^{LSJ+SK}$.*

Proof: *By following almost the same procedure as that of Duffie and Shafer (1985)] (Proposition 1, Proposition 2,Theorem 1 and Theorem 2), I am able to obtain the result. The only point left to be checked is as follows.*

Consider the map $K_\delta : \boldsymbol{R}_{++}^{L(S+1)I} \times W_\delta \times \boldsymbol{R}^{LSJ+SK} \to \boldsymbol{R}^{(S-(J+K))(J+K)}$ *defined by*

$$K_\delta(p, L, a, b) = (I,\ \phi_\delta(L))P_\delta(P_{\mathbf{1}}(a,\ b))$$

where $(W_\delta,\ \phi_\delta)$ *is a chart of the Grassmanian manifold* $G_{J+K,S}$, *I is a* $(S-(J+K)) \times (S-(J+K))$ *unit matrix and* P_δ *is the* $S \times S$ *permutation matrix corresponding to* δ, *which is a permutation of* $\{1, \ldots, S\}$. *Note that the map is defined chart by chart. In order to establish the existence of equilibria, I have only to check that the derivative of* K_δ *with respect to* (a, b) *(denoted by* $D_{a,b}K_\delta(p, L, a, b)$*) is of full rank.*

For simplicity, let δ *be the identity permutation. Then the submatrix of* $D_{a,b}K_\delta(p, L, a, b)$ *corresponding to* $(a_1^1(s), a_1^2(s), \ldots, a_1^J(s), b_s^1, b_s^2, \ldots, b_s^k)$ $s = 1, \ldots, S-(J+K)$ *is represented as follows:*

$$\begin{pmatrix}
P_J(1) & O & O & O & O & O & O & O \\
O & P_K(1) & O & O & O & O & O & O \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
O & O & O & P_J(s) & O & O & O & O \\
O & O & O & O & P_K(s) & O & O & O \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
O & O & O & O & O & O & P_J(S-(J+K)) & O \\
O & O & O & O & O & \ddots & O & P_K(S-(J+K))
\end{pmatrix}$$

where $P_J(s)$ *(resp.* $P_K(s)$*) denotes a* $J \times J$ *(resp.* $K \times K$*) diagonal matrix with* p_1^s *as its diagonal* $(s = 1, \ldots, S-(J+K))$. *Thus the above square matrix is obviously nonsingular, which implies that* $rankD_{a,b}K_\delta = (S-(J+K))(J+K)$.

Let the generic set of (ω, a, b) in the above proposition be Θ.

It is worth noting that under assumption 1 the set of equilibrium $((\bar{x}^i)_i, \bar{p})$ for any $(\omega, a, b) \in \Theta$ is finite and each $((\bar{x}^i)_i, \bar{p})$ is locally differentiable in (ω, a, b). It also holds that $rank(\bar{P}_{\mathbf{1}}(a,\ b^*)) = J+K$ at each equilibrium $((\bar{x}^i)_i, \bar{p})$ for any $(\omega, a, b) \in \Theta$ (see Duffie and Shafer (1985)).

Let $(\bar{y}^i, \bar{z}^i)$ be agent i's equilibrium portfolio for $(\omega, a, b) \in \Theta,\ i = 1, \ldots, I$.

I make an assumption on the number of the agents and assets.

Assumption 2.2 $I > J + K$.

Proposition 2.3 *Under the assumptions,* $rank((\bar{y}^i, \bar{z}^i)_i^I) = J+K$ *for almost all* $(\omega, a, b) \in \boldsymbol{R}_{++}^{L(S+1)I} \times \boldsymbol{R}^{LSJ+SK}$.

Proof: *Note that I may normalize spot prices, restricting them to* $\mathbb{S}_{++}^{L(S+1)-1}$. *Consider the map* $g : \mathbb{S}_{++}^{L(S+1)-1} \times \boldsymbol{R}^{LSJ+SK} \to \boldsymbol{R}^{S(J+K)}$ *given by* $g(p, a, b) = (P_{\mathbf{1}}a,\ P_{\mathbf{1}}b^*)$. *It is* *easily shown by calculation that g is a submersion. Since the whole set of full rank matrix constitutes an open and dense set in the $\boldsymbol{R}^{S(J+K)}$, the inverse image of the set by g is also open and dense in the domain. Let the inverse image be Δ. Notice that Δ is a $L(S+1)-1+LSJ+SK$-dimensional manifold in $\mathbb{S}_{++}^{L(S+1)-1} \times \boldsymbol{R}^{LSJ+SK}$. For simplicity, denote the initial endowment space $\boldsymbol{R}_{++}^{L(S+1)I}$ by Π. I consider an individual demand function x^i defined on $\Delta \times \Pi$. It is easily shown that x^i is smooth on $\Delta \times \Pi$. Hence, a smooth aggregate excess demand function can be obtained, which is admittedly truncated to a $L(S+1)-1$-vector valued function by Walras' law. Let the truncated function be $F(p,a,b,\omega)$. It is readily checked that the derivative $D_\omega F$ is full rank, so F is a submersion. Then I consider each agent's portfolio $(\boldsymbol{y}^i, \boldsymbol{z}^i)$ that is uniquely determined on $\Delta \times \Pi$. Note that $(\boldsymbol{y}^i, \boldsymbol{z}^i)$ is smooth by a smoothness of x^i. Let h be a nonzero linear combination of $J+K$ agents' portfolios, that is*

$$h(p,a,b,\omega,\alpha) = \sum_{i=2}^{J+K+1} \alpha_i(\boldsymbol{y}^i(p,a,b,\omega), \boldsymbol{z}^i(p,a,b,\omega))$$

where $\alpha = (\alpha_2, \ldots, \alpha_{J+K+1})$ is an element of the $J+K-1$-dimensional unit sphere $\mathbb{S}^{J+K-1}$ (for the reason why I start from agent 2, see Duffie and Shafer (1985)). Now consider the product map $H : \Delta \times \Pi \times \mathbb{S}^{J+K-1} \to \boldsymbol{R}^{L(S+1)-1} \times \boldsymbol{R}^{J+K}$ defined by

$$H(p,a,b,\omega,\alpha) = (F(p,a,b,\omega), g(p,a,b,\omega,\alpha)).$$

Since asset demands of the agents can be freely varied without affecting the demands for goods, I may presume through an appropriate alteration of their initial endowments that H is a submersion and has 0 as a regular value. Thus the set

$$M = \{(p,a,b,\omega,\alpha) \in \Delta \times \Pi \times \mathbb{S}^{J+K-1} \mid H(p,a,b,\omega,\alpha) = 0\}$$

is a manifold. On this manifold M I consider a projection $\pi(p,a,b,\omega,\alpha) = (\omega,a,b)$. Since π is proper by the boundary condition of each agent's utility function, the set of regular values forms an open and dense set (denoted by T) in $\Pi \times \boldsymbol{R}^{LSJ+SK}$ (note that Δ is open and dense in $\mathbb{S}_{++}^{L(S+1)-1} \times \boldsymbol{R}^{LSJ+SK}$). Defining $\Theta^* = T \cap \Theta$, then it is easily seen that for any element (ω,a,b) in Θ^*, $E((u^i,\omega^i),a,b^*) \neq \emptyset$ but $\pi^{-1}(\omega,a,b)$ is empty since the number of equations exceeds the number of unknowns. Noting that Θ^* is open and dense in $\Pi \times \boldsymbol{R}^{LSJ+SK}$, this implies that for almost all (ω,a,b), $rank((\bar{\boldsymbol{y}}^i, \bar{\boldsymbol{z}}^i)_i^I) = J+K$.

In the following the generic set of (ω,a,b) specified from the above proposition is denoted by Θ^* as is designated in the proof.

Now I turn to the central issue. It is clear that the structure of Γ depends on the property of the matrix $(A,\ (\Lambda B)^*)$. Admittedly, for almost all (ω,A,B), $E((u^i,\omega^i)_i, A, B^*)$ exists and is finite by proposition 2.2, but it is not enough to determine the structure of Γ .

In order to investigate the structure of Γ, I consider a particular set of A and B. Let $\mathcal{A}$ be the set of $LS \times J$ matrix of full rank and $\mathcal{B}$ be also the set of $S \times K$ matrix of full rank. $\mathcal{A}$ is considered to be a topological subspace of $\boldsymbol{R}^{LSJ}$. For a matrix $A \in \boldsymbol{R}^{LSJ}$, let A_{-1} be a $(L-1)S \times J$ matrix which is constructed by excluding every $(s-1)L+1$-th row from A $(s = 1, \ldots, S)$. Consider the set $\bar{\mathcal{A}}$ defined by

$$\bar{\mathcal{A}} = \{A \in \mathcal{A} \mid A_{-1} \textit{ has rank } J\}.$$

Lemma 2.1 *$\bar{\mathcal{A}}$ is an open and dense set of $\boldsymbol{R}^{LSJ}$.*

Proof: *Noting that generally the set of $n \times m$ matrix of full rank is open and dense in $\boldsymbol{R}^{nm}$, $\mathcal{A}$ is open and dense in $\boldsymbol{R}^{LSJ}$. Consider the map $\pi : \boldsymbol{R}^{LSJ} \to \boldsymbol{R}^{(L-1)SJ}$ given by*

$$\pi(A) = A_{-1}.$$

Let $\mathcal{A}_{-1}$ be the set of $(L-1)S \times J$ matrix of rank$J$ in $\boldsymbol{R}^{(L-1)SJ}$. Since $\mathcal{A}_{-1}$ is obviously open and dense in $\boldsymbol{R}^{(L-1)SJ}$ and π is a projection, $\pi^{-1}(\mathcal{A}_{-1})$ is open and dense in $\boldsymbol{R}^{LSJ}$. Hence $\bar{\mathcal{A}}$ is an open and dense set of $\boldsymbol{R}^{LSJ}$ since $\bar{\mathcal{A}} = \pi^{-1}(\mathcal{A}_{-1}) \cap \mathcal{A}$.

I now consider the product set $\bar{\mathcal{A}} \times \mathcal{B}$ which is obviously open and dense in $\boldsymbol{R}^{LSJ} \times \boldsymbol{R}^{SK}$. As for this set, I easily obtain the following result.

Lemma 2.2 *For any $(A, B) \in \bar{\mathcal{A}} \times \mathcal{B}$, rank$(A,\ B^*) = J + K$ and $(A, \Lambda B)$ is included in $\bar{\mathcal{A}} \times \mathcal{B}$ for any $\Lambda \in N$.*

Proof: *Since $B^*_{-1} = 0$ (zero matrix) by the property of Ω, rank$(A,\ B^*) =$ rank$A +$ rank$B^* = J + K$. It is easily seen from the properties concerning Ω and Λ that if B is included in $\mathcal{B}$, then both B^* and $(\Lambda B)^*$ has rank K for any $\Lambda \in N$. Thus $(A, \Lambda B) \in \bar{\mathcal{A}} \times \mathcal{B}$ for any $\Lambda \in N$.*

Let the intersection of $\boldsymbol{R}^{L(S+1)I}_{++} \times \bar{\mathcal{A}} \times \mathcal{B}$ and Θ^* be Υ, that is, $\Upsilon = (\boldsymbol{R}^{L(S+1)I}_{++} \times \bar{\mathcal{A}} \times \mathcal{B}) \cap \Theta^*$. I concentrate on this set in the following. Note that Υ is open and dense in $\boldsymbol{R}^{L(S+1)I}_{++} \times \boldsymbol{R}^{LSJ} \times \boldsymbol{R}^{SK}$.

In what follows I identify N with $\boldsymbol{R}^S_{++}$ and let $N_\epsilon(\boldsymbol{e})$ be an ϵ open neighborhood of the S-dimensional unit vector $\boldsymbol{e}$ $(= (1, \ldots, 1))$. I will consider an element of $N_\epsilon(\boldsymbol{e})$ to be a $S \times S$ diagonal matrix with the diagonal consisting of its components.

Proposition 2.4 *For any $(\omega, A, B) \in \Upsilon$ there exists an ϵ (> 0) such that $E((u^i, \omega^i)_i, A, (\Lambda B)^*) \neq \emptyset$ for all $\Lambda \in N_\epsilon(\boldsymbol{e})$.*

Proof: *Since Υ is open, there exists an ϵ (> 0) such that $(\omega, A, \Lambda B) \in \Upsilon$ for all $\Lambda \in N_\epsilon(\boldsymbol{e})$, which indicates that $E((u^i, \omega^i)_i, A, (\Lambda B)^*) \neq \emptyset$.*

Since $rank(\bar{P}_{\mathbf{1}}(a,\ b^*)) = J+K$ at each equilibrium $((\bar{x}^i)_i, \bar{p})$ for any $(\omega, a, b) \in \Theta$, I may presume $rank(P_{\mathbf{1}}(A,\ (\Lambda B)^*)) = J + K$ for any Λ in the proposition, where $P_{\mathbf{1}}$ is derived from an equilibrium price system of $\mathcal{E}((u^i, \omega^i)_i, A, (\Lambda B)^*)$. Then I obtain a proposition regarding how an equilibrium allocation of commodities depends on Λ.

Lemma 2.3 *For any given $(\omega, A, B) \in \Upsilon$ and $\Lambda, \Lambda' \in N_\epsilon(\boldsymbol{e})$ where ϵ is chosen in such a way as in proposition 2.4, let $((x^i)_i, p)$ and $((x^{i\prime})_i, p')$ be equilibria for, respectively, $\mathcal{E}((u^i, \omega^i)_i, A, (\Lambda B)^*)$ and $\mathcal{E}((u^i, \omega^i)_i, A, (\Lambda' B)^*)$. Then, under assumptions 2.1 and 2.2, if $\langle (A, (\Lambda B)^*) \rangle \neq \langle (A, (\Lambda' B)^*) \rangle$, $x^i \neq x^{i\prime}$ for some i.*

Proof: *Suppose that $x^i = x^{i\prime}$ for all i, then I must have $p = p'$ because of assumption 2.1 for each agent's utility function (especially, convexity and no corner solutions). Let $(\boldsymbol{y}^i, \boldsymbol{z}^i)_i$ and $(\boldsymbol{y}^{i\prime}, \boldsymbol{z}^{i\prime})_i$ be respectively corresponding equilibrium allocations of the portfolios. Thus, by the property of each agent's budget set (see definition 2.2), $(P_{\mathbf{1}}(A,\ (\Lambda B)^*))(\boldsymbol{y}^i, \boldsymbol{z}^i) = (P_{\mathbf{1}}(A, (\Lambda' B)^*))(\boldsymbol{y}^{i\prime}, \boldsymbol{z}^{i\prime}) \in \langle P_{\mathbf{1}}(A, (\Lambda B)^*) \rangle \cap \langle P_{\mathbf{1}}(A, (\Lambda' B)^*) \rangle$, $i = 1, \ldots, I$.*
Suppose $\langle (A, (\Lambda B)^*) \rangle \neq \langle (A, (\Lambda' B)^*) \rangle$. Since $\langle (A, (\Lambda B)^*) \rangle = \langle A \rangle + \langle (\Lambda B)^* \rangle$ where $+$ denotes a direct sum of vector spaces, $\langle (A, (\Lambda B)^*) \rangle \neq \langle (A, (\Lambda' B)^*) \rangle$ implies $\langle (\Lambda B)^* \rangle \neq \langle (\Lambda' B)^* \rangle$. It follows that $\langle P_{\mathbf{1}}(\Lambda B)^* \rangle \neq \langle P_{\mathbf{1}}(\Lambda' B)^* \rangle$. Then I obtain $\langle P_{\mathbf{1}}(A, (\Lambda B)^*) \rangle \neq \langle P_{\mathbf{1}}(A, (\Lambda' B)^*) \rangle$ since $rank P_{\mathbf{1}}(A, (\Lambda B)^*) = rank P_{\mathbf{1}}(A, (\Lambda' B)^*) = J + K$. This yields that $dim \langle P_{\mathbf{1}}(A, (\Lambda B)^*) \rangle \cap \langle P_{\mathbf{1}}(A, (\Lambda' B)^*) \rangle < J + K$. Considering all the agents, I have $(P_{\mathbf{1}}(A,\ (\Lambda B)^*))(\boldsymbol{y}^i, \boldsymbol{z}^i)$ (and $(P_{\mathbf{1}}(A, (\Lambda' B)^*))(\boldsymbol{y}^{i\prime}, \boldsymbol{z}^{i\prime})) \subset (\langle P_{\mathbf{1}}(A, (\Lambda B)^*) \rangle \cap \langle P_{\mathbf{1}}(A, (\Lambda' B)^*) \rangle)$ for all i, which is a contradiction since through proposition 2.3, $dim \langle (\boldsymbol{y}^1, \boldsymbol{z}^1), \ldots, (\boldsymbol{y}^I, \boldsymbol{z}^I) \rangle = dim \langle (\boldsymbol{y}^{1\prime}, \boldsymbol{z}^{1\prime}), \ldots, (\boldsymbol{y}^{I\prime}, \boldsymbol{z}^{I\prime}) \rangle = J + K$ where all $(\boldsymbol{y}^i, \boldsymbol{z}^i)$s and $(\boldsymbol{y}^{i\prime}, \boldsymbol{z}^{i\prime})$s are interpreted as column vectors.

Definition 2.3 *A matrix is in general position if every submatrix of it has full rank.*

Lemma 2.4 *Let (A, B) be an element of $\bar{\mathcal{A}} \times \mathcal{B}$. In addition, let B be in general position and Λ, Λ' be in N. If $\Lambda \neq \alpha \Lambda'$ for any α (> 0), then $\langle (A, (\Lambda B)^*) \rangle \neq \langle (A, (\Lambda' B)^*) \rangle$.*

Proof: *By lemma 2.2, $\langle (A, B^*) \rangle = \langle A \rangle + \langle B^* \rangle$ and $\langle (A, (\Lambda B)^*) \rangle = \langle A \rangle + \langle (\Lambda B)^* \rangle$ where $+$ denotes a direct sum of vector spaces. Thus, $\langle (A, (\Lambda B)^*) \rangle \neq \langle (A, (\Lambda' B)^*) \rangle \langle (A, (\Lambda B)^*) \rangle \neq \langle (A, (\Lambda' B)^*) \rangle$ if and only if $\langle A \rangle + \langle (\Lambda B)^* \rangle \neq \langle A \rangle + \langle (\Lambda' B)^* \rangle$, which is equivalent to $\langle (\Lambda B)^* \rangle \neq \langle (\Lambda' B)^* \rangle$. Note that through the property of the real asset operator Ω, $\langle C \rangle = \langle D \rangle$ for any two matrices C and D both with S rows. Therefore $\langle (\Lambda B)^* \rangle \neq \langle (\Lambda' B)^* \rangle$ if and only if $\langle (\Lambda B) \rangle \neq \langle (\Lambda' B) \rangle$. Then, the claim follows from lemma 4 of Geanakoplos and Mas-Colell (1989).*

By way of these lemmas, I obtain the following result.

Theorem 2.1 *Under assumptions 2.1 and 2.2, for almost all (ω, A, B) there are $S - 1$ dimensions of real indeterminacy of equilibria.*

Proof: *The set of B in general position constitutes an open and dense set in $\boldsymbol{R}^{SK}$. Thus the intersection of this set and $\mathcal{B}$ is also open and dense in $\boldsymbol{R}^{SK}$. Let the intersection be $\bar{\mathcal{B}}$. Pick any $(\omega, A, B) \in \Upsilon$ where $B \in \bar{\mathcal{B}}$ and take $N_\epsilon(\boldsymbol{e})$ in such a way as in proposition 2.4. Let the intersection of $N_\epsilon(\boldsymbol{e})$ and the $S-1$-dimensional strictly positive sphere through $\boldsymbol{e}$ be N^*. Then any different pair of Λ in N^* yields different equilibrium allocations of goods among the agents by lemmas 2.3 and 2.4. Since for any Λ in N^* equilibrium allocations of goods are finite and each allocation is locally differentiable in Λ, $S-1$-dimensional real indeterminacy occurs in Γ.*

So far I have concentrated solely on the case of incomplete asset markets; that is, I presume that $J+K<S$.

Thus, I consider, finally, the case of potentially complete markets where $S \leq J+K$. I shall show that in this case the generic property of real indeterminacy of equilibria disappears and the set of equilibrium allocation of goods is generically at most countable.

Definition 2.4 *Let the number of real assets be H. A real asset structure $A \in \boldsymbol{R}^{LSH}$ is regular if for each state s, a row a_s can be selected from the $L \times H$ matrix $A_s = (A_s^1, \ldots, A_s^H)$ such that the collection $(a_s)_s^S$ is linearly independent (Magill and Shafer (1990, 1991)).*

It is worth noting that if the asset structure is regular, the equilibrium allocations of the asset market coincide with those of the contingent markets, so that I have generically a finite set of equilibrium allocations (see Magill and Shafer (1990, 1991)).

Lemma 2.5 *If $S \leq J+K$, for each $(A,B) \in \bar{\mathcal{A}} \times \mathcal{B}$ the asset structure $(A, (\Lambda B)^*)$ is regular for any $\Lambda \in N$.*

Proof: *I consider $\Omega^t(A, (\Lambda B)^*)$ where the superscript 't' indicates the transpose. Then I have $\Omega^t(A, (\Lambda B)^*) = \Omega^t(A, \Omega\Lambda B) = (\Omega^t A, \Lambda B)$, which is a $S \times (J+K)$ matrix obtained by picking up each $(s-1)L+1$-th row ($s = 1, \ldots, S$) from $(A, (\Lambda B)^*)$. Since it is easy to see that $(\Omega^t)^{-1}\{0\} + (A, (\Lambda B)^*)(\boldsymbol{R}^{J+K}) = \boldsymbol{R}^{LS}$ where (Ω^t) and $(A, (\Lambda B)^*)$ are considered to be linear transformations, rank $\Omega^t(A, (\Lambda B)^*)$ = rank $(\Omega^t) = S$. Thus, all the rows of $(\Omega^t A, \Lambda B)$ are linearly independent, which implies that $(A, (\Lambda B)^*)$ is regular for any $\Lambda \in N$.*

Theorem 2.2 *If $S \leq J+K$, for each $(A,B) \in \bar{\mathcal{A}} \times \mathcal{B}$ there exists a dense set $W \subset \boldsymbol{R}_{++}^{L(S+1)I}$ such that for any $\omega \in W$, the equilibrium allocation set $\{(x^i)_i\}$ of Γ is at most countable, where Γ denotes the union $\bigcup_{\Lambda \in N} E((u^i, \omega^i)_i, A, (\Lambda B)^*)$.*

Proof: *Fix any given $(A,B) \in \bar{\mathcal{A}} \times \mathcal{B}$. Since $(A, (\Lambda B)^*)$ is regular for any $\Lambda \in N$, there exists an open and dense set $W(\Lambda)$ in $\boldsymbol{R}_{++}^{L(S+1)I}$ such that for any $\omega \in W(\Lambda)$ the set of equilibrium allocations (denoted by $X_\Lambda(\omega)$) is finite (see the note provided just before lemma 2.5). Moreover, by an invariance property of financial structure (see Magill* *and Shafer (1991), Theorem 6) there exists an open neighborhood of Λ (denoted by $N(\Lambda)$) such that for any $\Lambda' \in N(\Lambda), X_{\Lambda'}(\omega) = X_\Lambda(\omega)$ for all $\omega \in W(\Lambda)$. Since $N = \boldsymbol{R}^S_{++}$, the open covering $\{N(\Lambda)\}_{\Lambda \in N}$ of N has a countable open subcovering $\{N(\Lambda_i)\}_{i=1,2,\ldots}$. Let W denote $\bigcap_i W(\Lambda_i)$. Then $X_{\Lambda_i}(\omega)$ is obviously finite for any $\omega \in W$. The set of equilibrium allocations of Γ $(= \bigcup_{\Lambda \in N} E((u^i, \omega^i)_i, A, (\Lambda B)^*))$ for any given $\omega \in W$ is nothing but $\bigcup_{\Lambda \in N} X_\Lambda(\omega)$, which is equal to $\bigcup_i X_{\Lambda_i}(\omega)$. Thus the set is at most countable. Finally, the density of W is derived from the fact that N $(= \boldsymbol{R}^S_{++})$ is a Baire space.*

Note that the only assumption I need in theorem 2.2 is assumption 2.1 for a utility function.

2.4 Concluding Remarks

This chapter has analyzed the real indeterminacy of equilibria when the economy has both real and nominal assets. In contrast to Geanakoplos and Mas-Colell's work, I have delt with ordinary real assets in the model. In incomplete markets it has been shown that as long as the number of agents is larger than the total number of assets, there is real indeterminacy with the degree of $S-1$ generically in asset structures and initial endowments. This result is independent of the distribution of both assets. Hence, from the generic viewpoint it is irrelevant to conjecture that the larger the proportion of real assets, the smaller the indeterminacy. Otherwise stated, it is not typical that the dimension of the equilibrium set is less than $S-1$ when the asset structure consists of a mixture of real and nominal assets. Indeed, such a case is generically negligible.

I also addressed the problem of indeterminacy in potentially complete markets with both types of assets. In this case, contrary to incomplete markets, it has been proved that the number of equilibria is generically at most countable, so that real indeterminacy does not make sense. This result also does not depend on the proportional distribution of those two assets. Moreover, it also holds even if the number of agents is smaller than the total number of assets.

Finally, a comparison of the result obtained here with that of Geanakoplos and Mas-Colell is in order. As I have stated, their argument is based on real numeraire assets which promise to pay only in commodity 1 (the numeraire) at each state. The reason why they particularly chose real numeraire assets is that any nominal asset is transformed into a real numeraire asset, which obviously makes the matter simple. Their result is as follows. Suppose that $B \leq 2$, $S > 2(A+B)$ and $H > A+B$, where A is the number of real numeraire assets, B the number of nominal assets and H the number of agents. Then for a generic choice of real numeraire and nominal assets, there is a generic set of initial endowments such that each of the corresponding economies has $S-1$ dimensions of real indeterminacy. In contrast, theorem 1 provided here does not require that $S > 2(A+B)$ but only $S > J+K$ ($S > A+B$ in their notation). The difference stems from the fact that the real assets they consider are quite different from those I considered here. Let us show how different they are. As I have noted, they only considered the real *numeraire assets*. In contrast, I have first taken account of all real assets and then shown that I may confine myself to a particular set of them denoted by $\bar{\mathcal{A}}$, which is open and dense in the space of all real asset structures. However, there is a remarkable relation between $\bar{\mathcal{A}}$ and the set of real numeraire asset structures. Since any real numeraire asset structure is transformed into a $LS \times J$ matrix with all elements zero except each $nL+1$-th row ($n = 0, 1, \ldots, S-1$), the set of those structures is included in the complement of $\bar{\mathcal{A}}$ relative to the space of all real asset structures. Thus, the real assets they considered are outside of those I considered here, which causes the difference concerning the required conditions alluded to above. However, it is worth noting that real numeraire asset structures are in the set of Lebesgue measure 0 of the space of all real asset structures. Otherwise stated, I have considered almost all structures, while they have only considered a very limited, almost negligible, part of them.

Chapter 3

Non-convexity and Indeterminacy of Equilibria with Nominal Assets

In this chapter[1], I investigates the real indeterminacy of equilibria in an incomplete market model from a different viewpoint than that of the previous chapter. In chapter 2, I examined how real indeterminacy of equilibria with nominal assets is affected by introduction of real assets, ending up with the establishment of the robustness of the indeterminacy. In this chapter, I consider how real indeterminacy of equilibria with nominal assets depends on choice of a utility function. Specifically, I inquire how the 'degree' of real indeterminacy with nominal assets varies through generalization of a utility function, where the generalization particularly indicates a freedom from convexity. As a result, the robustness of the indeterminacy is shown to be still persistent.

[1]This chapter is based on R.Nagata, "The degree of indeterminacy of equilibria with incomplete markets", *Journal of Mathematical Economics*, 29, 109-123, 1998.

3.1 Introduction

When the number of nominal assets that pay some fixed amounts in units of accounts as returns is lower than the number of states of nature, then it has been shown that competitive equilibria (financial equilibria) are not locally unique and generically contain a smooth manifold whose dimensional magnitude depends on whether the prices of nominal assets are endogenous or exogenous (Balasko and Cass (1989), Geanakoplos and Mas-Colell (1989), Werner (1986,1990)).

In earlier works, initial endowments of agents are taken as a parameter in considering the generic property of financial equilibria. I shall add utility functions of agents to a list of parameters. Then, what kind of utility functions should be considered? In the literature, financial equilibria have been characterized by excess demand functions, which crucially depend on some convexity of utility functions. Then, what if utility functions possess no convexity? This is the motivation of this chapter.

Acceptable utility functions in this chapter are assumed to be smooth, strictly monotone and satisfy some other technical conditions, but no convexity is required of them. It is true that even in these circumstances with incomplete markets the excess demand function itself is viable in the spirit of Sonnenscein, Debreu and Mantel (see Bottazzi and Hens (1996), Detemple and Gottardi (1998)), but it would be hopeless to obtain such an excess demand function as can characterize the set of financial equilibria which is tractable from the differential viewpoint. Thus, instead of relying on the excess demand function, I shall directly use utility functions and work with the first-order conditions to grasp those equilibria. This idea is attributed to Smale (1974). According to this approach, however, a resulting equilibrium set becomes an extended one which contains the intrinsic financial equilibria. Since the extended equilibria would constitute a very huge set, it may be expected that the degree of their indeterminacy would increase proportionally. But I demonstrate in the text that this is not the case from a generic viewpoint. Specifically, given the prices of nominal assets, it is proved that the extended financial equilibrium set is generically a manifold with the same dimension as the one that Balasko and Cass obtained generically for the equilibrium set in endowments on the basis of an economy with given nominal asset prices and *strictly quasi-concave utility functions*. They proved that the dimension is equal to the difference between the number of states and the number of assets, called deficiency in financial asset markets; otherwise stated, the degree of indeterminacy in this case is typically equal to the degree of incompleteness of the markets. It follows from the result of this chapter that as far as the extended equilibrium set is concerned, this fact generically carries over even when preferences are not required to be convex, which implies that the degree of indeterminacy of intrinsic financial equilibria without any convexity of utility functions is also at most the deficiency in financial asset markets. I also investigate how the extended set is related to a choice of parameter values and prove that generically in utility functions any extended financial equilibrium is locally continuous with respect to initial endowments.

This chapter is organized as follows. Section 2 describes the basic model. The main feature of the model is that the returns and the prices of nominal assets are exogenously fixed. In the model, each agent's utility function is taken to be a parameter, as alluded to above, and the functional space is accommodated so as to facilitate the analysis of genericity. Then, the extended financial equilibria are defined by using utility functions, not using demand functions. Section 3 is devoted to an analysis of the extended financial equilibrium set. There I present main results on genericity of the degree of indeterminacy for both the extended equilibrium set and the intrinsic equilibrium set. The dependence of the extended equilibrium set on an economic parameter is also investigated and some continuity relation is derived. Concluding remarks are stated in section 4.

3.2 The model

I consider a pure exchange economy expanding over two periods ($t = 0, 1$) with uncertainty over the state of nature in the second period. As I have done in the previous chapter, I assume that there are I agents ($i = 1, \ldots, I$) and L goods ($l = 1, \ldots, L$), all of which have spot markets at date 0 and in any S states ($s = 1, \ldots, S$) at date 1. I consider date $t = 0$ to be $s = 0$ so that there exists $S + 1$ states. There are K types of financial assets ($k = 1, \ldots, K$) that are traded at date 0. I only consider nominal assets in this chapter, thus each asset pays a fixed amount in units of account in each state at date 1. Hence, the asset structure is represented by the following $S \times K$ matrix B:

$$B = [B^1, \ldots, B^K] = \begin{pmatrix} b_1^1 & \ldots & b_1^K \\ \vdots & \ddots & \vdots \\ b_S^1 & \ldots & b_S^K \end{pmatrix}$$

where $B^k (= (b_1^k, \ldots, b_S^k))$ is a vector of returns denominated in the unit of account in each state that a nominal asset k promises to pay ($k = 1, \ldots, K$). Since I am concerned with incomplete asset markets, I am allowed to assume that $K < S$ in the following. Spot prices of goods and assets are denoted by vectors $p = (\boldsymbol{p}^0, \boldsymbol{p}^1, \ldots, \boldsymbol{p}^S) \in \boldsymbol{R}_{++}^{L(S+1)} (with\ \boldsymbol{p}^s = (p_1^s, \ldots, p_L^s))$ and $\boldsymbol{r} = (r_1, \ldots, r_K) \in \boldsymbol{R}_{++}^n$, respectively. I make the following assumptions on the assets.

Assumption 3.1 $\quad rank\ B = K$ (No redundancy).

Assumption 3.2 $\quad \boldsymbol{r}$ is a non-arbitrage price vector for $\boldsymbol{B}$ (Non-arbitrage).

I take any $\boldsymbol{B}$ and $\boldsymbol{r}$ fulfilling assumptions 3.1 and 3.2 and fix them. Thus, the prices of the assets are considered to be exogenous. The consumption vector of each agent i will be denoted by $x^i = (\boldsymbol{x}_0^i, \boldsymbol{x}_1^i, \ldots, \boldsymbol{x}_S^i)$. Note that x^i is taken as a vector in $\boldsymbol{R}^{L(S+1)}$. Its preference is described by a utility function u^i defined on consumption vectors. An allocation of consumption vectors of all agents will be presented by an $LI(S+1)$-vector $x = (x^1, \dots, x^I)$ and an array of utility functions by $u = (u^1, \dots, u^I)$. The portfolio of agent i will be denoted by $\boldsymbol{z}^i = (z^i_1, \dots, z^i_K)$, where $z^i_k > 0 (< 0)$ implies the purchase (sale) of asset k by agent i. A portfolio allocation is presented by $z = (\boldsymbol{z}^1, \dots, \boldsymbol{z}^I)$. Each agent i also has an initial endowment vector $\omega^i = (\boldsymbol{\omega}^i_0, \dots, \boldsymbol{\omega}^i_S)$ an allocation of which will be denoted by $\omega = (\omega^1, \dots, \omega^I)$, which is also taken as an $LI(S+1)$-vector.

A state of economy is characterized by a tuple (x, p, z), while an economy itself is parameterized by (ω, u). Some assumptions are put on these parameters.

Assumption 3.3 $\omega \in \boldsymbol{R}^{LI(S+1)}_{++}$.

Assumption 3.4 Each utility function u^i $(i = 1, \dots, I)$ satisfies the following conditions:

1. $u^i \in C\big(\boldsymbol{R}^{L(S+1)}_{+}, \boldsymbol{R}\big)$, $u^i \in C^r\big(\boldsymbol{R}^{L(S+1)}_{++}, \boldsymbol{R}\big)$ for large enough r.
2. $Du^i_{\boldsymbol{x}} \in \boldsymbol{R}^{L(S+1)}_{++}$ for each $\boldsymbol{x} \in \boldsymbol{R}^{L(S+1)}_{++}$.
3. first r derivatives of u^i are all bounded (C^r-boundedness).
4. u^i has non-singular Hessian.

Note that no convexity is contained in the above conditions concerning u^i. The fourth condition of assumption 3.4 is included just for simplicity of arguments and can be relaxed (see the proof of Lemma 3.1).

Let the set of utility functions fulfilling above conditions be U. Then, one parameter space composed of each agent's utility function should be the I-product of U, which will be denoted by $\mathcal{U}$. The other parameter space composed of each agent's endowments, i.e., $\boldsymbol{R}^{LI(S+1)}_{++}$ is denoted by Ω. The product of these two parameter spaces, i.e., $\mathcal{U} \times \Omega$ is called the economy space, while the product of the admissible sets of x, p and z is called the state space. Thus, the state space is nothing but $\boldsymbol{R}^{LI(S+1)}_{++} \times \boldsymbol{R}^{L(S+1)}_{++} \times \boldsymbol{R}^{IK}$, which will be denoted by $X \times P \times Z$ for simplicity of notation.

It is worth noting that the set of C^r-bounded functions forms a Banach space with a canonical norm (see Abraham and Robbin (1967), p.24). Since the set of functions satisfying all conditions in the assumption is obviously open in the Banach space, U is a Banach manifold. Consequently, $\mathcal{U}$ is also a Banach manifold.

Now, the formalization of a financial equilibrium is in order. Given an economy $(u^i, \omega^i)_i$, which will be simly denoted by (u, ω) in the following, each agent i faces the following optimization problem $(i = 1, \dots, I)$ for its own.

$$
\begin{array}{ll}
\max & u^i(\boldsymbol{x}^i) \\
s.t. & \boldsymbol{p}^0 \boldsymbol{x}^i_0 - \boldsymbol{p}^0 \boldsymbol{\omega}^i_0 = -\boldsymbol{r}\boldsymbol{z}^i \\
 & \boldsymbol{p}^s \boldsymbol{x}^i_s - \boldsymbol{p}^s \boldsymbol{\omega}^i_s = (b^1_s, b^2_s, \dots, b^k_s)\boldsymbol{z}^i, \quad s = 1, \dots, S
\end{array}
$$

In order for the solutions of these problems to form a financial equilibrium, they have to satisfy the followng market-clearing conditions.

$$\begin{aligned}\sum_{i=1}^{I} \boldsymbol{x}_s^i &= \sum_{i=1}^{I} \omega_s^i, \quad s = 0, 1, \ldots, S \\ \sum_{s=1}^{I} \boldsymbol{z}^i &= 0.\end{aligned}$$

Thus, an intrinsic financial equilibrium for a given economy (u, ω) is formally defined to be a tuple (x, p, z) satisfying both the optimization problem for all i and market-clearing conditions. Next step is to analytically characterize an equilibrium. It is worth noting here that demand, i.e., the solution of each agent's optimization problem, might not even be a well-defined function with respect to p since no convexity is required for a utility function. Thus, one is not allowed to turn the market-clearing condition into a usual equation system on equilibrium prices by way of demand functions. In these circumstances, one might as well use calculus to derive the analytical representation of an equilibrium. It is, however, readily seen that the resulting one, if any, would be too involved to work with. So I give up the exact characterization of an equilibrium, considering instead a broader notion of an equilibrium. To this end, I will use the necessary conditions for the solution of the above optimization problem. Since the constraint functions of the problem are all linear with respect to $(x^i, \boldsymbol{z}^i)$, the well-defined first-order necessary conditions can be obtained for the solution as follows.

$$\begin{aligned}u_{0l}^i - \lambda p_l^0 &= 0, \quad l = 1, \ldots, L \\ u_{sl}^i - \mu_s p_l^s &= 0, \quad l = 1, \ldots, L, s = \ldots, S \\ -\lambda r_k + \sum_{s=1}^{S} \mu_s b_s^k &= 0, \quad k = 1, \ldots, K \\ \boldsymbol{p}^0 \boldsymbol{x}_0^i - \boldsymbol{p}^0 \boldsymbol{\omega}_0^i &= -\boldsymbol{r}\boldsymbol{z}^i \\ \boldsymbol{p}^s \boldsymbol{x}_s^i - \boldsymbol{p}^s \boldsymbol{\omega}_s^i &= (b_s^1, b_s^2, \ldots, b_s^k)\boldsymbol{z}^i, \quad s = 1, \ldots, S\end{aligned}$$

where u_{sl}^i is short for $\frac{\partial u^i}{\partial x_{sl}^i}(\boldsymbol{x}^i), (s = 0, \ldots, S,\ l = 1, \ldots, L)$ and λ, μ_s $(s = 1, \ldots, S)$ are Lagrange multipliers.

Using these conditions, I have an extension of an intrinsic equilibrium.

Definition 3.1 *(extended financial equilibrium) An extended financial equilibrium for a given economy (u, ω) is a tuple (x, p, z) satisfying both the first-order necessary conditions for all i and the market-clearing conditions.*

The set of extended financial equilibria for (u,ω) will be denoted by $FE_{ex}(u,\omega)$, which obviously contains all intrinsic financial equilibria.I investigate some properties of $FE_{ex}(u,\omega)$ in what follows. To this end, I first exclude Lagrange multipliers from the necessary conditions to obtain

$$\begin{aligned}
\frac{u^i_{sl}}{p^s_l} - \frac{u^i_{s1}}{p^s_1} &= 0, \quad s = 0,\dots,S,\ l = 1,\dots,L \\
\sum_{s=1}^{S} (\frac{u^i_{s1}}{p^s_1}) b^k_s - (\frac{u^i_{01}}{p^0_1}) r_k &= 0, \quad k = 1,\dots,K, \\
\boldsymbol{p}^0 \boldsymbol{x}^i_0 - \boldsymbol{p}^0 \boldsymbol{\omega}^i_0 &= -\boldsymbol{r}\boldsymbol{z}^i \\
\boldsymbol{p}^s \boldsymbol{x}^i_s - \boldsymbol{p}^s \boldsymbol{\omega}^i_s &= (b^1_s, b^2_s, \dots, b^k_s)\boldsymbol{z}^i, \quad s = 1,\dots,S.
\end{aligned}$$

The resulting equation system consisting of the above conditions for all i and the market-clearing conditions is linearly homogeneous in (p,z). Thus, I normalize the variables by taking the first good at date 0 to be the numeraire, i.e., I put $p^0_1 = 1$. The normalized variables $\frac{p^s_l}{p^0_1}, \frac{z}{p^0_1}$ will be denoted by $\bar{p}^s_l, \bar{z}$, respectively in the following. This normalization requires transformation of the state space $X \times P \times Z$ into $X \times \bar{P} \times \bar{Z}$ where $\bar{P} = \{p \in \boldsymbol{R}\ L(S+1)_{++} \mid p^0_1 = 1\}$, which is equivalent to $\boldsymbol{R}^{L(S+1)-1}_{++}$ and $\bar{Z}$ is substantially nothing but $\boldsymbol{R}^{IK}$. In accordance with this normalization, $FE_{ex}(u,\omega)$ should also be interpreted to be defined on the altered state space.

In order to facilitate a genericity analysis on the property of $FE_{ex}(u,\omega)$, I decompose the equation system provided above into two parts. One consists of the following equations for all i.

$$\begin{aligned}
\frac{u^i_{sl}}{p^s_l} - \frac{u^i_{s1}}{p^s_1} &= 0, \quad s = 0,\dots,S,\ l = 1,\dots,L \\
\sum_{s=1}^{S} (\frac{u^i_{s1}}{p^s_1}) b^k_s - (\frac{u^i_{01}}{p^0_1}) r_k &= 0, \quad k = 1,\dots,K.
\end{aligned}$$

Note that the set of $(x,\bar{p})$ satisfying these equations only depends on u, thus I denote the set by $\Gamma(u)$ in the following.

The other is, of course, composed of the rest.

$$\begin{aligned}
\boldsymbol{p}^0 \boldsymbol{x}^i_0 - \boldsymbol{p}^0 \boldsymbol{\omega}^i_0 &= -\boldsymbol{r}\boldsymbol{z}^i, \quad i = 1,\dots,I, \\
\boldsymbol{p}^s \boldsymbol{x}^i_s - \boldsymbol{p}^s \boldsymbol{\omega}^i_s &= (b^1_s, b^2_s, \dots, b^k_s)\boldsymbol{z}^i, \quad i = 1,\dots,I,\ s = 1,\dots,S, \\
\sum_{i=1}^{I} \boldsymbol{x}^i_s &= \sum_{i=1}^{I} \boldsymbol{\omega}^i_s, \quad s = 0,1,\dots,S \\
\sum_{s=1}^{I} \boldsymbol{z}^i &= 0.
\end{aligned}$$

Then the set of $(x, \bar{p}, \bar{z})$ satisfying these equations is, in turn, solely dependent on ω, thus it will be denoted by $\Theta(\omega)$. It is obvious that $FE_{ex}(u, \omega) = (\Gamma(u) \times \bar{Z}) \cap \Theta(\omega)$. I first investigate properties of these two sets generically in u and ω respectively, then proceed to their intersection, i.e., $FE_{ex}(u, \omega)$.

3.3 The structure and generic properties of $FE_{ex}(u, \omega)$

First of all, I investigate the generic properties of $\Gamma(u)$. Let Y denote $\boldsymbol{R}^{I(L-1)(S+1)} \times \boldsymbol{R}^{I}K$. I consider the map $\sigma : \mathcal{U} \to C^r(X \times \bar{P}, Y)$ given by

$$\sigma(u)(x^1, \ldots, x^I, \boldsymbol{p}^0, \ldots, \boldsymbol{p}^S) = \begin{pmatrix} (\frac{u^i_{sl}}{p^s_l} - \frac{u^i_{s1}}{p^s_1})_{i=1,\ldots,I,\ s=0,\ldots,S,\ l=2,\ldots,L} \\ (\sum_{s=1}^{S} (\frac{u^i_{s1}}{p^s_1}) b^k_s - u^i_{01} r_k)_{i=1,\ldots,I,\ k=1,\ldots,K} \end{pmatrix}$$

where the degree of differentiability r is assumed to be large enough. For simplicity, $\sigma(u)$ will be denoted by σ_u in the following. For this map the evaluation map $ev\ \sigma : \mathcal{U} \times (X \times \bar{P}) \to Y$ is defined as follows (for the evaluation map, see Abraham and Robbin (1967), Chap. 2).

$$ev\ \sigma(u, (x, \bar{p})) = \sigma_u(x, \bar{p}).$$

For this evaluation map I have

Lemma 3.1 *0 is a regular value of ev σ.*

Proof: *Let the set of C^r-bounded functions in $C^r(\boldsymbol{R}^{L(S+1)}_{++}, \boldsymbol{R})$ be F. Then note that $\Pi^I F$ is the tangent space of $\mathcal{U}$ at each u of $\mathcal{U}$. First, consider the derivative of ev σ at $(u, (x, \bar{p}))$. By the construction of σ_u, I obtain*

$$\begin{aligned} D\ ev\ \sigma(u, (x, \bar{p}))(v, (y, w)) &= D_1 ev\ \sigma(u, (x, \bar{p}))v + D_2 ev\ \sigma(u, (x, \bar{p}))(y, w) \\ &= \sigma_v(x, \bar{p}) + D_{(x,\bar{p})}\sigma_u(y, w) \end{aligned}$$

where v is an element of $\Pi^I F$ and (y, w) is an element of $\boldsymbol{R}^{(IL(S+1)} \times T_{\bar{p}}\bar{P}$ which is the tangent space of $X \times \bar{P}$ at $(x, \bar{p})$. Note that an element of $T_{\bar{p}}\bar{P}$ is represented by a vector of the form $(0, a_2, \ldots, a_{L(S+1)})$. In order to prove the claim of the lemma, I need to show that D ev $\sigma(u', (x', \bar{p}'))$ is surjective at an arbitrary point $(u', (x', \bar{p}')) \in ev\ \sigma^{-1}(0)$. To this end, I shall demonstrate that for any element (α, β) of $\boldsymbol{R}^{I(L-1)(S+1)} \times \boldsymbol{R}^{IK}$ which is the tangent space of Y at $0 \in Y$, there exists $v \in \Pi^I F$ and $(y, w) \in \boldsymbol{R}^{(IL(S+1)} \times T_{\bar{p}}\bar{P}$ such that $\sigma_v(x', \bar{p}') + D_{(x',\bar{p}')}\sigma_{u'}(y, w) = (\alpha, \beta)$. In the following, α is written in component form as $(\alpha^i_{sl})_{i=1,\ldots,I,s=0,\ldots,S,l=2,\ldots,L}$ and β as $(\beta^i_k)_{i=1,\ldots,I,k=1,\ldots,K}$.

Step 1. First consider (y, w). It is shown through calculation that the matrix representaion of $D_{(x,\bar{p})}\sigma_u$ amounts to be the following $(I(L-1)(S+1)+IK) \times ((IL(S+1)+L(S+1))$

matrix.

$$\left(\begin{array}{cccc|c} & A_1 & \dots & 0 & \dots \\ & \vdots & \ddots & \vdots & \dots \\ & 0 & \dots & A_I & \dots \\ \hline & [B, -\boldsymbol{r}^t]G_{\bar{p}}[U^0, \dots, U^S]_1 & \dots & 0 & \dots \\ & \vdots & \ddots & \vdots & \dots \\ & 0 & \dots & [B, -\boldsymbol{r}^t]G_{\bar{p}}[U^0, \dots, U^S]_I & \dots \end{array}\right)$$

where A_i is a $(L-1)(S+1) \times L(S+1)$ matrix $(i = 1, \dots, I)$ whose form is

$$\begin{aligned} A_i \quad = \quad & (\frac{u^{i\prime}_{s1,01}}{\bar{p}^s_1} - \frac{u^{i\prime}_{s1,01}}{\bar{p}^s_1}, \dots, \frac{u^{i\prime}_{sl,0L}}{\bar{p}^s_l} - \frac{u^{i\prime}_{s1,0L}}{\bar{p}^s_1}, \dots, \frac{u^{i\prime}_{sl,S1}}{\bar{p}^s_l} - \frac{u^{i\prime}_{s1,S1}}{\bar{p}^s_1}, \\ & \dots, \frac{u^{i\prime}_{sl,SL}}{\bar{p}^s_l} - \frac{u^{i\prime}_{s1,SL}}{\bar{p}^s_1})_{s=0,\dots,S,l=2,\dots,L}, \end{aligned}$$

and $G_{\bar{p}}$ is a diagonal matrix of the form

$$\begin{pmatrix} \frac{1}{\bar{p}^1_1} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\bar{p}^2_1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{\bar{p}^S_1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and finally U^s $(s = 0, \dots, S)$ of $[U^0, \dots, U^S]_i$ is a $(S+1) \times L$ whose form is as follows.

$$\begin{pmatrix} u^{i\prime}_{11,s1} & \dots & u^{i\prime}_{11,sL} \\ \vdots & \vdots & \vdots \\ u^{i\prime}_{S1,s1} & \dots & u^{i\prime}_{S1,sL} \\ u^{i\prime}_{01,s1} & \dots & u^{i\prime}_{01,sL} \end{pmatrix}$$

Through assumption 3.1 and condition (4) of $\mathcal{U}$, each $K \times L(S+1)$ submatrix $[B, -\boldsymbol{r}^t]G_{\bar{p}}[U^0, \dots, U^S]_i$ has a maximal rank K. It is worth noting that the latter condition is only a sufficient condition for $[U^0, \dots, U^S]_i$ to have maximal rank, thus it is unnecessarily tight as such. Take $w = 0$, then there exists an appropriate $\tilde{y} = (\tilde{y}^1, \dots, \tilde{y}^I)$ with $\tilde{y}^i \in \boldsymbol{R}^{L(S+1)}$ $(i = 1, \dots, I)$ such that

$$([B, -\boldsymbol{r}^t]G_{\bar{p}}[U^0, \dots, U^S]_1(\tilde{y}^1)^t, \dots, [B, -\boldsymbol{r}^t]G_{\bar{p}}[U^0, \dots, U^S]_I(\tilde{y}^I)^t) = \beta.$$

Then, $(\tilde{y}, 0)$ is the desired tangent vector.

Step 2. The task here is to choose such a v as can make $\sigma_v(x', \bar{p}')$ equal to the defficiency $\alpha - (A_1(\tilde{y}^1)^t, \ldots, A_I(\tilde{y}^I)^t)$. To this end, pick up and fix any C^r function $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ satisfying (i) $Df_x \neq 0$, $\forall \boldsymbol{x} \in \boldsymbol{R}_+$, and (ii) C^r-boundedness. Then, define a C^r function $v^i : \boldsymbol{R}^{L(S+1)}_{++} \to \boldsymbol{R}$ $(i = 1, \ldots, I)$ given by

$$\begin{aligned} v^i(x^i) &= u^{i\prime}(x^i) + \mu^i_{02} f(x^i_{02}) + \mu^i_{03} f(x^i_{03}) + \ldots + \mu^i_{0L} f(x^i_{0L}) + \mu^i_{12} f(x^i_{12}) + \ldots \\ &\quad + \mu^i_{1L} f(x^i_{1L}) + \ldots + \mu^i_{S2} f(x^i_{S2}) + \ldots + \mu^i_{SL} f(x^i_{SL}) \end{aligned}$$

where $\mu^i_{sl} = \bar{p}^s_l(\alpha^i_{sl} - \sum_{j=0}^{S}\sum_{h=1}^{L}(\frac{u^{i\prime}_{sl,jh}}{\bar{p}^s_l} - \frac{u^{i\prime}_{sl,jh}}{\bar{p}^s_1}))/Df_x(x^i_{sl})$, $s = 0, \ldots, S, l = 2, \ldots, L$ with $\bar{p}^0_1 = 1$. Note that v^i is an element of F since it is obviously C^r-bounded $(i = 1, \ldots, I)$. The desired v is formed of these v^is because $\sigma_v(x', \bar{p}')$ is expressed by the following matrix.

$$\begin{pmatrix} (\frac{u^{i\prime}_{sl}}{\bar{p}^s_l} - \frac{u^{i\prime}_{s1}}{\bar{p}^s_1} + Df_x(x'_{sl})\frac{\mu^i_{sl}}{\bar{p}^s_l})_{i=1,\ldots,I,\ s=0,\ldots,S,\ l=2,\ldots,L} \\ (\sum_{s=1}^{S}(\frac{u^{i\prime}_{s1}}{\bar{p}^s_1})b^k_s - u^{i\prime}_{01} r_k)_{i=1,\ldots,I,\ k=1,\ldots,K} \end{pmatrix}$$

which is equal to

$$\begin{pmatrix} (Df_x(x'_{sl})\frac{\mu^i_{sl}}{\bar{p}^s_l})_{i=1,\ldots,I,\ s=0,\ldots,S,\ l=2,\ldots,L} \\ \boldsymbol{0} \end{pmatrix}$$

Thus, $\sigma_v(x', \bar{p}') + D_{(x',\bar{p}')}\sigma_{u'}(\tilde{y}, 0) = (\alpha, \beta)$.

This lemma leads to the following proposition.

Proposition 3.1 *There exists an open and dense set $\hat{\mathcal{U}}$ of $\mathcal{U}$ such that for any $u \in \hat{\mathcal{U}}$, $\Gamma(u)$ constitutes an $(L + I)(S + 1) - IK - 1$ dimensional submanifold of $X \times \bar{P}$.*

Proof: *By the Transversality Theorem (see Abraham and Robbin (1967), pp.46-48) there is an open and dense set $\hat{\mathcal{U}}$ of $\mathcal{U}$ such that for any $u \in \hat{\mathcal{U}}$, 0 is a regular value of σ_u. Therefore, by the Preimage Theorem (see Guillemin and Pollack (1974), p.21) $\sigma_u^{-1}(0)$ is a submanifold of $X \times \bar{P}$, the dimension of which amounts to be $(L + I)(S + 1) - IK - 1$ (= dim $(X \times \bar{P})$ − dim Y). It is obvious that $\sigma_u^{-1}(0)$ is equal to $\Gamma(u)$.*

Next, consider the generic property of $\Theta(\omega)$. First note that the equations generating $\Theta(\omega)$ include redundant equations. Though there are many ways of excluding them, the way taken here is to omit the market-clearing conditions for the first good at every state. For a given $\omega \in \Omega$, I define a map $\phi_\omega : X \times \bar{P} \times \bar{Z} \to \boldsymbol{R}^{(I+L-1)(S+1)+K}$ given by

$$\phi_\omega(x, \bar{p}, \bar{x}) = \begin{pmatrix} (\bar{\boldsymbol{p}}^0\boldsymbol{\omega}^i_0 - \bar{\boldsymbol{p}}^0\boldsymbol{x}^i_0 - \boldsymbol{r}\bar{\boldsymbol{z}}^i)_{i=1,\ldots,I} \\ (\bar{\boldsymbol{p}}^s\boldsymbol{\omega}^i_s - \bar{\boldsymbol{p}}^s\boldsymbol{x}^i_s + (b^1_s, b^2_s, \ldots, b^k_s)\bar{\boldsymbol{z}}^i)_{i=1,\ldots,I,\ s=1,\ldots,S} \\ (\sum_{i=1}^{I} x^i_{sl} - \sum_{i=1}^{I}\omega^i_{sl})_{l=2,\ldots,L,\ s=0,1,\ldots,S} \\ \sum_{s=1}^{I}\bar{\boldsymbol{z}}^i \end{pmatrix}$$

Lemma 3.2 *For any $\omega \in \Omega$, ϕ_ω is a submersion.*

Proof: *Direct calculation reveals that for any given $\omega \in \Omega$, the derivative of ϕ_ω at any $(x, \bar{p}, \bar{z})$, when represented by a matrix, has partially those $I(S+1) + (L-1)(S+1) + K$ columns that lead to the following square matrix that has no operating part on $T_{\bar{p}}\bar{P}$.*

$$\begin{pmatrix} H_0 & H_1 & \dots & H_S & 0 & J \\ 0 & 0 & \dots & 0 & M & 0 \\ N & 0 & \ddots & 0 & 0 & 0 \\ 0 & N & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E \end{pmatrix}$$

where submatrices H_s, J, M, N and E are as follows. H_s is a $(S+1) \times L$ matrix in which the s-th row is $(-\bar{p}_1^s, \dots, -\bar{p}_L^s)$ and all other entries are 0 $(s = 0, 1, \dots, S)$. Then,

$$J = \begin{pmatrix} -r_1 & \dots & -r_K \\ b_1^1 & \dots & b_1^K \\ \vdots & \ddots & \vdots \\ b_S^1 & \dots & b_S^K \end{pmatrix}.$$

M is a diagonal matrix of order $(I-1)(S+1)$ whose diagonal elements are $(-1, -\bar{p}_1^1, \dots, -\bar{p}_1^S, -1, -\bar{p}_1^1, \dots, -\bar{p}_1^S, \dots, \dots, -1, -\bar{p}_1^1, \dots, -\bar{p}_1^S)$. N is a $(L-1) \times L$ matrix of the following form.

$$N = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Finally E is the identity matrix of order K. Thus, it is easily seen that the above square matrix is non-singular. On the other hand, obviously $T_x\boldsymbol{R}_{++}^{IL(S+1)} = \boldsymbol{R}^{IL(S+1)}$ and $T_{\bar{z}}\bar{Z} = \boldsymbol{R}^{IK}$. So the derivative of ϕ_ω at $(x, \bar{p}, \bar{z})$ is surjective. Since $(x, \bar{p}, \bar{z})$ is taken arbitrarily, the claim follows.

From this lemma, I obtain the following proposition.

Proposition 3.2 *For any $\omega \in \Omega$, $\Theta(\omega)$ constitutes a $(IL + I - 1)(S+1) + IK - K$ dimensional submanifold of $X \times \bar{P} \times \bar{Z}$.*

Proof: *Since 0 is a regular value of ϕ_ω by the above lemma, the claim is immediately followed by the Preimage Theorem.*

It is worth noting that for any $\omega \in \Omega$, $\phi_\omega^{-1}(0) \neq \emptyset$ since for any $\bar{p} \in \bar{P}$, $\phi_\omega(\omega, \bar{p}, 0) = 0$.

Now that I have had generic properties of $\Gamma(u)$ and $\Theta(\omega)$, I proceed to the investigation of $FE_{ex}(u, \omega)$. Since $FE_{ex}(u, \omega) = (\Gamma(u) \times \bar{Z}) \cap \Theta(\omega)$, the structure of $FE_{ex}(u, \omega)$ depends on how $\Theta(\omega)$ and $\Gamma(u) \times \bar{Z}$ intersect each other. To examine the way of the intersection, take any u among $\hat{\mathcal{U}}$ and fix it. Considering proposition 3.1, $\Gamma(u) \times \bar{Z}$ for u is obviously a submanifold of $X \times \bar{P} \times \bar{Z}$ with dimension $(I + L)(S + 1) - 1$. Then, it matters how $\Theta(\omega)$ intersects with $\Gamma(u) \times \bar{Z}$. I consider the problem from the viewpoint of the genericity in ω.

Proposition 3.3 *Given $u \in \hat{\mathcal{U}}$, $\Theta(\omega)$ for almost all ω of Ω intersects transversally with $\Gamma(u) \times \bar{Z}$, where 'almost all' means all except for a closed set of measure 0.*

Proof: *For any $\omega \in \Omega$, define a map $F : \Theta(\omega) \times \boldsymbol{R}_{++}^{IL(S+1)} \to X \times \bar{P} \times \bar{Z}$ given by $F((x, \bar{p}, \bar{z}), \varepsilon) = (x + \varepsilon, \bar{p}, \bar{z})$.*

Step 1. Consider a composite map $\tilde{F} \equiv \Pi \circ F : \Theta(\omega) \times \boldsymbol{R}_{++}^{IL(S+1)} \to X \times \bar{P}$ consisting of a projection map $\Pi : X \times \bar{P} \times \bar{Z} \to X \times \bar{P}$ and F. I show that $\tilde{F}$ is a submersion. It is easily checked by calculation that the matrix representation of the derivative of $\tilde{F}$ at any $((x, \bar{p}, \bar{z}), \varepsilon) \in \Theta(\omega) \times \boldsymbol{R}_{++}^{IL(S+1)}$ is as follows.

$$\begin{pmatrix} E^* & \mathbf{0}(IL(S+1) \times L(S+1)) & \mathbf{0}(IL(S+1) \times IK) & E^* \\ \mathbf{0}(L(S+1) \times IL(S+1)) & E^\dagger & \mathbf{0}(L(S+1) \times IK) & \mathbf{0}(L(S+1) \times IL(S+1)) \end{pmatrix}$$

where E^* is the identity matrix of order $IL(S + 1)$ and $E^\dagger$ is also the identity matrix of order $L(S + 1)$ while $\mathbf{0}(\cdot)$s are all zero matrix with the corresponding size $(\cdot)$. I have to show that the image of $(T_{(x,\bar{p},\bar{z})}\Theta_\omega) \times \boldsymbol{R}^{IL(S+1)}$ by the above matrix as a linear transformation coincides with $\boldsymbol{R}^{IL(S+1)} \times T_{\bar{p}}\bar{P}$, which is the tangent space of $X \times \bar{P}$ at $(x + \varepsilon, \bar{p})$. It is obvious from the construction of the matrix that the image covers $\boldsymbol{R}^{IL(S+1)}$, thus it only remains to see that $T_{\bar{p}}\bar{P}$ is also covered. To this end, consider the construction of $T_{(x,\bar{p},\bar{z})}\Theta_\omega$. It is easily seen that on the one hand $T_{(x,\bar{p},\bar{z})}\Theta_\omega \subset T_xX \times T_{\bar{p}}\bar{P} \times T_{\bar{z}}\bar{Z}$, but on the other hand $T_{(x,\bar{p},\bar{z})}\Theta_\omega = Ker\ D\phi_{\omega,(x,\bar{p},\bar{z})}$ where Ker indicates the kernel. Recall that $D\phi_{\omega,(x,\bar{p},\bar{z})}$ contains a non-singular matrix presented in the proof of lemma 3.2 which has no operating part on $T_{\bar{p}}\bar{P}$. Thus, $T_{(x,\bar{p},\bar{z})}\Theta_\omega$ includes $T_{\bar{p}}\bar{P}$. Since tha operating part of $D\tilde{F}_{((x,\bar{p},\bar{z}),\varepsilon)}$ on $T_{\bar{p}}\bar{P}$ is the identity map by the construction of the above matrix, the image covers $T_{\bar{p}}\bar{P}$.

Step 2. Through the Transversality Theorem (Guillemin and Pollack (1974), p.68) for almost all ε, $\tilde{F}_\varepsilon : \Theta(\omega) \to X \times \bar{P}$ given by $\tilde{F}_\varepsilon(x, \bar{p}, \bar{z}) = \tilde{F}((x, \bar{p}, \bar{z}), \varepsilon)$ is transversal to $\Gamma(u)$. Therefore, $F_\varepsilon : \Theta(\omega) \to X \times \bar{P} \times \bar{Z}$ defined by $F_\varepsilon(x, \bar{p}, \bar{z}) = F((x, \bar{p}, \bar{z}), \varepsilon)$ is also transversal to $\Gamma(\omega) \times \bar{Z}$ for almost all ε. Since $\Theta(\omega)(= \phi_\omega^{-1}(0))$ consists of those $(x, \bar{p}, \bar{z}) \in X \times \bar{P} \times \bar{Z}$ that satisfy

$$\begin{pmatrix} (\bar{\boldsymbol{p}}^0\boldsymbol{\omega}_0^i - \bar{\boldsymbol{p}}^0\boldsymbol{x}_0^i = \boldsymbol{r}\bar{\boldsymbol{z}}^i)_{i=1,\dots,I} \\ (\bar{\boldsymbol{p}}^s\boldsymbol{\omega}_s^i - \bar{\boldsymbol{p}}^s\boldsymbol{x}_s^i = -(b_s^1, b_s^2, \dots, b_s^k)\bar{\boldsymbol{z}}^i)_{i=1,\dots,I,\ s=1,\dots,S} \\ (\sum_{i=1}^I x_{sl}^i = \sum_{i=1}^I \omega_{sl}^i)_{l=2,\dots,L,\ s=0,1,\dots,S} \\ \sum_{s=1}^I \bar{\boldsymbol{z}}^i = 0 \end{pmatrix},$$

it is readly seen that $F_\varepsilon(\Theta(\omega)) = \Theta(\omega+\varepsilon)$. By lemma 3.2, $\Theta(\omega+\varepsilon)$ constitutes a manifold. Thus, it is transversal to $\Gamma(\omega) \times \bar{Z}$ for almost all ε. Since ω is arbitrarily chosen among Ω, the claim follows.

I am now in a position to state the main results.

Theorem 3.1 *Generically in endowments and utility functions, the set of extended equilibria constitutes a $S-K$ dimensional manifold.*

Proof: *The intersection of two transversal submanifolds is itself a manifold, and the codimension of the intersection is the sum of the codimensions of the two submanifolds (see Guillemin and Pollack (1974), p.30, theorem). Thus, by proposition 3.3, for any u of $\hat{\mathcal{U}}$ and almost all ω of Ω, $FE_{ex}(u,\omega)$ constitutes a submanifold in $X \times \bar{P} \times \bar{Z}$ the dimension of which is equal to dim $(X \times \bar{P} \times \bar{Z}) - ($dim $(\Gamma(u) \times \bar{Z}) +$ dim $\Theta(\omega)) = S-K$.*

Note that $S-K$ is known as the deficiency in the financial market (Balasko and Cass (1989), Cass (1992)).

Considering the characteristics of the notion of equilibrium under investigation, some implications are given by the above theorem. First, note that the equilibrium under consideration consists of three factors, namely, an allocation of consumption vector (x), a price vector $(\bar{p})$ and a portfolio allocation $(\bar{z})$, so that the real phase of the equilibrium is clarified by focusing only on the first factor. The set of equilibrium allocations of goods is, hence, represented by the image of the projection of the equilibrium set on X. The theorem implies that the dimension of the image is at most $S-K$. Otherwise stated, real indeterminacy of extended financial equilibria cannot be above the deficiency in the financial market. The same is true for price indeterminacy of equilibria for the same reason. Secondly, note that there is no requirement of convexity in a utility function and that I only consider the necessary ccondition for utility maximization. It is, therefore, intuitively expected that among others the set of equilibrium allocations of goods would be increased to a great extent. However, it is shown by the theorem that the dimension of the set has the maximum $S-K$.

Now, let me turn to another issue, namely, the correspondence between economic parameters and equilibrium. I consider $FE_{ex}(u,\omega)$ to be a correspondence, i.e., $(u,\omega) \to FE_{ex}(u,\omega)$, and invesitgate its structure. To this end, take any u among $\mathcal{U}$ and fix it. I only consider the relation between ω and equilibrium. Let $\Omega(u)$ denote the set of $\omega \in \Omega$ such that $\Gamma(u) \times \bar{Z}$ and $\Theta(\omega)$ are transversal. Note that $\Omega(u)$ is open and dense in Ω. Then, a particular relation is obtainable between $\omega \in \Omega(u)$ and an element of $FE_{ex}(u,\omega)$.

Theorem 3.2 *Given any u of $\mathcal{U}$, then, for any ω' of $\Omega(u)$ and any $(x',\bar{p}',\bar{z}')$ of $FE_{ex}(u,\omega')$ there exists some neighborhoods $N(\omega')$ of ω', $N(x',\bar{p}',\bar{z}')$ of $(x',\bar{p}',\bar{z}')$ and a C^r map $\eta : N(\omega') \to N(x',\bar{p}',\bar{z}')$ such that $\eta(\omega)$ is an element of $FE_{ex}(u,\omega)$ for all $\omega \in N(\omega')$ and $\eta(\omega') = (x',\bar{p}',\bar{z}')$.*

Proof: *Let V denote $\boldsymbol{R}^{(I+L-1)(S+1)+K}$ in what follows. Consider a C^r map $\xi : \Omega(u) \times (X \times \bar{P} \times \bar{Z}) \to Y \times V$ given by $\xi(\omega, (x, \bar{p}, \bar{z})) = (\sigma_u(x, \bar{p}), \phi_\omega(x, \bar{p}, \bar{z}))$. First note that the component $(x', \bar{p}', \bar{z}')$ of any $(\omega', (x', \bar{p}', \bar{z}'))$ satisfying $\xi(\omega, (x, \bar{p}, \bar{z})) = 0$ is an element of $FE_{ex}(u, \omega')$. Then, a C^r map $\xi(\omega', \cdot) : X \times \bar{P} \times \bar{Z} \to Y \times V$ is submersive at $(x', \bar{p}', \bar{z}')$, so that by the Local Submersion Theorem (Guillemin and Pollack (1974), p.20) there exist local coordinates around $(x', \bar{p}', \bar{z}')$ and 0 $(\in Y \times V)$ such that $\xi(\omega', (a_1, \ldots, a_T)) = (a_1, \ldots, a_H)$, where $T = dim(X \times \bar{P} \times \bar{Z}) = (IL + L)(S + 1) + IK - 1$ and $H = dim(Y \times V) = (IL + L - 1)(S + 1) + IK + K$. Since the preimage of 0 by $\xi(\omega', \cdot)$ is nothing but $FE_{ex}(u, \omega')$, $FE_{ex}(u, \omega')$ is locally represented around $(x', \bar{p}', \bar{z}')$ by the set of those points that have coordinates $(0, \ldots, 0, a_{H+1}, \ldots, a_T)$ (note that $T - H = S - K$). Thus, I restrict $\xi(\omega', \cdot)$ on the complement space of $FE_{ex}(u, \omega')$ near $(x', \bar{p}', \bar{z}')$ to obtain the identity map. Then, consider the same restriction of $\xi(\cdot, \cdot)$ with respect to the second argument $(x, \bar{p}, \bar{z})$ near $(\omega', (x', \bar{p}', \bar{z}'))$. It is easily seen that for the restricted ξ, $D_2\xi(\omega', (x', \bar{p}', \bar{z}'))$, i.e., the partial derivative in the second argument at $(\omega', (x', \bar{p}', \bar{z}'))$, is an isomorphism. Hence, through the Implicit Function Theorem there exist (i) an open neighborhood of ω', (ii) a relatively open neighborhood of $(x', \bar{p}', \bar{z}')$ in $X \times \bar{P} \times \bar{Z}$ that does not intersect $FE_{ex}(u, \omega')$ except for $(x', \bar{p}', \bar{z}')$, and (iii) a unique C^r map η such that $\eta(\omega') = (x', \bar{p}', \bar{z}')$ and $\xi(\omega, \eta(\omega)) = 0$. It is obvious by the definition that $FE_{ex}(u, \omega)$ includes $\eta(\omega)$.*

The theorem says that for almost all u, everywhere on the set of ω that makes $FE_{ex}(u, \omega)$ a $S - K$ dimensional manifold, any equilibrium point $(x, \bar{p}, \bar{z})$ of $FE_{ex}(u, \omega)$ is continuous in ω. Roughly speaking, every point included in the extended financial equilibrium set is continuous in initial endowments.

3.4 Concluding remarks

In this chapter, I examined some generic properties of a financial equilibrium set in utility functions as well as initial endowments. The basic result established is that the extended financial equilibrium set generically consitutes a manifold with the dimension less than the deficiency in the financial market and that every point of the set is continuous in initial endowments.

Balasko and Cass (1989) investigated the generic properties of the set of financial equilibrium price vectors and allocations for a given asset price vector and a given return matrix, which is the same framework as adopted in this chapter. They were able to sort out intrinsic financial equilibria to consider because they assumed that every utility function satisfies strict quasi-concavity. The main result they obtained was as follows. That is, generically in endownents the set of financial equilibrium allocations contains a manifold with the dimension equal to the deficiency in the financial market. Comparing their result with the one obtained here, a suggestive fact is drawn out. Namely, even if an agent's utility function is not strictly quasi-concave, for almost all cases real indeterminacy of financial equilibria is not above the deficiency. Note that the set of financial equilibria would be intuitively expected to grow considerably without strict quasi-concavity on a utility function. However, as long as one keeps a generic viewpoint, such a dimensional expansion has been shown not to take place.

In order to establish the results of this chapter, I rely on some assumptions concerning a utility function. Though almost all assumptions are harmless, it seems that C^r boundedness needs to be commented on. This condition is only a technical qualification to facilitate the analysis and not crucial for the results obtained. Indeed, suppose not C^r bounded. Then, only if a consumption vector has an ifinite norm, the corresponding utility could be infinite since C^r-differntiability and strict monotonicity are assumed on a utility function. But, except for an insignificant case of infinity of ω (initial endowments), an allocation including an element of an infinite norm cannot be an equilibrium, thus a non-C^r-bounded utility function can be replaced by some C^r-bounded function without any change in the set of equilibria.

I have only argued the case of fixed asset prices. But the approach provided here has such a wide validity that it can be successfully applied to the case of variable asset prices, though I omit the analysis for the latter case.

In the previous chapter, I established the robustness of indeterminacy caused by nominal assets for introduction of real assets. In this chapter, I also verified the robustness of indeterminacy caused by nominal assets for choice of a utility function.

Chapter 4

Inefficiency of Equilibria with Incomplete Markets I

In contrast to the arguments in the previous chapters, I will discuss in the subsequent chapters another very important issue concerning incomplete markets: that is, efficiency of equilibria. As is well known, equilibria with incomplete markets are generically Pareto inefficient. The arguments here focus on the cause of Pareto inefficiency of equilibria with incomplete markets, showing that Pareto inefficiency of equilibria occurs in incomplete markets in a very different way than in other market failures. That is to say, I reveal in this chapter[1] the leading role of a budget constraint in the occurrence of Pareto inefficiency. Specifically, on the basis of the classical two-period one-good pure exchange model I prove that so long as a budget constraint is met for all agents, equilibria with incomplete markets are generically Pareto inefficient in initial endowments and utility functions regardless of the optimization behavior of each agent. All I require of utility functions is a very weak hypothesis called current monotonicity. A simple unified method is presented which is applicable to both a real asset case and a nominal asset case.

[1]This chapter is based on R.Nagata, "Inefficiency of equilibria with incomplete markets", *Journal of Mathematical Economics*, 41, 887-897, 2005.

4.1 Introduction

It is well known that with incomplete asset markets, the equilibrium allocations need not be Pareto optimal, as Hart (1975) first suggested. Since Hart's work, many efforts have been made to investigate if a competitive outcome with incomplete markets is constrained optimal in some sense. The idea of constrained optimality itself has been formally shaped through the works of Diamond (1967), Stiglitz (1982) and Geanakoplos and Polemarchakis (1986) into the following notion. An allocation with incomplete asset markets is constrained optimal (or constrained Pareto optimal) if and only if it is not Pareto dominated by any other allocations that can be obtained by a social planner who can control only existing asset markets. With respect to this notion, however, it has been shown that except for a restricted case (i.e. the one-good two-period case) competitive equilibria with incomplete asset markets are generically constrained suboptimal, which substantially means that they are typically not constrained optimal (see esp. Geanakoplos and Polemarchakis (1986)). The generic constrained suboptimality has subsequently been investigated in variously elaborated contexts in the literature (see Geanakoplos, Magill, Quinzii and Drèze (1990), Werner (1991), Kajii (1994), Elul (1995, 1999), Cass and Citanna (1998) and Citanna, Kajii and Villanacci (1998)).

In contrast to many works concerning a modified optimality concept of equilibria with incomplete markets, there is only a small amount of literature that deals with Pareto inefficiency of equilibria with incomplete markets itself (Magill and Quinzii (1996a), Villanacci et al. (2002)). To this more fundamental issue, this literature has shown that such allocations are generically Pareto inefficient with respect to the agents' initial endowments with the assumption of concavity of utility functions of agents.

In this chapter, I consider the latter problem from a different viewpoint. The concern I have here is what determines Pareto inefficiency of equilibria with incomplete markets. With respect to this issue, I show that Pareto inefficiency of equilibria occurs in incomplete markets in a very different way than in other market failures. Indeed, it is shown in the case of incomplete markets that such inefficiency is not dependent on the optimization behavior of agents. More specifically, such inefficiency happens to those equilibria not because an objective equilibrium (market clearance) is accompanied by a specific subjective equilibrium (optimization) of each agent but because it is accompanied by a budget constraint of each agent. Alternatively put, so long as a budget constraint is met for all agents, those equilibria are generically inefficient regardless of each agent's optimization behavior based on its own consumption. Thus it may be safely said, though in a generic sense, that once the agents participate in incomplete markets, they are kept away from Pareto optimal allocations before they declare their demand.

In order to prove the claim effectively, I must bear some aspects in mind. One is to adopt less specified utility functions for agents. To this end, I consider a very weak monotonicity called current monotonicity as an assumption. This only requires monotonicity of utility with respect to consumption at present. I do not set other assumptions, particularly any concavity, on utility functions as other authors do. The type of assets should also be concerned. As is well known, assets are conceptually classified into two groups, that is, real assets and nominal assets. A real asset promises to deliver a bundle of goods at each state in the future, whereas a nominal asset promises to deliver a given stream of units of account across the states. It is noteworthy about these two kinds of assets that the structure of the set of equilibrium allocations is very different between them. It is shown, though on the basis of the concavity assumption, that in a real asset model the equilibrium set is generically finite (Duffie and Shafer (1985)) whereas in a nominal asset model there generically exists real indeterminacy of equilibria (Cass (1984, 1985), Werner (1985)). The real indeterminacy of equilibria indicates that the set of equilibrium allocations constitutes a continuum. Moreover, it has been shown that the continuum set generically contains a definite dimensional manifold and that the dimension of the manifold is $S-1$ or $S-J$ according as to whether the nominal asset prices are taken to be endogenous or exogenous (Geanakoplos and Mas-Colell (1989) and Balasko and Cass (1989)), where S indicates the number of states of nature in the future and J indicates the number of assets. In addition, I have shown in the previous chapter that those properties of equilibria in a nominal asset model carry over even when a utility function has nonconvexity. In view of these results, it is naturally inferred that the consequence to be obtained depends on which type the assets under consideration are; thus I have to deal with both cases.

I am going to present a simple unified approach applicable to both a real and a nominal asset model, investigating the viability of efficiency of equilibria with incomplete markets from a generic viewpoint with regard to both utility functions and initial endowments. To make matters simple, the argument is based on the classical two-period one-good pure exchange model which is described in section 2. In this section, as has been noted, I have specified current monotone utility functions and topologized their space by the compact open topology. In addition, in order to unify the real and nominal asset cases, I have considered a specific price matrix through which I have defined an asset market equilibrium. In section 3, after characterizing the Pareto efficient allocations based on current monotone utility functions, I have introduced a specific optimum called a budget optimum, which is the key concept of the following argument. It has been shown by means of a finite dimensional parameterization of utility functions that the set of budget optima is generically empty with regard to initial endowments and utility functions, which implies the generic nonexistence of asset equilibria with Pareto efficiency. This consequence also proves the leading role of a budget constraint in the occurrence of Pareto inefficiency of asset equilibria. Finally, in section 4, I address some implications derived from the results obtained here.

4.2 The Model

The model to be considered here is the simplest two-period one-good pure exchange economy under uncertainty. The reason why I won't consider multi-goods is that Pareto inefficiency has been generically shown to prevail among equilibria even in a one-good economy with incomplete markets (Magill and Quinzii (1996a)). The first and second periods are each specified by $t = 0$ and 1 and one of S states of nature ($s = 1, \dots, S$) occurs at date 1. For simplicity, I call date $t = 0$, state $s = 0$, so that in total there are $S + 1$ states. The economy consists of I consumers ($i = 1, \dots, I$) and a single consumption good. Thus, the commodity space for each consumer is $\boldsymbol{R}^{S+1}$. The characteristics of each agent i consist of three ingredients, that is, a consumption set X^i, a utility function u^i and a initial endowment $\boldsymbol{\omega}^i$. I make assumptions on those ingredients as follows. For each $i, (i = 1, \dots, I)$,

Assumption 4.1 X^i is R^{S+1}_{++}.

Assumption 4.2 u^i satisfies

1. $u^i \in C^r\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big)$ $(r \geq 2)$.
2. current monotonicity: $Du^i_{x_0}(\boldsymbol{x}) \in \boldsymbol{R}_{++}$ for each $\boldsymbol{x} \in \boldsymbol{R}^{S+1}_{++}$

Assumption 4.3 $\boldsymbol{\omega}^i \in \boldsymbol{R}^{S+1}_{++}$.

Note that monotonicity with respect to the good at date 0 is the only requirement for a utility function except for the differentiability. For simplicity, I denote $Du^i_{x_0}(\boldsymbol{x})$ by $Du^i_0(\boldsymbol{x})$ and let $u \equiv (u^1, \dots, u^I)$ and $\omega \equiv (\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^I)$ in the following.

First, I consider the efficiency of allocations. Given u and ω, a Pareto optimal allocation is defined as follows.

Definition 4.1 *An allocation* $x = (\bar{\boldsymbol{x}}^1, \dots, \bar{\boldsymbol{x}}^I) \in \boldsymbol{R}^{(S+1)I}_{++}$ *is a Pareto optimum if*

(i) $\sum_{i=1}^I \bar{\boldsymbol{x}}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$

(ii) *there does not exist* $x = (\boldsymbol{x}^1, \dots, \boldsymbol{x}^I) \in \boldsymbol{R}^{(S+1)I}_{++}$ *such that* $\sum_{i=1}^I \boldsymbol{x}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$ *and* $u^i(\boldsymbol{x}^i) \geq u^i(\bar{\boldsymbol{x}}^i), i = 1, \dots, I$ *with a strict inequality for at least one* i.

Then I work on modeling the market economy with assets. Let p^s be a spot price of the good in state s ($s = 0, 1, \dots, S$). There are J real assets ($j = 1, \dots, J$) in the economy. Since I am interested in the case of incomplete asset markets, I may assume that $J < S$. Each asset j can be purchased for the price q_j at date 0. For simplicity of notation, I set $\boldsymbol{p_1} = (p^1, \dots, p^S)$ in the following. As is well known, the assets are conceptually classified into two groups, that is, real assets and nominal assets. A real asset promises to deliver a bundle of goods at each state in the future, whereas a nominal asset promises to deliver a given stream of units of account across the states. It is noteworthy about these two kinds of assets that the structure of the set of equilibrium allocations is very different among them. In a real asset model, the equilibrium set is shown to be generically finite (Duffie and Shafer

(1985)), whereas in a nominal asset model there generically exists real indeterminacy of equilibria (Cass (1984, 1985), Werner (1985)). I deal with both models, so that if I denote a return of asset j across the states at date 1 by $\boldsymbol{v}^j = (v_1^j, \ldots, v_S^j)$, then each v_s^j indicates a certain amount of good in a real asset model, while it indicates a given units of accounts in a nominal asset model. In the following, I see each $\boldsymbol{v}^j$ ($j = 1, \ldots, J$) as a column vector and combine them to form an $S \times J$ matrix of returns $V = [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^J]$. Let $\mathcal{E}(u, \omega; V)$ denote the economy composed of u, ω and V. I investigate inefficiency of equilibria with incomplete markets from a generic viewpoint with respect to u and ω. Thus the asset structure V is fixed on which I may assume that rank $V = J$ without loss of generality.

Given the asset structure V, each agent has a chance to purchase some amounts of J assets and adjust its income stream so that it can optimize its intertemporal consumptions. Let $\boldsymbol{z}^i = (z_1^i, \ldots, z_J^i) \in \boldsymbol{R}^J$ denote the number of units of the J assets purchased by agent i. $\boldsymbol{z}^i$ is called a portfolio of agent i. In order to unify the real and nominal cases, I consider the following matrix $P \in \boldsymbol{R}^{(S+1)\times(S+1)}$ for given commodity prices $\boldsymbol{p_1} \in \boldsymbol{R}^S_{++}$ at date 1.

$$P = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \frac{1}{p_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \frac{1}{p_S} \end{pmatrix}$$

Then, given asset prices $\boldsymbol{q}$, agent i faces the following optimization problem.

$$\begin{array}{ll} \max\limits_{\boldsymbol{x}^i, \boldsymbol{z}^i} & u^i(\boldsymbol{x}^i) \\ s.t. & \boldsymbol{x}^i - \boldsymbol{\omega}^i = PW\boldsymbol{z}^i, \;\; \boldsymbol{z}^i \in \boldsymbol{R}^J \end{array}$$

where

$$W = \begin{pmatrix} -q_1 & \ldots & -q_J \\ v_1^1 & \ldots & v_1^J \\ \vdots & \vdots & \vdots \\ v_S^1 & \ldots & v_S^J \end{pmatrix}$$

and if there is no further constraint on $\boldsymbol{p_1}$, assets are nominal, but if there is the further constraint $\boldsymbol{p_1} = (1, \ldots, 1) \in \boldsymbol{R}^S$, then assets are real. Note that there is only one good in the economy, so that the good at date 0 is interpreted as the numeraire.

Now, focusing on the asset demand, I define the equilibrium for the economy $\mathcal{E}(u, \omega; V)$ (see Magill and Quinzii (1996a), Chap. 2, §11).

Definition 4.2 *An asset market equilibrium $((\boldsymbol{z}^i)_i, \boldsymbol{p_1}, \boldsymbol{q})$ for $\mathcal{E}(u, \omega; V)$ with $\boldsymbol{p_1} \in \boldsymbol{R}^S_{++}$ is such that*

(i) $\boldsymbol{z}^i$ *solves* $\quad max_{\boldsymbol{z}}\; u^i(\boldsymbol{\omega}^i + PW\boldsymbol{z}), \qquad i = 1, \ldots, I.$

(ii) $\sum_{i=1}^{I} \boldsymbol{z}^i = 0$

It is worth noting in this equilibrium that the assumption of current monotonicity of utility functions not only covers a large variety of optimization behaviors but also leads to a strict budget constraint among agents, resulting in $\boldsymbol{x}^i = \boldsymbol{\omega}^i + PW\boldsymbol{z}^i,\ i = 1, \ldots, I$.

Since u and ω are only parameters that specify the economy, I consider the space that consists of admissible u and ω. Let U be the set of functions satisfying assumption 4.2 and let $\mathcal{U}$ be I-product of U, that is, $\Pi^I U$. Then the space of admissible u and ω is $\mathcal{U} \times \boldsymbol{R}_{++}^{(S+1)I}$, which is called the space of economies. To $\boldsymbol{R}_{++}^{(S+1)I}$ a standard Euclidean topology is given, whereas $C^r\left(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\right)$ is endowed with the compact open C^r topology ($r \geq 2$), which confers the induced topology on U.

4.3 Main Results

I demonstrate that the asset equilibria are generically inefficient regardless of each agent's optimization behavior based on its own consumption. The key for our claim is a budget constraint for each agent. I will aim at showing how inconsistent a Pareto efficient allocation is with budget constraints for all agents. First, I characterize Pareto efficient allocations on the basis of our utility functions.

Proposition 4.1 *Given u and ω fulfilling assumption 4.2 and 4.3 and let an allocation $\bar{x} = (\bar{\boldsymbol{x}}^1, \ldots, \bar{\boldsymbol{x}}^I) \in \boldsymbol{R}_{++}^{(S+1)I}$ be Pareto optimal. Then the following equation holds.*

$$Du^1_{\mathbf{1}}(\bar{\boldsymbol{x}}^1)/Du^1_0(\bar{\boldsymbol{x}}^1) = \cdots = Du^I_{\mathbf{1}}(\bar{\boldsymbol{x}}^I)/Du^I_0(\bar{\boldsymbol{x}}^I)$$

where $Du^i_{\mathbf{1}}(\bar{\boldsymbol{x}}^i) \equiv (Du^i_1(\bar{\boldsymbol{x}}^i), \ldots, Du^i_S(\bar{\boldsymbol{x}}^i))$.

Proof: *First note that the condition makes sense since $Du^i_0(\bar{\boldsymbol{x}}^i) > 0$ according to assumption 2 ($i = 1, \ldots, I$). I denote $Du^i_s(\bar{\boldsymbol{x}}^i)$ by λ^i_s for simplicity of notation in the following ($i = 1, \ldots, I, s = 0, \ldots, S$). Suppose that some agents h, k have $\frac{\lambda^h_s}{\lambda^h_0} > \frac{\lambda^n_s}{\lambda^n_0}$ in some state s. I need to classify two cases since no sign condition is set on any λ^i_s but λ^i_0. (1) $\lambda^h_s > 0$. Let $\Delta \in \boldsymbol{R}^{S+1}$ be the direction $(-\frac{\lambda^h_s}{\lambda^h_0}, 0, \ldots, 0, (1+\epsilon), 0, \ldots, 0)$ where $(1+\epsilon)$ is in the $s+1-th$ place and $\epsilon > 0$. Then the directional derivative of u^h at $\bar{\boldsymbol{x}}^h$ in the direction of Δ is $-\lambda^h_s + (1+\epsilon)\lambda^h_s = \epsilon\lambda^h_s$, which is strictly positive. On the other hand, the directional derivative of u^n at $\bar{\boldsymbol{x}}^n$ in the direction of $-\Delta$ is $\frac{\lambda^h_s}{\lambda^h_0}\lambda^n_0 - (1+\epsilon)\lambda^n_s$ which is also strictly positive if $\epsilon (> 0)$ is small enough. Thus, by definition of derivative as a limit, transferring a small enough fraction of Δ increases the welfare of both agents (h, k) without any change for all others, which is a contradiction. (2) $\lambda^h_s \leq 0$. In this case, obviously $\lambda^n_s < 0$. Let $\Delta \in \boldsymbol{R}^{S+1}$ be the direction $(\frac{\lambda^n_s}{\lambda^n_0}, 0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is in the $s+1-th$ place. Then, the directional derivative of u^n at $\bar{\boldsymbol{x}}^n$ in the direction of Δ is $\lambda^n_s - \lambda^n_s = 0$. however, the* *directional derivative of u^h at $\bar{\boldsymbol{x}}^h$ in the direction of $-\Delta$ is $-\frac{\lambda_s^n}{\lambda_0^n}\lambda_0^h + \lambda_s^h > -\frac{\lambda_s^h}{\lambda_0^h}\lambda_0^h + \lambda_s^h = 0$. Thus, according to the same argument as in* (1)*, I have the situation that agent h becomes better off without any change for all others, which is also a contradiction.*

In the following, I investigate the generic analysis with respect to initial endowments and utility functions for the agents, the latter among which causes an analytical complexity since the space $\mathcal{U}$ is itself infinite dimensional. In order to cope with this problem, I adopt a simplification method called the perturbation technique to look at a finite dimensional subset of $\mathcal{U}$ (for the perturbation technique, see Cass and Citanna (1998), Citanna, A., A. Kajii and A. Villanacci (1998)).

Fix an arbitrary $u \in \mathcal{U}$ and consider the following function parameterized by $\boldsymbol{\mu}^i \in \boldsymbol{R}^S$.

$$u^i(\boldsymbol{x}^i; \boldsymbol{\mu}^i) = u^i(\boldsymbol{x}^i) + \boldsymbol{\mu}^i \cdot \boldsymbol{x}^i_{\mathbf{1}}.$$

Since $u^i(\boldsymbol{x}^i; 0) = u^i(\boldsymbol{x}^i)$, I choose an open neighborhood of zero in $\boldsymbol{R}^S$, say a unit open ball B. Noting that $Du^i_0(\boldsymbol{x}^i; \boldsymbol{\mu}^i) = Du^i_0(\boldsymbol{x}^i)$, $u^i(\boldsymbol{x}^i; \boldsymbol{\mu}^i)$ is an element of U for any $\boldsymbol{\mu}^i \in B$ according to assumption 4.2. Thus, if I denote $(u^i(\boldsymbol{x}^i; \boldsymbol{\mu}^i))_{i=1}^I$ by $u(x; \mu)$, $u(x; \mu)$ is an element of $\mathcal{U}$ for any $(\boldsymbol{\mu}^i)_i \in \Pi^I B$. Let $\mathcal{B} \equiv \Pi^I B$. Then, regarding $u(x; \mu)$ defined on $\boldsymbol{R}^{(S+1)I}_{++} \times \mathcal{B}$, it is easily seen that if μ_n is a sequence of vectors in $\mathcal{B}$ such that $\lim_{n\to\infty} \mu_n = \bar{\mu} \in \mathcal{B}$, $\lim_{n\to\infty} u^i(\cdot; \boldsymbol{\mu}^i_n) = u^i(\cdot; \bar{\boldsymbol{\mu}}^i_n)$ in the compact open topology ($i = 1, \ldots, I$). Thus, $\mathcal{B}$ can be seen as a finite dimensional submanifold of $\mathcal{U}$.

It follows from the construction that for each i

$$\frac{Du^i_{\mathbf{1}}(\boldsymbol{x}^i; \boldsymbol{\mu}^i)}{Du^i_0(\boldsymbol{x}^i; \boldsymbol{\mu}^i)} = \frac{Du^i_{\mathbf{1}}(\boldsymbol{x}^i) + \boldsymbol{\mu}^i}{Du^i_0(\boldsymbol{x}^i)}.$$

Now I consider the following situation. That is, all agents are supposed to participate in the incomplete markets to trade, though the only thing required of them is to strictly fulfill their budget constraints. I inquire whether a feasible allocation under this condition will be Pareto efficient or not. To this end, let me call a Pareto optimal allocation under this condition a budget optimal allocation (or a budget optimum for short), which is formally defined as follows.

Definition 4.3 *A budget optimum $((\boldsymbol{z}^i)_i, \boldsymbol{p}_{\mathbf{1}}, \boldsymbol{q})$ for $\mathcal{E}(u, \omega; V)$ with $\boldsymbol{p}_{\mathbf{1}} \in \boldsymbol{R}^S_{++}$ is such that*

(i) $\boldsymbol{\omega}^i + PW\boldsymbol{z}^i \in \boldsymbol{R}^{S+1}_{++}$, $i = 1, \ldots, I$

(ii) $(\boldsymbol{\omega}^i + PW\boldsymbol{z}^i)_i$ *constitutes a Pareto optimum*

(iii) $\sum_{i=1}^I \boldsymbol{z}^i = 0$

Note that $\boldsymbol{\omega}^i + PW\boldsymbol{z}^i$ in (i) of the definition forms a permissible demand vector $\boldsymbol{x}^i$ for each i by virtue of current monotonicity of utility functions, which immediately implies the fulfilment of a budget constraint for each agent. In addition, (iii) of the definition assures the feasibility for a Pareto optimum.

It is obvious that in order for an asset market equilibrium to be a Pareto optimum, it must be also a budget optimum. On the other hand, compatibility of a Pareto efficient allocation with budget constraints for all agents is dependent upon whether the set of budget optima is empty or not. To this question, I obtain a decisive result as follows.

Theorem 4.1 *Under assumptions 4.1∼ 4.3, if $I(S-J) > S$ with real assets and $I(S-J) > 2S$ with nominal assets, then generically in ω and u, there exists no budget optimum, thus every asset market equilibrium is Pareto inefficient.*

Proof: *It is easily seen that for an arbitrary economy $\mathcal{E}(u,\omega;V)$ a tuple $((\boldsymbol{z}^i)_i, \boldsymbol{p}_{\boldsymbol{1}}, \boldsymbol{q})$ is a budget optimum if and only if there exists $(\boldsymbol{x}^i)_i \in \boldsymbol{R}_{++}^{(S+1)I}$ such that*

$$
\begin{aligned}
\frac{Du^i_{\boldsymbol{1}}(\boldsymbol{x}^i)}{Du^i_0(\boldsymbol{x}^i)} - \frac{Du^I_{\boldsymbol{1}}(\boldsymbol{x}^I)}{Du^I_0(\boldsymbol{x}^I)} &= 0 \qquad i = 1, \dots, I-1 \\
\textstyle\sum_{i=1}^I \boldsymbol{z}^i &= 0 \\
\boldsymbol{x}^i - (\boldsymbol{\omega}^i + PW\boldsymbol{z}^i) &= 0 \qquad i = 1, \dots, I.
\end{aligned}
$$

Regarding this u of the economy, I construct the open set $\mathcal{B} \subset \boldsymbol{R}^{SI}$ according to the method explained above. Then, I may locally confine utility functions to this finite dimensional subset by means of the parameterization of those functions. Thus, locally I am allowed to be only concerned with $u(x;\mu)$, $\mu \in \mathcal{B}$ as conceivable utility functions.

In view of this parameterization, I can naturally consider the map $F : \boldsymbol{R}_{++}^{(S+1)I} \times \boldsymbol{R}^{JI} \times \boldsymbol{R}_{++}^{S} \times \boldsymbol{R}^J \times \boldsymbol{R}_{++}^{(S+1)I} \times \mathcal{B} \to \boldsymbol{R}^{S(I-1)} \times \boldsymbol{R}^J \times \boldsymbol{R}^{(S+1)I}$ given by

$$
F((\boldsymbol{x}^i, \boldsymbol{z}^i)_i, p_{\boldsymbol{1}}, \boldsymbol{q}; \omega, \mu) = \begin{pmatrix} (\frac{Du^i_{\boldsymbol{1}}(\boldsymbol{x}^i;\boldsymbol{\mu}^i)}{Du^i_0(\boldsymbol{x}^i);\boldsymbol{\mu}^i} - \frac{Du^I_{\boldsymbol{1}}(\boldsymbol{x}^I;\boldsymbol{\mu}^I)}{Du^I_0(\boldsymbol{x}^I;\boldsymbol{\mu}^I)} \\ \sum_{i=1}^I \boldsymbol{z}^i \\ (\boldsymbol{x}^i - (\boldsymbol{\omega}^i + PW\boldsymbol{z}^i))_{i=1}^I \end{pmatrix}
$$

I shall show that F is transversal to zero. For simplicity of notation, put $f_i((\boldsymbol{x}^i)_i, \mu) \equiv \frac{Du^i_{\boldsymbol{1}}(\boldsymbol{x}^i;\boldsymbol{\mu}^i)}{Du^i_0(\boldsymbol{x}^i);\boldsymbol{\mu}^i} - \frac{Du^I_{\boldsymbol{1}}(\boldsymbol{x}^I;\boldsymbol{\mu}^I)}{Du^I_0(\boldsymbol{x}^I;\boldsymbol{\mu}^I)}$, $i = 1, \dots, I-1$, $z \equiv \sum_{i=1}^I \boldsymbol{z}^i$ and $g_i((\boldsymbol{x}^i, \boldsymbol{z}^i)_i, p_{\boldsymbol{1}}, \boldsymbol{q}, \omega) \equiv \boldsymbol{x}^i - (\boldsymbol{\omega}^i + PW\boldsymbol{z}^i)$, $i = 1, \dots, I$. Noting that

$$
f_i((\boldsymbol{x}^i)_i, \mu) = \frac{Du^i_{\boldsymbol{1}}(\boldsymbol{x}^i)}{Du^i_0(\boldsymbol{x}^i)} - \frac{Du^I_{\boldsymbol{1}}(\boldsymbol{x}^I)}{Du^I_0(\boldsymbol{x}^I)} + \frac{1}{Du^i_0(\boldsymbol{x}^i)}\boldsymbol{\mu}^i - \frac{1}{Du^I_0(\boldsymbol{x}^I)}\boldsymbol{\mu}^I
$$

the computation of the derivative at any $((\bar{\boldsymbol{x}}^i, \bar{\boldsymbol{z}}^i)_i, \bar{p}_{\mathbf{1}}, \bar{\boldsymbol{q}}; \bar{\omega}, \bar{\mu}) \in F^{-1}(0)$ *is described as follows.*

$$
\begin{array}{c|ccccccccccccccc}
 & \boldsymbol{x}^1 & \cdots & \boldsymbol{x}^I & \boldsymbol{z}^1 & \cdots & \boldsymbol{z}^I & p_{\mathbf{1}} & \boldsymbol{q} & \boldsymbol{\omega}^1 & \cdots & \boldsymbol{\omega}^I & \boldsymbol{\mu}^1 & \cdots & \boldsymbol{\mu}^{I-1} & \boldsymbol{\mu}^I \\
\hline
f_1 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 I^S & 0 & 0 & * \\
\vdots & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & * \\
f_{I-1} & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{I-1} I^S & * \\
z & 0 & 0 & 0 & I^J & \cdots & I^J & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_1 & I^{S+1} & 0 & 0 & * & 0 & 0 & * & * & -I^{S+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & \ddots & 0 & 0 & \ddots & 0 & * & * & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
g_I & 0 & 0 & I^{S+1} & 0 & 0 & * & * & * & 0 & 0 & -I^{S+1} & 0 & 0 & 0 & 0
\end{array}
$$

where I^n *designates the identity matrix of order* n *and* $a_i = \frac{1}{Du_0^i(\bar{\boldsymbol{x}}^i)} (> 0), i = 1, \dots, I-1.$

By picking out the submatrix corresponding to $\boldsymbol{z}^I, \omega, \boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^{I-1}$ *from the above one, I obtain the following square matrix of order* $S(I-1) + J + (S+1)I$.

$$
\begin{array}{c|ccccccc}
 & \boldsymbol{z}^I & \boldsymbol{\omega}^1 & \cdots & \boldsymbol{\omega}^I & \boldsymbol{\mu}^1 & \cdots & \boldsymbol{\mu}^{I-1} \\
\hline
f_1 & 0 & 0 & 0 & 0 & a_1 I^S & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\
f_{I-1} & 0 & 0 & 0 & 0 & 0 & 0 & a_{I-1} I^S \\
z & I^J & 0 & 0 & 0 & 0 & 0 & 0 \\
g_1 & 0 & -I^{S+1} & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
g_I & \dagger & 0 & 0 & -I^{S+1} & 0 & 0 & 0
\end{array}
$$

where $\dagger$ *denotes the following matrix.*

$$
\begin{pmatrix}
\bar{q}_1 & \cdots & \bar{q}_J \\
-v_1^1 & \cdots & -v_1^J \\
\vdots & \vdots & \vdots \\
-v_S^1 & \cdots & -v_S^J
\end{pmatrix}.
$$

Thus, it is easily seen that the submatrix is non-singular, which implies that F *is transversal to zero. Besides, the natural projection* $((\boldsymbol{x}^i, \boldsymbol{z}^i)_i, p_{\mathbf{1}}, \boldsymbol{q}; \omega, \mu) \mapsto (\omega, \mu)$ *is obviously proper. Thus, through the transversality theorem, generically in* $(\omega, \mu) \in \boldsymbol{R}_{++}^{(S+1)I} \times \mathcal{B}$, $F(\cdot, \cdot, \cdot; \omega, \mu) : \boldsymbol{R}_{++}^{(S+1)I} \times \boldsymbol{R}^{JI} \times \boldsymbol{R}_{++}^S \times \boldsymbol{R}^J \to \boldsymbol{R}^{S(I-1)} \times \boldsymbol{R}^J \times \boldsymbol{R}^{(S+1)I}$ *is also transversal to zero. Hence, the preimage of zero through the latter map, which includes the set of budget optima, constitutes a differentiable manifold whose dimension, denoted by* d, *is as follows. If assets are real,* $d = (S+1)I + JI + 0 + J - (J + S(I-1) + (S+1)I) = I(J-S) + S$, *while if assets are nominal, then* $d = (S+1)I + JI + S + J - (J + S(I-1) + (S+1)I) = I(J-S) + 2S$. *however, according to the hypotheses, either proves to be negative, which implies that there is no budget optimum in each case. Since* u *is arbitrarily chosen, this completes the proof.*

It follows from this theorem that so long as each agent keeps its budget constraint, a resulting asset market equilibrium is generically Pareto inefficient regardless of the agent's optimization behavior.

4.4 Concluding Remarks

Some implications are obtainable from the previous arguments. As I have shown above, so long as each agent trades on incomplete asset markets keeping its budget constraints, Pareto optimal allocations are hardly attainable *whether it is engaged in the individual optimization or not.* This consequence implies that I can dispense with any concavity assumption on a utility function for each agent. If a concavity assumption is required for a utility function at all, it will be mainly because some significant demand function is obtained by the assumption. Thus I can say through the above claim that the agents are kept away from Pareto optimal allocations before they declare their demand. Otherwise stated, in order to obtain generic inefficiency of equilibrium allocations with incomplete markets, I do not even need any demand behavior of an agent. Incidentally, if I additionally require the subjective equilibrium on the basis of concave utility functions, then a resulting asset market equilibrium is generically Pareto inefficient under a weaker condition concerning the number of agents, assets and states of nature.

Another suggestive remark can be made about the likelihood of inefficiency of equilibrium allocations. When $J = S$, then the incomplete markets model turns into a complete markets model, so that every equilibrium allocation is Pareto optimal by the first fundamental theorem of welfare economics. Therefore, even if the asset markets are incomplete, the likelihood of inefficiency of equilibrium allocations is intuitively expected to be dependent on how much J differs from S. It is thereby conjectured that equilibrium allocations with incomplete markets are more likely to be Pareto optimal when J is close to S than when J is far below S. But it is revealed from the above argument that such an intuition is wrong. As long as $I > S$ in the real asset case and $I > 2S$ in the nominal asset case, resulting equilibrium allocations are generically inefficient regardless of the difference between S and J. This consequence reinforces one of Lipsey and Lancaster's points, that once a distortion takes place, amplifying it does not necessarily exacerbate the situation in a Pareto sense (Lipsey and Lancaster (1956)).

The third and last remark is concerned with a nominal asset model. It has been shown, though on the basis of concave utility functions, that in a nominal asset model the structure of the set of equilibrium allocations is different between endogenous asset prices and exogenous asset prices. That is to say, generically the set consists partly of an $S-1$ dimensional manifold in the former case (Geanakoplos and Mas-Colell (1989)), whereas it partly constitutes an $S-J$ dimensional manifold in the latter case (Balasko and Cass (1989)). Moreover, those properties of equilibria have been shown to carry over even when one does not require any concavity but only monotonicity of utility functions (see chapter 3 of this dissertation). Considering these facts, it may be conjectured that the consequence obtained here depends on whether the asset prices are endogenous or exogenous. The difference does not alter the intrinsic claim described above but slightly change a condition concerning the number of agents, assets and states of nature in the theorem. Actually, it is easily verified that the map F considered in the proof of the theorem is transversal to zero whether the asset prices are endogenous or exogenous. I have considered them to be endogenous in the text, whereas if they are exogenous, then the hypothesis in the theorem should be altered to $(S - J)(I - 1) > S$.

Chapter 5

Inefficiency of Equilibria with Incomplete Markets II

This chapter is devoted to the mathematical erabolation of the arguments in the previous chapter, which is motivated by the improvement of the treatment for function spaces. It also turns out that this sophistication improves the result of the previous chapter in some respects, e.g., the condition concerning the number of agents, goods and states of nature, which is necessary for the desired outcome. Specifically, I replace the compact open topology with the Whitney topology, which is more appropriate for the space of utility functions. However, such a refinement necessitates abandonment of the foregoing approach since the Whitney topology does not permit the use of the perturbation technique. Thus, I develop a very different way of proving the generic inefficiency of equilibria. That is to say, in order to reach the goal, I make use of the geometrical relation of two specific sets: one is the set dependent on initial endowments which contains all Pareto efficient allocations, and the other is the set dependent on utility functions which contains all equilibrium allocations. Then, possibility of efficiency of financial equilibria depends on whether these two sets intersect or not. I shall show that through two kinds of transversality theorem their intersection is generically empty, which implies the desired result, i.e., the generic inefficiency of equilibria. In the process of the argument, another mathematical tool, i.e., fiber bundle, is effectively used particularly for a nominal asset case.

5.1 Introduction

In the previous chapter, I demonstrated the generic inefficiency of equilibria with incomplete markets by checking compatibility of Pareto efficiency with budget constraints. Here in this chapter, I approach the same issue, i.e., Pareto inefficiency of equilibria with incomplete markets, from a different angle, which is caused by a mathematical refinement of the treatment for function spaces. Specifically, I consider the Whitney topology for the space of utility functions, though I endowed the space with the compact open topology in the previous chapter. It is true that the compact open topology has nice features: for instance it has a complete metric and a countable base, moreover, it allows the perturbation technique described in the previous chapter so that one has only to look at a finite dimensional subset of the space. But, noting that the domain ($\boldsymbol{R}^{S+1}_{++}$) of a utility function is not compact, that topology is not adequate since it does not control the behavior of a function "at infinity" very well. In other words, the generic outcome in chapter 4 only holds under the condition that nothing but a local behavior of a function is covered. On the contrary, if one wishes to cover a global behavior of a function for the genericity analysis, one cannot but resort to the Whitney topology. Thus, I consider here the Whitney topology for the space of utility functions.

However, the cost of substituting the Whitney topology for the compact open topology is not small. Above all, I have to abondan the finite dimensional transformation through the perturbation technique, which implies that the approach previously provided is not valid any more. Therefore, in this chapter, I adopt another method which is similar in spirit to the one described in chapter 3. The basic idea is as follows. First, separate two types of parameters: namely, utility functions (u) and initial endowments (ω). Then, it turns out that on the one hand there exists an appropriate set ($A(u)$) of allocations which depends only on utility functions *and* includes all Pareto efficient allocations, on the other hand there exists an appropriate set ($F(\omega)$) of allocations which depends only on initial endowments *and* includes all financial equilibrium allocations. It follows from this formalization that I have only to check the intersection of these two sets in order to examine possibility of efficient financial equilibrium with incomplete markets.

There are some merits in using this formula. First, this formula is independent on choice of a utility function. Indeed, I still consider a utility function free from convexity. Second, following this procedure, I can make use of the Thom transversality theorem which makes it possible to perform the genercity analysis for the the space of utility functions with the Whitney topology. Third, this formula admits both real and nominal assets, though the latter assets need an extra mathematical treatment through a fiber bundle. Lastly, what may be the most important in economical sense, I can improve the result previously obtained in some respects by means of this formula. In particular, it is shown through this formula that the condition to assure the generic inefficiency of equilibria is relaxed.

In section 2, I describe the model to be considered, which is substantially the same as the one in the previous chapter. Namely, I have chosen the one-good two-period model, based on a pure excange economy, where only monotonicity is assumed on each agent's utility function. In section 3, after presenting a specific method for examining inefficiency of equilibria with incomplete markets, I apply it to both a real asset model and a nominal asset model. The analytical key in the method consists in using two kinds of transversality theorem, the Thom transversality theorem and the standard transversality theorem, alternately to attain generic impossibility of efficient equilibrium allocations with incomplete markets. As a result, for both real asset and nominal asset models, I show that equilibrium allocations with incomplete markets are generically inefficient with regard to utility functions and intial endowments for agents. Finally, in section 4, I address some economical implications derived from the arguments provided here.

5.2 The Model

The model to be considered here is substantially the same as the one in the previous chapter: namely, the simplest one-good two-period exchange economy under uncertainty. The first and second period are each specified by $t = 0$ and 1 and one of S states of nature ($s = 1, \dots, S$) occurs at date 1. For simplicity, I call date $t = 0$, state $s = 0$, so that in total there are $S + 1$ states. The economy consists of I consumers ($i = 1, \dots, I$) and a single consumption good. Thus, the commodity space for each consumer is $\boldsymbol{R}^{S+1}$. The characteristics of each agent i consist of three ingredients, that is, its consumption set X^i, its utility function u^i and its initial endowment $\boldsymbol{\omega}^i$. I make assumptions on those ingredients as follows. For each $i, (i = 1, \dots, I)$,

Assumption 5.1 X^i is R^{S+1}_{++}.

Assumption 5.2 u^i satisfies

1. $u^i \in C^\infty\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big)$.
2. $Du^i(\boldsymbol{x}) \in \boldsymbol{R}^{S+1}_{++}$ for each $\boldsymbol{x} \in \boldsymbol{R}^{S+1}_{++}$.

Assumption 5.3 $\boldsymbol{\omega}^i \in \boldsymbol{R}^{S+1}_{++}$.

As in the previous chapter, I take both utility functions and initial endowments among agents as the parameters characterizing an economy. Note that monotonicity is required of a utility function, though current monotonicity is the only requirement in the previous chapter. Just as previously, let $u = (u^1, \dots, u^I)$ and $\omega = (\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^I)$ in the following.

First consider efficiency of allocations. Given u and ω, a Pareto optimal allocation is defined as follows.

Definition 5.1 *An allocation* $x = (\bar{\boldsymbol{x}}^1, \dots, \bar{\boldsymbol{x}}^I) \in \boldsymbol{R}^{(S+1)I}_{++}$ *is a Pareto optimum if*

(i) $\sum_{i=1}^I \bar{\boldsymbol{x}}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$

(ii) *there does not exist* $x = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^I) \in \boldsymbol{R}_{++}^{(S+1)I}$ *such that* $\sum_{i=1}^I \boldsymbol{x}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$ *and* $u^i(\boldsymbol{x}^i) \geq u^i(\bar{\boldsymbol{x}}^i), i = 1, \ldots, I$ *with a strict inequality for at least one* i.

Let p^s be a spot price of the good in state s $(s = 0, 1, \ldots, S)$ as before. I follow the assumption on the number of assets of the former arguments; namely, there are J assets $(j = 1, \ldots, J)$ in the economy. I may assume that $J < S$ because of my interest in incomplete markets. Let q_j be the price of asset j at date 0 as before. I make the following assumptions on the asset prices and spot prices of the good.

Assumption 5.4 $\quad \boldsymbol{p} \in \boldsymbol{R}_{++}^{S+1}, \ \boldsymbol{q} \in \boldsymbol{R}_{++}^J$
where $\boldsymbol{p} = (p^0, p^1, \ldots, p^S)$ and $\boldsymbol{q} = (q_1, \ldots, q_J)$.

For simplicity of notation, I set $\boldsymbol{p_1} = (p^1, \ldots, p^S)$ as before.

As I have stated before, the assets are conceptually classified into two groups, that is, real assets and nominal assets. A real asset promises to deliver a bundle of goods at each state in the future, whereas a nominal asset promises to deliver a given stream of units of account across the states. Regarding returns of these assets, I will use the same notaion as before; namely, let $\boldsymbol{v}^j = (v_1^j, \ldots, v_S^j)$ be a return vector of asset j, $j = 1, \ldots, J$, where v_s^j is interpreted as a certain amount of good in a real asset model while it is regarded as a given units of accounts in a nominal asset model. In addition, just as before, I see each $\boldsymbol{v}^j$ $(j = 1, \ldots, J)$ as a column vector and combine them to form $S \times J$ matrix of returns $V = [\boldsymbol{v}^1, \ldots, \boldsymbol{v}^J]$. Let $\mathcal{E}(u, \omega; V)$ denote the economy composed of u, ω and V. The object of this chapter is the same as the one in chapter 4. That is to say, I investigate inefficiency of equilibria with incomplete markets from a generic viewpoint with respect to u and ω. Thus the asset structure V is fixed on which I may assume that rank $V = J$ without loss of generality.

Given the asset structure V, each agent has a chance to purchase some amounts of J assets and adjust its income stream so that it can optimize its intertemporal consumptions. Let $\boldsymbol{z}^i = (z_1^i, \ldots, z_J^i) \in \boldsymbol{R}^J$ be a portfolio of agent i. Then the problem the agent has to solve in a *nominal* asset case is as follows.

$$\begin{aligned} \max_{\boldsymbol{x}^i, \boldsymbol{z}^i} \quad & u^i(\boldsymbol{x}^i) \\ s.t. \quad & x_0^i = \omega_0^i - \boldsymbol{q} \cdot \boldsymbol{z}^i, \ \boldsymbol{z}^i \in \boldsymbol{R}^J \qquad \ldots\ldots (*) \\ & p^s x_s^i = \sum_{j=1}^J v_s^j z_j^i + p^s \omega_s^i, \ s = 1, \ldots, S. \end{aligned}$$

Note that the good at date 0 is interpreted as a numeraire. If I put $p^s = 1, \ s = 1, \ldots, S$ in the above, then I have the problem of a real asset case because a real asset j, $j = 1, \ldots, J$ provides a nominal return of $p^s v_s^j$ at each state s, $s = 1, \ldots, S$ per one unit. In this sense, I can consider a real asset model to be a special case of a nominal asset model, so that I shall confine ourselves to a nominal asset case for the time being.

Now I define the equilibrium for the economy $\mathcal{E}(u, \omega; V)$ which is the same as before.

Definition 5.2 *An asset market equilibrium for $\mathcal{E}(u, \omega; V)$ is a tuple $((\boldsymbol{x}^i, \boldsymbol{z}^i)_i, \boldsymbol{p_1}, \boldsymbol{q})$ such that*

(i) $(\boldsymbol{x}^i, \boldsymbol{z}^i)$ *is a solution of the problem* $(*)$, $i = 1, \ldots, I$

(ii) $\sum_{i=1}^{I} \boldsymbol{x}^i = \sum_{i=1}^{I} \boldsymbol{\omega}^i$

(iii) $\sum_{i=1}^{I} \boldsymbol{z}^i = 0$

I follow the same notation for the space of economies. Namely, let U be the set of functions satisfying assumption 5.2 and let $\mathcal{U}$ be I-product of U, that is, U^I. Then the space of admissible u and ω is $\mathcal{U} \times \boldsymbol{R}_{++}^{(S+1)I}$, which is the space of economies. To $\boldsymbol{R}_{++}^{(S+1)I}$ a standard Euclidean topology is given, whereas $C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$ is endowed with the Whitney C^∞ topology. Then, it can be shown that U has a specific property as a subset of $C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$.

Proposition 5.1 *U is an open subset of $C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$ in the Whitney C^∞ topology.*

Proof: *Let $J^1\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$ be the 1-jet space generated by $C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$ (for the r-jet space see chapter 6). Consider the set W defined by $\{(\boldsymbol{a}, b, c_1, \ldots, c_{S+1}) \in J^1\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big) \mid c_i > 0, i = 1, \ldots, S+1\}$. Then obviously $U = \{f \in C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big) \mid J^1 f(\boldsymbol{R}_{++}^{S+1}) \subset W\}$. Noting that $J^1\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$ is substantially equal to $\boldsymbol{R}_{++}^{S+1} \times \boldsymbol{R} \times \boldsymbol{R}^{S+1}$, it is obvious that W is open in $J^1\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$, which implies that U is an open subset of $C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)$ (Golubitsky and Guillemin (1973), p.42).*

Thus, $\mathcal{U}$ is an open subset of a product topological space $C^\infty\big(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R}\big)^I$.

Before starting the investigation, I should give a brief explanation as to how our arguments will proceed in the following. I consider two specific sets of allocations for any given economy. One is the set that contains all efficient allocations. The other one is the set that includes all incomplete market equilibrium allocations. Thus, the intersection of these sets, if any, could contain any allocation that is an efficient equilibrium with incomplete markets. In other words, if there exists an incomplete market equilibrium that is also efficient, these two sets must intersect. First I am going to investigate from the generic viewpoint how these two sets behave with respect to the economic parameters. Then I shall consider the possibility of their intersection from the viewpoint of transversality.

5.3 Main results

5.3.1 Generic property of A(u)

First, I characterize the set that contains all efficient allocations. In order to obtain meaningful conditions, I add a harmless assumption regarding utility functions, which is only needed for the technical purpose.

Assumption 5.5 there exists an $i \in \{1, \ldots, I\}$ such that at every $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^I)$ satisfying $u^j(\boldsymbol{x}^j) = a_j \ (\forall j \neq i)$, $\sum_{i=1}^I \boldsymbol{x}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$ for any $a_j (j \neq i) \in \boldsymbol{R}$, the set $\{(\boldsymbol{y}^1, \ldots, \boldsymbol{y}^I) \in \boldsymbol{R}^{(S+1)I} \mid Du^i \cdot \boldsymbol{y}^i > 0,\ Du^j \cdot \boldsymbol{y}^j = 0 \ (\forall j \neq i),\ \sum_{i=1}^I \boldsymbol{y}^i = 0\}$ is empty.

Then I obtain the first order necessary conditions of Pareto optimal allocations.

Proposition 5.2 *Given u and ω fulfilling assumption 5.2, 5.3 and 5.5 and let an allocation $\bar{x} = (\bar{\boldsymbol{x}}^1, \ldots, \bar{\boldsymbol{x}}^I) \in \boldsymbol{R}_{++}^{(S+1)I}$ be Pareto optimal. Then the following equation holds.*

$$Du^1(\bar{\boldsymbol{x}}^1) / \sum_{i=0}^{S+1} Du_i^1(\bar{\boldsymbol{x}}^1) = \ldots = Du^I(\bar{\boldsymbol{x}}^I) / \sum_{i=0}^{S+1} Du_i^I(\bar{\boldsymbol{x}}^I)$$

Proof: *A Pareto optimal allocation can be characterized as a solution of the next maximization problem,*

$$\begin{aligned} \max_{\boldsymbol{x}^i} \quad & u^i(\boldsymbol{x}^i) \\ s.t. \quad & u^j(\boldsymbol{x}^j) \geq \bar{u}^j \quad \forall j \neq i \\ & \sum_{i=1}^I \boldsymbol{x}^i = \sum_{i=1}^I \boldsymbol{\omega}^i \end{aligned}$$

where i is the index specified in assumption 5.5. Since the inequality conditions for all u^j can be replaced by the equality conditions by virtue of assumption 5.2, assumption 5.5 ensures the existence of the Lagrange multipliers (Sakawa (1986)). Thus the consequence is readily deduced by eliminating the Lagrange multipliers in the Kuhn-Tucker conditions

For any economy $\mathcal{E}(u, \omega; V)$, I particularly specify the set of allocations that only satisfy the equation in the proposition. Because no feasibility is required for the set, the set does not coincide with the set of Pareto optimal allocations but contains all efficient allocations. It is also worth noting that the set in question is only dependent on u, thus I denote it by $A(u)$ in what follows.

Then it is shown through a modified Thom transversality theorem that $A(u)$ possesses a specific structure generically in u, which is as follows.

Proposition 5.3 *There exists a dense subset $\mathcal{U}^*$ of $\mathcal{U}$ such that for any u in $\mathcal{U}^*$, $A(u)$ constitutes a S+I dimensional submanifold in $\boldsymbol{R}_{++}^{(S+1)I}$.*

Proof: *I give the sketch of the entire proof. For details see the mathematical appendix (chapter 6).*

(1) For any $u = (u^1, \ldots, u^I) \in \mathcal{U}$, define $j_I^1 u : \boldsymbol{R}_{++}^{(S+1)I} \to J^1(\boldsymbol{R}_{++}^{S+1}, \boldsymbol{R})^I$ by

$$j_I^1 u(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^I) = (j^1 u^1(\boldsymbol{x}^1), \ldots, j^1 u^I(\boldsymbol{x}^I))$$

where j^1u^i is the 1-jet extension of u^i. Then, through a variation of the Thom transversality theorem, I have that for any submanifold $Q \subset J^1(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R})^I$

$$\mathcal{T} = \{u \in \mathcal{U} \mid j^1_I u \text{ is transversal to } Q\}$$

is dense in $\mathcal{U}$ in the Whitney C^∞ topology.

(2) Consider the map $\Phi : J^1_+\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big)^I \to (\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \Delta^S_+)^I$ given by

$$\Phi(y^1, \ldots, y^I) = (\phi(y^1), \ldots, \phi(y^I))$$

where $J^1_+(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R})$ is substantially equal to $\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \boldsymbol{R}^{S+1}_{++}$ (while $J^1_(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R})$ is equivalent to $\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \boldsymbol{R}^{S+1}$), Δ^S_+ indicates a strictly positive S-simplex and $\phi : J^1_+\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big) \to \boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \Delta^S_+$ is given by

$$\phi(\boldsymbol{a}, b, c_1, \ldots, c_{S+1}) = (\boldsymbol{a}, b, c_1 / \sum_{i=1}^{S+1} c_i, \ldots, c_{S+1} / \sum_{i=1}^{S+1} c_i).$$

Then, it is shown that for any submanifold W of $(\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \Delta^S_+)^I$

$$\mathcal{T} = \{u \in \mathcal{U} \mid \Phi \circ j^1_I u \text{ is transversal to } W\}$$

is dense in $\mathcal{U}$ in the Whitney C^∞ topology.

(3) Let W be the set $\{\, (\boldsymbol{a}^i, b^i, \boldsymbol{c}^i)_i \in (\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \Delta^S_+)^I \,|\, \boldsymbol{c}^1 = \boldsymbol{c}^2 = \ldots = \boldsymbol{c}^I\}$. It is obvious that W is a $(S+1)I + I + S$ dimensional submanifold of $(\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times \Delta^S_+)^I$. There exists a dense subset $\mathcal{U}^*$ of $\mathcal{U}$ such that for any u in $\mathcal{U}^*$, $\Phi \circ j^1_I u$ is transversal to W . Thus for those u, $\Phi \circ j^1_I u^{-1}(W)$ constitutes a $S + I$ dimensional submanifold of $\boldsymbol{R}^{(S+1)I}_{++}$. Since $\Phi \circ j^1_I u^{-1}(W) = A(u)$, the proof is completed.

5.3.2 Fibration of $\boldsymbol{F}_\lambda(\omega)$

Secondly, I specify the set that contains all equilibrium allocations with incomplete markets. To this end, I consider a particular feasibility for any economy.

Definition 5.3 *For any given strictly positive S-vector $\boldsymbol{\lambda}$ $(= (\lambda_1, \ldots, \lambda_S))$, an allocation $x = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^I) \in \boldsymbol{R}^{(S+1)I}$ is pseudo-λ-feasible if*

(i) $\sum_{i=1}^I \boldsymbol{x}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$

(ii) $\boldsymbol{x}^i_{\mathbf{1}} \in \langle \Lambda V \rangle + \boldsymbol{\omega}^i_{\mathbf{1}}, i = 1, \ldots, I$

where Λ is a $S \times S$ diagonal matrix with $\boldsymbol{\lambda}$ for its diagonal and $\langle \Lambda V \rangle$ indicates a vector subspace spanned by the columns of a $S \times J$ matrix ΛV.

For any given economy $\mathcal{E}(u,\omega;V)$, the set of pseudo-λ-feasible allocations is only dependent on ω, thus I denote it by $F_\lambda(\omega)$. Then, $F_\lambda(\omega)$ has significant characteristics as follows.

Proposition 5.4 *$F_\lambda(\omega)$ constitutes a $(J+1)(I-1)$ dimensional submanifold of $\boldsymbol{R}^{(S+1)I}$ for any $\lambda \in \boldsymbol{R}^S_{++}$ and $\omega \in \boldsymbol{R}^{(S+1)I}_{++}$.*

Proof: *First I consider the following set.*

$$Z = \{(\boldsymbol{z}^1, \dots, \boldsymbol{z}^I) \in \boldsymbol{R}^{(J+1)I} \mid \boldsymbol{z}^1 + \cdots + \boldsymbol{z}^I = 0\}.$$

It is easily verified that Z is a $(J+1)(I-1)$ dimensional linear submanifold of $\boldsymbol{R}^{(J+1)I}$.

Now I extend matrices Λ, V as follows.

$$\hat{\Lambda} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_S \end{pmatrix}$$

and

$$\hat{V} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ v_1^1 & \cdots & \cdots & v_1^J \\ \vdots & \vdots & \vdots & \vdots \\ v_S^1 & \cdots & \cdots & v_S^J \end{pmatrix}$$

By using these extended matrices, I define the map $F : Z \to \boldsymbol{R}^{(S+1)I}$ given by

$$F(\boldsymbol{z}^1, \dots, \boldsymbol{z}^I) = (\hat{\Lambda}\hat{V}\boldsymbol{z}^i + \boldsymbol{\omega}^i)_i.$$

Then it readily follows from the construction of F that $F(Z) = F_\lambda(\omega)$. Since F is an affine map and $\hat{\Lambda}\hat{V}$ is of full rank, dim $F(Z) =$ dim Z, which completes the proof.

It is easily seen from the proof that $F_\lambda(\omega)$ is virtually a $(J+1)(I-1)$ dimensional affine subspace of $\boldsymbol{R}^{(S+1)I}$, so that if I particularly take $F_{\boldsymbol{e}}(\omega)$ where $\boldsymbol{e} = \overbrace{(1, \dots, 1)}^{S}$, i.e. $\boldsymbol{e}$ is the sum vector of order S, then $F_{\boldsymbol{e}}(\omega)$ is diffeomorphic to $F_\lambda(\omega)$ for any $\lambda \in \Delta^{S-1}_{++}$.

The second feature of $F_\lambda(\omega)$ is as follows.

Proposition 5.5 *For any equilibrium $((\bar{\boldsymbol{x}}^i, \bar{\boldsymbol{z}}^i)_i, \bar{\boldsymbol{p}}_{\mathbf{1}}, \bar{\boldsymbol{q}})$ for $\mathcal{E}(u,\omega;V)$, there exists a vector $\bar{\boldsymbol{\lambda}}$ of Δ^{S-1}_{++} such that the equilibrium allocation $(\bar{\boldsymbol{x}}^i)_i$ is pseudo-$\bar{\lambda}$-feasible where Δ^{S-1}_{++} designates the strictly positive $S-1$ dimensional simplex in $\boldsymbol{R}^S$.*

Proof: *It is sufficient to show that* $(\bar{\boldsymbol{x}}^i)_i$ *satisfies that* $\sum_{i=1}^I \bar{\boldsymbol{x}}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$ *and* $\bar{\boldsymbol{x}}^i_{\mathbf{1}} \in \langle \bar{\Lambda} V \rangle + \boldsymbol{\omega}^i_{\mathbf{1}}, i = 1, \ldots, I$. *Set* $c = \sum_{s=1}^S 1/\bar{p}^s$, $\tilde{z}^i = \bar{z}^i/c$, $i = 1, \ldots, I$, $\tilde{\boldsymbol{p}}_{\mathbf{1}} = \big(1/\bar{p}^1, \ldots, 1/\bar{p}^S\big)$ *and* $\tilde{\boldsymbol{q}} = c\bar{\boldsymbol{q}}$. *Then it is easily seen that* $((\bar{\boldsymbol{x}}^i, \tilde{\boldsymbol{z}}^i)_i, \tilde{\boldsymbol{p}}_{\mathbf{1}}, \tilde{\boldsymbol{q}})$ *is also an equilibrium for the economy. Now I set* $\bar{\boldsymbol{\lambda}} = c\big(1/\bar{p}^1, \ldots, 1/\bar{p}^S\big)$ *which is obviously an element of* Δ^{S-1}_{++}. *Considering the property of the new equilibrium, the following equations hold.* $\sum_{i=1}^I \bar{\boldsymbol{x}}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$, $\bar{\boldsymbol{x}}^i_{\mathbf{1}} = \bar{\Lambda} V \cdot \tilde{\boldsymbol{z}}^i + \boldsymbol{\omega}^i_{\mathbf{1}}, i = 1, \ldots, I$.

In view of the above proposition, it is obvious that all the equilibrium allocations for $\mathcal{E}(u, \omega; V)$ are included in the union of $F_\lambda(\omega)$ over $\lambda \in \Delta^{S-1}_{++}$ (i.e. $\bigcup_{\lambda \in \Delta^{S-1}_{++}} F_\lambda(\omega)$). However, it should be noted that in spite of the property of each $F_\lambda(\omega)$ (see proposition 5) the union $\bigcup_{\lambda \in \Delta^{S-1}_{++}} F_\lambda(\omega)$ itself is not necessarily a manifold.

Here, invoke the notion of fibre bundles. As I have shown, $F_{\boldsymbol{e}}(\omega)$ is diffeomorphic to $F_\lambda(\omega)$ for any $\lambda \in \Delta^{S-1}_{++}$. Thus, there exists a smooth map $G : F_{\boldsymbol{e}}(\omega) \times \Delta^{S-1}_{++} \to \boldsymbol{R}^{(S+1)I}$ such that $G(\,\cdot, \lambda) : F_{\boldsymbol{e}}(\omega) \to \boldsymbol{R}^{(S+1)I}$ is an into-diffeomorphism and $G(F_{\boldsymbol{e}}(\omega), \lambda) = F_\lambda(\omega)$ for any λ in Δ^{S-1}_{++}. Now Let $\boldsymbol{F}(\omega)$ be the disjoint union of $F_\lambda(\omega)$, $\lambda \in \Delta^{S-1}_{++}$. Then, by using the idea of fibration, I can make it a manifold.

Proposition 5.6 *For any* ω *in* $\boldsymbol{R}^{(S+1)I}_{++}$, $\boldsymbol{F}(\omega)$ *constitutes a* $(J+1)(I-1) + S - 1$ *dimensional manifold.*

Proof: *I am going to give a smooth manifold structure as well as a topological structure to* $\boldsymbol{F}(\omega)$ *in such a way that it becomes a total space with* Δ^{S-1}_{++} *as a base space. Let* $\pi : \boldsymbol{F}(\omega) \to \Delta^{S-1}_{++}$ *be a projection defined by* $\pi(x^*) = \lambda$ *where* $x^* = ((x, \lambda) \mid x \in F_\lambda(\omega), \lambda \in \Delta^{S-1}_{++})$. *Let* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ *be the atlas of* Δ^{S-1}_{++}. *Consider* $\pi^{-1}(U_\alpha)$ *for any chart* $\varphi_\alpha : U_\alpha \to \boldsymbol{R}^{S-1}$ *and define the map* $\tilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \boldsymbol{R}^{(J+1)(I-1)} \times \boldsymbol{R}^{S-1}$ *by*

$$\tilde{\varphi}_\alpha(x^*) = (\phi_\beta \circ G^{-1}(x, \lambda), \, \varphi_\alpha(\lambda))$$

where ϕ_β *is an appropriate chart* $\phi_\beta : W_\beta \to \boldsymbol{R}^{(J+1)(I-1)}$ *of the atlas* $(W_\beta, \phi_\beta)_{\beta \in B}$ *for the manifold* $F_{\boldsymbol{e}}(\omega)$. *Let a subset* Z *of* F_ω *be called an open set if for any* α, $\tilde{\varphi}_\alpha(Z \cap \pi^{-1}(U_\alpha))$ *is open in* $\boldsymbol{R}^{(J+1)I-(S+1)} \times \boldsymbol{R}^{S-1}$. *Then it is easily seen that those sets constitutes a system of open sets, which gives a topological structure to* F_ω.

Next, consider the property of the collcetion $(\pi^{-1}(U_\alpha, \tilde{\varphi}_\alpha))_{\alpha \in A}$. *Suppose that for* $\alpha, \alpha' \in A, \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_{\alpha'}) \neq \emptyset$. *Then for any point* (a, b) *of* $\tilde{\varphi}_\alpha(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_{\alpha'}))$ *I have*

$$\begin{aligned} \tilde{\varphi}_{\alpha'} \circ \tilde{\varphi}_\alpha^{-1}(a, b) &= \tilde{\varphi}_{\alpha'}((G(\phi_\beta^{-1}(a), \lambda)) = (\phi_\beta \circ G^{-1}(G(\phi_\beta^{-1}(a), \lambda), \lambda), \varphi_{\alpha'}(\varphi_\alpha^{-1}(b))) \\ &= (a, \varphi_{\alpha'} \circ \varphi_\alpha^{-1}(b)) \end{aligned}$$

where $\lambda = \varphi_\alpha^{-1}(b)$. *Thus,* $\tilde{\varphi}_{\alpha'} \circ \tilde{\varphi}_\alpha^{-1} : \tilde{\varphi}_\alpha(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_{\alpha'})) \to \tilde{\varphi}_{\alpha'}(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_{\alpha'}))$ *is a diffeomorphism. Since it is obvious that* $\pi^{-1}(U_\alpha)_{\alpha \in A}$ *covers* $\boldsymbol{F}(\omega)$, *which implies that* $\boldsymbol{F}(\omega)$ *is regarded as a smooth manifold whose dimension is* $(J+1)(I-1) + (S-1)$.

5.3.3 Generic infefficiency of financial equilibria

As I have seen, an equilibrium allocation for an economy $\mathcal{E}(u, \omega; V)$ with incomplete markets, if any, must be in the intersection of $A(u)$ and $\boldsymbol{F}(\omega)$. In order to see how these two sets intersect, I investigate transversality between $A(u)$ and $\boldsymbol{F}(\omega)$.

I fix an $A(u)$ for any given u of $\mathcal{U}^*$ and consider the inclusion map $\iota : A(u) \to \boldsymbol{R}^{(S+1)I}$. Then the image of ι, i.e. $A(u)$, can be interpreted as a $(S+I)$ dimensional submanifold of $\boldsymbol{R}^{(S+1)I}$. I am going to check if $\boldsymbol{F}(\omega)$ is transversal to $A(u)$ generically in ω. To this end, I first pick an arbitrary ω out of $\boldsymbol{R}^{(S+1)I}_{++}$ and fix it. Then define the map ψ : $\boldsymbol{F}(\omega) \times N_1^+(\omega) \to \boldsymbol{R}^{(S+1)I}$ by $\psi(x^*, y) = x + y$ where $N_1^+(\omega) = \{y \in \boldsymbol{R}^{(S+1)I}_{++} \mid \|y - \omega\| < 1\}$ and $x^* = \{(x, \lambda) \mid x \in F_\lambda(\omega),\ \lambda \in \Delta^{S-1}_{++}\}$. Then I have

Proposition 5.7 *ψ is a smooth submersion.*

Proof: *It suffices for us to check the claim locally. Let $(\pi^{-1}(U_\alpha, \tilde{\varphi}_\alpha) \times (N_1^+(\omega), i))$ be a chart including any given point (x^*, y) of $\boldsymbol{F}(\omega) \times N_1^+(\omega)$ where i is the identity map. Let $\tilde{\varphi}_\alpha(x^*) = (a, b)$ where obviously $a \in \boldsymbol{R}^{(J+1)(I-1)}$ and $b \in \boldsymbol{R}^{S-1}$. Then I have the following local parameterization of ψ at (x^*, y).*

$$G(\phi_\beta^{-1}(a),\ \varphi_\alpha^{-1}(b)) + y,$$

which is obviously differentiable at (a, b, y) since G is a smooth map, thus ψ is also differentiable at (x^*, y). In addition, it is obvious from the form of the local parameterization that ψ is a submersion.

Then I can obtain some useful propsitions concerning ψ.

Proposition 5.8 *For almost all $y \in N_1^+(\omega)$, $\psi(\cdot, y) : \boldsymbol{F}(\omega) \to \boldsymbol{R}^{(S+1)I}$ is transversal to $A(u)$.*

Proof: *$\psi : \boldsymbol{F}(\omega) \times N_1^+(\omega) \to \boldsymbol{R}^{(S+1)I}$ is transversal to $A(u)$ since it is shown in the above proposition to be a submersion. Thus, by applying the Transversality Theorem (Guillemin and Pollack (1974), p.68) to ψ, I obtain the desired result.*

Proposition 5.9 *For each $y \in N_1^+(\omega)$, $\psi(\boldsymbol{F}(\omega), y) = \bigcup_{\lambda \in \Delta^{S-1}_{++}} F_\lambda(\omega + y)$.*

Proof: *Since $\boldsymbol{F}(\omega)$ is the disjoint union of $F_\lambda(\omega)$, $\lambda \in \Delta^{S-1}_{++}$, obtaining its image by $\psi(\cdot, y)$ requires us to consider all the $F_\lambda(\omega)$ with regard to $\lambda \in \Delta^{S-1}_{++}$. Recall that $F_\lambda(\omega)$ is the set of x $(= (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^I) \in \boldsymbol{R}^{(S+1)I})$ which satisfies (1) $\sum_{i=1}^I \boldsymbol{x}^i = \sum_{i=1}^I \boldsymbol{\omega}^i$ and (2) $\boldsymbol{x}^i_{\mathbf{1}} \in \langle \Lambda V \rangle + \boldsymbol{\omega}^i_{\mathbf{1}}, i = 1, \ldots, I$. Thus, for any $\lambda \in \Delta^{S-1}_{++}$ and $y \in N_1^+(\omega)$ the set $\{(x+y) \in \boldsymbol{R}^{(S+1)I} \mid x \in F_\lambda(\omega)\}$ is obviously equal to the set $\{x \in \boldsymbol{R}^{(S+1)I} \mid x \in F_\lambda(\omega + y)\}$. Since $\psi(\cdot, y)$ substantially transforms any x into $x + y$, the claim immediately follows.*

In order to obtain the final outcome, I need the following assumption regarding the numbers of agents, assets and states.

Assumption 5.6 $I > 2$ and $S > \frac{I-1}{I-2}J + 1$

Now I am in a position to state our main claim.

Theorem 5.1 *Under assumptions 5.1~5.6, for almost all u and ω, each equilibrium allocation of the economy $\mathcal{E}(u, \omega; V)$ is Pareto inefficient.*

Proof: *Let u and ω be respectively arbitrary elements of $\mathcal{U}^*$ and $\boldsymbol{R}_{++}^{(S+1)I}$. Let y be an element of $N_1^+(\omega)$ such that $\psi(\cdot, y)$ is transversal to $A(u)$. Note that from proposition 5.8 such a y is an element of a dense set of $N_1^+(\omega)$. Now suppose that $\psi(\boldsymbol{F}(\omega), y) \cap A(u) \neq \emptyset$. Then for any $x \in \psi(\boldsymbol{F}(\omega), y) \cap A(u)$, I have*

$$d\psi_{z^*}(T_{z^*}(\boldsymbol{F}(\omega)) + T_x(A(u)) = \boldsymbol{R}^{(S+1)I}$$

where T designates a tangent space and z^* is an element of $\psi^{-1}(\cdot, y)(x)$.

However, by propositions 5.3 and 5.6 and assumption 5.6, I have the following inequality.

$$\begin{aligned} dim\boldsymbol{R}^{(S+1)I} &- (dimT_{z^*}(\boldsymbol{F}(\omega)) + dimT_x(A(u))) \\ &= (S+1)I - ((J+1)(I-1) + S - I + S + 1) \\ &= S(I-2) - J(I-1) - (I-2) \\ &> 0, \end{aligned}$$

which is a contradiction to the previous equation. Thus $\psi(\boldsymbol{F}(\omega), y) \cap A(u) = \emptyset$, which implies that $\bigcup_{\lambda \in \Delta_{++}^{S-1}} F_\lambda(\omega + y) \cap A(u) = \emptyset$ by proposition 5.9. Noting that $\bigcup_{\lambda \in \Delta_{++}^{S-1}} F_\lambda(\omega + y)$ includes the whole set of equilibrium allocations for $\mathcal{E}(u, \omega + y; V)$ and that an equilibrium allocation cannot be Pareto efficient unless it belongs to $A(u)$, it turns out that any equilibrium allocation for $\mathcal{E}(u, \omega + y; V)$ is Pareto inefficient. Since ω is arbitrarily taken from $\boldsymbol{R}_{++}^{(S+1)I}$ and u and y are respectively arbitrary elements of the dense sets, the claim follows.

Finally, I refer to the real asset case. As I have already seen in section 2, a real asset model can be considered to be a special case of a nominal asset model. That is to say, if I put $p^s = 1,\ s = 1, \ldots, S$ in the equations system in $(*)$, I obtain the optimization problem for each agent in a real asset model, which implies that if $\lambda_s = 1,\ s = 1,\ \ldots,\ S$, then $F_\lambda(\omega)$ includes all equilibria for an economy $\mathcal{E}(u, \omega; V)$ with real assets. Thus, in the real asset case, I have only to consider one particular set $F_{\boldsymbol{e}}(\omega)$ instead of $\bigcup_{\lambda \in \Delta_{++}^{S-1}} F_\lambda(\omega)$, where $\boldsymbol{e}$ denotes the sum vector of order S, i.e. $\boldsymbol{e} = \overbrace{(1, \ldots, 1)}^{S}$. This fact facilitates the analysis, resulting in the similar result to the above one with much weaker conditions regarding the number of agents, assets and states.

Theorem 5.2 *If $I \geq 2$, $S > J+1$ and the rest of the assumptions are unchanged, then for almost all u and ω, each equilibrium allocation of the economy $\mathcal{E}(u, \omega; V)$ with real assets is Pareto inefficient.*

Proof: *I am allowed to follow the procedure described in the proof of the foregoing theorem. That is, fix an $A(u)$ for any given u of $\mathcal{U}^*$ and consider the inclusion map $\iota : A(u) \to \boldsymbol{R}^{(S+1)I}$. Then the image of ι, i.e. $A(u)$, can be interpreted as a $(S+I)$ dimensional submanifold of $\boldsymbol{R}^{(S+1)I}$. I am going to check if $F_{\boldsymbol{e}}(\omega)$ is transversal to $A(u)$ generically in ω. To this end, I first pick an arbitrary ω out of $\boldsymbol{R}^{(S+1)I}_{++}$ and fix it. Then define the map $\psi : F_{\boldsymbol{e}}(\omega) \times N_1^+(\omega) \to \boldsymbol{R}^{(S+1)I}$ by $\psi(x, y) = x + y$ where $N_1^+(\omega) = \{y \in \boldsymbol{R}^{(S+1)I}_{++} \mid \|y - \omega\| < 1\}$. As for this map, I can obtain through the transversality theorem that for almost all $y \in N_1^+(\omega)$, $\psi(\cdot, y) : F_{\boldsymbol{e}}(\omega) \to \boldsymbol{R}^{(S+1)I}$ is transversal to $A(u)$. In addition, it can be easily verified that for each $y \in N_1^+(\omega)$, $\psi(F_{\boldsymbol{e}}(\omega), y) = F_{\boldsymbol{e}}(\omega + y)$.*

Thus, let y be an element of $N_1^+(\omega)$ such that $\psi(\cdot, y)$ is transversal to $A(u)$. Suppose that $F_{\boldsymbol{e}}(\omega + y) \cap A(u) \neq \emptyset$. Since $\psi(\cdot, y)$ is transversal to $A(u)$, I have that $T_z F_{\boldsymbol{e}}(\omega) + T_x A(u) = \boldsymbol{R}^{(S+1)I}$ at any $x \in F_{\boldsymbol{e}}(\omega + y) \cap A(u)$, where $z = \psi(\cdot, y)^{-1}(x)$. But considering proposition 5.3, proposition 5.4 and the hypothesis on the number of agents, assets and states, I have

$$\begin{aligned} dim F_{\boldsymbol{e}}(\omega) + dim A(u) &= (J+1)(I-1) + S + I \\ &< S(I-1) + S + I \\ &= (S+1)I \\ &= dim \boldsymbol{R}^{(S+1)I}, \end{aligned}$$

which is a contradiction. Thus $F_{\boldsymbol{e}}(\omega + y) \cap A(u) = \emptyset$, that is to say, there exist no Pareto optimal allocations fulfilling pseudo-$\boldsymbol{e}$-feasibility. Since each equilibrium allocation of $\mathcal{E}(u, \omega; V)$ is obviously pseudo-$\boldsymbol{e}$-feasible, it is Pareto inefficient. Noting that ω is arbitrarily taken from $\boldsymbol{R}^{(S+1)I}_{++}$ and that u and y are respectively arbitrary elements of the dense sets, the claim follows.

5.4 Concluding Remarks

Some implications are obtainable from the arguments provided above. First, recall the proofs of the theorems. Then I find that generic inefficiency of equilibrium allocations is caused by the fact that for almost all u and ω, $A(u)$ is disjoint with $F_{\boldsymbol{e}}(\omega)$ (resp. $\bigcup_{\lambda \in \Delta^{S-1}_{++}} F_\lambda(\omega)$) which contains the set of equilibrium allocations in a real (resp. nominal) asset model. Considering the definion of pseudo-λ-feasibility, I immediately have that $F_{\boldsymbol{e}}(\omega)$ (resp. $\bigcup_{\lambda \in \Delta^{S-1}_{++}} F_\lambda(\omega)$) is derived from allocations that satisfy every agent's budget constraints of date 1, which are realized only by trade on asset markets. Noting that $F_{\boldsymbol{e}}(\omega)$ (resp. $\bigcup_{\lambda \in \Delta^{S-1}_{++}} F_\lambda(\omega)$) is independent of the optimization behavior of each agent, it may safely be said that as long as each agent trades on incomplete asset markets, Pareto optimal allocations are hardly attainable *whether he is engaged in the individual optimization or not*. This is just what I obtained in the previous chapter. To sum up, I have demonstrated in a different way incompatibility of Pareto efficiency with budget constraints for equilibria with incomplete markets.

Regarding the likelihood of inefficiency of equilibrium allocations, I also have the same conclution as before. That is to say, the intuition that equilibrium allocations with incomplete markets are more likely to be Pareto optimal when J is close to S than when J is far below S proves to be wrong according to the arguments provided here. It is because as long as $J < S$ (resp. $J < S+1$ and $I > J$), the transversality of $\psi(\cdot, y)$ to $A(u)$ prevents $F_{\boldsymbol{e}}(\omega)$ (resp. $\bigcup_{\lambda \in \Delta_{++}^{S-1}} F_\lambda(\omega)$) from intersecting $A(u)$, which necessarily yields that equilibrium allocations are inefficient.

Compared to the result in the previous chapter, the ones obtained here have an advantage in specifying the condition for the numbers of agents, assets and states. Namely, for the real asset case, the previous result requires that $I(S-J) > S$ for the generic inefficiency of equilibria, while here substantially $S > J+1$ is the only requirement, which is almost necessarily satisfied as long as incomplete markets are concerned. For the nominal asset case, admitterly, I cannot tell which condition is more plausible, but it is worth noting that according to the previous result I have to accept different conditions among endogenous asset prices and exogenous asset prices (see the concluding remarks of chapter 4), while the condition here is common to both cases. This is because the pseudo-feasible set which is combined with $A(u)$ to prove the impossibility of Pareto efficiency includes any equilibrium allocation regardless of whether it is based on endogenous asset prices or exogenous asset prices. Indeed, the pseudo-feasible set is derived from all agents' budget constraints of each state at *date 1* which are specified independently of asset prices (see definition 5.3). Obviously, the asset prices are only relevant to every agent's budget constraint at *date 0*.

Chapter 6

Mathematical Appendix

This chapter illustrates specific mathematical techniques that have facilitated the arguments in the text.[1] They consists of two parts: one is the intersection-based approach to genericity analysis for the equilibrium set, and the other is a variation of the Thom transversality theorem.

[1]This chapter is based on R. Nagata, *Theory of Regular Economies*, World Scientific, 2004 and R. Nagata, "An Intersection-based Approach to Genericity Analysis for the Equilibrium Set, with an Application to the Lindahl Equilibrium", *The Japanese Economic Review*, vol 51, no.3, 431-447, 2000.

6.1 Intersection-based approach to genericity analysis

The analysis of this dissertation is directed toward researching specific properties of the equilibrium set from the generic viewpoint in some economic parameters which are of different nature. What is the very suitable for this sort of issue is the intersection-based approach to genericity analysis I developed. Indeed, the arguments in chapter 3 and chapter 5 are basically owing to this approach. The basic idea of this approach is simple, which gives the approach itself a wide applicability. Thus, I first describe the basic idea in an abstract form, then I pick up the most standard Walrasian equilibria as a concrete example, applying this approach to the equilibria to generically investigate some of their properties. I believe that such an illustration articulates the idea of the approach.

6.1.1 Basic idea

Consider an abstract economy. There exists two types of notions concerning an economy. One is the collection (say, e) of parameters which specify the framework of an economy, the other is the collection (say, x) of endogenous variables which describe the state of an economy. Usually, one can restrict admissible e's to a certain set E, which is called the parameter space (or the economy space), while x is restricted to a certain set X called the state space. Given e of E, one is normally concerned with a particular state of the economy corresponding to e, which is denoted by $x^*(e)$. In most cases $x^*(e)$ is interpreted as an equilibrium. One aims at investigating some properties of $x^*(e)$. If $x^*(e)$ is considered as an equilibrium, existence, local uniqueness, stability etc. are included in such properties. In general, $x^*(e)$ is not uniquely determined, which implies that one has to consider the set $X^*(e) = (x^*(e)^1, x^*(e)^2, \dots)$. If the set $X^*(e)$ possesses a certain property for almost all e of E, then the property is called generic. (An inquiry into generic properties of equilibria with incomplete markets is just what this dissertation is concerned with.) A standard method of investigating generic properties in economic literature is as follows. First, characterize the set $(e, X^*(e))$ in $E \times X$. Secondly restrict a projection $pi : E \times X \to E$ giben by $pi(e, x) = e$ to the set. Then apply Sard's theorem to the restricted projection, gainning some generic properties of $X^*(e)$ (for details, see Nagata 2001b). This method, however, needs more technical process when e includes different type of parameters. The intersection-based approach has the advantage of systematically treating such a complexity. The basic idea of this approach is as follows.

Suppose that there exist two sets E_1, E_2 such that $E = E_1 \times E_2$ (thus, for all $e \in E,\ e = (e_1, e_2),\ e_1 \in E_1, e_2 \in E_2$) and that $X^*(e) = X^1(e_1) \cap X^2(e_2)$ for all e of E where $X^i(e_i)$ is the set of states corresponding to $e_i,\ i = 1, 2$. Needless to say, e_1 and e_2 indicate the collections of different types of parameters. In these circumstances, generic properties of $X^*(e)$ is explored through two specific procedures: namely, first investigating how the structure of $X^i(e_i)$ responds to $e^i,\ i = 1, 2$, second ascertaining how $X^1(e_1)$ and $X^2(e_2)$ intersect each other. It is worth noting that one can cover the case in which $X^1(e_1)$ and

$X^2(e_2)$ do not intersect, which implies that $X^*(e) = \emptyset$. Indeed, essentially speaking, the arguments of chapter 5 in this dissertation centers on the possibility of $X^*(e) = \emptyset$.

The success of this method rests on the three points: (1) one can find appropriate classes E_1, E_2, (2) one can analytically specify the correspondence of $X^i(e_i)$ to e^i, and (3) one can simply analyze the way of intersection of $X^1(e_1)$ and $X^2(e_2)$ in response to e_1 and e_2. In order to see that these matters are not as tough as they look, I pick up the standard Warlasian equilibrium and show how smoothly the method is applied to the genericity analysis for the equilibrium.

6.1.2 The Walrasian model

There exist L goods $(l = 1, \ldots, L)$, I consumers $(i = 1, \ldots, I)$ and no producers, which implies that I consider a pure exchange economy. Each agent is characterized by its consumption set X^i, its utility function u^i defined over the consumption set, and its initial endowment vector $\boldsymbol{\omega}^i$ $(i = 1, \ldots, I)$. The consumption vector for agent i is denoted by $\boldsymbol{x}^i$. Each good has its own competitive market, where a certain price is realised by the auctioneer. A price system is denoted by $\boldsymbol{p}(= (p_1, \ldots, p_L))$.

Under these circumstances, each agent's decision-making problem is summarized by

$$\begin{array}{ll} \max\limits_{\boldsymbol{x}^i} & u^i(\boldsymbol{x}^i) \\ s.t. & \boldsymbol{p} \cdot \boldsymbol{x}^i \leq \boldsymbol{p} \cdot \boldsymbol{\omega}^i. \end{array}$$

The solution of the above problem is called a subjective equilibrium for agent i. In order to have a Walrasian equilibrium, these subjective equilibria among agents are required to fulfill the so-called market equilibrium, that is

$$\sum_{i=1}^{I} \boldsymbol{x}^i = \sum_{i=1}^{I} \boldsymbol{\omega}^i.$$

In view of its framework, the state of a Walrasian equilibrium is naturally described by $(\boldsymbol{p}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^I)$ satisfying both the subjective equilibrium and the market equilibrium. It is worth noting that these states of equilibrium depend on a distribution of u^i and $\boldsymbol{\omega}^i$. Thus $(u^i, \boldsymbol{\omega}^i)_i$ is called economic parameters. In the following, $(u^i, \boldsymbol{\omega}^i)_i$ is denoted by e and called just an economy, whereas $(\boldsymbol{p}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^I)$ is denoted by s and called just a state for simplicity. Then, a Warlasian equilibrium is formally given as follows.

Definition 6.1 *A state $s = (\boldsymbol{p}, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^I)$ under an economy e, is a Walrasian equilibrium if and only if*

(i) $\boldsymbol{x}^i$ *is a subjective equilibrium for i, $i = 1, \ldots, I$*

(ii) $\sum_{i=1}^{I} \boldsymbol{x}^i = \sum_{i=1}^{I} \boldsymbol{\omega}^i$

The set of Walrasian equilibria under an economy e is denoted by $E(e)$.

According to the arguments of the forgoing subsection, I have to characterize every variable and function so as to obtain the parameter space and the state space. First, I specify the parameters which consist of utility functions and initial endowmets among agents.

Assumption 6.1 u^i satisfies

1. $u^i \in C^\infty\big(\boldsymbol{R}^L_{++}, \boldsymbol{R}\big)$.
2. $Du^i(\boldsymbol{x}) \in \boldsymbol{R}^L_{++}$ for each $\boldsymbol{x} \in \boldsymbol{R}^L_{++}$

Assumption 6.2 $\boldsymbol{\omega}^i \in \boldsymbol{R}^L_{++}$

In order to constrain a utility function as loosely as possible, I only require differentiability and monotonicity of the function. Obviously, u^i and $\boldsymbol{\omega}^i$ are of different nature, which correspond to e_1 and e_2 in the above explanation. Thus, let the set of a utility function satisfying assumption 1 be U and the set of an admissible $\boldsymbol{\omega}^i$ be Ω (obviously, $\Omega = \boldsymbol{R}^{S+1}_{++}$). Then, the economy space (denoted by E) is expressed by $\mathcal{U} \times \mathcal{W}$, where $\mathcal{U} = \Pi^I U$ and $\mathcal{W} = \Pi^I \Omega$. Next, I refer to a state $s = (\boldsymbol{p}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^I)$.

Assumption 6.3 $\boldsymbol{p} \in \boldsymbol{R}^L_{++}$

Assumption 6.4 $\boldsymbol{x}^i \in \boldsymbol{R}^L_{++},\ i = 1, \dots, I$

Noting the budget constraint of each agent, an admissible price vector can be normalized and confined to the $L-1$ dimensional strictly positive unit simplex, denoted by P. Thus, the state space (denoted by S) is $P \times \boldsymbol{R}^L I_{++}$. For simplicity of notation, set $u = (u^1, \dots, u^I)$, $\omega = (\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^I)$ and $x = (\boldsymbol{x}^1, \dots, \boldsymbol{x}^I)$. Thus, $e = (u, \omega)$ and $s = (\boldsymbol{p}, x)$.

In order to perform the genericity analysis, I have to specify a topology for relevant spaces. I may endow $\mathcal{W}$ and S with the usual Euclidean topology. On the other hand, two different types of topologies can be used for the function space, $\mathcal{U}$: namely, the compact open topology and the Whitney topology. The former topology is much more tractable than the latter one. It is, however, worth noting that the latter topology is more suitable for $\mathcal{U}$ than the former one because of the non-compactness of the domain of a utility function. I choose to use here the more tractable one, i.e., the compact open topology, since the subject of this section is to explain the basic procedure of the approach. To the case in which $\mathcal{U}$ is endowed with the Whitney topology the next section is devoted. Thus, I first set the compact open C^∞ topology on U, then endow $\mathcal{U}$ with the deduced product topology.

6.1.3 Specification of the approach

I consider the equilibrium set in relation to $e = (u, \omega)$. For this purpose, first of all I have to characterize the equilibrium set so as to facilitate further analysis. It is, however, worth noting that a utility function is too loosely constrained to give rise to the full specification of the set in the analytical sense. Therefore, I consider in S such a subset as includes the intrinsic equilibrium set and coincides with it if a certain convexity condition (i.e., strict quasi-concavity) is added to a utility function. To this end, take the first-order necessary conditions for the solution of each agent's optimization problem. Noting the linearity of the constraint function of the problem, I can obtain well-defined equations as the condition, which are as follows.

$$\begin{aligned} Du^i_{\boldsymbol{x}} &= \mid Du^i_{\boldsymbol{x}} \mid \boldsymbol{p} \\ \boldsymbol{p} \cdot \boldsymbol{x}^i &= \boldsymbol{p} \cdot \boldsymbol{\omega}^i, \quad i = 1, \dots, I \end{aligned}$$

where $\mid Du^i_{\boldsymbol{x}} \mid = \sum_{l=1}^{L} \partial u^i / \partial x_l$. For simplicity, I denote the set of above conditions by (C). Then I have an extended version of an equilibrium.

Definition 6.2 *A state $s = (\boldsymbol{p}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^I)$ under an econmy e, is an extended Walrasian equilibrium if and only if*

(i) $\boldsymbol{x}^i$ *satisfies (C),* $i = 1, \dots, I$

(ii) $\sum_{i=1}^{I} \boldsymbol{x}^i = \sum_{i=1}^{I} \boldsymbol{\omega}^i$

The set of extended Walrasian equilibria under an economy e is denoted by $E_{ex}(e)$. In the following, I focus on $E_{ex}(e)$ because it turns out that $E_{ex}(e)$ gives rich information about $E(e)$.

Next step is to find specific subsets $S^1(u)$ and $S^2(\omega)$ in S such that $E_{ex}(e) = S^1(u) \cap S^2(\omega)$ for any $e = (u, \omega)$. First, consider $S^1(u)$. To this end, it is convenient to construct the following map $\psi : P \times \boldsymbol{R}^{LI}_{++} \to \Pi^I H$ parameterized by u.

$$\psi_u(\boldsymbol{p}, x) = (Du^i_{\boldsymbol{x}} - \mid Du^i_{\boldsymbol{x}} \mid \boldsymbol{p})_{i=1}^{I}$$

where $H = \{\boldsymbol{z} \in \boldsymbol{R}^L \mid \sum_{l=1}^{L} z_l = 0\}$, which is an $(L-1)$-dimensional linear subspace (submanifold) of $\boldsymbol{R}^L$. Then, the zeros $(\psi_u^{-1}(0))$ of the map form the set of the state that satisfies the differential condition of (C) for every agent. This set $\psi_u^{-1}(0)$ corresponds to $S^1(u)$. On the other hand, $S^2(\omega)$ can be constructed through the following map $\phi_\omega : P \times \boldsymbol{R}^{LI}_{++} \to \boldsymbol{R}^{I-1} \times \boldsymbol{R}^L$ parameterized by ω.

$$\phi_\omega(\boldsymbol{p}, x) = \big((\boldsymbol{p} \cdot \boldsymbol{x}^i - \boldsymbol{p} \cdot \boldsymbol{\omega}^i)_{i=1}^{I-1}, \ \sum_{i=1}^{I} \boldsymbol{x}^i - \sum_{i=1}^{I} \boldsymbol{\omega}^i\big).$$

The zeros ($\phi_\omega^{-1}(0)$) of the map consist of the states satisfying both the budget constraints of each agent and the market clearance. It is worth noting that the budget constraint of I-th agent is excluded among the value of the map ϕ_ω, which is due to the fact that the constraint is automatically satisfied in $\phi_\omega^{-1}(0)$. Like $\psi_u^{-1}(0)$, $\phi_\omega^{-1}(0)$ corresponds to $S^2(\omega)$. Indeed, it is easily seen that $E_{ex}(e) = \psi_u^{-1}(0) \cap \phi_\omega^{-1}(0)$.

Thus, there remain two problems to be solved: (1) how does the structure of $\psi_u^{-1}(0)$ (resp. $\phi_\omega^{-1}(0)$) respond to u (resp. ω)? (2) how do these sets intersect each other? I can consider these problems from the viewpoint of genericity in (u, ω).

6.1.4 Genericity analysis in (u, ω)

As for problem (1), I can obtain the following answers. First, a decisive outcome can be derived on $\phi_\omega^{-1}(0)$.

Proposition 6.1 *For all $\omega \in \boldsymbol{R}^{LI}_{++}$, $\phi_\omega^{-1}(0)$ constitutes a $(L-1)I$-dimensional submanifold in $P \times \boldsymbol{R}^{LI}_{++}$.*

Proof: *I shall show that ϕ_ω is a submersion for any $\omega \in \boldsymbol{R}^{LI}_{++}$. By computaion, the derivative $D\phi_{\omega,(\boldsymbol{p},x)} : T_{(\boldsymbol{p},x)}(P \times \boldsymbol{R}^{LI}_{++}) \to \boldsymbol{R}^{I-1} \times \boldsymbol{R}^L$ proves to have the following sub-square matrix of degree $(I-1)+L$ at any point $(\boldsymbol{p}, x) \in P \times \boldsymbol{R}^{LI}_{++}$.*

$$\begin{pmatrix} A & 0 \\ B & I \end{pmatrix}$$

where A is a diagonal matrix of order $I-1$ with $(-p_1, \dots, -p_1)$ as its diagonal, I is the identity matrix of order L, 0 is a zero matrix and B is a $L \times (I-1)$ matrix of the following form.

$$\begin{pmatrix} 1 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

Thus,the above submatrix is obviously non-singular and $T_{(\boldsymbol{p},x)}(P \times \boldsymbol{R}^{LI}_{++})$ has a vector whose subvector corresponding to this submatrix takes any real elements, so that $D\phi_{\omega,(\boldsymbol{p},x)}$ is surjective.

On the other hand, the corresponding property of $\psi_u^{-1}(0)$ turns out to be generic in u.

Proposition 6.2 *There exists an open and dense set $\tilde{\mathcal{U}}$ in $\mathcal{U}$ such that for every $u \in \tilde{\mathcal{U}}$, $\psi_u^{-1}(0)$ constitutes a LI-dimensional submanifold in $P \times \boldsymbol{R}^{LI}_{++}$.*

Proof: *I shall show that the set of u such that 0 is a regular value of ψ_u is qualified as $\tilde{\mathcal{U}}$. Since 0 is a regular value of ψ_u, it is easily seen that $\psi_u^{-1}(0)$ constitures a LI-dimensional submanifold. Thus, there remain openness and density of the set in question to be shown. For simplicity, denote the set by $\tilde{\mathcal{U}}$ in the following. First, demonstrate its openness in $\mathcal{U}$. For any $u \in \mathcal{U}$, the derivative of ψ_u at any point $(\boldsymbol{p}, x) \in \psi_u^{-1}(0)$ is represented by the following $LI \times L(I+1)$ matrix.*

$$\begin{pmatrix} A_1 & [u^1] & 0 & \dots & 0 \\ A_2 & 0 & [u^2] & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_I & 0 & 0 \dots & [u^I] & \end{pmatrix}$$

where A_i is a diagonal matrix of order L with $(-\sum_{l=1}^L \partial u^i/\partial x_l, \dots, -\sum_{l=1}^L \partial u^i/\partial x_l)$ as its diagonal and $[u^i]$ is a square matrix of order L whose (n,m) element is $\partial u^i/\partial x_n \partial x_m - p_n \sum_{h=1}^L \partial u^i/\partial x_h \partial x_m$, $i = 1, \dots, I, n, m = 1, \dots, L$. Fix any u of $\tilde{\mathcal{U}}$. Obviously, the above matrix is surjective from $H \times \boldsymbol{R}^{LI}$ onto $\Pi^I H$ for the u. Let v be any point in a sufficiently small neibourhood of the u. Then, by the property of the compact open topology, v gives rise to the derivative matrix whose elements are sufficiently close to those of u at each point $(\boldsymbol{p}, x) \in \psi_v^{-1}(0)$, which in turn is sufficiently close to each point of $\psi_u^{-1}(0)$. Hence, the matrix is also surjective, which implies that $v \in \tilde{\mathcal{U}}$. Since u is taken arbitrarily chosen, the openness of $\tilde{\mathcal{U}}$ is followed.

The remaining task is to show that $\tilde{\mathcal{U}}$ is dense in $\mathcal{U}$. It is worth noting that the function space with the compact open C^∞ topology is a Banach space if the domain of a function is compact (Hirsch (1976), p. 35). Thus I consider a monotone sequence of compact sets C_k ($\subset C_{k+1}$) in $\boldsymbol{R}_{++}^L$ such that $\boldsymbol{R}_{++}^L = \cup_{k=1}^\infty C_k$. Then $U_k \equiv \{u|_{C_k} \mid u \in U\}$ is a Banach space, accordingly, $\Pi^I U_k$ can be seen as a Banach manifold. Let u_k denote $(u^1, \dots, u^I)$, $u^i \in U_k$, $i = 1, \dots, I$. I will show that the set of u_k for which $\psi_{uk} : P \times \Pi^I Int(C_k) \to \Pi^I H$ given by $\psi_{uk}(\boldsymbol{p}, x) = (Du^i_{\boldsymbol{x}} - \mid Du^i_{\boldsymbol{x}} \, |\boldsymbol{p})_{i=1}^I$ takes 0 as a regular value is open and dense in $\Pi^I U_k$, where $Int(\cdot)$ indicates the interior. Openness is obtained in the same manner as the one provided above. To prove the density, it suffices to show that 0 is a regular value of the map $ev : \Pi^I U_k \times P \times \Pi^I Int(C_k) \to \Pi^I H$ given by $ev(u, \boldsymbol{p}, x) = \psi_{u_k}(\boldsymbol{p}, x)$ since ev is thought of as an evaluation map in this case (see Abrahame and Robbin (1967),p. 48). To this end, I show that ev is a submersion. The derivative of ev at any point $(u_k, \boldsymbol{p}, x)$ is represented as follows (see Abrahame and Robbin (1967),p. 25).

$$Dev_{(u_k,(\boldsymbol{p},x))}(v, (\boldsymbol{p}', x') = D_1 ev_{(u_k,(\boldsymbol{p},x))}(v) + D_2 ev_{(u_k,(\boldsymbol{p},x))}(\boldsymbol{p}', x')$$

where $v = (v^1, \dots, v^I)$, $v^i \in C^\infty(C_k, \boldsymbol{R})$, $i = 1, \dots, I$ and $\boldsymbol{p}' \in H$, $x' \in \boldsymbol{R}^{LI}$. For a fixed u_k, ev is equal to $\psi_{u_k}(\boldsymbol{p}, x)$. Thus, $D_2 ev_{(u_k,(\boldsymbol{p},x))}$ is expressed by the matrix which appears in the proof of the openness of $\tilde{\mathcal{U}}$. However, for a fixed $(\boldsymbol{p}, x)$, ev is the map $u_k \to (Du^i_{\boldsymbol{x}} - \mid Du^i_{\boldsymbol{x}} \, |\boldsymbol{p})_{i=1}^I$, which is linear. Now define $v^i(x) = \sum_{l=1}^L a_l^i x_l$, $i = 1, \dots, I$ for *any given point* $(a^i_1,\dots,a^i_L)^I_{i=1}\in\Pi^I H$. *Then,* $(Dv^i_{\boldsymbol{x}}-\mid Dv^i_{\boldsymbol{x}}\mid\boldsymbol{p})^I_{i=1}=(a^i_1,\dots,a^i_L)^I_{i=1}$ *since* $\mid Dv^i_{\boldsymbol{x}}\mid\boldsymbol{p}=(\sum_{l=1}^L a^i_l)\boldsymbol{p}=0,\ i=1,\dots,I$. *Thus,* $Dev_{(u_k,(\boldsymbol{p},x))}(v,0,0)=(a^i_1,\dots,a^i_L)^I_{i=1}$, *which implies that ev is a submersion, which proves the density of the relevant set of* u_k. *Then, noting that* $\cup_{k=1}^\infty C_k=\cup_{k=1}^\infty Int(C_k)=\boldsymbol{R}^L_{++}$, *the density of* $\tilde{\mathcal{U}}$ *immediately follows.*

In order to treat problem (2), one lemma is needed. Fix any $u\in\tilde{\mathcal{U}}$, and consider a map $F_\omega:\phi_\omega^{-1}(0)\times\boldsymbol{R}^{LI}_{++}\to P\times\boldsymbol{R}^{LI}_{++}$ for any $\omega\in\boldsymbol{R}^{LI}_{++}$ as follows.

$$F_\omega((\boldsymbol{p},x),\varepsilon)=(\boldsymbol{p},\boldsymbol{x}^1+\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{x}^I+\boldsymbol{\varepsilon}^I)$$

where $\varepsilon=(\boldsymbol{\varepsilon}^1,\dots,\boldsymbol{\varepsilon}^I),\ \boldsymbol{\varepsilon}^i\in\boldsymbol{R}^L_{++},\ i=1,\dots,I$. Then, I have

Lemma 6.1 *For all* $\omega\in\boldsymbol{R}^{LI}_{++},\ F_\omega$ *is transversal to* $\psi_u^{-1}(0)$.

Proof: *I prove this lemma by showing that* F_ω *is a submersion (note that a submersion is transversal to any submanifold in the range space). Noting that* $T_{(\boldsymbol{p},x)}\phi_\omega^{-1}(0)$ *is the kernel of the derivative* $D\phi_{\omega,(\boldsymbol{p},x)}$, *it follows from computaion that* $T_{(\boldsymbol{p},x)}\phi_\omega^{-1}(0)\times\boldsymbol{R}^{LI}_{++}$ *contains a vector of the following form.*

$$\begin{aligned}&(a_1,\dots,a_L,(1/p_1)\sum_{l=1}^L a_l(\omega^1_l-x^1_l),0,\dots,0,(1/p_1)\sum_{l=1}^L a_l(\omega^2_l-x^2_l),0,\dots,0,\\&(1/p_1)\sum_{l=1}^L a_l(\omega^{I-1}_l-x^{I-1}_l),0,\dots,0,-(1/p_1)\sum_{i=1}^I\sum_{l=1}^L a_l(\omega^i_l-x^i_l),0,\dots,0,\\&b^1_1,\dots,b^1_L,b^2_1,\dots,b^2_L,\dots,b^I_1,\dots,b^I_L)\end{aligned}$$

where $a_1,\dots,a_L$ *are real numbers satisfying* $\sum_{l=1}^L a_l=0$ *and* b^i_l *is any real number,* $i=1,\dots,I,l=1,\dots,L$. *Hence, applying* $DF_{\omega,((\boldsymbol{p},x),\varepsilon)}$ *to the vector leads to*

$$\begin{aligned}&(a_1,\dots,a_L,(1/p_1)\sum_{l=1}^L a_l(\omega^1_l-x^1_l)+b^1_1,b^1_2,\dots,b^1_L,(1/p_1)\sum_{l=1}^L a_l(\omega^2_l-x^2_l)+b^2_1,\\&b^2_2,\dots,b^2_L,\dots,(1/p_1)\sum_{l=1}^L a_l(\omega^{I-1}_l-x^{I-1}_l)+b^{I-1}_1,b^{I-1}_2,\dots,b^{I-1}_L,\\&-(1/p_1)\sum_{i=1}^I\sum_{l=1}^L a_l(\omega^i_l-x^i_l)+b^I_1,b^I_2,\dots,b^I_L)\end{aligned}\tag{6.1}$$

On the other hand, any vector contained in $T_{F_\omega((\boldsymbol{p},x),\varepsilon)}(P\times\boldsymbol{R}^{LI}_{++})$ *is as follows.*

$$(\alpha_1,\dots,\alpha_L,\beta^1_1,\dots,\beta^1_L,\beta^2_1,\dots,\beta^2_L,\dots,\beta^I_1,\dots,\beta^I_L)\tag{6.2}$$

where α_l s are real numbers satisfying $\sum_{l=1}^{L} \alpha_l = 0$ and β_l^i is any real number, $i = 1, \ldots, I$, $L = 1, \ldots, L$. Thus, taking the a_l and b_l such that $a_l = \alpha_l, l = 1, \ldots, L, b_l^i = \beta_l^i, l = 2, \ldots, L$ and

$$\begin{aligned} b_1^i &= \beta_1^i - (1/p_1) \sum_{l=1}^{L} a_l(\omega_l^i - x_l^i), i = 1, \ldots, I-1, \\ b_1^I &= \beta_1^I + (1/p_1) \sum_{i=1}^{I} \sum_{l=1}^{L} a_l(\omega_l^i - x_l^i), \end{aligned}$$

then, (6.1) is equal to (6.2), which implies that $DF_{\omega,((\boldsymbol{p},x),\varepsilon)}$ is surjective.

From the above lemma, I can obtain the following result for problem (2).

Proposition 6.3 *For all u of $\tilde{\mathcal{U}}$ and almost all ω of $\boldsymbol{R}_{++}^{LI}$, $\psi_u^{-1}(0)$ and $\phi_\omega^{-1}(0)$ are transversal in $P \times \boldsymbol{R}_{++}^{LI}$.*

Proof: *Since F_ω is transversal to $\psi_u^{-1}(0)$ for all $\omega \in \boldsymbol{R}_{++}^{LI}$, the transversality theorem (Guillemin and Pollack (1974), p.68) ensures that for almost all ε, $F_\omega(\cdot, \varepsilon) : \phi_\omega^{-1}(0) \to P \times \boldsymbol{R}_{++}^{LI}$ is transversal to $\psi_u^{-1}(0)$ for all ω. Consequently, for almost all ε, the image of $F_\omega(\cdot, \varepsilon)$ transversally intersects $\psi_u^{-1}(0)$ since the image constitutes a submanifold by the construction of F_ω. However, that image is represented as follows.*

$$\begin{aligned} F_\omega(\phi_\omega^{-1}(0), \varepsilon) &= \{(\boldsymbol{p}', x' \in P \times \boldsymbol{R}_{++}^{LI} \mid \boldsymbol{p}' = \boldsymbol{p}, \boldsymbol{x}^{i\prime} = \boldsymbol{x} + \varepsilon^i, i = 1, \ldots, I, (\boldsymbol{p}, x) \in \phi_\omega^{-1}(0)\} \\ &= \{(\boldsymbol{p}', x' \in P \times \boldsymbol{R}_{++}^{LI} \mid (\boldsymbol{p}' \cdot \boldsymbol{\omega}^i = \boldsymbol{p}' \cdot (\boldsymbol{x}' - \varepsilon^i))_{i=1}^{I-1}, \sum_{i=1}^{I} (\boldsymbol{x}^{i\prime} - \varepsilon^i) = \sum_{i=1}^{I} \boldsymbol{\omega}^i\} \\ &= \phi_{\omega+\varepsilon}^{-1}(0). \end{aligned}$$

Since ω is arbitrary and ε covers almost all strictly positive LI-vectors, the claim follows.

The above propositions lead to significant claims about properties of $E(e)$.

Theorem 6.1 *There exists an open and dense set $\tilde{E}$ in E such that the set of Walrasian equilibria is locally unique for every $e \in \tilde{E}$.*

Proof: *Through the propositions, for all u of $\tilde{\mathcal{U}}$ and almost all ω of $\boldsymbol{R}_{++}^{LI}$, $\psi_u^{-1}(0) \cap \phi_\omega^{-1}(0)$ is 0 dimensional submanifold, which implies that $E_{ex}(e)$ is locally unique for almost all e. Since $E(e) \subset E_{ex}(e)$, the claim immediately follows.*

Theorem 6.2 *A Walrasian equilibrium is locally continuous in e in $\tilde{E}$.*

Proof: *Consider the product map* $\phi_\omega \times \psi_u : E \times P \times \boldsymbol{R}^{LI}_{++} \to \boldsymbol{R}^{I-1} \times \boldsymbol{R}^L \times \Pi^I H$ *and restrict it on* $\tilde{E} \times P \times \boldsymbol{R}^{LI}_{++}$*. It follows from the above propositions that the derivative of the restricted map with respect to* $(\boldsymbol{p}, x)$ *is non-singular at any* $(e, \boldsymbol{p}, x) \in \tilde{E} \times E_{ex}(e)$*. By the implicit function theorem, there exists a neighbourhood* $N \times M (\subset \tilde{E} \times P \times \boldsymbol{R}^{LI}_{++})$ *of* $(e, \boldsymbol{p}, x)$ *and a unique continuous function* $f : N \to M$ *such that* $\phi_\omega \times \psi_u|_{\tilde{E} \times P \times \boldsymbol{R}^{LI}_{++}}(e', f(e')) = \phi_\omega \times \psi_u|_{\tilde{E} \times P \times \boldsymbol{R}^{LI}_{++}}(e, f(e)) = 0$ *for* $e' \in N$*. Since* $f(e') \in E_{ex}(e')$ *for* $e' \in N$ *by the construction of* ϕ_ω *and* ψ_u*, a Walrasian equilibrium is locally continuous in* e *of* $\tilde{E}$*.*

Further generic properties are obtainable if some other constraints are added to a utility function. Through those constraints, existence, finiteness etc., prove to be qualified as a generic property. For details, see Nagata (2000, 2001b, 2004).

6.2 Variation of the Thom transversality theorem

As I have stated, the Whitney topology for the function space, which is more appropriate in the economic model adopted here than the compact open topology, is not tractable. Indeed, a space with the Whitney topology is not metrizable and does not have a countable base at any point unless the domain of a function is compact. Here I develop a method which successfully deals with the relevant economical question in the basis of the whitney topology. The key consists in making use of the Thom transversality theorem, which, however, needs mathematical preliminaries to understand. Thus, after giving necessary mathematical notions, I present the theorem and its application to the economical problem under consideration. In the process of proof of some claims, one partly needs to rely on the relation between the compact open topology and the Whitney topology which I cannot afford to comment on; for details, see Nagata (2004b), chap. 6, §1 and chap. 8, §1.

6.2.1 Jet Spaces

The goal in this subsection is to introduce some topologies into a function space. For the sake of simplicity, a function to be considered here is limited to a smooth map from $\boldsymbol{R}^m$ to $\boldsymbol{R}^n$. Thus, I solely deal with the space $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. However, the argument provided below is easily extended to the case of a differentiable map between two manifolds.

An efficient way to introduce some topology into $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is to make use of an equivalence class called a jet in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. As I shall see, it turns out that the space derived by classifying $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ through jets, which is called the jet space, has a very simple structure but provides necessary informations to topologize $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

A jet is defined by the following equivalence relation. Let f, g be two maps in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ and $\bar{x}$ be a point of $\boldsymbol{R}^m$. If $f(\bar{x}) = g(\bar{x})$ and all the partial derivatives at $\bar{x}$ up to r-th order are equal between f and g, then I write it as $f \sim_r g$ at $\bar{x}$. It is easily seen that this relation is an equivalence relation. An equivalence class represented by $f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ according to this relation is called the r-jet of f at $\bar{x}$, which is denoted by $j^r f(\bar{x})$. In particular, the whole set of the equivalence classes for all the maps carrying 0 to 0 is written by $J^r(m,n)$.

To begin with, let's consider the structure of $J^r(m,n)$. It is obvious that $J^r(m,n) = J^r(m,1) \times \ldots \times J^r(m,1)$, which is the n-product of $J^r(m,1)$; thus I have only to think of $J^r(m,1)$. Pick an element out of $J^r(m,1)$ and denote it by $j^r f(0)$. Then, consider Taylor's series about 0 generated by $f : \boldsymbol{R}^m \to \boldsymbol{R}$, i.e.

$$f(x) = \sum_{0 \le \|\alpha\| \le r} \frac{x^\alpha}{\alpha!} \frac{\partial^{\|\alpha\|} f}{\partial x^\alpha}(0) + R(x)$$

where $\alpha = (\alpha_1, \ldots, \alpha_m), \alpha_i$: *integer*, $i = 1, \ldots, m$ and $\|\alpha\| = \alpha_1 + \ldots + \alpha_m$. Thus, $f \sim_r g$ at 0 if and only if their Taylor's series up to r-th order are equal with each other. Accordingly, $j^r f(0)$ itself can be represented by the coefficients of Taylor polynomial up to r-th order as follows.

$$\frac{1}{\alpha!} \frac{\partial^{\|\alpha\|} f}{\partial x^\alpha}(0), \quad 1 \le \|\alpha\| \le r.$$

Conversely, an allocation of coefficients for the polynomial of degree r in m variables without a constant term determines a jet. It is easily seen that there is one to one correspondence between those allocations and $J^r(m,1)$. Since such an allocation always consists of ${}_{m+r}C_r - 1$ real numbers, $J^r(m,1)$ can be seen as equipotent to $\boldsymbol{R}^H$ where $H = {}_{m+r}C_r - 1$. Therefore, $J^r(m,n)$ is equipotent to $\boldsymbol{R}^{nH}$, which implies that I can identify $J^r(m,n)$ with $\boldsymbol{R}^{nH}$.

Now pick any point $(x,y) \in \boldsymbol{R}^m \times \boldsymbol{R}^n$. Let $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)_{(x,y)}$ be the set of equivalent classes, which is derived by the relation "$\sim_r$ at x ", for the maps carrying x to y in which x is called a source and y called a target.

Definition 6.3 *The disjoint union* $\bigcup_{(x,y) \in \boldsymbol{R}^m \times \boldsymbol{R}^n} J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)_{(x,y)}$ *is called the r-jet space on* $\boldsymbol{R}^m \times \boldsymbol{R}^n$ *and is denoted by* $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

Proposition 6.4 $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ *is equipotent to* $\boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m,n)$.

Proof: *For any* $(\bar{x}, \bar{y}) \in \boldsymbol{R}^m \times \boldsymbol{R}^n$, *consider the following coordinate transformations* $\varphi_{\bar{x}} : \boldsymbol{R}^m \to \boldsymbol{R}^m$, $\upsilon_{\bar{y}} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ *given by*

$$\begin{aligned} \varphi_{\bar{x}}(x) &= (x_1 + \bar{x}_1, \ldots, x_m + \bar{x}_m) \\ \upsilon_{\bar{y}}(y) &= (y_1 - \bar{y}_1, \ldots, y_n - \bar{y}_n). \end{aligned}$$

Then, the map $\Psi : J^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \to \boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m,n)$ *defined by*

$$\Psi(j^r f(\bar{x})) = (\bar{x}, \bar{y} = f(\bar{x}), j^r(\upsilon_{\bar{y}} \circ f \circ \varphi_{\bar{x}})(0))$$

is easily proved to be a bijection.

Through this proposition, I can identify the r-jet space $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ with $\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R}^{nH}$. Furthermore, by considering a projection $\pi : J^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \to \boldsymbol{R}^m \times \boldsymbol{R}^n$ defined by $\pi(j^r f(x)) = (x, f(x))$, a triple $(J^r(\boldsymbol{R}^m, \boldsymbol{R}^n), \pi, \boldsymbol{R}^m \times \boldsymbol{R}^n)$ turns out to be a vector bundle with fibre $J^r(m, n)$, so that $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is often called the r-jet bundle (for vector bundles, see Golubitsky and Guillemin (1973), chap.1 §5).

Definition 6.4 *For any map $f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$, the map $j^r f : \boldsymbol{R}^m \to J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ which carries $x \in \boldsymbol{R}^m$ to $j^r f(x)$ (the r-jet of f at x) is called the r-jet extension of f.*

Noting the above proposition, the r-jet extension of f can be represented in component form as follows. That is, for $f = (f_1, \ldots, f_n) \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ I have

$$j^r f : \boldsymbol{R}^m \to \boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m, n)$$
$$j^r f(x) = (x, f(x), (\frac{\partial^{\|\alpha\|} f_i}{\partial x^\alpha}(x))_{1 \le \|\alpha\| \le r, 1 \le i \le n}).$$

Through this observation, I immediately obtain the following claim.

Proposition 6.5 *The r-jet extension of $f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is itself C^∞ class.*

Proof: *Since $f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$, $(\frac{\partial^{\|\alpha\|} f_i}{\partial x^\alpha}(x))_{1 \le \|\alpha\| \le r, 1 \le i \le n}$ are all C^∞ class.*

Note that if r is 0 then the image $j^0 f(\boldsymbol{R}^m)$ of the 0-jet extension of f is the graph of f since $j^0 f(x) = (x, f(x))$.

6.2.2 Whitney Topology

Since I may presume that the r-jet space $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is nothing but $\boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R}^{nH}$, it allows a metric derived from the Euclidean norm. I denote the metric given to $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ by $d(\cdot, \cdot)$.

The basic idea provided below to topologize $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$, roughly speaking, consists in determining how close f is to g by evaluating the distance of their r-jets on $\boldsymbol{R}^m$. To be precise, let f be an arbitrary map in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ and δ be a continuous map from $\boldsymbol{R}^m$ to $\boldsymbol{R}_+$. Then consider the following set.

$$B_\delta(f) = \{g \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n) \mid d(j^r(x), j^r g(x)) < \delta(x), \forall x \in \boldsymbol{R}^m\}$$

I can construct a topology with the family of the sets $B_\delta(f)$ parameterized by δ as a fundamental system of neighborhoods at f.

Definition 6.5 *The topology with $B_\delta(f)$ as a neighborhood basis at each f is called the Whitney C^r topology for $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$.*

Thus, according to this topology, a map g is in a neighborhood of a map f if and only if all of the first r partial derivatives of g are δ-close to f over the whole domain $(\boldsymbol{R}^m)$. In other words, this topology does control the behavior of a map "at infinity". Therefore this is often called the strong topology $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

It is also possible to construct the Whitney C^r topology through a basis. That is, for an open subset U of $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ $(= \boldsymbol{R}^m \times \boldsymbol{R}^n \times \boldsymbol{R}^{nH})$ consider the following set $M(U)$.

$$M(U) = \{f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n) \mid j^r f(\boldsymbol{R}^m) \subseteq U\}.$$

Note that $M(U) \cap M(V) = M(U \cap V)$.

Then the family of sets $\{M(U)\}_U$ form a basis for a topology which is equivalent to the Whitney C^r topology defined above (for proof of the equivalence of those topologies, see Golubitsky and Guillemin (1973), p.43).

Let W_r be the family of all open subsets of $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ in the Whitney C^r topology. Then it is easily shown that $W_r \subset W_l$ whenever $r \leq l$. Thus, I can construct another topology on $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ with $W = \bigcup_{r=0}^\infty W_r$ as a basis, which is called the Whitney C^∞ topology. If a subset of $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is open for the Whitney C^r topology, then obviously so is it for the Whitney C^∞ topology.

From the viewpoint of applicability, the Whitney C^r (or C^∞) topology is not necessarily tractable since it is not metrizable and in fact does not satisfy the first axiom of countability. It is too strong for those properties. However, it does have one nice feature.

Definition 6.6 *A topological space X is a Baire space if the intersection of each countable family of open dense sets in X is dense.*

In general, the countable intersection of open dense sets is called a residual set. Thus, I can say that a Baire space is a topological space in which every residual set is dense.

Proposition 6.6 *The function space $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is a Baire space in the Whitney C^r (and C^∞) topology.*

For proof of this proposition, e.g. see Hirsch (1976), Chap.2, §4.

Given the Whitney C^r (C^∞) topology, I can determine the continuity of various maps between two function spaces. In particular, I give the two propositions for further argument. To this end, let $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)^s$ be the s-product of $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$.

Proposition 6.7 *Let $\theta : C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)^s \to C^\infty(\boldsymbol{R}^{ms}, \boldsymbol{R}^{ns})$ be the map given by*

$$\theta(f_1, \ldots, f_s) = (f_1 \times \ldots \times f_s).$$

Then θ is continuous in the Whitney C^r (C^∞) topology where $(f_1 \times \ldots \times f_s)(x_1, \ldots, x_s) = (f_1(x_1), \ldots, f_s(x_s))$.

For proof of this proposition, see Golubitsky and Guillemin (1974), pp.49-50.

For the other proposition, recall the r-jet extension of $f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$, which can be seen as the map from $\boldsymbol{R}^m$ to $\boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m,n)$. Thus, I newly consider the function space $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m,n))$. Then I have

Proposition 6.8 *Let j^r be the map from $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ to $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m,n))$ given by $j^r(f) = j^r f$. Then j^r is continuous in the Whitney C^r (C^∞) topology.*

For proof of this proposition, see Golubitsky and Guillemin (1974), pp.46-48.

6.2.3 Thom Transversality Theorem and its variation

In this subsection, I state the Thom transversality theorem and its variation. I begin with the following lemma.

Lemma 6.2 *Let $F : P \times M \to N$ be a smooth family of maps and Q be a submanifold in N. If F is transversal to Q, then the set*

$$\{p \in P \mid F_p : M \to N \textit{ is transversal to } Q\}$$

is equal to the set

$$\{p \in P \mid p \textit{ is a regular value of } \pi|_{F^{-1}(Q)}\}$$

where $\pi|_{F^{-1}(Q)}$ is a restriction of the projection $\pi : P \times M \to P$ to $F^{-1}(Q)$.

Proof: *Since F is transversal to Q, for any $(p,x) \in F^{-1}(Q)$ I have*

$$\begin{aligned} DF_{(p,x)}(T_{(p,x)}(P \times x)) + DF_{(p,x)}(T_{(p,x)}(p \times M)) + T_{F(p,x)}Q & \\ = T_{F(p,x)}N & \qquad (6.3) \\ DF_{(p,x)}(T_{(p,x)}(F^{-1}(Q))) = T_{F(p,x)}Q & \qquad (6.4) \end{aligned}$$

Set $P_1 = \{p \in P \mid F_p : M \to N \text{ is transversal to } Q\}$ and $P_2 = \{p \in P \mid p \text{ is a regular value of } \pi|_{F^{-1}(Q)}\}$.

Suppose that $p \in P_1$, then for any $x \in F_p^{-1}(Q)$ I have

$$DF_{p,x}(T_x M) + T_{F_p(x)}Q = T_{F_p(x)}N.$$

Since $DF_{p,x}(T_x M) = DF_{(p,x)}(T_{(p,x)}(p \times M))$, (6.3) and (6.4) yield that

$$DF_{(p,x)}(T_{(p,x)}(P \times x)) \subset DF_{(p,x)}(T_{(p,x)}(p \times M)) + DF_{(p,x)}(T_{(p,x)}(F^{-1}(Q))),$$

hence

$$T_{(p,x)}(P \times x) \subset T_{(p,x)}(p \times M) + T_{(p,x)}(F^{-1}(Q)). \qquad (6.5)$$

Then, consider the derivative $D\pi_{(p,x)} : T_{(p,x)}(P \times M) \to T_pP$ *of the projection* $\pi : P \times M \to P$ *at any* $(p, x) \in P \times M$*. It is obvious that*

$$D\pi_{(p,x)}(T_{(p,x)}(p \times M) + T_{(p,x)}(F^{-1}(Q))) = D\pi_{(p,x)}(T_{(p,x)}(F^{-1}(Q)))$$

and

$$D\pi_{(p,x)}(T_{(p,x)}(P \times x)) = T_pP.$$

Therefore, applying $d\pi_{(p,x)}$ *to both sides of (6.5), I have*

$$T_pP \subset D\pi_{(p,x)}(T_{(p,x)}(F^{-1}(Q))),$$

which means that the derivative of the restriction $\pi|_{F^{-1}(y)}$ *at* (p, x) *is surjective; that is,* p *is a regular value of* $\pi|_{F^{-1}(Q)}$*. Since I can inversely follow this reasoning, the lemma is proved.*

This lemma leads to the following important lemma.

Lemma 6.3 *Let* M, N *and* P *be manifolds and* $Q \subset N$ *be a submanifold of* N*. Let* G *be a topological space and* j *be a map from* G *to* $C^\infty(M, N)$*. Suppose that for each* $g \in G$ *there exists* $p_0 \in P$ *and a continuous map* $\varphi : P \to G$ *such that* $\varphi(p_0) = g$ *and a map* $\Phi : P \times M \to N$ *defined by* $\Phi(p, x) = j(\varphi(p))(x)$ *is smooth and transversal to* Q*. Then the set*

$$\{g \in G \mid j(g) \text{ is transversal to } Q\}$$

is dense in G*.*

Proof: *Set* $Z = \{ g \in G \mid j(g) \text{ is transversal to } Q\}$*. Fix any* $g \in G$*. Noting that the map* $\Phi : P \times M \to N$ *corresponding to* g *is actually a smooth family of maps, it follows from the above lemma that the set*

$$\{p \in P \mid j(\varphi(p)) \text{ is transversal to } Q\}$$

is equal to the set

$$\{p \in P \mid p \text{ is a regular value of } \pi|_{\Phi^{-1}(Q)}\}$$

where $\pi|_{\Phi^{-1}(Q)}$ *is a restriction of the projection* $\pi : P \times M \to P$ *to* $\Phi^{-1}(Q)$*.*

Then consider any open neighborhood $U(g)$ *of* g *in* G*. Since* φ *is continuous,* $\varphi^{-1}(U(g))$ *is an open neighborhood of* p_0 *in* P*. Since the set of regular values of* $\pi|_{\Phi^{-1}(Q)}$ *is, by Sard's theorem, dense in* P*, there exists some regular value* p *of* $\pi|_{\Phi^{-1}(Q)}$ *in* $\varphi^{-1}(U(g))$*. For such a* p*, through the above observation,* $j(\varphi(p))$ *is transversal to* Q*. Thus,* $\varphi(p) \in U(g) \cap Z$*, which implies that* Z *is dense in* G *since* g *is arbitrarily chosen.*

This lemma has very rich applicability, thus called the fundamental lemma. In fact, if G is equal to P and φ is the identity map, then this lemma yields the usual transversality theorem (Guillemin and Pollack (1974), p. 68). Furthermore, it is worth noting that the G in the lemma is only required to be a topological space. Thus I am allowed to choose a function space with Whitney topology as such. The Thom transversality theorem is induced according to this way of thinking.

Before stating the theorem, I need to review jet spaces. For simplicity, I consider the function space $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$. Then the r-jet space on $\boldsymbol{R}^m \times \boldsymbol{R}^n$ denoted by $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is defined as the disjoint union $\bigcup_{(x,y)\in \boldsymbol{R}^m \times \boldsymbol{R}^n} J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)_{(x,y)}$ where an equivalent class $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)_{(x,y)}$ consists of all the maps in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ which carry x to y and have the same partial derivatives at x up to r-th order. Recall that $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is identified with $\boldsymbol{R}^m \times \boldsymbol{R}^n \times J^r(m,n)$ where $J^r(m,n)$ is substantially equal to $\boldsymbol{R}^q$ where $q = k({}_{m+r}C_r - 1)$. Thus $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ itself is considered to be a manifold.

In this connection, there is another important concept; that is, the r-jet extension j^r, which associates each map $f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ with the map $j^r f : \boldsymbol{R}^m \to J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ that carries $x \in \boldsymbol{R}^m$ to $j^r f(x)$ (the r-jet of f at x). I should point out that the above arguments hold for the case in which $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ are respectively replaced by M (m-dimensional manifold) and N (n-dimensional manifold).

Now I am in a position to state the Thom transversality theorem. Since the full proof is tedious, I only give a sketch of it.

Theorem 6.3 *Let M, N be manifolds and Q be a submanifold of the r-jet space $J^r(M,N)$. Then the set*

$$\{f \in C^\infty(M,N) \quad | \quad j^r f : M \to J^r(M,N) \ (the\ r-jet\ extension\ of\ f) \\ is\ transversal\ to\ Q\}$$

is dense in $C^\infty(M,N)$ in the Whitney C^∞ topology.

Proof: *Here is a sketch of the proof. I may concentrate on the case in which $M = \boldsymbol{R}^m$ and $N = \boldsymbol{R}^n$. The point of proof consists in the use of the fundamental lemma along with the Baire property of $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ in the Whitney C^∞ topology.*

To begin with, set $Z = \{f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n) \mid j^r f$ is transversal to $Q\}$.

(1) First observe that there exists a countable compact covering $\{K_j\}$ of Q; that is, $Q = \cup_{j=1}^\infty K_j$ where K_j is compact in $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ for each j. This is because $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is, after all, considered to be a finite dimensional Euclidean space.

(2) Set $Z_j = \{f \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n) \mid j^r f$ is transversal to Q on $K_j\}$ where '$j^r f$ is transversal to Q on K_j' means that

$$Dj^r f_x(\boldsymbol{R}^m) + T_{j^r f(x)} Q = T_{j^r f(x)}(J^r(\boldsymbol{R}^m, \boldsymbol{R}^n))$$

for all x for which $j^r f(x) \in Q$. Note that $Z = \bigcap_{j=1}^\infty Z_j$

(3) I show that Z_j is open and dense in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ in the Whitney C^∞ topology for all j. First, to show the openness of Z_j, consider the set $\tilde{Z}_j$ defined as follows.

$$\{g \in C^\infty(\boldsymbol{R}^m, J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)) \mid g \textit{ is transversal to } Q \textit{ on } K_j\}.$$

In general, I have that for any two manifolds X, Y and a submanifold $W \subset Y$, if W is closed in Y, then the set

$$\{f \in C^\infty(X, Y) \mid f \textit{ is transversal to } W\}$$

is open in $C^\infty(X, Y)$ in the Whitney C^∞ topology (see Golubitsky and Guillemin (1973), Cap II, proposition 4.5).

Since K_j is closed and contained in Q, this claim can be easily applied to show that $\tilde{Z}_j$ is open. Obviously $Z_j = (j^r)^{-1}(\tilde{Z}_j)$, which implies that Z_j is open since the map $j^r : C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n) \to C^\infty(\boldsymbol{R}^m, J^r(\boldsymbol{R}^m, \boldsymbol{R}^n))$ is continuous in the Whitney C^∞ topology.

(4) Secondly I show the density of Z_j. To this end, first consider the projection $\pi : J^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \to \boldsymbol{R}^m$ and take $\pi(K_j)$ which is obviously compact in $\boldsymbol{R}^m$. Then there exists a smooth map $\rho_j : \boldsymbol{R}^m \to \boldsymbol{R}$ such that $\rho_j(x) = 1$ for all x in a neighborhood of $\pi(K_j)$ and that $\rho_j(x) = 0$ for all x outside a compact subset including $\pi(K_j)$. Define the map $\varphi_j : P(m, k; r) \to C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ as follows.

$$\varphi_j(p)(x) = \rho_j(x)p(x) + f(x).$$

Then it is easily seen that φ_j is continuous in the Whitney C^∞ topology. Thus, the fundamental lemma is readily adapted to show that Z_j is dense in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ in the Whitney C^∞ topology.

(5) Since $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ is a Baire space in the Whitney C^∞ topology, I obtain that Z $(= \bigcap_{j=1}^\infty Z_j)$ is dense in $C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^n)$ in the Whitney C^∞ topology.

A remark from the viewpoint of application is in order. As is shown through the above arguments, the Thom transversality theorem is basically *a special case* of the fundamental lemma, which is, however, just the reason that the theorem is important for our practical purposes. To be precise, the theorem is useful for the relevant economical analysis not only because it is concerned with a family of maps accompanying a function space as its parametric space, but also because *the family of maps in the theorem involves the derivatives of a parametric function*. The meaning of this statement will be seen later.

For the sake of the economical application, however, I need to partly modify this theorem. This is because I am required to consider multiple parameters at a time when considering the trade among many agents.

Now, let $C^\infty\big(M, N\big)^k$ be k-product of $C^\infty\big(M, N\big)$ and $J^r\big(M, N\big)^k$ be k-product of r-jet space $J^r\big(M, N\big)$ where M and N are differential manifolds. k-product function of

$f_1 \times \ldots \times f_k \in C^\infty\big(M,N\big)^k$ and its r-jet extension $j^r_k(f_1 \times \ldots \times f_k)$ are defined respectively as follows. That is, $f_1 \times \ldots \times f_k : M^k \to N^k$ is given by

$$f_1 \times \ldots \times f_k(x_1, \ldots, x_k) = (f_1(x_1), \ldots, f_k(x_k))$$

and $j^r_k(f_1 \times \ldots \times f_k) : M^n \to J^r\big(M,N\big)^k$ is given by

$$j^r_k(f_1 \times \ldots \times f_k)(x_1, \ldots, x_k) = (j^r f_1(x_1), \ldots, j^r f_k(x_k)).$$

Then I have a product functional version of the original Thom Transversality Theorem.

Theorem 6.4 *Let Q be a submanifold of $J^r\big(M,N\big)^k$. Then*

$$\mathcal{T} = \Big\{ f_1 \times \ldots \times f_k \in C^\infty\big(M,N\big)^k \,|\, j^r_k(f_1 \times \ldots \times f_k) \text{ is transversal to } Q \Big\}$$

is a residual subset of $C^\infty\big(M,N\big)^k$ in the Whitney C^∞ product topology. So $\mathcal{T}$ is dense in $C^\infty\big(M,N\big)^k$.

Proof: *It is sufficient to prove the theorem when $M = \boldsymbol{R}^m$ and $N = \boldsymbol{R}^n$. Consider the set of a polynominal function from $\boldsymbol{R}^m$ to $\boldsymbol{R}^n$ the order of which is less than r. Let the set be $P(m,n;r)$. It is easily shown that $P(m,n;r)$ constitutes a manifold of dimension $m({}_{m+r}C_r)$. Put $V = M^k$, $W = J^r(M,N)^k$, $F = C^\infty(M,K)^k$ and $j = j^r_k$ in the fundamental lemma. Consider n-product of $P(m,n;r)$ (denoted by $P(m,n;r)^k$) and a continuous mapping $\varphi : P(m,n;r)^k \to C^\infty(M,N)^k$ defined by*

$$\varphi(e)(x) = \big(e_1(x_1) + f_1(x_1), e_2(x_2) + f_2(x_2), \ldots, e_k(x_k) + f_k(x_k)\big)$$

for any $f = (f_1, \ldots, f_k)$ of $C^\infty(M,N)^k$ where $e = (e_1, \ldots, e_k) \in P(m,n;r)^k$ and $x = (x_1, \ldots, x_k) \in M^k$.

Then the map $\Phi : P(m,n;r)^k \times M^k \to J^r(M,N)^k$ defined by

$$\Phi(e,x) = j(\varphi(e))(x) = \big(j^r(e_1 + f_1)(x_1), j^r(e_2 + f_2)(x_2), \ldots, j^r(e_k + f_k)(x_k)\big)$$

is obviously a submersion, so transversal to any submanifold of $J^r(M,N)^k$. Thus, we may apply the fundamental lemma and obtain the desired consequence.

6.2.4 Application to economic analysis

Here I present a method of genericity analysis to economic issues in the basis of the Whitney topology. However, one should particularly notice that the method is *not* independent of the intersection-based approach provided above *but* complementary to it. Though this method is applicable to various situations, I take as an illustration the same problem as in the previous section for the sake of consistency and simplicity of the argument. It is, however, worth noting that the analysis of chapter 6 of this dissertaion substantially follows the same procedure as the one described here.

Thus, the problem is to investigate generic properties of $E_{ex}(e)$ (the set of extended Walrasian equilibria). This time, however, I have to work with the utility function space endowed with the Whitney topology, which requires some modification in the analysis. Among the two maps, ψ_u and ϕ_ω which are used to characterize $E_{ex}(e)$, I modify ψ_u as follows, leaving ϕ_ω as it is. For the sake of simplicity, I denote $\Pi^I P$ by P^I etc.

$$\tilde{\psi}_u : P \times \boldsymbol{R}^{LI}_{++} \to P^{I+1}$$
$$\tilde{\psi}_u(\boldsymbol{p}, x) = (Du^1_{\boldsymbol{x}}/|Du^1_{\boldsymbol{x}}|, \dots, Du^I_{\boldsymbol{x}}/|Du^I_{\boldsymbol{x}}|, \boldsymbol{p})$$

where $|\cdot|$ indicates the sum of the elements and one should recall that P is the $L-1$ dimensional strictly positive unit simplex. Let $\mathcal{D}$ be the diagonal set of P^{I+1}, i.e., $\mathcal{D} \equiv \{(\boldsymbol{a}^1, \dots, \boldsymbol{a}^{I+1}) \in P^{I+1} \mid \boldsymbol{a}^1 = \dots = \boldsymbol{a}^{I+1}\}$. Then, obviously $\psi_u^{-1}(0) = \tilde{\psi}_u^{-1}(\mathcal{D})$. Thus, in order to obtain a generic consequence similar to the one in the previous section, I have only to show that there exists a dense set in $\mathcal{U}$ such that for any u in the set, $\tilde{\psi}_u$ is transversal to $\mathcal{D}$. Indeed, if $\tilde{\psi}_u$ is transversal to $\mathcal{D}$, $\tilde{\psi}_u^{-1}(\mathcal{D})$ constitutes a LI dimensional submanifold (Guillemin and Pollack (1974), p. 60, Theorem). It is worth noting that I may truncate $\tilde{\psi}_u$ to this end. That is, it is easily shown that if the map $\varphi_u : \boldsymbol{R}^{LI}_{++} \to P^I$ given by

$$\varphi_u(\boldsymbol{x}^1, \dots, \boldsymbol{x}^I) = (Du^1_{\boldsymbol{x}}/|Du^1_{\boldsymbol{x}}|, \dots, Du^I_{\boldsymbol{x}}/|Du^I_{\boldsymbol{x}}|)$$

is transversal to $\tilde{\mathcal{D}}$, then $\tilde{\psi}_u$ is transversal to $\mathcal{D}$, where $\tilde{\mathcal{D}}$ is the diagonal set of P^I. Thus, I may concentrate on φ_u instead of $\tilde{\psi}_u$.

It is worth noting that φ_u only involves first order derivatives of u^i. Thus, I properly consider the 1-jet space $J^1\big(\boldsymbol{R}^L_{++}, \boldsymbol{R}\big)$, which is naturally identified with $\boldsymbol{R}^L_{++} \times \boldsymbol{R} \times \boldsymbol{R}^L$. Let $J^1_+\big(\boldsymbol{R}^L_{++}, \boldsymbol{R}\big)$ denote $\boldsymbol{R}^L_{++} \times \boldsymbol{R} \times \boldsymbol{R}^L_{++}$ which is obviously an open submanifold of $J^1\big(\boldsymbol{R}^L_{++}, \boldsymbol{R}\big)$. Then I define the map $\phi : J^1_+\big(\boldsymbol{R}^L_{++}, \boldsymbol{R}\big) \to \boldsymbol{R}^L_{++} \times \boldsymbol{R} \times P$ by

$$\phi(\boldsymbol{a}, b, c_1, \dots, c_L) = (\boldsymbol{a}, b, c_1/\sum_{l=1}^{L} c_i, \dots, c_L/\sum_{l=1}^{L} c_i).$$

Furthermore, I consider the I-product function of ϕ (denoted by $\Phi : J^1_+\big(\boldsymbol{R}^L_{++}, \boldsymbol{R}\big)^I \to (\boldsymbol{R}^L_{++} \times \boldsymbol{R} \times P)^I)$ given by

$$\Phi(y^1, \dots, y^I) = (\phi(y^1), \dots, \phi(y^I)).$$

Then, it is easily seen that the set $\{u \in \mathcal{U} \mid \varphi_u \textit{ is transversal to } \tilde{\mathcal{D}}\}$ is equal to the set $\{u \in \mathcal{U} \mid \Phi \circ j^1_I u \textit{ is transversal to } W\}$, where $W = \{(\boldsymbol{a}^i, b^i, \boldsymbol{c}^i)_i \in (\boldsymbol{R}^L_{++} \times \boldsymbol{R} \times P)^I \mid \boldsymbol{c}^1 = \boldsymbol{c}^2 = \dots = \boldsymbol{c}^I\}$. Thus, in order to reach the goal, I have only to show that the latter set is dense in $\mathcal{U}$. To this end, I prepare two lemmas.

Lemma 6.4 *Let X, Y and Z be smooth manifolds with W a submanifold of Z and $f : X \to Y$ be a smooth mapping. Assume that there exists a submersion $g : Y \to Z$ and f is transversal to $g^{-1}(W)$. Then $g \circ f$ is transversal to W.*

Proof: *When g is a submersion, g is transversal to W and $g^{-1}(W)$ constitutes a submanifold of Y. So I obtain that (i) $dg_y(T_yY) + T_{g(y)}W = T_{g(y)}Z$ and (ii) $dg_y\big(T_y g^{-1}(W)\big) = T_{g(y)}W$. Now assume further that f is transversal to $g^{-1}(W)$, then for every $x \in (g \circ f)^{-1}(W)$ I have*

$$Df_x(T_xX) + T_{f(x)}g^{-1}(W) = T_{f(x)}Y.$$

Applying $dg_{f(x)}$ to both sides of the above equation, I obtain

$$Dg_{f(x)} \circ df_x(T_xX) + T_{g\circ f(x)}W = Dg_{f(x)}(T_{f(x)}Y). \qquad \text{(by (ii))}$$

Thus I have

$$\begin{aligned} Dg_{f(x)} \circ Df_x(T_xX) + T_{g\circ f(x)}W &= Dg_{f(x)}(T_{f(x)}Y) + T_{g\circ f(x)}W \\ &= T_{g\circ f(x)}Z, \qquad \text{(by (i))} \end{aligned}$$

which implies that $g \circ f$ is transversal to W.

Lemma 6.5 *A smooth map $f : \boldsymbol{R}^{LI}_{++} \to P^I$ defined by*

$$f(\boldsymbol{x}^1, \dots, \boldsymbol{x}^I) = (\boldsymbol{x}^1/|\boldsymbol{x}^1|, \dots, \boldsymbol{x}^I/|\boldsymbol{x}^I|)$$

is a submersion.

Proof: *Considering the structure of f, it is sufficient to show that a map $h : \boldsymbol{R}^L_{++} \to P$ defined by $h(\boldsymbol{x}) = \boldsymbol{x}/|\boldsymbol{x}|$ is a submersion. To this end, I show that at any $\boldsymbol{x}(\in \boldsymbol{R}^L_{++})$, $Dh_{\boldsymbol{x}} : \boldsymbol{R}^L \to \boldsymbol{R}^{L-1}$ is surjective. It is easily seen that i-th colum vector of $Dh_{\boldsymbol{x}}$ is*

$$\left(\frac{x_1}{(\sum_{l=1}^L x_l)^2}, \dots, \frac{x_{i-1}}{(\sum_{l=1}^L x_l)^2}, \frac{\sum_{j\neq i}^L x_j}{(\sum_{l=1}^L x_l)^2}, \frac{x_{i+1}}{(\sum_{l=1}^L x_l)^2}, \dots, \frac{x_L}{(\sum_{l=1}^L x_l)^2} \right)^t$$

where t designates the transpose. Thus I see that first $L-1$ column vectors of $Dh_{\boldsymbol{x}}$ are linearly independent. Indeed, set

$$\sum_{l=1}^{L-1} \lambda_i \left(\frac{x_1}{(\sum_{l=1}^L x_l)^2}, \dots, \frac{x_{i-1}}{(\sum_{l=1}^L x_l)^2}, \frac{\sum_{j\neq i}^L x_j}{(\sum_{l=1}^L x_l)^2}, \frac{x_{i+1}}{(\sum_{l=1}^L x_l)^2}, \dots, \frac{x_L}{(\sum_{l=1}^L x_l)^2} \right)^t = 0$$

where each λ_i is a scalar ($i = 1, \dots, L-1$). Then its L-th equation yields $\sum_{l=1}^{L-1} \lambda_l = 0$, which in turn makes each k-th equation turn into $\lambda_k/(\sum_{i=l}^L x_l) = 0$ ($k = 1, \dots, L-1$). It follows that $\lambda_1 = \lambda_2 = \dots = \lambda_{L-1} = 0$. So $Dh_{\boldsymbol{x}}$ is surjective.

In light of these lemmas and the fact that W is actually a submanifold of $(\boldsymbol{R}^L_{++} \times \boldsymbol{R} \times P)^I$, I obtain the desired result through the following proposition.

Proposition 6.9 *Let V be a submanifold of $(\boldsymbol{R}^L_{++} \times \boldsymbol{R} \times P)^I$. Then*

$$\mathcal{T} = \left\{ u \in \mathcal{U} \,|\, \Phi \circ j^1_I u \textit{ is transversal to } V \right\}$$

is dense in $\mathcal{U}$.

Proof: *Note that when u is an element of $\mathcal{U}$, then the range of $j^1_I u$ is not $J^1\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big)^I$ but $J^1_+\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big)^I$. Take $\boldsymbol{R}^{(S+1)I}_{++}$, $J^1_+\big(\boldsymbol{R}^{S+1}_{++}, \boldsymbol{R}\big)^I$ and $(\boldsymbol{R}^{S+1}_{++} \times \boldsymbol{R} \times P)^I$ for X,Y and Z in lemma 6.4. Then if I show that Φ is a submersion, the proof is completed by lemma 6.4 and theorem 6.2. But by the structure of Φ, to show its submersiveness I have only to show that the map $\zeta : \boldsymbol{R}^L I_{++} \to P^I$ defined by*

$$\zeta(\boldsymbol{c}^1, \boldsymbol{c}^2, \dots, \boldsymbol{c}^I) = \left(\frac{\boldsymbol{c}^1}{\sum_{l=1}^L c^1_l}, \frac{\boldsymbol{c}^2}{\sum_{l=1}^L c^2_l}, \dots, \frac{\boldsymbol{c}^I}{\sum_{l=1}^L c^I_l} \right)$$

is a submersion. By applying lemma 6.5, I have that this map is indeed a submersion.