A Note on the Pre-Kernel and Pre-Nucleolus for Bankruptcy Games

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A commonly held belief of game theorists is that the pre-nucleolus of the modest bankruptcy game is identical to the pre-nucleolus of its dual game, the greedy bankruptcy game. In this paper we provide some results that indicate that this belief is false. We establish some non-coincidence results of the pre-kernel and pre-nucleolus of the modest bankruptcy game with respect to the pre-kernel and pre-nucleolus of its corresponding greedy bankruptcy game. By clarifying the game theoretical solution concepts applied in the literature we are able to show that the pre-kernel and the pre-nucleolus coincide with the anti-pre-kernel and the anti-pre-nucleolus of its associated greedy bankruptcy game, respectively. Probably this is the cause of the belief that both nucleoli are identical. At the end of the paper, we highlight the results with some illustrative examples.

1. Introduction

In Driessen (1998) the relationship between the pre-kernel solution related to the modest bankruptcy game, as it was introduced by O'Neill (1982), and some related pre-kernel-like solutions for its dual game representation were investigated. As his first step, Driessen introduced a dual representation of the modest bankruptcy game that provides an alternative approach to model general bankruptcy problems: the greedy bankruptcy game. Then he developed a new solution concept to solve in a consistent way the greedy bankruptcy game that was henceforth known in the literature as the greedy-pairwise-bargained consistent allocation. This solution is a generalization of the pairwise-bargained consistency solution that solves a general two-creditor bankruptcy problem. Driessen showed the coincidence of the greedy-pairwise consistent allocation with a pre-kernel-like solution of the greedy bankruptcy problem, the anti-pre-kernel solution of a transferable utility game (TU-game). According to its formal definition, the anti pre-kernel of a

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‡ The research of this paper was done while the second author was visiting the Waseda University. The second author also gratefully acknowledges financial support of the Waseda University provided by the COE-21 GLOPE project.

TU-game is simply the reflection of the pre-kernel solution, since instead of balancing the maximal excesses among all possible pairs of players, it balances the minimal excesses between each pair of players. A further result of Driessen was that the set of all greedy-pairwise-bargained consistent allocations coincides with the intersection of the anti-pre-kernel and the anti-core of the greedy bankruptcy game, which had been considered as the intersection of the pre-kernel and core of the greedy bankruptcy game.

According to this interpretation of solution concepts, Driessen came up with the conclusion that the nucleolus of the greedy bankruptcy game must be contained in the intersection of the both solution concepts. However, it is actually contained in the intersection of the anti-pre-kernel and the anti-core of the greedy bankruptcy game. Thus, rather than conclude that the unique greedy-pairwise-bargained consistent allocation has to coincide with the nucleolus, it turns out, as it will become clearer in the sequel of our paper, that it actually coincides with the anti-nucleolus of the greedy bankruptcy game. Moreover, it is not difficult to establish that the anti-nucleolus of the greedy bankruptcy game coincides with the nucleolus of its associated modest bankruptcy game. This stands in contrast to the commonly held belief that the nucleoli of the modest and greedy bankruptcy games are identical.

The remainder of the paper is organized as follows: In Section 2, we introduce the definitions of the different solution concepts related to the pre-kernel and pre-nucleolus, as well as the definitions of the general bankruptcy problem and the modest and greedy bankruptcy games. Then in Section 3, we provide our coincidence results. The first coincidence result is related to the dual pre-kernel of a transferable utility game and its anti-pre-kernel, whereas our second result establishes a coincidence of the dual pre-nucleolus and its anti-pre-nucleolus. Some illustrative examples related to the non-coincidence of the nucleolus for the modest bankruptcy game and the nucleolus of the associated greedy bankruptcy game are given in Section 4. Section 5, the final section, provides some concluding remarks.

2. Definitions and Notations

An n-person cooperative game with side payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1, 2, \cdots, n\}$ represents the player set and v is the characteristic function with $v: 2^N \rightarrow \mathbb{R}$ and the convention that $v(\emptyset) := 0$. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. Formally, we identify a cooperative game by the vector $v := (v(S))_{S \subseteq N} \in \mathbb{R}^{2^{|N|}}$. The *dual* $v^*: 2^N \rightarrow \mathbb{R}$ of the game v is defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$. A vector $\vec{x} := (x_k)_{k \in N} \in \mathbb{R}^N$ represents a payoff distribution on the player set N, where the payoff of player k is denoted by x_k for all $k \in N$. A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a vector $\vec{x} \in \mathbb{R}^N$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\vec{x}$ with the corresponding measure on S, such that $x(S) = \sum_{k \in S} x_k$. The set of vectors $\vec{x} \in \mathbb{R}^N$ which satisfy the efficiency principle

$v(N) = x(N)$ is called the *preimpu-tation set* and it is defined by

$$\mathcal{I}'(v) := \{\vec{x} \in \mathbb{R}^N | x(N) = v(N)\},$$

where an element $\vec{x} \in \mathcal{I}'(v)$ is called a pre-imputation. A vector $\vec{x}$ that satisfies in addition individual rationality, that is, $x_i \geq v(\{i\})$ for all $i \in N$, is called an imputation. The set of all imputations is denoted by $\mathcal{I}(v)$.

Given a vector $\vec{x} \in \mathcal{I}'(v)$, we define the *excess* of coalition S with respect to the imputation $\vec{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S, \vec{x}) := v(S) - x(S).$$

A non-negative (non-positive) excess of S at $\vec{x}$ in the game $\langle N, v\rangle$ represents a gain (loss) to the members of the coalition S, if the members of S do not accept the payoff distribution $\vec{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game v. For any pair of players $i, j \in N, i \neq j$, the *maximum surplus* of player i over player j with respect to the pre-imputation $\vec{x} \in \mathcal{I}'(v)$, is given by the maximum excess at $\vec{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\vec{x}, v) := \max_{S \in G_{ij}} e^v(S, \vec{x})$$

where

$$G_{ij} := \{S | S \ni i \text{ and } S \not\ni j\}.$$

The expression $s_{ij}(\vec{x}, v)$ describes the maximal amount at the pre-imputation $\vec{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\vec{x} \in \mathcal{I}'(v)$ that balance the maximum surpluses for each distinct pair of player $i, j \in N, i \neq j$ is called the *pre-kernel* of the game v, and is defined by

$$\mathcal{PrK}(v) := \{\vec{x} \in \mathcal{I}'(v) | s_{ij}(\vec{x}, v) = s_{ji}(\vec{x}, v) \text{ for all } i, j \in N, i \neq j\}. \qquad (1)$$

The pre-kernel of the dual game v^*, known as the *dual pre-kernel*, is denoted by $\mathcal{PrK}^*(v)$.Here $\mathcal{PrK}^*(v) = \mathcal{PrK}(v^*)$ for any v.

Now consider the reverse relationship for a game v, such that for any pair of players $i,j \in N$, $i \neq j$, the minimum surplus (maximum loss) of player i over player j with respect to the pre-imputation $\vec{x} \in \mathcal{I}'(v)$, is given by the minimum excess at $\vec{x}$ over the set of coalitions containing player i but not player j. The *anti-surplus* (minimum surplus) is defined by

$$s^{\#}_{ij}(\vec{x}, v) := \min_{S \in G_{ij}} e^v(S, \vec{x})$$

The expression $s^{\#}_{ij}(\vec{x}, v)$ describes the minimal amount at the pre-imputation $\vec{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\vec{x} \in \mathcal{I}'(v)$ that balance the minimum surpluses for each distinct pair of player $i, j \in N, i \neq j$, is called the *anti-pre-kernel* of the game v, and is defined by

$$\mathcal{PrK}^{\#}(v) := \{\vec{x} \in \mathcal{I}'(v) | s^{\#}_{ij}(\vec{x}, v) = s^{\#}_{ji}(\vec{x}, v) \text{ for all } i, j \in N, i \neq j\}. \qquad (2)$$

As the solution concept of the dual pre-kernel, we introduce the notion of an anti-pre-kernel of the dual game v^*, which will be called the *dual anti-pre-kernel*. This is denoted by $(\mathcal{P}r\mathcal{K}^{\#})^{*}(v)$. Here $(\mathcal{P}r\mathcal{K}^{\#})^{*}(v) = \mathcal{P}r\mathcal{K}^{\#}(v^{*})$ for any v.

Consider a TU-game v and a closed set X of vectors in $\mathbb{R}^N$. For each vector $\vec{x}$, we define a vector $\theta^v(\vec{x}) \in \mathbb{R}^{2^n}$ as

$$\theta^v(\vec{x}) := (e^v(S_1, \vec{x}), e^v(S_2, \vec{x}), \cdots, e^v(S_{2^n}, \vec{x})),$$

where the excesses for all coalitions are given in decreasing order. The vector $\theta^v(\vec{x})$ is said to be *lexicographically smaller than* $\theta^v(\vec{y})$, that is $\theta^v(\vec{x}) < \theta^v(\vec{y})$ if there is a positive integer q such that $\theta^v_k(\vec{x}) = \theta^v_k(\vec{y})$ whenever $k < q$ and $\theta^v_q(\vec{x}) < \theta^v_q(\vec{y})$. The *general nucleolus* of the arbitrary set X, denoted by $\mathcal{N}(X, v)$, is the set of vectors in X whose θ^v's have lexicographically minimized the excesses, hence

$$\mathcal{N}(X, v) := \{\vec{x} \in X \mid \theta^v(\vec{x}) \leq \theta^v(\vec{y}) \text{ for all } \vec{y} \in X\}.$$

If $X = \mathcal{I}'(v)$ or $X = \mathcal{I}(v)$, then the general nucleolus is called the *pre-nucleolus*, respectively the *nucleolus of the game* v, that is denoted by $\mathcal{P}r\mathcal{N}(v)$ and $\mathcal{N}(v)$. On the analogy of the definition of the dual pre-kernel, the dual pre-nucleolus is denoted by $\mathcal{P}r\mathcal{N}^{*}(v)$. Here $\mathcal{P}r\mathcal{N}^{*}(v) = \mathcal{P}r\mathcal{N}(v^{*})$ for any v again.

Now consider for each vector $\vec{x}$ the reverse excess order by defining a vector $\theta^{\# v}(\vec{x}) \in \mathbb{R}^{2^n}$ as

$$\theta^{\# v}(\vec{x}) := (e^v(S_1, \vec{x}), e^v(S_2, \vec{x}), \cdots, e^v(S_{2^n}, \vec{x})),$$

where the excesses for all coalitions are given in increasing order. The vector $\theta^{\# v}(\vec{x})$ is said to be *lexicographically larger than* $\theta^{\# v}(\vec{y})$, that is $\theta^{\# v}(\vec{x}) > \theta^{\# v}(\vec{y})$ if there is a positive integer q such that $\theta^{\# v}_k(\vec{x}) = \theta^{\# v}_k(\vec{y})$ whenever $k < q$ and $\theta^{\# v}_q(\vec{x}) > \theta^{\# v}_q(\vec{y})$. The *general anti-nucleolus* of the arbitrary set X, denoted by $\mathcal{N}^{\#}(X, v)$, is the set of vectors in X whose $\theta^{\#}$'s have lexicographically maximized the excesses, hence

$$\mathcal{N}^{\#}(X, v) := \{\vec{x} \in X \mid \theta^{\# v}(\vec{x}) \geq \theta^{\# v}(\vec{y}) \text{ for all } \vec{y} \in X\}.$$

If $X = \mathcal{I}'(v)$, then the general antinucleolus is called the anti-pre-nucleolus of the game v, that is denoted by $\mathcal{P}r\mathcal{N}^{\#}(v)$. To complete our definitions, the dual anti-pre-nucleolus is denoted by $(\mathcal{P}r\mathcal{N}^{\#})^{*}(v)$. Here $\mathcal{P}r\mathcal{N}^{\#}(v^{*}) = (\mathcal{P}r\mathcal{N}^{\#})^{*}(v)$ for any v.

Remark 2.1.

It is not difficult to prove that for any two person games v, $\mathcal{P}r\mathcal{N}^{\#}(v) = \mathcal{P}r\mathcal{K}(v) = \mathcal{P}r\mathcal{N}^{\#}(v) = \mathcal{P}r\mathcal{N}(v)$. These solutions become what is known as the *standard solution* for two person games, which is given by

$$x_i = \frac{v(\{i\}) - v(\{j\}) + v(\{i, j\})}{2} \quad \text{and}$$
$$x_j = \frac{v(\{j\}) - v(\{i\}) + v(\{i, j\})}{2}$$
$$\text{for } N = \{i, j\},\ i \neq j.$$

A *general bankruptcy problem* is an ordered pair $(E, \vec{d})$, where E is a real number and $\vec{d}$ is a vector in $\mathbb{R}^N$ such that $d_k \geq 0$ for all $k \in N$ and

$$0 < E \leq \sum_{k \in N} d_k.$$

One can think of a bankrupt firm with an estate of value E which has to be divided among n creditors of the firm with claim $d_k \geq 0$ for each claimant $k \in N$. The amount of estate E of the firm is insufficient to match the total amount of claims. The nonnegative surplus of debts is defined by $\Delta := \sum_{k \in N} d_k - E$ and specifies the amount of debts that can not be covered by the estate. Since $\Delta \geq 0$, the problem arises how to allocate the estate E to the claimants of the bankrupt firm. A general solution $\vec{x} \in \mathbb{R}^N$ related to a bankruptcy situation $(E, \vec{d})$ must satisfy at least the efficiency principle $\sum_{k \in N} x_k = E$, where the number x_k represents the amount given to claimant k.

A *modest bankruptcy game* $v_{E,d} : 2^N \rightarrow \mathbb{R}$ corresponding to a general bankruptcy problem $(E, \vec{d})$ is defined by

$$v_{E,d}(S) := \max[0, E - \sum_{k \in N \setminus S} d_k] \quad \text{for all } S \subseteq N, S \neq \emptyset. \quad (3)$$

In the literature, there is also a dual game representation of a bankruptcy problem, that is known as the *greedy bankruptcy game*. A greedy bankruptcy game $w_{E,d} : 2^N \rightarrow \mathbb{R}$ corresponding to a general bankruptcy problem $(E, \vec{d})$ is defined by

$$w_{E,d}(S) := \min[E, \sum_{k \in S} d_k] \quad \text{for all } S \subseteq N, S \neq \emptyset. \quad (4)$$

Notice that the modest $v_{E,d}$ and greedy bankruptcy game $w_{E,d}$ are dual to each other, since, for all $S \subseteq N$, $S \neq \emptyset$,

$$\begin{aligned}(v_{E,d})^*(S) &= v_{E,d}(S) - v_{E,d}(N \setminus S) \\ &= E - \max[0, E - d(S)] \\ &= \min[E, d(S)] \\ &= w_{E,d}(S)\end{aligned}$$

and, for all $S \subseteq N$, $S \neq \emptyset$,

$$\begin{aligned}(w_{E,d})^*(S) &= (v_{E,d})^{**}(S) \\ &= v_{E,d}(S).\end{aligned}$$

3. Main Results

Theorem 3.1. *Let v be a transferable utility game. Then the dual pre-kernel of game v coincides with the anti-pre-kernel of v, that is*

$$\mathcal{P}r\mathcal{K}^*(v) = \mathcal{P}r\mathcal{K}^\#(v).$$

Proof. Using the definition of the dual game v^* and the definition of the pre-kernel (1), then we obtain

$$\begin{aligned}s_{ij}(\vec{x}, v^*) &= \max_{S \in G_{ij}}[v^*(S) - x(S)] \\ &= \max_{S \in G_{ik}}[v(N) - v(N \setminus S) - x(S)] \\ &= \max_{N \setminus S \in G_{ij}}[-v(N \setminus S) + x(N \setminus S)] \\ &= -\min_{T \in G_{ij}}[v(T) - x(T)] \\ &= -s^\#_{ji}(\vec{x}, v).\end{aligned}$$

Thus, the maximal surplus of player i against player j in the dual game v^* is equal to the negative of the anti-surplus of player j against player i in the game v. Applying the definition of the pre-kernel, we get

$$\begin{aligned}& s_{ij}(\vec{x}, v^*) = s_{ji}(\vec{x}, v^*) \\ \Longleftrightarrow\ & s^\#_{ji}(\vec{x}, v) = s^\#_{ij}(\vec{x}, v).\end{aligned}$$

Thus, the dual pre-kernel coincides with the anti-pre-kernel. □

Remark 3.1.

It should be obvious that due to $v^{**} = v$, we have $\mathcal{P}r\mathcal{K}(v^{**}) = (\mathcal{P}r\mathcal{K}^*)^*(v) = \mathcal{P}r\mathcal{K}(v)$, and the pre-kernel of the game v is identical to the anti-pre-kernel of the dual game v^*, hence it holds that $\mathcal{P}r\mathcal{K}(v) = (\mathcal{P}r\mathcal{K}^\#)^*(v)$.

Corollary 3.1. *Let $v_{E,d}$ be the modest bankruptcy game and let $w_{E,d}$ be the greedy bankruptcy game as given by* (3) *and* (4), *respectively. Then the pre-kernel of the greedy bankruptcy game coincides with the anti-pre-kernel of the modest bankruptcy game, that is*

$$\mathcal{P}r\mathcal{K}(w_{E,d}) = \mathcal{P}r\mathcal{K}^\#(v_{E,d}).$$

Theorem 3.2. *Let v be a transferable utility game. Then the dual pre-nucleolus of game v coincides with the anti-pre-nucleolus of the game v, that is*

$$\mathcal{P}r\mathcal{N}^*(v) = \mathcal{P}r\mathcal{N}^\#(v).$$

Proof. The dual pre-nucleolus is the unique vector in $\mathcal{I}'(v)$ whose θ^{v^*}'s have lexicographically minimized the excesses, hence

$$\mathcal{P}r\mathcal{N}^*(v) := \{\vec{x} \in \mathcal{I}'(v) | \theta^{v^*}(\vec{x}) \leq \theta^{v^*}(\vec{y}) \text{ for all } \vec{y} \in \mathcal{I}'(v)\}. \quad (5)$$

Using the definition of the dual game v^*, we obtain for an arbitrary vector $\vec{y}$ the following equality:

$$\begin{aligned}\theta^{v^*}(\vec{y}) &= (e^{v^*}(S_1, \vec{y}), e^{v^*}(S_2, \vec{y}), \cdots, e^{v^*}(S_{2^n}, \vec{y})) \\ &= (v^*(S_1) - y(S_1), v^*(S_2) - y(S_2), \cdots, v^*(S_{2^n}) - y(S_{2^n})) \\ &= (v(N) - v(N \backslash S_1) - y(S_1), v(N) - v(N \backslash S_2) - y(S_2), \cdots, v(N) - v(N \backslash S_{2^n}) - y(S_{2^n})) \\ &= -(e^v(T_1, \vec{y}), e^v(T_2, \vec{y}), \cdots, e^v(T_{2^n}, \vec{y})) \\ &= -\theta^{\# v}(\vec{y}).\end{aligned}$$

Plugging in this result in the set (5), thus we get

$$\begin{aligned}\mathcal{P}r\mathcal{N}^*(v) &= \{\vec{x} \in \mathcal{I}'(v) | -\theta^{\# v}(\vec{x}) \leq -\theta^{\# v}(\vec{y}) \text{ for all } \vec{y} \in \mathcal{I}'(v)\} \\ &= \{\vec{x} \in \mathcal{I}'(v) | \theta^{\# v}(\vec{x}) \geq \theta^{\# v}(\vec{y}) \text{ for all } \vec{y} \in \mathcal{I}'(v)\} \\ &= \mathcal{P}r\mathcal{N}^\#(v),\end{aligned}$$

whose $\theta^\#$'s have lexicographically maximized the excesses. Hence the unique vector $\vec{x}$ that determines the dual pre-nucleolus of the game v coincides with the set of vectors $\vec{x}$ that specify the anti-pre-nucleolus of the game v. □

Remark 3.2.
By analogy with Remark 3.1., it should be clear that $\mathcal{P}r\mathcal{N}(v^{**}) = (\mathcal{P}r\mathcal{N}^*)^*(v) = \mathcal{P}r\mathcal{N}(v)$, and that the pre-nucleolus of the game v is identical to the anti-pre-nucleolus of the dual game v^*. Hence it holds that $\mathcal{P}r\mathcal{N}(v) = (\mathcal{P}r\mathcal{N}^\#)^*(v)$.

Corollary 3.2. *Let $v_{E,d}$ be the modest bankruptcy game and let $w_{E,d}$ be the greedy bankruptcy game as given by* (3) *and* (4), *respectively. Then the pre-nucleolus of the greedy bankruptcy game coincides with the anti-pre-nucleolus of the modest bankruptcy game, that is*

$$\mathcal{P}r\mathcal{N}(w_{E,d}) = \mathcal{P}r\mathcal{N}^\#(v_{E,d}).$$

Remark 3.3.
By combining Corollaries 3.1., 3.2. and Remark 2.1. in the previous section, for

any two person games v, it holds that $\mathcal{P}r\mathcal{K}^{*}(v) = \mathcal{P}r\mathcal{K}^{\#}(v) = \mathcal{P}r\mathcal{K}(v) = \mathcal{P}r\mathcal{N}^{*}(v) = \mathcal{P}r\mathcal{N}^{\#}(v) = \mathcal{P}r\mathcal{N}(v)$.

4. Examples

In this section, we want to discuss some examples to demonstrate that the pre-nucleolus of a modest bankruptcy game does not equal the pre-nucleolus of its dual game, the greedy bankruptcy game. The examples will highlight the results provided by Theorem 3.2. and Corollary 3.2. that the pre-nucleolus of the greedy bankruptcy game equals the anti-pre-nucleolus of the associated modest bankruptcy game. In order that the reader can better follow our arguments and for an ease of computation, we introduce first the Talmudic solution, that is a solution for a generalized bankruptcy situation. This solution is identical to the pre-nucleolus solution for the associated modest bankruptcy game. A proof of the coincidence of both solution concepts was provided in Aumann and Maschler (1985).

Before going into details, we should provide the reader with a short note that, for a two person bankruptcy game, the pre-kernel of a modest bankruptcy game coincides with the pre-kernel of a greedy bankruptcy game because of Remark 3.3. This also holds for the pre-nucleolus. This result can also be found in Aumann and Maschler (1985).

A unique solution that solves a bankruptcy problem is known as the Talmudic rule. We distinguish two cases. In case that the half sum of the debts exceeds the estate, that is, $(1/2)\cdot\sum_{k\in N} d_k \geq E$, the Talmudic rule should not give each creditor $k\in N$ more than half of his claim, that is $y_k \leq (1/2)\cdot d_k$ for all $k\in N$. And indeed, in case that $(1/2)\cdot\sum_{k\in N} d_k \geq E$, the Talmudic solution is given by $y_k = \min[\lambda, (1/2)\cdot d_k]$ for all $k\in N$, where λ is a real number that is determined by $\sum_{k\in N} y_k = E$. But if $(1/2)\cdot\sum_{k\in N} d_k \leq E$, then the Talmudic solution of a bankruptcy problem is specified by $y_k = \max[d_k - \lambda, (1/2)\cdot d_k]$ for all $k\in N$ and $\lambda\in\mathbb{R}$ is chosen such that $\sum_{k\in N} y_k = E$ (cf. Thomson (2002)). The unique Talmudic solution from a bankruptcy problem coincides with the unique pre-kernel solution, that is, the pre-nucleolus of the corresponding modest bankruptcy game, since this type of game belongs to the class of convex games.

Example 4.1. Consider a three claimants $N = \{1, 2, 3\}$ bankruptcy problem with a claim vector $\vec{d} = (60, 80, 120)$ and an estate of $E = 150$. Observe first that the half sum of the debts is less than the estate, i.e., $(1/2)\cdot 260 = 130 < 150 = E$. Thus the Talmudic solution must be determined by $y_k = \max[d_k - \lambda, (1/2)\cdot d_k]$ for all $k\in N$ and $\lambda\in\mathbb{R}$ must be chosen such that $\sum_{k\in N} y_k = 150$. By some calculations, it turns out that the Talmudic solution is $\vec{y} = (30, 40, 80)$. From the specified bankruptcy problem, we can derive its associated modest bankruptcy game of the form (3). This game is specified by

$$\begin{aligned} &v_{E,d}(\{1\}) = 0, \quad v_{E,d}(\{2\}) = 0, \\ &v_{E,d}(\{3\}) = 10, \\ &v_{E,d}(\{1,2\}) = 30, \quad v_{E,d}(\{1,3\}) = 70, \\ &v_{E,d}(\{2,3\}) = 90, \\ &v_{E,d}(\emptyset) = 0, \quad v_{E,d}(\{N\}) = 150. \end{aligned} \tag{6}$$

A modest bankruptcy game belongs to the class of convex games, thus, the pre-kernel solution coincides with the pre-nucleolus. According to the forgoing discussion, we know that the unique pre-kernel of the game (6) equals the Talmudic solution. This means we obtain again $\vec{y} = \vec{x}_1 = (30, 40, 80)$. In order to see that the computed solution is indeed a pre-kernel solution of the game, let us verify that the maximal excesses for each pair of players $i, j \in N$, $i \neq j$ are balanced. By some calculations, we get the following results: $s_{1,2}(\vec{x}_1, v_{E,d}) = s_{2,1}(\vec{x}_1, v_{E,d}) = s_{1,3}(\vec{x}_1, v_{E,d}) = s_{3,1}(\vec{x}_1, v_{E,d}) = -30$ and $s_{2,3}(\vec{x}_1, v_{E,d}) = s_{3,2}(\vec{x}_1, v_{E,d}) = -40$. As required, the maximal excesses for each pair of players $i, j \in N$, $i \neq j$ are equalized. Thus the computed solution is a pre-kernel solution of the game $v_{E,d}$, which is unique and coincides with the pre-nucleolus due to the convexity of the game.

Similarly, from the bankruptcy problem above we can derive its associated greedy bankruptcy game of the form (4). This game is given by

$$w_{E,d}(\{1\}) = 60, \quad w_{E,d}(\{2\}) = 80,$$
$$w_{E,d}(\{3\}) = 120,$$
$$w_{E,d}(\{1,2\}) = 140, \quad w_{E,d}(\{1,3\}) = 150,$$
$$w_{E,d}(\{2,3\}) = 150,$$
$$w_{E,d}(\emptyset) = 0, \quad w_{E,d}(N) = 150.$$

The associated greedy bankruptcy game belongs to the class of concave games, thus, the core is empty[1] and the pre-kernel solution need not anymore to be unique. According to the forgoing discussion, we know that the pre-nucleolus of the game (6) equals the Talmudic solution $\vec{x}_1 = (30, 40, 80)$. But this solution does not coincide with the pre-kernel of the dual game $w_{E,d}$, as can be easily checked by considering the maximal excesses for each pair of players. According to our result stated in Remark 3.2., the Talmudic solution (pre-nucleolus) equals the anti-pre-nucleolus of the dual game $w_{E,d}$. A pre-kernel solution of the game is now given by $\vec{x}_2 = (85, 85, 130)/2$. In order to see that the computed solution is indeed a pre-kernel of the greedy bankruptcy game (of the dual game), let us again check that the maximal excesses for each pair of players $i, j \in N$, $i \neq j$ are balanced. By some calculations, we get the following results: $s_{1,2}(\vec{x}_2, w_{E,d}) = s_{2,1}(\vec{x}_2, w_{E,d}) = 85/2$ and $s_{1,3}(\vec{x}_2, w_{E,d}) = s_{3,1}(\vec{x}_2, w_{E,d}) = s_{2,3}(\vec{x}_2, w_{E,d}) = s_{3,2}(\vec{x}_2, w_{E,d}) = 55$. As required, the maximal excesses for each pair of players $i, j \in N$, $i \neq j$ are equalized. Thus the computed solution is a pre-kernel element of the dual game $w_{E,d}$. Observe now that this solution is an element of the anti-pre-kernel of the game $v_{E,d}$, too. To see that, notice that the minimal excesses between each pair of players $i, j \in N$, $i \neq j$ must be equalized in accordance to definition (2). Indeed, we get $s^{\#}_{1,2}(\vec{x}_2, v_{E,d}) = s^{\#}_{2,1}(\vec{x}_2, v_{E,d}) = -85/2$ and $s^{\#}_{1,3}(\vec{x}_2, v_{E,d}) = s^{\#}_{3,1}(\vec{x}_2, v_{E,d}) = s^{\#}_{2,3}(\vec{x}_2, v_{E,d}) = s^{\#}_{3,2}(\vec{x}_2, v_{E,d}) = -55$. We see that the minimal excesses for each pair of players $i, j \in N$, $i \neq j$ are equalized. Thus the dual pre-kernel solution is the also an element of the anti-pre-kernel of the game $v_{E,d}$.

Example 4.2. Finally, let us discuss a more complex example. For this purpose consider a four claimants $N = \{1, 2, 3, 4\}$ bankruptcy problem with a claim vector $\vec{d} = (50, 80, 150, 250)$ and an estate of $E = 300$. Observe first that the half sum of the

TABLE 4.1 Coalitional values for the modest bankruptcy game

S	$v_{E,d}(S)$	S	$v_{E,d}(S)$	S	$v_{E,d}(S)$
∅	0	{1, 3}	0	{1, 2, 4}	150
{1}	0	{1, 4}	70	{1, 3, 4}	220
{2}	0	{2, 3}	0	{2, 3, 4}	250
{3}	0	{2, 4}	100	{1, 2, 3, 4}	300
{4}	20	{3, 4}	170		
{1, 2}	0	{1, 2, 3}	50		

TABLE 4.2 Coalitional values for the greedy bankruptcy game

S	$w_{E,d}(S)$	S	$w_{E,d}(S)$	S	$w_{E,d}(S)$
∅	0	{1, 3}	200	{1, 2, 4}	300
{1}	50	{1, 4}	300	{1, 3, 4}	300
{2}	80	{2, 3}	230	{2, 3, 4}	300
{3}	150	{2, 4}	300	{1, 2, 3, 4}	300
{4}	250	{3, 4}	300		
{1, 2}	130	{1, 2, 3}	280		

debts is less than the estate, i. e., $(1/2)\cdot 530 = 265 < 300 = E$. As Example 4.1., the Talmudic solution must be determined by $y_k = \max[d_k - \lambda, (1/2)\cdot d_k]$ for all $k \in N$ and $\lambda \in \mathbb{R}$ must be chosen such that $\sum_{k\in N} y_k = 300$. The solution is now given by $\vec{y} = (25, 40, 75, 160)$. From this bankruptcy problem, we derive again its associated modest bankruptcy game of the form (3). This game is specified TABLE 4.1 above.

Recall that a modest bankruptcy game belongs to the class of convex games, thus, the pre-kernel solution coincides with the pre-nucleolus. According to the forgoing discussion we know that the unique pre-kernel, hence the pre-nucleolus also, of the game (6) equals the Talmudic solution. this means we obtain again $\vec{x} = (25, 40, 75, 160)$. According to the convexity of the game, we can apply the definition of the pre-kernel to check that the computed solution is the pre-nucleolus of the game $v_{E,d}$. The maximal excesses for each pair of players $i, j \in N$, $i \neq j$ are balanced and are listed next:

$$\begin{aligned} s_{1,2}(\vec{x}_1, v_{E,d}) &= s_{2,1}(\vec{x}_1, v_{E,d}) = -25, \\ s_{1,3}(\vec{x}_1, v_{E,d}) &= s_{3,1}(\vec{x}_1, v_{E,d}) = s_{1,4}(\vec{x}_1, v_{E,d}) \\ &= s_{4,1}(\vec{x}_1, v_{E,d}) = -25, \\ s_{2,3}(\vec{x}_1, v_{E,d}) &= s_{3,2}(\vec{x}_1, v_{E,d}) = s_{2,4}(\vec{x}_1, v_{E,d}) \\ &= s_{4,2}(\vec{x}_1, v_{E,d}) = -40, \\ s_{3,4}(\vec{x}_1, v_{E,d}) &= s_{4,3}(\vec{x}_1, v_{E,d}) = -75. \end{aligned}$$

As required the maximal excesses for each pair of players $i, j \in N$, $i \neq j$ are equalized. Thus, the computed solution is the pre-kernel and, therefore the pre-nucleolus of the game $v_{E,d}$.

Similarly, from the bankruptcy problem above we can derive its associated greedy bankruptcy game of the form (4). This gome is specified in TABLE 4.2 above.

Now, let us discuss what is a pre-kernel solution of the greedy bankruptcy game $w_{E,d}$. The dual pre-kernel is now given by $\vec{x}_2 = (100, 115, 115, 270)/2$. In order to see that the computed solution is indeed a pre-kernel element of the greedy bankruptcy game (of the dual game), let us again check that the maximal excesses for each pair of players $i, j \in N, i \neq j$ are balanced. Again by some calculations, we get the following results:

$$
\begin{aligned}
s_{1,2}(\vec{x}_2, w_{E,d}) &= s_{2,1}(\vec{x}_2, w_{E,d}) = 115,\\
s_{1,3}(\vec{x}_2, w_{E,d}) &= s_{3,1}(\vec{x}_2, w_{E,d}) = s_{1,4}(\vec{x}_2, w_{E,d})\\
&= s_{4,1}(\vec{x}_2, w_{E,d}) = 115,\\
s_{2,4}(\vec{x}_2, w_{E,d}) &= s_{4,2}(\vec{x}_2, w_{E,d}) = s_{3,4}(\vec{x}_2, w_{E,d})\\
&= s_{3,4}(\vec{x}_2, w_{E,d}) = 115\\
s_{2,3}(\vec{x}_2, w_{E,d}) &= s_{3,2}(\vec{x}_2, w_{E,d}) = 215/2.
\end{aligned}
$$

As required, the maximal excesses for each pair of players $i, j \in N, i \neq j$ are equalized. Thus the computed solution is a pre-kernel of the dual game $w_{E,d}$. Observe now that this solution is an element of the anti-pre-kernel of the game $v_{E,d}$. To see that, notice that the minimal excesses between each pair of players $i, j \in N, i \neq j$ must be equalized in accordance to definition (2). Indeed, we get

$$
\begin{aligned}
s^{\#}_{1,2}(\vec{x}_2, v_{E,d}) &= s^{\#}_{2,1}(\vec{x}_2, v_{E,d}) = -115,\\
s^{\#}_{1,3}(\vec{x}_2, v_{E,d}) &= s^{\#}_{3,1}(\vec{x}_2, v_{E,d}) = s^{\#}_{1,4}(\vec{x}_2, v_{E,d})\\
&= s^{\#}_{4,1}(\vec{x}_2, v_{E,d}) = -115,\\
s^{\#}_{2,4}(\vec{x}_2, v_{E,d}) &= s^{\#}_{4,2}(\vec{x}_2, v_{E,d}) = s^{\#}_{3,4}(\vec{x}_2, v_{E,d})\\
&= s^{\#}_{4,3}(\vec{x}_2, v_{E,d}) = -115\\
s^{\#}_{2,3}(\vec{x}_2, v_{E,d}) &= s^{\#}_{3,2}(\vec{x}_2, v_{E,d}) = -215/2.
\end{aligned}
$$

We see that the minimal excesses for each pair of players $i, j \in N, i \neq j$ are equalized. Thus, the dual pre-kernel solution is the also an element of the anti-pre-kernel of the game $v_{E,d}$.

5. Concluding Remarks

We proved the coincidence between the dual pre-kernel and the anti-pre-kernel, and so between the dual pre-nucleolus and the anti-pre-nucleolus. If we consider about the kernel and the nucleolus, we need to define *anti-imputation set*. The anti-imputation set $\mathcal{I}^{\#}(v)$ is specified by

$$
\mathcal{I}^{\#}(v) := \Big\{ \vec{x} \in \mathcal{I}'(v) \Big| \sum_{k \in N\setminus\{i\}} x_k \leq v(N\setminus\{i\}) \text{ for all } i \in N \Big\}.
$$

Then the *dual kernel* $\mathcal{K}^*(v)$ of game v denoted by $\mathcal{K}^*(v) = \mathcal{K}(v^*)$, for any v, satisfies:

$$
\mathcal{K}^*(v) = \{ \vec{x} \in \mathcal{I}^{\#}(v) | (s^{\#}_{ij}(\vec{x}, v) - s^{\#}_{ji}(\vec{x}, v)) \cdot \Big(\sum_{k \in N\setminus\{i\}} x_k - v(N\setminus\{k\}) \Big) \leq 0 \text{ for all } i, j \in N, i \neq j \}.
$$

and the *dual nucleolus* $\mathcal{N}^*(v)$ of game v denoted by $\mathcal{N}^*(v) = \mathcal{N}(v^*)$ satisfies:

$$
\begin{aligned}
\mathcal{N}^*(v) &= \mathcal{N}^{\#}(\mathcal{I}^{\#}, v)\\
&= \{ \vec{x} \in \mathcal{I}^{\#}(v) | \theta^{\# v}(\vec{x}) \geq \theta^{\# v}(\vec{y}) \text{ for all } \vec{y} \in \mathcal{I}^{\#}(v) \}.
\end{aligned}
$$

One of the other well-known solutions for a game theory is the *Shapley value*. It is easy to check that the dual solution of the Shapley value is the Shapley value itself. Hence, the Shapley value of the moderate bankruptcy game coincides with the one of the greedy bankruptcy game.

For the other well-known solution the

core, the dual core is given by the following definition, that can also be considered as the *anti-core*,

$$\mathscr{C}^{\#}(v) := \{\vec{x} \in \mathcal{I}^{\#}(v) | \sum_{k \in S} x_k \leq v(S) \text{ for all } S \subset N\}.$$

The core of the modest bankruptcy game corresponds to the anti-core of the greedy bankruptcy game.

Notes

1 This is true simply by the fact that the anti-core of the greedy bankruptcy game is not empty, which corresponds to the core of the modest bankruptcy game. A formal proof of the non-emptiness of the anti-core was given by Driessen (1998). The reader should notice that Driessen called the anti-core, the core of the greedy bankruptcy game. See also the final discussion presented in section 5 at the end of this paper.

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