ON ASYMPTOTICALLY IDEAL INVARIANT EQUIVALENCE OF DOUBLE SEQUENCES

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Abstract. In this study, the concepts of asymptotically $\mathcal{I}_2^\sigma$-equivalent, asymptotically invariant equivalent, strongly asymptotically invariant equivalent and p-strongly asymptotically invariant equivalent for double sequences are defined. Also, we investigate relationships among these new type equivalence concepts.

1. Introduction and Background

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$. It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \quad \text{uniformly in } n \right\}.$$

Several authors have studied invariant convergent sequences (see, [11–15, 19–21, 23–25]). The concept of strongly σ-convergence was defined by Mursaleen in [12]:

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in n. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.

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In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strongly σ-convergence was generalized by Savaş [20] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$.

If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

The idea of statistical convergence was introduced by Fast [6] and studied by many authors. The concept of σ-statistically convergent sequence was introduced by Savaş and Nuray in [23]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [8] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Similar concepts can be seen in [7, 14].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Recently, the concepts of σ-uniform density of subset A of the set $\mathbb{N}$ and corresponding $\mathcal{I}_\sigma$-convergence for real number sequences was introduced by Nuray et al. [14]. Marouf [10] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the concept of asymptotically equivalence has been developed by many other researchers (see, [16, 17, 22]).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if $\lim_k \frac{x_k}{y_k} = 1$. It is denoted by $x \sim y$.

Convergence and $\mathcal{I}$-convergence of double sequences in a metric space and some properties of this convergence, and similar concepts which are noted following can be seen in [1, 2, 9, 18].

A double sequence $x = (x_{kj})$ is said to be bounded if $\sup_{k,j} x_{kj} < \infty$. The set of all bounded double sequences of sets will be denoted by ℓ_∞^2.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Let (X, ρ) be a metric space and $\mathcal{I}_2$ be a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$

$$A(\varepsilon) = \big\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon \big\} \in \mathcal{I}_2.$$

It is denoted by $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mn} := \min_{k,j} \left| A \cap \left\{ \big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big) \right\} \right|$$

and

$$S_{mn} := \max_{k,j} \left| A \cap \left\{ \left(\sigma(k), \sigma(j)\right), \left(\sigma^2(k), \sigma^2(j)\right), ..., \left(\sigma^m(k), \sigma^n(j)\right) \right\} \right|.$$

If the following limits exists

$$\underline{V_2}(A) := \lim_{m,n\to\infty} \frac{s_{mn}}{mn} \quad \text{and} \quad \overline{V_2}(A) := \lim_{m,n\to\infty} \frac{S_{mn}}{mn},$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.

Dündar et al. [3] studied the concepts of invariant convergence, strongly invariant convergen, p-strongly invariant convergen and ideal invariant convergence of double sequences.

A double sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_2^\sigma$-convergent to L if for every $\varepsilon > 0$

$$A(\varepsilon) = \left\{(k,j) : |x_{kj} - L| \geq \varepsilon\right\} \in \mathcal{I}_2^\sigma,$$

that is, $V_2\big(A(\varepsilon)\big) = 0$. In this case, we write $\mathcal{I}_2^\sigma - \lim x = L$ or $x_{kj} \to L(\mathcal{I}_2^\sigma)$.

The set of all $\mathcal{I}_2$-invariant convergent double sequences will be denoted by $\mathfrak{I}_2^\sigma$.

A double sequence $x = (x_{kj})$ is said to be strongly invariant convergent to L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right| = 0,$$

uniformly in s, t. In this case, we write $x_{kj} \to L\big([V_\sigma^2]\big)$.

A double sequence $x = (x_{kj})$ is said to be p-strongly invariant convergent to L, if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p = 0,$$

uniformly in s, t, where $0 < p < \infty$. In this case, we write $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

The set of all p-strongly invariant convergent double sequences will be denoted by $[V_\sigma^2]_p$.

Hazarika [4] introduced the notion of asymptotically $\mathcal{I}$-equivalent sequences and investigated some properties of it. Definitions of P-asymptotically equivalence, asymptotically statistical equivalence and asymptotically $\mathcal{I}_2$-equivalence of double sequences were presented by Hazarika and Kumar [5] as following:

Two nonnegative double sequences $x = (x_{kl})$ and $x = (y_{kl})$ are said to be P-asymptotically equivalent if

$$P - \lim_{k,l} \frac{x_{kl}}{y_{kl}} = 1,$$

denoted by $x \sim^P y$.

Two nonnegative double sequences $x=(x_{kl})$ and $x=(y_{kl})$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon>0$

$$P-\lim_{m,n}\frac{1}{mn}\left|\left\{k\leq m, l\leq n:\left|\frac{x_{kl}}{y_{kl}}-L\right|\right\}\right|=0,$$

denoted by $x\sim^{\mathcal{S}^L}y$ and simply asymptotically statistical equivalent if $L=1$.

Two nonnegative double sequences $x=(x_{kl})$ and $x=(y_{kl})$ are said to be asymptotically $\mathcal{I}_2$-equivalent of multiple L provided that for every $\varepsilon>0$

$$\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\in\mathcal{I}_2.$$

denoted by $x\sim^{\mathcal{I}^L}y$ and simply asymptotically statistical equivalent if $L=1$.

2. Asymptotically $\mathcal{I}_2^{\sigma}$-Equivalence

In this section, the concepts of asymptotically $\mathcal{I}_2^{\sigma}$-equivalent, asymptotically σ_2-equivalent, strongly asymptotically σ_2-equivalent and p-strongly asymptotically σ_2-equivalent for double sequences are defined. Also, we investigate relationships among these new type equivalence concepts.

Definition 2.1. Two nonnegative double sequences $x=(x_{kl})$ and $y=(y_{kl})$ are said to be asymptotically invariant equivalent or asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}=L,$$

uniformly in s,t. In this case, we write $x\overset{V_{2(L)}^{\sigma}}{\sim}y$ and simply σ_2-asymptotically equivalent, if $L=1$.

Definition 2.2. Two nonnegative double sequences $x=(x_{kl})$ and $y=(y_{kl})$ are said to be asymptotically $\mathcal{I}_2^{\sigma}$-equivalent of multiple L if for every $\varepsilon>0$,

$$A_{\varepsilon}:=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\left|\frac{x_{kl}}{y_{kl}}-L\right|\geq\varepsilon\right\}\in\mathcal{I}_2^{\sigma},$$

i.e., $V_2(A_{\varepsilon})=0$. In this case, we write $x\overset{\mathcal{I}_{2(L)}^{\sigma}}{\sim}y$ and simply asymptotically $\mathcal{I}_2^{\sigma}$-equivalent, if $L=1$.

The set of all asymptotically $\mathcal{I}_2^{\sigma}$-equivalent of multiple L sequences will be denoted by $\mathfrak{I}_{2(L)}^{\sigma}$.

Theorem 2.3. *Suppose that $x=(x_{kl})$ and $y=(y_{kl})$ are bounded double sequences. If x and y are asymptotically $\mathcal{I}_2^{\sigma}$-equivalent of multiple L, then these sequences are σ_2-asymptotically equivalent of multiple L.*

Proof. Let $m,n,s,t\in\mathbb{N}$ be arbitrary and $\varepsilon>0$. Now, we calculate

$$u(m,n,s,t):=\left|\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|.$$

We have

$$u(m,n,s,t) \leq u^{(1)}(m,n,s,t) + u^{(2)}(m,n,s,t),$$

where

$$u^{(1)}(m,n,s,t) := \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| \geq \varepsilon}}^{m,n} \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right|$$

and

$$u^{(2)}(m,n,s,t) := \frac{1}{mn} \sum_{\substack{k,l=1,1 \\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| < \varepsilon}}^{m,n} \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right|.$$

We get $u^{(2)}(m,n,s,t) < \varepsilon$, for every $s,t = 1,2,\ldots$. The boundedness of $x = (x_{kl})$ and $y = (y_{kl})$ implies that there exists a $M > 0$ such that

$$\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| \leq M,$$

for $k,l = 1,2,...,\ \ s,t = 1,2,\ldots$. Then, this implies that

$$\begin{aligned} u^{(1)}(m,n,s,t) &\leq \frac{M}{mn}\left|\left\{1 \leq k \leq m,\ 1 \leq l \leq n : \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| \geq \varepsilon\right\}\right| \\ &\leq M\frac{\max_{s,t}\left|\left\{1 \leq k \leq m,\ 1 \leq l \leq n : \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| \geq \varepsilon\right\}\right|}{mn} = M\frac{S_{mn}}{mn}, \end{aligned}$$

hence x and y are σ_2-asymptotically equivalent to multiple L. □

The converse of Theorem 2.3 does not hold. For example, $x = (x_{kl})$ and $y = (y_{kl})$ are the double sequences defined by following;

$$x_{kl} := \begin{cases} 2 \ , & \text{if } \ k+l \ \text{ is an even integer,} \\ 0 \ , & \text{if } \ k+l \ \text{ is an odd integer.} \end{cases}$$

$$y_{kl} := 1$$

When $\sigma(m) = m+1$ and $\sigma(n) = n+1$, this sequences are asymptotically σ_2-equivalent but they are not asymptotically $\mathcal{I}_2^\sigma$-equivalent.

Definition 2.4. Two nonnegative double sequence $x = (x_{kl})$ and $y = (y_{kl})$ are said to be strongly asymptotically invariant equivalent or strongly asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}} - L\right| = 0,$$

uniformly in s,t. In this case, we write $x \overset{[V^{\sigma}_{2(L)}]}{\sim} y$ and simply strongly asymptotically σ_2-equivalent if $L=1$.

Definition 2.5. Let $0<p<\infty$. Two nonnegative double sequence $x=(x_{kl})$ and $y=(y_{kl})$ are said to be p-strongly asymptotically invariant equivalent or p-strongly asymptotically σ_2-equivalent of multiple L if

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p=0,$$

uniformly in s,t. In this case, we write $x \overset{[V^{\sigma}_{2(L)}]_p}{\sim} y$ and simply p-strongly asymptotically σ_2-equivalent if $L=1$.

The set of all p-strongly asymptotically σ_2-equivalent of multiple L sequences will be denoted by $[\mathcal{V}^{\sigma}_{2(L)}]_p$.

Theorem 2.6. *Let $0<p<\infty$. Then, $x \overset{[\mathcal{V}^{\sigma}_{2(L)}]_p}{\sim} y \Rightarrow x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y$.*

Proof. Let $x \overset{[\mathcal{V}^{\sigma}_{2(L)}]_p}{\sim} y$ and given $\varepsilon>0$. Then, for every $s,t\in\mathbb{N}$ we have

$$\begin{aligned}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p &\geq \sum_{\substack{k,l=1,1\\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon}}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p\\ &\geq \varepsilon^p\left|\left\{1\leq k\leq m, 1\leq l\leq n: \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon\right\}\right|\\ &\geq \varepsilon^p\max_{s,t}\left|\left\{1\leq k\leq m, 1\leq l\leq n: \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon\right\}\right|\end{aligned}$$

and

$$\begin{aligned}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p &\geq \varepsilon^p\frac{\max\limits_{s,t}\left|\left\{1\leq k\leq m, 1\leq l\leq n: \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon\right\}\right|}{mn}\\ &= \varepsilon^p\frac{S_{mn}}{mn}\end{aligned}$$

for every $s,t=1,2,\ldots$. This implies $\lim\limits_{m,n\to\infty}\frac{S_{mn}}{mn}=0$ and so $x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y$. □

Theorem 2.7. *Let $0<p<\infty$ and $x,y\in\ell^2_\infty$. Then, $x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y \Rightarrow x \overset{[V^{\sigma}_{2(L)}]_p}{\sim} y$.*

Proof. Suppose that $x,y\in\ell^2_\infty$ and $x \overset{\mathcal{I}^{\sigma}_{2(L)}}{\sim} y$. Let $\varepsilon>0$. By assumption, we have $V_2(A_\varepsilon)=0$. The boundedness of x and y implies that there exists a $M>0$ such that

$$\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\leq M$$

for $k,l=1,2,...,\ s,t=1,2,...$. Observe that, for every $s,t\in\mathbb{N}$ we have

$$\begin{aligned}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p &= \frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon}}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p\\ &\quad+\frac{1}{mn}\sum_{\substack{k,l=1,1\\ \left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|<\varepsilon}}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p\\ &\leq M\frac{\max\limits_{s,t}\left|\left\{1\leq k\leq m,1\leq l\leq n:\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon\right\}\right|}{mn}+\varepsilon^p\\ &\leq M\frac{S_{mn}}{mn}+\varepsilon^p.\end{aligned}$$

Hence, we obtain

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,l=1,1}^{m,n}\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|^p=0$$

uniformly in s,t.

□

Theorem 2.8. *Let* $0<p<\infty$*. Then,* $\mathfrak{I}^{\sigma}_{2(L)}\cap\ell^2_{\infty}=[\mathcal{V}^{\sigma}_{2(L)}]_p\cap\ell^2_{\infty}$.

Proof. This is an immediate consequence of Theorem 2.6 and Theorem 2.7. □

Now we give definition of asymptotically S^{σ}_2-equivalent for double sequences and we shall state a theorem that gives a relationship between asymptotically $\mathcal{I}^{\sigma}_2$-equivalence and asymptotically S^{σ}_2-equivalence of double sequences.

Definition 2.9. Two nonnegative double sequences $x=(x_{kl})$ and $y=(y_{kl})$ are said to be asymptotically S^{σ}_2-equivalent of multiple L provided that for every $\varepsilon>0$

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|\left\{k\leq m,l\leq n:\left|\frac{x_{\sigma^k(s),\sigma^l(t)}}{y_{\sigma^k(s),\sigma^l(t)}}-L\right|\geq\varepsilon\right\}\right|=0,$$

uniformly in $s,t=1,2,...,$ (denoted by $x\overset{S^{\sigma}_{2(L)}}{\sim}y$) and simply asymptotically S^{σ}_2-equivalent, if $L=1$.

Theorem 2.10. *The double sequences* $x=(x_{kl})$ *and* $y=(y_{kl})$ *are asymptotically* $\mathcal{I}^{\sigma}_2$*-equivalent to multiple* L *if and only if they are asymptotically* S^{σ}_2*-equivalent of multiple* L.

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