A planar single-facility competitive location and design problem under the multi-deterministic choice rule

José Fernández*

Dpt. of Statistics and Operations Research, University of Murcia, Murcia, Spain

Boglárka G.- Tóth

Dpt. Differential Equations, Budapest University of Technology and Economics, Budapest, Hungary

Juana L. Redondo and Pilar M. Ortigosa

Dpt. of Informatics, University of Almería, ceiA3, Almería, Spain

Aránzazu Gila Arrondo

Indra, Madrid, Spain

Abstract

A new customer choice rule, which may model in some cases the actual patronising behaviour of customers towards the facilities closer to reality than other existing rules, is proposed. According to the new rule, customers split their demand among the firms in the market by patronising only one facility from each firm, the one with the highest utility, and the demand is split among those facilities proportionally to their attraction. The influence of the choice rule in the location of facilities is investigated. In particular, a new continuous competitive single-facility location and design problem using this new rule is proposed. Both exact and heuristic methods are proposed to solve it. A comparison with the classical proportional (or Huff) choice rule when solving the location model reveals that both the location and the

*Corresponding author

Email addresses: josefdez@um.es (José Fernández), bog@math.bme.hu (Boglárka G.- Tóth), {jlredondo, ortigosa}@ual.es (Juana L. Redondo and Pilar M. Ortigosa), aranzazugila@gmail.com (Aránzazu Gila Arrondo)

URL: http://www.um.es/geloca/gio/josemain.html (José Fernández)

quality of the new facility to be located may be quite different depending on the patronising behaviour of customers. Most importantly, the profit that the locating chain may lose if a wrong choice is made can be quite high in some instances.

Keywords: Facility location, patronising behaviour, competition, global optimization, computational study

1. Introduction and notation

The estimation of the market share that can be captured by a facility in a competitive environment where there exist other facilities offering the same product is a topic of major concern for managers, as the survival of a facility depends on the revenues it can obtain, and those revenues largely depend on the market share. Where to locate a facility is a strategic decision which cannot be easily altered as the location of a facility usually requires a massive investment. But how do we choose the right location for a new facility?

Competitive location problems concerning optimally placing facilities in a competitive environment have been widely developed for a number of contextual applications in the traditional retail sector, see for instance the survey papers of **??** and **?** and the references therein. They vary in the ingredients which form the model. For instance, the *location space* may be the plane, a network or a discrete set. We may want to locate just one or more than one new facility. The competition may be *static*, which means that the competitors are already in the market and the owner of the new facility knows their characteristics, or *with foresight*, in which the competitors are not in the market yet but they will be soon after the new facility enters. Demand is usually supposed to be concentrated in a discrete set of points, called *demand points*, and it can be either *inelastic* or *elastic*, depending on whether the goods are essential or inessential.

It is also necessary to specify what the *attraction (or utility) function* of a customer towards a given facility is. Usually, the attraction function depends on the distance between the customer and the facility, as well as on other characteristics of the facility which determine its *quality*.

The *patronising behaviour* of the customers must also be taken into account, since the *market share* captured by the facilities depends on it. This is the topic this paper is devoted to. For instance, it is not uncommon to see in the literature papers where consumers shop at the closest store supplying a specific product or service. But, does this assumption reflect consumer behaviour? It seems more realistic to admit that consumers do not merely consider distance when choosing retail outlets. Also, consumers may patronise more than one facility to satisfy their demand. Consumer choice behaviour literature studies the key variables that a customer takes into account to patronise one or another facility, and how these variables interact.

A common classification of the consumer choice behaviour states that this can be done in three groups (**?**):

- The first one includes models that rely on some "normative assumption" regarding consumer travel behaviour. This hypothesis is too simple and is useful only in a limited number of applications. The classic example is the so-called *deterministic rule*, which says that "consumers patronise the nearest outlet that provides the required goods or service". This hypothesis has not found much empirical support, except in areas where shopping opportunities are limited and transportation is difficult.

- The second group uses information revealed by past behaviour to understand the dynamics of retail competition and how consumers choose among alternative shopping opportunities. **?** was the first one to use the revealed preference approach to study retail store choice. The Huff probability formulation, known as the *probabilistic rule*, uses distance (or travel time) from consumer zones to retail centres and the size of retail centres as inputs to find the probability of consumers shopping at a given retail outlet.

- The third group of models estimates the consumer utility function from simulated choice data using information integration, conjoint or logit techniques. Instead of observing past choices, these methods use consumer evaluations of hypothetical store descriptions to calibrate the utility function. The best representative model of this group is the one developed by **?** based on game theory.

? have pointed out that firms prefer the revealed preference approach to model consumer store-choice behaviour. This approach is preferred to normative models since it more faithfully reflects real consumer behaviour, and to the direct utility approach because it is simpler since it uses surveys and linear regression instead of conjoint, logit techniques or game theory. We follow the revealed preference approach in this paper.

The two customer choice rules commonly used in literature are the following:

Deterministic (or binary) rule: it assumes that the full demand of a customer is satisfied by *only one* centre, the one to which he/she is attracted most, disregarding all other facilities which are less attractive, even those whose difference in attraction is very small.

Probabilistic rule: it assumes that a customer splits his/her demand probabilistically over *all* facilities in the market proportionally to his/her attraction to each facility.

? was the first to propose the deterministic choice rule for a simple model on a line. That is why competitive location models using this rule are also referred to as Hotelling models. The first two papers that introduced location models in a more general space assuming that customers patronize the closest facility were **?** in the plane and **?** on a network.

?? described the gravity model suggested by **?**, although he did not investigate any location problem. The first paper that considered the location problem based on the Huff rule was **?**. Later on, **?** and **?** introduced the design as an additional variable of the model, although an earlier version of location and design was already introduced in **?**.

The aim of this paper is twofold. First, we present a new choice rule, named multi-deterministic choice rule, which may, in some cases, model the patronising behaviour of customers closer to reality than other existing rules in many practical applications. In particular, we introduce a new single-facility location and design problem on the plane which considers this rule. Second, we investigate up to what extent the selection of the choice rule may affect the location decisions of a firm that wants to expand its presence in a given geographical region by opening new facilities. In particular, we will compare the outputs provided by models using the probabilistic and the multi-deterministic rules on the same input data sets.

In the rest of the paper, in order to fix ideas, we assume the following scenario (notice, however, that the main conclusion from the paper, i.e., that the selection of the right customer choice rule is a critical issue for the location decisions of a firm that wants to set up new facilities, remains valid for other competitive location models as well): A chain wants to locate a new single facility in a given area of the plane, where there already exist other facilities in the vicinity offering the same goods or product. Some of those facilities may belong to the locating chain. The demand is supposed to be fixed and concentrated at given demand points, whose locations and buying powers are known, as well as the location and quality of the existing facilities. The attraction of a demand point towards a facility is modelled multiplicatively as quality divided by perceived distance. This generalizes the law of retail gravitation of **?**, who considered the perceived distance to be the squared distance. Quality was first estimated as store surface by **?**, and later several other store characteristics were incorporated by **?** and **?**. For details see **?**.

The objective is to maximize the *profit* obtained by the chain after the location of the new facility, to be understood as the income due to the market share captured by the chain minus its operational costs. Both the location and the quality of the new facility are to be found.

In order to give a mathematical formulation of location models using the different customer choice rules, the following notation will be used:

Indices

- i index of demand points, $i = 1, \ldots, i_{\max}$.
- c index of competing chains, $c = 1, \ldots, c_{\max}$ (chain $c = 1$ is the locating chain).
- j index of existing facilities, $j = 1, \ldots, j_{\max}$ (we assume that from $j = j^1_{\min}(= 1)$ to $j^1_{\max}$ the facilities belong to chain $c = 1$ ($j^1_{\max} < j_{\max}$); from $j = j^2_{\min}(= j^1_{\max} + 1)$ to $j^2_{\max}$ belong to chain $c = 2$,..., from $j = j^{c_{\max}}_{\min}(= j^{c_{\max}-1}_{\max} + 1)$ to $j^{c_{\max}}_{\max}(= j_{\max})$ to chain $c = c_{\max}$).

Variables

- x location of the new facility, $x = (x_1, x_2)$.
- α quality of the new facility.

Input data

p_i	location of demand point i.
w_i	demand (or buying power) at p_i, $w_i > 0$.
f_j	location of existing facility j.
d_{ij}	distance between demand point p_i and facility f_j, $d_{ij} > 0$.
α_j	quality of facility f_j, $\alpha_j > 0$.
γ_i	weight for the quality of the facilities as perceived by demand point p_i, $\gamma_i > 0$.
$d_i^{\min}$	minimum distance from p_i at which the new facility can be located, $d_i^{\min} > 0$.
$\alpha_{\min}$	minimum level of quality for the new facility, $\alpha_{\min} > 0$.
$\alpha_{\max}$	maximum level of quality for the new facility, $\alpha_{\max} \geq \alpha_{\min}$.
S	region of the plane where the new facility can be located.

Miscellaneous

$g_i(\cdot)$	a continuous non-negative non-decreasing function, which modulates the decrease in attractiveness as a function of distance.
$d_i(x)$	distance between demand point p_i and the new facility.
$u_{i0}(x, \alpha)$	attraction that p_i feels for the new facility; $u_{i0}(x, \alpha)$ $= \gamma_i \alpha / g_i(d_i(x))$.

Computed parameters

u_{ij}	attraction that p_i feels for f_j (or utility of f_j perceived by the people at p_i). In this paper, $u_{ij} = \gamma_i \alpha_j / g_i(d_{ij})$.
u_i^c	maximum attraction that p_i feels for any of the existing facilities of chain c, $u_i^c = \max\{u_{ij} : j = j_{\min}^c, \ldots, j_{\max}^c\}$

Based on these assumptions, the market share captured by the chain when a deterministic rule is used is

$$M_D(x, \alpha) = \sum_{\{i \in \{1,\ldots,i_{\max}\} : \max\{u_i^1, u_{i0}(x,\alpha)\} \geq \max\{u_i^c : c=2,\ldots,c_{\max}\}\}} w_i.$$

In the previous formula we have assumed that, in case of ties in the attraction, customers choose the locating chain. Notice that in the deterministic rule it is assumed that the attraction of the customers at p_i towards a chain is determined only by the facility to which they are attracted most. The rest of the facilities do not play any role.

When a probabilistic rule is used, the market share captured by the chain is given by

$$M_P(x,\alpha) = \sum_{i=1}^{i_{\max}} w_i \frac{u_{i0}(x,\alpha) + \sum_{j=j^1_{\min}}^{j^1_{\max}} u_{ij}}{u_{i0}(x,\alpha) + \sum_{j=1}^{j_{\max}} u_{ij}}.$$

In the probabilistic rule the attraction of the customers at p_i towards a chain is determined by all the facilities belonging to the chain. As we can see, it is assumed that the utility is *additive*: for instance, the utility for the first chain is given by $U_i^1(x,\alpha) = u_{i0}(x,\alpha) + \sum_{j=j^1_{\min}}^{j^1_{\max}} u_{ij}$.

The problem to be solved is then

$$\begin{cases} \max & \Pi(x,\alpha) = \Pi(M(x,\alpha), G(x,\alpha)) \\ \text{s.t.} & d_i(x) \geq d_i^{\min} \ \forall i \\ & \alpha \in [\alpha_{\min}, \alpha_{\max}] \\ & x \in S \subset \mathbb{R}^2 \end{cases} \tag{1}$$

where $M(x,\alpha)$ stands for either $M_D(x,\alpha)$ or $M_P(x,\alpha)$, $G(x,\alpha)$ is a function which gives the operating cost of a facility located at x with quality α, and $\Pi(x,\alpha)$ is the profit obtained by the chain. This profit depends on x and α through the functions $M(x,\alpha)$ and $G(x,\alpha)$, with $\Pi(M,G)$ increasing and decreasing, respectively, in its two arguments M and G (see **?**). In all the computational studies done in this paper we have assumed, following **?**, that $\Pi(M,G) = F(M) - G$, where $F(\cdot)$ is a strictly increasing function which transforms the market share into expected sales. See **?** for possible expressions for F and G. Note that this profit disregards the operating costs of the existing facilities of the own chain, since these are considered to be constant. The parameters $d_i^{\min} > 0$ and $\alpha_{\min} > 0$ are given thresholds, which guarantee that the new facility is not located over a demand point and that it has a minimum level of quality, respectively. The parameter $\alpha_{\max}$ is the maximum value that the quality of a facility may take in practice. By S we denote the region of the plane where the new facility can be located. Distances are assumed to be computed with the help of a distance predicting function induced by a norm (see **?**). As for $g_i(d)$, it is usually considered to be of the form $g_i(d) = d^\lambda$ for some $\lambda > 0$ (see **???**) or $g_i(d) = \exp(\kappa d^\tau)$ for some $\kappa, \tau > 0$ (see **??**). Notice that since $d_i^{\min} > 0$, then $g_i(d_i(x))$ is strictly positive for any feasible location x.

In the following section, the multi-deterministic choice rule is introduced, and the corresponding continuous competitive facility *location and design* problem is also formulated for the same scenario. Different approaches to solve the problem are presented in Section 3. In particular, an exact branch-and-bound algorithm, a multi-start strategy whose local optimizer is a Weiszfeld-like algorithm, and an evolutionary algorithm will be discussed. The exact B&B method will be used in Section 4 to research up to what extent the use of a particular customer choice rule may affect the decision about the optimal location of a new facility. Some computational studies to investigate the effectiveness and efficiency of the methods are reported in Section 5. The paper ends in Section 6 with some conclusions.

2. The multi-deterministic choice rule and the corresponding location model

Although in some cases customers patronise facilities according to the deterministic or the probabilistic choice rules (or at least, those rules provide good estimations of the market share captured by the facilities), there are also other cases in which those choice rules do not represent customer behaviour properly. In order to have a better estimation of the market share captured by each facility or chain, new customer choice rules which model customer behaviour closer to reality are needed. Consider, for instance, the case of a customer who needs to do his/her weekly shopping. There are five supermarkets around his/her home, two of them belonging to chain A, and the other three to a different chain, B. Most likely, he/she will not do all the weekly shopping in a single supermarket, as some products may not be available there, or their price is lower in the supermarkets of the other chain. However, he/she will not go to all the supermarkets either, as he/she will find the same products, even with the same price, in all the supermarkets belonging to the same chain. So, he/she will decide to go to one of the supermarkets in chain A and to one of the supermarkets in chain B. In particular, the supermarket from each chain that he/she will choose will be the one for which he/she is attracted most. And he/she will do his/her weekly shopping in those two supermarkets not on a 50% basis: most likely, he/she will spend more money in the supermarket for which he/she feels more attraction. The multi-deterministic rule that we introduce next in this section tries to model this behaviour.

Hakimi already proposed something like this back in 1990, see Section 10.4 in **?**. He called it 'partially binary rule' (a name that we consider a bit misleading). Following this idea, Serra and colleagues (**??**) presented a *discrete* location model using the multi-deterministic rule, but in which the utility of a facility for a demand point was determined by the distance between them only. The multi-deterministic rule has also been addressed in *networks* in papers by Suárez-Vega and colleagues (**??**), where some discretization results are shown. However, to the extent of our knowledge, this is the first paper to address the problem in a *continuous* setting, and we do it using a general attraction function and including the quality as a third variable to be determined in the problem.

As in the previous section, we consider the problem of locating a single facility in the plane, with static competition and inelastic demand, where the attraction function depends on both the location and the quality of the facilities. These two last factors are the variables of the problem. The objective is again to maximize the *profit* obtained by the chain, to be understood as the income due to the market share captured by the chain minus its operational costs. As before, several firms are present in the market, but now customers split their demand among the firms by patronising only one facility from each firm, the one with the highest utility, and the demand is split among those facilities proportionally to their attraction.

The market share captured by the locating chain (chain $c = 1$) is

$$M_M(x,\alpha) = \sum_{i=1}^{i_{\max}} w_i \frac{\max\{u_{i0}(x,\alpha), u_i^1\}}{\max\{u_{i0}(x,\alpha), u_i^1\} + \sum_{c=2}^{c_{\max}} u_i^c}. \tag{2}$$

As can be seen in the formula, it is assumed here that the attraction of the customers at p_i towards a chain is determined only by the facility of the chain to which they are attracted most. The rest of the facilities of the chain do not play any role. But unlike the deterministic rule, now all the chains capture part of the demand at p_i.

The market share captured by the new facility is

$$m_{M_0}(x,\alpha) = \sum_{i=1}^{i_{\max}} w_i \frac{\tilde{u}_{i0}(x,\alpha)}{\max\{u_{i0}(x,\alpha), u_i^1\} + \sum_{c=2}^{c_{\max}} u_i^c},$$

where

$$\tilde{u}_{i0}(x,\alpha) = \begin{cases} u_{i0}(x,\alpha) & \text{if } u_{i0}(x,\alpha) \geq u_i^1 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding continuous competitive facility location and design problem is given by (1), where $M(x,\alpha)$ is given by (2). Figure 1 gives the graph of the objective function on the location domain for a problem with setting $(i_{max} = 71, j_{max} = 5, c = 2, j^1_{max} = 2, j^2_{max} = 3)$ for a fixed value of the variable α. Figure 2 is the corresponding contour graph in location space. The white holes in the graphs correspond to the forbidden regions around the demand points. As can be seen, this problem is a highly nonlinear optimisation problem which requires global optimisation techniques to be solved. Notice that when the number of facilities of each competing chain is equal to one, and the locating chain is a newcomer, the model reduces to the standard probabilistic model introduced in **?**.

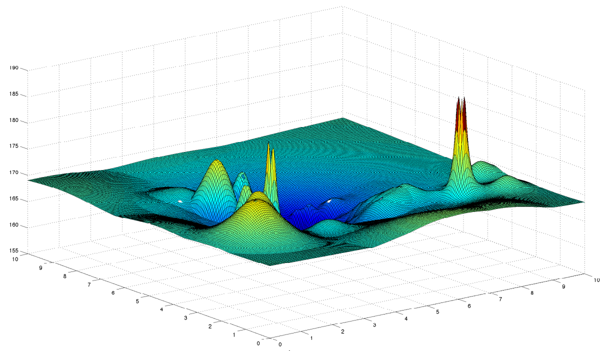

Figure 1: Objective function of an instance with setting $(i_{max} = 71, j_{max} = 5, c = 2, j^1_{max} = 2, j^2_{max} = 3)$ when $\alpha = 0.5$.

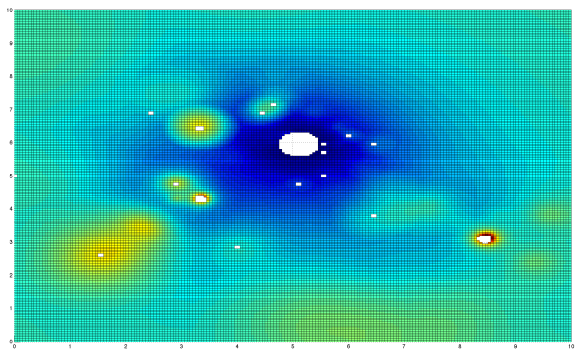

Figure 2: Contour projected in the 2-dimensional location space of an instance with setting $(i_{max} = 71, j_{max} = 5, c = 2, j^1_{max} = 2, j^2_{max} = 3)$ when $\alpha = 0.5$.

3. Solving the multi-deterministic location model

As stated above, problem (1) with the multi-deterministic choice rule (2) (and also with the other rules) is very difficult to solve due to the non-convexity of the objective function (see Fig. 1) and the non-convexity (maybe even non-connectedness) of the feasible set. Hence, it requires global optimisation techniques to be solved. Next, both exact and heuristic methods are suggested to cope with it.

3.1. An exact interval branch-and-bound method

Branch-and-bound (B&B) algorithms are probably the most used exact methods to cope with difficult problems. Their success relies on the goodness of the bounds obtained through the process. Interval analysis tools can be used both to compute bounds automatically and to discard suboptimal regions. Essential reading, including useful references in this area, can be found in the books by **??** and **?**.

Interval B&B methods have been successfully applied to solve location problems (see for instance **??** and the references therein). In particular, in **?** (see also **?**) an exact interval branch-and-bound method (iB&B in what follows) was proposed and applied to solve the corresponding location problem with probabilistic patronising behaviour of customers described in Section 1. The method produces a list of 3-dimensional intervals which contain any global optimal solution. The same method can handle the multi-deterministic model thanks to the use of the interval tools employed to compute the bounds. However, only up to medium size instances can be solved with iB&B, as we will see. One of the challenges in our problem is that the objective function is given as a sum of functions which in turn are defined by maximum of functions. To overcome this difficulty, we have employed a similar strategy to the one used in **?** for piece-wise functions.

3.2. A multi-start heuristic

A local-search method is proposed in **?** (see also **?**) for addressing the problem with probabilistic patronising behaviour of customers. The algorithm is a steepest descent type method which takes discrete steps along the search directions and, usually, converges to a local optimum. In this method, the derivatives of the objective function are equated to zero and the next iterate is obtained by implicitly solving these equations. In location literature these types of methods are known as Weiszfeld-like methods, in honour of E. Weiszfeld, who first proposed that strategy (**?**).

When solving the probabilistic case, all the demand points are taken into account as the new facility captures some demand from all of them. A similar algorithm can be applied to the multi-deterministic case, but taking into account only the demand points actually served by the new facility.

In order to detail the steps of the algorithm, we need to have specific expressions for functions F and G. In what follows, for instance, we will assume F to be linear, $F(M(x,\alpha)) = c \cdot M(x,\alpha)$, and G to be separable, in the form $G(x,\alpha) = G_1(x) + G_2(\alpha)$, where $G_1(x) = \sum_{i=1}^{i_{\max}} \Phi_i(d_i(x))$, with $\Phi_i(d_i(x)) = w_i/((d_i(x))^{\phi_{i0}} + \phi_{i1})$, $\phi_{i0}, \phi_{i1} > 0$ and $G_2(\alpha) = e^{\frac{\alpha}{\beta_0}+\beta_1} - e^{\beta_1}$, with $\beta_0 > 0$ and β_1 given values. Other possible expressions can be found in **?**. Depending on the particular problem the most suitable functions should be ascertained. We also need g_i and d_i to be differentiable.

Consider in formula (2) only the demand points to be served by the new facility. The market share that it captures is given by

$$m_0(x,\alpha) = m_{M_0}(x,\alpha) = \sum_{\{i:u_{i0}(x,\alpha)\geq u_i^1\}} w_i \frac{u_{i0}(x,\alpha)}{u_{i0}(x,\alpha) + \sum_{c=2}^{c_{\max}} u_i^c}.$$

To develop a Weiszfeld-like algorithm, the function

$$\pi_0(x,\alpha) = F(m_0(x,\alpha)) - \sum_{i=1}^{i_{\max}} \Phi_i(d_i(x)) - G_2(\alpha)$$

is used as a *surrogate* for the objective function Π. We use π_0 instead of Π because the function giving the market share captured by the chain (see formula (2)) is not differentiable (it includes a maximum function), whereas the market share captured by the new facility (see $m_0(x,\alpha)$) is differentiable at any feasible point.

In the Appendix, the partial derivatives of π_0 are equated to zero. From (4), (6) and (7) a Weiszfeld-like algorithm similar to the one described in **?** can be constructed. Notice that it is designed to improve the location and quality of the new facility when π_0 is considered as objective function. But as we sill see, Π usually improves as π_0 does (the computational studies corroborate that this is usually the case).

Also notice that as the algorithm goes on, the demand points to be served by the new facility may vary. This is taken into account as follows: when the

Weiszfeld-like algorithm stops, we check whether the set of demand points served by the facility has changed. If so, the algorithm is called again, but considering the new set of demand points served by the facility. Otherwise, the process stops.

Observe that this is just a local procedure. Thus, in order to have a good chance of finding the optimal solution, one should apply the algorithm repeatedly using different starting points, and then select the solution that obtains the maximum profit. When generating the seed points to start the procedure described above, only those points which serve at least one demand point should be taken into account. This multi-start strategy will be one of the procedures studied to solve problem (1). Next, we present another heuristic procedure, UEGO, introduced in **?**, to cope with the problem.

3.3. An evolutionary algorithm

In **?** an evolutionary algorithm called UEGO was studied for solving the corresponding model with probabilistic choice rule. It has also been applied to other competitive location problems as well (**??**), as it is a general algorithm able to solve many global optimisation problems. Only the local search procedure used within UEGO needs to be adapted for each particular problem. UEGO has also been used in this paper to solve the new location model introduced above, using the Weiszfeld-like algorithm developed in Subsection 3.2 as local search. Additionally, the parameters that control UEGO have to be tuned to this new problem. They have been set to $L = 10$, $R_L = 0.03$, $M = 400$ and $N = 2 \cdot 10^6$. The reader is referred to **??** for a more detailed description of UEGO.

4. The influence of the choice rule on the location and costs

Since we have several ways of modelling the patronising behaviour of customers, and hence, of estimating the market share captured by the facilities, the following question arises: how much does the choice of a particular patronising behaviour affect the location decision of the new facility and the profit obtained by it and by the whole locating chain?

We will study this point by solving some location problems with both the probabilistic and the multi-deterministic choice rules. We will solve them using the interval branch-and-bound algorithm iB&B described in Subsection 3.1 (see also Section 5 for details about the implementation), as the optimal solution is required to have a fair comparison.

Let us denote by $\Pi_P(\cdot)$ the objective function of location problem (1) when the probabilistic choice rule is employed, by $\mathcal{L}_\mathcal{P}$ the list of solution boxes provided by iB&B, and by (x_P^*, α_P^*) the best point found by iB&B during the execution, and by $\Pi_M(\cdot)$, $\mathcal{L}_\mathcal{M}$ and (x_M^*, α_M^*) the corresponding items when the multi-deterministic choice rule is employed.

We will compute the Euclidean distance between x_P^* and x_M^*, denoted by $dist_{loc}$, and the difference between the qualities, $dist_{qual} = |\alpha_P^* - \alpha_M^*|$, to measure the difference between the optimal solutions.

We will also compute the relative profit loss incurred when the probabilistic choice rule is assumed in a problem where the multi-deterministic rule should have been chosen,

$$loss(P|M) = 100 \cdot (\Pi_M(x_M^*, \alpha_M^*) - \Pi_M(x_P^*, \alpha_P^*))/\Pi_M(x_M^*, \alpha_M^*),$$

and the relative profit loss incurred when the multi-deterministic choice rule is assumed in a problem where the probabilistic rule should have been chosen,

$$loss(M|P) = 100 \cdot (\Pi_P(x_P^*, \alpha_P^*) - \Pi_P(x_M^*, \alpha_M^*))/\Pi_P(x_P^*, \alpha_P^*),$$

to measure the cost of choosing the wrong model for the chain as a whole.

Finally, in order to measure the cost of choosing the wrong model in the profit increment because of the new facility, the relative profit lost due to the new facility only when the probabilistic choice rule is assumed in a problem where the multi-deterministic rule should have been chosen,

$$loss(P|M)_0 = 100 \cdot (Incr\Pi_M(x_M^*, \alpha_M^*) - Incr\Pi_M(x_P^*, \alpha_P^*))/Incr\Pi_M(x_M^*, \alpha_M^*),$$

is computed, where $Incr\Pi_M(x_M^*, \alpha_M^*) = \Pi_M(x_M^*, \alpha_M^*) - \Pi_M(before)$ and $Incr\Pi_M(x_P^*, \alpha_P^*) = \Pi_M(x_P^*, \alpha_P^*) - \Pi_M(before)$, and $\Pi_M(before)$ stands for the profit obtained by the chain before the location of the new facility. Analogously, $loss(M|P)_0$ will be computed too.

4.1. A case study

First we will research a quasi-real example dealing with the location of a shopping mall in an area around the city of Murcia, in south-eastern Spain. A working radius of 25 km around Murcia was considered. 632558 people live within the circle, distributed over $i_{\max} = 71$ population centres, with population varying between 1138 and 178013 inhabitants. In this study we have considered each population centre as a demand point, with buying power proportional to its total population (one unit of buying power per 17800 inhabitants). Their position and population can be seen in Figure 3: each demand point is shown as a grey circle (or a black dot), whose radius is proportional to the buying power. Note that here the grey circles also show the forbidden regions. There are five shopping malls present in the area: two from a first chain A (marked with a red •, and three from another chain B, marked with a green ×. Figure 3 shows the location of each mall. The feasible set S was taken exactly as depicted in Figure 3, i.e. the smallest rectangle containing all demand points. This is approximately a square centred in Murcia and of sides close to 45 Km.

The coordinates of the population centres and the malls were obtained with the geographical information system called VisualMap **?**, and were rescaled from coordinates $([200, 245], [243, 285])$ to an approximate standard square $([0, 10], [0, 10])$. Thus, the units correspond approximately to 4.5 Km. The minimum distance $d_i^{\min}$ at which the new facility must be from the population centre i was chosen to be $w_i/30$. The qualities of the existing facilities lie in the interval $[0.4, 4]$ and for the new facility in the interval $[0.5, 5]$. And the parameter γ_i modulating the quality of the facilities as perceived by the demand point p_i in the interval $[0.75, 1.25]$. For more details about the data set, the interested reader is referred to **?**.

The basic data described above have been used to define several different competitive market structures:

Scenario 'newcomer 1': $c_{\max} = 2$ (number of chains), and the number of existing facilities belonging to each chain is 0 and 5, respectively. Notice that in this case the locating chain (chain $c = 1$) has no existing facilities and all the existing facilities are assumed to belong to the same chain.

Scenario 'newcomer 2': $c_{\max} = 3$, and the number of existing facilities belonging to each chain is 0, 3 and 2, respectively. Again, in this case, the locating chain has no existing facilities, so it is a new entering firm. But now the existing facilities are assumed to belong to two different chains.

Scenario 'small chain A': $c_{\max} = 2$, and the number of existing facilities belonging to each chain is 2 and 3, respectively. The locating chain is the small one, chain A.

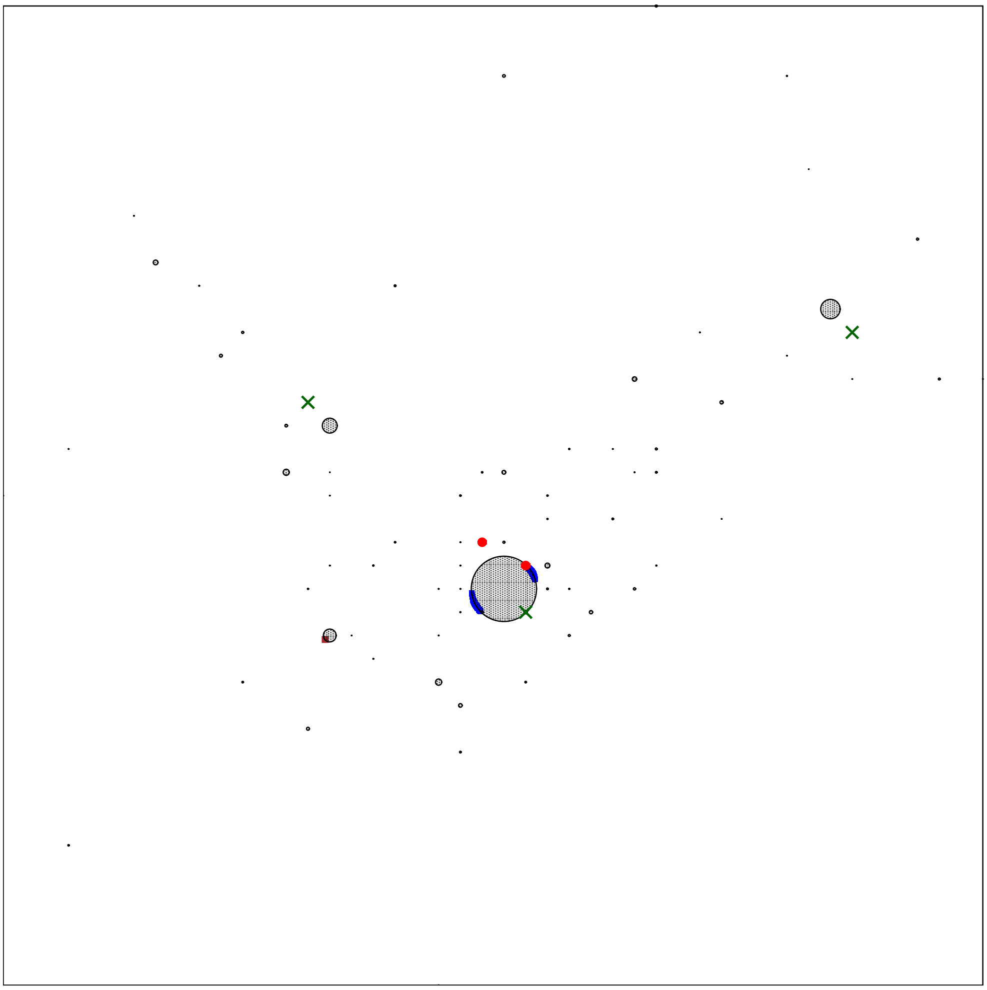

Figure 3: Case study: scenario large chain B.

Scenario 'large chain B': $c_{\max} = 2$, and the number of existing facilities belonging to each chain is 3 and 2, respectively. The locating chain is the greater one, chain B.

The results obtained are shown in Table 1. As we can see, for the second and third scenarios the differences are rather slight. However, in the 'newcomer 1' scenario, the relative profit loss incurred when the multi-deterministic choice rule is assumed instead of the probabilistic rule is more than 6%.

For the 'large chain B' scenario the differences are much higher. The relative profit loss incurred for the chain when the probabilistic choice rule is assumed instead of the multi-deterministic rule is more than 10%. This is a rather high loss, especially taking into account that the locating chain, chain B, is dominant in the market, and after the location will have four facilities, against the two facilities of the competitor. But the loss for the new facility is much higher, more than 400%, which clearly shows that the location chosen for the new facility in that case is completely wrong. If the patronising behaviour of customers was probabilistic, then the corresponding relative profit loss incurred for the chain and for the facility will be 1.13% and 23.51%, respectively. As we can see, although the loss for the chain as a whole is not that big, the loss for the new facility is rather high, too.

In Figure 3 we can see a picture of this last scenario projected onto the 2-dimensional locational space. When the probabilistic choice rule is used, the boxes of the solution list $\mathcal{L}_{\mathcal{P}}$, marked in dark blue in the picture, are around the city of Murcia, the big grey circle where most of the inhabitants of the region live. Even though the chain already has a facility in the South-East of the city, it is still more advantageous for the chain to locate the new facility there, either close to one of the existing facilities of the competitor or opposite it, and also from the second facility of the competitor chain. However, when the multi-deterministic choice rule is used, those areas are no longer an optimal solution, as the existing facility of the chain already captures a large part of the demand from Murcia. Locating the new facility close to Murcia will not increase the captured demand too much (most of the demand that the new facility will capture will be stolen from its own existing facility, an effect known as cannibalisation). That is why the optimal solution in this case is to locate the new facility in the surroundings of the fourth most populated city of the region, where the locating chain does not have any facility yet. The area covered by the list $\mathcal{L}_{\mathcal{M}}$ is drawn in brown

Table 1: Case study: differences in the solutions obtained by the probabilistic and multi-deterministic choice rules.

Scenario	$dist_{loc}$	$dist_{qual}$	$loss(P\|M)$	$loss(M\|P)$	$loss(P\|M)_0$	$loss(M\|P)_0$
newcomer 1	0.67	0.14	1.06	6.31	1.06	6.31
newcomer 2	0.09	0.14	0.39	0.49	0.39	0.49
small chain A	0.15	0.07	0.02	0.02	0.16	0.17
large chain B	2.23	3.44	10.06	1.13	418.53	23.51

colour, and is located in the South-West of the fourth most populated city. As we can see in the example, the probabilistic choice rule is more prone to the concentration of facilities around the areas with more demand, whereas the multi-deterministic choice rule favours the dispersion of facilities more.

Not only is the location for the new facility different, the quality is too. When the probabilistic rule is used, the facility has to be located in Murcia, where there already exist other facilities. So, the new facility needs to have a high quality in order to capture more demand (the optimal value for the parameter α lies in the interval $[3.35, 4.72]$). However, when the multi-deterministic rule is employed, the facility is located in an area where there are no facilities around, hence a small quality is enough to capture most of the demand of the area (the optimal value of the variable α lies in this case in the interval $[0.60, 0.68]$).

As we have seen in the case study, the assumption of a wrong customer choice rule may, depending on the location of demand points and existing facilities, provoke high losses in profit.

4.2. Random problems

Next, we will research the differences between the solutions obtained by the probabilistic and multi-deterministic choice rules in a set of random problems. We have generated location problems with different $(i_{\max}, c_{\max}, j_{\max})$ settings. Let us denote by $\lfloor z \rfloor$ the greatest integer lower than or equal to z. The number of existing facilities belonging to each chain has been obtained as $\lfloor j_{\max}/c_{\max} \rfloor$, and in case $\lfloor j_{\max}/c_{\max} \rfloor < j_{\max}/c_{\max}$ the remaining $j_{\max} - c_{\max}\lfloor j_{\max}/c_{\max} \rfloor$ facilities have been assigned to the first chains, one facility to each of those chains. The settings employed can be seen in the first column of Table 2. For every setting, five problems were generated by randomly choosing the parameters of the problems uniformly within the following intervals:

- $p_i, f_j \in S$,
- $\omega_i \in [1, 10]$,
- $\gamma_i \in [0.75, 1.25]$,
- $a_j \in [0.4, 4]$,
- $G(x, \alpha) = \sum_{i=1}^{i_{\max}} \Phi_i(d_i(x)) + G_2(\alpha)$ where

- $\Phi_i(d_i(x)) = w_i \frac{1}{(d_i(x))^{\phi_{i0}+\phi_{i1}}}$ with $\phi_{i0} = \phi_0 = 2, \phi_{i1} \in [0.5, 2]$
- $G_2(\alpha) = e^{\frac{\alpha}{\beta_0}+\beta_1} - e^{\beta_1}$ with $\beta_0 \in [5, 7], \beta_1 \in [4, 5]$

- $c \in [2, 3.5]$, the parameter for $F(M(x, \alpha)) = c \cdot M(x, \alpha)$,
- $b_1, b_2 \in [1, 2]$, parameters for $d_i(x) = \sqrt{b_1(x_1 - p_{i1})^2 + b_2(x_2 - p_{i2})^2}$

Those intervals were obtained by varying the value of the parameters of the quasi-real problem studied in Subsection 4.1 up and down.

The meaning of the columns in Table 2 correspond to those of Table 1. However, in order to avoid giving detailed results of each problem, we give for each setting the average value of the five problems, followed by the corresponding maximum.

Table 2: Random problems: differences in the solutions obtained by the probabilistic and multi-deterministic choice rules.

Setting	$dist_{loc}$	$dist_{qual}$	$loss(P\|M)$	$loss(M\|P)$	$loss(P\|M)_0$	$loss(M\|P)_0$
(100,2,5)	(0.49;1.20)	(0.22;0.73)	(0.6;2.0)	(0.4;1.0)	(13.0;40.6)	(8.6;26.3)
(100,2,10)	(0.13;0.62)	(0.10;0.48)	(0.0;0.2)	(0.0;0.2)	(7.5;37.0)	(2.6;11.4)
(100,3,10)	(0.13;0.43)	(0.06;0.15)	(0.1;0.4)	(0.0;0.2)	(6.0;20.7)	(1.8;6.3)
(100,3,15)	(1.83;4.91)	(1.01;2.36)	(1.1;3.6)	(1.1;3.7)	(109.8;454.9)	(33.1;120.0)
(100,4,15)	(3.97;10.00)	(1.05;3.45)	(1.3;2.4)	(1.2;3.8)	(18.7;40.4)	(67.8;301.9)
(1000,3,15)	(0.17;0.31)	(0.00;0.00)	(0.1;0.1)	(0.0;0.0)	(4.2;15.4)	(0.8;1.5)
(1000,4,30)	(0.51;1.13)	(0.55;2.76)	(0.5;1.0)	(0.4;0.8)	(16.9;33.3)	(13.8;37.7)
(1000,6,30)	(7.40;14.14)	(0.30;1.48)	(2.0;5.9)	(0.6;0.8)	(31.8;131.3)	(7.7;15.6)
(1000,5,40)	(4.23;10.29)	(0.00;0.00)	(1.5;4.6)	(1.2;4.9)	(31.0;65.1)	(28.0;53.2)
(1000,8,40)	(2.71;10.47)	(0.52;1.35)	(1.0;2.0)	(1.7;5.4)	(5.8;13.6)	(7.9;18.9)

From the results in column $dist_{loc}$, we can see that the location of the facility may vary considerably depending on the behaviour of customers assumed when solving the problem. This is also true for the quality of the facility to be located (see the results in $dist_{qual}$).

But for a decision maker it is the profit that makes a difference. As we can see, the *average* relative profit loss for the locating chain is not too big, regardless the choice rule assumed. Since the problems are generated at random, with the demand points and the existing facilities uniformly distributed over the feasible set, the chances of having clusters of points with high demand concentration and with facilities belonging to the locating chain around those clusters (as in the case study) is small. Still, notice that the relative profit loss is greater than 4.5% in at least 4 of the problems (see the maximum values of columns $loss(P|M)$ and $loss(M|P)$).

The differences are much clearer regarding the average relative profit loss for the new facility. In this case, it is over 5% in most of the settings, regardless the customer choice rule assumed. And in 6 of them the average loss is over 25%. Concerning the maximum values, they are greater than 25% in at least 12 settings.

5. Solving large instances: a computational study

As we have seen how important to select the right choice rule is, it is clear that we need methods for solving the corresponding location problems accurately. Interval branch-and-bound algorithms can manage small size problems, as we have seen in the previous subsection, and also medium size problems, as we will see in this section. For large size problems, heuristics procedures are required. In particular, the multi-start heuristic (denoted in what follows as MSH) and the evolutionary algorithm UEGO described in Section 3 will be analysed in this section.

All the computational studies have been carried out in a cluster with 18 nodes of shared memory and 8 GPUs. Each node has 16 cores (Intel Xeon E5 2650) and 64 GB of memory and 128 GB of solid-state drive. In total, 288 cores, 1151 GB of memory and 2304 GB of SSD. The interconnection networks are Infiniband and Ethernet. In our computational studies, each problem was run in one of the cores of the nodes (one problem at a time). The algorithms have been implemented in C++. For the interval branch-and-bound method (iB&B) we used the interval arithmetic in the PROFIL/BIAS library (**?**), and the automatic differentiation of the C++ Toolbox library (**?**).

In order to evaluate the performance of the algorithms, a new set of location problems has been generated, by increasing the number $i_{\max}$ of demand points, and varying the number $c_{\max}$ of firms and the number $j_{\max}$ of existing facilities accordingly. The settings $(i_{\max}, c_{\max}, j_{\max})$ employed in the problems can be seen in Table 3.

For every setting, one problem was generated by randomly choosing the parameters of the problems uniformly within the intervals described in Subsection 4.2.

For the problems with $i_{\max} = 2000$ demand points the multi-start algorithm (denoted by MSH) performed the local search from 1000 different starting points, and for the problems with $i_{\max} \geq 5000$ from 500 points. As

Table 3: Settings of the test problems.

$i_{\max}$	2000			5000			10000		
$c_{\max}$	30	40	50	40	50	60	60	70	80
$j_{\max}$	5,10	8,15	10,25	8,15	10,15	15,25	15,25	15,30	23,30
S	$x \in ([0,100],[0,100]), \alpha \in [0.5,5]$								
$i_{\max}$	20000			30000			50000		
$c_{\max}$	80	90	100	80	90	100	80	90	100
$j_{\max}$	23,30	32,40	40,50	23,30	32,40	40,50	32,40	40,50	
S	$x \in ([0,200],[0,200]), \alpha \in [0.5,5]$								

the heuristic algorithms may produce different results in each run, each problem has been solved 10 times with MSH and UEGO, and average values have been considered.

It is important to mention that the exact iB&B method is not able to solve all the instances. On the contrary, it starts experiencing difficulties for problems with $i_{\max} = 20000$. In particular, the computer run out memory for the cases with $i_{\max} = 20000$ and $c_{\max} = 100$ and for problems with $i_{\max} \geq 30000$

Table 4 summarizes the results obtained by the three algorithms for the instances where iB&B was able to provide a solution. The first column refers to the algorithm employed and the second one to the number of demand points. The third one gives the average CPU time employed by the algorithm when considering all the problems with the same $i_{\max}$ value. Max Dist gives the maximum Euclidean distance (in locational space) between any pair of solutions given by the algorithm in different runs. Then the minimum (Min), average (Av), maximum (Max) and standard deviation (Dev) of the objective function value is given. To check whether UEGO and MSH have obtained the global optimal solution, we have also solved the problems using the exact interval branch-and-bound method iB&B described in **?**. The problems were solved only once with this method, as it is a deterministic one. Column Imp(T) gives the reduction in CPU time obtained by the heuristics as compared to iB&B, in percentage. Finally, column %Succ. gives the number of times that the heuristic algorithm has obtained the global optimal solution, in percentage. We say that a heuristic algorithm has obtained the global optimal solution when the solution provided by the algorithm is included in the list of 3-dimensional intervals provided by iB&B as a solution.

Table 4: Computational results. iB&B has been able to solve these problems.

Alg.	$i_{\max}$	Av(T) Secs.	Max Dist	Objective Function Min	Av	Max	Dev	Imp(T)	%Succ.
UEGO	2000	19.705	0.004	1529.175	1529.175	1529.175	0.000	80	100
	5000	55.861	0.004	3661.786	3661.786	3661.786	0.000	83	100
	10000	125.125	0.005	6340.107	6340.107	6340.107	0.000	83	100
	20000	268.605	0.001	9465.572	9465.572	9465.572	0.000	78	100
MSH	2000	34.654	0.094	1529.162	1529.173	1529.175	0.004	65	94
	5000	60.849	0.022	3661.783	3661.783	3661.786	0.001	81	85
	10000	215.191	0.796	6338.288	6339.719	6340.097	0.573	71	43
	20000	312.385	0.254	9465.362	9465.545	9465.572	0.066	74	87
iB&B	2000	99.440	-	1529.175	-	1529.176	-		
	5000	319.733	-	3661.786	-	3661.787	-		
	10000	752.847	-	6340.107	-	6340.108	-		
	20000	1223.935	-	9465.572	-	9465.572	-		

As we can see, although the average objective function value obtained by both heuristic algorithms is quite close to the optimal one, UEGO is more reliable, as it always obtains the global optimal solution with 100% success, i.e., it has obtained the global optimal solution in all the problems and in all the runs, whereas the percentage of success of the multi-start strategy is smaller, achieving just 43% for the problems with 10000 demand points.

Table 5 shows the behaviour of the heuristic algorithms for those problems which could not be solved by iB&B. The columns have the same meaning as in Table 4, although in this case, columns Imp(T) and %Succ have been omitted, since we do not have the exact solutions. As can be seen, UEGO is the algorithm providing the best quality results. In fact, the maximum Euclidean distance (Max Dist) and the standard deviation (Dev) are very small, showing the high reliability of this algorithm. The objective values of the solutions obtained by MSH are also fairly good, although higher Max Dist values are obtained. This means that MSH may get trapped in a local optima in some of the runs.

6. Conclusions and future research

The estimation of the market share captured by a facility in a competitive environment depends largely on both the utility of the facility perceived by the customers and how the customers decide to patronise among the existing facilities. The influence of the second issue in the location of new facilities has been researched in this paper. In particular, the problem of locating a single new facility in the *plane* has been considered. The new customer choice rule employed in this paper, named multi-deterministic choice rule, assumes

Table 5: Computational results. iB&B has not been able to solve these problems.

Alg.	$i_{\max}$	Av(T) Secs.	Max Dist	Objective Function Min	Av	Max	Dev
UEGO	20000	359.849	0.000	6531.324	6531.324	6531.324	0.000
	30000	443.475	0.016	10404.803	10404.804	10404.805	0.001
	50000	934.622	0.015	25168.374	25168.375	25168.376	0.001
MSH	20000	375.729	3.003	6527.321	6530.647	6531.324	1.470
	30000	533.178	1.588	10403.647	10404.656	10404.804	0.350
	50000	1639.625	0.734	25167.186	25168.288	25168.376	0.289

that customers split their demand among all the firms by patronising only one facility from each firm, the one with the highest utility, and the demand is then split among those facilities proportionally to their attraction. The corresponding location problem for profit maximisation has been formulated, and an exact interval branch-and-bound method, as well as a multi-start heuristic and an evolutionary algorithm have been developed to solve the problem. The interval branch-and-bound algorithm can solve problems with up to 20000 demand points exactly. For larger instances, the evolutionary algorithm UEGO (using a Weiszfeld-like algorithm as local search) provides better and more robust results than a multi-start algorithm based on the same local search.

According to the computational results, the optimal location of the new facility as well as the profit obtained by the chain and by the new facility may vary considerably depending on the customer choice rule employed. Hence, the selection of the choice rule to be used in real applications should be made with care.

As already stated in **?**, competitive location is a difficult field not only because it involves rather complex mathematical models, but also because customer behaviour cannot easily be transcribed into neat equations. Models provide only an approximation to reality. More research on customer behaviour modelling, as we have done here, is required. In particular, the influence of the attraction function in the location of facilities deserves its own study. The extension of the single facility location model with multi-deterministic choice rule to the case of the location of more than one facility (**?**), and to the case where competitors react by locating new facilities too (**?**), should also be studied. Variable demand (**?**) is also an extension that should be researched in the future. From a computational point of view, the design of new algorithms able to solve bigger instances, or the parallelisation of the introduced ones, is another field for research.

Acknowledgments

This research has been supported by grants from the Spanish Ministry of Economy and Competitiveness (MTM2015-70260-P, and TIN2015-66680-C2-1-R), in part financed by the European Regional Development Fund (ERDF), the Hungarian National Research, Development and Innovation Office - NKFIH (OTKA grant PD115554), Fundación Séneca (The Agency of Science and Technology of the Region of Murcia, 19241/PI/14), and Junta de Andalucía (P11-TIC7176 and P12-TIC301). Juana López Redondo is a fellow of the Spanish 'Ramón y Cajal' contract program.

Appendix

When using the multi-deterministic choice rule, the market share captured by the new facility is given by

$$\begin{aligned} m_0(x,\alpha) = m_{M_0}(x,\alpha) = & \sum_{\{i: u_{i0}(x,\alpha) \geq u_i^1\}} w_i \frac{u_{i0}(x,\alpha)}{u_{i0}(x,\alpha) + \sum_{c=2}^{c_{\max}} u_i^c} \\ = & \sum_{\{i: u_{i0}(x,\alpha) \geq u_i^1\}} w_i \left(1 - \frac{\sum_{c=2}^{c_{\max}} u_i^c}{u_{i0}(x,\alpha) + \sum_{c=2}^{c_{\max}} u_i^c}\right). \end{aligned}$$

If we set

$$\tilde{w} = \sum_{\{i: u_{i0}(x,\alpha) \geq u_i^1\}} w_i, \quad r_i = \sum_{c=2}^{c_{\max}} u_i^c, \quad t_i = w_i r_i$$

then

$$m_0(x,\alpha) = \tilde{w} - \sum_{\{i: u_{i0}(x,\alpha) \geq u_i^1\}} \frac{t_i g_i(d_i(x))}{\gamma_i \alpha + r_i g_i(d_i(x))}.$$

We will develop a Weiszfeld-like algorithm using the function

$$\pi_0(x,\alpha) = F(m_0(x,\alpha)) - \sum_{i=1}^{i_{\max}} \Phi_i(d_i(x)) - G_2(\alpha)$$

as a surrogate for the objective function Π.

Assuming an unconstrained problem, a necessary condition for a vector (x^*, α^*) to be a local or global maximum of

$$\pi_0(x,\alpha) = F(m_0(x,\alpha)) - \sum_{i=1}^{i_{\max}} \Phi_i(d_i(x)) - G_2(\alpha)$$

is that the partial derivatives of π_0 at that point must vanish.

- With regard to the first variable, x_1, we have that

$$\frac{\partial \pi_0}{\partial x_1} = 0 \Longleftrightarrow \frac{\mathrm{d}F}{\mathrm{d}m_0} \cdot \frac{\partial m_0}{\partial x_1} - \sum_{i=1}^{i_{\max}} \frac{\partial \Phi_i}{\partial x_1} = 0.$$

  Doing some calculations the previous expression is equivalent to

$$\frac{\mathrm{d}F}{\mathrm{d}m_0} \cdot \sum_{\{i:u_{i0}(x,\alpha)\geq u_i^1\}} \frac{-\alpha\gamma_i t_i g_i'(d_i(x))}{(\gamma_i\alpha + r_i g_i(d_i(x)))^2} \cdot \frac{\partial d_i(x)}{\partial x_1} - \sum_{i=1}^{i_{\max}} \frac{\mathrm{d}\Phi_i}{\mathrm{d}d_i(x)} \frac{\partial d_i(x)}{\partial x_1} = 0,$$

  which can be rewritten as

$$\sum_{i=1}^{i_{\max}} H_i(y) \frac{\partial d_i(x)}{\partial x_1} = 0,$$

  where $y = (x, \alpha)$ and

$$H_i(y) = \begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}m_0} \cdot \dfrac{\alpha\gamma_i t_i g_i'(d_i(x))}{(\gamma_i\alpha + r_i g_i(d_i(x)))^2} + \dfrac{\mathrm{d}\Phi_i}{\mathrm{d}d_i(x)} & \text{if } u_{i0}(x,\alpha) \geq u_i^1 \\ \dfrac{\mathrm{d}\Phi_i}{\mathrm{d}d_i(x)} & \text{otherwise} \end{cases}.$$

  Assume now that $d_i(x)$ is a distance function such that

$$\frac{\partial d_i(x)}{\partial x_1} = x_1 A_{i1}(x) - B_{i1}(x), \tag{3}$$

  where $A_{i1}(x)$ and $B_{i1}(x)$ are functions of x, then

$$\frac{\partial \Pi}{\partial x_1} = 0 \Longleftrightarrow x_1 = \frac{\sum_{i=1}^{i_{\max}} H_i(y) B_{i1}(x)}{\sum_{i=1}^{i_{\max}} H_i(y) A_{i1}(x)}. \tag{4}$$

- Analogously, if
$$\frac{\partial d_i(x)}{\partial x_2} = x_2 A_{i2}(x) - B_{i2}(x), \tag{5}$$
then
$$\frac{\partial \pi_0}{\partial x_2} = 0 \Longleftrightarrow x_2 = \frac{\sum_{i=1}^{i_{\max}} H_i(y) B_{i2}(x)}{\sum_{i=1}^{i_{\max}} H_i(y) A_{i2}(x)}. \tag{6}$$

- Finally,
$$\frac{\partial \pi_0}{\partial \alpha} = 0 \Longleftrightarrow \frac{\mathrm{d}F}{\mathrm{d}m_0} \cdot \sum_{\{i: u_{i0}(x,\alpha) \geq u_i^1\}} \frac{\gamma_i t_i g_i(d_i(x))}{(\gamma_i \alpha + r_i g_i(d_i(x)))^2} - \frac{\mathrm{d}G_2}{\mathrm{d}\alpha} = 0. \tag{7}$$

  Notice that this last equation, when we fix $x = (x_1, x_2)$, has just one variable, α. Thus we could solve it by using any algorithm for solving equations of a single variable. Furthermore, π_0, as a function of the α variable alone, is concave: regardless the strictly increasing differentiable function $F(\cdot)$ considered, the second derivative of π_0 with regard to α is negative, since
$$\frac{\partial^2 \pi_0}{\partial \alpha^2} = -\frac{\mathrm{d}F}{\mathrm{d}m_0} \cdot \sum_{\{i: u_{i0}(x,\alpha) \geq u_i^1\}} \frac{2\gamma_i^2 t_i g_i(d_i(x))(\gamma_i \alpha + r_i g_i(d_i(x)))}{(\gamma_i \alpha + r_i g_i(d_i(x)))^4} - \frac{\mathrm{d}^2 G_2}{\mathrm{d}\alpha^2},$$
  and $\frac{\mathrm{d}^2 G_2}{\mathrm{d}\alpha^2} > 0 \ \forall \alpha > 0$. So, the solution of the previous equation is guaranteed to be a global maximum.

The l_{2b} distance, given by
$$d_i(x) = \sqrt{b_1(x_1 - p_{i1})^2 + b_2(x_2 - p_{i2})^2},$$
satisfies the conditions (3) and (5) given above. Furthermore, it has proved to be a good distance predicting function (see ?), and it is therefore a good distance function to be used in competitive location models, as it measures distances (or travel times) as they are perceived by customers on their ways to and from facilities.