UNIVERSITY OF CANBERRA

This is the author(s) refereed version of a paper that was accepted for publication:

Lisle, I., & Reid, G. (2006). Symmetry Classification Using Noncommutative Invariant Differential Operators. *Foundations of Computational Mathematics*, 6, 353-386. https://doi.org/10.1007/s10208-005-0186-x

Symmetry Classification Using Non-commutative Invariant Differential Operators

I.G. Lisle* and G.J. Reid†‡

Abstract

Given a class F of differential equations, the symmetry classification problem is to determine for each member $f \in F$ the structure of its Lie symmetry group G_f, or equivalently of its Lie symmetry algebra. The components of the symmetry vector fields of the Lie algebra are solutions of an associated over-determined 'defining system' of differential equations. The usual computer classification method which applies a sequence of total derivative operators and eliminations to this associated system often fails on problems of interest due to the excessive size of expressions generated in intermediate computations.

We provide an alternative classification method which exploits the knowledge of an equivalence group $\mathcal{G}$ preserving the class. A non-commutative differential elimination procedure due to Lemaire, Reid and Zhang, where each step of the procedure is invariant under $\mathcal{G}$, can be applied and an existence and uniqueness theorem for the output used to classify the structure of symmetry groups for each $f \in F$.

The method is applied to a class of nonlinear diffusion convection equations $v_x = u$, $v_t = B(u)u_x - K(u)$ which is invariant under a large but easily determined equivalence group $\mathcal{G}$. In this example the complexity of the calculations is much reduced by the use of $\mathcal{G}$-invariant differential operators.

AMS Classification: 35N10, 58J70, 53A55, 13P10, 12H05.

*School of Information Sciences and Engineering, University of Canberra, ACT Australia. 2600. Email: `Ian.Lisle@canberra.edu.au`

†Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada. Email: `reid@uwo.ca`, Web: `www.apmaths.uwo.ca/~reid`

‡GJR gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada. He also acknowledges many helpful discussions with Elizabeth Mansfield on the topic of this paper, and support of an ESPRC grant from the U.K. government for a visit to the University of Kent in Canterbury. Support from University of Canberra is also acknowledged.

Keywords: Lie symmetries of PDE, equivalence group, symmetry classification of PDE, moving frames, noncommutative invariant differential operators, reduced involutive form, diffusion convection equation

1 Introduction

This article falls under the general area of Geometric Integration [14] and specifically in the development of tools that share symmetry properties of the problems to which they are applied [24, 26]. See [6] for applications to computer vision and [25, 14] for applications to group invariant numerical integrators. In this paper, we show how to perform symmetry classification for a class of PDE in a $\mathcal{G}$-invariant way, where $\mathcal{G}$ is a known Lie group which leaves the class invariant (i.e. $\mathcal{G}$ is an equivalence group for the class). The advantages of this approach are demonstrated in the reduction of complexity of computations for problems to which they are applied.

The Lie symmetry group of a differential equation [23] preserves its family of solutions and is the basis of several useful techniques in applied mathematics. Such techniques include finding invariant solutions, solving ordinary differential equations in formula, finding conservation laws and linearizations [27, 23, 4, 24] and the development of symmetry invariant numerical integrators [14].

Some symmetries are known a priori on physical grounds – for instance an isotropic medium will have rotational symmetries – but in general a given PDE must be analysed to find all its symmetries. The analysis proceeds by seeking vector fields $\mathbf{X} = \sum_i \xi^i(w)\frac{\partial}{\partial w^i}$ on the space of independent and dependent variables, such that the PDE are invariant under the action of $\mathbf{X}$. The unknown components ξ^i of the vector fields satisfy a linear homogeneous and generally over-determined system of PDE, the *symmetry system* (or defining system) for the algebra of symmetry vector fields of G.

Several programs are available [34, 28, 35] for reducing the symmetry system by applying a sequence of eliminations and commutative differentiations to the symmetry system. An existence and uniqueness theorem is available for the output form of some of the programs [32, 33] and allows determination of the size of the group. In addition the formal series solution for ξ^i can be constructed up a given order, allowing computation of the structure of the unknown group without explicitly determining ξ [29, 30]. Some programs focus on explicit determination of the ξ^i by employing integration [34, 35] in addition to differentiations and eliminations. The above programs can often successfully determine symmetry properties of a single differential equation.

Often one wishes to find the structure of the symmetry group G_f of each member f of a class F of differential equations. For instance, one might be interested in second order ODE

$$F = \{y'' = h(x, y, y') \mid h : \mathbb{R}^3 \to \mathbb{R}\}.$$

Specification of the arbitrary element h picks out a differential equation from F, with an associated symmetry group G_h.

Example 1.1. Consider the class F of scalar nonlinear diffusion equations

$$u_t = \big(B(u)u_x\big)_x \tag{1}$$

which is assumed to be nonlinear so that $B \neq 0$, $B_u(u) \neq 0$. Different functional forms of the diffusivity $B(u)$ will lead to different symmetry groups G_B. The symmetry operators $\mathbf{X} = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u$ which generate the Lie algebra $\mathcal{L}_B$ of symmetry vector fields for G_B have components ξ, τ, η which obey the symmetry system [27, eq.(6.7.3)]

$$\tau_x = \tau_u = \xi_u = \eta_{uu} = 0 \tag{2a}$$
$$B(2\xi_x - \tau_t) - B_u\eta = 0 \tag{2b}$$
$$B(2\eta_{xu} - \xi_{xx}) + 2B_u\eta_x + \xi_t = 0 \tag{2c}$$
$$B\eta_{xx} - \eta_t = 0. \tag{2d}$$

For given $B(u)$, this is an over-determined linear homogeneous system. We will consider the overall system to consist of not only the symmetry system $\mathcal{S}$ given by (2), but also the 'classifying system' $\mathcal{C}$ satisfied by B, in this case $\mathcal{C} = \{B_x = B_t = 0,\ B \neq 0,\ B_u \neq 0\}$. The overall system is $\mathcal{S} \cup \mathcal{C}$.

In analysing such systems containing arbitrary elements, new problems arise. As the defining equations are manipulated, derivatives of the arbitrary elements (e.g. B_u, B_{uu}, $\dots$) accumulate in the coefficients, significantly increasing the algebraic complexity of the equations. One also uncovers *case splittings*, conditional on the arbitrary elements obeying certain *classification conditions*.

Example 1.1 *(cont.)*. Assuming that $B_u \neq 0$, (2b) implies that $\frac{B}{B_u}(2\xi_x - \tau_t) - \eta = 0$; differentiation and reduction modulo (2a) then gives $(B/B_u)_{uu}\ (2\xi_x - \tau_t) = 0$. There are two cases:

$$(B/B_u)_{uu} = 0 \tag{3}$$

or $(B/B_u)_{uu} \neq 0$, $2\xi_x - \tau_t = 0$. In this second case, further differentiations and eliminations quickly show that the diffusion equation (1) admits only the obvious symmetries [27, §6.7]

$$G_B = \{g : (x,t,u) \mapsto (\alpha x + \mu, \alpha^2 t + \nu, u)\}. \tag{4}$$

Thus if equation (1) is to have any non-obvious symmetries, $B(u)$ must satisfy classification condition (3) [27, §6.7].

Such conditions are what investigators aim for in classification problems.

The computational labor for symmetry classification is greater than for symmetry analysis of one system, mainly because algebraic complexity of the symmetry and classification systems can build up explosively as the equations are manipulated. Programs such as [34, 35, 31, 22] can handle classes of PDE, but can fail due to exhaustion of memory. Even when answers are returned, the case splitting criteria may be so complex as to defy interpretation. To overcome this, various workers [1, 2] follow Ovsiannikov [27] in using *equivalence transformations*, which map equations in a class to other equations in the same class.

Example 1.1 *(cont.).* For the diffusion equation (1), the transformations given by

$$\mathcal{G} = \{g : (x,t,u) \mapsto (\beta x, t, \gamma u + \delta), \qquad \beta\gamma \neq 0\} \tag{5}$$

form an equivalence group for the class, since these transformations map the diffusion equation to $\gamma u'_{t'} = \gamma \beta^{-2}\big(B(\gamma u' + \delta)u'_{x'}\big)_{x'}$. Hence the coefficient $B(u)$ is mapped to a new coefficient, given by $B'(u) = \beta^{-2}B(\gamma u + \delta)$, and transformations (5) along with (4) constitute an equivalence group $\mathcal{G}$ for the class. These transformations are usually not symmetries since in general $B \neq B'$.

When classifying symmetries, the equivalence group can be used to eliminate parameters from cases.

Example 1.1 *(cont.).* The classification equation (3) has solutions

$$B(u) = ae^{mu}, \qquad B(u) = (au+b)^m, \qquad (a, m \neq 0).$$

Choosing class representatives with respect to the equivalence group (5) gives normal forms

$$B(u) = e^u, \qquad B(u) = u^m, \qquad (m \neq 0).$$

This use of equivalence transformations to 'clean up' at the end of a symmetry classification does not address the problem of algebraic complexity arising in intervening calculations. In the current paper, we present a method which takes advantage of equivalence transformations at the outset rather than at the end of the analysis. The goal is to provide a method which has the equivalence group built into it, in the sense that two equations connected by an equivalence transformation will automatically be identified throughout the calculation. Significant clarification and simplification of symmetry classifications are thereby achieved.

Our method is based on differential elimination procedures developed by Reid et al. [28, 29, 30, 31] and which complete the symmetry system by adjoining compatibility conditions [5]. After a finite number of steps, the system reaches a reduced form where a local existence-uniqueness theorem can be used to deduce properties of the defining system, such as the dimension of its solution space. The differential elimination methods of [28] gave algorithms for exhibiting symmetry classifying conditions with the restriction that the classifying conditions remained linear in their highest derivatives. The restriction of leading linearity is removed in the symmetry classification problem by Mansfield [20]. Similar methods are now employed in several programs [34, 35, 22]. However these methods are subject to severe expression swell, and can fail for this reason. In addition, the programs often give rise to spurious case splits: the user pursues first one then another branch, only to find that the branches have no special symmetry properties.

Now suppose one has available an equivalence group $\mathcal{G}$ for the class F. By writing the defining equations in a form which is $\mathcal{G}$-invariant, we are able to carry out calculations which lessen these computational difficulties. The virtue of using such a $\mathcal{G}$-invariant formulation is that equations which are connected by a transformation from $\mathcal{G}$ are identified, leading to results which are easier to interpret, and also a reduction in their complexity.

To use such a $\mathcal{G}$-invariant formulation one must compute the differential invariants of the equivalence group $\mathcal{G}$, and rewrite the defining system in $\mathcal{G}$-invariant form. Fortunately, over the last decade Fels, Olver and others [9, 10, 24, 26] have significantly developed and generalized Cartan's theory of moving frames. As a result there are now systematic methods for finding differential invariants and other associated objects such as invariant differential operators. These methods are important prerequisites for the applications described in our paper.

Writing the systems in $\mathcal{G}$-invariant form requires the use of *non-commuting* invariant bases of differential operators (BDO) and methods for systematically manipulating and simplifying such systems. There is available a non-commutative differential elimination procedure due to Lemaire, Reid and Zhang [18, 39] to bring such systems to a reduced form enjoying properties crucial to our article (also see the approach of Mansfield [21]). In particular, a local existence and uniqueness theorem [18, 39] is available for the output non-commutative reduced involutive form (RIF-form). This theorem enables us to algorithmically find the dimension (§6) and structure (§7) of symmetry algebras $\mathcal{L}_f$ for each member f of the class F. The reduction procedure also enables classifying conditions to be determined algorithmically. Thus our approach combines classical methods of symmetry analysis [4, 23, 27], the theory of $\mathcal{G}$-invariant objects [9, 10, 26] and differential elimination methods for PDE [18, 39].

The remainder of this paper is organized as follows. In §2 non-commuting differential operators and their structure relations are introduced. In §3 we introduce notation and discuss the process that puts the non-commutative derivations in a normalized order. Rankings of derivations and reduction of PDE are discussed in §4. Using the equivalence group to find suitable frames is discussed in §5 and applied to the example of the nonlinear diffusion equation (1) in §5.3. The calculation of structure constants from a frame defining system is discussed in §7. A substantial example (nonlinear diffusion convection equations) of symmetry classification is presented in §8. Concluding remarks are given in §10.

2 Derivations and Structure Relations

We consider systems of partial differential equations (PDE) with independent variables $x = (x^1, x^2, ..., x^n) \in \mathbb{F}^n$ and dependent variables $u = (u^1, ..., u^m) \in \mathbb{F}^m$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Differential elimination methods, such as those of [16, 28, 31, 5], are stated in a coordinate system, with commuting derivations. We now relax this and permit an arbitrary, possibly non-commutative, basis for first order derivations. See [18] for details and a careful treatment.

We restrict our attention to (mainly over-determined) systems which are $\mathbb{F}$-analytic functions of their independent, dependent variables and total differential operators applied to u where the differential operators are compositions of derivations of the form

$$\widetilde{D}_i = \sum_{j=1}^{n} a_{ij}(x, u) D_{x^j} , \quad i = 1, \ldots, n. \tag{6}$$

Here D_{x^j} is the usual (commutative) total derivative operator, the $a_{ij}(x, u)$ are $\mathbb{F}$-analytic functions and the $n \times n$ matrix $A(x, u) = (a_{ij}(x, u))$ is invertible (i.e. $\det(A(x,u)) \neq 0$) at points (x,u) in $\mathbb{F}^n \times \mathbb{F}^m$. Since $A(x,u)$ is invertible it follows that $D_{x^i} = \sum_{j=1}^n b_{ij}(x,u)\widetilde{D}_j$ where $(b_{ij}(x,u)) = A(x,u)^{-1}$. Thus analytic PDE systems written in terms x, u and D_i can be written in terms of x, u and $\widetilde{D}_i$.

The $\widetilde{D}_i$ satisfy commutation relations of the form:

$$[\,\widetilde{D}_i, \widetilde{D}_j\,] = \widetilde{D}_i\widetilde{D}_j - \widetilde{D}_j\widetilde{D}_i = \sum_{k=1}^{n} \gamma_{ij}^k \widetilde{D}_k\ , \quad 1 \leq i,j \leq n. \tag{7}$$

Here the γ_{ij}^k are functions of x,u and first order derivations of u, and can be easily calculated from $A(x,u)$ using (6) to express the commutator in terms of D_j and then applying the inverse to express the result in terms of $\widetilde{D}_j$. Thus the $\widetilde{D}_i$ are generally non-commuting, in contrast to the usual commuting total derivatives D_{x^i} (abbreviated as D_i).

Example 2.1. The following non-commutative derivations arise from the nonlinear diffusion convection potential system discussed in §8. Let $\{D_v, D_x, D_t, D_u\}$ be commuting derivations on $\mathbb{R}^4$ with coordinates (v,x,t,u). Let K,q be dependent variables in the simple system of PDE $K_v = K_x = K_t = 0, K_u = q$. We will denote sets of derivations by upper case Greek letters. For example here we use Δ where $\widetilde{D}_i = \Delta_i$, $i = 1,2,3,4$ to denote:

$$\Delta_1 = D_v, \quad \Delta_2 = D_x, \quad \Delta_3 = D_t + qD_x + \big(uq - K\big)D_v, \quad \Delta_4 = D_u\ . \tag{8}$$

The matrix $A(v,x,t,u)$ has determinant 1, so Δ satisfies the determinant condition. The derivations Δ (8) have $[\Delta_i, \Delta_j] = 0$ apart from the relations involving Δ_3, for instance

$$[\Delta_3, \Delta_4] = -q_{,4}\Delta_2 - (uq_{,4} + q - K_{,4})\Delta_1\ .$$

We shall have frequent need to manipulate vector fields. If $\mathbf{X} = \sum_{i=1}^n \xi^i D_i$ is a vector field then its components θ^j with respect to $\widetilde{D}_j$ are

$$\mathbf{X} = \sum_{j=1}^{n} \theta^j \widetilde{D}_j, \qquad \text{where} \qquad \sum_{i=1}^{n} a_{ij}(x,u)\theta^i = \xi^j. \tag{9}$$

Invertibility of $A(x,u)$ guarantees that θ^j can be expressed in terms of ξ^i.

Example 2.1. *(cont.)* Let $\mathbf{X} = \chi D_v + \xi D_x + \tau D_t + \eta D_u$ be a vector field, with χ, ξ, τ, η functions of (v,x,t,u). Resolving with respect to Δ, i.e. $\mathbf{X} = \theta^i \Delta_i$ gives

$$\theta^1 = \chi - \big(uq - K\big)\tau, \quad \theta^2 = \xi - q\tau, \quad \theta^3 = \tau, \quad \theta^4 = \eta. \tag{10}$$

3 Derivations, Normalization and Bijection to the Commutative Case

We briefly discuss the normalization of non-commutative operators and a bijection to the commutative case. See [18] for a detailed treatment which is a minor variation on the standard one (in which only the independent variables appear in the coefficients). The main point is that basic operations, such as chain rules, work in the expected way.

As is usual in the commutative case we define [32, 5] a set of indeterminates $\Omega = \{v^i_\alpha \mid \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n, i = 1, \cdots, m\}$ where each indeterminate of Ω corresponds to a partial derivative by:

$$v^i_\alpha \leftrightarrow (\partial_n)^{\alpha_n} \cdots (\partial_1)^{\alpha_1} u^i(x^1, \cdots, x^n) := \partial^\alpha u^i(x^1, \cdots, x^n)\ .$$

Commutative total derivative operators act on members of Ω by a unit increment of the i-th index of their vector subscript: $D_i v^k_\alpha := v^k_{\alpha+1_i}$ where $\alpha + 1_i = (\alpha_1, ..., \alpha_i + 1, ..., \alpha_n)$. The usual (commutative) total derivative $D_{x^i} \equiv D_i$ action on functions of $\{x\} \cup \Omega$ is then given by: $D_i = \partial_i + \sum_{v\in\Omega}(D_i v)\frac{\partial}{\partial v}$.

In the non-commutative case, n total derivations $\widetilde{D}_1, ..., \widetilde{D}_n$ act on formal power series in the x^i. A general total derivation operator of order p where $p \in \mathbb{N}$ has form $\widetilde{D}_{i_1}\widetilde{D}_{i_2}...\widetilde{D}_{i_p}$ where $i_j \in \{1, 2, ..., n\}$ for $j = 1, ..., p$. In a similar manner to the commutative case let $K^n = \{(i_p, ..., i_2, i_1) : i_j \in \{1, 2, ..., n\}$ and $p \in \mathbb{N}\}$ and define $\widehat{\Omega} = \{\hat{v}^k_I \mid I \in K^n, k = 1, \cdots, m\}$ where $\widetilde{D}_{i_1}\widetilde{D}_{i_2}...\widetilde{D}_{i_p} u^k \leftrightarrow \hat{v}^k_{(i_p,...,i_2,i_1)}$. Then a formal total derivation operator $\widetilde{D}_j$ acts on members of $\hat{v}^k_I \in \widehat{\Omega}$ where $I = (i_p, ..., i_2, i_1)$ by appending j to their index $\widetilde{D}_j \hat{v}^k_I := \hat{v}^k_{I,j} \equiv \hat{v}^k_{i_p,...,i_2,i_1,j}$. One demonstrates easily (e.g. see [18]) that the chain rule works as expected on analytic functions f of $\{x\} \cup \widehat{\Omega}$, namely

$$\widetilde{D}_i f = \sum_{j=1}^{n} (\widetilde{D}_i x^j)\frac{\partial}{\partial x^j} f + \sum_{\hat{v}\in\widehat{\Omega}} (\widetilde{D}_i \hat{v})\frac{\partial}{\partial \hat{v}} f\ . \qquad (11)$$

We will use the natural notation $\widetilde{\partial}_i := \sum_{j=1}^n (\widetilde{D}_i x^j)\frac{\partial}{\partial x^j}$ and call $\widetilde{\partial}_i$ the partial derivation. Using this notation we can write $\widetilde{D}_i = \widetilde{\partial}_i + \sum_{\hat{v}\in\widehat{\Omega}}(\widetilde{D}_i\hat{v})\frac{\partial}{\partial\hat{v}}$.

Define $\widetilde{\Omega} = \{\tilde{v}^i_\alpha \mid \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n, i = 1, \cdots, m\}$ where each element of this set corresponds to a derivation by:

$$\tilde{v}^i_\alpha \leftrightarrow (\widetilde{D}_n)^{\alpha_n} \cdots (\widetilde{D}_1)^{\alpha_1} u^i(x^1, \cdots, x^n) := \widetilde{D}^\alpha u^i(x^1, \cdots, x^n)\ .$$

In contrast to the commutative case this correspondence only gives a subset of the set of all derivations $(\widehat{\Omega})$. However the commutation relations (7) enable us to extend this correspondence to the whole set.

Hypothesis 3.1 (Finiteness and Normalization Assumption for Derivations). In this article we only consider functions of finitely many indeterminates (i.e. the differential equations considered are always of finite order). Also, each time a derivation is applied to such a function of $\{x\} \cup \tilde{\Omega}$ we assume that the commutation rules are applied to get an expression only involving elements of $\{x\} \cup \widetilde{\Omega}$.

It follows [18] from our normalization that $\widetilde{D}_i = \widetilde{\partial}_i + \sum_{\tilde{v} \in \widetilde{\Omega}} (\widetilde{D}_i \tilde{v}) \frac{\partial}{\partial \tilde{v}}$. The other vital property [18, 39] underlying our method is:

Theorem 3.2 (Bijection between derivations and derivatives). *Under the analyticity and invertibility assumption on a_{ij} in §2 each order q derivation operator (resp. differential operator) can be expressed as a linear combination of differential operators (resp. derivation operators) of order q or less with coefficients being analytic functions of x and commutative (resp. non-commutative) derivation variables of order $q-1$ or less.*

In summary three different sets of derivations are used in this paper: the non-normalized derivations $\widehat{\Omega}$, the normalized derivations $\tilde{\Omega}$ and the commutative derivatives Ω. They simply correspond to the following spaces: $\hat{J}^q(\mathbb{F}^n, \mathbb{F}^m)$, $\tilde{J}^q(\mathbb{F}^n, \mathbb{F}^m)$ and $J^q(\mathbb{F}^n, \mathbb{F}^m)$.

The relationship between these spaces is shown below:

$$\hat{J}^q(\mathbb{F}^n, \mathbb{F}^m) \xrightarrow{\Psi^{(q)}} \tilde{J}^q(\mathbb{F}^n, \mathbb{F}^m) \xleftrightarrow{\Phi^{(q)}} J^q(\mathbb{F}^n, \mathbb{F}^m)\,. \tag{12}$$

Here $J^q(\mathbb{F}^n, \mathbb{F}^m)$ is the usual q-th order jet space. As usual the zero set of the functions defining the differential equations is a locus of points in that space. In a similar way a differential equation expressed in terms of non-normalized derivations from $\widehat{\Omega}$ is represented by a zero set in $\hat{J}^q(\mathbb{F}^n, \mathbb{F}^m)$. Finally $\tilde{J}^q(\mathbb{F}^n, \mathbb{F}^m)$ represents the space of non-commutative normalized derivations. The normalization map $\Psi^{(q)}$ at order q quotients by the commutation relations, mapping points in $\hat{J}^q(\mathbb{F}^n, \mathbb{F}^m)$ to ones in $\tilde{J}^q(\mathbb{F}^n, \mathbb{F}^m)$. Because normalization is applied, we do not see the space $\hat{J}^q(\mathbb{F}^n, \mathbb{F}^m)$ in calculations of the paper. The bijection to order q denoted by $\Phi^{(q)}$ is obtained through the correspondence $\widetilde{D}_i = \sum_{j=1}^n a_{ij}(x,u) D_{x^j}$. The theory, which is given elsewhere [18, 39], essentially exploits this as a map to the commutative case.

4 Rankings and Reduction of Classification Systems

The application of the algorithms of [18, 39] to symmetry classification problems in this paper, require an input ranking of derivations satisfying certain properties. In §4.1 and §4.2 we show that such rankings exist and are easily constructed. As discussed in §4.3 reduction with respect to such rankings enables the application of a formal existence and uniqueness theorem [18, 39] to the output.

4.1 Rankings

We direct the reader to [18] for background on rankings in the non-commutative case and to [32] for the commutative case.

Suppose $\prec$ is a total order on the set of (normalized) derivations $\widetilde{\Omega}$. Consider a function f on $\tilde{J}^q(\mathbb{F}^n, \mathbb{F}^m)$ which is analytic at $(x, \tilde{v}^i_\alpha) \in \tilde{J}^q(\mathbb{F}^n, \mathbb{F}^m)$. Let $\textsc{hd}\, f$ denote the greatest derivation in f with respect to the ranking $\prec$. The definition of a positive ranking in the non-commutative case as given in [18] is:

Definition 4.1. A *positive ranking* $\prec$ of $\widetilde{\Omega}$ is a total ordering on $\widetilde{\Omega}$ satisfying:

$$\textsc{hd}(\widetilde{D}^\beta \tilde{v}^i_\alpha) = \tilde{v}^i_{\alpha+\beta} \tag{13a}$$

$$\tilde{v}^i_\alpha \prec \tilde{v}^j_\beta \quad \Rightarrow \quad \textsc{hd}\, \widetilde{D}^\gamma \tilde{v}^i_\alpha \prec \textsc{hd}\, \widetilde{D}^\gamma \tilde{v}^j_\beta \tag{13b}$$

$$\tilde{v}^i_\alpha \prec \textsc{hd}\, \widetilde{D}^\gamma \tilde{v}^i_\alpha \qquad \text{for } |\gamma| \neq 0 \tag{13c}$$

where as usual all derivations are normalized (so $\widetilde{D}^\beta \tilde{v}^i_\alpha$, $\widetilde{D}^\gamma \tilde{v}^i_\alpha$, etc are normalized before $\textsc{hd}$ is applied).

Property (13a) is trivially satisfied in the commutative case. It implies that (13b) and (13c) become

$$\tilde{v}^i_\alpha \prec \tilde{v}^j_\beta \Rightarrow \tilde{v}^i_{\alpha+\gamma} \prec \tilde{v}^j_{\beta+\gamma}, \qquad \tilde{v}^i_\alpha \prec \tilde{v}^i_{\alpha+\gamma}\,. \tag{14}$$

Since these are just the requirements of positive rankings in the commutative case [32] it implies that every positive ranking in the non-commutative case is also a positive ranking for the commutative case. All such positive rankings have been classified [32]. The condition (13a) for a given problem restricts the available positive rankings to a subclass of the commutative ones. In particular, commutation relations (7) and the fact that the γ_{ij}^k are functions of at most first order derivations imply that

$$\widetilde{D}^\beta \tilde{v}_\alpha^i = \tilde{v}_{\alpha+\beta}^i + R \tag{15}$$

where R represents remainder terms of total derivation order strictly lower than $|\alpha + \beta|$. Thus if we choose a ranking satisfying (14) and compatible with total derivation order i.e. satisfying:

$$|\alpha| < |\beta| \implies \tilde{v}_\alpha^i \prec \tilde{v}_\beta^j \text{ for any } 1 \le i, j \le n \tag{16}$$

then property (13a) is ensured. Properties (13b) and (13c) follow immediately yielding a positive ranking.

4.2 Rankings for decoupling Classification Systems

For our class of problems, we need to construct positive rankings slightly more general than those satisfying (16). Note that symmetry defining systems are linear in the symmetry components ξ^i and have coefficients which are in general non-linear functions of the classification functions f^j. To preserve this quasi-linear structure, and decouple equations in the classification variables alone, we need to show that there exist positive 'elimination' rankings that satisfy:

$$\xi^i \succ \widetilde{D}^\beta f^j \ , \ \text{ for all } \ i, j, \beta \, . \tag{17}$$

Proposition 4.2. *Suppose $\prec$ is a ranking satisfying (14). Let the dependent variables v be partitioned as $v = (f, \xi)$, where f has μ components and ξ has $m - \mu$ components, and suppose that the structure functions γ_{ij}^k given in* (7) *are functions only of (x, f) and first order derivations of f. Suppose that $\prec$ also satisfies the conditions*

(i) $|\alpha| < |\beta| \implies f_\alpha^k \prec f_\beta^\ell$

(ii) $|\alpha| < |\beta| \implies \xi_\alpha^i \prec \xi_\beta^i$

for $1 \le k, \ell \le \mu$ and $1 \le i \le m - \mu$. Then $\prec$ is a positive ranking, that is, it satisfies (13).

The proposition asserts that a ranking $\prec$ need only be compatible with total derivation order on the subset f and on each ξ^i variable individually.

Proof. Suppose that $\prec$ satisfies the conditions of the theorem. From (15) it follows that

$$\widetilde{D}^{\beta} f^{\ell}_{\alpha} = f^{\ell}_{\alpha+\beta} + R \tag{18}$$

where R represents normalization terms of strictly lower derivation order than $|\alpha+\beta|$. Now, because γ^k_{ij} depends only on f, R can depend only on f-variables and their derivatives, and because $\prec$ satisfies (i), this means that all terms in R are lower ranked than $f^{\ell}_{\alpha+\beta}$. Hence

$$\text{HD}\, \widetilde{D}^{\beta} f^{\ell}_{\alpha} = f^{\ell}_{\alpha+\beta} \,. \tag{19}$$

Now suppose ξ^i is one of the other variables. Again by (15)

$$\widetilde{D}^{\beta} \xi^{i}_{\alpha} = \xi^{i}_{\alpha+\beta} + R' \tag{20}$$

where the normalization terms R' have derivations of strictly lower derivation order than $|\alpha+\beta|$. But the γ^k_{ij} do not depend on ξ, so R' only includes terms of the form $b\xi^i_{\delta}$ where b depends on derivatives of f. Since $\prec$ satisfies (ii), it follows that all terms in R' are lower ranked than $\xi^i_{\alpha+\beta}$, so

$$\text{HD}\, \widetilde{D}^{\beta} \xi^{i}_{\alpha} = \xi^{i}_{\alpha+\beta} \,. \tag{21}$$

The two statements (19,21) amount to property (13a). The other parts of (13) follow immediately. □

Proposition 4.2 ensures the existence of rankings satisfying (17), which is what we will require for the applications considered in our paper. The proof also shows that if ξ^i satisfy linear homogeneous equations, then normalization preserves this feature.

4.3 Algorithmic Reduction to non-commutative Rif-form

In previous work algorithms were developed for transforming systems of PDE by applying a sequence of commutative differentiations and eliminations, enabling existence and uniqueness theorems to be stated for their output. That work had its roots in the Riquier-Janet Theory [16, 37]. Subsequently [31, 33, 32] the method was generalized to nonlinear systems. Later the theory was adapted to linear [12] and nonlinear [18, 39] systems written in terms of non-commuting derivations.

In particular we can algorithmically apply the results of [18, 39] to systems with polynomially nonlinear dependence on the classification functions and their derivations over the rational numbers. The output non-commutative reduced involutive form obtained by this method has an associated existence and uniqueness theorem [18, Theorem 9.11], which is sufficient for our purposes in this article.

In the application of this algorithmic process, whenever derivation operators are applied, the normalization Φ is automatically invoked to place the derivation operators in normalized order. The bijection Ψ to the commutative case implies that the algorithms are a direct translation of those for the commutative case. The algorithms have been implemented, extensively documented and described in detail in [38, 39, 32] so we do not describe them here.

5 Invariant Form of Classification

5.1 Classification Procedure

In [29] it was shown that symmetry classification can be performed *without* solving the symmetry equations. The method algorithmically finds classification conditions by appending integrability conditions to the symmetry system (also see [20, 5]). The non-commutative RIF algorithm can be adapted to effect a classification for a systems containing arbitrary elements.

For symmetry classification the overall system $\mathcal{C} \cup \mathcal{S}$ consists of:

1. A classifying system $\mathcal{C}$, consisting of equations to be satisfied by the arbitrary elements f of the original class of differential equations. Normally system $\mathcal{C}$ will be nonlinear.

2. The complement $\mathcal{S}$, consisting of equations which are linear and homogeneous in the symmetry vector field components θ^i.

The classifying system $\mathcal{C}$ is decoupled from the rest of the symmetry system $\mathcal{S}$, and this decoupling is maintained by using an elimination ranking. Observe that the structure functions γ_{ij}^k depend only on f^j and their first derivations, so that Proposition 4.2 applies. Choose a ranking in which any derivation of a vector field component θ^i is ranked higher than any derivation of the f^j. This is valid so long as the ranking obeys the restrictions of Proposition 4.2. Such an elimination ranking ensures that $\mathcal{C}$ remains decoupled, and that $\mathcal{S}$ remains linear and homogeneous in θ^i.

Reduction of the overall system to non-commutative RIF-form requires determination of the highest ranked derivation indeterminate in each equation. For the symmetry system $\mathcal{S}$ this involves determining whether certain coefficients (called *pivots*) are zero or nonzero, leading to a binary case splitting:

Case a. Adjoin $pivot = 0$ to $\mathcal{C}$

Case b. Adjoin $pivot \neq 0$ to $\mathcal{C}$

Each case is pursued separately (see [5] for an algebraic interpretation). A binary *tree* of possibilities is thereby built up. Each leaf of the tree has an associated non-commutative RIF-form of the system. From these non-commutative RIF-forms, one can apply the existence and uniqueness result [18, Theorem 9.11] to get counts of both the degree of arbitrariness in the classifying system $\mathcal{C}$, and the dimension of the symmetry algebra. Moreover the method of [30] can be adapted to find structure constants for the symmetry algebra associated with the leaf (see §7).

5.2 Invariant form of group classification

A class of differential equations often has a nontrivial associated equivalence group $\mathcal{G}$. Methods for finding a suitable $\mathcal{G}$ are described in [2, 19]. Each transformation in $\mathcal{G}$ maps each equation in the class F to another equation in F. It is desirable to perform symmetry classification in a way which is invariant with respect to $\mathcal{G}$, that is, in which $\mathcal{G}$-equivalent equations are identified. To achieve this, one seeks to express the system $\mathcal{C} \cup \mathcal{S}$ in terms of scalar differential invariants of $\mathcal{G}$ and with respect to derivations which are $\mathcal{G}$-invariant so far as is possible. Subsequent reduction to noncommutative RIF form is then guaranteed to give classifying equations which are $\mathcal{G}$-invariant. The structure constants of the symmetry algebra (§7) will likewise be $\mathcal{G}$-invariant.

The procedure is as follows:

1. Obtain the equivalence group $\mathcal{G}$ of the differential equations F.
2. Derive symmetry equations $\mathcal{S}$ for the symmetries $\mathcal{L}_f$ of the differential equations.
3. Construct invariants and invariant derivations of the equivalence group, along with their structure relations. Case splittings may arise during this process.

4. Rewrite the system $\mathcal{S} \cup \mathcal{C}$ in terms of differential invariants and $\mathcal{G}$-invariant derivations.

5. Invoke the classification procedure. Each leaf of the resulting tree has a non-commutative RIF-form of the system $\mathcal{S} \cup \mathcal{C}$: find the dimension and structure of the associated Lie symmetry algebra (§7).

This method for symmetry classification, which first appeared in Lisle's thesis [19], is therefore a $\mathcal{G}$-invariant generalization of [29] to a case where an equivalence group is available.

A $\mathcal{G}$-invariant formulation has a number of computational advantages beyond its obvious theoretical desirability. First, expression swell that plagues symmetry classification methods can be lessened, presumably because many symmetries are simply inherited from the equivalence group. Second, the number of case splittings can be much reduced. Against this is the expense of computing invariants, and then performing reduction of the symmetry defining system using non-commuting derivation operators.

5.3 Application to Nonlinear Diffusion Equation

Before exhibiting a substantial classification in §8, we give a simpler example. Consider again the nonlinear diffusion equation (1), with symmetry equations $\mathcal{S}$ given by (2) [27, eq.(6.7.3)].

Setting aside the case $B \neq 0, B_u = 0$ (linear heat equation) we initialize the classifying system

$$\mathcal{C} \ = \ \{B_x = B_t = 0,\ B \neq 0,\ B_u \neq 0\}\ .$$

We seek to rewrite the system $\mathcal{S} \cup \mathcal{C}$ in a form invariant under the action of the 3-parameter equivalence group $\mathcal{G}$ given by (5).

Using methods such as those of Fels and Olver [9, 10], or otherwise, one finds $\mathcal{G}$-invariant derivations:

$$\Delta_1 = B^{1/2}\,\partial_x \qquad \Delta_2 = \partial_t \qquad \Delta_3 = B/B_u\,\partial_u\ . \tag{22}$$

But note that some of the coefficients a_{ij} in (6) above depend on derivatives of the dependent variables. We introduce a new dependent variable p satisfying $p = B_u/B$ and include it in $\mathcal{C}$ to remove this dependence:

$$\mathcal{C} \ = \ \{B_x = B_t = 0,\ B_u = Bp,\ B \neq 0,\ p \neq 0\}\ . \tag{23}$$

The derivations and vector field components are

$$\begin{array}{lll} \Delta_1 = B^{1/2}\,\partial_x & \Delta_2 = \partial_t & \Delta_3 = 1/p\,\partial_u \\ \theta^1 = B^{-1/2}\xi & \theta^2 = \tau & \theta^3 = p\,\eta\,. \end{array} \tag{24}$$

The structure relations for Δ are

$$[\Delta_1, \Delta_2] = -\tfrac{1}{2}\frac{B_{,2}}{B}\,\Delta_1, \quad [\Delta_1, \Delta_3] = -\tfrac{1}{2}\frac{B_{,3}}{B}\Delta_1 - \frac{p_{,1}}{p}\,\Delta_3, \quad [\Delta_2, \Delta_3] = -\frac{p_{,2}}{p}\,\Delta_3. \tag{25}$$

A scalar differential invariant is

$$J = \frac{p_u}{p^2}\,. \tag{26}$$

Appending this to classifying system $\mathcal{C}$ given by (23), rewriting in terms of derivations (24), and completing to RIF-form using a ranking compatible with total derivation order gives the updated classifying system $\mathcal{C}$:

$$\begin{array}{llll} B_{,1} = 0 & B_{,2} = 0 & B_{,3} = B & B \neq 0 \end{array} \tag{27a}$$
$$\begin{array}{llll} p_{,1} = 0 & p_{,2} = 0 & p_{,3} = pJ & p \neq 0 \end{array} \tag{27b}$$
$$\begin{array}{ll} J_{,1} = 0 & J_{,2} = 0. \end{array} \tag{27c}$$

In fact the non-invariant quantities B, p disappear from the symmetry system, and (27a,27b) play little further role. Rewriting symmetry system (2) in terms of Δ, θ using (24) and reducing modulo (27) gives $\mathcal{S}$ as

$$\begin{array}{lll} \theta^1_{,3} = -\tfrac{1}{2}\theta^1 & \theta^2_{,1} = 0 & \theta^3_{,11} = \theta^3_{,2} \\ & \theta^2_{,2} = 2\theta^1_{,1} - \theta^3 & \theta^3_{,31} = \tfrac{1}{2}\theta^1_{,11} + (J-1)\theta^3_{,1} - \tfrac{1}{2}\theta^1_{,2} \\ & \theta^2_{,3} = 0 & \theta^3_{,33} = J\theta^3_{,3} + J_{,3}\theta^3\,. \end{array} \tag{28}$$

We now reduce the overall system $\mathcal{S} \cup \mathcal{C}$ (28,27) to RIF-form. We use a ranking as follows:

(a) Any derivation of $\theta^1, \theta^2, \theta^3$ is ranked higher than any derivation of B, p, J.

(b) If tied after (a), a derivation of higher order is ranked higher.

(c) If tied after (b), rank $\theta^1 \prec \theta^2 \prec \theta^3$ and $B \prec p \prec J$ lexicographically.

(d) If tied after (c), rank $\theta^i_{\alpha_1} \prec \theta^i_{\alpha_2}$ lexicographically by the α's.

During the reduction, derivations $\theta^i_{,i_1\ldots i_p}$ may need to be normalized using structure relations (25). Any terms arising are then reduced mod $\mathcal{C}$, so for this purpose we can reduce structure relations (25) mod $\mathcal{C}$ once and for all, as

$$[\Delta_1, \Delta_2] = 0 \qquad [\Delta_1, \Delta_3] = -\tfrac{1}{2}\Delta_1 \qquad [\Delta_2, \Delta_3] = 0\,. \tag{29}$$

(Note that although it is valid to use these reduced structure relations while working on $\mathcal{S}$, the original relations (25) are needed while working on $\mathcal{C}$.)

Integrability conditions for (28) are needed during the reduction of (28) to noncommutative RIF-form. For example one finds

$$(\theta^2_{,3})_{,2} - (\theta^2_{,2})_{,3} = -(2\theta^1_{,1} - \theta^3)_{,3}\,.$$

Normalizing the left hand side modulo the reduced structure relations (29), this becomes $0 = 2\theta^1_{,13} - \theta^3_{,3}$. Reduction mod $\mathcal{S}$ (28) gives $\theta^3_{,3} = 0$, which is adjoined to $\mathcal{S}$. Further reduction of $\mathcal{S}$ gives $J_{,3}\theta^3 = 0$, and hence the case splitting $J_{,3} \neq 0$, $J_{,3} = 0$.

Case A: $\mathbf{J_{,3} \neq 0}$. In this case, non-commutative RIF-form of the classifying system is (27) with the inequation $J_{,3} \neq 0$ appended. RIF-form of the symmetry system $\mathcal{S}$ is:

$$\begin{aligned} &\theta^1_{,11} = 0 && \theta^2_{,1} = 0 && \theta^3 = 0 \\ &\theta^1_{,2} = 0 && \theta^2_{,2} = 2\theta^1_{,1} && \\ &\theta^1_{,3} = -\tfrac{1}{2}\theta^1 && \theta^2_{,3} = 0. && \end{aligned} \tag{30}$$

Case B: $\mathbf{J_{,3} = 0}$. In this case (J constant), further integrability conditions and reductions lead to $(3-4J)\theta^3_{,1} = 0$. The pivot $(3-4J)$ gives two subcases: $J \neq 3/4$ and $J = 3/4$.

Subcase C: $\mathbf{J_{,3} = 0,\ J \neq 3/4}$. In this case, the non-commutative RIF-form of $\mathcal{C}$ consists of (27a,27b) along with

$$J_{,1} = J_{,2} = J_{,3} = 0 \qquad J \neq 3/4\,. \tag{31}$$

Splitting from Case B shows $\theta^3_{,1} = 0$, and $\mathcal{S}$ reaches RIF-form:

$$\theta^1_{,11} = 0 \qquad \theta^2_{,1} = 0 \qquad \theta^3_{,1} = 0 \tag{32a}$$
$$\theta^1_{,2} = 0 \qquad \theta^2_{,2} = 2\theta^1_{,1} - \theta^3 \qquad \theta^3_{,2} = 0 \tag{32b}$$
$$\theta^1_{,3} = -\tfrac{1}{2}\theta^1 \qquad \theta^2_{,3} = 0 \qquad \theta^3_{,3} = 0\,. \tag{32c}$$

Subcase D: J = 3/4. Non-commutative RIF-form of the classifying system $\mathcal{C}$ is:

$$B_{,1} = 0 \qquad B_{,2} = 0 \qquad B_{,3} = B \qquad B \neq 0 \tag{33a}$$
$$p_{,1} = 0 \qquad p_{,2} = 0 \qquad p_{,3} = \tfrac{3}{4}p \qquad p \neq 0 \tag{33b}$$
$$J = 3/4. \tag{33c}$$

Symmetry system $\mathcal{S}$ reaches RIF-form consisting of

$$\theta^1_{,11} = \theta^3_{,1}/2 \qquad \theta^2_{,1} = 0 \qquad \theta^3_{,11} = 0 \tag{34}$$

along with (32b,32c).

6 Computation of Dimension and Initial Data

The determination of symmetry structure depends on our ability to give an existence and uniqueness theorem for local solutions of the output non-commutative RIF-form. In [18], methods originally due to Riquier are extended to the non-commutative case. In what follows, as usual all derivations are assumed to be normalized.

Given a ranking $\prec$ then the output system has two parts: a finite set $\mathcal{M}$ of functions which are linear in their highest derivatives with respect to $\prec$, and its complement $\mathcal{N}$ which is nonlinear in its highest derivations. Then as in [18, §6] the *principal derivations* of $\mathcal{M}$ are defined as

$$\operatorname{Prin}\mathcal{M} := \{\tilde{v} \in \widetilde{\Omega} |\ \text{there exist } f \in \mathcal{M} \text{ and } \alpha \in \mathbb{N}^n \text{ with } \tilde{v} = \text{HD}\, \widetilde{D}^\alpha f\}.$$

The *parametric derivations* of $\mathcal{M}$, denoted by $\operatorname{Par}\mathcal{M}$, are those derivations that are not principal. Denoting the set of highest derivations of $\mathcal{M}$ by $\text{HD}\,\mathcal{M}$, then

$$\operatorname{Par}\mathcal{M} := \{\tilde{v} \in \widetilde{\Omega} \mid \tilde{v} \neq \text{HD}\, \widetilde{D}^\beta \tilde{w} \text{ for any } \tilde{w} \in \text{HD}\,\mathcal{M}\}. \tag{35}$$

Example 6.1 *(Initial Data for Case A: $J_{,3} \neq 0$).* In this case, the non-commutative RIF-form $\mathcal{S} \cup \mathcal{C}$ is given by (30,27). The system $\mathcal{S} \cup \mathcal{C}$ is linear in its highest derivations so $\mathcal{M}$ consists of all equations from (30,27), while $\mathcal{N} = \emptyset$. The highest derivations are

$$\begin{aligned}\text{HD}\,\mathcal{M} =& \{\theta^1_{,11}, \theta^1_{,2}, \theta^1_{,3}, \theta^2_{,1}, \theta^2_{,2}, \theta^2_{,3}, \theta^3\} \\ & \cup \{B_{,1}, B_{,2}, B_{,3}, p_{,1}, p_{,2}, p_{,3}, J_{,1}, J_{,2}\}.\end{aligned} \tag{36}$$

Note how we have separated the highest derivations into those from the symmetry system $\mathcal{S}$ and those from the classifying system $\mathcal{C}$. Computation of initial data proceeds exactly as in the commutative case. We have

$$\operatorname{Par}\mathcal{M} = \{\theta^1_{,1}, \theta^1, \theta^2\} \cup \{B, p, J, J_{,3}, J_{,33}, \ldots\}. \tag{37}$$

Since $\mathcal{N} = \emptyset$, we only need the leading linear form [18, Theorem 7.6] of the non-commutative existence and uniqueness theorem. According to this theorem, if initial data are prescribed by assigning values to the parametric derivatives (37) at $w_0 = (x_0, t_0, u_0)$

$$\begin{aligned} &\mathrm{ID}(\mathcal{S}) \qquad \theta^1_{,1}(w_0) = a_1, \theta^1(w_0) = a_2, \theta^2(w_0) = a_3 \qquad (38)\\ &\mathrm{ID}(\mathcal{C}) \qquad B(w_0) = h_1, p(w_0) = h_2, J(w_0) = h_3, J_{,3}(w_0) = h_4, J_{,33}(w_0) = h_5, \ldots \end{aligned}$$

then there is a unique formal solution to the system $\mathcal{S} \cup \mathcal{C}$ (30,27) with these initial data. Note that there are infinitely many parameters to assign in the initial data. However for the application of this paper we are interested in the the properties of the symmetry algebras $\mathcal{L}_f$ corresponding to some fixed formal solution f of the classifying system $\mathcal{C}$. With f fixed, the dimension of $\mathcal{L}_f$ is the number of parametric derivations for the symmetry system $\mathcal{S}_f$. Thus for the example above, any diffusivity B solving $\mathcal{C}$ has an associated 3-dimensional symmetry algebra $\mathcal{L}_B$, corresponding to the three parametric derivations from $\mathcal{S}$. The classifying equations $\mathcal{C}$ have infinite initial data, reflecting the fact that symmetries (4) are present for arbitrary diffusivity functions B.

The following examples illustrate that as the symmetry algebra becomes larger the classifying system has fewer degrees of freedom.

Example 6.2 *(Initial Data for Subcase C: $J \neq 3/4$).* The non-commutative RIF-form (27a,27b,31) of the classifying system has parametric derivations B, p, J, so this subcase represents a 3-parameter family of diffusivities. There are four parametric derivations $(\theta^1, \theta^2, \theta^3, \theta^1_{,1})$ in symmetry system (32), so the symmetry algebra $\mathcal{L}_B$ is of dimension four.

Example 6.3 *(Initial Data for Subcase D: $J = 3/4$).* The parametric derivations for the RIF-form of $\mathcal{C}$ given in (33) are B, p, with associated initial data $B(w_0) = h_1$, $p(w_0) = h_2$. Hence this subcase represents a 2-parameter family of diffusivities. The parametric derivations for the RIF-form of $\mathcal{S}$ given by (32b,32c,34) are $\theta^1, \theta^2, \theta^3, \theta^1_{,1}, \theta^3_{,1}$, giving a 5-dimensional symmetry algebra.

7 Computation of Structure Constants

Consider a symmetry system $\mathcal{S}$ for the components of a Lie algebra of vector fields. A finite-dimensional Lie algebra is characterized up to isomorphism by structure constants C_{ij}^k. We show how to find C_{ij}^k directly from the symmetry system without solving the system. The method is a generalization to the non-commuting case of the method of Reid et al. [30].

The formal solutions of $\mathcal{S}$ are components of a vector field, and with the usual commutator bracket on vector fields, the local solutions at a point w_0 are therefore also a Lie algebra. The commutator bracket on solutions can be used to induce a bracket on initial data as in [30].

We adopt the notation $\mathrm{Par}(\theta)(w_0) = \mathbf{a}$ to represent the initial data corresponding to symmetry components. Similarly $\mathrm{Par}(f)(w_0) = \mathbf{h}$ represents the classification initial data.

Example 7.1. Consider the system from Subcase D ($J = 3/4$) of §6, for which $\mathcal{S} \cup \mathcal{C}$ given by (32b,32c,34,33) is in non-commutative RIF-form. So $\mathrm{Par}(\theta) = (\theta^1, \theta^2, \theta^3, \theta^1_{,1}, \theta^3_{,1})$ and $\mathrm{Par}(\theta)(w_0) = \mathbf{a} = (a_1, a_2, a_3, a_4, a_5)$ represents the symmetry component initial data. Similarly $\mathrm{Par}(f) = (B, p)$ and $\mathrm{Par}(f)(w_0) = \mathbf{h} = (h_1, h_2)$.

Consider two formal solutions of $\mathcal{S}_f$ with associated initial data vectors $\mathbf{a}$, $\mathbf{b}$ and the same classification initial data $\mathbf{h}$. Let Sol be the invertible linear map that takes initial data values for $\mathcal{S}_f$ to formal solutions of $\mathcal{S}_f$ at w_0. Then Sol^{-1} is the map that evaluates initial data for a formal solution at w_0. We define a commutator bracket on initial data by

$$[\mathbf{a}, \mathbf{b}]_{\text{ID}} := \mathrm{Sol}^{-1}\big([\mathrm{Sol}(\mathbf{a}), \mathrm{Sol}(\mathbf{b})]\big)$$

That is, the commutator of two initial data vectors is found by constructing the two associated formal solutions, taking commutators, then evaluating the initial data.

Let the (unique) formal solutions of $\mathcal{S}_f$ associated with initial data $\mathbf{a}, \mathbf{b}$ be ϕ, ψ, where $\phi = \sum_{i=1}^n \phi^i \Delta_i$, $\psi = \sum_{i=1}^n \psi^i \Delta_i$. Thus

$$\mathrm{Par}(\phi)(w_0) = \mathbf{a} \qquad \mathrm{Par}(\psi)(w_0) = \mathbf{b}$$

where both solutions have the same classification initial data $\mathrm{Par}(f)(w_0) = \mathbf{h}$. The commutator of solutions is

$$\omega := [\phi, \psi] = \sum_{k=1}^n (\phi^i \psi^k_{,i} - \psi^i \phi^k_{,i}) \Delta_k + \sum_{i=1}^n \sum_{j=1}^n \phi^i \psi^j [\Delta_i, \Delta_j] \,. \tag{39}$$

If the invariant BDO Δ has structure relations $[\Delta_i, \Delta_j] = \sum_{k=1}^{n} \gamma_{ij}^k \Delta_k$, the k-th component of the commutator is

$$\omega^k = \sum_{i=1}^{n} \left(\phi^i \psi^k_{,i} - \psi^i \phi^k_{,i}\right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij}^k \phi^i \psi^j \,.$$

Let the initial data for $\mathcal{S}_f$ associated with ω be $\mathbf{c}$, that is, $\mathbf{c} = \mathrm{Par}(\omega)(w_0)$. Each component of $\mathbf{c}$ is therefore $(\omega^k)_\alpha(w_0)$ for some k, α. Now, the expression

$$\Big(\sum_{i=1}^{n} \left(\phi^i \psi^k_{,i} - \psi^i \phi^k_{,i}\right) + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij}^k \phi^i \psi^j\Big)_\alpha$$

can be reduced modulo $\mathcal{S} \cup \mathcal{C}$ to an expression involving only $\mathrm{Par}(\phi)$, $\mathrm{Par}(\psi)$ and $\mathrm{Par}(f)$. Evaluation at w_0 gives an expression involving only $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{h}$. By doing so, each component of the commutator initial data $\mathbf{c}$ is expressed as a skew-symmetric bilinear function of $\mathbf{a}, \mathbf{b}$:

$$c^k = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^k(\mathbf{h}) a^i b^j$$

where $C_{ji}^k = -C_{ij}^k$. By construction, the quantities C_{ij}^k are structure constants of the Lie algebra induced on initial data space, and therefore of the desired Lie algebra of vector fields $\mathcal{L}_f$. Thus C_{ij}^k is the k-th piece of initial data for the commutator of solution i with solution j of $\mathcal{S}_f$.

Example 7.1 *(cont.).* The parametric derivations for this case are listed above. The θ^i are components of a vector field $\mathbf{Y} = \sum_{i=1}^{3} \theta^i \Delta_i$ referred to an invariant BDO $\Delta_1, \Delta_2, \Delta_3$ with structure relations (25). Let

$$\phi = \phi^1 \Delta_1 + \phi^2 \Delta_2 + \phi^3 \Delta_3 \qquad \psi = \psi^1 \Delta_1 + \psi^2 \Delta_2 + \psi^3 \Delta_3$$

be two solutions of $\mathcal{S}_B$, and let $\omega = [\phi, \psi]$. Taking commutators (39) and using the reduced structure relations (29), gives for instance

$$\omega^1 = (\phi^1 \psi^1_{,1} - \psi^1 \phi^1_{,1}) + (\phi^2 \psi^1_{,2} - \psi^2 \phi^1_{,2}) + (\phi^3 \psi^1_{,3} - \psi^3 \phi^1_{,3}) - \tfrac{1}{2}(\phi^1 \psi^3 - \psi^1 \phi^3) \,.$$

After reducing modulo $\mathcal{S} \cup \mathcal{C}$ (32b,32c,34,33), we obtain expressions for three of the parametric derivatives of ω:

$$\begin{aligned}
\omega^1 &= \phi^1 \psi^1_{,1} - \psi^1 \phi^1_{,1} \\
\omega^2 &= 2(\phi^2 \psi^1_{,1} - \psi^2 \phi^1_{,1}) - (\phi^2 \psi^3 - \psi^2 \phi^3) \\
\omega^3 &= \phi^1 \psi^3_{,1} - \psi^1 \phi^3_{,1} \,.
\end{aligned}$$

Further differentiation and reduction modulo $\mathcal{S} \cup \mathcal{C}$ gives

$$\omega^1_{,1} = \tfrac{1}{2}(\phi^1\psi^3_{,1} - \psi^1\phi^3_{,1}) \qquad \omega^3_{,1} = \phi^1_{,1}\psi^3_{,1} - \psi^1_{,1}\phi^3_{,1}\,.$$

Evaluation of the initial data $\mathrm{Par}(\theta)(\omega) = \mathbf{c} = (c_1, c_2, c_3, c_4, c_5)$ gives

$$\begin{aligned}
c^1 &= a^1b^4 - b^1a^4 && (C^1_{14} = 1)\\
c^2 &= 2(a^2b^4 - b^2a^4) - (a^2b^3 - b^2a^3) && (C^2_{24} = 2,\ C^2_{23} = -1)\\
c^3 &= a^1b^5 - b^1a^5 && (C^3_{15} = 2)\\
c^4 &= \tfrac{1}{2}(a^1b^5 - b^1a^5) && (C^4_{15} = \tfrac{1}{2})\\
c^5 &= a^4b^5 - b^4a^5 && (C^5_{45} = 1)\,.
\end{aligned}$$

The Lie algebra $\mathcal{L}_B$ therefore has commutation relations

$$\begin{array}{lll}
[\mathbf{Y}_1, \mathbf{Y}_4] = \mathbf{Y}_1 & [\mathbf{Y}_1, \mathbf{Y}_5] = \mathbf{Y}_3 + \frac{1}{2}\mathbf{Y}_4 & [\mathbf{Y}_2, \mathbf{Y}_3] = -\mathbf{Y}_2\\
[\mathbf{Y}_2, \mathbf{Y}_4] = 2\mathbf{Y}_2 & [\mathbf{Y}_4, \mathbf{Y}_5] = \mathbf{Y}_5. &
\end{array}$$

Note that it was not necessary to construct explicit solutions during this process, nor was it necessary to know expressions for Δ_i in terms of commuting derivations. An alternative and equivalent method for evaluating structure constants by selecting numerical values for the a^i, b^i is given in [30].

8 Potential Diffusion Convection System

We now present a substantial computational example, applying the invariant BDO method to the diffusion convection system

$$v_x = u, \qquad v_t = Bu_x - K. \tag{40}$$

The arbitrary elements $B(u)$ (diffusivity) and $K(u)$ (convection) obey the classifying system

$$B_v = B_x = B_t = 0, \qquad K_v = K_x = K_t = 0, \qquad B \neq 0\,. \tag{41}$$

8.1 Equivalence group

A calculation detailed in [19] shows that the class of equations (40) is preserved by a 10-parameter equivalence group, generated as the product of subgroups:

$$(a)\begin{cases}\bar v &= v+\varepsilon_1\\ \bar x &= x+\varepsilon_2\\ \bar t &= t+\varepsilon_3\end{cases}\quad (b)\begin{cases}\bar v &= v-\kappa_1 t\\ \bar x &= x+\kappa_2 t\\ \bar K &= K+\kappa_1+\kappa_2 u\end{cases}\quad (c)\begin{cases}\bar v &= \alpha v+\beta x\\ \bar x &= \gamma v+\delta x\\ \bar u &= \frac{\alpha u+\beta}{\gamma u+\delta}\\ \bar K &= \frac{K}{\gamma u+\delta}\\ \bar B &= (\gamma u+\delta)^2 B\end{cases}$$

$$(d)\begin{cases}\bar v &= v/a\\ \bar x &= x/a\\ \bar t &= t/a^2\\ \bar K &= aK\end{cases}\quad (e)\begin{cases}\bar v &= bv\\ \bar x &= bx\\ \bar t &= bt\\ \bar B &= bB\end{cases}\tag{42}$$

where $ab \neq 0$ and $\alpha\delta - \beta\gamma = 1$. We will consider both the variables in the diffusion-convection equation (40) and the parameters of the equivalence group to be real-valued.

8.2 Symmetry system

We start with classifying system $\mathcal{C}$ (41). Computation shows that the invariant BDO has coefficients depending on B_u and K_u. In order to apply our approach we introduce new dependent variables: $B_u/B = p$ and $K_u = q$ and then the classifying system $\mathcal{C}$ is:

$$B_x = 0 \qquad B_t = 0 \qquad B_v = 0 \qquad B_u = Bp \qquad B \neq 0 \tag{43a}$$

$$K_x = 0 \qquad K_t = 0 \qquad K_v = 0 \qquad K_u = q. \tag{43b}$$

The components χ, ξ, τ, η of the symmetry vector field

$$\mathbf{Y} = \chi\,\partial_v + \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u$$

satisfy the symmetry system $\mathcal{S}$:

$$\begin{gathered}\tau_v = \tau_x = \tau_u = \xi_u = \chi_u = 0\\ p(\,\partial_x + u\,\partial_v)(\chi - u\xi) - 2B(\,\partial_x + u\,\partial_v)\xi + B\,\partial_t\tau = 0\\ \big(\,\partial_t + q(\,\partial_x + u\,\partial_v) - K\,\partial_v\big)(\chi - u\xi) + K\,\partial_t\tau - B(\,\partial_x + u\,\partial_v)^2(\chi - u\xi) = 0\\ \eta = (\,\partial_x + u\,\partial_v)(\chi - u\xi).\end{gathered}\tag{44}$$

8.3 Construction of invariant BDO

Invariants and invariant BDO for the full equivalence group can be constructed, by successively enlarging subgroups (42) (a), (a,b), . . . , (a,b,c,d,e) as per theory described by Kogan [17]. At each step the following elements are constructed:

- Scalar differential invariants.
- Invariant BDO and vector field components, along with structure relations of the BDO.
- Classifying system $\mathcal{C}$ in invariant form.
- Symmetry system in invariant form.

To avoid trivialities, note that a simple time reflection (42e) ensures that $B > 0$, and hence has a real square root.

Subgroups (a,b,c) Carrying out the above is straightforward for subgroups (a), (b), (c). We find a scalar invariant L

$$L = \frac{p_u - \frac{1}{2}p^2}{B^2} \tag{45}$$

and an immediate case splitting on the quantity $q_u (= K_{uu})$. If $q_u = 0$ then no other invariant exists. If $q_u \neq 0$, a simple reflection (42d) ensures that $q_u > 0$, and we assume this has been done. A second invariant is then

$$I = q_u B^{-3/2}. \tag{46}$$

An invariant BDO Λ is:

$$\begin{aligned} &\Lambda_1 = B^{1/2}(\partial_x + u\partial_v) && \Lambda_2 = B^{-1/2}\big(2\partial_v + p(\partial_x + u\partial_v)\big) \\ &\Lambda_3 = \partial_t - K\partial_v + q(\partial_x + u\partial_v) && \Lambda_4 = \tfrac{1}{B}\partial_u, \end{aligned} \tag{47}$$

and vector field components λ:

$$\begin{aligned} &\lambda^1 = -\tfrac{1}{2}B^{-1/2}\big(p(\chi + K\tau - u\xi) - 2(\xi - q\tau)\big) && \lambda^3 = \tau \\ &\lambda^2 = \tfrac{1}{2}B^{1/2}(\chi + K\tau - u\xi) && \lambda^4 = B\eta. \end{aligned} \tag{48}$$

Case 1. $q_u > 0$.

All derivatives below represent differentiation with respect to BDO Λ (47). The classifying system $\mathcal{C}$ (43) is enlarged via (45,46) to non-commutative RIF-form:

$$f_{,\ell} = 0 \qquad \text{for} \qquad f = B, K, p, q, L, I; \quad \ell = 1, 2, 3 \tag{49a}$$

$$B_{,4} = p \qquad B\,K_{,4} = q \qquad p_{,4} = BL + \frac{1}{2}\frac{p^2}{B} \qquad q_{,4} = B^{1/2} I \tag{49b}$$

$$B > 0 \qquad I > 0. \tag{49c}$$

Symmetry system $\mathcal{S}$ (44) becomes

$$\begin{array}{llll} \lambda^3_{,1} = 0 & \lambda^1_{,1} = \frac{1}{2}\lambda^3_{,3} & \lambda^2_{,11} = \lambda^2_{,3} & \lambda^4 = 2\lambda^2_{,1} \\ \lambda^3_{,2} = 0 & & & \\ \lambda^3_{,4} = 0 & \lambda^1_{,4} = -L\lambda^2 - I\lambda^3 & \lambda^2_{,4} = -\frac{1}{2}\lambda^1 \,. & \end{array} \tag{50}$$

In this beautiful form only two terms have nonconstant coefficients, and the simplicity of structure of symmetry system (44) is revealed.

Integrability conditions of (50) can be found with the help of structure relations (reduced modulo $\mathcal{C}$):

$$[\Lambda_1, \Lambda_4] = -\tfrac{1}{2}\Lambda_2, \qquad [\Lambda_2, \Lambda_4] = -L\Lambda_1, \qquad [\Lambda_3, \Lambda_4] = -I\Lambda_1 \,. \tag{51}$$

Case 2. $q_u = 0$.
This is the case of equations equivalent to diffusion equations $K = 0$. The BDO Λ (48) still serves here. The classifying system $\mathcal{C}$ amounts to removing the equations for I from (49a), setting $I = 0$ in (49b) and dropping $I > 0$ from (49c). The symmetry system $\mathcal{S}$ is (50) with I set to 0, similarly for structure relations (51).
(d) Scaling group – convection So far, only subgroups (42a,b,c) have been accounted for. The scaling subgroup (42d) acts on the invariants L (45), I (46) and Λ (47) by

$$\begin{array}{llll} \Lambda_1' = a\Lambda_1, & \Lambda_2' = a\Lambda_2, & \Lambda_3' = a^2\Lambda_3, & \Lambda_4' = \Lambda_4 \\ L' = L, & I' = aI \,. & & \end{array}$$

Case 1 (cont.). $I \neq 0$. The quantity L (45) is an invariant of the enlarged subgroup (42a,b,c,d). An additional invariant is

$$M = \Lambda_4 I \,. \tag{52}$$

An invariant BDO denoted by Γ exists, with corresponding vector field components ζ:

$$\begin{array}{llll} \Gamma_1 = I^{-1}\Lambda_1 & \Gamma_2 = I^{-1}\Lambda_2 & \Gamma_3 = I^{-2}\Lambda_3 & \Gamma_4 = \Lambda_4 \\ \zeta^1 = I\lambda^1 & \zeta^2 = I\lambda^2 & \zeta^3 = I^2\lambda^3 & \zeta^4 = \lambda^4 \,. \end{array} \tag{53}$$

With respect to this BDO Γ, classifying system $\mathcal{C}$ (49) is updated via (52) to include the equations

$$L_{,\ell} = I_{,\ell} = M_{,\ell} = 0 \qquad \text{for} \quad \ell = 1, 2, 3 \tag{54a}$$

$$I_{,4} = I\,M \qquad I > 0. \tag{54b}$$

In addition, $\mathcal{C}$ contains the B, K, p, q equations from (49a,49b) (but rewritten in terms of Γ). These equations are present in all versions of our classifying systems, and have four associated parametric derivatives B, K, p, q. From now on we decline to write them explicitly.

Symmetry system $\mathcal{S}$ (50) becomes

$$\begin{array}{llll} \zeta^3_{,1} = 0 & \zeta^1_{,1} = \frac{1}{2}\zeta^3_{,3} & \zeta^2_{,11} = \zeta^2_{,3} & \zeta^4 = 2\zeta^2_{,1} \\ \zeta^3_{,2} = 0 & & & \\ \zeta^3_{,4} = 2M\zeta^3 & \zeta^1_{,4} = M\zeta^1 - L\zeta^2 - \zeta^3 & \zeta^2_{,4} = -\frac{1}{2}\zeta^1 + M\zeta^2 \, . & \end{array} \tag{55}$$

The structure relations of Γ (reduced modulo $\mathcal{C}$ (54)) are

$$[\Gamma_1, \Gamma_4] = -\tfrac{1}{2}\Gamma_2 + M\Gamma_1, \quad [\Gamma_2, \Gamma_4] = -L\Gamma_1 + M\Gamma_2, \quad [\Gamma_3, \Gamma_4] = -\Gamma_1 + 2M\Gamma_3 . \tag{56}$$

(e) Scaling group – diffusion The final subgroup (42e) of the equivalence group acts on Λ, L, I as follows:

$$\begin{array}{llll} \Lambda_1' = b^{-1/2}\Lambda_1 & \Lambda_2' = b^{-3/2}\Lambda_2 & \Lambda_3' = b^{-1}\Lambda_3 & \Lambda_4' = b^{-1}\Lambda_4 \\ L' = b^{-2}L & I' = b^{-3/2}I \, . & & \end{array} \tag{57}$$

We take $b > 0$ here so as to preserve the condition $B > 0$. Cases 1. ($I \neq 0$) and 2. ($I = 0$) from above are further split by this subgroup action.

Case 1. (cont.). $I \neq 0$.

The action of (42e) on Γ, L, M is:

$$\begin{array}{llll} \Gamma_1' = b\Gamma_1 & \Gamma_2' = \Gamma_2 & \Gamma_3' = b^2\Gamma_3 & \Gamma_4' = b^{-1}\Gamma_4 \\ L' = b^{-2}L & M' = b^{-1}M \, . & & \end{array} \tag{58}$$

The calculation splits into two subcases, depending whether L vanishes.

Subcase 3. $I \neq 0$, $L \neq 0$. In this subcase of Case 1, invariants of the whole group (42) exist. The scalar invariants are

$$P = |L|^{-3/2}\Gamma_4 L \qquad Q = M|L|^{-1/2} \qquad \sigma = \operatorname{sgn} L \, . \tag{59}$$

The sign σ is genuinely invariant, so long as transformations are real-valued. An invariant BDO Σ and associated vector field components β exist:

$$\begin{array}{llll} \Sigma_1 = |L|^{1/2}\Gamma_1 & \Sigma_2 = \Gamma_2 & \Sigma_3 = L\Gamma_3 & \Sigma_4 = |L|^{-1/2}\Gamma_4 \\ \beta^1 = |L|^{-1/2}\zeta^1 & \beta^2 = \zeta^2 & \beta^3 = L^{-1}\zeta^3 & \beta^4 = |L|^{1/2}\zeta^4. \end{array} \tag{60}$$

Classifying system $\mathcal{C}$ (54) is modified to

$$L_{,\ell} = I_{,\ell} = P_{,\ell} = Q_{,\ell} = 0 \qquad \text{for} \quad \ell = 1,2,3 \tag{61a}$$
$$I_{,4} = I\,Q \qquad L_{,4} = |L|\,P \tag{61b}$$

where derivatives are with respect to BDO Σ. The note after eq.(54) still applies. Symmetry system (55) becomes

$$\begin{array}{llll}
\beta^3_{,1} = 0, & \beta^1_{,1} = \frac{1}{2}\beta^3_{,3}, & \beta^2_{,11} = \sigma\beta^2_{,3}, & \beta^4 = 2\beta^2_{,1} \\
\beta^3_{,2} = 0, & & \beta^2_{,4} = -\frac{1}{2}\beta^1 + Q\beta^2 & \\
\beta^3_{,4} = (2Q-\sigma P)\beta^3 & \beta^1_{,4} = \frac{1}{2}(2Q-\sigma P)\beta^1 - \sigma\beta^2 - \sigma\beta^3 \;. & &
\end{array} \tag{62}$$

The structure relations of Σ (reduced modulo $\mathcal{C}$ (61)) are

$$\begin{array}{ll}
[\Sigma_1, \Sigma_4] = \frac{1}{2}(2Q-\sigma P)\Sigma_1 - \frac{1}{2}\Sigma_2 & [\Sigma_2, \Sigma_4] = -\sigma\Sigma_1 + Q\Sigma_2 \\
 & [\Sigma_3, \Sigma_4] = -\sigma\Sigma_1 + (2Q-\sigma P)\Sigma_3 \;.
\end{array} \tag{63}$$

Subcase 4. $I \neq 0$, $L = 0$. Returning to (58), this subcase of Case 1 gives a case splitting on M.
Subcase 5. $I \neq 0$, $L = 0$, $M \neq 0$. For this subcase of Subcase 4, invariants of the whole group (42) exist. A scalar invariant is

$$R = M^{-2}\,\Gamma_4 M. \tag{64}$$

An invariant BDO Ξ and associated vector field components ξ are:

$$\begin{array}{llll}
\Xi_1 = M\,\Gamma_1 & \Xi_2 = \Gamma_2 & \Xi_3 = M^2\,\Gamma_3 & \Xi_4 = M^{-1}\,\Gamma_4 \\
\xi^1 = M^{-1}\,\zeta^1 & \xi^2 = \zeta^2 & \xi^3 = M^{-2}\,\zeta^3 & \xi^4 = M\,\zeta^4.
\end{array} \tag{65}$$

Classifying system $\mathcal{C}$ (54) is updated to

$$I_{,\ell} = M_{\ell} = R_{,\ell} = 0 \qquad \text{for} \quad \ell = 1,2,3. \tag{66a}$$
$$I_{,4} = I \qquad M_{,4} = M\,R \qquad I > 0 \tag{66b}$$

where derivatives are with respect to BDO Ξ and the note from (54) applies. Symmetry system (55) becomes, with respect to Ξ,

$$\begin{array}{llll}
\xi^3_{,1} = 0 & \xi^1_{,1} = \frac{1}{2}\xi^3_{,3} & \xi^2_{,11} = \xi^2_{,3} & \xi^4 = 2\xi^2_{,1} \\
\xi^3_{,2} = 0 & & & \\
\xi^3_{,4} = 2(1-R)\xi^3 & \xi^1_{,4} = (1-R)\xi^1 - \xi^3 & \xi^2_{,4} = -\frac{1}{2}\xi^1 + \xi^2 \;. &
\end{array} \tag{67}$$

The structure relations of Ξ (reduced modulo $\mathcal{C}$ (66)) are

$$
\begin{aligned}
&[\Xi_1, \Xi_4] = -\tfrac{1}{2}\Xi_2 + (1-R)\Xi_1 \qquad && [\Xi_2, \Xi_4] = \Xi_2 \\
& && [\Xi_3, \Xi_4] = -\Xi_1 + 2(1-R)\Xi_3 \,. \qquad (68)
\end{aligned}
$$

Subcase 6. $I \neq 0$, $L = 0$, $M = 0$.
For this subcase of Subcase 4., we retain the BDO Γ (53). Note that the conditions $L = 0$, $M = 0$, $I > 0$ gives the diffusion convection equations which are equivalent to Burgers' equation $B(u) = 1$, $K(u) = \frac{1}{2}u^2$. These equivalent systems include an equation $B(u) = u^{-2}$, $K(u) = u^{-1}$ analysed by Fokas and Yortsos [11]. It is interesting that the $\mathcal{G}$-invariant BDO calculations pick this out as a singular case even though the Cole-Hopf linearizing transformation that takes Burgers' to the heat equation is not in the equivalence group (42), and hence not detected.
Case 2. (cont.) $I = 0$.
The action of scaling group (42e) on Λ is given by (57). With $I = 0$, the calculation splits on L.

Subcase 7. $I = 0$, $L \neq 0$.
This subcase of Case 2 picks out those diffusion equations which are genuinely nonlinear in the sense that they are not equivalent to the linear heat equation via (42).

Scalar invariants of the group action exist:

$$
P = |L|^{-3/2}\Lambda_4 L \qquad \sigma = \operatorname{sgn} L. \tag{69}
$$

This P is the same as (59), merely rewritten in new notation.

An invariant set of DO denoted by Ω and vector field components denoted by ω are:

$$
\begin{aligned}
&\Omega_1 = |L|^{-1/4}\Lambda_1 \quad && \Omega_2 = |L|^{-3/4}\Lambda_2 \quad && \Omega_3 = |L|^{-1/2}\Lambda_3 \quad && \Omega_4 = |L|^{-1/2}\Lambda_4 \\
&\omega^1 = |L|^{1/4}\lambda^1 && \omega^2 = |L|^{3/4}\lambda^2 && \omega^3 = |L|^{1/2}\lambda^3 && \omega^4 = |L|^{1/2}\lambda^4. \qquad (70)
\end{aligned}
$$

With respect to this BDO Ω, the classifying system (49) becomes after adjoining P (69)

$$
L_{,\ell} = P_{,\ell} = 0 \qquad \text{for} \quad \ell = 1, 2, 3; \qquad\qquad L_{,4} = \sigma\, L\, P \tag{71}
$$

where the comment after (54) again applies. Symmetry system $\mathcal{S}$ (50) becomes

$$
\begin{aligned}
&\omega^3_{,1} = 0 \qquad && \omega^1_{,1} = \tfrac{1}{2}\omega^3_{,3} \qquad && \omega^2_{,11} = \omega^2_{,3} \qquad && \omega^4 = 2\omega^2_{,1} \\
&\omega^3_{,2} = 0 && && && \\
&\omega^3_{,4} = \tfrac{1}{2}\sigma P \omega^3 && \omega^1_{,4} = \tfrac{1}{4}\sigma P \omega^1 - \sigma\omega^2 && \omega^2_{,4} = -\tfrac{1}{2}\omega^1 + \tfrac{3}{4}\sigma P \omega^2 \,. && (72)
\end{aligned}
$$

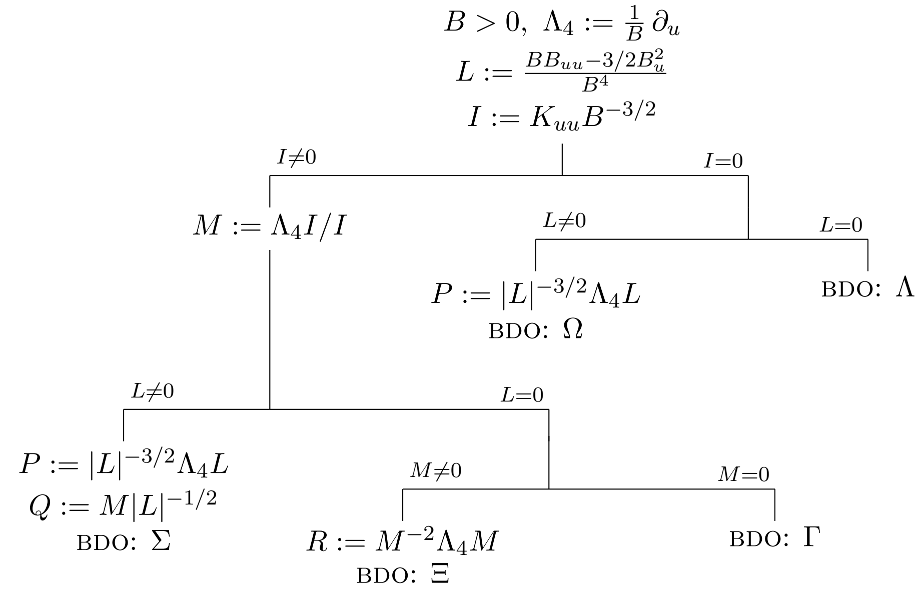


Figure 1: Preliminary classification tree for potential diffusion convection system (40). Branchings are on the basis of whether or not particular invariant BDO exist where BDO = Basis of Differential Operators

The structure relations of Ω, reduced modulo (71) are

$$[\Omega_1, \Omega_4] = \tfrac{1}{4}\sigma P \Omega_1 - \tfrac{1}{2}\Omega_2 \qquad [\Omega_2, \Omega_4] = -\sigma\Omega_1 + \tfrac{3}{4}\sigma P \Omega_2$$
$$[\Omega_3, \Omega_4] = \tfrac{1}{2}\sigma P \Omega_3\,. \tag{73}$$

Subcase 8. $I = 0$, $L = 0$.

This subcase of Case 2 picks out diffusion convection equations that are equivalent to the linear heat equation $B = 1$, $K = 0$. This includes the equation $B = u^{-2}$ studied by [36, 3] and found to be equivalent to the linear heat system. In this case the linearizing transformation is in the equivalence group $\mathcal{G}$ (42), so it is expected that this equation be picked out as singular.

8.4 Completion to non-commutative rif-form

So far we have an incomplete classification tree, as shown in Figure 1. For each leaf of this tree we now complete the defining system to non-commutative RIF-form, giving rise to further splittings. Note that three common translation symmetries (42a) are always present, and we do not present results for any branch with only a three-dimensional solution space.

Subcase 3. $I \neq 0$, $L \neq 0$.
We have symmetry system $\mathcal{S}$ (62) and classifying system $\mathcal{C}$ (61), referred to Σ. The ranking is as follows:

(a) Rank any derivative of P or Q lower than any derivative of any β^i.

(b) If tied after (a), rank any derivative of $\beta^1, \beta^2, \beta^3$ lower than β^4

(c) If tied after (b), rank by order of derivative.

(d) If tied after (c), rank lexicographically $\beta^3 \prec \beta^2 \prec \beta^1$ and $P \prec Q$.

(e) If tied after (d), rank $\beta^i_{\alpha_1} \prec \beta^i_{\alpha_2}$ lexicographically by the α's.

According to Proposition 4.2, conditions (a,b,c) ensure this is a positive ranking (Definition 4.1). Case splittings arising during the completion process show that for symmetry beyond the minimal translations (42a), it is necessary that $P_{,4} = Q_{,4} = 0$, so that (from (61a)) P, Q are constants. In this case, the system $\mathcal{S} \cup \mathcal{C}$ reduces to the non-commutative RIF-form (61) and

$$\begin{array}{llll} \beta^3_{,1} = 0 & \beta^2_{,11} = 0 & \beta^1_{,1} = -(2Q - \sigma P)\beta^2_{,1} & \\ \beta^3_{,2} = 0 & \beta^2_{,2} = -2Q\beta^2_{,1} & \beta^1_{,2} = 2\sigma\beta^2_{,1} & \beta^4 = 2\beta^2_{,1} \\ \beta^3_{,3} = -2(2Q - \sigma P)\beta^2_{,1} & \beta^2_{,3} = 0 & \beta^1_{,3} = 2\sigma\beta^2_{,1} & \\ \beta^3_{,4} = (2Q - \sigma P)\beta^3 & \beta^2_{,4} = -\frac{1}{2}\beta^1 + Q\beta^2 & \beta^1_{,4} = \frac{1}{2}(2Q - \sigma P)\beta^1 - \sigma\beta^2 - \sigma\beta^3\,. & \end{array} \tag{74}$$

The four parametric derivatives β^1, β^2, β^3, $\beta^2_{,1}$ give a 4-parameter symmetry group (§6). Application of the method (§7) for finding structure constants gives a Lie algebra of symmetry operators $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$, $\mathbf{Y}_4$ with commutation relations

$$\begin{aligned} [\mathbf{Y}_1, \mathbf{Y}_4] = -(2Q - \sigma P)\mathbf{Y}_1 + \mathbf{Y}_2 \quad & [\mathbf{Y}_2, \mathbf{Y}_4] = 2\sigma\mathbf{Y}_1 - 2Q\mathbf{Y}_2 \\ & [\mathbf{Y}_3, \mathbf{Y}_4] = 2\sigma\mathbf{Y}_1 - 2(2Q - \sigma P)\mathbf{Y}_2\,. \end{aligned}$$

Subcase 5. $I \neq 0$, $L = 0$, $M \neq 0$.
Applying the non-commutative differential elimination method to system (67,66) with BDO Ξ (65) (and ranking similar to above) shows that additional symmetry only arises only if $R_{,4} = 0$, in which case a non-commutative rif

form is:

$$
\begin{array}{lll}
\xi^3_{,1} = 0 & \xi^2_{,1} = -\frac{1}{2}\xi^2_{,2} & \xi^1_{,1} = (1-R)\xi^2_{,2} \\
\xi^3_{,2} = 0 & \xi^2_{,22} = 0 & \xi^1_{,2} = 0 \\
\xi^3_{,3} = 2(1-R)\xi^2_{,2} & \xi^2_{,3} = 0 & \xi^1_{,3} = -\xi^2_{,2} \\
\xi^3_{,4} = 2(1-R)\xi^3 & \xi^2_{,4} = -\frac{1}{2}\xi^1 + \xi^2 & \xi^1_{,4} = (1-R)\xi^1 - \xi^3 \\
 & & \xi^4 = -\xi^2_{,2}\,.
\end{array}
\tag{75}
$$

The parametric derivatives ξ^1, ξ^2, ξ^3, $\xi^2_{,2}$ give a four-dimensional symmetry algebra with commutation relations

$$[\mathbf{Y}_1, \mathbf{Y}_4] = (1-R)\mathbf{Y}_1 - \tfrac{1}{2}\mathbf{Y}_2, \quad [\mathbf{Y}_2, \mathbf{Y}_4] = \mathbf{Y}_2, \quad [\mathbf{Y}_3, \mathbf{Y}_4] = -\mathbf{Y}_1 + 2(1-R)\mathbf{Y}_3\,.$$

Subcase 6. $I \neq 0$, $L = 0$, $M = 0$.

No further case splittings arise for this case, which includes Burgers' equation and Fokas-Yortsos' equation [11]. The Cole-Hopf transformation connects these diffusion-convection equations to the linear heat system, and the non-commutative RIF-form has infinitely many parametric derivatives. It is not reproduced here.

Subcase 7. $I = 0$, $L \neq 0$.

These are the genuinely nonlinear diffusion equations. Completion of system (72,71) (with ranking similar to above) yields a case splitting on $P_{,4}$. If $P_{,4} \neq 0$ the non-commutative RIF-form is

$$
\begin{array}{lll}
\omega^3_{,1} = 0 & \omega^2_{,1} = 0 & \omega^1_{,1} = \omega^2_{,2} \\
\omega^3_{,2} = 0 & \omega^2_{,22} = 0 & \omega^1_{,2} = 0 \\
\omega^3_{,3} = 2\omega^2_{,2} & \omega^2_{,3} = 0 & \omega^1_{,3} = 0 \\
\omega^3_{,4} = \frac{1}{2}\sigma P\omega^3 & \omega^2_{,4} = -\frac{1}{2}\omega^1 + \frac{3}{4}\sigma P\omega^2 & \omega^1_{,4} = \frac{1}{4}\sigma P\omega^1 - \sigma\omega^2 \\
 & & \omega^4 = 0\,.
\end{array}
\tag{76}
$$

The parametric derivatives ω^1, ω^2, ω^3, $\omega^2_{,2}$ give the four-dimensional symmetry algebra common to all diffusion systems. The commutation relations are

$$[\mathbf{Y}_1, \mathbf{Y}_4] = \mathbf{Y}_1, \qquad [\mathbf{Y}_2, \mathbf{Y}_4] = \mathbf{Y}_2, \qquad [\mathbf{Y}_3, \mathbf{Y}_4] = \mathbf{Y}_3.$$

If $P_{,4} = 0$, we obtain the non-commutative RIF-form

$$
\begin{array}{lll}
\omega^3_{,1} = 0 & \omega^2_{,11} = 0 & \omega^1_{,1} = \sigma P\omega^2_{,1} + \omega^2_{,2} \\
\omega^3_{,2} = 0 & \omega^2_{,12} = 0 & \omega^1_{,2} = 2\sigma\omega^2_{,1} \\
\omega^3_{,3} = 2\sigma P\omega^2_{,1} + 2\omega^2_{,2} & \omega^2_{,22} = 0 & \omega^1_{,3} = 0 \\
\omega^3_{,4} = \frac{1}{2}\sigma P\omega^3 & \omega^2_{,3} = 0 & \omega^1_{,4} = \frac{1}{4}\sigma P\omega^1 - \sigma\omega^2 \\
 & \omega^2_{,4} = -\frac{1}{2}\omega^1 + \frac{3}{4}\sigma P\omega^2 & \omega^4 = 2\omega^2_{,1}\,.
\end{array}
\tag{77}
$$

The parametric derivatives ω^1, ω^2, ω^3, $\omega^2_{,1}$, $\omega^2_{,2}$ give a five-dimensional symmetry algebra, with structure relations

$$\begin{array}{lll} [\mathbf{Y}_1, \mathbf{Y}_4] = \frac{1}{2}\mathbf{Y}_1 & [\mathbf{Y}_1, \mathbf{Y}_5] = \mathbf{Y}_2 & [\mathbf{Y}_2, \mathbf{Y}_4] = \frac{1}{2}\mathbf{Y}_2 \\ [\mathbf{Y}_2, \mathbf{Y}_5] = 2\sigma\mathbf{Y}_1 - \sigma P\mathbf{Y}_2 & [\mathbf{Y}_3, \mathbf{Y}_4] = \mathbf{Y}_3. & \end{array}$$

Subcase 8. $I = 0$, $L = 0$

There is no further case splitting. The non-commutative RIF-form for (50) is

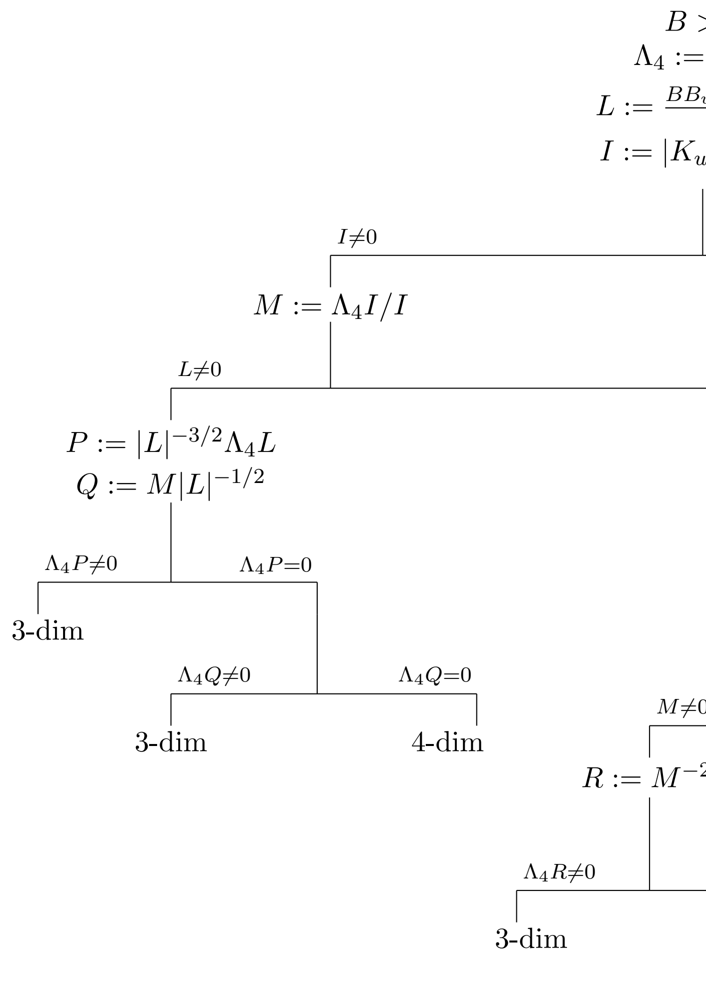
$$\begin{array}{lll} \lambda^3_{,1} = 0 & \lambda^2_{,4} = -\frac{1}{2}\lambda^1 & \lambda^1_{,1} = \frac{1}{2}\lambda^3_{,3} \\ \lambda^3_{,2} = 0 & \lambda^2_{,11} = \lambda^2_{,3} & \lambda^1_{,2} = 0 \\ \lambda^3_{,4} = 0 & \lambda^2_{,12} = -\frac{1}{2}\lambda^1_{,3} & \lambda^1_{,4} = 0 \\ \lambda^3_{,33} = 0 & \lambda^2_{,22} = \frac{1}{2}\lambda^1_{,3} & \lambda^1_{,33} = 0 \\ & \lambda^2_{,23} = 0 & \lambda^4 = 2\lambda^2_{,1}. \end{array} \tag{78}$$

There are infinitely many parametric derivatives: λ^1, λ^3, $\lambda^1_{,3}$, $\lambda^2_{,2}$, $\lambda^3_{,3}$, and the sequences λ^2, $\lambda^2_{,3}$, $\lambda^2_{,33}$,... and $\lambda^2_{,1}$, $\lambda^2_{,13}$, $\lambda^2_{,133}$,.... The symmetry algebra is therefore infinite-dimensional: this subcase consists of equations which can be mapped to the heat equation by an equivalence transformation, so its symmetry properties can be regarded as known (e.g. [27, p.82]).

8.5 Summary of classification

The calculations of this section yield the classification tree shown in Figure 2. In this compact diagram is present all the information required to decide the symmetry properties of a diffusion convection potential system. The elegance of the result is apparent when compared with the classifying equations produced by the 'raw' version of Riquier–Janet [29].

In Figure 2, all the splittings are (by construction) invariant under the action of the equivalence group. Hence two equations connected by an equivalence transformation always occur on the same branch. This greatly cuts down on spurious splittings. Note that equations occurring on different branches of the tree could be equivalent with respect to a transformation *not* in the group (42). Indeed Burgers' and linear heat equations occur on different branches, yet are connected by the Cole-Hopf transformation.


$B >$
$\Lambda_4 :=$
$L := \frac{BB_u}{}$
$I := |K_u$
$I \neq 0$
$M := \Lambda_4 I / I$
$L \neq 0$
$P := |L|^{-3/2} \Lambda_4 L$
$Q := M|L|^{-1/2}$
$\Lambda_4 P \neq 0$
$\Lambda_4 P = 0$
3-dim
$\Lambda_4 Q \neq 0$
$\Lambda_4 Q = 0$
3-dim
4-dim
$M \neq 0$
$R := M^{-2}$
$\Lambda_4 R \neq 0$
3-dim

9 Transformations and Useful Forms of the Output Systems

We have shown that the non-commutative RIF-form of $\mathcal{C} \cup \mathcal{S}$ is sufficient for finding structure and dimension of Lie symmetry algebras of classes of PDE. However for some applications, such as finding explicit group invariant solutions, further processing and integration of these non-commutative RIF-forms may be needed. We briefly consider this topic, using results which are for the most part simple consequences of the classical Frobenius theory.

Suppose that we have a non-commutative RIF-form with a finite set of parametric derivations $\mathrm{Par}\mathcal{M} = \{w^1, ..., w^k\}$. Then consider the unique formal power series solution about x_0 with initial data $w^1(x_0) = w_0^1, ..., w^k(x_0) = w_0^k$ at x_0, with the initial values satisfying the leading non-linear PDE $\mathcal{N}$. For $i = 1, ..., n$ any $\widetilde{D}_i w^\ell \in \mathrm{Prin}\mathcal{M}$ can be completely reduced by $\mathcal{M}$ to an analytic function f_i^ℓ of $\{x\} \cup \mathrm{Par}\mathcal{M}$ such that $\widetilde{D}_i w^\ell = f_i^\ell$. In addition the leading non-linear PDE have form $g = 0$, where g is a (vector) function of $\{x\} \cup \mathcal{N}$: such that

$$\mathcal{M} = \{\widetilde{D}_i w^\ell = f_i^\ell\}, \qquad \mathcal{N} = \{g = 0\}. \tag{79}$$

It follows from (6) that (79) is equivalent to

$$\mathcal{M}' = \{D_i w^\ell = \sum_j b_{ij}(x, w) f_j^\ell\}, \qquad \mathcal{N}' = \{g = 0\}. \tag{80}$$

where $b(x, w)$ is the inverse matrix of $A(x, u)$, and u, g are expressed in terms of $\{x\} \cup \mathrm{Par}\,\mathcal{M}$.

Any non-trivial compatibility conditions of (80) would contradict the existence and uniqueness theorem for the non-commutative RIF-form (79) so (80) is in commutative RIF-form. By the standard commutative (Frobenius) theory it has a formal power series solution with the given data, which is analytic at x_0.

Directly from the classical Frobenius Theory it follows that the integration of $\mathcal{M}' \cup \mathcal{N}'$ is equivalent to integrating a system of differential algebraic equations along analytic curves $x(\tau) = x^i(\tau)$, with $x(0) = x_0$ (e.g $x^i(\tau) = a^i\tau + x_0^i$, $0 \le \tau \le 1$). In particular $\frac{dw^\ell}{d\tau} = \sum_i \frac{dx^i}{d\tau}\frac{\partial w^\ell}{\partial x^i}$ which from (80) yields the system of (index 0 or 1) differential algebraic equations on a manifold $\mathcal{N}'$:

$$\{\frac{dw^\ell}{d\tau} = \sum_i \frac{dx^i}{d\tau} \sum_j b_{ij}(x, w) f_j^\ell\}, \qquad \mathcal{N}' = \{g = 0\}\ . \tag{81}$$

10 Concluding Remarks

We have given a method for classifying symmetry groups for a class of differential equations. Our $\mathcal{G}$-invariant classification is not only more satisfying theoretically, but also obviates the problem of expression swell. The example presented in §8 demonstrates the usefulness of our approach on nontrivial problems.

The method here resembles in some respects Cartan's method of equivalence [8, 13, 24], in that equations are being cast in a $\mathcal{G}$-invariant form and cases with special symmetry are being picked out. However, the Cartan method will exhibit only those symmetries which are part of $\mathcal{G}$. To perform symmetry classification by the Cartan method for classes F of PDE arising in practice, one has two choices. One could enlarge the class F to one that has a suitably large $\mathcal{G}$ (e.g. so that $\mathcal{G}$ is the pseudogroup of all point transformations); the Cartan method would then provide a complete symmetry classification for the enlarged F. Unfortunately this choice can lead to overwhelming computational difficulties. Alternatively, one could apply the Cartan method to the *given* class F and its equivalence group $\mathcal{G}$. Further calculations would then be needed for a full classification of point or contact symmetries for the class F. This mirrors our two-part process of first writing the symmetry system $\mathcal{S}$ in $\mathcal{G}$-invariant form, and then completing to find the symmetry algebras $\mathcal{L}_f$.

In §9 we indicated how the output non-commutative RIF-forms can be easily converted to commutative RIF-forms which open these systems to the application of traditional commutative packages for solving PDE. The systems can also be transformed to equivalent systems of ODE on a manifold (differential algebraic equations), possessing subsystems which are $\mathcal{G}$-invariant. It will be interesting to explore to what extent $\mathcal{G}$-invariant numerical (geometric) integrators [14, 7], can be fruitfully applied to such systems.

Our work takes place against a backdrop of revitalized work in invariantization methods such as Cartan's method of moving frames and its generalizations. In particular see [24] and the review paper [26], and especially the recent work [9, 10] which provides a foundation for flexible and powerful computational approaches to moving frames theory. Such approaches require the development of automatic Gröbner-type methods for manipulating non-commutative operators.

The non-commutative reduction of PDE used here was first applied in Lisle's thesis [19] but was deficient in several respects. It lacked an existence-uniqueness result, and did not give an adequate treatment of the generally non-linear classification systems. The difficulty of establishing a non-commutative Gröbner basis theory for the moving frames case has become apparent since the seminal work of Mansfield [21], which produces interesting results, but like the less ambitious work of Lisle, lacks an existence and uniqueness theorem. The work of Lemaire et al. [18, 39] remedies the deficiencies of [19] and [12].

We note also that the reduction method is not dependent on $\mathcal{G}$-invariance of the BDO: correct dimension and structure for symmetry algebras $\mathcal{L}_f$ result regardless of how much or how little invariance is built into the BDO.

Hubert has given differential elimination algorithms [15] for systems of differential polynomials using non-commutative derivations. In particular in [15] she establishes necessary and sufficient conditions for a non-commutative differential algebra to be isomorphic to a commutative one. She obtains methods for algorithmically representing the radical of a differential ideal, generated by non-commutative derivations, generalizing those of [5]. The approach of Lemaire, Reid and Zhang [18, 39] deals with analytic systems, and is algorithmic for the subclass of differential polynomials. In the example of §8 we chose to compute with real-analytic differential equations and group actions and were successful in completing the calculations (although this is not guaranteed to be algorithmically effective in general). In actual applications one is often interested in classification up to real-valued transformations – in some cases a much more difficult task. Indeed symmetry classification problems are often intrinsically non-polynomial (analytic) since at their inception (e.g. such as for $u_t = (B(u)u_x)_x$) classical differential algebra may not directly apply.

An alternative approach, is to embed, if possible, the problem into a complex differential polynomial one at the onset. If that is successful then after the application of effective differential elimination algorithms (such as [18, 39] restricted to differential polynomials, or that of [15]), the information for the real case is extracted at the end (also often a non-algorithmic and difficult task). The dichotomies of analytic versus differential polynomial algebra, and complex versus real computation, raise important questions, for which there is no panacea.

Acknowledgements

We thank Yang Zhang for rejuvenating this project. We especially also thank Elizabeth Mansfield, Evelyne Hubert and Peter Olver.

References

[1] I. Akhatov, R. Gazizov, and N. Ibragimov, *Group classification*

of the equations of nonlinear filtration, Soviet Math. Dokl., 35 (1987), pp. 384–386.

[2] ——, *Nonlocal symmetries: heuristic approach*, J. Soviet Math., 55 (1991), pp. 1401–1450.

[3] G. Bluman and S. Kumei, *On the remarkable nonlinear diffusion equation* $\frac{\partial}{\partial x}\left[a(u+b)^{-2}\frac{\partial u}{\partial x}\right]-\frac{\partial u}{\partial t}=0$, J. Math. Phys., 21 (1980), pp. 1019–1023.

[4] ——, *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.

[5] F. Boulier, D. Lazard, F. Ollivier, and P. Petitot, *Representation for the radical of a finitely generated differential ideal*, Proceedings of ISSAC'95, (1995), pp. 158–166.

[6] M. Boutin, *Numerically invariant signature curves*, Int. J. Computer Vision, 40 (2000), pp. 235–248.

[7] C. Budd and A. Iserles, *Geometric integration: numerical integration of differential equations on manifolds*, Phil. Trans. Roy. Soc: London A, 357 (1999), pp. 945–956.

[8] E. Cartan, *La Méthode du Repère Mobile, la Théorie des Groupes Continus, et les Espaces Généralisés*, vol. 5 of Exposés de Géométrie, Hermann, Paris, 1935.

[9] M. Fels and P. Olver, *Moving Coframes. I. A practical algorithm*, Acta. Appl. Math., 51 (1998), pp. 161–213.

[10] ——, *Moving Coframes. II. Regularization and theoretical foundations*, Acta. Appl. Math., 55 (1999), pp. 127–208.

[11] A. Fokas and Y. Yortsos, *On the exactly solvable equation* $S_t = [(\beta S+\gamma)^{-2}S_x]_x+\alpha(\beta S+\gamma)^{-2}S_x$ *occurring in two phase flow in porous media*, SIAM J. Appl. Math., 42 (1982), pp. 318–332.

[12] M. Giesbrecht, G. Reid, and Y. Zhang, *Non-commutative Gröbner Bases in Poincaré-Birkhoff-Witt Extensions*, in Proc. of the Fifth International Workshop on Computer Algebra in Scientific Computing (CASC 2002, Yalta, Ukraine), V. Ganzha, E. Mayr, and E. Vorozhtsov, eds., Technical University of Munich, 2002, pp. 97–106.

[13] P. Griffiths, *On Cartan's method of Lie Groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J., 41 (1974), pp. 775–814.

[14] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration*, Springer-Verlag, New York, 2002.

[15] E. Hubert, *Differential algebra for derivations with nontrivial commutation rules*, 2004. To appear in the Journal of Pure and Applied Algebra.

[16] M. Janet, *Sur les systèmes d'équations aux dérivées partielles*, J. de Math, 3 (1920), pp. 65–151.

[17] I. Kogan, *Inductive construction of moving frames*, Contemp. Math., 285 (2001), pp. 157–170.

[18] F. Lemaire, G. Reid, and Y. Zhang, *Non-commutative Existence and Uniqueness Theory for Analytic Systems of Nonlinear Partial Differential Equations*, 2004. www-address: www.orcca.on.ca/~reid/ReidPapers/LemaireReidZhang05.ps.

[19] I. Lisle, *Equivalence Transformations for Classes of Differential Equations*, PhD thesis, Univ. of British Columbia, 1992. Available at: http://www.ise.canberra.edu.au/MathStat/StaffPages/LisleDissertation.pdf.

[20] E. Mansfield, *Computer algebra for the classification problem*, Mars Publishers, Trondheim, Norway, 1999, pp. 211–217.

[21] ——, *Algorithms for symmetric differential systems*, Foundations of Computational Math., 1 (2001), pp. 335–383.

[22] E. Mansfield and P. Clarkson, *Applications of the differential algebra package* `diffgrob2` *to classical symmetries of differential equations*, J. Symb. Comp., 23 (1997), pp. 517–533.

[23] P. Olver, *Application of Lie groups to differential equations*, Springer-Verlag, New York, 1986.

[24] ——, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, 1995.

[25] ——, *Geometric foundations of numerical algorithms and symmetry*, Appl. Alg. Engin. Comput., 11 (2001), pp. 417–436.

[26] ——, *Moving frames*, 2003. Preprint available at http://www.math.umn.edu/~olver/xtra.html.

[27] L. Ovsiannikov, *Group analysis of differential equations*, Academic Press, New York, 1982.

[28] G. Reid, *Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution*, Euro. J. Appl. Maths., 2 (1991), pp. 293–318.

[29] ——, *Finding abstract Lie symmetry algebras of differential equations without integrating determining equations*, Euro. J. Appl. Maths., 2 (1991), pp. 319–340.

[30] G. Reid, I. Lisle, A. Boulton, and A. Wittkopf, *Algorithmic determination of commutation relations for Lie symmetry algebras of PDEs,*, in Proc. ISSAC '92, New York, 1992, ACM Press, pp. 63–68.

[31] G. Reid, A. Wittkopf, and A. Boulton, *Reduction of systems of nonlinear partial differential equations to simplified involutive forms*, Euro. J. Appl. Math., 7 (1996), pp. 604–635.

[32] C. Rust, *Rankings of Derivatives for Elimination Algorithms and Formal Solvability of Analytic Partial Differential Equations*, PhD thesis, Univ. Chicago, 1998. www-address: www.cecm.sfu.ca/~rust.

[33] C. Rust, G. Reid, and A. Wittkopf, *Formal existence and uniqueness theorems for formal power series solutions of analytic differential systems*, in Proc. ISSAC '99, Vancouver, S. Dooley, ed., ACM Press, 1999, pp. 105–112.

[34] F. Schwarz, *Automatically determining symmetries of differential equations*, Computing, 34 (1985), pp. 91–106.

[35] J. Sherring and G. Prince, *DIMSYM—symmetry determination and linear partial differential equations package*, Dept. of Mathematics Preprint, LaTrobe Univ, Australia, 1992.

[36] M. Storm, *Heat conduction in simple metals*, J. Appl. Phys., 22 (1951), pp. 940–951.

[37] J. Thomas, *Riquier's existence theorems*, Annals of Math., 30 (1929), pp. 285–310.

[38] A. Wittkopf, *Algorithms and Implementations for Differential Elimination*, PhD thesis, Simon Fraser Univ., 2004. www-address: www.cecm.sfu.ca/~wittkopf.

[39] Y. Zhang, *Algorithms for Non-Commutative Differential Operators*, PhD thesis, University of Western Ontario, 2004. www-address: www.orcca.on.ca/~reid/Zhang/YangZhangPhDThesis.pdf.