A comparative study of Tarski's fixed point theorems with the stress on commutative sets of **L**-fuzzy isotone maps with respect to transitivities

František Včelař[a,*], Zuzana Pátíková[a]

[a] *Department of Mathematics, Faculty of Applied Informatics, Tomas Bata University in Zlín, Nad Stráněmi 4511, Zlín, 76005, Czech Republic*

Abstract

The paper deals mainly with a fuzzification of the classical Tarski's theorem for commutative sets of isotone maps (the so-called generalized theorem) in a sufficiently rich fuzzy setting on general structures called **L**-complete propelattices. Our concept enables a consistent analysis of the validity of single statements of the generalized Tarski's theorem in dependence on assumptions of relevant versions of transitivity (weak or strong). The notion of the **L**-complete propelattice was introduced in connection with the fuzzified more famous variant of Tarski's theorem for a single **L**-fuzzy isotone map, whose main part holds even without the assumption of any version of transitivity. These results are here extended also to the concept of the so-called **L**-fuzzy relatively isotone maps and then additionally compared to the results, which are achieved for the generalized theorem and which always need a relevant version of transitivity. Wherever it is possible, facts and differences between both the theorems are demonstrated by appropriate examples or counterexamples.

Keywords: Fuzzy relation; Fixed point; Complete lattice; **L**-fuzzy isotone map; Commutative set of maps; Relatively isotone map; Transitivity (weak or strong)

1. Introductory remark

Alfred Tarski in his paper [14] in 1955 presented two important theorems on fixed points of isotone maps on complete lattices. The first one was for a single map (named *lattice-theoretical fixpoint theorem*) and the second one for a commutative set of maps (*generalized lattice-theoretical fixpoint theorem*). A fuzzification of the first theorem and its generalization in a sufficiently rich fuzzy

*Corresponding author, Phone number: +420576035060

Email addresses: vcelar@fai.utb.cz (František Včelař), patikova@fai.utb.cz (Zuzana Pátíková)

setting were considered in the paper [16]. There we proved that major parts of this theorem remain valid even under the absence of any version of transitivity. In this paper, we consider mainly a fuzzification of the generalized theorem. Our previous results are extended onto the concept of **L**-*fuzzy relatively isotone maps* (Definition 15) and compared to newly obtained results. The basic structure to work with is going to be the so-called **L**-*complete propelattice* (Definition 14) in addition with some other necessary assumptions.

We will not repeat all connections to other results from the literature, as they were presented in [16], we only remark that different fuzzifications of Tarski's theorem started to appear in literature shortly after Lotfi A. Zadeh in [17] in 1971 introduced the notion of *fuzzy ordering*. These attempts were more or less successful according to the strength of chosen assumptions on the supporting structure and on the fuzzy setting. In our case, we consider *weak*, resp. *strong version of transitivity* (Definition 13). *Antisymmetry* (Definition 11) is always used in the weakest possible version. At the same time, it is known that elimination of antisymmetry is not possible in general, otherwise undesirable cycles could appear, which could result in the nonexistence of fixed points.

In the paper, we adopt the notional apparatus from [16], where we considered the question if the very fundamental assumption of transitivity can be possibly fully eliminated. The answer, which is for the case of a single **L**-*fuzzy isotone map* (Definition 15) given in [16], is surprisingly positive. The set of fixed points of an **L**-fuzzy isotone map is nonempty and forms a so-called *complete propelattice* (Definition 10), i.e., a structure which has all the properties of a complete lattice except transitivity. Nevertheless, the strong version of transitivity is necessary for the validity of the complete fuzzification of Tarski's theorem including the validity of explicit formulas for the least and the greatest fixed point. We show that the same also holds for the generalized theorem (Theorem 4).

What can be observed is that literature is very stingy with fuzzifications of the generalized Tarski's theorem for a commutative set of isotone maps, unfortunately, we are not able to provide any references (which, however, can be only an imperfection of the authors). Therefore this paper is devoted mainly to the fuzzification of the generalized Tarski's theorem. Simultaneously, our approach to a fuzzy setting is framed by the acceptance or generalizations of standard definitions known from literature [2–4, 8]. We also observe differences between necessary assumptions in the case of a single **L**-fuzzy isotone map and in the case of a commutative set (Definition 1) of **L**-fuzzy isotone maps. Mainly, we show that now we cannot get by with at least some version of transitivity (Theorem 3). Further, we deal with the validity or invalidity of single statements about fuzzifications of both Tarski's theorems for **L**-fuzzy relatively isotone maps, which generalize the so-called *relatively isotone maps* (Definition 8) [5, 7].

Before we recall the original generalized Tarski's theorem, let us remind the following simple but here fundamental definition.

Definition 1. A set of maps $\emptyset \neq F \subseteq X^X$, i.e., maps of the form $f : X \to X$, is *commutative* if $f \circ g = g \circ f$ holds for every $f, g \in F$, i.e., $f(g(x)) = g(f(x))$ for any $x \in X$.

Now we present the original version of the generalized Tarski's theorem (only with adapted symbolism).

Theorem 1. ([14]) *Let*

(i) $\mathcal{X} = \langle X, \leq \rangle$ *be a complete lattice,*
(ii) F *be any commutative set of increasing functions on* X *to* X*,*
(iii) P *be the set of all common fixpoints of all the functions* $f \in F$*.*

Then

$$\textit{the set } P \textit{ is not empty} \tag{1}$$

and

$$\textit{the system } \langle P, \leq \rangle \textit{ is a complete lattice}; \tag{2}$$

in particular, we have

$$\bigvee P = \bigvee \{x \in X \mid f(x) \geq x \textit{ for every } f \in F\} \in P \tag{3}$$

and

$$\bigwedge P = \bigwedge \{x \in X \mid f(x) \leq x \textit{ for every } f \in F\} \in P. \tag{4}$$

Obviously, the core of Theorem 1 is in statements (1) and (2), even if the formulas (3), (4) create the basis of the original proof. Nevertheless, making use of the explicit formulas for the least and the greatest fixed point can be in concrete cases somewhat problematic. Naturally, the existence of fixed points itself is the most important fact.

Let us point out that the motivation of many problems solved in this study and in the paper [16] has arisen from concrete research tasks out of mathematics. To mention in brevity the most important one, we turn our attention to genetics. Some parts of genetic code can evince nontransitive relationships which at the same time show a certain form of completeness. The formal description of this fact can be represented by the notion of a complete propelattice. If a modification of a gene sequence is described by an isotone map on a propelattice, there arises a question of the existence of some reference gene, where the gene sequence can be fixed on a medium, i.e., the question of the existence of a fixed point. The issue of interchangeability of manipulations with gene sequences leads to a search for conditions which would be sufficient for the existence of a common fixed point. The mentioned relationships usually have different strength, which is the reason for solving all the tasks in a fuzzy setting. (One can say that concrete real problems have indicated a deeper understanding of the significance and strength of one of the most fundamental properties of the entire mathematics - transitivity.)

2. Prerequisities, basic notions and facts

Now we briefly present all necessary facts and notation concerning residuated lattices and fuzzy sets, i.e., **L**-sets in our terminology. For details and proofs, we refer to reliable resources: [2, 10, 11]. Some of the needed facts from ordinal or cardinal arithmetics can be found in [9, 13] and from lattice theory in [1, 6]. However, we try to be as self-sufficient as possible in the sense that we introduce all necessary fundamental notions, which can be found in the literature under the same names in different variants and usually with stronger assumptions.

The following notion is in our case just auxiliary, but with respect to its importance in fuzzy set theory, we recall its definition. It delimits all the needed properties of the fuzzy setting, where our results hold.

Definition 2. An algebra $\mathbf{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, \mathbf{0}, \mathbf{1}\rangle$ is called a *complete residuated lattice* if the following conditions are fulfilled:

(i) $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle = \langle L, \leq\rangle$ is a complete lattice;
(ii) $\langle L, \otimes, \mathbf{1}\rangle$ is a commutative monoid;
(iii) binary operations $\otimes, \rightarrow$ satisfy the so-called *adjointness condition*, i.e., for every $a, b, c \in L$ the following equivalence holds:

$$a \otimes b \leq c \;\Leftrightarrow\; a \leq b \rightarrow c. \tag{5}$$

Let us point out that the assumption of completeness is fundamental in the whole paper, because the existence of $\inf(A)$ for any set $\emptyset \neq A \subseteq L$ (from which follows also the existence of $\sup(A)$ [1, 6]) is necessary in all the following statements. Both the operations *multiplication* $\bullet \otimes \bullet : L \times L \rightarrow L$ and *residuum* $\bullet \rightarrow \bullet : L \times L \rightarrow L$ form in $\mathbf{L}$ the so-called *adjoint couple*. As a consequence of completeness of $\mathbf{L}$ we have for operations $\otimes$ and $\rightarrow$ the following identities:

$$a \rightarrow b = \bigvee\{c \in L \,|\, a \otimes c \leq b\}, \quad a \otimes b = \bigwedge\{c \in L \,|\, a \leq b \rightarrow c\}. \tag{6}$$

Immediately from adjointness (5) the following important relations follow for every $a, b \in L$:

$$a \otimes b \leq a \wedge b, \tag{7}$$

$$a = \mathbf{1} \rightarrow a, \quad a \leq b \;\Leftrightarrow\; a \rightarrow b = \mathbf{1}. \tag{8}$$

Finally, the operation $\otimes$ is *mototone* and the operation $\rightarrow$ is *antitone* in relevant operands in the following way: for every $a, b, c \in L$ such that $a \leq b$ the following holds:

$$\begin{aligned} &a \otimes c \leq b \otimes c, \quad c \otimes a \leq c \otimes b; \\ &c \rightarrow a \leq c \rightarrow b, \quad a \rightarrow c \geq b \rightarrow c. \end{aligned} \tag{9}$$

The classical logical setting is in the following represented by the *Boolean algebra* $\mathbf{L} = \mathbf{2}$ with the support $\{\mathbf{0}, \mathbf{1}\}$ (which is obviously a trivially complete residuated lattice, where $\otimes \equiv \wedge$ and $\rightarrow \equiv \Rightarrow$). A richer logical setting is then represented by some complete residuated lattice $\mathbf{L} \neq \mathbf{2}$.

Everywhere in the following X denotes an arbitrary but fixed nonempty *universe set* and $\mathbf{L}$ denotes an arbitrary but fixed complete residuated lattice.

Any element $\Phi \in L^X$, i.e., any map $\Phi : X \to L$, is called an **L**-*set*. For the special cases $\emptyset, X \in L^X$ we have $\emptyset(x) \equiv \mathbf{0}$ and $X(x) \equiv \mathbf{1}$ $(x \in X)$. For $\Phi \in L^X$ and $\alpha \in L$ the set ${}^{\alpha}\Phi := \{x \in X \,|\, \Phi(x) \geq \alpha\}$ is the α-*cut* of Φ; especially $\mathrm{Ker}(\Phi) := {}^{\mathbf{1}}\Phi = \{x \in X \,|\, \Phi(x) = \mathbf{1}\}$ is the *kernel* of Φ. If $\mathrm{Ker}(\Phi) \neq \emptyset$, then the **L**-set Φ is *normal*, otherwise Φ is *subnormal*. For fixed $x \in X$ and $\alpha \in L$, $\alpha \neq \mathbf{0}$ we denote as $\{{}^{\alpha}/_{x}\} \in L^X$ such an **L**-set that $\{{}^{\alpha}/_{x}\}(x) = \alpha$ and $\{{}^{\alpha}/_{x}\}(y) = \mathbf{0}$ for every $y \neq x$.

Because **L** is complete, we introduce for the indexed system $\emptyset \neq \mathcal{F} = \{\Phi_\lambda \in L^X \,|\, \lambda \in \Lambda\} \subseteq L^X$ the operations of *intersection* $\bigcap \mathcal{F} = \bigcap_{\lambda\in\Lambda} \Phi_\lambda$ and *union* $\bigcup \mathcal{F} = \bigcup_{\lambda\in\Lambda} \Phi_\lambda$ for every $x \in X$ in the following way:

$$\left(\bigcap \mathcal{F}\right)(x) = \left(\bigcap_{\lambda\in\Lambda} \Phi_\lambda\right)(x) := \bigwedge_{\lambda\in\Lambda} \Phi_\lambda(x),$$

$$\left(\bigcup \mathcal{F}\right)(x) = \left(\bigcup_{\lambda\in\Lambda} \Phi_\lambda\right)(x) := \bigvee_{\lambda\in\Lambda} \Phi_\lambda(x). \tag{10}$$

If $\emptyset \neq A \subseteq X$ is a crisp subset, we identify it sometimes (not consistently) thanks to (10) with its membership function, i.e., we write $A \equiv \bigcup_{x\in A}\{{}^{\mathbf{1}}/_{x}\}$. For a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq X$ we also write

$$\{{}^{\alpha_1}/_{x_1}, {}^{\alpha_2}/_{x_2}, \ldots, {}^{\alpha_n}/_{x_n}\} := \bigcup_{i=1}^{n} \{{}^{\alpha_i}/_{x_i}\}.$$

For any $\Phi, \Psi \in L^X$ the relation of *inclusion* $\Phi \subseteq \Psi$ holds if for every $x \in X$ we have $\Phi(x) \leq \Psi(x)$. Then Φ is an **L**-*subset* of Ψ. Obviously for any $\Phi, \Psi \in L^X$ the following holds:

$$\begin{aligned} &\emptyset \subseteq X, \quad \emptyset \subseteq \Phi, \quad \Phi \subseteq X; \\ &\Phi \subseteq \Psi \ \ \& \ \ \Psi \subseteq \Phi \quad \Leftrightarrow \quad \Psi = \Phi. \end{aligned} \tag{11}$$

It is substantial that with respect to completeness of **L** also $\langle L^X, \cup, \cap, \emptyset, X\rangle = \langle L^X, \subseteq\rangle$ has a structure of a **complete lattice**.

The following definition introduces the most general variant of a singleton, which is for our purposes not only sufficient but also favorable.

Definition 3. ([8]) An **L**-set $\Phi \in L^X$ is an *SC-singleton at the point* $x_0 \in X$ if $\mathrm{Ker}(\Phi) = \{x_0\}$. An arbitrary *SC*-singleton at x_0 is denoted by the constant term $\mathcal{S}[x_0]$.

Remark 1. In literature, there are many variants of the definition of a singleton. In extreme cases it can be a subnormal **L**-set or one member crisp set $\{{}^{\mathbf{1}}/_{x_0}\}$ for $x_0 \in X$. Other definitions of a singleton, see for example [2–4, 10, 11], usually require further conditions which are reasonable or useful in the relevant context. This, however, means also some restrictions on the form of the membership function of the singleton Φ. For our purposes, it is important that its form is restricted just by the only condition, i.e., that $|\mathrm{Ker}(\Phi)| = 1$. This enables us to use in all our considerations the simple symbol $\mathcal{S}[x_0]$. (On the other hand, this arbitrariness of the form of the membership function means that the structure given by Definition 10 is not uniquely determined by its **1**-cut, cf. [4]). The only case, where the form of the membership function is for us important, is in Example 4.

Any element $\triangle \in L^{X\times X}$ is called an **L**-(*binary*) *relation*. For $x, y \in X$ we write $(x \triangle y) := \triangle(x, y)$. For $\alpha \in L$ the relation ${}^{\alpha}\triangle := \{(x, y) \in X \times X \mid (x \triangle y) \geq \alpha\}$ is the α-*cut* of $\triangle$. For our purposes we find important mainly **1**-cuts ${}^{\mathbf{1}}\triangle := \{(x, y) \in X \times X \mid (x \triangle y) = \mathbf{1}\}$. We write

$$x\, {}^{\mathbf{1}}\triangle\, y \quad \overset{df}{\Leftrightarrow} \quad (x, y) \in {}^{\mathbf{1}}\triangle \quad \Leftrightarrow \quad (x \triangle y) = \mathbf{1}.$$

The following definition is effective for introducing or testing the properties which have no analogies in the crisp setting [15]. It enables us to construct simple but not trivial (i.e., where $\mathbf{L} \neq \mathbf{2}$) examples and counterexamples.

Definition 4. ([15]) An element $\mathcal{N}(\mathbf{L}) \in L$ is called a *neutral of* **L** if $\mathcal{N}(\mathbf{L}) = \mathcal{N}(\mathbf{L}) \to \mathbf{0}$. The set of all neutrals is denoted as $\mathbb{N}(\mathbf{L})$.

Obviously, the cases $\mathcal{N}(\mathbf{L}) = \emptyset$ but also $|\mathcal{N}(\mathbf{L})| \geq 2$ are possible. Until something else is pointed out, anytime we speak about a neutral $\mathcal{N}(\mathbf{L})$, we mean any fixed element of $\mathbb{N}(\mathbf{L})$. Every neutral $\mathcal{N}(\mathbf{L}) \in \mathbb{N}(\mathbf{L})$ has with respect to (5) the following useful property:

$$\mathcal{N}(\mathbf{L}) \leq \mathcal{N}(\mathbf{L}) \to \mathbf{0} \;\Rightarrow\; \mathcal{N}(\mathbf{L}) \otimes \mathcal{N}(\mathbf{L}) \leq \mathbf{0} \;\Rightarrow\; \mathcal{N}(\mathbf{L}) \otimes \mathcal{N}(\mathbf{L}) = \mathbf{0}. \qquad (12)$$

If $\Phi(x) = \mathcal{N}(\mathbf{L})$ for $\Phi \in L^X$, we interpret such a situation naturally as that we occupy a neutral attitude towards the statement "x is an element of Φ".

Remark 2. Let us notice that any two neutrals are incomparable in **L**. Let $\mathcal{N}_1(\mathbf{L}), \mathcal{N}_2(\mathbf{L}) \in \mathbb{N}(\mathbf{L})$ be different and for example $\mathcal{N}_1(\mathbf{L}) < N_2(\mathbf{L})$. Then the following chain of implications holds: $\mathcal{N}_1(\mathbf{L}) \leq \mathcal{N}_2(\mathbf{L}) \Rightarrow \mathcal{N}_1(\mathbf{L}) \leq \mathcal{N}_2(\mathbf{L}) \to \mathbf{0} \Rightarrow \mathcal{N}_1(\mathbf{L}) \otimes \mathcal{N}_2(\mathbf{L}) \leq \mathbf{0} \Rightarrow \mathcal{N}_2(\mathbf{L}) \otimes \mathcal{N}_1(\mathbf{L}) \leq \mathbf{0} \Rightarrow \mathcal{N}_2(\mathbf{L}) \leq \mathcal{N}_1(\mathbf{L}) \to \mathbf{0} \Rightarrow \mathcal{N}_2(\mathbf{L}) \leq N_1(\mathbf{L})$, which is a contradiction, and hence $\mathcal{N}_1(\mathbf{L}) \parallel \mathcal{N}_2(\mathbf{L})$.

The following definition introduces nothing else but 3-*valued Łukasiewicz algebra* (see [2, 11]), only its notation is adapted to the concept of complete residuated lattices and to the specific role of neutrals in the following examples and counterexamples. These demonstrate some important properties or contrarily the fact that relevant assumptions cannot be weakened.

Definition 5. Let us have $L_N = \{\mathbf{0}, N, \mathbf{1}\}$ with the order $\mathbf{0} < N < \mathbf{1}$. Then the residuated lattice $\mathbf{L}_N := \langle L_N, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ for which $\mathbb{N}(\mathbf{L}_N) = \{N\}$ is called a 3-*valued Łukasiewicz algebra*.

With respect to (12), (6) and the fact that N is the only neutral, $\mathbf{L}_N$ is obviously a uniquely determined complete residuated lattice.

One of the most important standard definitions in the paper is the following one [2–4, 8].

Definition 6. An **L**-relation $\approx\, \in L^{X\times X}$ is called an **L**-*equality* if for every $x, y, z \in X$ the following four conditions are fulfilled:

$$\begin{aligned} &\text{(i)} && (x \approx x) = \mathbf{1}\,(\textit{reflexivity}); \\ &\text{(ii)} && (x \approx y) = (y \approx x)\,(\textit{symmetry}); \\ &\text{(iii)} && (x \approx y) \otimes (y \approx z) \leq (x \approx z)\,(\textit{transitivity}); && (13) \\ &\text{(iv)} && (x \approx y) = \mathbf{1} \Rightarrow x = y\,(\textit{separation}). && (14) \end{aligned}$$

Remark 3. (a) Obviously, thanks to (7) (and transitivity of the order in $\langle L, \leq\rangle$), the following implication holds:

$$[(x \approx y) \wedge (y \approx z) \leq (x \approx z)] \;\Rightarrow\; [(x \approx y) \otimes (y \approx z) \leq (x \approx z)].$$

According to this implication, relation (13) introduces *weak version* of transitivity $\approx$, whereas the left side of this implication introduces its *strong version.* Analogous terminology is used also in the rest of the paper.

(b) In connection with the above-mentioned, it is suitable to notice that our prior effort is to use as minimal number of as weak assumptions as possible. Definition 6 fulfills both these requirements. Let us remark that in many concepts known from literature, the **L**-equality is usually being substituted by other notions, whose conditions are however in total stronger.

Throughout the paper, we use only such notions, which are *conservative extensions* of "crisp" notions, more precisely, the notions which for $\mathbf{L} = \mathbf{2}$ automatically reduce onto the standard notions. From this point of view, the **L**-equality from Definition 6 is an excellent example of a conservative extension of the mathematical identity on the set X.

Remark 4. From a strictly logical point of view, conservative extensions are hence such notions which have the following property: whatever can be proved in a fuzzy setting which is represented by a logically richer setting $\mathbf{L} \neq \mathbf{2}$, that can be proved even in the crisp setting of the classical logic $\mathbf{L} = \mathbf{2}$ only with the use of its own means. This fact implies that it is enough to construct all counterexamples only in the crisp setting, which is often easier. Nevertheless, later we present a universal method (see Lemma 1), which enables to transfer every (counter)example into a simple but not trivial fuzzy setting.

Concerning the notation, let us add that if the universe set X is considered together with a fixed **L**-equality $\approx\, \in L^{X\times X}$, this fact is underlined by the notation as we write it as the ordered pair $\langle X, \approx\rangle$. Especially, $\langle X, =\rangle$ means that we consider the classical identity on X.

3. Fundamental crisp notions and illustrative examples

In this part, basic notions from [16] are reviewed and extended and a few examples, which are then used also in further parts of the paper, are presented. At first, analogies of notions from poset theory, which terminologically reflect the absence of the assumption of transitivity, are introduced. These point out that still there is some idea of "orderliness". For the majority of the presented notions, we bring in the next section also their conservative extensions. Nevertheless, these notions are important also themselves because they appear in the following statements and theorems. (For terminology see Remark 7.)

Definition 7. ([16]) Let a binary relation $\triangle \subseteq X \times X$ be reflexive and antisymmetric. Then $\triangle$ is a *propeorder* on X and the pair $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ is a *propeordered set.* If $A \subseteq X$ then

(p1) $\mathcal{L}(A) := \{a \in X \mid \forall x \in A : a \triangle x\}$ is a *lower propecone of* A; any $a \in \mathcal{L}(A)$ is a *lower propebound of* A;

(p2) $\mathcal{U}(A) := \{a \in X \mid \forall x \in A : x \triangle a\}$ is an *upper propecone of* A; any $a \in \mathcal{U}(A)$ is an *upper propebound of* A;

(p3) an element $a \in X$ is the *propeleast element of* A iff $a \in A \cap \mathcal{L}(A)$; we write $p\min(A) := a$;

(p4) an element $a \in X$ is the *propegreatest element of* A iff $a \in A \cap \mathcal{U}(A)$; we write $p\max(A) := a$;

(p5) an element $a \in X$ is the *propeinfimum of* A if $a = p\max(\mathcal{L}(A))$; we write $p\inf(A) := a$, especially we denote $\bot := p\inf(X) = p\min(X)$;

(p6) an element $a \in X$ is the *propesupremum of* A if $a = p\min(\mathcal{U}(A))$; we write $p\sup(A) := a$, especially we denote $\top := p\sup(X) = p\max(X)$.

Remark 5. (a) The conditions in parts (p1) and (p2) can be also vacuously true, i.e., $\mathcal{L}(\emptyset) = \mathcal{U}(\emptyset) = X$. If elements in (p3) and (p4) exist, then it automatically means that the set A is nonempty and hence the assumption $A \neq \emptyset$ is not needed in Definition 7.

None of the elements defined in (p3)–(p6) needs to exist but if it exists, then it is unique thanks to antisymmetry. Therefore their notation could have been introduced directly in the definition.

(b) It is not hard to check that for every $A \subseteq X$ if $p\inf(A)$, or $p\sup(A)$, exists, then

$$\{p\inf(A)\} = \mathcal{L}(A) \cap \mathcal{U}(\mathcal{L}(A)), \quad \{p\sup(A)\} = \mathcal{U}(A) \cap \mathcal{L}(\mathcal{U}(A)), \tag{15}$$

respectively. The same relations, which depend only on reflexivity and antisymmetry but not on transitivity, are well known also from lattice theory.

The following definition is here recalled above all because the second part does not need to be well-known. Conservative extensions of these notions into the fuzzy setting are fundamental in the next.

Definition 8. Let $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ be a propeordered set. A map $f : X \to X$ is *isotone on* $\mathbf{X}$ if for any $x, y \in X$ the following implication holds:

$$x \triangle y \quad \Rightarrow \quad f(x) \triangle f(y). \tag{16}$$

A map $f : X \to X$ is *relatively isotone on* $\mathbf{X}$ if for every $x, y \in X$ the following implication holds:

$$f(x) \triangle y \ \ \& \ \ x \triangle y \ \ \& \ \ x \triangle f(y) \quad \Rightarrow \quad f(x) \triangle f(y). \tag{17}$$

The notion of a relatively isotone map is adopted from poset theory [5, 7]. Condition (17) requires the fact that images are comparable only in the situation when not only their preimages but also the preimages with the relevant images are comparable; in the opposite case the condition does not require anything. Thus, condition (17) is weaker than (16). Whenever the existence of a fixed point for an arbitrary relatively isotone map $f \in X^X$ is guaranteed, the existence of fixed points of isotone maps is automatically ensured too. The major advantage of condition (17) is that its antecedent is usually simply or even trivially fulfilled if its consequent is fulfilled too, but condition (16) does not have to be satisfied. (Let us notice that relatively isotone maps enable an alternative formulation of the famous theorem by A. Davis, which is a converse of Tarski's theorem.) Mainly by looking for needed counterexamples it appears that the difference between conditions (16) and (17) is bigger than one would await at first glance.

Remark 6. The following can be said about the history of condition (17). The authors first met the condition (exactly in this form) in [12] in relation with the so-called *double superinduction* principle from set theory (for more details see also [13]). We do not know if the same condition appeared in lattice theory independently.

The following definition appears here because of the codification of notation.

Definition 9. The *set of all fixed points of a map* $f \in X^X$ is denoted as $\mathrm{Fix}(f) := \{x \in X \,|\, x = f(x)\}$. The *set of all common fixed points of a set of maps* $\emptyset \neq F \subseteq X^X$ is denoted as $\mathrm{Fix}(F) := \bigcap_{f \in F} \mathrm{Fix}(f)$.

The fundamental role is in the following played by a conservative extension of the next definition, which afterwards arises from successive extensions of notions from Definition 7.

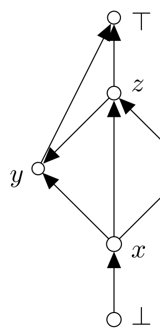
Definition 10. ([16]) A propeordered set $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ is called a *complete propelattice*, if for any set $A \subseteq X$ both $p\inf(A)$ and $p\sup(A)$ exist simultaneously. Then $\triangle$ is called a *complete propeorder on* X.

Remark 7. Considering the formation of the terminology, one can see that for transitive structures, especially lattices, all usual prefixes with Latin or Greek origins are occupied: semilattice, pseudolattice, etc. Especially in connection with Definition 10 and the following Definition 14 we have chosen short and easily pronounceable Latin adverbium "prope" [prope], which means "close to". Our terminology should emphasize that the introduced notions have many properties in common with complete lattices.

The following examples illustrate the introduced notions and are then used further in the text. Obviously, for a complete propelattice which is not automatically transitive, $card(X) \geq 5$ has to hold. Hence the first two examples belong to the simplest ones (because there $card(X) = 6$). Lattices are demonstrated by "Hasse diagrams", however, it is clear that it is transitivity what is the key property which enables their unique interpretation and application of all the rules for their construction. Therefore our figures are only helping and they can be made even in different readable ways.

Example 1. Let a complete propelattice $\mathbf{X} = \langle\langle X, =\rangle, \vartriangle\rangle$ have the "Hasse diagram" given in Figure 1.

Figure 1: Diagram for Example 1



Here $X = \{\bot, x, y, w, z, \top\}$ and the propeorder $\vartriangle$ is given as follows:

$$\begin{aligned}\vartriangle = \{&(\bot, x), (\bot, y), (\bot, w), (\bot, z), (\bot, \top), (x, y), (x, w),\\ &(x, z), (z, y), (w, z), (x, \top), (y, \top), (w, \top), (z, \top)\} \cup id_X.\end{aligned}$$

The relation $\vartriangle \subseteq X \times X$ is obviously not transitive, because $w \vartriangle z$ and $z \vartriangle y$ but $(w, y) \notin \vartriangle$. Also it is easy to see that $\mathbf{X}$ is a complete propelattice; e.g., for $A = \{x, y, z\}$ we have:

$$\begin{aligned}p\inf(A) &= p\max(\mathcal{L}(A)) = p\max(\{\bot, x\}) = x,\\ p\sup(A) &= p\min(\mathcal{U}(A)) = p\min(\{\top\}) = \top.\end{aligned}$$

Further, let a map $f : X \to X$ be given by the following list:

$$f := [\bot \mapsto x, x \mapsto z, y \mapsto y, w \mapsto z, z \mapsto z, \top \mapsto y].$$

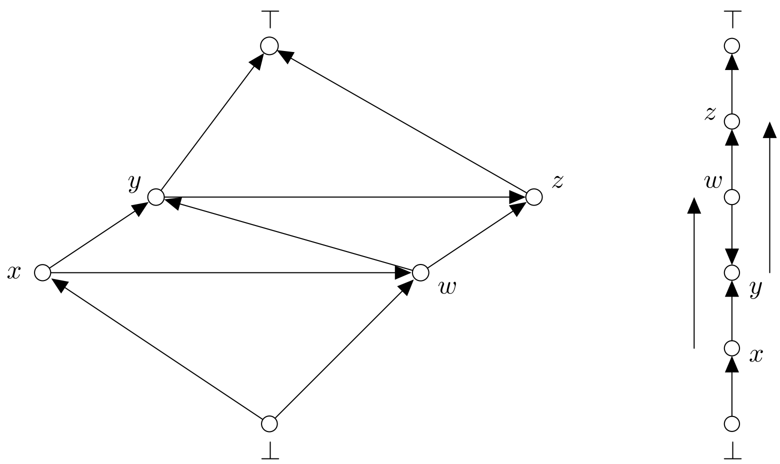
It can be easily seen that f is isotone on $\mathbf{X}$; for example we have:

$$\triangle \ni (\bot, x) \overset{f\times f}{\longmapsto} (x, z) \in \triangle, \quad \triangle \ni (x, w) \overset{f\times f}{\longmapsto} (z, z) \in \triangle,$$
$$\triangle \ni (w, \top) \overset{f\times f}{\longmapsto} (z, y) \in \triangle.$$

Here $\mathrm{Fix}(f) = \{y, z\} \neq \emptyset$ and $p\min(\mathrm{Fix}(f)) = z$ and $p\max(\mathrm{Fix}(f)) = y$. In the following we will see that these equalities are not coincidental but legitimate.

Example 2. Even here let us have $X = \{\bot, x, y, w, z, \top\}$. Let now the complete propelattice $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ have one of the following diagrams in Figure 2. The first one follows all the rules for constructing Hasse diagrams which can be used for nontransitive relations; the second one represents only a "skeleton" of the relation and even if it has nothing common with Hasse diagrams, it is probably better readable.

Figure 2: Diagrams for Example 2



Here the propeorder $\triangle$ is given by the following:

$$\triangle = \{(\bot, x), (\bot, y), (\bot, w), (\bot, z), (\bot, \top), (x, y), (w, y), (w, z), (x, w), (y, z), (x, \top), (y, \top), (w, \top), (z, \top)\} \cup id_X.$$

Clearly, $\mathbf{X}$ is a complete lattice, however even here the relation $\triangle \subseteq X \times X$ is not transitive, because $x \triangle w$ and $w \triangle z$ and also $x \triangle y$ and $y \triangle z$ but $(x, z) \notin \triangle$. Further, let us define a map $f : X \to X$ by the following list:

$$f := [\bot \mapsto w, x \mapsto w, y \mapsto y, w \mapsto w, z \mapsto y, \top \mapsto y].$$

Again, it is trivial to check that f is isotone on $\mathbf{X}$, for example:

$$\triangle \ni (\bot, z) \overset{f\times f}{\longmapsto} (w, y) \in \triangle, \quad \triangle \ni (x, w) \overset{f\times f}{\longmapsto} (w, w) \in \triangle,$$
$$\triangle \ni (x, \top) \overset{f\times f}{\longmapsto} (w, y) \in \triangle.$$

Here $\mathrm{Fix}(f) = \{y, w\} \neq \emptyset$ and simultaneously $p\min(\mathrm{Fix}(f)) = w$ and $p\max(\mathrm{Fix}(f)) = y$. Even here it is not coincidental, as we show in the next.

Further, let us define a map $g : X \to X$ by the following list:

$$g := [\bot \mapsto x, x \mapsto x, y \mapsto w, w \mapsto x, z \mapsto z, \top \mapsto w].$$

This map is not isotone on $\mathbf{X}$, but it is relatively isotone on $\mathbf{X}$. Still for example the following holds:

$$\triangle \ni (x, y) \overset{g\times g}{\longmapsto} (x, w) \in \triangle, \quad \triangle \ni (y, \top) \overset{g\times g}{\longmapsto} (w, w) \in \triangle,$$
$$\triangle \ni (w, y) \overset{g\times g}{\longmapsto} (x, w) \in \triangle.$$

Nevertheless $\triangle \ni (z, \top) \overset{g\times g}{\longmapsto} (z, w) \notin \triangle$, however the following trivial implication is exactly of the form (17):

$$z \triangle \top \;\; \& \;\; z \triangle \top \;\; \& \;\; z \triangle w \quad \Rightarrow \quad z \triangle w;$$

in a similar way for example $\triangle \ni (w, z) \overset{g\times g}{\longmapsto} (x, z) \notin \triangle$, but again (17) holds trivially:

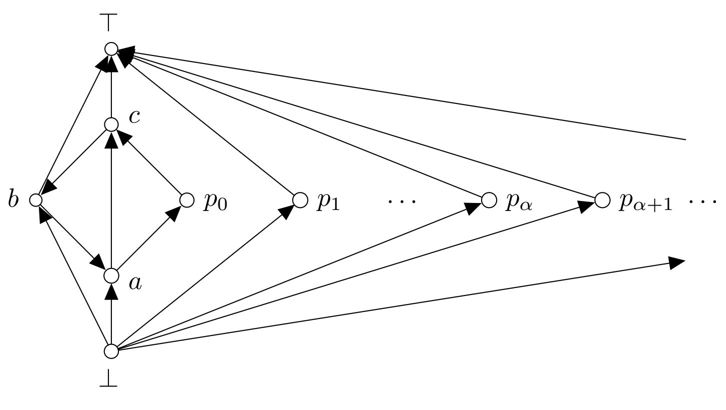
$$x \triangle z \;\; \& \;\; w \triangle z \;\; \& \;\; w \triangle z \quad \Rightarrow \quad x \triangle z.$$

If we check all the possibilities, we can see that $g : X \to X$ is really relatively isotone. Here $\mathrm{Fix}(g) = \{x, z\} \neq \emptyset$ but x and z are incomparable.

The last example has in the fuzzy setting an important meaning as a counterexample. Naturally, the universe set has to be richer here.

Example 3. Let the complete propelattice $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ be infinite, i.e. $card(X) \geq \aleph_0$. The situation in $\mathbf{X}$ is demonstrated by the diagram in Figure 3.

Figure 3: Diagram for Example 3



Let $\Omega \in On$ be an arbitrary fixed limit ordinal and let $X = \{\bot, a, b, c, \top\} \cup \{p_\alpha \,|\, \alpha \in \Omega\}$, where $\{p_\alpha \,|\, \alpha \in \Omega\}$ is an antichain. Now for fixed $\beta \in \Omega$ we define the map $f_\beta : X \to X$ in the following way: for a set $A_\beta := \{\bot, a, b, c, \top\} \cup \{p_\alpha \,|\, \alpha \leq \beta\}$ be $f''_\beta(A_\beta) := \{p_\beta\}$; for $\alpha \geq \beta + 1$ then be $f_\beta(p_\alpha) := p_\alpha$. The map f_β is obviously not isotone on $\mathbf{X}$, because for arbitrary p_α (where $\alpha \geq \beta+1$) we have $\bot \,\triangle\, p_\alpha$ and even $p_\alpha \,\triangle\, \top$ but not $p_\beta \,\triangle\, p_\alpha$ nor $p_\alpha \,\triangle\, p_\beta$ hold. On the other hand, f_β is relatively isotone on $\mathbf{X}$ de facto trivially. Here for every $x, y \in A_\beta$ the implication $x \,\triangle\, y \Rightarrow p_\beta \,\triangle\, p_\beta$ clearly holds. For $\alpha \geq \beta+1$ then p_α is comparable only with $\bot$ and $\top$ and the implications in the form of (17) hold:

$$
\begin{aligned}
p_\beta \,\triangle\, p_\alpha \;\;\&\;\; \bot \,\triangle\, p_\alpha \;\;\&\;\; \bot \,\triangle\, p_\alpha \quad &\Rightarrow \quad p_\beta \,\triangle\, p_\alpha, \\
p_\alpha \,\triangle\, \top \;\;\&\;\; p_\alpha \,\triangle\, \top \;\;\&\;\; p_\alpha \,\triangle\, p_\beta \quad &\Rightarrow \quad p_\alpha \,\triangle\, p_\beta.
\end{aligned}
$$

Hence $f_\beta : X \to X$ is indeed relatively isotone on $\mathbf{X}$. At the same time $\mathrm{Fix}(f_\beta) = \{p_\alpha \,|\, \alpha \geq \beta\} \neq \emptyset$ is a nonempty antichain.

Moreover, $F = \{f_\beta \,|\, \beta \in \Omega\} \subseteq X^X$ is a commutative set of maps. Let $\beta_1 < \beta_2 \in \Omega$ and $x \in A_{\beta_2} = \{\bot, a, b, c, \top\} \cup \{p_\alpha \,|\, \alpha \leq \beta_2\}$. Then

$$
f_{\beta_1}\left(f_{\beta_2}(x)\right) = f_{\beta_1}(p_{\beta_2}) = p_{\beta_2} = f_{\beta_2}(p_{\beta 2}) = f_{\beta_2}\left(f_{\beta_1}(x)\right).
$$

For p_α, where $\alpha \geq \beta_2 + 1$, we then have $f_{\beta_1}\left(f_{\beta_2}(p_\alpha)\right) = f_{\beta_2}\left(f_{\beta_1}(p_\alpha)\right) = p_\alpha$.

Now it can be easily shown by a contradiction that

$$
\mathrm{Fix}(F) = \bigcap_{\beta \in \Omega} \mathrm{Fix}(f_\beta) = \emptyset.
$$

Obviously $\mathrm{Fix}(F) \subseteq \{p_\alpha \,|\, \alpha \in \Omega\}$. Let $\alpha_0 \in \Omega$ be such that $p_{\alpha_0} \in \mathrm{Fix}(f_\beta)$ for every $\beta \in \Omega$. Because Ω is limit and $\alpha_0 < \Omega$, there exists $\beta_0 \in \Omega$ such that $\alpha_0 < \beta_0$ and hence $p_{\alpha_0} \notin \mathrm{Fix}(f_{\beta_0})$, which is a contradiction. Further, if $\gamma < \Omega$ and $F_\gamma := \{f_\beta \,|\, \beta \leq \gamma\} \subset F$, then also $\mathrm{Fix}(F_\gamma) = \bigcap_{\beta \leq \gamma} \mathrm{Fix}(f_\beta) = \{p_\alpha \,|\, \gamma \leq \alpha\} \neq \emptyset$, hence there exists a subset of F such that the set of common fixed points is a nonempty antichain, i.e., especially it has not a structure of a complete propelattice. Moreover, if $\Omega > \omega$ (ordinal type of the set of natural numbers) and $\omega < \gamma < \Omega$, then $\mathit{card}\,(\mathrm{Fix}(F_\gamma)) \geq \aleph_0$, which means that the set of common fixed points $\mathrm{Fix}(F_\gamma)$ is infinite.

The following facts are important for further argumentation. If we turn all the arrows up and leave the unnecessary ones in Figure 3, then we obtain a standard Hasse diagram of a complete lattice. But everything above mentioned remains **valid without any changes** even in this case, where $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ is a complete lattice.

All the examples in this part demonstrate special cases of facts which hold in a much general fuzzy setting and which are partially presented in [16] or are derived in the following. Here we have in mind the following: every relatively isotone map on a complete propelattice has its fixed point, but the set of fixed points does not need to have a structure of a complete propelattice or complete lattice (Examples 2 and 3); an isotone map on a complete propelattice then has fixed points and the structure of their set is again a complete propelattice (Examples 1 and 2).

Example 3 shows among others two important things. Firstly, relatively isotone maps are indeed a fundamental extension of isotone maps, because often a map is trivially relatively isotone but is not isotone. However, the cost for this extension is the fact that for a commutative set of relatively isotone maps on a complete propelattice, as well as on a complete lattice, none of the statements (1)–(4) from Theorem 1 holds. As we use only conservative extensions of standard notions, no analogy of Tarski's theorem for a commutative set of further introduced **L**-fuzzy relatively isotone maps can be derived in the fuzzy setting.

4. Conservative extensions, some fundamental and auxiliary facts, another example

In this section we present conservative extensions of fundamental notions from Definitions 7, 10 and 8. Of course, the main part is taken over from [16], where it is justified that all of the notions are really conservative (even if it is factually evident). The first definition extends the notions of propeorder and propeordered set.

Definition 11. ([16]) Let $\approx \in L^{X\times X}$ be an **L**-equality on X, then the **L**-relation $\triangle \in L^{X\times X}$ is *reflexive* and *antisymmetric w.r.t.* $\approx$, if for every $x, y \in X$

$$\text{(i)}\quad (x \triangle x) = \mathbf{1} \quad (\textit{reflexivity}); \tag{18}$$

$$\text{(ii)}\quad (x \triangle y) \otimes (y \triangle x) \le (x \approx y) \quad (\textit{antisymmetry (w.r.t} \approx \textit{)}). \tag{19}$$

An **L**-*propeorder on* X *(w.r.t.* $\approx$*)* is an **L**-relation $\triangle \in L^{X\times X}$ which is reflexive and antisymmetric (w.r.t. $\approx$). The pair $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ is called an **L**-*propeordered set*.

Remark 8. Let us emphasize that all the definitions in this section have of course their analogies in existing literature, where different kinds of fuzzy orders are considered. All of them just need in addition some version of transitivity (usually one of those in Definition 13) and they can more or less differ from each other [4, 8, 17]. Our definitions only fix and effectively shorten the terminology. As mentioned in the introduction, our main effort was to choose as the basis of each definition the variant with the weakest assumptions. Therefore, for example, the basis of Definition 11 was adopted from [8], whereas in [4] there is antisymmetry in spite of (19) specified by probably a more frequent but stronger condition, that for every $x, y \in X$ the inequality $(x \triangle y) \wedge (y \triangle x) \le (x \approx y)$ holds.

Especially with respect to (15), we can de facto without any formal change take over the following definitions from Bělohlávek's work [2–4]. Our terminology is only adapted to the generally supposed absence of transitivity.

Definition 12. ([16]) Let $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ be an **L**-propeordered set and $\Phi \in L^X$ be an arbitrary **L**-set. Then $\mathcal{L}(\Phi) \in L^X$ is the *lower propecone of* Φ if for every $x \in X$

$$\mathcal{L}(\Phi)(x) := \bigwedge_{y \in X} (\Phi(y) \to (x \triangle y)); \tag{20}$$

$\mathcal{U}(\Phi) \in L^X$ is the *upper propecone of* Φ if for every $x \in X$

$$\mathcal{U}(\Phi)(x) := \bigwedge_{y \in X} (\Phi(y) \to (y \triangle x)). \tag{21}$$

For any $x \in X$ we denote:

$$\mathcal{L}(x) := \mathcal{L}\left(\{{}^{\mathbf{1}}/_{x}\}\right) = \mathcal{L}(\{x\}), \quad \mathcal{U}(x) := \mathcal{U}\left(\{{}^{\mathbf{1}}/_{x}\}\right) = \mathcal{U}(\{x\}). \tag{22}$$

Propeinfimum and *propesupremum* of Φ are successively the following **L**-sets:

$$p\inf(\Phi) := \mathcal{L}(\Phi) \cap \mathcal{U}(\mathcal{L}(\Phi)), \quad p\sup(\Phi) := \mathcal{U}(\Phi) \cap \mathcal{L}(\mathcal{U}(\Phi)). \tag{23}$$

Especially we denote:

$$\bot := p\inf(X), \quad \top := p\sup(X). \tag{24}$$

Remark 9. In relation with following notions let us notice that none of the **L**-sets defined by relations (20)–(24) needs to be normal in general.

It is mentioned in Remark 3 that many notions used here have their weak as well as strong versions. Up to now we have used the weak versions of transitivity of $\approx$ in (13) and of antisymmetry of $\triangle$ in (19) and we continue in this manner. But for transitivity of $\triangle$, which is in the center of our attention, the situation is different, because the validity of our statements substantially depends on the used version of transitivity. In the same way as in Remark 3 we obtain from (7) and transitivity of the order in $\mathbf{L} = \langle L, \leq \rangle$ immediately the implication:

$$(x \triangle y) \wedge (y \triangle z) \leq (x \triangle z) \;\Rightarrow\; (x \triangle y) \otimes (y \triangle z) \leq (x \triangle z).$$

This implication justifies the following important definition (even if it has mainly terminological and systematical reasons).

Definition 13. An **L**-relation $\triangle \in L^{X \times X}$ is *weakly transitive* if for every $x, y, z \in X$

$$(x \triangle y) \otimes (y \triangle z) \leq (x \triangle z); \tag{25}$$

$\triangle \in L^{X \times X}$ is *strongly transitive* if for every $x, y, z \in X$

$$(x \triangle y) \wedge (y \triangle z) \leq (x \triangle z). \tag{26}$$

One of the main questions being solved in the paper is if transitivity can be eliminated or not and if not, then if weak transitivity is sufficient or strong transitivity is necessary. As already mentioned, every strongly transitive **L**-relation is also weakly transitive.

The following auxiliary statement, which is, except for the last part, proved in [16], is needed in the following and summarizes basic properties of all presented notions.

Proposition 1. ([16]) *Let* $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ *be an* $\mathbf{L}$*-propeordered set. Then the following statements hold.*

1. *Let* $x \in X$ *and* $\Phi \in L^X$*. Then the following implications hold:*

$$\Phi \subseteq \mathcal{L}(x) \Rightarrow p\sup(\Phi) \subseteq \mathcal{L}(x), \quad \Phi \subseteq \mathcal{U}(x) \Rightarrow p\inf(\Phi) \subseteq \mathcal{U}(x). \tag{27}$$

2. *The following equalities hold:*

$$\bot = p\inf(X) = p\sup(\emptyset) = \mathcal{L}(X), \top = p\sup(X) = p\inf(\emptyset) = \mathcal{U}(X). \tag{28}$$

3. *Let* $x \in X$ *be arbitrary. Then the inclusions hold*

$$\bot \subseteq \mathcal{L}(x), \quad \top \subseteq \mathcal{U}(x). \tag{29}$$

4. *Let* $\Phi \in L^X$ *be arbitrary. Then the following implications hold:*

$$\mathrm{Ker}(p\inf(\Phi)) \neq \emptyset \;\Rightarrow\; \exists\, x_0 \in X\colon p\inf(\Phi) = \mathcal{S}[x_0], \tag{30}$$
$$\mathrm{Ker}(p\sup(\Phi)) \neq \emptyset \;\Rightarrow\; \exists\, x_1 \in X\colon p\sup(\Phi) = \mathcal{S}[x_1]. \tag{31}$$

5. *Let for* $\Phi, \Psi \in L^X$ *be* $\Phi \subseteq \Psi, p\inf(\Psi) = \mathcal{S}[x_0], p\inf(\Phi) = \mathcal{S}[x_1], p\sup(\Phi) = \mathcal{S}[x_2]$ *and* $p\sup(\Psi) = \mathcal{S}[x_3]$*. Then* $x_0 \,{}^{\mathbf{1}}\!\triangle\, x_1$ *and* $x_2 \,{}^{\mathbf{1}}\!\triangle\, x_3$*.*

6. *Let* $\Phi \in L^X$ *be normal and* $\triangle \in L^{X\times X}$ *be weakly transitive. If* $p\inf(\Phi) = \mathcal{S}[x_0]$ *and* $p\sup(\Phi) = \mathcal{S}[x_1]$*, then* $x_0 \,{}^{\mathbf{1}}\!\triangle\, x_1$*. All the more so, the relation holds even if* $\triangle$ *is strongly transitive.*

Proof. We prove only statement 6, because the other proofs can be found in [16].

According to the assumptions and (23) we have:

$$p\inf(\Phi)(x_0) = \mathcal{L}(\Phi)(x_0) \wedge \mathcal{U}(\mathcal{L}(\Phi))(x_0) = \mathbf{1},$$
$$p\sup(\Phi)(x_1) = \mathcal{U}(\Phi)(x_1) \wedge \mathcal{L}(\mathcal{U}(\Phi))(x_1) = \mathbf{1}.$$

From here according to (20) and (21) we have

$$\mathcal{L}(\Phi)(x_0) = \bigwedge_{y\in X} (\Phi(y) \to (x_0 \triangle y)) = \mathbf{1},$$
$$\mathcal{U}(\Phi)(x_1) = \bigwedge_{y\in X} (\Phi(y) \to (y \triangle x_1)) = \mathbf{1}.$$

For every $y \in X$ hence $\Phi(y) \to (x_0 \triangle y) = \mathbf{1}$ and $\Phi(y) \to (y \triangle x_1) = \mathbf{1}$. Because $\Phi \in L^X$ is normal, i.e., $\mathrm{Ker}(\Phi) \neq \emptyset$, there exists $y_0 \in X$ such that $\Phi(y_0) = \mathbf{1}$. With respect to (8) then $(x_0 \triangle y_0) = \mathbf{1}$ and $(y_0 \triangle x_1) = \mathbf{1}$. From weak transitivity we then obtain:

$$(x_0 \triangle y_0) \otimes (y_0 \triangle x_1) = \mathbf{1} \otimes \mathbf{1} \leq (x_0 \triangle x_1) = \mathbf{1} \;\Rightarrow\; x_0 \,{}^{\mathbf{1}}\!\triangle\, x_1.$$

The same would hold also for strong transitivity. The proof is finished. □

Let us point out that any **L**-set appearing in (27)–(29) generally does not need to be normal.

Remark 10. In relation to Proposition 1, statement 5, it is suitable to realize that the relation $x_0 \, {}^{\mathbf{1}}\!\vartriangle x_3$, resp. $x_1 \, {}^{\mathbf{1}}\!\vartriangle x_2$, which would be awaited analogously to relations from lattice theory, generally does not hold. These relations are in fact substantially dependent on transitivity (see statement 6).

The following fundamental definition formulates a conservative extension of Definition 10.

Definition 14. ([16]) Let $\mathbf{X} = \langle\langle X, \approx\rangle, \vartriangle\rangle$ be an **L**-propeordered set. Then $\mathbf{X}$ is called an **L**-*complete propelattice* if for any **L**-set $\Phi \in L^X$ both the **L**-sets $p\inf(\Phi)$ and $p\sup(\Phi)$ are normal, i.e., $\mathrm{Ker}(p\inf(\Phi)) \neq \emptyset \neq \mathrm{Ker}(p\sup(\Phi))$. The **L**-relation $\vartriangle \in L^{X\times X}$ is then called an **L**-*complete propeorder on* X.

From this definition, (30) and (31) it follows that in an **L**-complete propelattice for every $\Phi \in L^X$ there exist elements $x_0, x_1 \in X$ such that

$$p\inf(\Phi) = \mathcal{S}[x_0] \quad \text{and} \quad p\sup(\Phi) = \mathcal{S}[x_1]. \tag{32}$$

The last fundamental definition formulates a conservative extension of Definition 8. Its first part is presented in [16].

Definition 15. Let $\mathbf{X} = \langle\langle X, \approx\rangle, \vartriangle\rangle$ be an **L**-propeordered set. Then a map $f : X \to X$ is **L**-*fuzzy isotone on* $\mathbf{X}$, if for every $x, y \in X$

$$(x \vartriangle y) \leq (f(x) \vartriangle f(y)). \tag{33}$$

A map $f : X \to X$ is **L**-*fuzzy relatively isotone on* $\mathbf{X}$, if for every $x, y \in X$

$$(f(x) \vartriangle y) \wedge (x \vartriangle y) \wedge (x \vartriangle f(y)) \leq (f(x) \vartriangle f(y)). \tag{34}$$

Naturally, (33) implies (34), hence every **L**-fuzzy isotone map on $\mathbf{X}$ is also **L**-fuzzy relatively isotone; the opposite does not hold.

Remark 11. It would not make a great sense to substitute in inequality (34) its left side by expression $(f(x) \vartriangle y) \otimes (x \vartriangle y) \otimes (x \vartriangle f(y))$, i.e., to consider its "weak" version. If any two members in this expression would equal to some fixed neutral $\mathcal{N}(\mathbf{L}) \in \mathbb{N}(\mathbf{L})$, then its value would be $\mathbf{0}$ according to (12), whatever the value of the third member would be.

The following theorem formulates complete a fuzzification of Tarski's theorem for a single isotone map and its great part is proved in [16]. As we will see, it presents significantly different results in comparison with those, which can be achieved by a fuzzification of Tarski's generalized theorem. Here we prove especially its first part. Moreover, its proof demonstrates the basic general idea which also applies to further parts.

Theorem 2. ([16]) *Let* $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ *be an* $\mathbf{L}$*-complete propelattice. Then the following statements hold.*

1. *If a map* $f : X \to X$ *is* $\mathbf{L}$*-fuzzy relatively isotone on* $\mathbf{X}$*, then* $\mathrm{Fix}(f) \neq \emptyset$*.*

2. *If a map* $f : X \to X$ *is* $\mathbf{L}$*-fuzzy isotone on* $\mathbf{X}$*, then* $\mathrm{Fix}(f) \neq \emptyset$*. Moreover, the system*

$$\mathbf{F}_f := \langle\langle \mathrm{Fix}(f), =\rangle, {}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(f) \times \mathrm{Fix}(f)]\rangle \tag{35}$$

   is a complete propelattice. Especially, for $\mathbf{L}$*-fuzzy isotone map* f *on* $\mathbf{X}$ *there exist its propeleast and propegratest fixed point.*

3. *If a map* $f : X \to X$ *is* $\mathbf{L}$*-fuzzy isotone on* $\mathbf{X}$ *and the* $\mathbf{L}$*-complete propeorder* $\triangle \in L^{X\times X}$ *is weakly transitive, then the complete propelattice* $\mathbf{F}_f$ *in* (35) *is a complete lattice.*

4. *If a map* $f : X \to X$ *is* $\mathbf{L}$*-fuzzy isotone on* $\mathbf{X}$ *and the* $\mathbf{L}$*-complete propeorder* $\triangle \in L^{X\times X}$ *is strongly transitive, then* $\mathbf{F}_f$ *in* (35) *is a complete lattice. Futher, let us define* $\mathbf{L}$*-sets* $\Phi, \Psi \in L^X$ *for every* $x \in X$ *in this way:*

$$\Phi(x) := (f(x) \triangle x), \quad \Psi(x) := (x \triangle f(x)). \tag{36}$$

   Then the $\mathbf{L}$*-sets* Φ *and* Ψ *are normal. If* $p\inf(\Phi) = \mathcal{S}[x_0]$ *and* $p\sup(\Psi) = \mathcal{S}[x_1]$*, then*

$$x_0 = \min(\mathrm{Fix}(f)), \quad x_1 = \max(\mathrm{Fix}(f)), \tag{37}$$

   where the minimum and maximum are taken with respect to the order ${}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(f) \times \mathrm{Fix}(f)]$*.*

Proof. We prove only statements 1 and 3, the proofs of the other parts can be found in [16].

Ad 1. For an arbitrary but fixed $\mathbf{L}$-fuzzy relatively isotone map $f : X \to X$ on $\mathbf{X}$ we construct a point $x_0 \in \mathrm{Fix}(f)$, which means $\mathrm{Fix}(f) \neq \emptyset$.

Let $\mathcal{F} := \{\Phi_\lambda \in L^X \mid \lambda \in \Lambda\} \subseteq L^X$ be the indexed system of **all** the $\mathbf{L}$-sets such that every $\Phi_\lambda \in \mathcal{F}$ fulfills the following two conditions:

(a) $\Phi_\lambda \subseteq \Phi_\lambda \circ f$, i.e., $\forall y \in X : \Phi_\lambda(y) \leq \Phi_\lambda(f(y))$;

(b) $\forall \phi \in L^X : \phi \subseteq \Phi_\lambda \Rightarrow p\inf(\phi) \subseteq \Phi_\lambda$.

Firstly, $\mathcal{F} \neq \emptyset$. For every $y \in X$ we have $X(y) = X(f(y)) = \mathbf{1}$ and for an arbitrary $\phi \in L^X$ we have $\phi \subseteq X$ as well as $p\inf(\phi) \subseteq X$, i.e., $X \in \mathcal{F}$. Now let us define the following $\mathbf{L}$-set:

$$\Phi := \bigcap \mathcal{F} = \bigcap_{\lambda\in\Lambda} \Phi_\lambda \neq \emptyset. \tag{38}$$

Indeed, $\Phi \neq \emptyset$. If we choose $\phi := \emptyset$ then according to (b) $\top = p\inf(\emptyset) \subseteq \Phi_\lambda$ for every $\lambda \in \Lambda$. Then, because $\mathbf{X}$ is an $\mathbf{L}$-complete propelattice, according to (32) there exists $x^* \in X$ such that $\top = \mathcal{S}[x^*] \neq \emptyset$, i.e., $\emptyset \neq \top \subseteq \Phi$.

Now we show that Φ fulfills both the conditions (a) and (b), which means that $\Phi \in \mathcal{F}$ is the least element of $\mathcal{F}$ in the complete lattice $\langle L^X, \subseteq \rangle = \langle L^X, \cup, \cap, \emptyset, X \rangle$. For any $\lambda \in \Lambda$ and $y \in X$ we have

$$\Phi(y) = \left(\bigcap_{\lambda \in \Lambda} \Phi_\lambda \right)(y) = \bigwedge_{\lambda \in \Lambda} \Phi_\lambda(y) \leq \Phi_\lambda(y) \leq \Phi_\lambda(f(y)),$$

which leads to

$$\Phi(y) \leq \bigwedge_{\lambda \in \Lambda} \Phi_\lambda(f(y)) = \left(\bigcap_{\lambda \in \Lambda} \Phi_\lambda \right)(f(y)) = \Phi(f(y)), \tag{39}$$

and hence $\Phi \subseteq \Phi \circ f$, i.e., the condition (a) is fulfilled by Φ. Further, let $\phi \subseteq \Phi$. Then according to (38) $\phi \subseteq \Phi_\lambda$ for every $\lambda \in \Lambda$ and thus $p\inf(\phi) \subseteq \bigcap_{\lambda \in \Lambda} \Phi_\lambda = \Phi$. Hence Φ satisfies also the condition (b). In total one can observe that in the complete lattice $\langle L^X, \subseteq \rangle$ the following fundamental equality holds:

$$\Phi = \inf_{\subseteq}(\mathcal{F}) = \min_{\subseteq}(\mathcal{F}) \in \mathcal{F}. \tag{40}$$

Because $\Phi \subseteq \Phi \in \mathcal{F}$, from the property (b) we get $p\inf(\Phi) \subseteq \Phi$. According to (32) there exists $x_0 \in X$ such that $p\inf(\Phi) = \mathcal{S}[x_0]$. Thus according to (b) and (a) we have $\mathbf{1} = p\inf(\Phi)(x_0) \leq \Phi(x_0) \leq \Phi(f(x_0))$, which gives $\Phi(f(x_0)) = \mathbf{1}$. In addition to that, according to (23)

$$\mathcal{S}[x_0] = p\inf(\Phi) = \mathcal{L}(\Phi) \cap \mathcal{U}(\mathcal{L}(\Phi)). \tag{41}$$

From here with respect to (20) we have

$$\mathcal{L}(\Phi)(x_0) = \bigwedge_{y \in X} (\Phi(y) \to (x_0 \vartriangle y)) = \mathbf{1},$$

i.e., $\Phi(y) \to (x_0 \vartriangle y) = \mathbf{1}$ for any $y \in X$. With respect to relation (8) we get the following important inequality:

$$\Phi(y) \leq (x_0 \vartriangle y), \quad \forall y \in X. \tag{42}$$

For the special choice $y := f(x_0)$ in (42) we obtain

$$\mathbf{1} = \Phi(f(x_0)) \leq (x_0 \vartriangle f(x_0)),$$

and

$$(x_0 \vartriangle f(x_0)) = \mathbf{1}. \tag{43}$$

From here with respect to antisymmetry (19) and separation (14) it is obvious that now it is enough to prove the equality $(f(x_0) \vartriangle x_0) = \mathbf{1}$. For this purpose let us introduce the **L**-set (see Definition 12, (22))

$$\Phi^{\mathcal{U}} := \Phi \cap \mathcal{U}(f(x_0)). \tag{44}$$

We have to show that $\Phi = \Phi^{\mathcal{U}}$. The inclusion $\Phi^{\mathcal{U}} \subseteq \Phi$ is obvious. The opposite inclusion is to be proved in such a way that we show that $\Phi^{\mathcal{U}}$ satisfies both the conditions (a) and (b). Then with respect to minimality of Φ in $\mathcal{F}$ according to (40) the relation $\Phi \subseteq \Phi^{\mathcal{U}}$ has to hold.

According to (44) and (39) for every $y \in X$ we have:

$$\begin{aligned} \Phi^{\mathcal{U}} &= (\Phi \cap \mathcal{U}(f(x_0)))\,(y) = \Phi(y) \wedge \mathcal{U}(f(x_0))(y) \leq \\ &\leq \Phi(y) \leq \Phi(f(y)). \end{aligned} \tag{45}$$

Further, according to (45) and (42) we obtain:

$$\Phi^{\mathcal{U}}(y) \leq \Phi(y) \leq \Phi(f(y)) \leq (x_0 \triangle f(y)). \tag{46}$$

Now with respect to (22), (21) and (8) for any $y \in X$ the following holds:

$$\begin{aligned} \mathcal{U}(f(x_0))(y) &= \bigwedge_{z \in X} \left(\{\mathbf{1}/_{f(x_0)}\}(z) \to (z \triangle y)\right) = \\ &= \bigwedge_{z \neq f(x_0)} (\mathbf{0} \to (z \triangle y)) \wedge (\mathbf{1} \to (f(x_0) \triangle y)) = \\ &= \bigwedge_{z \neq f(x_0)} \mathbf{1} \wedge (\mathbf{1} \to (f(x_0) \triangle y)) = (f(x_0) \triangle y). \end{aligned} \tag{47}$$

Hence according to (44) and (47) for every $y \in X$ we have:

$$\Phi^{\mathcal{U}}(y) \leq \mathcal{U}(f(x_0))(y) = (f(x_0) \triangle y). \tag{48}$$

Finally with respect to (44) and (42) obviously:

$$\Phi^{\mathcal{U}}(y) \leq \Phi(y) \leq (x_0 \triangle y). \tag{49}$$

From (48), (49) and (46) we obtain

$$\Phi^{\mathcal{U}}(y) \leq (f(x_0) \triangle y) \wedge (x_0 \triangle y) \wedge (x_0 \triangle f(y)). \tag{50}$$

Because the map $f : X \to X$ is **L**-fuzzy relatively isotone on **X**, we get with respect to (34) the following inequality:

$$(f(x_0) \triangle y) \wedge (x_0 \triangle y) \wedge (x_0 \triangle f(y)) \leq (f(x_0) \triangle f(y)). \tag{51}$$

From (50) and (51) with respect to (47) we finally obtain the inequality:

$$\Phi^{\mathcal{U}}(y) \leq (f(x_0) \triangle f(y)) = \mathcal{U}(f(x_0))(f(y)). \tag{52}$$

For every $y \in X$ hence according to (52) and the fact that the **L**-set Φ satisfies the condition (a), the following inequality holds:

$$\begin{aligned} \Phi^{\mathcal{U}}(y) &= \Phi(y) \wedge \mathcal{U}(f(x_0))(y) \leq \Phi(f(y)) \wedge \mathcal{U}(f(x_0))(f(y)) = \\ &= (\Phi \cap \mathcal{U}(f(x_0)))(f(y)) = \Phi^{\mathcal{U}}(f(y)), \end{aligned}$$

which means that the **L**-set $\Phi^{\mathcal{U}}$ fulfills the condition (a).

Now we have to justify for $\Phi^{\mathcal{U}}$ the condition (b). Let for some $\phi \in L^X$ hold: $\phi \subseteq \Phi^{\mathcal{U}} = \Phi \cap \mathcal{U}(f(x_0))$. Then at the same time $\phi \subseteq \Phi$ and also $\phi \subseteq \mathcal{U}(f(x_0))$. Because $\Phi \in \mathcal{F}$, then $p\inf(\phi) \subseteq \Phi$. According to implications (27) even $p\inf(\phi) \subseteq \mathcal{U}(f(x_0))$, which gives

$$p\inf(\phi) \subseteq \Phi \cap \mathcal{U}(f(x_0)) = \Phi^{\mathcal{U}}$$

and this means that $\Phi^{\mathcal{U}}$ satisfies also the condition (b).

Because the **L**-set $\Phi^{\mathcal{U}}$ fulfills both the conditions (a) and (b), we have that $\Phi^{\mathcal{U}} \in \mathcal{F}$. From (40) we then get the inclusion $\Phi \subseteq \Phi^{\mathcal{U}}$ and because the opposite inclusion is trivial, we finally obtain the equality:

$$\Phi^{\mathcal{U}} = \Phi. \tag{53}$$

From (41), (53) and (44) we then have the following chain of inclusions:

$$\mathcal{S}[x_0] = p\inf(\Phi) = p\inf(\Phi^{\mathcal{U}}) \subseteq \Phi^{\mathcal{U}} = \Phi \cap \mathcal{U}(f(x_0)) \subseteq \mathcal{U}(f(x_0)).$$

From here according to (47) for the special choice $y := x_0$ we obtain:

$$\mathcal{S}[x_0](x_0) = \mathbf{1} \leq \mathcal{U}(f(x_0))(x_0) = (f(x_0) \bigtriangleup x_0),$$

hence

$$(f(x_0) \bigtriangleup x_0) = \mathbf{1}. \tag{54}$$

As a final consequence, according to (43) and (54) with respect to antisymmetry of $\bigtriangleup$ w.r.t $\approx$ (19) and to the subsequent separation (14), we obtain the next chain of implications:

$$\begin{aligned} &\mathbf{1} = \mathbf{1} \otimes \mathbf{1} = (x_0 \bigtriangleup f(x_0)) \otimes (f(x_0) \bigtriangleup x_0) \leq (x_0 \approx f(x_0)) \quad \Rightarrow \\ &\Rightarrow \quad (x_0 \approx f(x_0)) = \mathbf{1} \quad \Rightarrow \quad x_0 = f(x_0). \end{aligned}$$

Hence $x_0 \in \mathrm{Fix}(f)$, so $\mathrm{Fix}(f) \neq \emptyset$ and statement 1 from Theorem 2 is proved.

For the sake of completeness let us add that a fixed point of the map $f : X \to X$ could be constructed by a dual approach in the following way. At first we would introduce $\mathcal{P} := \{\Psi_\lambda \in L^X \mid \lambda \in \Lambda\} \subseteq L^X$ as the indexed system of **all** the **L**-sets such that every Ψ_λ satisfies the following two conditions:

(aa) $\Psi_\lambda \subseteq \Psi_\lambda \circ f$, i.e., $\forall y \in X : \Psi_\lambda(y) \leq \Psi_\lambda(f(y))$;

(bb) $\forall \phi \in L^X : \phi \subseteq \Psi_\lambda \Rightarrow p\sup(\phi) \subseteq \Psi_\lambda$.

Then we would introduce in the same manner the **L**-sets $\Psi := \bigcap \mathcal{P} = \bigcap_{\lambda \in \Lambda} \Psi_\lambda \neq \emptyset$ and $\Psi^{\mathcal{L}}$. All further considerations would then be analogous and they would lead to the existence of a certain (generally different) fixed point $x_1 \in \mathrm{Fix}(f)$, i.e., again we would find that $\mathrm{Fix}(f) \neq \emptyset$.

Ad 3. With respect to statement 2 of the theorem and the fact that in Definitions 7 and 10 none of the notions is dependent on transitivity, it is sufficient to show that the propeorder ${}^{\mathbf{1}}\triangle \subseteq X \times X$ is transitive (in the usual sence). Let for $x, y, z \in X$ be $x\, {}^{\mathbf{1}}\triangle\, y$ and $y\, {}^{\mathbf{1}}\triangle\, z$. Then according to the definition, $(x \triangle y) = \mathbf{1}$ and $(y \triangle z) = \mathbf{1}$ and according to inequality (25) we get

$$\mathbf{1} = \mathbf{1} \otimes \mathbf{1} = (x \triangle y) \otimes (y \triangle z) \leq (x \triangle z).$$

From here $(x \triangle z) = \mathbf{1}$, i.e., $x\, {}^{\mathbf{1}}\triangle\, z$. Hence ${}^{\mathbf{1}}\triangle$ is a transitive relation and $\mathbf{F}_f$ is a complete lattice.

As it was already mentioned, the other two statements 2 and 4 are proved in [16]. Thus the proof is finished. □

Remark 12. (a) An interesting question arises in relation to the proof of statement 1 from Theorem 2, i.e., if the fixed points x_0 and x_1 are different from each other in the case, where $card(\mathrm{Fix}(f)) \geq 2$. As proved in [16], if $f : X \to X$ is $\mathbf{L}$-fuzzy isotone on $\mathbf{X}$, then $x_0 = p\max(\mathrm{Fix}(f))$ and $x_1 = p\min(\mathrm{Fix}(f))$, where the propegreatest and propeleast elements are taken in the propeorder ${}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(f)]^2$ and thus necessarily $x_0 \neq x_1$. However the situation is different in Example 3. Let us consider any from maps $f_\beta : X \to X$. With respect to minimality of the sets Φ and Ψ from the proof (in this case in $\langle \mathrm{Exp}(X), \subseteq \rangle$) obviously $A_\beta \in \mathcal{F}$, $A_\beta \in \mathcal{P}$ and $\Phi, \Psi \subseteq A_\beta$ and in this case for both the fixed points we have $p\inf(\Phi), p\sup(\Psi) \in A_\beta$. Because $\mathrm{Fix}(f_\beta) \cap A_\beta = \{p_\beta\}$, the following has to hold: $p\inf(\Phi) = p_\beta = p\sup(\Psi)$. Hence both the fixed points blend in one. (Although now $\mathbf{L} = \mathbf{2}$, the same could be observed also in the fuzzy setting, see the following Lemma 1.)

(b) The proof of statement 1 demonstrates a general idea of all the proofs of the other statements from Theorem 2. Always an $\mathbf{L}$-set of desired properties is introduced, which is the least in the complete lattice $\langle L^X, \subseteq \rangle = \langle L^X, \cup, \cap, \emptyset, X \rangle$. Completeness of this lattice is here fundamental. This is the main reason why $\mathbf{L} = \langle L, \vee, \wedge, \otimes, \to, \mathbf{0}, \mathbf{1} \rangle$ has to be a complete lattice. The same method is also used in the proof of the following Proposition 2.

Theorem 2 implies the following facts. If $f : X \to X$ is $\mathbf{L}$-fuzzy *relatively* isotone on an $\mathbf{L}$-complete propelattice $\mathbf{X} = \langle \langle X, \approx \rangle, \triangle \rangle$, then $\mathrm{Fix}(f) \neq \emptyset$. However, as it is shown in Example 3, no specific structure of $\mathrm{Fix}(f)$ can be awaited, whatever the $\mathbf{L}$-propeorder $\triangle \in L^{X \times X}$ is like, regardless if it is weakly or strongly transitive or even not transitive at all.

It is obvious that statements 2 and 4 of Theorem 2 provide a complete fuzzification of Tarski's theorem on fixed points of a single isotone map. These statements are surprising because of the following reasons. Let us start with statement 4. For validity of this statement strong transitivity is necessary and it cannot be replaced by the weak one, because, as mentioned in [16], there exist weakly transitive $\mathbf{L}$-complete propeorders, where equalities (37) do not hold. Obviously, for the $\mathbf{L}$-sets $\Phi, \Psi \in L^X$ in (36), with respect to reflexivity (18), inclusions $\mathrm{Fix}(f) \subseteq \Phi$ as well as $\mathrm{Fix}(f) \subseteq \Psi$ hold. Then regardless if the $\mathbf{L}$-relation $\triangle \in L^{X \times X}$ is weakly transitive or is not transitive at all, the following relations always hold, according to Proposition 1 (statement 5) and the definition of the propeleast and propegreatest element:

$$x_0 \, {}^{\mathbf{1}}\!\triangle \, p\min(\mathrm{Fix}(f)), \quad p\max(\mathrm{Fix}(f)) \, {}^{\mathbf{1}}\!\triangle \, x_1, \qquad (55)$$
$$p\min(\mathrm{Fix}(f)) \, {}^{\mathbf{1}}\!\triangle \, p\max(\mathrm{Fix}(f)).$$

Statement 2 of Theorem 2 is much more interesting, especially because it holds surprisingly without any transitivity. In addition, $\mathbf{F}_f$ in (35) is a **crisp** complete propelattice. This its "exceptionality" however can be seen only in a logically richer setting; in the setting determined by $\mathbf{L} = \mathbf{2}$ it cannot be observed at all. It might seem that this exceptionality has a connection to the result [4], which shows that every *lattice fuzzy order* is uniquely determined by its **1**-cut. Nevertheless, we need to realize that its author utilizes in the proofs three fundamental properties: firstly the "strong version" of antisymmetry in place of the weak one (19); secondly weak transitivity, whereas we generally suppose no transitivity; thirdly we do not suppose the so-called *compatibility of* the **L**-*relation* $\triangle \in L^{X\times X}$ *w.r.t.* $\approx$ (for more detail see the following Remark 13). As we show in the following, an **L**-complete propeorder is not determined uniquely by its **1**-cut (Example 4).

Finally let us point out that $\mathbf{F}_f = \langle\langle \mathrm{Fix}(f), =\rangle, {}^{\mathbf{1}}\!\triangle \cap [\mathrm{Fix}(f) \times \mathrm{Fix}(f)]\rangle$ is according to statement 3 a complete lattice, if $\triangle \in L^{X\times X}$ is weakly transitive. However, as we already know, weak transitivity still does not suffice for the validity of equalities (37).

To sum up the previous, we can say that the most important statement in Theorem 2 is statement 2. This statement fuzzifies and at the same time widely generalizes fundamental parts of the classic Tarski's theorem for a single isotone map. It says that **without any assumption of transitivity** the set of fixed points is nonempty and its structure is a complete propelattice. Especially, the propeleast, as well as propegreatest, element exist. (The only cost which is paid for such a generality is the fact that we do not have explicit formulas for their expression. Nevertheless, the usability of such explicit formulas can be sometimes problematic even in the crisp setting.)

The following auxiliary statement enables to construct trivially illustrative examples of **L**-complete propelattices and among others to demonstrate that an **L**-complete propeorder does not need to be uniquely determined by its **1**-cut. Moreover, together with the following Proposition 2 we will be able to present a (almost) complete discussion of the necessity of the assumption of a relevant version of transitivity for the validity of single statements of Theorem 1 in a fuzzy setting. (For the notation used in the following lemma, see Definition 5).

Lemma 1. *Let* $\mathbf{L} = \mathbf{L}_N$ *be a* 3*-valued Lukasiewics algebra and* $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ *be a complete propelattice. Let the* $\mathbf{L}_N$*-equality* $\approx_N \in (L_N)^{X\times X}$ *be defined for any* $x, y, z \in X$ *in the following way:*

$$(x \approx_N y) := \mathbf{1} \;\Leftrightarrow\; x = y,$$
$$(x \approx_N y) := N \;\Leftrightarrow\; x \neq y; \qquad (56)$$

and further the $\mathbf{L}_N$*-propeorder* $\triangle_N \in (L_N)^{X\times X}$ *(w.r.t.* $\approx_N$*) on* X *by the following way:*

$$\begin{aligned} &(x \triangle_N x) := \mathbf{1}, \\ &(x \triangle_N y) := \mathbf{1} \;\&\; (y \triangle_N x) := N \;\;\Leftrightarrow\;\; x \triangle y \;\&\; x \neq y, \\ &(x \triangle_N y) := N \;\&\; (y \triangle_N x) := N \;\;\Leftrightarrow\;\; x \parallel y. \end{aligned} \tag{57}$$

Then $\mathbf{X}_N := \langle\langle X, \approx_N\rangle, \triangle_N\rangle$ *is an* $\mathbf{L}_N$*-complete propelattice, where* $\triangle = {}^{\mathbf{1}}\triangle_N$*. Simultaneously, for any* $\Phi \in (L_N)^X$ *we have*

$$p\inf_{\triangle_N}(\Phi) = \mathcal{S}[p\inf_{\triangle}(\mathrm{Ker}(\Phi))], \quad p\sup_{\triangle_N}(\Phi) = \mathcal{S}[p\sup_{\triangle}(\mathrm{Ker}(\Phi))]. \tag{58}$$

If $\triangle \subseteq X \times X$ *is transitive, then* $\triangle_N \in (L_N)^{X\times X}$ *is strongly transitive; if* $\triangle$ *is not transitive, then* $\triangle_N$ *is not even weakly transitive. Moreover, an arbitrary map* $f : X \to X$*, which is (relatively) isotone on* $\mathbf{X}$*, is* $\mathbf{L}_N$*-fuzzy (relatively) isotone on* $\mathbf{X}_N$*.*

Proof. Firstly, $\approx_N \in (L_N)^{X\times X}$ is with respect to relations (56) and according to Definition 6 reflexive and symmetric and obviously even the condition of separation (14) holds. Transitivity of $\approx_N$ then holds according to (12). The $\mathbf{L}_N$-relation $\triangle_N \in (L_N)^{X\times X}$ is with respect to (57) reflexive in the form of (18). With respect to (57) and (12), the relation $\triangle_N$ is also antisymmetric w.r.t $\approx_N$ in the form of (19).

For the sake of brevity, we will no more specify the lower indices of propeinfima and propesuprema. If we show that equalities (58) hold, at the same time we prove also that $\mathbf{X}_N$ is an $\mathbf{L}_N$-complete propelattice. We verify the validity of the first equality, the second one is dual. Let $\Phi := \bigcup_{x\in A}\{{}^{\mathbf{1}}/_x\} \cup \bigcup_{x\in B}\{{}^{N}/_x\} \in (L_N)^X$, where $A, B \subseteq X$, $A \cap B = \emptyset$ and eventually $A \neq \emptyset \neq B$. Here $\mathrm{Ker}(\Phi) = A$. We show that if $\{x_0\} = p\inf(A) = \mathcal{L}(A) \cap \mathcal{U}(\mathcal{L}(A))$ in $\mathbf{X}$, then $p\inf(\Phi) = \mathcal{S}[x_0]$. With respect to (57), the N-cut is ${}^{N}(\triangle_N) = X \times X$ and then for every $y \in X$ according to (8) we have:

$$\begin{aligned} \mathcal{L}(\Phi)(y) &= \bigwedge_{x\in A} (\Phi(x) \to (y \triangle x)) \wedge \bigwedge_{x\in B} (\Phi(x) \to (y \triangle x)) = \\ &= \bigwedge_{x\in A} (A(x) \to (y \triangle x)) \wedge \mathbf{1} = \bigwedge_{x\in A} (A(x) \to (y \triangle x)). \end{aligned}$$

From here for $y \in \mathcal{L}(A)$ immediately $\mathcal{L}(\Phi)(y) = \mathbf{1}$ and for $y \notin \mathcal{L}(A)$ then $\mathcal{L}(\Phi)(y) = N$ and hence

$$\mathcal{L}(\Phi) = \bigcup_{x\in\mathcal{L}(A)} \{{}^{\mathbf{1}}/_x\} \;\cup \bigcup_{x\notin\mathcal{L}(A)} \{{}^{N}/_x\}. \tag{59}$$

By an analogous consideration one would obtain also

$$\mathcal{U}(\mathcal{L}(\Phi)) = \bigcup_{x\in\mathcal{U}(\mathcal{L}(A))} \{{}^{\mathbf{1}}/_x\} \;\cup \bigcup_{x\notin\mathcal{U}(\mathcal{L}(A))} \{{}^{N}/_x\}. \tag{60}$$

From (59) and (60) we have that $p\inf(\Phi)(x_0) = \mathcal{L}(\Phi)(x_0) \wedge \mathcal{U}(\mathcal{L}(\Phi))(x_0) = \mathbf{1}$ and thus with respect to implication (30) also $p\inf(\Phi) = \mathcal{S}[x_0] = \mathcal{S}[p\inf(\mathrm{Ker}(\Phi))]$ has to hold. Finally, we observe that $\mathbf{X}_N = \langle\langle X, \approx_N\rangle, \triangle_N\rangle$ is an $\mathbf{L}_N$-complete propelattice and according to (57) clearly $\triangle = {}^{\mathbf{1}}\triangle_N$.

Further, let $\triangle \subseteq X \times X$ be transitive and $x, y, z \in X$ be arbitrary. If $(x \triangle_N y) = N$ or if $(y \triangle_N z) = N$, then because ${}^N(\triangle_N) = X \times X$ holds, also (26) holds. Of course, if $x \triangle y$ and $y \triangle z$ then inequality (26) holds too.

Contrarily, if the relation $\triangle$ is not transitive, then there exist $x, y, z \in X$ such that $x \triangle y$ and $y \triangle z$; but either $z \triangle x$ or $x \parallel z$. In both the cases hence $(x \triangle_N z) = N$ and at the same time $(x \triangle_N y) = \mathbf{1}$ and $(y \triangle_N z) = \mathbf{1}$. Hence the following inequality holds:

$$(x \triangle_N y) \otimes (y \triangle_N z) = \mathbf{1} \otimes \mathbf{1} > N = (x \triangle_N z).$$

Thus inequality (25) does not hold and therefore $\triangle_N \in (L_N)^{X\times X}$ is not weakly transitive.

Finally, let $f : X \to X$ be relatively isotone on $\mathbf{X}$. The only case which is worth mentioning arises in the situation when all the three relations hold together: $f(x) \triangle y, x \triangle y$ and $x \triangle f(y)$. Then we have

$$(f(x) \triangle_N y) \wedge (x \triangle_N y) \wedge (x \triangle_N f(y)) = \mathbf{1} \leq (f(x) \triangle_N f(y)) = \mathbf{1}.$$

The other eventualities follow from (57), or more precisely from the fact that ${}^N(\triangle_N) = X \times X$, and hence f is $\mathbf{L}_N$-fuzzy relatively isotone. The case, where $f : X \to X$ is isotone, is even simpler. The proof is complete. □

The following example together with Lemma 1 show above others that an **L**-complete propeorder is not uniquely determined by its **1**-cut.

Example 4. Let us have $X = \{\bot, x, y, z, \top\}$ and let $\mathbf{X} = \langle\langle X, \approx_N\rangle, \triangle\rangle$ be an $\mathbf{L}_N$-complete propelattice. A diagram of its "skeleton" is given in Figure 4.

Now, the $\mathbf{L}_N$-complete propeorder $\triangle \in (L_N)^{X\times X}$ is uniquely given by the following two conditions (not according to Lemma 1):

$${}^{\mathbf{1}}\triangle = \{(\bot, x), (\bot, y), (\bot, z), (\bot, \top), (x, y), (y, z), (x, \top), (y, \top), (z, \top)\} \cup id_X;$$

for every $u, v \in X$ we have

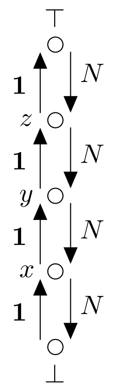
$$\begin{aligned}
&(u \triangle u) := \mathbf{1}, \\
&(u \triangle v) := \mathbf{1} \quad \& \quad (v \triangle u) := N \quad \Leftrightarrow \quad u\,{}^{\mathbf{1}}\triangle\, v \quad \& \quad u \neq v, \\
&(x \triangle z) := \mathbf{0} \quad \& \quad (z \triangle x) := \mathbf{0}.
\end{aligned}$$

Obviously, the $\mathbf{L}_N$-propeorder $\triangle$ is not strongly nor weakly transitive.

It can be easily seen that $\mathbf{X} = \langle\langle X, \approx_N\rangle, \triangle\rangle$ is $\mathbf{L}_N$-complete. We simply compute for example that:

$$\begin{aligned}
\Phi &= \{{}^{\mathbf{1}}/_{\bot}, {}^{\mathbf{1}}/_{x}, {}^{\mathbf{1}}/_{y}\} \quad \Rightarrow \quad p\inf(\Phi) = \mathcal{S}[\bot], \\
\Phi &= \{{}^{N}/_{\bot}, {}^{\mathbf{1}}/_{x}, {}^{\mathbf{1}}/_{y}\} \quad \Rightarrow \quad p\inf(\Phi) = \mathcal{S}[x], \\
\Phi &= \{{}^{N}/_{\bot}, {}^{N}/_{x}, {}^{\mathbf{1}}/_{y}\} \quad \Rightarrow \quad p\inf(\Phi) = \mathcal{S}[y], \\
\Phi &= \{{}^{N}/_{\bot}, {}^{N}/_{x}, {}^{N}/_{y}\} \quad \Rightarrow \quad p\inf(\Phi) = p\inf(\emptyset) = \mathcal{S}[\top].
\end{aligned}$$

Figure 4: Diagram of the "skeleton" of $\mathbf{X}$ for Example 4



It can be justified that in this case also relations (58) hold.

Nevertheless, for example for the $\mathbf{L}_N$-set $\Phi := \{^{\mathbf{1}}/_x, ^{N}/_y, ^{\mathbf{1}}/_z\}$ we have in $\mathbf{X}$:

$$p\sup(\Phi) = \mathcal{S}[\top] = \{^{N}/_\bot, ^{\mathbf{0}}/_x, ^{N}/_y, ^{\mathbf{0}}/_z, ^{\mathbf{1}}/_\top\}. \tag{61}$$

But on the other hand, in $\mathbf{X}_N = \langle\langle X, \approx_N\rangle, (^{\mathbf{1}}\triangle)_N\rangle$ we have according to (59) and (60) that

$$p\sup(\Phi) = \mathcal{S}[\top] = \{^{N}/_\bot, ^{N}/_x, ^{N}/_y, ^{N}/_z, ^{\mathbf{1}}/_\top\}. \tag{62}$$

In (61) and (62) there are two SC-singletons at the same point $\top \in X$, but they have different membership functions.

Only for the sake of completeness of the demonstration of introduced notions let us notice that the map $f : X \to X$ with the list

$$f := [\bot \mapsto x, x \mapsto y, y \mapsto z, z \mapsto \top, \top \mapsto \top]$$

is clearly $\mathbf{L}_N$-fuzzy isotone on $\mathbf{X}$. The map $g : X \to X$ given by the list

$$g := [\bot \mapsto \bot, x \mapsto x, y \mapsto z, z \mapsto \bot, \top \mapsto \top]$$

is not $\mathbf{L}_N$-fuzzy isotone on $\mathbf{X}$ (which is shown for example by inequality (63), because for the same elements $x, y \in X$ the inequality analogous to (33) does not hold), however, it is $\mathbf{L}_N$-fuzzy relatively isotone. Clearly, the following relations successively hold (and the other cases are in fact trivial):

$$\begin{aligned}
&(x \triangle y) \wedge (x \triangle y) \wedge (x \triangle z) \leq (x \triangle z), \\
&(z \triangle z) \wedge (y \triangle z) \wedge (y \triangle \bot) = \mathbf{1} \wedge \mathbf{1} \wedge N \leq (z \triangle \bot) = N, \\
&(z \triangle x) \wedge (y \triangle x) \wedge (y \triangle x) \leq (z \triangle x), \\
&(\bot \triangle y) \wedge (z \triangle y) \wedge (z \triangle z) = \mathbf{1} \wedge N \wedge \mathbf{1} \leq (\bot \triangle z) = \mathbf{1}.
\end{aligned} \tag{63}$$

In all these and also other cases hence inequality (34) holds.

Further simple examples of extensions of the notions which were introduced in this part into a fuzzy setting could be obtained by the application of Lemma 1 onto Examples 1–3. This lemma also implies that in Example 4 the propeordered sets $\mathbf{X} = \langle\langle X, \approx_N\rangle, \triangle\rangle$ and $\mathbf{X}_N = \langle\langle X, \approx_N\rangle, ({}^{\mathbf{1}}\triangle)_N\rangle$ are two different $\mathbf{L}_N$-complete propelattices, especially $\triangle \neq ({}^{\mathbf{1}}\triangle)_N$ are two different $\mathbf{L}_N$-complete propeorders. Nevertheless, at the same time ${}^{\mathbf{1}}\triangle = {}^{\mathbf{1}}\big(({}^{\mathbf{1}}\triangle)_N\big)$. From here it follows that an **L**-complete propeorder generally is not uniquely determined by its **1**-cut. Above all this is the fact that represents the difference between our concept and the concept of the *completely lattice* **L***-order* from [2–4].

Remark 13. It is important to realize that the $\mathbf{L}_N$-equality $\approx_N \in (L_N)^{X\times X}$ in Lemma 1 is introduced by relations (56) in such a way, that the $\mathbf{L}_N$-propeorder $\triangle_N \in (L_N)^{X\times X}$ in (57) is then antisymmetric w.r.t. $\approx_N$ according to (19) (let us point out that it is not antisymmetric w.r.t. the classical identity $= \equiv id_X$ on X). In general, the **L**-relation $\triangle \in L^{X\times X}$ is *compatible w.r.t.* **L***-equality* $\approx \in L^{X\times X}$ [2–4, 8], if for any $x, y, u, v \in X$ holds:

$$(u \approx x) \otimes (x \triangle y) \otimes (y \approx v) \leq (u \triangle v).$$

This property certainly has very epistemiological core but it is also technically very effective and the proof of the result in [4] would not be possible without it. However, compatibility of the **L**-propeorder is not supposed and a consequence of this fact is that the **L**-complete propeorder is not uniquely determined by its **1**-cut. For example the $\mathbf{L}_N$-propeorder $\triangle_N$ from Lemma 1 is obviously compatible w.r.t $\approx_N$, but $\triangle$ from Example 4 is not compatible, because here the following inequalities hold:

$$(x \approx x) \otimes (x \triangle y) \otimes (y \approx z) = \mathbf{1} \otimes \mathbf{1} \otimes N > (x \triangle z) = \mathbf{0},$$
$$(x \approx y) \otimes (y \triangle z) \otimes (z \approx z) = N \otimes \mathbf{1} \otimes \mathbf{1} > (x \triangle z) = \mathbf{0}.$$

The following proposition is of great importance for further results and for discussion of the necessity of relevant versions of transitivity for the validity of single statements of Theorem 1 in a richer fuzzy setting. In its proof, we use a method which appeared to be very effective in proofs of all the statements of Theorem 2 in connection with **L**-complete propelattices, i.e., generally under the absence of transitivity (compare to Remark 12 (b)).

Proposition 2. *Let* $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ *be an* **L***-complete propelattice. Then* ${}^{\mathbf{1}}\mathbf{X} := \langle\langle X, =\rangle, {}^{\mathbf{1}}\triangle\rangle$ *is a complete propelattice. Moreover, for an arbitrary* ***crisp*** *set* $A \subseteq X$ *the following equalities generally hold:*

$$p\inf_{\triangle}(A) = \mathcal{S}[p\inf_{{}^{\mathbf{1}}\triangle}(A)], \quad p\sup_{\triangle}(A) = \mathcal{S}[p\sup_{{}^{\mathbf{1}}\triangle}(A)]. \tag{64}$$

If in addition $\triangle \in L^{X\times X}$ *is weakly transitive, then* ${}^{\mathbf{1}}\mathbf{X}$ *is a complete lattice. All the more so, the same holds if* $\triangle$ *is strongly transitive.*

Proof. Let $\emptyset \neq P \subseteq X$ (for $P = \emptyset$ it is sufficient to use Proposition 1 and relations (28) for the case $\mathbf{L} = \mathbf{2}$, because all here introduced notions are conservative extensions.) We show for example that for P there exists $p\sup(P)$ in ${}^{\mathbf{1}}\mathbf{X}$. The case of $p\inf(P)$ can be justified dually.

Let $\mathcal{P} := \{\Pi_\lambda \in L^X \mid \lambda \in \Lambda\} \subseteq L^X$ be the indexed system of **all** the **L**-sets such that every $\Pi_\lambda \in \mathcal{P}$ satisfies the following two conditions:

(a) $P \equiv \bigcup_{p\in P}\{{}^{\mathbf{1}}/_{p}\} \subseteq \Pi_\lambda$;

(b) $\forall\, \omega \in L^X : \omega \subseteq \Pi_\lambda \Rightarrow p\sup(\omega) \subseteq \Pi_\lambda$.

Obviously $\mathcal{P} \neq \emptyset$. Certainly $P \subseteq X$ and for any $\omega \in L^X$ we have $\omega \subseteq X$ and even $p\sup(\omega) \subseteq X$, i.e., $X \in \mathcal{P}$. Now let us introduce the following **L**-set:

$$\Pi := \bigcap \mathcal{P} = \bigcap_{\lambda\in\Lambda} \Pi_\lambda \neq \emptyset. \tag{65}$$

Indeed, $\Pi \neq \emptyset$. Let us choose $\omega := \emptyset$, then according to (b) we have $\bot = p\sup(\emptyset) \subseteq \Phi_\lambda$ for every $\lambda \in \Lambda$. Now because $\mathbf{X}$ is an **L**-complete propelattice, then according to (32) there exists $x^* \in X$ such that $\bot = \mathcal{S}[x^*] \neq \emptyset$, which gives $\emptyset \neq \bot \subseteq \Pi$. Because the **L**-set $\Pi \in L^X$ itself obviously satisfies both the conditions (a) and (b), we obtain the following equality in the complete lattice $\langle L^X, \subseteq\rangle = \langle L^X, \cup, \cap, \emptyset, X\rangle$:

$$\Pi = \inf_{\subseteq}(\mathcal{P}) = \min_{\subseteq} \mathcal{P} \in \mathcal{P}. \tag{66}$$

Now let $p\sup(\Pi) = \mathcal{S}[x_0]$ for a certain $x_0 \in X$. We show that $x_0 = p\sup(P)$ in ${}^{\mathbf{1}}\mathbf{X}$. Above all

$$\mathcal{S}[x_0](x_0) = (\mathcal{U}(\Pi) \cap \mathcal{L}(\mathcal{U}(\Pi))\,(x_0) = \mathbf{1}, \tag{67}$$

which gives

$$\mathcal{U}(\Pi)(x_0) = \bigwedge_{y\in X} (\Pi(y) \to (y \mathbin{\triangle} x_0)) = \mathbf{1}.$$

From here $\Pi(y) \to (y \mathbin{\triangle} x_0) = \mathbf{1}$ for every $y \in X$ and according to (8) for $y := p \in P$ we have

$$\mathbf{1} = \Pi(p) \leq (p \mathbin{\triangle} x_0),$$

which means that $(p \mathbin{\triangle} x_0) = \mathbf{1}$ and so $p \mathbin{{}^{\mathbf{1}}\triangle} x_0$. As its consequence we have in ${}^{\mathbf{1}}\mathbf{X} = \langle\langle X, =\rangle, {}^{\mathbf{1}}\triangle\rangle$ that

$$x_0 \in \mathcal{U}(P). \tag{68}$$

We have to show that $x_0 = p\min(\mathcal{U}(P))$, i.e., $\forall x^* \in \mathcal{U}(P) : x_0 \mathbin{{}^{\mathbf{1}}\triangle} x^*$. Let us choose an arbitrary but fixed $x^* \in \mathcal{U}(P)$. Then for every $p \in P$ we have $p \mathbin{{}^{\mathbf{1}}\triangle} x^*$. We introduce the **L**-set

$$\Pi^{\mathcal{L}} := \Pi \cap \mathcal{L}(x^*) \tag{69}$$

and show that $\Pi^{\mathcal{L}} \in \mathcal{P}$.

Analogously to (47), according to (22) and (20), we get for any $y \in X$:

$$\mathcal{L}(x^*)(y) = \bigwedge_{z\in X} \left(\{{}^{\mathbf{1}}/_{x^*}\}(z) \to (y \vartriangle z)\right) = \mathbf{1} \to (y \vartriangle x^*) = (y \vartriangle x^*). \tag{70}$$

The next equalities for every $p \in P$ then follow from (70):

$$\Pi^{\mathcal{L}}(p) = (\Pi \cap \mathcal{L}(x^*))(p) = \Pi(p) \wedge \mathcal{L}(x^*)(p) = \mathbf{1} \wedge (p \vartriangle x^*) = \mathbf{1} \wedge \mathbf{1} = \mathbf{1}.$$

From here obviously $P \subseteq \Pi^{\mathcal{L}}$ and $\Pi^{\mathcal{L}}$ fulfills the condition (a).

Let now be $\omega \in L^X$ arbitrary such that $\omega \subseteq \Pi^{\mathcal{L}} = \Pi \cap \mathcal{L}(x^*)$. Then $\omega \in \Pi$ and according to the assumptions $p\sup(\omega) \subseteq \Pi$. Because also $\omega \subseteq \mathcal{L}(x^*)$, then according to implications (27) also $p\sup(\omega) \subseteq \mathcal{L}(x^*)$. Finally $p\sup(\omega) \subseteq \Pi \cap \mathcal{L}(x^*)$ and $\Pi^{\mathcal{L}}$ fulfills also the condition (b).

From the previous facts it follows that $\Pi^{\mathcal{L}} \in \mathcal{P}$ and according to (66) the inclusion $\Pi \subseteq \Pi^{\mathcal{L}}$ holds. However, according to (69) also the opposite inclusion $\Pi^{\mathcal{L}} \subseteq \Pi$ holds trivially and with respect to (11) we have the equality

$$\Pi = \Pi^{\mathcal{L}}. \tag{71}$$

From (71) we then successively obtain that

$$\mathcal{S}[x_0] = p\sup(\Pi) \subseteq \Pi = \Pi \cap \mathcal{L}(x^*) \subseteq \mathcal{L}(x^*).$$

Hence according to (67) and (70) we have

$$\mathbf{1} = \mathcal{S}[x_0](x_0) \leq \mathcal{L}(x^*)(x_0) = (x_0 \vartriangle x^*),$$

which means $(x_0 \vartriangle x^*) = \mathbf{1}$ and hence $x_0 \,{}^{\mathbf{1}}\!\vartriangle x^*$. Because $x^* \in \mathcal{U}(P)$ was arbitrary, we get that in ${}^{\mathbf{1}}\mathbf{X} = \langle\langle X, =\rangle, {}^{\mathbf{1}}\!\vartriangle\rangle$ really

$$x_0 = p\min(\mathcal{U}(P)) \tag{72}$$

and thus there exists $p\sup(P) = x_0$ in ${}^{\mathbf{1}}\mathbf{X}$. Because the existence of $p\inf(P)$ can be shown analogously, it is clear that ${}^{\mathbf{1}}\mathbf{X} = \langle\langle X, =\rangle, {}^{\mathbf{1}}\!\vartriangle\rangle$ is a complete propelattice.

Equalities (64) are an immediate consequence of the facts that all used notions are conservative extensions and that propeinfima, resp. propesuprema, are SC-singletons.

If $\vartriangle \in L^{X\times X}$ would be in addition weakly transitive, then the proof of the fact that ${}^{\mathbf{1}}\mathbf{X} = \langle\langle X, =\rangle, {}^{\mathbf{1}}\!\vartriangle\rangle$ is a complete propelattice is identical to the proof of statement 3 of Theorem 2. The more so, the statement holds even for strong transitivity. The proof is complete.

□

5. Analysis of Tarski's theorem for commutative sets of maps in a fuzzy setting

We attempt to make here an almost complete analysis. The word "almost" is written intentionally, because one part of the argumentation has to be based on a fact, which has not been published yet. The mentioned fact is almost obvious, but a relevant construction is rather sophisticated (more precisely, the authors do not know a simple construction, even if they do not exclude its existence), but above all, it does not relate directly to the topic of our paper. Now we sum up all the results about Tarski's generalized theorem in connection with transitivity.

(**A**) It immediately follows from Example 3 and Lemma 1 that for a commutative set $\emptyset \neq F \subseteq X^X$ of **L**-fuzzy relatively isotone maps on an **L**-complete propelattice $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ none of the statements (1)–(4) of Theorem 1 holds. This is valid even if the **L**-propeorder $\triangle \in L^{X\times X}$ is strongly transitive. According to Theorem 2 we can say only that every $f \in F$ has necessarily a fixed point, but no common fixed point needs to exist.

(**B**) In the case if this part is not observed as trustworthy, it can be eventually considered as a so far unsolved problem, even if its usefulness can appear to be much smaller than the effort to solve it. Simplified versions of consequences of the above-mentioned construction are summed up in the following proposition. The main problem is the fact that whereas a construction of a commutative set of relatively isotone maps, for which Theorem 1 does not hold (Example 3), is in fact trivial, the situation is different for isotone maps.

Proposition 3. *There exists a nontransitive complete propelattice* $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ *and a commutative set* $\emptyset \neq F \subseteq X^X$ *of isotone maps on* $\mathbf{X}$ *such that none of the statements* (1)–(4) *of Theorem 1 holds.*

Lemma 1 and Proposition 3 then imply that there exists an $\mathbf{L}_N$-complete propelattice $\mathbf{X}_N = \langle\langle X, \approx_N\rangle, \triangle_N\rangle$ and there a commutative set $\emptyset \neq F \subseteq X^X$ of $\mathbf{L}_N$-fuzzy isotone maps such that no analogy of statements (1)–(4) of Theorem 1 holds. Naturally, the $\mathbf{L}_N$-propeorder $\triangle_N \in L^{X\times X}$ cannot be according to Lemma 1 even weakly transitive.

(**C**) The case where the **L**-propeorder $\triangle \in L^{X\times X}$ is weakly transitive is solved by the following result.

Theorem 3. *Let* $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ *be an* **L***-complete propelattice and let* $\triangle \in L^{X\times X}$ *be weakly transitive. Then for an arbitrary commutative set* $\emptyset \neq F \subseteq X^X$ *of isotone maps the following statements hold.*

1. *The set of common fixed points is nonempty:*

$$\mathrm{Fix}(F) \neq \emptyset. \tag{73}$$

2. *The system*

$$\mathbf{F}_F := \langle\langle \mathrm{Fix}(F), =\rangle, {}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(F) \times \mathrm{Fix}(F)]\rangle \tag{74}$$

is a complete lattice.

Proof. If $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ is an **L**-complete propelattice and if $\triangle \in L^{X\times X}$ is weakly transitive, then according to Proposition 2 ${}^{\mathbf{1}}\mathbf{X} = \langle\langle X, =\rangle, {}^{\mathbf{1}}\triangle\rangle$ is a complete lattice and also every map $f \in F$ is isotone on ${}^{\mathbf{1}}\mathbf{X}$. According to Theorem 1, (1) we have $\mathrm{Fix}(F) \neq \emptyset$ and $\mathbf{F}_F := \langle\langle \mathrm{Fix}(F), =\rangle, {}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(F) \times \mathrm{Fix}(F)]\rangle$ is a complete lattice according to Theorem 1, (2). The proof is finished. □

We noticed in [16] that there exist counterexamples, which show that weak transitivity is not sufficient for the validity of equalities (37). Because every map is commutative with itself, even analogies of relations (3)–(4) from Theorem 1 cannot be awaited in this case, but only corresponding analogies of relations (55), which are presented further in the form (94).

(**D**) Finally, we solve the case where the **L**-propeorder $\triangle \in L^{X\times X}$ is strongly transitive. It is the following theorem which possesses the complete fuzzification of Tarski's generalized theorem (i.e., Theorem 1).

Theorem 4. *Let $\mathbf{X} = \langle\langle X, \approx\rangle, \triangle\rangle$ be an **L**-complete propelattice and let $\triangle \in L^{X\times X}$ be strongly transitive. Then for an arbitrary commutative set $\emptyset \neq F \subseteq X^X$ of **L**-fuzzy isotone maps the following statements hold.*

1. *The set of common fixed points is nonempty:*

$$\mathrm{Fix}(F) \neq \emptyset. \tag{75}$$

2. *The system*

$$\mathbf{F}_F := \langle\langle \mathrm{Fix}(F), =\rangle, {}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(F) \times \mathrm{Fix}(F)]\rangle \tag{76}$$

is a complete lattice.

3. *Further we define **L**-sets $\Phi, \Psi \in L^X$ for every $x \in X$ in the following way:*

$$\Phi(x) := \bigwedge_{f\in F} (f(x) \triangle x), \tag{77}$$

$$\Psi(x) := \bigwedge_{f\in F} (x \triangle f(x)). \tag{78}$$

*Then the **L**-sets Φ and Ψ are normal. If $\mathcal{S}[x_0] = p\inf(\Phi)$ and $\mathcal{S}[x_1] = p\sup(\Psi)$, then*

$$x_0 = \min(\mathrm{Fix}(F)), \quad x_1 = \max(\mathrm{Fix}(F)) \tag{79}$$

(where minimum and maximum are taken with respect to the order ${}^{\mathbf{1}}\triangle \cap [\mathrm{Fix}(F) \times \mathrm{Fix}(F)]$).

Proof. If $\triangle \in L^{X\times X}$ is strongly transitive, then it is also weakly transitive and the first two statements follow immediately from Theorem 3. So now the third statement, i.e., the validity of equations (79), is to be proved.

At first we show that Φ is normal (where the **L**-set $\Phi \in L^X$ is defined by equality (77)), which means $\mathrm{Ker}(\Phi) \neq 0$. Primarily, with respect to (28) and (8) we have for any $y \in X$:

$$\top(y) = \mathcal{U}(X)(y) = \bigwedge_{z \in X} (\mathbf{1} \to (z \bigtriangleup y)) =$$
$$= \bigwedge_{z \in X} (z \bigtriangleup y) \leq \bigwedge_{f \in F} (f(y) \bigtriangleup y) = \Phi(y).$$

From here we see that the inclusion $\top = \mathcal{U}(X) \subseteq \Phi$ holds. Because $\mathbf{X} = \langle\langle X, \approx\rangle, \bigtriangleup\rangle$ is an **L**-complete propelattice, according to (32) there exists $\tilde{x} \in X$ such that $\mathcal{S}[\tilde{x}] = p\sup(X) = \mathcal{U}(X) \subseteq \Phi$. From here we have that $\tilde{x} \in \mathrm{Ker}(\Phi) \neq \emptyset$ and thus Φ is normal.

Now we show that there exists $x_0 \in \mathrm{Fix}(F) = \bigcap_{f \in F} \mathrm{Fix}(f) \neq \emptyset$ and if $\mathcal{S}[x^*] = p\inf(\Phi)$, then at the same time $x_0 = \min(\mathrm{Fix}(F))$ and $x_0 = x^*$, that is the first equality in (79) holds. The second equality in (79) is then proved by a dual argumentation.

Let $\mathcal{G} := \{\Gamma_\lambda \in L^X \mid \lambda \in \Gamma\} \subseteq L^X$ be the indexed system of **all** the **L**-sets such that every $\Gamma_\lambda \in \mathcal{G}$ fulfills the following three conditions:

(a) $\Gamma_\lambda \subseteq \Gamma_\lambda \circ f$, i.e., $\forall y \in X : \Gamma_\lambda(y) \leq \Gamma_\lambda(f(y))$, for every $f \in F$;

(b) $\forall \phi \in L^X : \phi \subseteq \Gamma_\lambda \Rightarrow p\inf(\phi) \subseteq \Gamma_\lambda$;

(c) $\Phi \subseteq \Gamma_\lambda$ for every $\Gamma_\lambda \in \mathcal{G}$.

Primarily, $\mathcal{G} \neq \emptyset$. For every $y \in X$ and $f \in F$ we have $X(y) = X(f(y)) = \mathbf{1}$ and for any **L**-set $\phi \in L^X$ we have $\phi \subseteq X$, $p\inf(\phi) \subseteq X$ and $\Phi \subseteq X$, which gives $X \in \mathcal{G}$. Now let us define the following **L**-set whose role is fundamental in the following:

$$\Gamma := \bigcap \mathcal{G} = \bigcap_{\lambda \in \Lambda} \Gamma_\lambda \neq \emptyset. \tag{80}$$

Indeed, $\Gamma \neq \emptyset$. If we choose $\phi := \emptyset$ then according to (b) and (28) we have $\top = p\inf(\emptyset) \subseteq \Gamma_\lambda$ for every $\lambda \in \Lambda$. Further, because $\mathbf{X}$ is an **L**-complete propelatice, with respect to (32) there exists such $x^* \in X$ that $\top = \mathcal{S}[x^*] \neq \emptyset$, which implies $\emptyset \neq \mathcal{S}[x^*] \subseteq \Gamma$.

Analogously to the proof of statement 1 from Theorem 2 one could easily show that Γ fulfills all the three conditions (a)–(c) and hence in the complete lattice $\langle L^X, \cup, \cap, \emptyset, X\rangle = \langle L^X, \subseteq\rangle$ the following holds:

$$\Gamma = \min_{\subseteq}(\mathcal{G}) \in \mathcal{G}. \tag{81}$$

Because $\Gamma \subseteq \Gamma \in \mathcal{G}$, there exists $x_0 \in X$ such that

$$\mathcal{S}[x_0] = p\inf(\Gamma) \subseteq \Gamma \tag{82}$$

From (82) according to (a) we obtain for Γ equalities

$$\Gamma(x_0) = \Gamma(f(x_0)) = \mathbf{1}, \quad \forall f \in F. \tag{83}$$

Furthermore, according to (23) we have:

$$p\inf(\Gamma) = \mathcal{S}[x_0] = \mathcal{L}(\Gamma) \cap \mathcal{U}(\mathcal{L}(\Gamma)). \tag{84}$$

From here, specially at the point $x_0 \in X$, we have that

$$\mathbf{1} = \mathcal{S}[x_0](x_0) \leq \mathcal{L}(\Gamma)(x_0) = \bigwedge_{y \in X} (\Gamma(y) \to (x_0 \vartriangle y)),$$

that is $\Gamma(y) \to (x_0 \vartriangle y) = \mathbf{1}$ for every $y \in X$. Further, with respect to (8) we have

$$\Gamma(y) \leq (x_0 \vartriangle y), \quad \forall y \in X. \tag{85}$$

According to (83) for the choice $y := f(x_0)$ in (85) we then have

$$(x_0 \vartriangle f(x_0)) = \mathbf{1}, \quad \forall f \in F. \tag{86}$$

Now let us introduce the following **L**-set (see Definition 12, (22)):

$$\Gamma^{\mathcal{U}} := \Gamma \cap \bigcap_{g \in F} \mathcal{U}(g(x_0)). \tag{87}$$

We have to prove that $\Gamma^{\mathcal{U}} = \Gamma$. We show that the **L**-set $\Gamma^{\mathcal{U}}$ fulfills all the three conditions (a)–(c), which means $\Gamma^{\mathcal{U}} \in \mathcal{G}$.

Firstly, with respect to (86), because every map $g \in F$ is **L**-fuzzy isotone and F is commutative, the following holds:

$$(g(x_0) \vartriangle f(g(x_0))) = (g(x_0) \vartriangle g(f(x_0))) = \mathbf{1}, \quad \forall f, g \in F. \tag{88}$$

According to (47) and (88), with respect to strong transitivity of the complete propeorder $\vartriangle \in L^{X \times X}$, for any $y \in X$ we obtain the following inequalities:

$$\begin{aligned}\mathcal{U}(g(x_0))(y) &= (g(x_0) \vartriangle y) \leq (f(g(x_0)) \vartriangle f(y)) = \mathbf{1} \wedge (f(g(x_0)) \vartriangle f(y)) = \\ &= (g(x_0) \vartriangle f(g(x_0))) \wedge (f(g(x_0)) \vartriangle f(y)) \leq \\ &\leq (g(x_0) \vartriangle f(y)) = \mathcal{U}(g(x_0))(f(y)).\end{aligned}$$

These inequalities for every $y \in X$ and every $f \in F$ immediately imply

$$\bigcap_{g \in F} \mathcal{U}(g(x_0))(y) \leq \bigcap_{g \in F} \mathcal{U}(g(x_0))(f(y)).$$

From here we get the following inclusions:

$$\bigcap_{g \in F} \mathcal{U}(g(x_0)) \subseteq \bigcap_{g \in F} \mathcal{U}(g(x_0)) \circ f, \quad \forall f \in F. \tag{89}$$

With respect to the property (a) of the **L**-set Γ, according to (89) for any $f \in F$ finally holds:

$$\Gamma^{\mathcal{U}} = \Gamma \cap \bigcap_{g \in F} \mathcal{U}(g(x_0)) \subseteq \Gamma \circ f \cap \left[\bigcap_{g \in F} \mathcal{U}(g(x_0)) \right] \circ f =$$
$$= \left[\Gamma \cap \bigcap_{g \in F} \mathcal{U}(g(x_0)) \right] \circ f,$$

i.e., the **L**-set $\Gamma^{\mathcal{U}}$ fulfills the condition (a).

It is rather simple to show that $\Gamma^{\mathcal{U}}$ satisfies the condition (b). Let $\phi \in L^X$ be such that $\phi \subseteq \Gamma^{\mathcal{U}}$. Then according to (87) we have $\phi \subseteq \Gamma$ and simultaneously $\phi \subseteq \bigcap_{g \in F} \mathcal{U}(g(x_0))$, i.e., $\phi \subseteq \mathcal{U}(g(x_0))$ for every $g \in F$. Then according to assumptions $p\inf(\phi) \subseteq \Gamma$. Also, with respect to (27), we have $p\inf(\phi) \subseteq \mathcal{U}(g(x_0))$ for every $g \in F$, that is $p\inf(\phi) \subseteq \bigcap_{g \in F} \mathcal{U}(g(x_0))$. Altogether we have $p\inf(\phi) \subseteq \Gamma \cap \bigcap_{g \in F} \mathcal{U}(g(x_0)) = \Gamma^{\mathcal{U}}$ and hence $\Gamma^{\mathcal{U}}$ fulfills the condition (b).

Finally, we show that $\Gamma^{\mathcal{U}}$ fulfills also the condition (c). According to the condition (c) for Γ we have $\Phi \subseteq \Gamma$ and hence, with respect to (85) and (77), for any $y \in X$ the following inequality holds:

$$\Phi(y) = \bigwedge_{f \in F} (f(y) \triangle y) \leq (x_0 \triangle y). \tag{90}$$

Let now $g \in F$ be an arbitrary but fixed map. If we "multiply" inequality (90) by the expression $(g(y) \triangle y)$, then with respect to idempotency of the operation $\bullet \wedge \bullet : L \times L \to L$ and to the generalized associative law (see [1]), further with respect to strong transitivity of $\triangle \in L^{X \times X}$ and finally to the fact that $g : X \to X$ is **L**-fuzzy isotone on $\mathbf{X} = \langle \langle X, \approx \rangle, \triangle \rangle$, we successively obtain the following:

$$\begin{aligned}
(g(y) \triangle y) \wedge \bigwedge_{f \in F} (f(y) \triangle y) &= ((g(y) \triangle y) \wedge (g(y) \triangle y)) \wedge \bigwedge_{g \neq f \in F} (f(y) \triangle y) = \\
&= (g(y) \triangle y) \wedge \bigwedge_{g \neq f \in F} (f(y) \triangle y) = \bigwedge_{f \in F} (f(y) \triangle y) = \Phi(y) \leq \\
&\leq (g(y) \triangle y) \wedge (x_0 \triangle y) \leq (g(y) \triangle y) \wedge (g(x_0) \triangle g(y)) = \\
&= (g(x_0) \triangle g(y)) \wedge (g(y) \triangle y) \leq (g(x_0) \triangle y).
\end{aligned} \tag{91}$$

From here we immediately have that for any $y \in X$ and every $g \in F$ the following inequality holds

$$\Phi(y) \leq (g(x_0) \triangle y),$$

and hence for every $y \in X$ we have

$$\Phi(y) \leq \bigwedge_{g \in F} (g(x_0) \triangle y).$$

Thus the following inclusion holds:

$$\Phi \subseteq \bigcap_{g \in F} \mathcal{U}(g(x_0)). \tag{92}$$

From (87), the property (c) for Γ and from (92) we finally arrive at

$$\Phi \subseteq \Gamma \cap \bigcap_{g \in F} \mathcal{U}(g(x_0)) = \Gamma^{\mathcal{U}},$$

after which $\Gamma^{\mathcal{U}}$ finally fulfills also the condition (c).

Because the **L**-set $\Gamma^{\mathcal{U}}$ satisfies all the three conditions (a)–(c), we have $\Gamma^{\mathcal{U}} \in \mathcal{G}$. Thanks to this, according to (81), the inclusion $\Gamma^{\mathcal{U}} \supseteq \Gamma$ holds. The opposite inclusion $\Gamma^{\mathcal{U}} \subseteq \Gamma$ is trivial, so according to (11) we obtain the following equality:

$$\Gamma^{\mathcal{U}} = \Gamma.$$

From this equality, according to (82), we have

$$\mathcal{S}[x_0] = p\inf(\Gamma) \subseteq \Gamma = \Gamma^{\mathcal{U}} \subseteq \bigcap_{g \in F} \mathcal{U}(g(x_0)).$$

Specially at the point $x_0 \in X$ we obtain

$$\mathbf{1} = \mathcal{S}[x_0](x_0) \leq \left[\bigcap_{g \in F} \mathcal{U}(g(x_0)) \right](x_0) = \bigwedge_{g \in F} (g(x_0) \triangle x_0).$$

From here we see that the following equalities hold:

$$(g(x_0) \triangle x_0) = \mathbf{1}, \quad \forall g \in F. \tag{93}$$

No matter how a map from the commutative set F is denoted, from (86) and (93) in total follows that with respect to antisymmetry (19) and separation (14) for any $f \in F$ the next chain of implications holds:

$$\begin{aligned} \mathbf{1} = \mathbf{1} \otimes \mathbf{1} = (f(x_0) \triangle x_0) \otimes (x_0 \triangle f(x_0)) \leq (x_0 \approx f(x_0)) \quad \Rightarrow \\ \Rightarrow \quad (x_0 \approx f(x_0)) = \mathbf{1} \quad \Rightarrow \quad x_0 = f(x_0). \end{aligned}$$

It means that $x_0 \in \mathrm{Fix}(f)$ for every $f \in F$, so $x_0 \in \mathrm{Fix}(F) = \bigcap_{f \in F} \mathrm{Fix}(f) \neq \emptyset$.

If we denote the **L**-set $\Pi := \bigcup_{p \in \mathrm{Fix}(F)} \{^{\mathbf{1}}/_{p}\}$ then with respect to reflexivity of the propeorder $\triangle \in L^{X \times X}$ we have $\Pi \subseteq \Phi \subseteq \Gamma$, or less formally $\emptyset \neq \mathrm{Fix}(F) \subseteq \Phi \subseteq \Gamma$. Let $x_1 = \min(\mathrm{Fix}(F)) = p\inf(\mathrm{Fix}(F))$ be the propeinfimum with respect to the complete propeorder ${}^{\mathbf{1}}\triangle \subseteq X \times X$. The set $\mathrm{Fix}(F) = \Pi$ is crisp, thus according to (64) we have $p\inf(\Pi) = \mathcal{S}[x_1]$. Because $p\inf(\Gamma) = \mathcal{S}[x_0]$, then according to statement 5 from Proposition 1 we have $x_0 \, {}^{\mathbf{1}}\triangle \, x_1$. However $x_0 \in \mathrm{Fix}(F)$ and hence also $x_1 \, {}^{\mathbf{1}}\triangle \, x_0$ holds. If we denote $\mathcal{S}[x^*] = p\inf(\Phi)$, then by an analogous argumentation we obtain $x_0 \, {}^{\mathbf{1}}\triangle \, x^*$. But also $x^* \, {}^{\mathbf{1}}\triangle \, x_1 = x_0$ holds.

Again, from antisymmetry we finally have $x_0 = x^*$, that is $\min(\mathrm{Fix}(F)) = x^*$, which is in our notation the first equality in (79).

Finally, we present the dual argumentation which proves that also the second equality in (79) holds. At first we introduce the *dual* **L**-propeorder $\triangle^{-1} \in L^{X\times X}$ to the **L**-propeorder $\triangle \in L^{X\times X}$ for every $x, y \in X$ by a natural way:

$$(x \triangle^{-1} y) := (y \triangle x).$$

Now we consider the *dual* **L**-complete propelattice $\mathbf{X}_d = \langle\langle X, \approx\rangle, \triangle^{-1}\rangle$, where the **L**-propeorder $\triangle^{-1}$ is strongly transitive too. Clearly, every map $f \in F$ is isotone even on $\mathbf{X}_d$. If we denote the relevant operations in $\mathbf{X}_d$ by index d then, according to the previous, for $\mathcal{S}[x_1] = p\inf_d(\Phi_d)$ we have $x_1 = \min_d(\mathrm{Fix}(F))$. At the same time for every $y \in X$ we have (see (78))

$$\Phi_d(y) = \bigwedge_{f\in F} (f(y) \triangle^{-1} y) = \bigwedge_{f\in F} (y \triangle f(y)) = \Psi(y),$$

that is $\Phi_d = \Psi$. In addition to that, obviously $p\inf_d(\Phi_d) = p\sup(\Phi_d) = p\sup(\Psi)$ and $\min_d(\mathrm{Fix}(F)) = \max(\mathrm{Fix}(F))$ hold. From here we see that even the second equality in (79) holds.

Now the proof is complete. □

The last example demonstrates quite simply all the statements of Theorem 4.

Example 5. Let $0 \neq \Omega \in On$ be an arbitrary ordinal. Then $\Omega + 1$ is a well-ordered set with respect to inclusion and is in addition also a complete lattice. If we set $X := \Omega + 1$ and $\triangle := \subseteq$, then $\mathbf{X} = \langle\langle X, =\rangle, \triangle\rangle$ is a complete lattice (transitive complete propelattice). Further we define the commutative set of maps $\emptyset \neq F_\omega \subseteq X^X$ for fixed $0 < \omega \leq \Omega$ in the following way. For $0 < \beta \leq \omega$ let $f''_\beta(\beta + 1) := \{\beta\}$ and $f_\beta(\alpha) := \alpha$ for $\Omega \geq \alpha > \beta$. Obviously every map $f_\beta : X \to X$ is isotone on $\mathbf{X}$ and the set $F_\omega := \{f_\beta \,|\, 0 < \beta \leq \omega\}$ is commutative. Now according to Lemma 1 let $\mathbf{X}_N = \langle\langle X, \approx_N\rangle \triangle_N\rangle$ be a strongly transitive $\mathbf{L}_N$-complete propelattice, where all maps are $\mathbf{L}_N$-fuzzy isotone on $\mathbf{X}_N$. Because $\Omega + 1$ is a chain, for every $x, y \in X, x \neq y$ only $x \triangle y$ and simultaneously $(y \triangle_N x) = N$, or $y \triangle x$ and simultaneously $(x \triangle_N y) = N$ (because in $\Omega + 1$ there are no incomparable elements). From here it is clear that

$$0 \leq x < \beta \Rightarrow (f_\beta(x) \triangle x) = N; \quad \beta \leq x \leq \Omega \Rightarrow (f_\beta(x) \triangle x) = \mathbf{1},$$

and hence according to (77) and (78) we obtain

$$\Phi = \bigcup_{\omega\leq x\leq\Omega} \{{}^{\mathbf{1}}/_{x}\} \cup \bigcup_{x<\omega} \{{}^{N}/_{x}\} = [\omega, \Omega] \cup \bigcup_{x<\omega} \{{}^{N}/_{x}\}$$

and clearly

$$\Psi = \Omega + 1.$$

From the definitions of F_ω and f_β it is clear that $\mathrm{Fix}(F_\omega) = [\omega, \Omega]$. Further then $\min(\mathrm{Fix}(F_\omega)) = \min([\omega, \Omega]) = \omega$ and $\max(\mathrm{Fix}(F_\omega)) = \max([\omega, \Omega]) = \Omega$ in $\langle \Omega + 1, \subseteq \rangle$. Moreover, with respect to (58) we have

$$p\inf(\Phi) = \mathcal{S}[p\inf([\omega, \Omega])] = \mathcal{S}[\inf([\omega, \Omega])] = \mathcal{S}[\min([\omega, \Omega])] = \mathcal{S}[\omega],$$
$$p\sup(\Psi) = \mathcal{S}[p\sup([\omega, \Omega])] = \mathcal{S}[\sup([\omega, \Omega])] = \mathcal{S}[\max([\omega, \Omega])] = \mathcal{S}[\Omega].$$

From here we can see that equalities (79) hold.

In conclusion, let us mention some facts. As it was already said, if $\triangle \in L^{X\times X}$ is only weakly transitive, then equalities (79) do not need to hold. Nevertheless, with respect to reflexivity both the following inclusions clearly hold:

$$\mathrm{Fix}(F) \subseteq \Phi, \quad \mathrm{Fix}(F) \subseteq \Psi.$$

According to Proposition 1, statement 5, and the definitions of the propeleast and propegreatest elements, only the following relations hold in general even without any assumption of any transitivity:

$$x_0 \,{}^{\mathbf{1}}\!\triangle\, p\min(\mathrm{Fix}(F)), \quad p\max(\mathrm{Fix}(F)) \,{}^{\mathbf{1}}\!\triangle\, x_1, \qquad (94)$$
$$p\min(\mathrm{Fix}(F)) \,{}^{\mathbf{1}}\!\triangle\, p\max(\mathrm{Fix}(F)).$$

To have in relations (94) the equalities instead of ${}^{\mathbf{1}}\!\triangle$, strong transitivity is needed. As already mentioned in [16], the most important fact for the validity of equalities (37) is that the operation of meet $\bullet \wedge \bullet : L \times L \to L$, which appears in relation (26), is idempotent. Contrarily, multiplication $\bullet \otimes \bullet : L \times L \to L$ generally does not have this property. Also in the proof of Theorem 4 idempotency plays an irreplaceable role, even if it is maybe in the proof somehow hidden. However, fundamental relations (91), which are the basis of the proof, would not hold without idempotency.

It is clear that one of the major reasons why transitivity is more important for existence of common fixed points for a commutative set **L**-fuzzy isotone maps than for existence of fixed points of a single **L**-fuzzy isotone map – as formulated in Theorem 2 – is the fact that commutativity has no connection to the structure of the **L**-complete propelattice $\mathbf{X} = \langle \langle X, \approx \rangle, \triangle \rangle$ itself. On one hand, exactly this kind of properties is for the existence of common fixed points the most important, but it is clear from our examples that the system of commutative sets of **L**-fuzzy isotone maps is relatively narrow. On the other hand, this independence of commutativity on the structure of $\mathbf{X}$ enabled the analysis of the dependence of the validity of single statements on different versions of transitivity.

6. Conclusion: a mutual comparison of the both fuzzified Tarski's theorems

Primarily, we showed (Theorem 2, statement 1, and Example 3, or its fuzzification according to Lemma 1) that fixed points do always exist for **L**-fuzzy relatively isotone maps, but no other statement of any of Tarski's theorems holds.

However, the situation is diametrically different for **L**-fuzzy isotone maps. Theorem 2, statement 2, says that for a single **L**-fuzzy isotone map the most substantial part of Tarski's theorem holds even without the assumption of transitivity and surprisingly, the set of its fixed points is a "crisp" complete propelattice. Weak transitivity is sufficient for this set to be a complete lattice (Theorem 2, statement 3). And it is only strong transitivity, what enables the validity of the explicit formulas for the least and greatest fixed points (Theorem 2, statement 4). On the other hand, transitivity is necessary for any statement of the generalized theorem. For the existence of common fixed points of a commutative set of **L**-fuzzy isotone maps at least weak transitivity is needed and the set of fixed points then forms automatically a complete lattice (Theorem 3). For the validity of the explicit formulas for the least and greatest common fixed point strong transitivity is absolutely necessary (Theorem 4).

Summed up: Fuzzifications of both the theorems essentially differ. For the main part of the first theorem no transitivity is needed, for the second theorem transitivity is everywhere essential. On the contrary, their similarity consists in the fact that strong transitivity is in both the cases needed for the validity of the explicit formulas for the least and greatest (common) fixed point.

7. Acknowledgements

The authors express their thanks to the anonymous referee for his valuable suggestions which improved their manuscript.

References

[1] G. Birghof, Lattice Theory, 3rd ed., Amer. Math. Soc., Rhode Island, 1973.

[2] R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer, New York, 2002.

[3] R. Bělohlávek, Concept lattices and order in fuzzy logic, Annals of Pure and Applied Logic **128** (1–3) (2004), 277–298.

[4] R. Bělohlávek, Lattice-type fuzzy order is uniquely given by its 1-cut: proof and consequences, Fuzzy Sets and Systems **123** (2004), 447–458.

[5] M. Erné, W-completeness and fixpoint properties, Arch. Math. 24 (3)(1988) 147–155.

[6] G. Grätzer, General Lattice Theory, 2nd ed., Birkhäuser Verlag, Basel-Boston-Berlin, 2003.

[7] J. Klimeš, Fixed point characterization of completeness on lattices for relatively isotone mappings, Archivum Mathematicum **20** (3) (1984), 125–132.

[8] P. Martinek, On generalization of fuzzy concept lattices based on change of underlying fuzzy order, CLA 2008, Proc. of the Sixth International Conference on Concept Lattices and Their Applications, Palacky University, Olomouc (2008), 207–215.

[9] E. Mendelson, Introduction to Mathematical Logic, 4th ed., Chapman & Hall/CRC, Boca Raton, Florida, 2001.

[10] V. Novák, Fuzzy Sets and Their Applications, Adam-Hilger, Bristol, 1989.

[11] V. Novák, I. Portfilieva, J. Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer, Boston-Dordrecht-London, 1999.

[12] R. M. Smullyan, On transfinite recursion, Transactions of the N. Y. Acad. Sci. 30 (1965), 175–185.

[13] R. M. Smullyan, M. Fitting, Set Theory and the Continuum Problem, Oxford University Press, New York, 1996.

[14] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific Journal of Mathematics **5** (1955), 285–309.

[15] F. Včelař, On an approach to the fuzzification of classical Arrow‘s aggregation problem, Int. J. of General Systems **2** (1994), 139–154.

[16] F. Včelař, Z. Pátíková, On fuzzification of Tarski's fixed point theorem without transitivity, Fuzzy Sets and Systems 320 (2017), 93–113 .

[17] L. A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. **3** (1971), 177–200.