DISTANCE DIFFERENCE REPRESENTATIONS OF SUBSETS OF COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. Let (N, g) be a complete smooth Riemannian manifold with the distance function $d(x, y)$, $U \subset N$ be relatively compact, open subset with smooth boundary. We also assume that $\overline{U}$ is geodesically convex. Also, let $M \subset U$ be an open subset with smooth boundary such that $\overline{M} \subset U$. We assume that the topology of M and metric $g|_M$ are unknown. Let $F = \overline{U} \setminus M$ be the observation domain. For $x \in M$ we denote by D_x the distance difference function $D_x : F \times F \to \mathbb{R}$, given by $D_x(z_1, z_2) = d(x, z_1) - d(x, z_2)$, $z_1, z_2 \in F$. We show that the manifold M and the metric $g|_M$ on it can be determined uniquely, up to an isometry, when we are given the set F, the metric $g|_F$, and the collection $\mathcal{D}(M) = \{D_x;\ x \in M\}$ of distance difference functions. The embedded image $\mathcal{D}(M)$ of the manifold M, in the vector space $C(F \times F)$, is the distance difference representation of manifold M.

The inverse problem of determining (M, g) from $\mathcal{D}(M)$ arises for example in the study of the wave equation on $\mathbb{R} \times N$ when we observe in F the waves produced by spontaneous point sources at unknown points $(t, x) \in \mathbb{R} \times M$. The results presented in this paper generalize the earlier results where N is assumed to be compact that the observation domain F is assumed to be the whole complement of M in N.

Keywords: Inverse problems, distance functions, wave equation.

1. Introduction

1.1. **Formulation of problem and motivation.** Let us consider a body in which there are point sources that create propagating waves. Such point sources can either appear spontaneously, or they are caused by reflections of some propagating wave from small scatterers. In many applications one encounters a geometric inverse problem where we detect such waves emanating from point sources, either outside or at the boundary of the body, and we need to determine the unknown wave speed inside the body. As an example of such situation, one can consider the micro-earthquakes that appear very frequently near active faults. The related inverse problem is whether the surface observations of elastic waves produced by the micro-earthquakes can be used in the geophysical imaging of Earth's subsurface [21, 43], that is, to determine the speed of the elastic waves in the studied volume. In this paper we consider an idealized version of the above inverse problem: We consider the problem on an n dimensional manifold N with a Riemannian metric g that corresponds to the travel time of a wave between two points. The Riemannian distance of points $x, y \in N$ is denoted by $d(x, y)$. For simplicity we assume that the manifold N is complete and has no boundary. Here, completeness is considered in the sense of metric spaces, so also compact, closed manifolds are considered to be complete manifolds. We assume that the manifold contains an unknown part $M \subset N$ and the metric is known in a certain area F outside of the set M. When a spontaneous point source produces a wave at some unknown point $x \in M$ at some unknown time $t \in \mathbb{R}$, the produced wave is observed at the point $z \in F$ at time $T_{x,t}(z) = d(z, x) + t$. These observation times at two points $z_1, z_2 \in F$ determine the *distance difference function*

$$D_x(z_1, z_2) = T_{x,t}(z_1) - T_{x,t}(z_2) = d(z_1, x) - d(z_2, x). \tag{1}$$

Physically, this function corresponds to the difference of times when the wave produced by a point source at (x, t) is observed to arrive at points z_1 and z_2, see Fig 2. and Section 4. An assumption there is a large number point sources and that we do measurements over a long time can be modeled by the assumption that we are given the set F and the family of functions

$$\{D_x \ ; \ x \in X\} \subset C(F \times F),$$

where $X \subset M$ is either the whole manifold M or its dense subset,

1.2. **Definitions and the main result.** Let us consider a smooth complete Riemannian manifold (N, g) with dimension $n \geq 2$ (Here and below, smooth means C^∞-smooth).

Definition 1.1. *Let (N, g) be a smooth complete Riemannian manifold. Let $A \subset N$. We say that A is convex in N, if for every $x, y \in A$ any distance minimizing geodesic segment $\gamma : [0, d(x, y)] \to N$ from x to y is contained in A.*

Suppose that $M \subset N$ is a relatively compact open set such that ∂M is a smooth submanifold of dimension $(n-1)$. We also assume that there exists an open, relatively compact U that ∂U is smooth submanifold of N of dimension $(n-1)$, U contains $\overline{M}$ and $\overline{U}$ is convex. Notice, that M does not need to be convex and that both M and U may have non-trivial topology. We denote $F := \overline{U} \setminus M$. Note that then it holds that $F^{int} \neq \emptyset$ (see Figure 1).

We are interested about the following family of *Distance difference functions*

$$\mathcal{D}(M) := \{D_x \in C(F \times F) : x \in M\},$$

where for each $x \in N$ the corresponding distance difference function is

$$D_x(z_1, z_2) := d_g(x, z_1) - d_g(x, z_2), \ z_1, z_2 \in F.$$

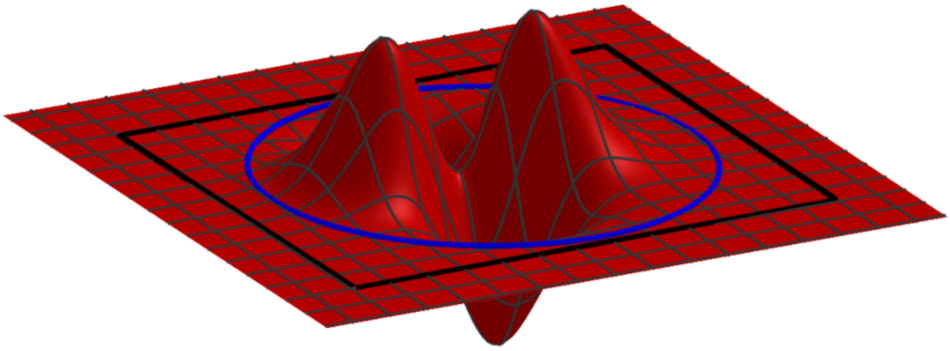

FIGURE 1. In the figure the boundary of the unknown domain M is the blue circle and M is contained in a larger domain U which boundary is the black rectangle. We make no assumptions on topology of the set M. The distance difference functions $D_x(z_1, z_2)$ of the points $x \in M$ are evaluated at the points $z_1, z_2 \in F = \overline{U} \setminus M$.

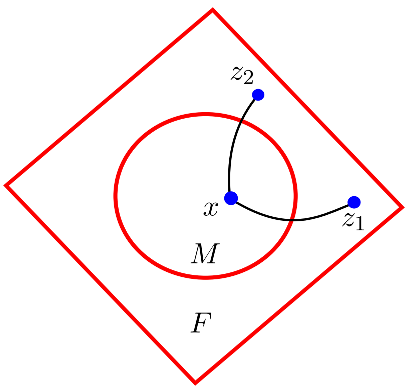

FIGURE 2. Distance difference function $D_x(z_1, z_2) = d(x, z_1) - d(x, z_2)$ of the point $x \in M$ is evaluated at the points $z_1, z_2 \in F = \overline{U} \setminus M$. Note that below we will assume for simplicity that the boundaries of M and U are smooth.

The main result considered in this paper is the following.

Theorem 1.2. *Let $n \geq 2$ and (N_i, g_i), $i = 1, 2$ be complete Riemannian manifolds. Also, let $U_i \subset N_i$ be a relatively compact open set with smooth boundary. Let $\overline{U}_i$ be convex and $M_i \subset U_i$ be an open subset which boundary is a smooth submanifold of dimension $(n-1)$ and $\overline{M}_i \subset U_i$. Denote $F_i = \overline{U_i} \setminus M_i$.*

Assume that there exists a diffeomorphism $\phi : F_1 \to F_2$ such that

$$g_1|_{F_1} = \phi^* g_2|_{F_2}. \tag{2}$$

Moreover, assume that the distance difference data for manifolds M_1 and M_2 are the same in the sense that

(3)
$$\{D_x^1 \in C(F_1 \times F_1) : x \in M_1\} = \{D_y^2(\phi(\cdot), \phi(\cdot)) \in C(F_1 \times F_1) : y \in M_2\}.$$

Here, $D^i_x(z_1,z_2) = d_{g_i}(x,z_1) - d_{g_i}(x,z_2)$ *for* $x \in N_i$ *and* $z_1, z_2 \in F_i$.

Then the Riemannian manifolds with boundary $(\overline{U}_1, g_1|_{\overline{U}_1})$ *and* $(\overline{U}_2, g_2|_{\overline{U}_2})$ *are isometric.*

This means that, if (N,g), M, U and F are as above and if we are given the following *Distance difference data*

$$\{(F, g|_F), \mathcal{D}(M)\}, \tag{4}$$

then the Riemannian structure of $(\overline{U}, g|_{\overline{U}})$ is uniquely determined. Note that the sets in (3) are given in unindexed sets, that is, for a given $D \in \{D^1_x \in C(F_1 \times F_1) : x \in M_1\}$ we do not know the point x for which $D = D^1_x$.

We start with recalling some known and related results. The main theorem is to be proved in parts after these.

1.3. **The distance function of a complete Riemannian manifold.** Here we recall some basic properties of a complete Riemannian manifolds and Riemannian distance function.

Let (N,g) be a smooth Riemannian manifold without boundary and let $d_g : N \times N \to (0,\infty)$ be the distance function related to the metric tensor g. Notice that for an arbitrary $q \in N$, the distance function $d_g(q,\cdot)$ in $N \setminus \{q\}$ is not necessarily smooth. We assume below that (N,g) is complete.

Let $p \in N$ and $\xi \in T_pN$ be such that $\|\xi\|_g = 1$. We denote by $\gamma_{p,\xi} : \mathbb{R} \to N$ the unique unit speed geodesic with initial conditions

$$\gamma_{p,\xi}(0) = p \text{ and } \dot{\gamma}_{p,\xi}(0) = \xi.$$

A general geodesic $\gamma_{x,\xi}$ is not a distance minimizer from x to $\gamma_{x,\xi}(t)$ for all $t \in \mathbb{R}$. We define a cut distance function

$$(x,\xi) \mapsto \tau(x,\xi) := \sup\{t > 0 : d_g(p, \gamma_{x,\xi}(t)) = t\} \in (0,\infty].$$

Function τ is continuous (see [24] Lemma 2.1.5.) and tells how long each geodesic is a distance minimizer. Actually, for a point $p \in N$ and a distance function $d_g(p,\cdot)$ the following holds: Function $d(p,\cdot)$ is smooth at $q \in N$ if and only if there exists $\xi \in T_pN$, $\|\xi\| = 1$ such that $q = \gamma_{p,\xi}(d(p,q))$ and $d(p,q) < \tau(p,\xi)$.

Let $S \subset N$ be a bounded, smooth $n-1$ dimensional submanifold of N. Therefore there exist, locally, precisely two vector fields ν_+ and ν_- on S that are orthogonal to S and of unit length. Let $q \in N$. Since S is compact, there exists a point $z_q \in S$ that is a closest point of S to q. Since (N,g) is complete, it holds that there exists a unit speed distance minimizing geodesic γ from z_q to q. It can be proved that geodesic γ is orthogonal to S (see [10], III.6). Then it must hold that $\gamma = \gamma_{z_q,\nu_+}$ or $\gamma = \gamma_{z_q,\nu_-}$. Suppose that $\gamma = \gamma_{z_q,\nu_+}$. Then it also holds that (see Lemma 7.7 of [2])

$$\tau(z_q,\nu_+) > d_g(q,z_q).$$

This means that the geodesic γ from z_q to q can be continued over the end point q to some point $p = \gamma(s)$, where $d_g(q, z_q) < s < \tau(z_q, \nu_+)$, so that it remain to be a distance minimizing curve between z_q and p. Note that the continued geodesic might not be a distance minimizing curve from p to ∂S. These topics are covered for instance in [30], [23], [10] and [24].

1.4. **Embeddings of a Riemannian manifold.** Often one is interested in embedding a manifold M to some Euclidean space that has as small dimension as possible. Two main examples of this nature are the well known Whitney and Nash embedding theorems. In our case we are interested in quite different kind of embeddings. We will embed M into infinite dimensional Banach-space using the distance difference functions. Similar techniques are also well known in literature. A classical distance function representation of a Riemannian manifold is the Kuratowski-Wojdyslawski embedding,

$$\mathcal{K} : x \mapsto \mathrm{dist}_M(x, \cdot\,),$$

from M to the space of continuous functions $C(M)$ on it. The mapping $\mathcal{K} : M \to C(M)$ is an isometry so that $\mathcal{K}(M)$ is an isometric representation of M in a vector space.

An other important example is the Berard-Besson-Gallot representation [9]

$$\mathcal{G} : M \to C(M \times \mathbb{R}_+), \quad \mathcal{G}(x) = \Phi_M(x, \cdot\,, \cdot\,)$$

where $(x, y, t) \mapsto \Phi_M(x, y, t)$ is the heat kernel of the manifold (M, g). The asymptotics of the heat kernel $\Phi_M(x, y, t)$, as $t \to 0$, determines the distance $d(x, y)$, and by endowing $C(M \times \mathbb{R}_+)$ with a suitable topology, the image $\mathcal{G}(M) \subset C(M \times \mathbb{R}_+)$ can be considered as an embedded image of the manifold M.

Theorem 1.2 implies that the set $\mathcal{D}(M) = \{D_x;\ x \in M\}$ can be considered as an embedded image (or a representation) of the manifold (M, g) in the space $C(F \times F)$ in the embedding $x \mapsto D_x$. Moreover, in the proof of Theorem 1.2 we show that $(F, g|_{F^{int}})$ and the set $\mathcal{D}(M)$ determine uniquely an atlas of differentiable coordinates and a metric tensor on $\mathcal{D}(M)$. These structures make $\mathcal{D}(M)$ a Riemannian manifold that is isometric to the original manifold M. Note that the metric is different than the one inherited from the inclusion $\mathcal{D}(M) \subset C(F \times F)$. Hence, $\mathcal{D}(M)$ can be considered as a representation of the manifold M, given in terms of the distance difference functions, and we call it the *distance difference representation* of the manifold of M in $C(F \times F)$.

The embedding $\mathcal{D}$ is different to the above embeddings $\mathcal{K}$ and $\mathcal{G}$ in the following way that makes it important for inverse problems: With $\mathcal{D}$ one does not need to know a prori the set M to consider the function space $C(F \times F)$ into which the manifold M is embedded. Similar type of embedding have been also considered in context of the boundary distance functions, see Subsection 1.5.2.

1.5. **Earlier results and the related inverse problems.** The inverse problem for the distance difference function is closely related to many other inverse problems. We review some results below:

1.5.1. *Determination of a compact Riemannian manifold from the distance difference functions.* This paper is closely related to the inverse problem of reconstructing a compact Riemannian manifold (N, g) from distance difference functions considered in [1]. There, the unknown set M is assumed to be an open subset of N with smooth boundary ∂M. The known measurement area is the compact set $F := N \setminus M$. It is also assumed that F^{int} is not empty. This is actually a crucial assumption since in [1] a counterexample is provided.

With this setup the distance difference data

$$\{(F, g|_F), \mathcal{D}(M)\}$$

determines uniquely, up to an isometry, the topological, smooth and Riemannian structure of (N, g).

1.5.2. *Boundary distance functions and the inverse problem for a wave equation.* The reconstruction of a compact Riemannian manifold (M, g) with boundary from distance information has been considered e.g. in [23, 25]. There, one defines for $x \in M$ the boundary distance function $r_x : \partial M \to \mathbb{R}$ given by $r_x(z) = d(x, z)$. Assume that one is given the boundary ∂M and the collection of boundary distance functions corresponding to all $x \in M$ that is,

$$\partial M \quad \text{and} \quad \mathcal{R}(M) := \{r_x \in C(\partial M);\ x \in M\}. \tag{5}$$

It is shown in [23, 25] that only knowing the boundary distance data (5) one can reconstruct the topology of M, the differentiable structure of M (i.e., an atlas of C^∞-smooth coordinates), and the Riemannian metric tensor g. Thus $\mathcal{R}(M) \subset C(\partial M)$ can be considered as an isometric copy of M, and the pair $(\partial M, \mathcal{R}(M))$ is called the boundary distance representation of M, see [23, 25]. Similar results for non-compact manifolds is considered in [14, 2]. Constructive solutions to determine the metric from the boundary distance functions have been developed in [12] using a Riccati equation [42] for metric tensor in boundary normal coordinates and in [41] using the properties of the conformal killing tensor.

The results of this paper are closely related to data (5): Knowing the distance difference functions $D_x^{\partial M} : \partial M \times \partial M \to \mathbb{R}$

$$D_x^{\partial M}(z_1, z_2) = d(x, z_1) - d(x, z_2), \quad (z_1, z_2) \in \partial M \times \partial M$$

is equivalent to knowing the boundary distance functions with error $\varepsilon(x)$, depending on $x \in M$, that is, the functions $z \mapsto r_x(z) + \varepsilon(x)$

where $x \in M$. Indeed, we can write $r_x(z) + \varepsilon(x) = D_x^{\partial M}(z, z_2)$ where $\varepsilon(x) = -d(x, z_2)$.

Physically speaking, functions r_x are determined by the wave fronts of waves produced by the delta-sources $\delta_{x,0}$ that take place at the point x at time $s = 0$. The distance difference functions $D_x^{\partial M}$ are determined by the wave fronts of waves produced by the delta-sources $\delta_{x,s}$ that take place at the point x at an unknown time $s \in \mathbb{R}$.

Many hyperbolic inverse problems with time-independent metric reduce to the problem of reconstructing the isometry type of the manifold from its boundary distance functions. Indeed, in [22, 23, 28, 38, 39] it has been shown that the boundary measurements for the scalar wave equation, Dirac equation, and for Maxwell's system (with isotropic scalar impedance) determine the boundary distance functions of the Riemannian metric associated to the wave velocity.

1.5.3. *Reconstruction from the Spherical surface data.* Let (N, g) be a complete or closed Riemannian manifold of dimension $n \in \mathbb{Z}_+$ and $M \subset N$ be an open subset of N with smooth boundary. We denote by $U := N \setminus \overline{M}$. Suppose that we are given the data

$$\{(\overline{U}, g|_{\overline{U}}), \mathcal{D}(M)\}, \tag{6}$$

where

$$\mathcal{D}(M) = \{D_x \in C(\overline{U} \times \overline{U}) : x \in M\}.$$

Let $D = D_x \in \mathcal{D}(M)$ and $w \in U$. Suppose that a point $z_0 \in U$ is not a cut point of x. Then it holds that there exists a neighborhood V of z_0 such that the function

$$f_w(z) := D(z, w) = d(x, z) - d(x, w),$$

for some $x \in M$, is smooth in V and sets

$$\mathcal{S}_{D,V,w,r} = \{z \in V : f_w(z) = r\} = \{z \in V : d(x, z) = r + d(x, w)\}$$

are metric spheres. Therefore, using the data (6) one can find the family of smooth hyper-surfaces

$$\begin{aligned} S \;\; = \;\; \{S_{D,V,w,r} :\; & D \in \mathcal{D}(M),\; w \in U,\; V \subset U \text{ is open}, \\ & D(\cdot\,, w)|_V \in C^\infty(V), r \in \mathbb{R}\}. \end{aligned}$$

In [12] one considers the *Spherical surface* data consisting of the set U and the collection of all pairs (Σ, r) where $\Sigma \subset U$ is a smooth $(n-1)$ dimensional submanifold that can we written in the form

$$\Sigma = \Sigma_{x,r,W} = \{\exp_x(rv) \in N : v \in W\},$$

where $x \in M$, $r > 0$ and $W \subset S_xN$ is an open and connected set. Such surface Σ is called spherical surfaces, or more precisely, subsets of generalised spheres of radius r. Also, in [12] one assumes that U is given with its C^∞-smooth coordinate atlas. The data (6) is close to Spherical surface data in the sense that data (6) contain only some of the spherical surfaces Σ and there is no information on the radii r of the spherical surfaces Σ. Notice that in general, the spherical surface Σ may be related to many centre points and radii. For instance consider the case where N is a two dimensional sphere.

In [12] it is shown that the Spherical surface data determine uniquely the Riemannian structure of U. However these data are not sufficient to determine (N, g) uniquely. In [12] a counterexample is provided. In [12] it is shown that the Spherical surface data determine uniquely the universal covering space of (N, g) up to an isometry.

1.5.4. *Inverse problems of micro-earthquakes.* The earthquakes are produced by the accumulated elastic strain that at some time suddenly produce an earthquake. As mentioned above, the small magnitude earthquakes (e.g. the micro-earthquakes of magnitude $1 < M < 3$) appear so frequently that the surface observations of the produced elastic waves have been proposed to be used in the imaging of the Earth near active faults [21, 43]. The so-called time-reversal techniques to study the inverse source and medium problems arising from the microseismology have been developed in [5, 13, 20].

In geophysical studies, one often approximates the elastic waves with scalar waves satisfying a wave equation. Let us also assume that the sources of such earthquakes are point-like and that one does measurements over so long time that the source-points are sufficiently dense in the studied volume. Then the inverse problem of determining the the speed of the waves in the studied volume from the surface observations of the microearthquakes is close to the problem studied in this paper. We note that the above assumptions are highly idealized: For example, considering the system of elastic equations would lead to a problem where travel times are determined by a Finsler metric instead of a Riemannian one.

1.5.5. *Broken scattering relation.* If the sign in the definition of the distance difference functions is changed in (1), we come to distance sum functions

$$D_x^+(z_1, z_2) = d(z_1, x) + d(z_2, x), \quad x \in M,\ z_1, z_2 \in F. \tag{7}$$

This function gives the length of the broken geodesic that is the union of the shortest geodesics connecting z_1 to x and the shortest geodesics connecting x to z_2. Also, the gradients of $D_x^+(z_1, z_2)$ with respect to z_1 and z_2 give the velocity vectors of these geodesics. The functions (7) appear in the study of the radiative transfer equation on manifold (N, g), see [11, 34, 35, 36, 40]. Also, the inverse problem of determining the manifold (M, g) from the broken geodesic data, consisting of the initial and the final points and directions, and the total length, of the broken geodesics, has been considered in [26].

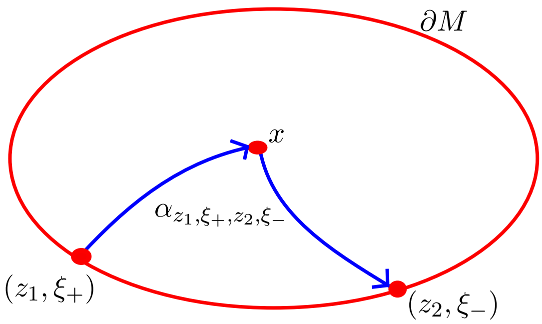

FIGURE 3. Broken geodesic data consists of initial direction $(z_1, \xi+)$, final direction (z_2, ξ_-) and total length of of broken geodesic $\alpha_{z_1,\xi_+,z_2,\xi_-}$.

2. Extension of the data and topological equivalence

We will prove Theorem 1.2 in several steps. In order to do this we start with a series of auxiliary results. Our first task is to extend the family $\mathcal{D}(M)$ to family

$$\mathcal{D}(\overline{U}) := \{D_x \in C(F \times F) : x \in \overline{U}\}.$$

Proposition 2.1. *The data* $\{(F, g|_F), \mathcal{D}(M)\}$ *determine*

$$d_g|_{F\times F} : F \times F \to \mathbb{R}.$$

Proof. Let $z_1, z_2 \in \partial M$. We start with finding $d_g(z_1, z_2)$. This can be obtained by using the triangular inequality and that $d_g(z_1, z_2) = D_{z_2}(z_1, z_2)$. Thus we see easily that

$$d_g(z_1, z_2) = \sup_{x\in M} D_x(z_1, z_2). \tag{8}$$

Let $z, w \in F$. As we are given the pair $(F, g|_F)$, we can determine the length of any smooth path $\alpha : [0, L] \to F$. Since manifold N is complete, it holds that there exists a distance minimizing geodesic segment γ from z to w. Moreover, as $\overline{U}$ is convex, the segment $\gamma([0, d_g(x, w)])$ is contained in $\overline{U}$.

When $\gamma([0, d_g(x, w)])$ does not intersect M, we can compute the length $\mathcal{L}(\gamma)$ of γ. Consider next the case when γ intersects M. Then it holds that

$$\begin{aligned} d_g(z, w) = \quad & \inf\{\mathcal{L}(\alpha) + d(q_1, q_2) + \mathcal{L}(\beta) : q_1, q_2 \in \partial M, \\ & \alpha \text{ is a smooth curve in } F \text{ connecting } z \text{ to } q_1, \\ & \beta \text{ is a smooth curve in } F \text{ connecting } q_2 \text{ to } w\}. \end{aligned}$$

This shows that we can determine the function $d_g|_{F\times F}$. □

Corollary 2.2. *The data* $\{(F, g|_F), \mathcal{D}(M)\}$ *determine* $\mathcal{D}(\overline{U})$.

Proof. Let $x \in \overline{U}$ and $z_1, z_2 \in F$. If $x \in M$, then $D_x \in \mathcal{D}(M)$. If $x \notin M$, then by Proposition 2.1, we can compute $d_g(x, z_1)$ and $d_g(x, z_2)$ from $\{(F, g|_F), \mathcal{D}(M)\}$. Thus we can determine $D_x(z_1, z_2)$. □

Proposition 2.3. *Suppose that mapping $\phi : F_1 \to F_2$ is as in Theorem 1.2 and (2)-(3) are valid. Then ϕ is a metric isometry, this is, $d_1(x, y) = d_2(\phi(x), \phi(y))$ for all $x, y \in F_1$.*

Proof. Let $x, y \in F_1$ and let $\gamma : [0, d(x, y)] \to N$ be a distance minimizing unit speed geodesic segment of g_1 from x to y contained in $\overline{U_1}$. If the geodesic segment $\gamma : [0, d_1(x, y)] \to N_1$ is contained in F_1, then we have that $\phi \circ \gamma$ is a curve connecting $\phi(x)$ to $\phi(y)$. Thus $d_1(x, y) \geq d_2(\phi(x), \phi(y))$.

Suppose that set $S := \gamma([0, d_1(x, y)] \cap \partial M \neq \emptyset$. Then there exist closest points $e_1, e_2 \in S$ to x and y respectively. We denote by α the part of geodesic segment γ from x to e_1 and β the part of geodesic segment γ from e_2 to y. Then α and β are contained in F_1. Using formulas (3) and (8) we can conclude that

$$d_1(x, y) = \mathcal{L}(\alpha) + d_1(e_1, e_2) + \mathcal{L}(\beta)$$

$$= \mathcal{L}(\phi \circ \alpha) + d_2(\phi(e_1), \phi(e_2)) + \mathcal{L}(\phi(\beta)) \geq d_2(\phi(x), \phi(y)).$$

Thus we have proved that $d_1(x, y) \geq d_2(\phi(x), \phi(y))$ in all cases. Switch the roles of x and $\phi(x)$ and y and $\phi(y)$ and notice that ϕ^{-1} also preserves the metric tensor. Therefore we can conclude that $d_1(x, y) = d_2(\phi(x), \phi(y))$. □

Since F is compact, it holds that $C(F \times F) \subset L^\infty(F \times F)$.

Corollary 2.4. *Suppose (N_i, g_i) $i = 1, 2$ U_i, M_i and F_i are as in the Theorem 1.2 and (2)-(3) are valid. Then*

$$\{D^1_x \in C(F_1 \times F_1) : x \in \overline{U}_1\} = \{D^2_y(\phi(\cdot), \phi(\cdot)) \in C(F_1 \times F_1) : y \in \overline{U}_2\}, \tag{9}$$

Proof. The claim follows from Corollary 2.2 and Proposition 2.3. □

We consider the mapping

$$\mathcal{D} : \overline{U} \to L^\infty(F \times F),\ \mathcal{D}(x) := D_x.$$

Theorem 2.5. *Mapping $\mathcal{D} : \overline{U} \to \mathcal{D}(\overline{U}) \subset L^\infty(F \times F)$ is a homeomorphism.*

Proof. Let $x, y \in \overline{U}$ and $z, w \in F$. By triangle inequality we have

$$|D_x(z, w) - D_y(z, w)| \leq |d(x, z) - d(y, z)| + |d(x, w) - d(y, w)| \leq 2d(x, y).$$

Thus $\|D_x - D_y\|_\infty \leq 2d(x, y)$ and therefore $\mathcal{D}$ is 2-Lipschitz. Hence, $\mathcal{D}$ is continuous.

Next we prove that $\mathcal{D}$ is one-to-one. To show this, assume $x, y \in \overline{U}$ are such that $D_x = D_y$. We split the proof into three different cases.

(1) If $x, y \in F$, then
$$d(x,y) = D_x(y,x) = D_y(y,x) = -d(x,y).$$
Thus $d(x,y) = 0$ and $x = y$.

(2) If $x \in F \backslash \partial M$ and $y \in M$, let $L = d(x,y)$ and $\gamma : [0,L] \to N$ be a distance minimizing geodesic from x to y. Since $\mathrm{dist}(\partial U, \partial M) > 0$ there exists $s \in (0,L)$ such that we have $z = \gamma(s) \in F$. Then $d(x,z) = s$ and $d(y,z) = L - s$. As $D_x = D_y$,
$$d(x,z) = D_x(z,x) = D_y(z,x) = d(y,z) - d(y,x) = d(y,z) - L.$$
These yield $L = d(y,z) - d(x,z) = L - 2s < L$. Hence, it is not possible that there are $x \in F \setminus \partial M$ and $y \in M$ satisfying $D_x = D_y$.

(3) Consider the case $x, y \in \overline{M}$. To show that x and y have to be equal, assume on the contrary that $x \neq y$. Let $z_x, z_y \in \partial M$ be some closest points of F to x and y, respectively. Since $D_x = D_y$, $d(x,z_x) - d(x,z_y) \leq 0$ and $d(y,z_x) - d(y,z_y) \geq 0$, it holds that
$$d(x,z_x) = d(x,z_y) \text{ and } d(y,z_y) = d(y,z_x).$$
Therefore z_x is also a closest point of F to y and z_y is a closest point of F to x.

Let $s_x = d(x,z_x) = d(x,z_y)$ and $s_y = d(y,z_x) = d(y,z_y)$. Without lost of generality we can assume that $s_x \leq s_y$. Since boundary ∂M is a smooth $(n-1)$ dimensional submanifold of N, there exists a unique inward pointing unit normal vector field ν of ∂M. Then it holds that $\gamma_{z_x,\nu}$ is the distance minimizing geodesic from ∂M to x and y.
$$x = \gamma_{z_x,\nu}(s_x) \text{ and } y = \gamma_{z_x,\nu}(s_y).$$
As geodesic segment $\gamma_{z_x,\nu} : [0,s_y] \to \overline{M}$ is a distance minimizing curve between all of its points,
$$d(x,y) = d(y,z_x) - d(x,z_x) = s_y - s_x. \tag{10}$$
Since $\gamma_{z_x,\nu}$ is orthogonal to boundary ∂M, there exists $s > 0$ such that $\gamma_{z_x,\nu}(-s,0) \cap \partial M = \emptyset$. Thus there exists a point $z \in \partial M \setminus \{z_x\}$ that is close to z_x, but the distance minimizing geodesic γ_x from z to x is not the same geodesic as $\gamma_{z_x,\nu}$, that is, the angle β of the curves γ_x and $\gamma_{z_x,\nu}$ at the point x is strictly between 0 and π. Let γ_y be a distance minimizing geodesic from y to z. Since $D_x(z,z_x) = D_y(z,z_x)$, we have $d(x,z) - d(x,z_x) = d(y,z) - d(y,z_x)$, that further yields
$$d(y,z) - d(x,z) = d(y,z_x) - d(x,z_x) = s_y - s_x = d(x,y).$$
Hence,
$$\mathcal{L}(\gamma_y) = d(y,z) = d(y,x) + d(x,z) = \mathcal{L}(\gamma_{z_x,\nu}|_{[s_x,s_y]}) + \mathcal{L}(\gamma_x).$$

Thus the union μ of the curves $\gamma_{z_x,\nu}([s_x, s_y])$ and γ_x is a distance minimising curve from z to y, and hence it is a geodesic. However, as the angle β defined above is strictly between 0 and π, the union μ of the curves $\gamma_{z_x,\nu}([s_x, s_y])$ and γ_x is not smooth at x, and hence it is not possible that μ is a geodesic. Thus the assumption $x \neq y$ led to a contradiction and hence x and y have to be equal.

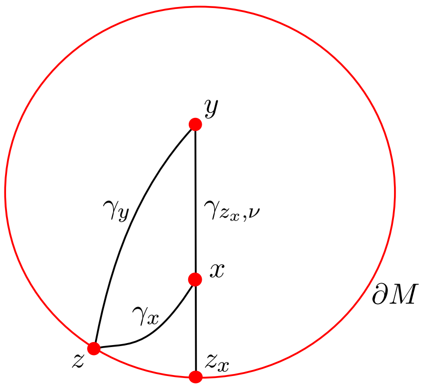

FIGURE 4. A schematic picture about the final setting of case (3).

We conclude that in all cases the assumption $D_x = D_y$ implies that $x = y$. Therefore $\mathcal{D}$ is one-to-one.

Since $\overline{U}$ is compact it follows from continuity that $\mathcal{D}$ is a closed mapping. This shows that $\mathcal{D} : \overline{U} \to \mathcal{D}(\overline{U})$ is a continuous and closed bijection that proves the claim. $\square$

We are now ready to define a mapping $\Psi : \overline{U}_2 \to \overline{U}_1$ that we will use to show that $(\overline{U}_1, g_1|_{\overline{U}_1})$ and $(\overline{U}_2, g_2|_{\overline{U}_2})$ are isometric Riemannian manifolds with boundary. Let $\mathcal{D}_i : \overline{U}_i \to C(F_i \times F_i)$, $i = 1, 2$ be defined as $\mathcal{D}_i(x) := D^i_x$. We also define a mapping

$$\Phi : C(F_2 \times F_2) \to C(F_1 \times F_1),\ \Phi(f) := f(\phi(\cdot), \phi(\cdot)).$$

Lemma 2.6. *Mapping Φ is a homeomorphism.*

Proof. Since $\phi : F_1 \to F_2$ is one-to-one and onto, it holds that mapping $f \mapsto f(\phi^{-1}(\cdot), \phi^{-1}(\cdot))$, $f \in C(F_1 \times F_1)$ exists and is the inverse of Φ.

Let $f, h \in C(F_2 \times F_2)$. Then it holds that

$$\|f(\phi(\cdot), \phi(\cdot)) - h(\phi(\cdot), \phi(\cdot))\|_\infty \leq \|f - h\|_\infty.$$

Thus Φ is continuous. The same arguments, with the mapping Φ being replaced by Φ^{-1}, show that Φ^{-1} is continuous. $\square$

By formula (9), Theorem 2.5 and Lemma 2.6 it holds that mapping

$$\Psi : \overline{U}_2 \to \overline{U}_1,\ \Psi := \mathcal{D}_1^{-1} \circ \Phi, \circ \mathcal{D}_2,$$

is well defined.

Theorem 2.7. *Mapping $\Psi : \overline{U}_2 \to \overline{U}_1$ is a homeomorphism and*

$$\Psi|_{F_2} = \phi^{-1}. \tag{11}$$

Proof. By formula (9), Theorem 2.5 and Lemma 2.6 it holds that mapping Ψ is a homeomorphism. The second claim follows from formula (9) and the definition of Ψ. $\square$

3. Smooth and Riemannian structures

Our next goal is to show that the mapping $\Psi : \overline{U}_2 \to \overline{U}_1$ is a diffeomorphism. The task at hand is to construct smooth atlases on $\overline{U}_2$ and $\overline{U}_1$ and show that with respect to these differential structures the mapping Ψ is a diffeomorphism.

Let $p \in U_2$. By [1] there exist points $\{y_i\}_{i=0}^n \in F_2^{int}$ such that mappings

$$x \mapsto (D_x^2(y_i, y_0))_{i=1}^n \text{ and } \widetilde{x} \mapsto (D_{\widetilde{x}}^1(\phi(y_i), \phi(y_0)))_{i=1}^n$$

are smooth local coordinate mappings defined in a sufficiently small neighborhood of p and $\Psi(p)$, respectively. It also holds that the local representation of Ψ in this coordinate system is an identity mapping of $\mathbb{R}^n$. Thus the following theorem holds.

Suppose that $p \in \partial U_2$. Since we assumed that $\overline{M} \subset U$ it holds that $\operatorname{dist}(\partial U, \partial M) > 0$. Therefore there exists some $r > 0$ such that sets $B_{g_2}(p, r) \subset N_2$ and $\overline{M}_2$ are disjoint. By Theorem 2.7 it holds that

$$\Psi|_{B_{g_2}(p,r)\cap\overline{U}} = \phi^{-1}|_{B_{g_2}(p,r)\cap\overline{U}}.$$

Therefore Ψ is also smooth at p. Thus we have proved the following theorem.

Theorem 3.1. *Mapping $\Psi : \overline{U}_2 \to \overline{U}_1$ is a diffeomorphism.*

The last step is to show that mapping $\Psi : \overline{U}_2 \to \overline{U}_1$ is a Riemannian isometry. This means that $\Psi^*(g_1|_{\overline{U}_1}) = g_2|_{\overline{U}_2}$. We denote $g := g_2$ and $\widetilde{g} := \Psi^*(g_1|_{\overline{U}_1})$. From now on we will use short hand notations $N_2 = N$, $F_2 = F$, $M_2 = M$ and $U_2 = U$. Next we consider the properties of the metric tensors g and $\widetilde{g}$ on $\overline{U}$. We recall that two metric tensors g and $\widetilde{g}$ defined on the same manifold $\overline{U}$ are said to be geodesically equivalent, if the geodesic curves corresponding these metric tensors are the same as unparametrized curves. In other words, any geodesic of $(\overline{U}, g)$ can be re-parametrized so that it becomes a geodesic of $(\overline{U}, \widetilde{g})$ and vice versa.

Definition 3.2. *Let $p \in F$ and $\xi \in T_pN$ be such that $\|\xi\|_g = 1$. Define a set.*

$$\sigma(p,\xi) := \{x \in U; \textit{ there is } w \in F \textit{ such that } D_x(\cdot, w) \textit{ is } C^1\textit{-smooth in some neighbourhood of } p \textit{ and } \nabla D_x(\cdot, w)|_p = \xi\}.$$

Let $p \in F$ and $\xi \in T_pN$ be such that $\|\xi\|_g = 1$. By [1] it holds that

$$\sigma(p,\xi) = \gamma_{p,-\xi}(\{s \ ; \ 0 < s < \tau(p,-\xi)\}) \cap U, \tag{12}$$

where $\tau : SN \to (0,\infty]$ is the cut distance function of metric g and $\gamma_{p,-\xi}$ is the unique unit speed geodesic of g with initial point p and initial direction $-\xi$, (see Figure 5). This means that we can find the geodesic $\gamma_{p,-\xi}(\{s \ ; \ 0 < s < \tau(p,-\xi)\}) \cap U$ as a point set. Note that the segments of geodesics of $(\overline{U}, g)$ we know as non-parametrized curves are not self-intersecting, since cut points occur before a geodesic stops to be one-to-one.

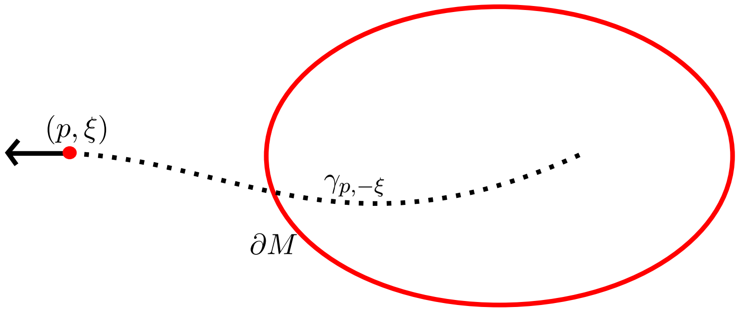

FIGURE 5. $\sigma(p,\xi)$ is the part image of geodesic $\gamma(p,-\xi)$ contained in U.

By formulas (9) and (12), it holds that set $\sigma(p,\xi)$ is also an image of some geodesic of $(\overline{U}, \widetilde{g})$. Furthermore, it is easy to see that there is a re-parametrization

$$s : \left[0, \widetilde{\tau}\Big(p, \frac{-\xi}{\|\xi\|_{\widetilde{g}}}\Big)\right) \to [0, \tau(p,-\xi))$$

such that $\gamma_{p,-\xi}(s(t))$, $t \in [0, t_1)$, is an unit speed geodesic of $(\overline{U}, \widetilde{g})$ and $\widetilde{\tau}$ is the is the cut distance function of metric $\widetilde{g}$.

Since $F^{int} \neq \emptyset$, it holds that for each $q \in M$, there exists an open cone Σ_q contained in T_qN such that for each $v \in \Sigma_q$ the corresponding geodesic segment $\gamma_{q,\frac{v}{\|v\|_g}}$ intersects F and this geodesic segment is also a pre-geodesic of metric $\widetilde{g}$, i.e., there exists a re-parametrization of the curve $\gamma_{q,\frac{v}{\|v\|_g}}$ that is a geodesic curve with respect to the metric $\widetilde{g}$. By using results of [33] for general affine connections, see [1, Lem. 2.13] for details, this is a sufficient condition for the metric tensors g and $\widetilde{g}$ on $\overline{U}$ to be geodesically equivalent.

We provide here a rough idea for the proof. Let (V, X) be a smooth local coordinate chart in M. We denote the Christoffel symbols of metrics g and $\widetilde{g}$ by Γ and $\widetilde{\Gamma}$, respectively. The first step is to show that there exists a smooth local 1-form φ on V such that the following equation holds.

$$\widetilde{\Gamma}^k_{i,j} = \Gamma^k_{i,j} + \delta^k_i \varphi_j + \delta^k_j \varphi_i, \quad \text{for } k, i, j \in \{1, 2, \ldots, n\}. \tag{13}$$

To do this we define, for each $p \in V$, a collection $\mathcal{C}(p)$ of geodesics γ of $(\overline{U}, g)$ and real numbers $t_0 \in \mathbb{R}$, given by

$$\begin{aligned}\mathcal{C}(p) \;=\; &\{(\gamma, t_0)\,;\ \gamma : (a,b) \to U \text{ is a geodesic of } (\overline{U}, g),\ \gamma(t_0) = p, \text{ and}\\ &\text{there are } z \in F^{int} \text{ and } \xi \in T_z U \text{ such that } \gamma((a,b)) = \sigma(z,\xi)\}.\end{aligned}$$

Here γ is given as a pair of the set $\mathrm{dom}(\gamma) = (a,b) \subset \mathbb{R}$, $-\infty \le a < b \le \infty$, where the mapping γ is defined and the function $\gamma : \mathrm{dom}(\gamma) \to U$. Also, $t_0 \in (a,b)$. Moreover, above $\gamma((a,b)) = \sigma(z,\xi)$ means that the sets $\gamma((a,b)) \subset U$ and $\sigma(z,\xi) \subset U$ are the same, or equivalently, that $\gamma((a,b))$ and $\sigma(z,\xi)$ are the same as unparameterized curves.

Then it holds that for each $(\gamma, t_0) \in \mathcal{C}(p)$ we have $\dot\gamma(t_0) \in \Sigma_p$. In [1] it is shown how one can use these observations and the fact that Σ_p is an open conic set to prove that equation (13) is valid for some 1-form φ.

The second step is to prove that equation (13) implies the geodesic equivalence of g and $\widetilde{g}$ on M. See [1] for details.

Finally we will introduce a function $I_0 : TU \to \mathbb{R}$ that is defined as

$$I_0((x,v)) := \Big(\frac{\det(g_x)}{\det(\widetilde{g}_x)}\Big)^{\frac{2}{n+1}} \widetilde{g}_x(v,v). \tag{14}$$

By the above, the metric tensors g and $\widetilde{g}$ are geodesically equivalent on open smooth manifold U. By [32] the geodesic equivalence of g and $\widetilde{g}$ implies that function I_0 is constant on every curve $t \mapsto (\gamma(t), \dot\gamma(t))$, where γ is a geodesic of metric $g|_U$. By (2) it holds $\widetilde{g}|_{F^{int}} = g|_{F^{int}}$. Thus we see that and $I_0(x,\xi) = 1$ for all $(x,\xi) \in TU$ such that $x \in F^{int}$. Denote by W_x the set of those $(x,\xi) \in TU$ having the property that the geodesic $\gamma_{x,\xi}$ intersects F^{int}. The invariance of $I_0(x,\xi)$ along geodesics implies that $I_0(x,\xi) = 1$ for all $(x,\xi) \in W_x$. As $W_x \subset T_0U \setminus \{0\}$ is an open set for all $x \in U$, can use the definition (14) of I_0 to see first that $\widetilde{g}$ and g are conformal on M and further that $\widetilde{g} = g$ on M (see [1, Lemma 2.16] for proof of this analysis). Since (2) and (11), hold we have proved Theorem 1.2.

4. Application for an inverse problem for a wave equation

Here we consider the application of Theorem 1.2 for an inverse problem for a wave equation with spontaneous point sources.

4.0.6. *Support sets of waves produced by point sources.* Let (N, g) be a complete Riemannian manifold. Denote the Laplace-Beltrami operator of metric g by Δ_g. (For definitions see [30, 10]). We consider a wave equation

$$\left\{\begin{array}{l}(\partial_t^2 - \Delta_g) G(\cdot,\cdot,y,s) = \kappa(y,s)\delta_{y,s}(\cdot,\cdot), \quad \text{in } \mathcal{N}\\ G(x,t,y,s) = 0, \quad \text{for } t < s,\ x \in N.\end{array}\right. \tag{15}$$

where $\mathcal{N} = N \times \mathbb{R}$ is the space-time. The solution $G(x, t, y, s)$ is the wave produced by a point source located at the point $y \in M$ and time $s \in \mathbb{R}$ having the magnitude $\kappa(y, s) \in \mathbb{R} \setminus \{0\}$. Above, we have $\delta_{y,s}(x, t) = \delta_y(x)\delta_s(t)$ corresponds to a point source at $(y, s) \in \mathcal{N}$.

4.0.7. *Inverse coefficient problem with spontaneous point source data.* Assume that there are two manifolds (N_1, g_1) and (N_2, g_2) satisfying the assumptions given in Section 1.2 for sets U_i, M_i and F_i, $i = 1, 2$. In addition we assume that there exists a diffeomorphism $\phi : F_1 \to F_2$ such that (2) is valid and

$$W_1 = W_2 \tag{16}$$

where W_1 and W_2 are collections of supports of waves produced by point sources taking place at unknown points at unknown time, that is,

$$W_1 = \{\mathrm{supp}\,(G^1(\cdot, \cdot, y_1, s_1)) \cap (F_1 \times \mathbb{R});\ y_1 \in M_1,\ s_1 \in \mathbb{R}\} \subset 2^{F_1 \times \mathbb{R}}$$

and

$$W_2 = \{\mathrm{supp}\,(G^2(\phi(\cdot), \cdot, y_2, s_2)) \cap (F_1 \times \mathbb{R});\ y_2 \in M_2,\ s_2 \in \mathbb{R}\} \subset 2^{F_1 \times \mathbb{R}}$$

where functions G^j, $j = \{1, 2\}$ solve equation (15) on manifold N_j. Here $2^{F_j \times \mathbb{R}} = \{F';\ F' \subset F_j \times \mathbb{R}\}$ is the power set of $F_j \times \mathbb{R}$. Roughly speaking, W_j corresponds to the data that one makes by observing, in the set F_j, the waves that are produced by spontaneous point sources that that go off, at an unknown time and at an unknown location, in the set M_j.

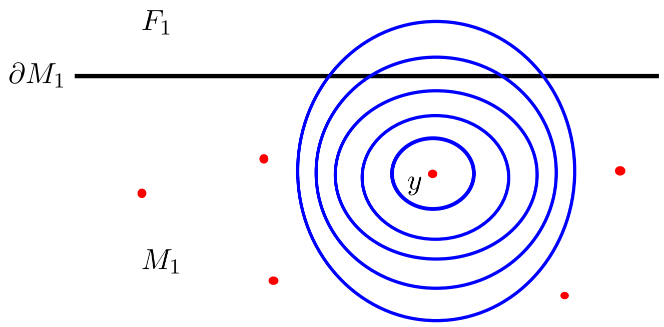

Figure 6. Illustration of $\mathrm{supp} G_1(\cdot, \cdot, y, s) \subset \mathcal{N}_1$.

Earlier, the inverse problem for the sources that are delta-distributions in time and localized also in the space has been studied in [13] in the case when the metric g is known. Theorem 1.2 yields the following result telling that the metric g can be determined when a large number of waves produced by the point sources is observed:

Proposition 4.1. *Let (N_j, g_j), $j = 1, 2$ be a complete compact Riemannian n-manifold, $n \geq 2$. Let $M_j \subset N_j$ be an open set whose closure is contained in open set U_j . Suppose also that ∂M_j is smooth, U_j is* *relatively compact, ∂U_j is smooth and $\overline{U}$ is convex. If the spontanuous point source data of $\overline{U}_1$ and $\overline{U}_2$ coincide, that is, we have* (2) *and* (16)*, then $(\overline{U}_1, g_1|_{\overline{U}_1})$ and $(\overline{U}_2, g_2|_{\overline{U}_2})$ are isometric Riemannian manifolds with boundary.*

Proof. We provide here a sketch of the proof (see [1] for the detailed proof). The main idea is to relate the numbers

$$\begin{aligned} \mathcal{T}_{y,s}(z) \;:=\; & \sup\{t \in \mathbb{R};\ \text{the point } (z,t) \text{ has a neighborhood} \\ & A \subset \mathcal{N} \text{ such that } G(\cdot,\cdot,y,s)|_A = 0\} \end{aligned}$$

$y \in M$, $s \in \mathbb{R}$, and $z \in F$, to the distance difference functions. The number $T_{y,s}(z)$ tells us, what is the first time when the wave $G(\cdot,\cdot,y,s)$ is observed near the point z. Using the finite velocity of the wave propagation for the wave equation, see [17], we see that the support of $G(\cdot,\cdot,y,s)$ is contained in the future light cone of the point $q = (y,s) \in \mathcal{N}$ given by

$$J^+(q) = \{(y',s') \in \mathcal{N};\ s' \geq d(y',y) + s\}.$$

Next, step is to show that a wave emanating from a point source (y,s) propagates along the geodesics of manifold (N,g) and the boundary

$$\partial J^+(q) = \{(\exp_y(t\eta), s+t) \in \mathcal{N}; \eta \in S_yN,\ t \geq 0\}.$$

See [15] and [16]. Therefore it can be shown that the function $G(\cdot,\cdot,y,s)$ vanishes outside $J^+(q)$ and is non-smooth, and thus non-zero, in a neighbourhood of arbitrary point of $\partial J^+(q)$. Thus, for $z \in F$ we have $\mathcal{T}_{y,s}(z) = d(z,y) - s$. Hence the distance difference functions satisfy equation

$$D_y(z_1,z_2) = \mathcal{T}_{y,s}(z_1) - \mathcal{T}_{y,s}(z_2). \tag{17}$$

Therefore we see using equation (17) that we have (2)-(3). Hence, the claim follows from Theorem 1.2. □

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