A learning path for rational numbers through different representations

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Abstract. *In this paper, we aim to understand, in a context of teachers' collaborative group, how emphasizing multiple representations can contribute to the learning of the rational numbers by elementary school students. We report part of a Design Based Research, within which a learning path for rational numbers was constructed and implemented in grade 3 classes. Data was collected through audio recordings of the collaborative group sessions, written records of the teachers' and students' interactions, as well as photo records of classroom work. We analyse two tasks focusing on students' rational number learning of two classes, through discussion and reflection in the collaborative group. The results show that enactive and iconic representations, used as models in a recurrent way, support an intertwined understanding of symbolic representations. We conclude that the collaborative group work was essential to bring research into the classroom.*

Keywords: Rational numbers, representations, elementary school, collaborative work.

Introduction

In mathematics education research, learning and understanding rational numbers is a very important and complex topic (Behr, Lesh, Post, & Silver, 1983; Tian & Siegler, 2018). This complexity relates to the multiple representations and meanings that rational numbers can assume. Although research provides clues about how different representations can be articulated, with understanding, at an early stage of students' learning (e.g., Moss & Case, 1999), its effective implementation in the work carried out in the classroom seems hard to reach. Thus, we seek to implement mathematics education innovations into practice within an implementation research perspective (Century & Cassata, 2016). In this paper, we aim to understand, in the context of the collaborative work of a group of teachers, how research-based ideas on emphasizing multiple representations and models (Guerreiro, Serrazina, & Ponte, 2018; Morais, Serrazina, & Ponte, 2018) may contribute to the learning of the rational numbers, by elementary school students.

Multiple representations in learning rational numbers

When learning rational numbers[1], students should realize that the same rational number might be expressed in different symbolic representations, such as decimal number, percentage or fraction.

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[1] We use the term "rational numbers" to designate non-negative rational numbers.

Besides these, enactive and iconic representations (Bruner, 1999), actions and images, respectively, and oral and written language (Ponte & Serrazina, 2000), a supporting mode of representation at the early grades, should also be considered. These types of representations are considered useful in the development of students' conceptual understanding as they help them to track ideas and inferences when reflecting and structuring a problem.

Moving across different types of representations is essential for the recognition that each representation presents a different perspective of rational numbers, and students' understanding develops as the number of perspectives increases (Ponte & Quaresma, 2011; Tripathi, 2008). Gravemeijer (1999) reinforces that a model emerges when it is underpinned by representations. In this emergent modelling process, representations become models, as they allow a direct modelling of a contextualized situation and support the development of more formal mathematical knowledge (Gravemeijer, 1999). Consequently, learning rational numbers through models, at the elementary grades, may be a dynamic process required to co-develop representation and conceptual understanding.

Contexts, within which representations can be perceived as models, are fundamental to understand and establish complex and meaningful relations (Brocardo, 2010). The number line, with an implied measure meaning, and the *decimat* (Roche, 2010) that emphasizes a part-whole meaning, are useful representations that highlight the multiplicative structure of rational numbers. Post, Cramer, Behr, Lesh, and Harel (1993) highlight the role of representations in understanding rational numbers, relating it to the flexibility with transformations between and within rational number representations. Thus, students' flexibility in making transformations involving different representations can show their understanding of the rational numbers involved (Post, Wachsmuth, Lesh, & Behr, 1985).

Recognizing the same rational number across different representations is an outcome of a global coordination of representations, which in turn empowers mathematical reasoning (Duval, 2006). The transformations of representations, which support translation between different representations of rational numbers, are central in the mathematical activity (Duval, 2006), and the analysis of these transformations provides a lens to access students' mathematical reasoning processes, as solving strategies and justification (Mata-Pereira & Ponte, 2017).

Methodology

This study follows a design-based research approach (Cobb, Jackson, & Dunlap, 2016). It focuses on two grade 3 classes (8 years old) students' learning. These classes belong to two teachers that participate on a collaborative group of five teachers, with the first author as a regular member. The group has been meeting weekly for about ten years to plan classes lessons together. In different ways, all teachers have connections to the research field, in which they seek to support their professional practice.

To face the demand of learning rational numbers in grade 3 (a national curriculum determination) with understanding, the group decided to construct a learning path, a trajectory for learning rational numbers, that promoted an active engagement of students in the construction of knowledge through meaningful tasks. Thus, along with the first author, the second author also integrated the group as invited researcher, with the purpose of sharing key ideas of ongoing studies to be discussed in the group (Guerreiro, Serrazina, & Ponte, 2018; Morais, Serrazina, & Ponte, 2018) and support the construction, implementation and reflection of tasks. Together, teachers and researchers, collaborated for developing the learning path, meeting those students, with the aim of implementing theoretical ideas identified as key in the learning of rational numbers, understood as innovations (Century & Cassata, 2016). All the three teachers put the learning path into action, but only two, Hélia and Sandra, actively participated in all the sessions of collaborative group, thus being participants in this research. These sessions were held once a week, according to the school calendar, between February and June of 2018.

The need to intertwine different symbolic representations of rational numbers, supported by several representations used as models, like common batteries icons or status bars, but also the multibase arithmetic blocks (MAB) or the decimat, which make explicit the “ten-ness” of the base ten place value system (Roche, 2010). This was one of the guiding principles of the learning path constructed and implemented, within this design-based research. This path privileged the symbolic representations of percentage (Phase 1), decimal number (Phase 2 and 3), and fraction (Phase 4), using part-whole and measure meanings, according to the sequence presented in Figure 1.

PHASE 1	PHASE 2	PHASE 3	PHASE 4
Percentage Decimal number Fraction	Percentage **Decimal number (hundredth)** **Fraction (decimal)**	Percentage **Decimal number** Fraction (decimal)	Percentage Decimal number **Fraction**
Status bar Batteries Number line (double and simple)	Status bar Bottles Number line (double and simple)	Number line (double and simple) MAB 10x10 grid Decimat	Number line (double and simple) Without models

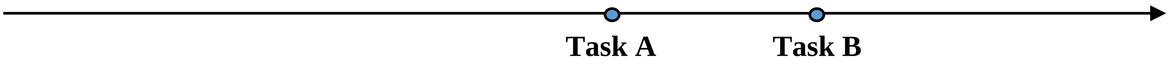

Figure 1: Implemented learning path of rational numbers

Data was collected through audio recordings of the group sessions, collection of written records of the teachers, the written works of students (brought by the teachers for the sessions), as well as photo records made by teachers during classroom work.

We focus the analysis on two tasks, Task A and Task B, solved by students, which were carried out in Phase 3 of the learning path. Data analysis is centered on students’ use of representations as models, and on how they use and relate different symbolic representations.

Results

Task A

Before task A, students were asked to fill and empty five bottles in order to establish relationships between their capacities, and associate labels written in decimal numbers to each bottle (Figure 2).

Figure 2: Labeling bottles task

Two bottles had the same capacity, and two labels represented the same quantity: 0.5 *l* and 0.50 *l*. At the group session when the work carried out in this task was discussed, the teachers identified the need to lead students to justify the equality between the representations 0.5 and 0.50:

Hélia: Now that we have finished [the task] the filling of the bottles and they realized that five tenths are equal to fifty hundredths but they don't know why. . . So, we could go from there...

Helena: To the equivalence with different representations.

. . .

Hélia: So, I will ask why fifty hundredths are the same as five tenths, because I want them to be able to justify...

Although Sandra suggested that the status bar could be used as a support representation for task A, the group discussed that it would be important to promote the use of the number line, recognizing the need to evolve to a more formal representation.

Sandra, referring to her students, suggested that the double number line could be presented with marks corresponding to tenths and, eventually, to hundredths:

Sandra: In my class... I usually focus their [students] attention on "Into how many [parts] is [the unit] divided? So, they are how many of how many?" That's why we came to this [suggested way to divide the number line] . . .

This suggestion stems from the fact that Sandra acknowledges that her students had already explored rational numbers mainly with a part-whole meaning, which had been associated with the fraction representation. Hélia recalls that her students used, above all, the decimal fraction with denominator one hundred, because percentage was a reference for them:

Hélia: When I call upon fraction it's always decimal fraction with hundredths . . . Because for them percentage is... Is their guiding line!

Thus, in anticipating how her students would relate 0.5 and 0.50, Hélia suggests that 50% should be already presented in the double number line:

Hélia: Let's just start by asking. They will realize that fifty is because it is divided... The percentages here... I think that it should be 50% already represented...

However, the group discussed that by dividing the number line in that way, in order to include other symbolic representations, could restrain students when choosing a representation to justify. Therefore, the group decided to present only the representation in decimal number to, on one hand, focus the question in this specific equivalence and, on the other hand, to allow the students to mobilize representations on their own.

The task implementation showed that the students used different kinds of justifications. Sandra mentioned that her students justified the equality between 0.5 and 0.50 by presenting arguments based on an enactive representation, related to what they had experienced in the labeling bottles task:

Sandra: They concluded that five tenths are equal to fifty hundredths because these bottles had the same capacity…

These arguments seem to show that, in this task, the students understood the symbolic representation of a decimal number as a label that identifies a certain amount of water, an understanding that is strongly linked to the context. The meaning that the students give to the decimal number symbolic representation was based on an enactive representation.

The arguments used by Hélia's students show an understanding of the symbolic representations involved without needing to refer to their experience in the task of labeling bottles.

Hélia: Almost all the groups [of students] concluded that in five tenths we have the unit divided in ten parts, and that in fifty hundredths we have the unit divided in one hundred parts…

The meaning that the students assign to the symbolic representation in decimals is supported by the interpretation of the number line and other representations, such as the decimal fraction (Figure 3).

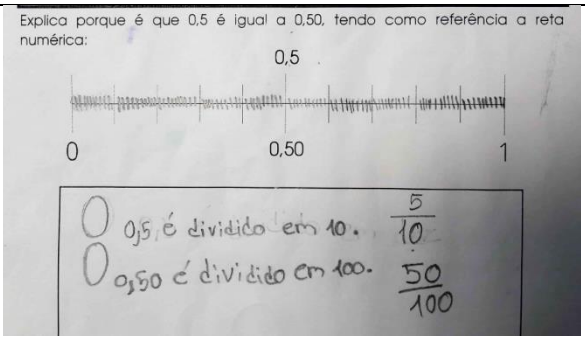 Explica porque é que 0,5 é igual a 0,50, tendo como referência a reta numérica: 0,5 0 0,50 1 0,5 é dividido em 10. 5/10 0,50 é dividido em 100. 50/100	Explain why 0.5 is equal to 0.50, considering the number line: *0.5 is divided in 10. 5/10* *0.50 is divided in 100. 50/100*

Figure 3: Written record of a student from Hélia's class – Task A

In this way, the students seem to recognize the same number in different representations, which they mobilize to justify the equivalence between 0.5 and 0.50. In this phase of the learning path, in addition to the enactive representation of the bottles and the number line iconic representation, the decimat was another important representation for the understanding of the decimal number representation.

Task B

In Task B, the students were asked to represent the shaded areas of the decimat, in percent, decimal number and fraction. The written record presented in Figure 4 illustrates how a student from Sandra's class used the decimat as a model, as the student gives meaning to this representation, using percentage to keep track of the relations established.

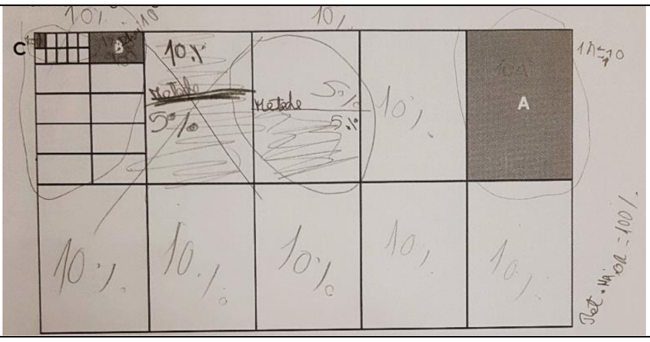	*10%, half 5%* [rectangle A] *1%, I have divided into 10 the 100* [hundredth, rectangle B] *1000* [rectangle C] *Larger rectangle = 100%*

Figure 4: Written record of a student from Sandra's class – Task B

The student interpreted the decimat that she named as the larger rectangle, as 100%, associating each rectangle A to 10%. She identified half of 10% as 5%, which she justified by tracing half of a rectangle A. She interpreted rectangle B as 1%, justifying it with the division of 100, which she seems to relate to one hundredth, by 10. She also pointed rectangle C as "1000", which seems to show that the student identified the relation 1/1000 of this rectangle with the larger one. Thus, the student seemed to have understood the relationship between each shaded area in the decimat.

Hélia agreed that this model allowed her students to visualize and mobilize different symbolic representations, facilitating their interrelation. This is evident when Hélia reflected on another task involving the decimat:

> Hélia: For my students it was a systematization… This construction was essential. For example, my students looked at the decimat and most of them said that it was [shaded] 30%.

The use of this iconic representation was highly valued by the teachers as a "systematization representation", as Hélia mentioned, considering the phase of the learning path in which it was used. Although the decimat representation is usually perceived as related to the decimal number representation, it supported and triggered other representations such as percentage.

Final remarks

Considering that the aim of this study was to understand how emphasizing multiple representations could contribute to the learning of rational numbers in the elementary school, this study shows that a work involving the interrelation of percentage, decimal number and fraction, is very promising in the initial approach of the learning of rational numbers.

We emphasize that the understanding of these symbolic representations, and their relations, is supported by the use of enactive and iconic representations (Webb, Boswinkel, & Dekker, 2008), in a recurrent way. Enactive and iconic representations were used as models, leading students into the establishment of relationships (Gravemeijer, 2004), between and within the same type of representation. Therefore, enactive and iconic representations allowed the emergence of symbolic representations with understanding.

Even though the learning path was constructed according to the sequence of symbolic representations, percentage – decimal number – fraction, the students mobilized the representation that they considered most appropriate to justify the equivalence relation. This ability shows an apparent confidence in working with rational numbers, considering the initial stage of these students' learning of this concept. In this way, a work focused on the understanding that the same number can assume different representations, not only symbolic, but also enactive and iconic, contributed to the students' conceptualization of rational number.

The fact that the Portuguese curriculum does not emphasize percentage at grade 3, as a rational number representation, could be considered an obstacle to the implementation of the intervention. However, the fact that the collaborative group was willing to develop a learning path comprised by key research findings overcame this obstacle. The teacher group already had a joint work routine prior to this study facilitated the sharing environment created. This was a factor that influenced the effectiveness of this intervention. Those teachers used to plan and reflect together and, with researchers, they got confidence in constructing and implementing the learning path, a common aim that provided a desirable gateway for implementing research in the classroom. A discussion about how to echo this experience on a larger scale is still much needed to enhance its beneficial effects in new contexts (Century & Cassata, 2016).

We highlight the relationship between theory and practice in this study: theory found a way to support practice, identifying the guiding principles of this intervention, and practice provided clues for problematizing theory, necessarily refining those principles. On one hand, the learning path of rational numbers was constructed based on previous research, which emphasizes the crucial articulation among symbolic representations of rational numbers (e.g., Moss & Case, 1999; Ponte & Quaresma, 2011; Tripathi, 2008). On the other hand, the practice informed that not only this articulation is possible at the elementary school grades, but also that it benefits from a continuous work that embeds other types of representations.

Acknowledgment

This research is supported by national funds through Universidade de Lisboa by grant to the first author and through FCT – Fundação para a Ciência e Tecnologia to the second author (SFRH/BD/108341/2015).

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