Enhancing students' generalizations: a case of abductive reasoning

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The aim of this paper is to understand how a path of teacher's actions leads to students' generalization. Generalization, as a main process of mathematical reasoning, may be inductive, abductive, or deductive. In this paper, we focus on an abductive generalization made by a student. The study is carried out in the third cycle of design of a design-based research involving lessons about linear equations in a grade 7 class. Data is gathered by classroom observations, video and audio recorded, and by notes made in a researcher's logbook. Data analysis focus on students' generalizations and on teacher's actions during whole-class mathematical discussions. The results show a path of teacher's actions, with a central challenging action, that allowed an extending abductive generalization, and also a subsequent deductive generalization.

Keywords: Generalization, Abductive reasoning, Mathematical reasoning, Teacher's actions.

Introduction

Generalizing in algebra is a highly relevant aspect of mathematical teaching and learning as it is an essential part of algebra (Kaput, 2008). In addition, generalizing is a central mathematical reasoning process. As such, what the teacher does in the classroom to enhance students' generalizations is of great importance. To enhance students' mathematical reasoning in the classroom, and hence generalizations, involves setting a challenging learning environment that goes beyond proposing exercises to solve using well-known procedures. In this research, we address mathematical whole-class discussions, unleashed by exploratory tasks (Ponte, 2005), as privileged moments to promote students' mathematical reasoning. Seeking to develop knowledge about how teachers can help students to engage in mathematical reasoning, we conduct a design-based research (Cobb, Jackson, & Dunlap, 2016). In this paper, we focus on a specific situation of the third cycle of design, aiming to understand how a path of teacher's actions, supported by design principles that focus on generalization, lead to students' generalization, particularly in a case of an abductive generalization.

Mathematical reasoning

There are several definitions of mathematical reasoning, but most of them gravitate around the idea of making justified inferences (e.g. Aliseda, 2003; Pólya, 1954; Rivera & Becker, 2009). What differs in those various definitions is the path that takes place from prior knowledge to new knowledge. As such, the perspectives on mathematical reasoning accommodate both logical and intuitive aspects, providing a scope that includes deductive, inductive and abductive inferences. Deductive inference, characterized by a logic perspective, has two main characteristics: (1) certainty, that refers to the necessary relationship between premises and conclusion, where the conclusion follows necessarily from a set of premises, and (2) monotonicity, related to the irrefutability of conclusions, i.e., a valid inference remains valid when additional premises are added (Aliseda, 2003). Deductive inference, despite being often presented as the paradigm of mathematical reasoning (Aliseda, 2003) is not necessarily the single path to carry out mathematical reasoning. There are other rigorous forms of reasoning, such as inductive and abductive inferences (Jeannotte & Kieran, 2017; Rivera & Becker, 2009), although they do not provide the same certainty and irrefutability as deductive inferences (Aliseda, 2003). Russell (1999) stresses that mathematical reasoning consists in thinking about properties of a mathematical object and developing generalizations that apply to a broad class of objects, thus underlining the inductive aspect of mathematical reasoning. On one hand, inductive inferences occur essentially when predictions are made or conjectures are formulated (Aliseda, 2003), and are also associated with generalization from the identification of a certain characteristic common to several cases (Rivera & Becker, 2009). On the other hand, abductive inferences have mainly an explanatory role, but also have a knowledge-building role. Thus, abductive inferences aim to construct hypotheses for unknown phenomena, being a reasoning used to explain something intriguing (Aliseda, 2003) or to discover something (Magnani, 2001). In this sense, abductive reasoning is identifiable with the formulation of a generalization based on relations between aspects of a given situation and its conclusions are plausible in the context of the situation (Rivera & Becker, 2009).

Generalizations

Given its complex nature, mathematically reasoning involves a variety of processes that are evidenced in the students' individual thinking and sense making, in their classroom work, and in the interactions that take place during whole-class discussions (Brodie, 2010). These processes include formulating questions and solving strategies, formulating and testing generalizations and other conjectures, and justifying them. From these reasoning processes, we hereby highlight generalization as a key process of mathematical reasoning and, hence, of algebraic thinking. Generalizing, by stating that an idea, property or procedure is valid to a given set of objects (Dörfler, 1991; Ellis, 2007), is the basis of many mathematical ideas and concepts. On a day-to-day basis, students are naturally predisposed to generalize (Becker & Rivera, 2005). However, it is important to note that in the classroom, generalizations may be incorrect or only implicit presented (Becker & Rivera, 2005; Reid, 2002). Thus, to promote students' mathematical reasoning, it is necessary to create situations in which generalization plays a central role (Kieran, 2007), in order to lead students to present generalizations based on mathematical ideas, concepts and properties.

Generalizations that students present or use in the classroom may emerge from different approaches and at different levels. To develop the capacity of formulating generalizations, both in empirical and deductive approaches, students may act at three levels: factual, contextual and symbolic (Radford, 2003). Factual generalization comes from empirical observation or particular cases that are applied to new cases in the same set of mathematical objects. Contextual generalization, also based on empirical observation or particular cases, presumes an extension to a new set of mathematical objects. Symbolic generalization emerges from the use and understanding of symbolic language. Within this scope of levels, students' generalizations may emerge from (a) relating, when students create a relation or make a connection between situations, ideas or objects; (b) searching, when students search for an element of similarity, or (c) extending, when students go beyond the situation or case, which originated the generalization (Ellis, 2007). Moreover, as highlighted by Jeannotte and Kieran (2017), generalizing is a process related with the search for similarities and differences. As such, when a student generalizes, it is possible to identify either a continuing phenomenon, an element of sameness, or a general principle (Ellis, 2007). Generalizations refer to a continuing phenomenon when the students identify properties that go beyond a particular instance. When an element of sameness is at stake, the students identify either a common property, the same objects or representations or the same situations. Regarding generalizing by stating a general principle, the students may identify general rules, patterns, strategies and global rules.

Teacher's actions during whole class discussions

To promote generalization at different levels, and subsequently to contribute to students' competence of a proper use of inductive, abductive and deductive reasoning, teacher's actions are a central aspect. These teacher's actions to promote mathematical reasoning in the classroom should consider the different moments of the lesson. In lessons framed by exploratory teaching (Ponte & Quaresma, 2016), whole-class discussion moments that stand out as very promising to enhance students' mathematical reasoning (Ponte, 2005). Ponte, Mata-Pereira, and Quaresma (2013) identify four main categories of teacher's actions that can be distinguished during whole class discussions and that are directly related to mathematical processes: (i) inviting actions – leading students to engage in the discussion, (ii) guiding/supporting actions – conducting students along the discussion in an implicit or explicit way in order to continue the discussion; (iii) informing/suggesting actions – introducing information, providing an argument or validating students' interventions; and (iv) challenging actions – leading students to add information, provide an argument or evaluate an argument or a solution. Guiding/supporting, informing/suggesting, and challenging actions, are main supports to develop whole class mathematical discussions, and involve key mathematical processes such as (i) representing – provide, revoice, use, change a representation (including procedures), (ii) interpreting – interpret a statement or idea, make connections, (iii) reasoning – raise a question about a claim or justification, generalize a procedure, a concept or a property, justify, provide an argument, and (iv) evaluating – make judgments about a method or solution, compare different methods.

Methodology

This paper reports part of the third cycle of a research study that follows a design-based research (Cobb et al., 2016) aiming to develop a local theory about enhancing students' mathematical reasoning in the classroom. Before this third cycle of design, a first cycle took place in lessons about sequences and a second cycle in lessons about linear equations. In order to achieve the overall aim of this research, we establish a set of design principles (Cobb et al., 2016) based on the literature and on previous cycles of design focusing on tasks and on teacher's actions to enhance students' mathematical reasoning, particularly emphasizing generalizing and justifying. Due to the focus of this paper, here we specifically focus on three principles for teacher's actions that aim to enhance students' generalizations, indicating that the teacher should (a) promote situations that prompt students to share ideas, namely considering and valuing invalid or partially valid contributions, deconstructing, complementing or clarifying them, (b) support or inform students in order to highlight reasoning processes, particularly generalizing, and (c) challenge students to go beyond the task.

The episodes reported in this paper took place in a Portuguese public school in a grade 7 class with 27 students (12-13 years old), and involved nine lessons about linear equations. These were students' first approach to equations, and connections between functions and equations had not yet been addressed. The particular episodes presented are from lesson 3 and lesson 6 and regard the number of solutions of a linear equation, with a specific focus on equations with no solutions. The main goal of lesson 3 was to introduce the property of invariance of equality by multiplying and the proposed exploratory task included solving an equation in order to generalize this property. In lesson 6, the main goal was to discuss the number of solutions of an equation, particularly in impossible equalities. In order to do this, students are proposed to solve some equations, including $3x + 6 - x - 15 = 2x + 9$. Both lessons were directly observed and video and audio recorded, and notes were made in a researcher's logbook. A detailed plan of each lesson, prepared by the first author and discussed in detail with the teacher, was made attending to the tasks to propose and considering teacher's actions to enhance students' mathematical reasoning. The participating teacher was selected because of her experience, commitment to professional development, and availability to consider changes in her practice. All participants in this study are volunteers, have fictitious names and have given their informed consent to participate. Data analysis is centered on students' generalizations and focus on the design principles and the conceptual framework regarding teacher's actions.

An unexpected generalization

After a whole class discussion about the task that aimed to introduce the property of invariance of equality by multiplication, the teacher begins to register this property on the board. However, while writing down the property, the teacher realizes that during the discussion, the exception of zero was not taken into account. As such, she poses a question to students, regarding possible exclusions:

Teacher: Let us register the multiplication property of equality that says that, if one multiplies or divides each member of the equation by the same number… Any number? Or do I have to ensure something? If we multiply or divide both members of the equation by the same number, my question is, by any number? Or is there any number that I have to exclude?

As the task that students had previously worked on and discussed did not include any question regarding this exclusion, by posing this question to students, the teacher is *challenging* them to go beyond the proposed task (principle c). This question receives an immediate answer from Clara, one of the students. However, the teacher decides on going further on the discussion by *challenging* students to present for a justification (principle c).

Clara: Zero.

Teacher: Why?

Gabriel: Because it is neutral, is neutral! Is the neutral element.

Gabriel provides an invalid justification based on his previous knowledge of the formal properties of operations. At this point, teacher opts on *guiding* students to deconstruct the invalid statement (principle a):

Teacher: Easy there, is the neutral element of which operation?

Several students: Addition.

Teacher: But are we talking about addition?

Leonardo: Oh, no, is about multiplication.

Teacher: Is the… Element, how is it called?

Gabriel: Neutral!

Leonardo: No, is the one that absorbs everything.

Clara: Absorbing.

After some students' interventions, one of them presents a justification that is considered by the teacher as being partially correct. However, the teacher keeps *guiding* students in order to go further on their justification (principle a).

Teacher: Absorbing element. So, can I… Can I divide by zero?

Several students: No.

Teacher: No, it doesn't make sense. Can I multiply? What is the problem of multiplying both members by zero?

Several students: Is going to be zero.

Teacher: I will get zero equals zero and I will not be able to move forward. So, [continuing to write the property] distinct from zero, the solution-set is preserved.

At this point of the discussion, the teacher *informs* students in order to conclude the introduction to the multiplication property of equality (principle b).

After clarifying why zero has to be excluded in this property, and straight after writing down the property, Clara asks to intervene:

Clara: Teacher, I don't know why, but after you wrote down that [the property of invariance of equality by multiplication] I believe… I have this feeling that not all equations have a solution.

Supported by the property of invariance of equality by multiplication, Clara *generalizes* that not all equations have a solution. In this generalization, Clara relates aspects of the property that is being discussed with what she knows so far about equations, without relying on a particular example, presenting an abductive generalization. By its relations to the particular property and also by expanding its scope, this generalization is a general principle of a contextual and extending nature. This was not an expected generalization at this moment, however, the teacher *challenges* Clara to elaborate on her statement (principle c).

Teacher: Why did this [multiplication property of equality] lead you to believe that not all [equations] have a solution?

Clara: I don't know, but…

Teacher: But I got curious, why did anything that I have said here…

Clara: I do not know it myself… I think it is because of zero… I don't know, but I get the feeling that not all of them have. . . .

Teacher: Look [Clara], hold back, if necessary take a register . . . Stating that this made you think that it might have equations that do not have a solution. When we discuss this issue, that won't be right now, we will see.

As Clara cannot go further on her justification, the teacher begins by *challenging* this student again to present a justification (principle c), but then quits to obtain such justification and *informs* the class that Clara's idea will be discussed later.

Validating Clara's generalization

A few lessons after, the teacher proposed a task to students in order to introduce the classification of linear equations according to the number of solutions. While the students were solving an equation proposed in the task using the properties of equality, the equation $0x = 18$ emerged, triggering the discussion that follows:

Leonardo: But that will keep having an infinity of solutions.

Teacher: Will it?

Leonardo: Oh, no, no, it won't work for every number.

Teacher: Won't work for every number? . . .

Leonardo: No, because no number times zero equals 18! . . . So, isn't this a false equality? Is, isn't it? This is a false equality, because any number times zero will equal zero.

Teacher: And?

Gustavo: This one doesn't have a solution, teacher. . . Is impossible! . . .

Leonardo: So, this means that there are equations with no solution!

As a previous equation of the task had infinite solutions, Leonardo wrongly *generalizes* that this equation also has it. By being wrong, this generalization is factual and of a searching nature, as the student considers that it belongs to the same set of mathematical objects, looking for an element of sameness. However, instead of telling the student that he is wrong, the teacher *challenges* him to evaluate his statement (principle a), which he properly does. Once more, the teacher questions the student *challenging* him to justify his statement (principle c), which he does. By *guiding* the student to continue (principle a), students *generalize* that there are equations with no solutions, based on the example of the equation that they are working with. This generalization, by going beyond the previous set of mathematical objects is of a contextual level and has a relating nature. Moreover, it refers to a continuing phenomenon as it identifies a property that does not apply only to this particular instance. After concluding this segment of the discussion, Clara intervenes, recalling that she had already sated this generalization:

Clara: Teacher, do you remember that lesson in which I had… That thing that we had regarding the multiplication property of equality, that I said that… Because I hear about zero and it was…

Teacher: Exactly, it was because of zero. That is why I asked you what made you think [that]. Because it was when we talked about zero that you did that observation. . .

Clara: It was because of zero, but I couldn't [explain].

At this point, teacher *informs* that it was valid and that the argument was also correct, despites incomplete, as Clara was not able to justify her statement (principle b).

Discussion and conclusion

Generalizations on both episodes emerged from a path of teacher's actions with a central challenging action. These paths include this central action followed by other challenging actions or by guiding actions and end up with an informing action from the teacher. These teacher's actions are strictly related to the design principles of the study as challenging. Guiding actions mostly relate to situations where a partially valid or invalid contributions emerge or to situations where the students are challenged to go beyond what was initially proposed. In addition, informing actions have a relationship with situations where the teacher highlights a generalization.

Whereas the paths of actions oriented by the design principles lead, in both episodes, to generalizations, the nature of the generalizations that emerged is not the same. In the first episode, Clara, supported by the discussion around the property of invariance of equality by multiplication, presents an abductive generalization as she relates aspects of the situation being presented, despite not being able to justify her statement. This generalization, is a contextual and extending generalization of a general principle. As Ellis (2007) indicates, this type of generalization is what researchers seek. In the second episode, the valid generalization that emerges is more deductive. This generalization, by its logical form, only needs an example to be valid. As such, the students properly realize that not all equations have a solution. This generalization, despites also being of a contextual level, is considered as a relating generalization of a continuing phenomenon.

This study shades light to what generalizations of different nature can look like in the classroom and particularly on what paths of teacher's actions can lead to such generalizations. In doing so, the framework for teacher's actions proposed by Ponte, Mata-Pereira, and Quaresma (2013) as well as the design principles have a central role. Another aspect highlighted by this study is the idea that developing algebraic thinking does not have to be immediate, as this competence should include the intellectual patience to embrace a partial understanding and the confidence that, with future actions, knowledge will advance (Arcavi, 2007). As such, not only opportunities for deductive generalizations should be considered, but also there should be opportunities for students to generalize abductively and inductively.

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References

Aliseda, A. (2003). Mathematical reasoning Vs. abductive reasoning: A structural approach. *Synthese, 134*(1/2), 25–44.

Arcavi, A. (2007). El desarrollo y el uso del sentido de los símbolos. *Revista Uno, 44*, 59–75.

Becker, J. R., & Rivera, F. (2005). Generalization strategies of beginning high school algebra students. In H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group* *for the Psychology of Mathematics Education* (Vol. 4, pp. 121–128). Melbourne: PME.

Brodie, K. (2010). *Teaching mathematical reasoning in secondary school classrooms*. Boston, MA: Springer.

Cobb, P., Jackson, K., & Dunlap, C. (2016). Design research: An analysis and critique. In Lyn D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 481–503). New York, NY: Routledge.

Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. Bishop, S. Mellin-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 61–85). Dordrecht: Springer.

Ellis, A. B. (2007). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *Journal of the Learning Sciences*, *16*(2), 221–262.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–16.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York, NY: Lawrence Erlbaum.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707–762). Reston, VA: NCTM.

Magnani, L. (2001). *Abduction, reason and science*. Boston, MA: Springer US.

Pólya, G. (1954). *Mathematics and plausible reasoning: Induction and analogy in mathematics. induction and analogy in mathematics* (Vol. 1). Princeton, NJ: Princeton University Press.

Ponte, J. P. (2005). Gestão curricular em matemática. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.

Ponte, J. P., Mata-Pereira, J., & Quaresma, M. (2013). Ações do professor na condução de discussões matemáticas. *Quadrante, XXII*(2), 55-81.

Ponte, J. P., &, Quaresma, M. (2016). Teachers' professional practice conducting mathematical discussions. *Educational Studies in Mathematics*, *93*(1), 51-66.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37–70.

Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, *33*(1), 5–29.

Rivera, F. D., & Becker, J. R. (2009). Algebraic reasoning through patterns. *Mathematics Teaching in the Middle School*, *15*(4), 212–221.

Russell, S. J. (1999). Mathematical reasoning in the elementary grades. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12* (pp. 1–12). Reston, VA: NCTM.