Mathematical and didactical knowledge about patterns and regularities mobilized by teachers in a professional learning task

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Investigating teachers' knowledge for teaching mathematics has been an important theme in mathematics teacher education. In this paper we aim to analyze which mathematical and didactical knowledge secondary mathematics teachers have about patterns and regularities. For this, we developed a teacher education process with mathematics teachers in Brazil during 15 weeks using professional learning tasks (PLT) specially built for this purpose. The research is qualitative-interpretative and data were collected by audio and video recording, with gathering of written documents. The results show that PLT allowed to recognize what mathematical and didactical knowledge teachers had in the begging of the teacher education process, and how the PLT enabled the development of new professional teacher knowledge.

Keywords: Professional learning task, mathematical and didactical knowledge, teaching of algebra.

Introduction

To unveil and to understand teachers' mathematical and didactical knowledge (Ponte, 1999) constitutes an important field of research in teacher education and, in particular, when practice is considered as a starting point (Lampert, 2010) for the construction of teacher professional knowledge. This underscores the importance of investigating the role of professional learning tasks (PLT) (Ball & Cohen, 1999; Smith, 2001; Swan, 2007) as a means to foster reflection on teachers' knowledge and to share their classroom experiences.

Regarding the specific field teaching of algebra, researches document the difficulties encountered by teachers in their teaching practice (Doerr, 2004; Ribeiro, 2012). At the same time, researches also highlights the importance of working with patterns and regularities as a promising path for the development of algebraic thinking (Mason, 1996; Twohill, 2015; Vale & Pimentel, 2015).

Thereby, our aim in this paper is *to* investigate the mathematical and didactical knowledge mobilized by mathematics teachers to solve individually and collectively a professional learning task related to teaching of algebra in basic education about patterns and regularities. To this end, we took a mathematics teachers' education programme as a research setting, aiming to answer the following research question: *In what way does a professional learning task allow (and favour) access to teachers' mathematical and didactical knowledge?*

Professional learning tasks and teachers' knowledge

In our research, teacher professional learning is a process anchored in classroom practice (Ball & Cohen, 1999; Ponte & Chapman, 2008; Smith, 2001) based on collective activity. Researches by Ball and Cohen (1999), and White, Jaworski, Agudelo-Valderrama and Gooya, (2013) emphasize the creation of opportunities for teachers to learn each other, in order to break with a type of isolation that is very present and usual when one considers the work of the teacher, thus increasing opportunities for them to start learning in a collective way. Thus, teacher professional learning, in this perspective, are mediated by professional learning tasks (PLT), assumed in our research as

> tasks that involve teachers in the work of teaching, can be developed in order to find a specific goal for teachers' learning and takes into account the previous knowledge and experience that teachers bring to their teaching (Ball & Cohen, 1999, p. 27).

PLT may support the access the professional knowledge of the teachers about patterns and regularities. In order to consider the different dimensions of professional teacher knowledge, which will be better discussed later, it must be considered in the composition of the TAP the use of records of practice (Ball, Ben-Peretz & Cohen, 2014), such as protocols of student solutions, parts of curriculum proposals and teaching plans, must be taken into account. These resources allow us to bring aspects of classroom practice into the context of teacher education processes as important components of professional learning tasks (Smith, 2001).

In the perspective of teacher professional knowledge, we are interested in Ponte's perspective (1999), which discusses a view of teacher professional knowledge strongly anchored in teaching practice, arguing that teacher knowledge is action-oriented. In his perspective, this knowledge unfolds in four domains: knowledge of teaching contents, knowledge of the curriculum, knowledge of students and knowledge of the teaching process. For the author, this knowledge

> is closely related to several aspects of the personal and informal teacher's knowledge of everyday life as the knowledge of the context (the school, the community, the society) and the knowledge that he/she has of himself (Ponte, 1999, p. 3).

Regarding the teachers' mathematical and didactical knowledge about patterns and regularities and their connections with mathematics teaching, we should consider how relevant it is that teachers mobilize knowledge that make possible to understand the students' algebraic thinking and to support the overcoming of difficulties they usually have regarding the generalisation of numerical and geometrical patterns (Orton & Orton, 2005). In order to be able to mobilize mathematical and didactical knowledge on the subject matter, it is necessary to consider, during the teacher education processes, professional learning tasks that contemplate mathematical situations involving different types of patterns in which algebraic expressions that generalise them are asked for (Zazkis & Liljedahl, 2002).

In order to promote discussions and reflections in the teacher education process regarding teachers' mathematical and didactical knowledge, teacher educators should stimulate the development of PLT using some specific practices for this purpose (Stein, Engle, Smith & Hughes, 2008).

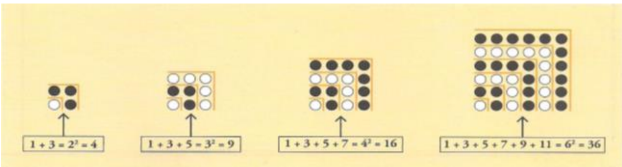
Stein *et al.*, (2008) propose five practices developed in classroom, which are called: anticipating, monitoring, selecting, sequencing and connecting. For this purpose, the authors argue that anticipating is the process in which the teacher, before taking the task to his/her classroom, anticipates the possible resolutions of the students and their possible difficulties. In the classroom, monitoring is the process in which the teacher will monitor the discussions that occur in small groups and thus he/she will select the most interesting resolutions to share with the classroom, whether they are unusually resolutions like those that may present errors and difficulties of the students. When the discussion is open to the whole classroom, the selected groups are presenting their resolutions within a sequencing, drawn up by the teacher. With this, the teacher should connect the student's resolutions with the mathematical knowledge proposed in the task in question. Thus, from these five practices, we conjecture that they can also help to foster discussions and mathematical and didactic reflections when used in teacher education process.

Research methodology

Context of the study. The teacher education process in which the data were collected had the general aim of developing and expanding participant teachers' mathematical and didactical knowledge regarding patterns and regularities in school mathematics. It was carried out during 15 weekly meetings of 4 hours, led by the first and second authors of this article. The meetings combined moments of (i) individual work, (ii) work in small groups, and (iii) plenary collective discussions. The participants were mathematics teachers (pre-service and in-service) and the activities were mostly carried out at the university, with 3 meetings held at basic education schools. The work sessions included moments of theoretical studies (a total of 8 hours) and *hands on* work, which were mediated by professional learning tasks developed by the leaders of the meetings.

Participants and developing of the PLT. The participants of our study were teachers who teach mathematics in Brazilian secondary schools. In relation to the PLT that we explore in this paper (Figure 1 and Figure 2), for the begging of the teacher education process we counted on the participation of 42 teachers, being 10 pre-service teachers and 32 in-service teachers (7 of these with no classroom experience). Of the 32 in-service teachers, 17 had been graduated for less than 5 years; 7 had between 5 and 10 years of teaching practice; 8 had more than 10 years of teaching practice. For the implementation of PLT, the 42 participants were distributed in 9 groups (3 to 6 participants), organized by the facilitators so that in all groups there were (i) teachers with and without classroom experience, and (ii) pre-service and in-service teachers. Teachers groups were built in this manner in order to promote the exchange of experiences and knowledge among participants. Of the three facilitators present, two were teacher educators and researchers in teacher education.

Observe the following squares and the strategy used to calculate the sum of the first odd numbers, starting from 1.



a) Inspired by the idea, calculate the sum of the 9 first odd numbers.
b) Generalise a formula to calculate the sum of the **n** first odd numbers, starting from 1.
Based on your classroom experience and considering the mathematical task solved previously, answer the following questions:
1) If your students were to resolve this mathematical task, which strategies do you think they could use?
2) Which difficulties do you think your students can have when solving this kind of task?
3) For which secondary school grade do you think this kind of task is appropriate?
4) Have you ever seen tasks of this nature in the curricular materials you use in the classroom? If you have, what type of curricular material is that?
5) Do you usually use this kind of mathematical tasks in your mathematics classes? If you do, in what secondary education grades? If you do not, please justify.

Figure 1: Example of Professional Learning Task[1]

Thus, PLT's main goal was to raise teachers' prior knowledge about patterns and regularities. It was planned for and developed in three moments: individual, small group discussion, and plenary collective discussion. These moments were accompanied and encouraged by the facilitators who, to favour the collective discussions, using five practices presented by Stein *et al*., (2008).

Data sources. Our study follows a qualitative research approach (Bogdan & Biklen, 1994), under the interpretive paradigm (Crotty, 1998), with the data collection taking place through video and audio recordings (both within each group as with the whole group) and through gathering of written documents resulting from the development of the PLT. The analyses considered: (i) teachers' individual notes; (ii) notes written by small groups of teachers; (iii) the audios of small group discussions; and (iv) the video of the collective discussion. The audio and video records were analysed in full, in articulation with the documents produced, and allowed the organisation and analysis of the data in order to identify the mathematical and didactical knowledge about patterns and regularities contemplated in PLT.

Results

Individual work. We begin the analysis considering teachers' individual work. It was observed, in general, that teachers found it difficult to mobilise mathematical knowledge to solve the mathematical situations presented in the PLT, regardless of their school year. The teachers sought to describe the observed mathematical pattern and to present an algebraic expression that represented the generalisation of this pattern. Afterwards, in the part of the PLT that explored the didactical knowledge – in relation to the students and in relation to the teaching processes – the teachers said that (i) they would have difficulties to solve the mathematical task by missing the perception of a pattern in the sequence, therefore having trouble in writing an algebraic generalisation; (ii) regardless of the school year, the students would use strategies for solving mathematical tasks by using counting processes or constructing the other elements of the sequence presented, just as the teachers themselves did, as can be seen in the statement[2]:

T7: [The students will have] The same difficulties I'm having.

Still with regard to didactical knowledge, the teachers said that, in their classroom, they do not propose mathematical tasks similar to those they were working on because (i) they have a tight schedule; (ii) students have difficulty in solving tasks of this kind; (iii) they do not have the adequate preparation to work with this type of tasks, as stated by teacher T8:

T8: I did not use this type of task because I am not used to let students build their own knowledge. So, my concern in learning how to support them build the formulas, and not going to the classroom with ready-made formulas.

Group work. At a second moment, the teachers begun working in small groups, comparing their own responses with those of their peers and, through collective discussions, reflecting on their mathematical and didactical knowledge regarding patterns and regularities. During the discussions

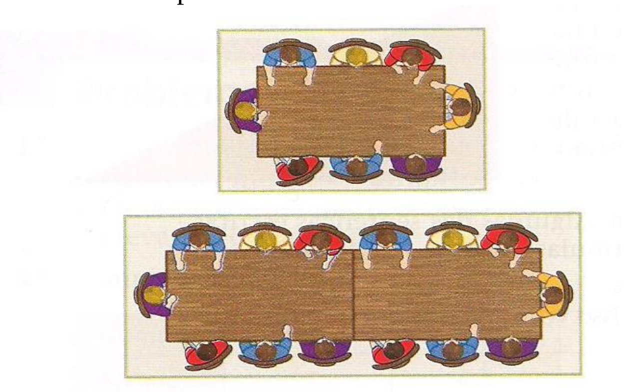
[1] Due to the space constraint, we present the PLT used with the teachers of grades 8-9, since the PLT structure of the other teachers (grades 6-7 and 10-12) was the same, changing only the mathematical situation of part 1 of the PLT.

[2] We use T for teacher-participant and TE for teacher educator/facilitator.

in small groups, many similarities were observed in the way they had worked individually, but, nevertheless, it is worth mentioning that the groups thought it difficult to find new mathematical strategies for solving tasks, even when working in groups. Only three groups stated that they found different solutions. During the small group discussions, the teachers emphasised that the students would be able to use counting processes or build the other elements of the sequence, but they would still have a hard time finding the pattern in the sequence, and even more in writing the generalisation in algebraic expression. In their discussions, the teachers recognised that, although they had difficulties in carrying out tasks similar to those in the classroom, this approach is very important for students to overcome their difficulties and to build their knowledge through tasks – and not just through "teacher's talk".

While teachers worked in small groups, the facilitators carried out the practices proposed by Stein *et al*., (2008) in order to prepare the next stage of the development of the PLT, namely, plenary collective discussions, which included three moments: (a) the first, aimed at sharing the teachers' mathematical knowledge and sharing unusual answers; (b) the second, aimed at sharing teachers' didactical knowledge regarding the students' difficulties; and (c) the third, aimed at sharing whether teachers used mathematical tasks of this kind in their classrooms.

Observe at the pictures and answer:



a) How many people can settle at the tables, if we arrange 3 tables as shown beside?
b) Write an expression that gives the number of people accommodated at 13 tables.
c) Complete the general conclusion: the number of people is ______
(Attention: Do not forget that the number of people depends on the number of tables).
d) Write the conclusion in a formula where the letter *p* represents the number of people, and the letter *m*, the number of tables.

Figure 2: Example of Professional Learning Task[3]

First part of the plenary collective discussion. Throughout the first part of the plenary collective discussion, it was perceived that, in fact, the teachers themselves thought it difficult to explain different solutions to the other participants, since they always tried to rely on the mathematical properties of Arithmetic Progressions (AP), exploring little other possible strategies. T1's statement shows us this:

T1: The exercise was exactly the same [as that of the previous group] and then we get pretty much the same idea. I even got into the same situation [as the teacher who spoke before]. I started with an AP, I did the AP formula, I did the computation with the AP and I said: "hold on... The grade 6 student will not think about the AP", [see Figure 2] and then we begin thinking in other ways (…) and we arrived

[3] Due to the space constraint, we present only mathematical situation of the PLT used with the teachers of grades 6-7, The complete PLT structure was the same presented in Figure 1

at $p = 6m + 2$ [p = people and m = tables] and thinking about AP, I got the relation $a_{13} = 8 + (13 - 1) \times 6$ and the result equals 80.

It can be seen in this statement, which was recurrent in the other groups, how teachers were not distant from the concept of arithmetic progression and also of what their students could think and do (although it had been reinforced by the facilitators that at that moment the mathematical knowledge of the teachers themselves was being discussed). In counterpoint to this idea, teacher T4 presenting the response of his group about the task assigned to students of grades 8-9 (Figure 1) said:

T4: The answer [of our group] is different from that here [the answer that was already written on board, $S_n = n^2$ regarding to Figure 1] (...) So it would start at zero and go up to n [the teacher then writes down on the board $\sum_{i=0}^{n} 2n - 1$].

T5: But this [solution] is for the 9th grade?

T4: We did not think about the student, this solution is ours.

It is interesting to note here that this group stated that it worked with the two solutions exposed in the dialogue, one that would possibly be that of the students, and another, that would be theirs. With this, facilitator TE2 asked:

TE2: For you, the meaning of n is the same in the two expressions [$S_n = n^2$ and $\sum_{i=0}^{n} 2n - 1$]?

T4: Yes! n indicates the natural numbers 0, 1, 2…

Such confusion presented by teachers about the meaning of the variable does not mean necessarily lack of mathematical knowledge, probably a difficulty orally expressing such knowledge.

With this, we perceive that the group of teachers present different representations about the generalization of patterns but they have difficulties in presenting their justifications of these representations.

Second part of the plenary collective discussion. Continuing with the plenary collective discussion, now focusing on the dimensions of teachers' didactical knowledge, teacher T2 pointed out that the students could construct a numerical table with the relation between the number of tables and people (Figure 2) and this would help him to realise the pattern. On the other hand, teachers T9, T10 and T6 had other concerns:

T9: At least in the few years that I have been in the classroom, the biggest problem I encounter is the algebraic part. (…) Teaching them [the students] that they have to solve the exercise in a way and still look at its generic part. I believe it is very difficult for the 6th and 7th graders.

T10: I think like this: If you do not insist... The student has difficulty but if the teacher has already been working this in the classroom, I think the student can get it.

T6: But that is the purpose of the activity, for the teacher to know the students' previous knowledge, so he can get to algebra.

At this point, the facilitator TE1 suggested that teachers were only thinking about the difficulty of the symbolic representation, but if students in any school year could explain the generalisation without the use of symbols and, even so, they already be a way of thinking algebraically.

This led to the establishment of a controversy in the plenary collective discussion, as pointed out by teacher T3:

T3: It may be the way we work with students. We give the contents and then the student finds it easy. We do not make the student an investigative being. We come and give him/her the formula to solve the problem. Wouldn't the lack of learning be one of the great problems in the classroom? The question that we don't go there and make the child investigate, search for things. Shouldn't we invest in this line of child's reasoning?

That thought - expressed by T3 - took the proportion of a challenge to be pursued by the group of teachers and facilitators throughout the teacher education process, that is, that the teachers' work with their students should become a more investigative exploratory approach rather than just follow a traditional approach to teaching.

Discussion and Conclusion

By developing the PLT, it was possible to realize that the teachers showed some difficulties in their mathematical knowledge regarding to the concept of variable what could implicate the recognition of patterns and regularities and the formulation of algebraic expressions to represent complex mathematical sequences (Zazkis & Liljedahl, 2002). In the dimension of didactical knowledge (Ponte, 1999), it can be noted that the teachers, possibly due to the fragility in mathematical knowledge, did not feel safe and comfortable to use tasks with their students such as those proposed in the PLT. From the point of view of teachers' knowledge of students, their solution strategies and difficulties, we noted that, throughout the development of the PLT, the teachers conjectured about students' algebraic thinking and how they could help them to overcome the difficulties they could present regarding the generalisation of numerical and geometrical patterns (Orton & Orton, 2005).

Regarding the structure of PLTs (Ball & Cohen, 1999; Smith, 2001), we conclude that the results show their potential to favour the mobilisation, connection and (re)construction of mathematical and didactical knowledge about patterns and regularities in the teaching of mathematics and, in special, to establish relations with classroom practice. In particular, we highlight as potential of the PLT proposed in this paper, the use of the five practices to prepare and lead (plenary) collective discussions (Stein *et al*., 2008). In addition, the proposal to use the five practices (Stein *et al*., 2008) was intended to show teachers a way to foster collective discussions in the classroom.

This framework enabled the group of teachers and facilitators to share knowledge and experiments that made teachers, for example, (i) wonder both why they do not provide investigative and challenging tasks for students to explore different kinds of numerical and geometric sequences, and (ii) how such work may contribute to the development of algebraic thinking in their students. Therefore, we believe that the use of PLT in teacher education process can help to unveil and understand the teachers' mathematical and didactical knowledge.

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