Pertanika J. Sci. & Technol. 27 (4): 2381 - 2398 (2019)

SCIENCE & TECHNOLOGY

Journal homepage: http://www.pertanika.upm.edu.my/

Two Point Block Multistep Methods with Trigonometric–Fitting for Solving Oscillatory Problems

Aini Fadhlina Mansor[1], Fudziah Ismail[1, 2]* and Norazak Senu [1, 2]

[1]*Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*
[2]*Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia*

ABSTRACT

In this paper, we present the absolute stability of the existing 2-point implicit block multistep step methods of step number $k = 3$ and $k = 5$ and solving special second order ordinary differential equations (ODEs). The methods are then trigonometrically fitted so that they are suitable for solving highly oscillatory problems arising from the special second order ODEs. Their explicit counterparts are also trigonometrically fitted so that in the implementation the methods can act as a predictor-corrector pairs. The numerical results based on the integration over a large interval are given to show the performance of the proposed methods. From the numerical results we can conclude that the new trigonometrically-fitted methods are superior in terms of accuracy and execution time, compared to the existing methods in the scientific literature when used for solving problems which are oscillatory in nature.

Keywords: Block method, multistep method, oscillatory problems, special second order ODEs, trigonometrically fitted

INTRODUCTION

In this research, we are concerned with the numerical methods for solving special second order ordinary differential equation (ODE) of the form as follows:

$$y'' = f(x,y),\ y(x_0) = y_0, y'(x_0) = y'_0, \qquad [1]$$

General second order ODE can be written as $y'' = f(x,y,y')$, special second order ODE does not depend on the derivative of the solution. This type of ordinary differential equations often appear in many scientific areas such as mechanics,

ARTICLE INFO

Article history:
Received: 06 December 2018
Accepted: 04 June 2019
Published: 21 October 2019

E-mail addresses:
aini_virgo92@yahoo.com (Aini Fadhlina Mansor)
fudziah_i@yahoo.com.my (Fudziah Ismail)
norazak@upm.edu.my (Norazak Senu)
* Corresponding author

ISSN: 0128-7680
e-ISSN: 2231-8526

astrophysics, quantum chemistry and electronics. Further details can be seen in Fang and Wu (2008).

The most common technique to numerically solve the second order problem is by reducing it to a system of first order ODEs. However it is more efficient if [1] can be solved directly without converting it to a system of first order ODEs. Such methods for directly solving the special second order ODEs are direct multistep method, Runge-Kutta-Nyström (RKN) method and hybrid method.

All the methods mentioned above, approximate the solution of the equation at only one point at a time step. Hence, to increase the efficiency of the numerical methods for solving ODEs, recently there is a lot of research has been done on block methods. Block multistep methods calculate the solutions of the ODEs at more than one point at a time, for example 2 point block method calculates the solution at two points concurrently, hence less execution time is needed to solve the ODEs. Fatunla (1995) constructed block methods for solving special second order ordinary differential equations. Then, Akinfewa et al. (2015) proposed a family of continuous third derivative block methods for numerical integration of first order system of ODEs. Ramos et al. (2015) developed an optimized two-step hybrid block method for solving general second order ODEs.

Quite often the solution of [1] exhibits a pronounced oscillatory character. Oscillatory problems are usually harder to solve than the non oscillatory problems. To obtain a more efficient process for solving oscillatory problems, numerical methods are constructed by taking into account the nature of the problem. This results in methods in which the coefficients depend on the frequency of the problem to be solved. Some important classes of the numerical methods are exponentially fitted, trigonometrically fitted or phase-fitted methods.

Simos (2003) in his work, had proposed exponentially-fitted and trigonometrically-fitted symmetric linear multistep methods for solving orbital problems. Then, Fang and Wu (2008) in their research, had proposed trigonometrically fitted explicit Numerov-type method. Phase fitted Runge-Kutta-Nystrom method had been studied by Papadopoulos et al. (2008), Ahmad et al. (2016), extended the work by developing trigonometrically-fitted hybrid method for solving oscillatory delay differential equations.

In the quest for methods that best approximate the solution of [1] which are oscillatory in nature, in this paper, we developed an efficient block multistep methods by trigonometrically-fitted the methods, in which to the best of our knowledge is the first work on trigonometrically fitted block multistep methods for solving highly oscillatory problems. The coefficients of the method depend on the frequency of the problem to be solved, hence the frequency of the problems must be priory known.

The research is based on the 2-point and 3-point explicit and implicit block multistep methods which have been derived in Mansor et al. (2017). First, the stability aspect of the 2-point and 3-point implicit block multistep methods are investigated. Then, both the explicit and implicit block multistep methods of step number $k = 3$ and $k = 5$, are trigonometrically fitted. The methods are then implemented as a predictor-corrector pairs. Numerical results of the proposed methods in the form of efficiency curves for solving five oscillatory problems are then presented.

MATERIALS AND METHODS

Absolute Stability

In our previous work, see Mansor et al. (2017), we derived the 2-point and 3-point explicit and implicit block multistep methods of step number $k = 3$ and $k = 5$ with orders three and five respectively for the explicit methods and orders four and six for the implicit methods. Here we are going to investigate the stability of the implicit block methods.

2-point implicit Block Methods for $k = 3$:

The first and second point of the implicit block methods that have been derived in Mansor et al. (2017) for step number $k = 3$ are given as follows:

$$y_{n+1} = 2y_n - y_{n-1} + h^2\left(\frac{1}{12}f_{n+1} + \frac{5}{6}f_n + \frac{1}{12}f_{n-1} + 0f_{n-2}\right), \quad [2]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2\left(0f_{n+2} + \frac{4}{3}f_{n+1} + \frac{4}{3}f_n + \frac{4}{3}f_{n-1}\right). \quad [3]$$

The implicit 2-point block methods can be presented in the matrix form as:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\end{bmatrix} = \begin{bmatrix}-1 & 2\\ 0 & 2\end{bmatrix}\begin{bmatrix}y_{n-1}\\ y_n\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}y_{n-3}\\ y_{n-2}\end{bmatrix}$$
$$+h^2\left(\begin{bmatrix}\frac{1}{12} & 0\\ \frac{4}{3} & 0\end{bmatrix}\begin{bmatrix}f_{n+1}\\ f_{n+2}\end{bmatrix} + \begin{bmatrix}\frac{1}{12} & \frac{5}{6}\\ \frac{4}{3} & \frac{4}{3}\end{bmatrix}\begin{bmatrix}f_{n-1}\\ f_n\end{bmatrix}\right)$$

Using the test equation $y'' = \lambda y$ on the method, gives

$$\begin{bmatrix}1-\frac{1}{12}\lambda h^2 & 0\\ -\frac{4}{3}\lambda h^2 & 1\end{bmatrix}\begin{bmatrix}y_{n+1}\\ y_{n+2}\end{bmatrix} = \begin{bmatrix}-1+\frac{1}{12}\lambda h^2 & 2+\frac{5}{6}\lambda h^2\\ \frac{4}{3}\lambda h^2 & 2+\frac{4}{3}\lambda h^2\end{bmatrix}\begin{bmatrix}y_{n-1}\\ y_n\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}y_{n-3}\\ y_{n-2}\end{bmatrix}.$$

Taking the determinant equals to zero, we have the stability polynomial $t^2\left(-\frac{1}{12}\hbar + 1\right) + t(-\hbar^2 - 4\hbar) + \left(-\hbar^2 - \frac{47}{12}\hbar - 1\right) = 0$

where $\hbar = \lambda h^2$.

Then, by solving the stability polynomial for values of $\hbar$ with $|t| \leq 1$, gives the absolute stability region of the method as shown in Figure 1, where the horizontal axis is the real part of $\hbar$ and the vertical axis is the imaginary part of $\hbar$.

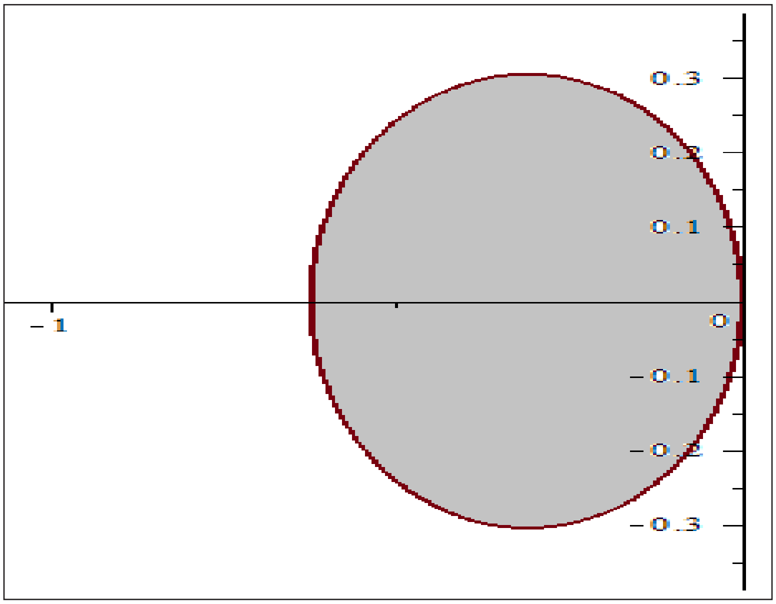

Figure 1. Stability region of the 2-point implicit block multistep method for $k = 3$

The 2-point implicit block multistep method for $k = 3$, has a small region of absolute stability, however it is still stable and can be used to solve the special second order ODEs.

2-point Block Methods for $k = 5$:

$$y_{n+1} = 2y_n - y_{n-1} + h^2\left(\frac{3}{40}f_{n+1} + \frac{209}{240}f_n + \frac{1}{60}f_{n-1} + \frac{7}{120}f_{n-2} - \frac{1}{40}f_{n-3} + \frac{1}{240}f_{n-4}\right)\Bigg] \quad [4]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2\left(\frac{1}{15}f_{n+2} + \frac{16}{15}f_{n+1} + \frac{26}{15}f_n + \frac{16}{15}f_{n-1} + \frac{1}{15}f_{n-2} + 0f_{n-3}\right) \quad [5]$$

The implicit 2-point block methods can be presented in the matrix form as:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\end{bmatrix} = \begin{bmatrix}-1 & 2\\0 & 2\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_n\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}y_{n-3}\\y_{n-2}\end{bmatrix} + h^2\left(\begin{bmatrix}\frac{3}{40} & 0\\\frac{16}{15} & \frac{1}{15}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\end{bmatrix} + \begin{bmatrix}\frac{1}{60} & \frac{209}{240}\\\frac{16}{15} & \frac{26}{15}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_n\end{bmatrix}\right.$$

$$\left. + \begin{bmatrix}-\frac{1}{40} & \frac{7}{120}\\0 & \frac{1}{15}\end{bmatrix}\begin{bmatrix}f_{n-3}\\f_{n-2}\end{bmatrix}\right).$$

Substituting the test equation $y'' = \lambda y$ into the method, gives,

$$\begin{bmatrix} 1-\frac{3}{40}\lambda h^2 & 0 \\ -\frac{16}{15}\lambda h^2 & 1-\frac{1}{15}\lambda h^2 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -1+\frac{1}{60}\lambda h^2 & 2+\frac{209}{240}\lambda h^2 \\ \frac{16}{15}\lambda h^2 & 2+\frac{26}{15}\lambda h^2 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}$$
$$+\begin{bmatrix} -\frac{1}{40}\lambda h^2 & \frac{7}{120}\lambda h^2 \\ 0 & -1+\frac{1}{15}\lambda h^2 \end{bmatrix}\begin{bmatrix} y_{n-3} \\ y_{n-2} \end{bmatrix}$$
$$+\begin{bmatrix} 0 & \frac{1}{240}\lambda h^2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_{n-5} \\ y_{n-4} \end{bmatrix}.$$

Taking the determinant equals to zero, we have the stability polynomial

$$t^2\left(\frac{1}{200}\hbar^2 - \frac{17}{120}\hbar + 1\right) + t\left(-\frac{31}{36}\hbar^2 - \frac{47}{12}\hbar\right) + \left(-\frac{1819}{1800}\hbar^2 - \frac{473}{120}\hbar - 1\right) = 0.$$

Solving the stability polynomial for values of $\hbar$ with $|t| \leq 1$, we obtained the absolute stability region of the method as shown in the Figure 2, where the horizontal axis is the real part of $\hbar$ and the vertical axis is the imaginary part of $\hbar$.

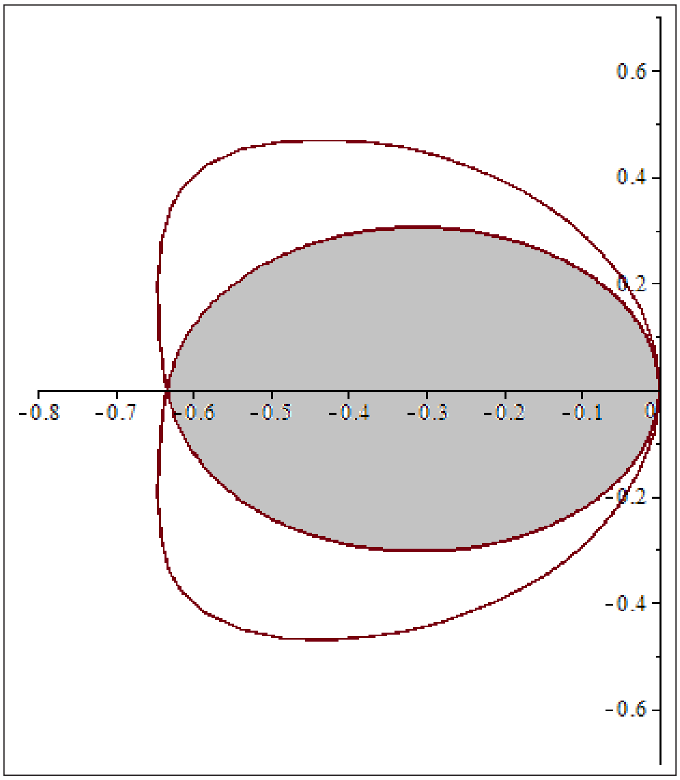

Figure 2. Stability region of the 2-point implicit block multistep method for $k = 5$

Hence we can conclude that both the k = 3 and k = 5 step implicit block multistep methods do have a substantial regions of absolute stability. In the next section, we will derive the trigonometrically ftted block methods.

Derivation of the Trigonomerically-Fitted Methods

Explicit block method for k = 3:

The first and second point of the explicit block methods that have been derived in Mansor et al. (2017) for step number k = 3 are given as follows:

$$y_{n+1} = 2y_n - y_{n-1} + h^2\left(\frac{13}{12}f_n - \frac{1}{6}f_{n-1} + \frac{1}{12}f_{n-2}\right) \qquad [6]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2\left(\frac{16}{3}f_n - \frac{8}{3} - f_{n1} + \frac{4}{3}f_{n-2}\right) \qquad [7]$$

In general form, the methods can be written as

$$y_{n+1} = 2y_n - y_{n-1} + h^2(p_2 f_n + p_1 f_{n-1} + p_0 f_{n-2}) \qquad [8]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2(q_2 f_n + q_1 f_{n-1} + q_0 f_{n-2}) \qquad [9]$$

Equation $y'' = f(t, y) = -\omega^2 y$ is the equation that most researchers used in the literature when they are dealing with oscillatory problems and trigonometric-fitting methods. For further details see Li et al. (2017). The method integrates exactly the differential equation whose solutions can be expressed as the linear combination of $\{\sin(\omega t), \cos(\omega t)\}$. Hence $y_n = \cos(\omega t_n)$, we have

$$y_n' = -\omega \sin(\omega t_n)$$
$$y_n'' = -\omega^2 \cos(\omega t_n)$$
$$y_n'' = f(t_n, y_n) = -\omega^2 y_n = f_n$$
$$t_{n+1} = t_n + h \text{ and } t_{n-1} = t_n - h.$$

Let $H = \omega h$ and taking $t_n = 0$, this is the same approach as in Fang and Wu (2008), thus we have

$y_n = \cos(0)$, $y_{n-1} = \cos(-H)$, and $y_{n+1} = \cos(H)$.

Substituting into [8] and [9], we obtain

$$2\cos(H) = 2 - H^2\big(p_2 + p_1\cos(H) + p_0\cos(2H)\big), \qquad [10]$$

$$2\cos(2H) = 2 - H^2\big(q_2 + q_1\cos(H) + q_0\cos(2H)\big). \qquad [11]$$

Then, letting $y_n = \sin(\omega t_n)$ and using the same technique, we obtain

$$H^2\big(p_1\sin(H) + p_0\sin(2H)\big) = 0, \qquad [12]$$

$$H^2\big(q_1\sin(H) + q_0\sin(2H)\big) = 0. \qquad [13]$$

There are six undetermined coefficients, they are $p_0, p_1, p_2, q_0, q_1, q_2$ and four equations to be solved. Letting p_0 and q_0 as free parameters where the values are obtained from the coefficients of the original methods ($p_0 = \frac{1}{12}$, $q_0 = \frac{4}{3}$) and solving equations [10] - [13] and rewriting in Taylor series expansion to avoid heavy cancellation in the implementation of the methods, we have

$$p_1 = -\frac{1}{6} + \frac{1}{12}H^2 - \frac{1}{144}H^4 + \frac{1}{4320}H^6 - \frac{1}{241920}H^8 + \frac{1}{21772800}H^{10} - \frac{1}{2874009600}H^{12} + O(H^{14}),$$

$$p_2 = \frac{13}{12} - \frac{1}{12}H^2 + \frac{1}{360}H^4 - \frac{1}{20160}H^6 + \frac{1}{1814400}H^8 - \frac{1}{239500800}H^{10} + O(H^{12}),$$

$$q_1 = -\frac{8}{3} + \frac{4}{3}H^2 - \frac{1}{9}H^4 + \frac{1}{270}H^6 - \frac{1}{15120}H^8 + \frac{1}{1360800}H^{10} - \frac{1}{179625600}H^{12} + O(H^{14}),$$

$$q_2 = \frac{16}{3} - \frac{4}{3}H^2 + \frac{8}{45}H^4 - \frac{4}{315}H^6 + \frac{8}{14175}H^8 - \frac{8}{467775}H^{10} + O(H^{12}),$$

Implicit block method for $k = 3$:

The implicit block methods for $k = 3$, is given as in equations [2] and [3] in the previous section. In general form the equations can be written as below:

$$y_{n+1} = 2y_n - y_{n-1} + h^2(P_3 f_{n+1} + P_2 f_n + P_1 f_{n-1} + P_0 f_{n-2}), \qquad [14]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2(Q_4 f_{n+2} + Q_3 f_{n+1} + Q_2 f_n + Q_1 f_{n-1}). \qquad [15]$$

Substituting $y_n = \cos(\omega t_n)$ and $y_n = \sin(\omega t_n)$ respectively into [14] qand [15], resulting in,

$$2\cos(H) = 2 - H^2\big(P_3\cos(H) + P_2 + P_1\cos(H) + P_0\cos(2H)\big), \qquad [16]$$

$$2\cos(2H) = 2 - H^2\big(Q_4\cos(2H) + Q_3\cos(H) + Q_2 + Q_1\cos(H)\big), \qquad [17]$$

$$H^2\big(-P_3\sin(H) + P_1\sin(H) + P_0\sin(2H)\big) = 0, \qquad [18]$$

$$H^2\left(-Q_4\sin(2H) - Q_3\sin(H) + Q_1\sin(H)\right) = 0. \qquad [19]$$

Solving equations [16 - [19] by letting P_2, P_3, Q_1 and Q_2 as the original values $(P_2 = \frac{5}{6}, P_3 = \frac{1}{12}, Q_1 = \frac{4}{3}, Q_2 = \frac{4}{3})$ and rewriting in Taylor series expansion, we have

$$P_0 = \frac{1}{240}H^4 - \frac{11}{60480}H^6 + \frac{13}{3628800}H^8 - \frac{1}{23950080}H^{10} + O(H^{12}),$$

$$P_1 = \frac{1}{12} - \frac{1}{120}H^4 + \frac{137}{30240}H^6 - \frac{139}{259200}H^8 + \frac{13}{427680}H^{10} + O(H^{12}),$$

$$Q_3 = \frac{4}{3} + \frac{2}{15}H^4 - \frac{16}{189}H^6 + \frac{1763}{113400}H^8 - \frac{1097}{748440}H^{10} + O(H^{12}),$$

$$Q_4 = -\frac{1}{15}H^4 + \frac{17}{1890}H^6 - \frac{113}{226800}H^8 + \frac{7}{427680}H^{10} + O(H^{12}).$$

Explicit block methods for $k = 5$:

The first and second point of the explicit block methods in Mansor et. al (2017) are given as follows:

$$y_{n+1} = 2y_n - y_{n-1} + h^2\left(\frac{299}{240}f_n - \frac{11}{15}f_{n-1} + \frac{97}{120}f_{n-2} - \frac{2}{5}f_{n-3} + \frac{19}{240}f_{n-4}\right). \qquad [20]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2\left(\frac{121}{15}f_n - \frac{184}{15}f_{n-1} + \frac{206}{15}f_{n-2} - \frac{104}{15}f_{n-3} + \frac{7}{5}f_{n-4}\right) \qquad [21]$$

In general form it can be written as

$$y_{n+1} = 2y_n - y_{n-1} + h^2(r_4 f_n + r_3 f_{n-1} + r_2 f_{n-2} + r_1 f_{n-3} + r_0 f_{n-4}), \qquad [22]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2(s_4 f_n + s_3 f_{n-1} + s_2 f_{n-2} + s_1 f_{n-3} + s_0 f_{n-4}). \qquad [23]$$

Substitute $y_n = \cos(\omega t_n)$ and $y_n = \sin(\omega t_n)$ respectively into [22] and [23], gives

$$2\cos(H) = 2 - H^2\left(r_4 + r_3\cos(H) + r_2\cos(2H) + r_1\cos(3H) + r_0\cos(4H)\right), \qquad [24]$$

$$2\cos(2H) = 2 - H^2\left(s_4 + s_3\cos(H) + s_2\cos(2H) + s_1\cos(3H) + s_0\cos(4H)\right), \qquad [25]$$

$$H^2\left(r_3\sin(H) + r_2\sin(2H) + r_1\sin(3H) + r_0\sin(4H)\right) = 0, \qquad [26]$$

$$H^2\left(s_3\sin(H) + s_2\sin(2H) + s_1\sin(3H) + s_0\sin(4H)\right) = 0. \qquad [27]$$

Then, solving equations [24] - [27] by letting r_0, r_1, r_4, s_0, s_1 and s_2 as the original values, we obtain

$$r_2 = \frac{97}{120} - \frac{3}{40}H^4 + \frac{787}{60480}H^6 - \frac{1789}{1814400}H^8 + \frac{10649}{239500800}H^{10} + O(H^{12}),$$

$$r_3 = -\frac{11}{15} + \frac{3}{40}H^4 - \frac{283}{30240}H^6 + \frac{131}{259200}H^8 - \frac{197}{11975040}H^{10} + O(H^{12}),$$

$$s_3 = -\frac{184}{15} - \frac{22}{15}H^4 + \frac{74}{45}H^6 - \frac{221}{600}H^8 + \frac{1181}{28350}H^{10} - \frac{40661}{13608000}H^{12} + O(H^{14}),$$

$$s_4 = \frac{121}{15} + \frac{22}{15}H^4 - \frac{229}{945}H^6 + \frac{2041}{113400}H^8 - \frac{859}{1069200}H^{10} + O(H^{12}).$$

Implicit block methods for $k = 5$:

The general form of the first and second point implicit block methods given in equations [4] and [5] in the previous section can be written as

$$y_{n+1} = 2y_n - y_{n-1} + h^2(R_5 f_{n+1} + R_4 f_n + R_3 f_{n-1} + R_2 f_{n-2} + R_1 f_{n-3} + R_0 f_{n-4}), \quad [28]$$

$$y_{n+2} = 2y_n - y_{n-2} + h^2(S_6 f_{n+2} + S_5 f_{n+1} + S_4 f_n + S_3 f_{n-1} + S_2 f_{n-2} + S_1 f_{n-3}). \quad [29]$$

Substituting $y_n = \cos(\omega t_n)$ and $y_n = \sin(\omega t_n)$ respectively into [28] and [29], resulting in

$$2\cos(H) = 2 - H^2\big(R_5\cos(H) + R_4 + R_3\cos(H) + R_2\cos(2H) + R_1\cos(3H) + R_0\cos(4H)\big), \quad [30]$$

$$2\cos(2H) = 2 - H^2\big(S_6\cos(2H) + S_5\cos(H) + S_4 + S_3\cos(H) + S_2\cos(2H) + S_1\cos(3H)\big), \quad [31]$$

$$H^2\big(-R_5\sin(H) + R_3\sin(H) + R_2\sin(2H) + R_1\sin(3H) + R_0\sin(4H)\big) = 0, \quad [32]$$

$$H^2\big(-S_6\sin(2H) - S_5\sin(H) + S_3\sin(H) + S_2\sin(2H) + S_1\sin(3H)\big) = 0. \quad [33]$$

Solving equations [30] - [33] by letting $R_0, R_2, R_3, R_4, S_1, S_3, S_4$ and S_5 as the original values we obtained values of R_1, R_5, S_2 and S_6 as follows:

$$R_1 = -\frac{1}{40} + \frac{473}{241920}H^6 - \frac{24223}{7257600}H^8 + \frac{370289}{68428800}H^{10} + O(H^{12}),$$

$$R_5 = \frac{3}{40} + \frac{137}{80640}H^6 + \frac{7823}{2419200}H^8 + \frac{57149}{10644480}H^{10} + O(H^{12}),$$

$$S_2 = \frac{1}{15} + \frac{1}{945}H^6 + \frac{221}{113400}H^8 + \frac{23971}{7484400}H^{10} + \frac{212498413}{40864824000}H^{12} + O(H^{14}),$$

$$S_6 = \frac{1}{15} + \frac{1}{945}H^6 + \frac{221}{113400}H^8 + \frac{23971}{7484400}H^{10} + \frac{212498413}{40864824000}H^{12} + O(H^{14}).$$

RESULTS AND DISCUSSION

The proposed methods were implemented using predictor-corrector technique with only one iteration. The 2-point trigonometrically-fitted block explicit method for $k = 3$ is taken as the predictor equation and the 2-point trigonomerically-fitted block implicit method for $k = 3$ as the corrector equation this pair is denoted as two-point trigonometrically-fitted method of order 4 (TF2PBM4). The same goes for $k = 5$, where the 2-point trigonometrically-fitted block explicit method acts as the predictor and the 2-point trigonometrically-fitted block implicit method as the corrector, this pair is denoted as 2-point trigonometrically-fitted method of order 6 (TF2PBM6). We solved five tested problems that were obtained from the literature. Total time taken and maximum error would be shown in the form of efficiency curves.

Problem 1 [Rabiei et al., 2012]

$$y''(x) = -y(x), \qquad y(0) = 0, \qquad y'(0) = 1,$$

and the fitted frequency, ω=1. Exact solution is $y(x) = \sin(x)$.

Problem 2 [Jikantoro et al., 2015a]

$$y''(x) = -100y(x), \qquad y(0) = 1, \qquad y'(0) = -2,$$

and the fitted frequency, ω=10. Exact solution is $y(x) = -\frac{1}{5}\sin(10x) + \cos(10x)$.

Problem 3 [Senu et al., 2015]

$$y''(x) = -\omega^2 y(x) + (\omega^2 - 1)\sin(x), \qquad y(0) = 1, \qquad y'(0) = (\omega + 1),$$

and the fitted frequency, ω=10. Exact solution is $y(x) = \cos(\omega x) + \sin(\omega x) + \sin(x)$.

Problem 4 [Simos, 2003]

$$y_1''(x) = -y_1(x) + \epsilon\cos(\psi x), \qquad y_1(0) = 1, \qquad y_1'(0) = 0,$$

$$y_2''(x) = -y_2(x) + \epsilon\sin(\psi x), \qquad y_2(0) = 0, \qquad y_2'(0) = 1,$$

where $\epsilon = 0.001, \psi = 0.01$ and the estimated frequency, ω=1. Exact solutions are:

$$y_1(x) = \frac{1 - \epsilon - \psi^2}{1 - \psi^2}\cos(x) + \frac{\epsilon}{1 - \psi^2}\cos(\psi x),$$

$$y_2(x) = \frac{1 - \epsilon\psi - \psi^2}{1 - \psi^2}\sin(x) + \frac{\epsilon}{1 - \psi^2}\sin(\psi x).$$

Problem 5 [Jikantoro et al., 2015b]

$$y_1''(x) = -y_1(x) + 0.001\cos(x), \qquad y_1(0) = 1, \qquad y_1'(0) = 0,$$

$$y_2''(x) = -y_2(x) + 0.001\sin(x), \qquad y_2(0) = 0, \qquad y_2'(0) = 0.9995$$

and the fitted frequency, ω=1. Exact solutions are $y_1(x) = \cos(x) + 0.0005x\sin(x)$ and $y_2(x) = \sin(x) - 0.0005x\cos(x)$.

The notations used are as follows:

ω	Frequency of the problem
h	Step size
TIME(s)	Time taken to compute the method in second
MAXERR	Maximum error
TF2PBM4	The fourth order trigonometrically fitted 2-point block multistep method derived in this paper
TF2PBM6	The sixth order trigonometrically fitted 2-point block multistep method derived in this paper
ETSHMs	The fourth order explicit two-step hybrid method by Franco (2006).
IRKNM	The fourth order improved Runge-Kutta-Nystrom method with three stages by Rabiei et al. (2012).
PFRKN	The fourth order phase fitted Runge-Kutta-Nystrom method by Papadopoulos et al. (2008).
MSHMs	The four-step multistep hybrid method by Li and Wang (2016).
ETSHM6	The sixth order explicit two-step hybrid method by Franco (2006).
PFHM6	The sixth order phase fitted hybrid method by Senu et al. (2015).
NTM6	The sixth order explicit Numerov-type method by Tsitouras (2003).

The above methods are chosen as comparison because those are the methods usually used by most reserchers who are working on numerically solving oscillatory problems. The maximum error is defined by

$$MAXERR = \max|y(x_n) - y_n|,$$

where $y(x_n)$ is the exact solution and y_n is the approximate solution.

Methods of the same orders or steps were compared for the integration intervals of [1, 1000]. for methods of order four the efficiency curves for problems 1-5 are given in Figures 3 -7 and for methods of order six, the efficiency curves are given in Figures 8-12.

Five tested problems have been solved using the trigonometrically fitted block methods TF2PBM4 and TF2PBM6 of order four and five respectively. The maximum error of the

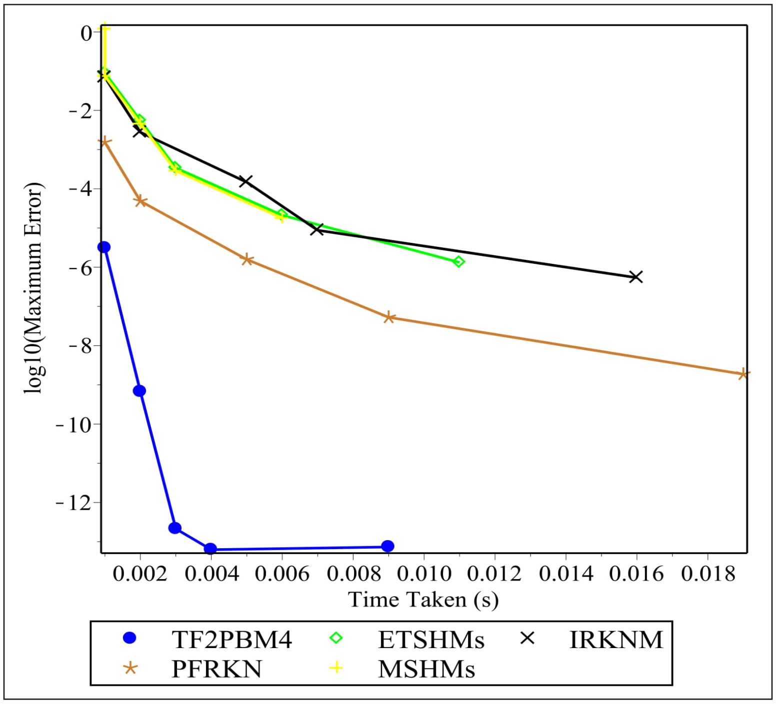

Figure 3. Efficiency curves of TF2PBM4 for Problem 1 with $xn = 1,000$ and $h = 0.5 / 2\hat{}i$, $i = 0, \ldots, 4$

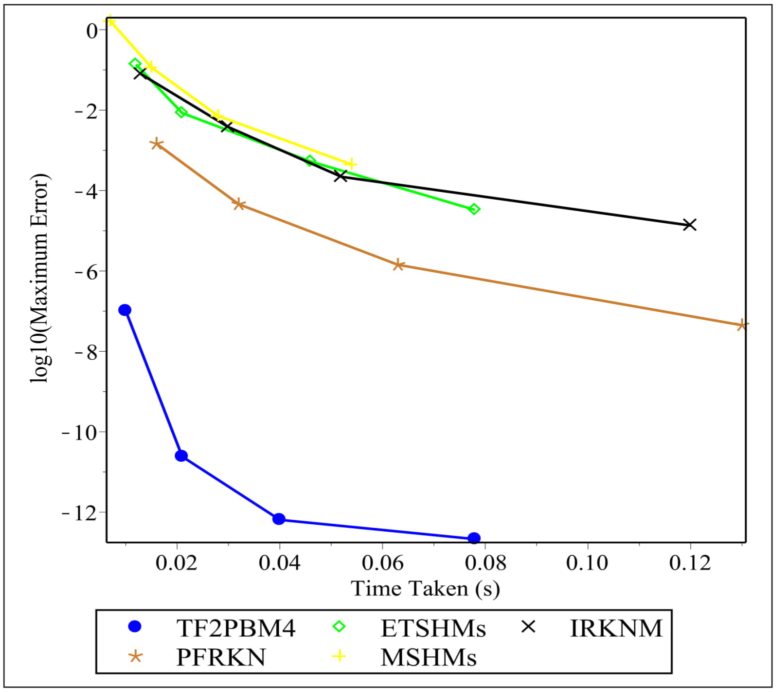

Figure 4. Efficiency curves of TF2PBM4 for Problem 2 with $xn = 1,000$ and $h = 0.125/2^{i}$, $i = 2, \ldots, 5$

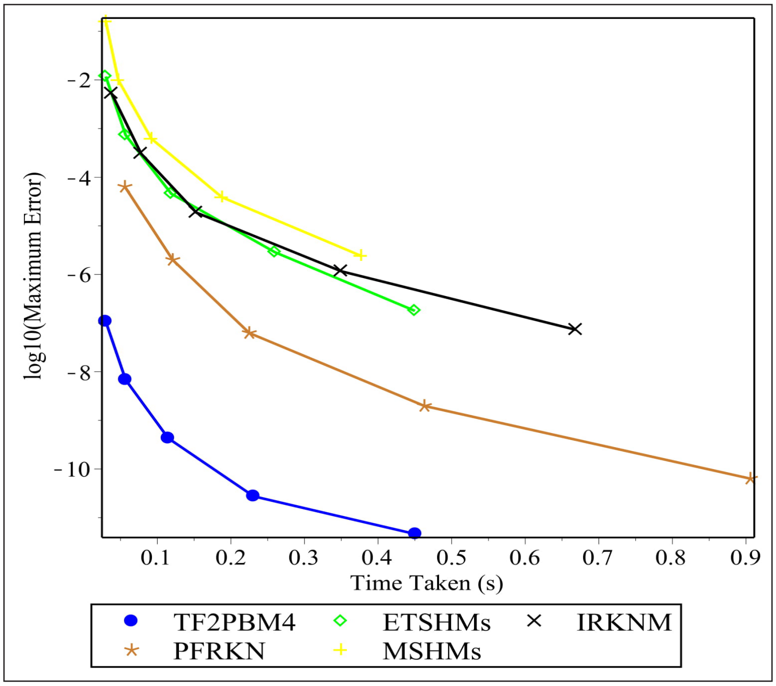

Figure 5. Efficiency curves of TF2PBM4 for Problem 3 with $xn = 1,000$ and $h = 0.125/2^i, i = 3, \ldots, 7$

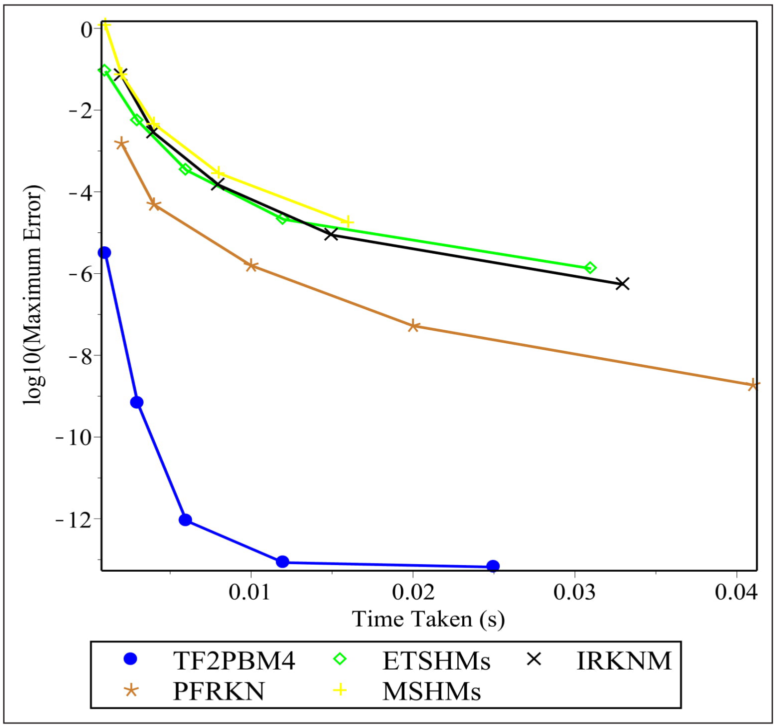

Figure 6. Efficiency curves of TF2PBM4 for Problem 4 with $xn = 1,000$ and $h = 0.5/2^i, i = 0, \ldots, 4$

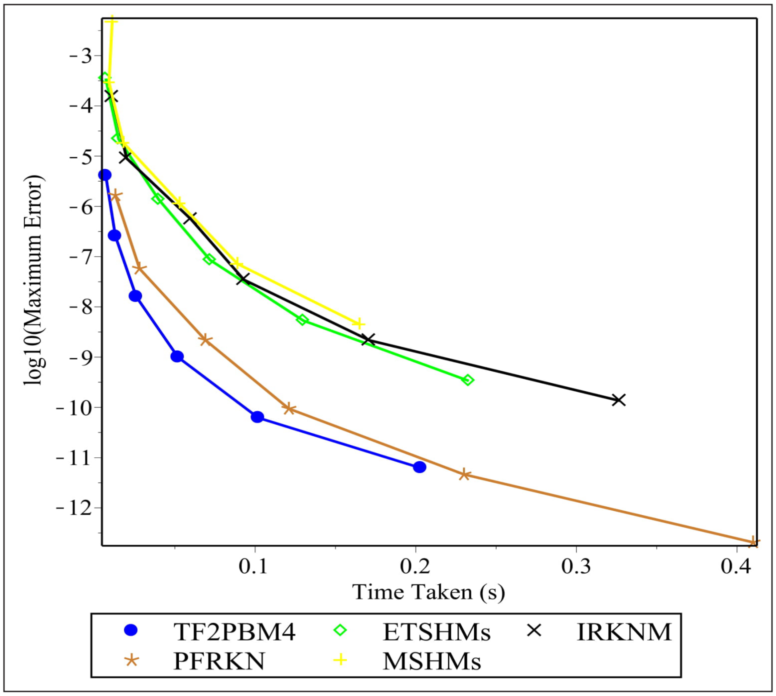

Figure 7. Efficiency curves of TF2PBM4 for Problem 5 with $xn = 1,000$ and $h = 0.125/2^i, i = 0, \ldots, 5$

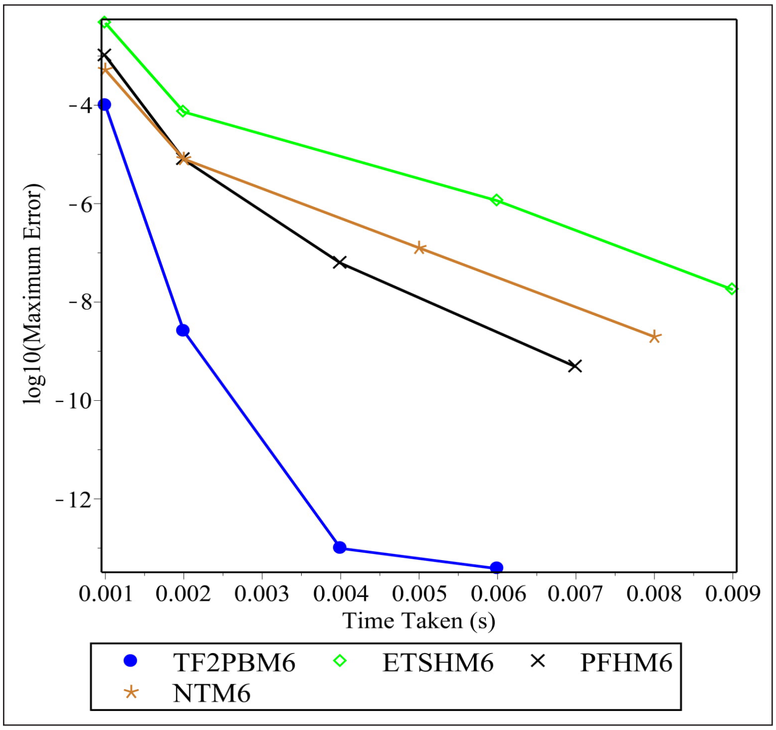

Figure 8. Efficiency curves of TF2PBM6 for Problem 1 with $xn = 1,000$ and $h = 0.5/2^i, i = 0, \ldots, 3$

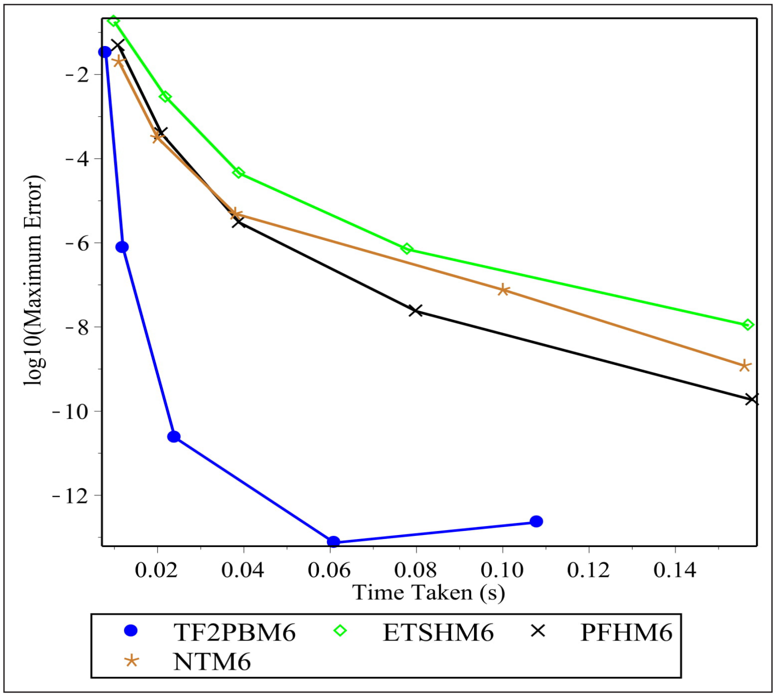

Figure 9. Efficiency curves TF2PBM6 for Problem 2 with $xn = 1,000$ and $h = 0.125/2^i, i = 1, \ldots, 5$

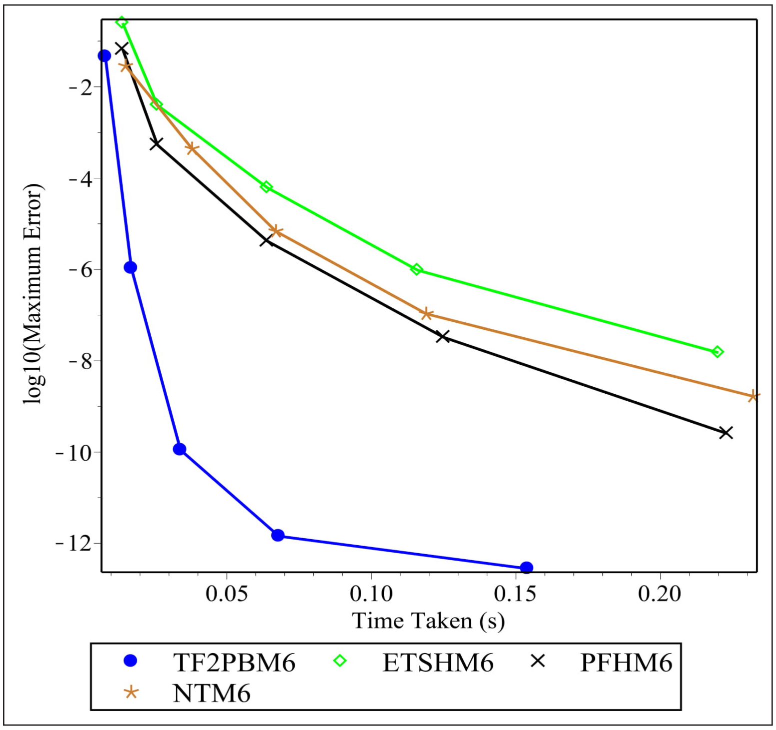

Figure 10. Efficiency curves TF2PBM6 for Problem 3 with $xn = 1,000$ and $h = 0.125/2^i, i = 3, \ldots, 7$

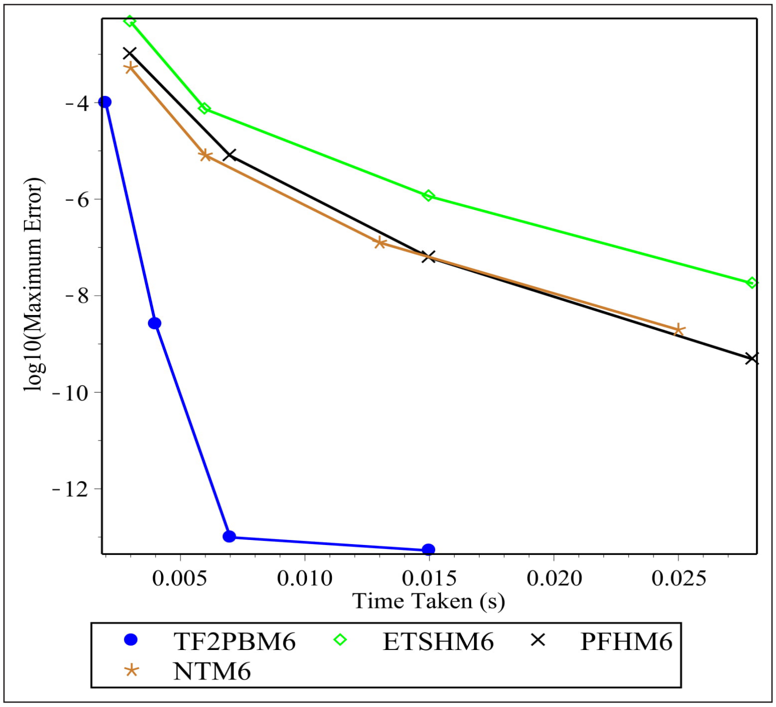

Figure 11. Efficiency curves TF2PBM6 for Problem 4 with $xn = 1,000$ and $h = 0.5/2^i$, $i = 0, \ldots, 3$

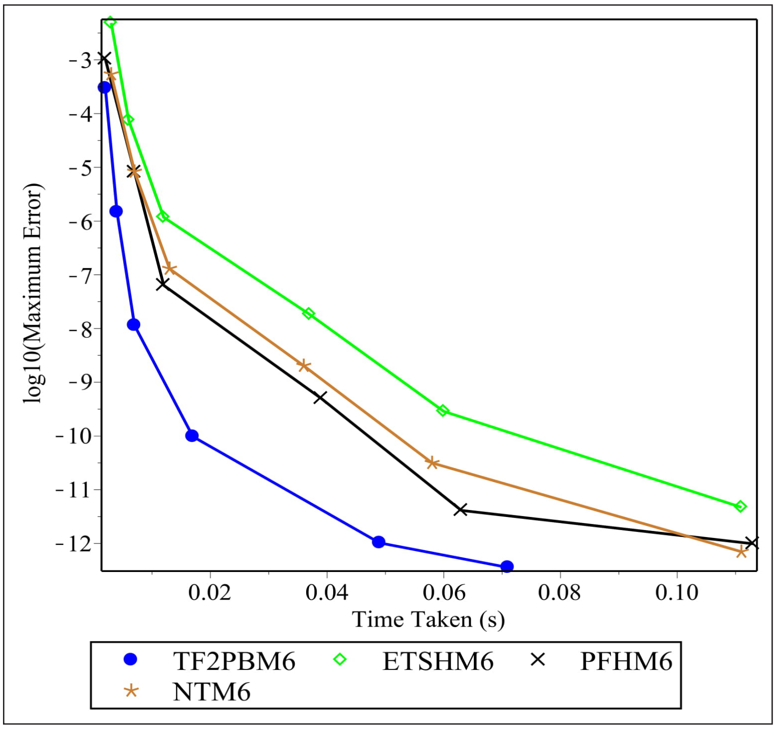

Figure 12. Efficiency curves TF2PBM6 for Problem 5 with $xn = 1,000$ and $h = 0.125/2^i$, $i = 0, \ldots, 5$

new methods are plotted against execution time and they are compared with the existing methods based on the order of the methods. Based on the efficiency curves presented, for the fourth order methods, even though the execution time taken by MSHMs is shorter than TF2PBM4, TF2PBM4 gives more accurate results compared tot MSHMs. It is observed also that the execution time taken by TF2PBM6 is the shortest compared to the other methods. TF2PBM4 and TF2PBM6 have the smallest maximum error indicating that the new methods are more accurate compared to the existing methods. However, it can be observed that for certain problems, when the value of h is too small, the maximum error became larger as shown in Figure 9. This is because the value of $H = \omega h$ will approach zero when the value of h is too small and hence the coefficients of the trigonometrically fitted methods will approach the original methods.

CONCLUSION

In this paper, the 2-point block multistep methods that have been derived in Mansor et al. (2017) is shown to be absolutely stable. The methods are trigonometrically-fitted so that they are suitable for solving oscillatory problems. Codes based on the methods are developed using C Programming Language and are used to solve all the problems. The numerical results are compared with the existing methods in the scientific literature to present the performance of the proposed methods in the form efficiency curves. In conclusions, the new methods are superior than the existing methods in terms of accuracy and execution time. Hence trigonometric-fitting approach, enhanced the performance of the methods when used for solving oscillatory problems.

ACKNOWLEDGEMENT

We gratefully acknowledged Universiti Putra Malaysia for the financial assistance received through Putra Research Grant vote number 9543500**.**

REFERENCES

Ahmad, S. Z., Ismail, F., & Senu, N. (2016). A four-stage fifth-order trigonometrically fitted semi-implicit hybrid method for solving second-order delay differential equations. *Mathematical Problems in Engineering, 2016*, 1-7.

Akinfenwa, O., Akinnukawe, B., & Mudasiru, S. (2015). A family of continuous third derivative block block methods for solving stiff systems of first order ordinary differential equations. *Journal of the Nigerian Mathematical Society,* 34(2), 160-168.

Fang, Y., & Wu, X. (2008). A trigonometrically fitted explicit numerov-type method for second-order initial value problems with oscillating solutions. *Applied Numerical Mathematics, 58*, 341-351.

Fatunla, S. O. (1995). A Class of Block Methods for Second Order IVPs. *International Journal of Computer Mathematics, 55*(1-2), 119-133.

Franco, J. M. (2006). A class of explicit two-step hybrid methods for second-order IVPs. *Journal of Computational and Applied Mathematics, 187*, 41-57.

Jikantoro, Y. D., Ismail, F., & Senu, N. (2015a). Zero-dissipative semi-implicit hybrid method for solving oscillatory or periodic problems. *Applied Mathematics and Computation, 252*, 388-396.

Jikantoro, Y. D., Ismail, F., & Senu, N., (2015b). Zero-dissipative trigonometrically fitted hybrid method for numerical solution of oscillatory problems. *Sains Malaysiana, 44*(3), 473-482.

Li, J., & Wang, X. (2016). Multi-step hybrid methods for special second-order differential equations $y''(t) = f(t, y(t))$. *Numerical Algorithms, 73*(3), 711-733.

Li, J., Lu, M., & Qi, X. (2017). Trigonometrically fitted multistep hybrid methods for oscillatory special second order initial value problems. *International Journal of Computer Mathematics*, *95*(5), 979-997.

Mansor, A. F., Ismail, F., & Senu, N. (2017). Explicit and implicit block multistep methods for solving special second order ordinary differential equations. *International Journal of Applied and Engineering Research, 12*(24), 14176-14189.

Papadopoulos, D. F., Anastassi, Z. A., & Simos, T. E. (2008). A phase-fitted runge-kutta-nyström method for the numerical solution of initial value problems with oscillating solutions. *Computer Physics Communications, 180*, 1839-1846.

Rabiei, F., Ismail, F., Senu, N., & Abasi, N. (2012). Construction of improved runge-kutta-nystrom method for solving second-order ordinary differential equations. *World Applied Science Journal, 20*, 1685-1695.

Ramos, H., Kalogiratou, Z., Monovasilis, T., & Simos, T. E. (2016). An optimized two-step hybrid block method for solving general second order initial-value problems. *Numerical Algorithms, 72*(4), 1089-1102.

Senu, N., Ismail, F., Ahmad, S. Z., & Suleiman, M. (2015). Optimize hybrid methods for solving oscillatory second order initial value problems. *Discrete Dynamics in Nature and Society, 2015*, 1-11.

Simos, T. E. (2003). Exponentially-fitted and trigonometrically-fitted symmetric linear multistep methods for the numerical integration of orbital problems. *Physics Letters A, 315*, 437-446.

Tsitouras, C. H. (2003). Explicit numerov type methods with reduced number of stages. *Computer and Mathematics with Applications, 45*, 37-42.