A design procedure for a single time-varying functional observer

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***Abstract*—The paper proposes an algorithm for the design of a single functional observer for a linear time-varying system. The proposed constructive procedure can be iterated to obtain a minmal order for the observer where the existence conditions are fufilled. As a specific feature, the proposed procedure does not require the solution of a differential Sylvester equation.**

***Index Terms*—Linear time-varying system, single functional observer, Luenberger observer, design algorithm.**

I. INTRODUCTION

Since Luenberger's seminal work [11] a significant amount of research is devoted to the problem of observing a linear functional in a time-invariant setting, see for instance [14], [26], [1], [24] and the references therein. Whereas, unlike the time-invariant counterpart, since [27], there are few papers dealing with the observer design for time-varying systems. The main part of the proposed developments are limited to the case of state observers design (see [18], [4], [17], [29], [23] and the references therein). The interest to consider linear time-varying systems is twofold [8], [6], [17] : on the one hand as general models of linear behaviour for a plant, on the other hand as linearized models of non linear systems about a given trajectory. As an example, in [30] a full order observer is used for the estimation of the imbalance in a speed-varying rotating machine.

When the observation of the whole state is not needed and in order to obtain reduced-order observers the observation of a linear functional of the state. To obtain a minimum stable observer is always an open problem even in the time-invariant case [24]. To simplify, we consider here the problem of observing a single linear functional

$$v(t) = l(t)x(t), \tag{1}$$

where, for every time t in $\mathbb{R}^+$, $l(t)$ is a differentiable vector, and $x(t)$ is the n-dimensional state vector of the state space system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned} \tag{2}$$

where $u(t)$ is the p-dimensional control, and $y(t)$ is the m-dimensional output. For every t in $\mathbb{R}^+$, $A(t)$, $B(t)$, and $C(t)$ are known matrices of appropriate dimensions. To avoid tedious counts and distracting lists of differentiability requirements, we assume every time-varying matrices and vectors are such that all derivatives that appear in the paper are continuous for all t. Without loss of generality and in order to avoid useless dynamic parts in the observer, we suppose $\begin{bmatrix} C(t) \\ l(t) \end{bmatrix}$ is a full row rank matrix for all t. Indeed if there exists $\lambda(t)$ such that $l(t) = \lambda(t)C(t)$, then, $w(t) = \lambda(t)C(t)$ is an observer for $v(t)$.

Let us define the observability matrix of (2) by

$$\Gamma(t) = \begin{bmatrix} \Gamma_0(t) \\ \Gamma_1(t) \\ \vdots \\ \Gamma_{n-1}(t) \end{bmatrix},$$

where $\Gamma_0(t) = C(t)$, and $\Gamma_j(t) = \Gamma_{j-1}(t)A(t) + \dot{\Gamma}_{j-1}(t)$ for $j = 1, 2, ...n-1$. System (2), or shortly $(A(t), C(t))$, is completely observable if rank $(\Gamma(t)) = n$ for some t in $\mathbb{R}^+$. It is uniformly observable if rank $(\Omega(t)) = n$ for every t in $\mathbb{R}^+$[21], [28].

Following [8], if r states of (2) are not observable there exists a transformation which induces the following partitions for $A(t)$ and $C(t)$

$$\begin{aligned} A(t) &= \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ 0_{(n-r)\times r} & A_{22}(t) \end{bmatrix}, \\ C(t) &= \begin{bmatrix} 0_{m\times r} & C_2(t) \end{bmatrix}, \end{aligned}$$

where $(A_{22}(t), C_2(t))$ is completely observable. Then the system is detectable when $A_{11}(t)$ is a Hurwitz matrix. A matrix $F(t)$ is said to be a Hurwitz (convergent in [25]) matrix if every solution $x(t, t_0, x_0)$ of the differential system

$$\dot{x}(t) = F(t)x(t), \; x(t_0) = x_0,$$

is such that $\lim_{t\to\infty} x(t, t_0, x_0) = 0$ for every t_0 and x_0. For example, in the case of a scalar system

$$\dot{x}(t) = f(t)x(t), \; x(t_0) = x_0,$$

where $\dim x(t) = 1$, we know that

$$x(t) = \exp\left(\int_{t_0}^{t} f(\tau)d\tau\right) x_0.$$

Thus the scalar function $f(t)$ is Hurwitz if and only if $\lim_{t\to\infty} \int_{t_0}^{t} f(\tau)d\tau = -\infty$.

Now, it is well known that the observation of $v(t)$ can be carried out with the design of the Luenberger observer

$$\begin{aligned} \dot{z}(t) &= F(t)z(t) + G(t)u(t) + H(t)y(t), \\ w(t) &= P(t)z(t) + V(t)y(t), \end{aligned} \tag{3}$$

where $z(t)$ is a q-dimensional state vector. The time-varying matrices $F(t)$, $G(t)$, $H(t)$, $P(t)$ and $V(t)$ must be determined such that (3) is an asymptotic observer of (1) for the system (2). Namely, they have to ensure

$$\lim_{t\to\infty}(v(t)-w(t))=0.$$

Following [25], [16], the completely observable system (3) is an asymptotic observer of linear functional (1) for system (2) if and only if there exists a continuously differentiable solution $T(t)$ of equations

$$G(t)=T(t)B(t),$$
$$T(t)A(t)-F(t)T(t)+\dot{T}(t)=H(t)C(t), \quad (4)$$
$$l(t)=P(t)T(t)+V(t)C(t), \quad (5)$$

and $F(t)$ is a Hurwitz matrix.

From the Cumming-Gopinath well known design procedure we can obtain a reduced-order state observer with $q=n-m$. Our main motivation is to give a simple procedure to design an asymptotic observer of the single linear functional with an order $q<n-m$. Several designs have been proposed (see for instance [19]) always to solve the fixed in the outset behaviour of the observation error. In the opposite, our purpose is to obtain a stable observer. Namely, we solve the stable observer design for the single functional (1). This standpoint leads to less order observers than those obtained to solve the fixed poles in the outset observer problem. Obviously, the pole notion must be understood here in the time-varying setting (see for instance [13]).

Let us notice that the existence conditions of the asymptotic observer require the solution $T(t)$ of the differential Sylvester equation (4) where $F(t)$ is unknown as well as the initial conditions for $T(t)$. The second motivation of our paper is to circumvent the determination of $T(t)$ as solution of the differential equation (4).

Related to our design problem we use the following derivative operator $\mathcal{D}$, for every time-varying matrix $M(t)$ with n columns

$$\mathcal{D}(M(t))=\dot{M}(t)+M(t)A(t).$$

Recursively, we define $\mathcal{D}^0(M(t))=M(t)$ and, for $i=1,\dots$

$$\begin{aligned}\mathcal{D}^i(M(t)) &= \mathcal{D}\left(\mathcal{D}^{i-1}(M(t))\right),\\ &= \mathcal{D}^{i-1}(\dot{M}(t))+\mathcal{D}^{i-1}(M(t))A(t).\end{aligned}$$

We use also two matrices, for $i=1,\dots$

$$\Sigma_i(t)=\begin{bmatrix}\mathcal{D}^0(C(t))\\ \mathcal{D}^0(l(t))\\ \mathcal{D}^1(C(t))\\ \mathcal{D}^1(l(t))\\ \vdots\\ \mathcal{D}^i(C(t))\end{bmatrix}, \quad (6)$$

and,

$$\overline{\Sigma}_i(t)=\begin{bmatrix}\mathcal{D}^0(C(t))\\ \mathcal{D}^0(l(t))\\ \mathcal{D}^1(C(t))\\ \mathcal{D}^1(l(t))\\ \vdots\\ \mathcal{D}^i(C(t))\\ \mathcal{D}^i(l(t))\end{bmatrix}. \quad (7)$$

The relationship between these two matrix will be the key point of the algorithm.

Due to tedious calculations the procedure will not be explained in a general case. Consequently, the paper is organised as follows. In the first section are detailled the procedure and conditions to obtain firstly a one-order observer and, secondly, a second-order observer. To generalize the previous steps, the second section is devoted to a discussion on several points of the procedure. In a final section an example is proposed on a time-varying system described with a canonical observable state space equation. Let us mention here that this example is a generic one which is used to detail and illustrate some points of the procedure in a rather general framework.

II. Iterative observer design

In this section we detail only the first iterative steps of our procedure. The existence of a second-order points out the main features and the basic principles for the design of the single functional observer.

A. Existence of a one-order observer

1) Design: Let us suppose that for every t we have $\operatorname{rank}(\Sigma_1(t)) = \operatorname{rank}\left(\overline{\Sigma}_1(t)\right)$. Thus, there exist $m_{C,0}(t)$, $m_{l,0}(t)$, and $m_{C,1}(t)$ which have m, 1, and m columns respectively,

$$\mathcal{D}^1(l(t))=\begin{bmatrix} m_{C,0}(t) & m_{l,0}(t) & m_{C,1}(t)\end{bmatrix}\Sigma_1(t). \quad (8)$$

For simplicity sake we suppose that the derivative of $m_{C,1}(t)$ exists.

Let us consider $v(t)=l(t)x(t)=\mathcal{D}^0(l(t))x(t)$. When we derivate this variable we get

$$\dot{v}(t)=\mathcal{D}^1(l(t))x(t)+\mathcal{D}^0(l(t))B(t)u(t). \quad (9)$$

The basic principle of our procedure is to detect in this expression known variables and their derivatives. For this purposeet us suppose we use the decomposition (8) which leads to

$$\begin{aligned}\dot{v}(t) &= m_{C,0}(t)\mathcal{D}^0(C(t))x(t)+m_{l,0}(t)\mathcal{D}^0(l(t))x(t)\\ &\quad +m_{C,1}(t)\mathcal{D}^1(C(t))x(t)+\mathcal{D}^0(l(t))B(t)u(t).\end{aligned}$$

We can include in this expression

$$\begin{aligned}y(t) &= \mathcal{D}^0(C(t))x(t),\\ \dot{y}(t) &= \mathcal{D}^1(C(t))x(t)+\mathcal{D}^0(C(t))B(t)u(t),\end{aligned}$$

which lead to

$$\begin{aligned}\dot{v}(t) &= m_{C,0}(t)y(t) + m_{l,0}(t)v(t) \\ &\quad + m_{C,1}(t)\left(\dot{y}(t) - \mathcal{D}^0(C(t))B(t)u(t)\right). \\ &\quad + \mathcal{D}^0(l(t))B(t)u(t).\end{aligned}$$

To eliminate the derivative of $y(t)$, we define $z(t) = v(t) - m_{C,1}(t)y(t)$, or

$$v(t) = z(t) + m_{C,1}(t)y(t). \tag{10}$$

Thus, we obtain

$$\begin{aligned}\dot{z}(t) &= m_{l,0}(t)z(t) + \\ &\quad (m_{C,0}(t) - \dot{m}_{C,1}(t) + m_{L,0}(t)m_{C,1}(t))y(t) \\ &\quad + (\mathcal{D}^0(l(t)) - m_{C,1}(t)\mathcal{D}^0(C(t)))B(t)u(t).\end{aligned} \tag{11}$$

When $m_{l,0}(t)$ is a Hurwitz matrix we have designed a one-order asymptotic observer for $l(t)$ given by (11) and the output (10).

2) Determination of $T(t)$: When we identify (11) and (3) we get

$$\begin{aligned}F(t) &= m_{l,0}(t), \\ G(t) &= (\mathcal{D}^0(l(t)) - m_{C,1}(t)\mathcal{D}^0(C(t)))B(t), \\ H(t) &= m_{C,0}(t) - \dot{m}_{C,1}(t) + m_{L,0}(t)m_{C,1}(t), \\ P(t) &= 1, \\ V(t) &= m_{C,1}(t).\end{aligned}$$

It is obvious that if we let

$$T(t) = \mathcal{D}^0(l(t)) - m_{C,1}(t)\mathcal{D}^0(C(t)),$$

this matrix fulfils the differential equation (4). This points ends the proofs of the design of the one-order observer.

B. Existence of a second-order observer

When the conditions $m_{l,0}(t)$ is a Hurwitz matrix or $\operatorname{rank}(\Sigma_1(t)) = \operatorname{rank}\left(\overline{\Sigma}_1(t)\right)$ are not fulfilled we can look for a second-order asymptotic observer. In order to design this observer we can derivate (9) which leads to

$$\ddot{v}(t) = \mathcal{D}^2(l(t))x(t) + \left(\mathcal{D}^1(l(t))B(t)u(t) + \mathcal{D}^0(l(t))\dot{B}(t)u(t)\right).$$

Let us suppose that $\operatorname{rank}(\Sigma_2(t)) = \operatorname{rank}\left(\overline{\Sigma}_2(t)\right)$. Thus, there exist $m_{C,0}(t)$, $m_{l,0}(t)$, $m_{C,1}(t)$, $m_{l,1}(t)$ and $m_{C,2}(t)$ which have m, 1, m, 1 and m columns respectively, such that

$$\begin{aligned}&\mathcal{D}^2(l(t)) \\ &= \begin{bmatrix} m_{C,0}(t) & m_{l,0}(t) & m_{C,1}(t) & m_{l,1}(t) & m_{C,2}(t) \end{bmatrix}\Sigma_2(t).\end{aligned}$$

Thus $\ddot{v}(t)$ can be written as

$$\begin{aligned}\ddot{v}(t) &= m_{C,0}(t)\mathcal{D}^0(C(t))x(t) + m_{l,0}(t)\mathcal{D}^0(l(t))x(t) \\ &\quad + m_{C,1}(t)\mathcal{D}^1(C(t))x(t) + m_{l,1}(t)\mathcal{D}^1(l(t))x(t) \\ &\quad + m_{C,2}(t)\mathcal{D}^2(C(t))x(t) + \\ &\quad \left(\mathcal{D}^1(l(t))B(t)u(t) + \mathcal{D}^0(l(t))\dot{B}(t)u(t)\right).\end{aligned}$$

To design the observer there are two steps :

1) to recognize known variables and their derivatives. Namely, $y(t)$, $\dot{y}(t)$, $\ddot{y}(t)$, $v(t)$ and $\dot{v}(t)$.
2) to realize the obtained differential input-output relationship.

1) Use of known variables: For the first step we use

$$\begin{aligned}y(t) &= \mathcal{D}^0(C(t))x(t), \\ \dot{y}(t) &= \mathcal{D}^1(C(t))x(t) + \mathcal{D}^0(C(t))B(t)u(t), \\ \ddot{y}(t) &= \mathcal{D}^2(C(t))x(t) + \\ &\quad \left(\mathcal{D}^1(C(t))B(t)u(t) + \mathcal{D}^0(C(t))\dot{B}(t)u(t)\right),\end{aligned}$$

and

$$\begin{aligned}v(t) &= \mathcal{D}^0(l(t))x(t), \\ \dot{v}(t) &= \mathcal{D}^1(l(t))x(t) + \mathcal{D}^0(l(t))B(t)u(t),\end{aligned}$$

which leads to

$$\begin{aligned}\ddot{v}(t) &= m_{C,0}(t)y(t) + m_{l,0}(t)v(t) \\ &\quad + m_{C,1}(t)\left(\dot{y}(t) - \mathcal{D}^0(C(t))B(t)u(t)\right) \\ &\quad + m_{l,1}(t)\left(\dot{v}(t) - \mathcal{D}^0(l(t))B(t)u(t)\right) \\ &\quad + m_{C,2}(t)\times \\ &\quad \left(\ddot{y}(t) - \mathcal{D}^1(C(t))B(t)u(t) - \mathcal{D}^0(C(t))\dot{B}(t)u(t)\right) \quad (12) \\ &\quad + \left(\mathcal{D}^1(l(t))B(t)u(t) + \mathcal{D}^0(l(t))\dot{B}(t)u(t)\right). \quad (13)\end{aligned}$$

2) Realization: To realize the previous input-output relationship we write $\ddot{v}(t)$ as

$$\begin{aligned}\ddot{v}(t) &= \mu_{C,0}(t)y(t) + \mu_{l,0}(t)v(t) + \beta_0(t)u(t) \\ &\quad + p\left[\mu_{C,1}(t)y(t) + \mu_{l,1}(t)v(t) + \beta_1(t)u(t)\right] \\ &\quad + p^2\left[\mu_{C,2}(t)y(t)\right],\end{aligned}$$

where the functions $\mu_{C,0}(t)$, $\mu_{l,0}(t)$, $\beta_0(t)$, $\mu_{C,1}(t)$, $\mu_{l,1}(t)$, $\beta_1(t)$ and $\mu_{C,2}(t)$ are defined by identification with (13) and p is the derivative operator with respect to time. For instance, we have

$$\begin{aligned}\mu_{C,0}(t) &= m_{C,0}(t) - \dot{m}_{C,1}(t) - \ddot{m}_{C,2}(t), \\ \mu_{C,1}(t) &= m_{C,1}(t) - \dot{m}_{C,2}(t), \\ \mu_{C,2}(t) &= m_{C,2}(t),\end{aligned}$$

and,

$$\begin{aligned}\mu_{l,0}(t) &= m_{l,0}(t) - \dot{m}_{l,1}(t), \\ \mu_{l,1}(t) &= m_{l,1}(t).\end{aligned}$$

The usual method for realization consists in writing the previous relationship as

$$\begin{aligned}v(t) &= \frac{1}{p}\left[\frac{1}{p}\left[\mu_{C,0}(t)y(t) + \mu_{l,0}(t)v(t) + \beta_0(t)u(t)\right]\right. \\ &\quad + \left.\left[\mu_{C,1}(t)y(t) + \mu_{l,1}(t)v(t) + \beta_1(t)u(t)\right]\right] \\ &\quad + \left[\mu_{C,2}(t)y(t)\right],\end{aligned}$$

and define the following state variables :

$$\begin{aligned} z_1(t) &= \frac{1}{p}\left[\mu_{C,0}(t)y(t) + \mu_{l,0}(t)v(t) + \beta_0(t)u(t)\right], \\ z_2(t) &= \frac{1}{p}\left[z_1 + \mu_{C,1}(t)y(t) + \mu_{l,1}(t)v(t) + \beta_1(t)u(t)\right]. \end{aligned}$$

With $v(t) = z_2(t) + \mu_{C,2}(t)y(t)$, and,

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

we get the following second-order observable realization :

$$\begin{aligned} \dot{z}(t) &= \begin{bmatrix} 0 & \mu_{l,0}(t) \\ 1 & \mu_{l,1}(t) \end{bmatrix} z(t) + \begin{bmatrix} \beta_0(t) \\ \beta_1(t) \end{bmatrix} u(t) \\ &\quad + \begin{bmatrix} \mu_{C,0}(t) + \mu_{l,0}(t)\mu_{C,2}(t) \\ \mu_{C,1}(t) + \mu_{l,1}(t)\mu_{C,2}(t) \end{bmatrix} y(t), \qquad (14) \\ v(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} z(t) + \mu_{C,2}(t)y(t). \end{aligned}$$

When the matrix

$$F(t) = \begin{bmatrix} 0 & \mu_{l,0}(t) \\ 1 & \mu_{l,1}(t) \end{bmatrix}$$

is a Hurwitz matrix we have designed a second-order asymptotic observer for $l(t)$.

Let us remark that this procedure does not need the determination of the matrix $T(t)$. But, if are interested in we can, through some calculations, obtain it. As an example this point is detailled in the appendix.

III. Discussion

For shortness sake, in order to cope with the design of high-order asymptotic single functional observers we will not detail the calculations but only discuss some points.

A. The general case

First of all the conditions for the existence of a q-order Luenberger observer are obtained through the calculus of the q-th derivative of $v(t)$ and the two steps :

1) Recognize known variables and their derivatives. Namely, $y(t), \ldots, y^{(q-1)}(t), y^{(q)}(t), v(t), \ldots, v^{(q-2)}(t)$, and, $v^{(q-1)}(t)$.
2) Realize the obtained differential input-output relationship.

Namely, the procedure we adopted for the second-order can be generalized and the conditions to design a q-order Luenberger observer are :

1) $\operatorname{rank}(\Sigma_q(t)) = \operatorname{rank}\left(\overline{\Sigma}_q(t)\right)$. Namely, there exist $m_{C,0}(t)$, $m_{l,0}(t)$, …, $m_{C,q-1}(t)$, $m_{l,q-1}(t)$ and $m_{C,q}(t)$ where the $m_{C,i}(t)$ have m columns and $m_{l,i}(t)$ are scalar such that
$$\begin{aligned} \mathcal{D}^q(l(t)) &= \left[\; m_{C,0}(t) \quad m_{l,0}(t) \quad \cdots \quad m_{l,q-1}(t) \right. \\ &\qquad \left. m_{l,q-1}(t) \quad m_{C,q}(t) \;\right] \Sigma_q(t). \qquad (15) \end{aligned}$$
2) Due to the realization step, from the scalar functions $m_{l,0}(t)$, …, $m_{l,q-1}(t)$ we deduce scalar functions $\mu_{l,0}(t)$, …, $\mu_{l,q-1}(t)$. To have an asymptotic observer the matrix
$$F(t) = \begin{bmatrix} & & & \mu_{l,0}(t) \\ 1 & & & \mu_{l,1}(t) \\ & \ddots & & \vdots \\ & & 1 & \mu_{l,q-1}(t) \end{bmatrix}$$
must be an Hurwitz matrix.

When $\operatorname{rank}(\Sigma_q(t)) < \operatorname{rank}\left(\overline{\Sigma}_q(t)\right)$ or $F(t)$ is not an Hurwitz matrix we must iterate again by another derivation of $v(t)$, namely, $v^{(q+1)}(t)$. Obviously, q is upper bounded with $n-m$.

B. The decomposition of $\mathcal{D}^q(l(t))$.

The design procedure lays on the solution of the linear equation (15) which can be solved by means of time-varying generalized inverses [2], [9], [10]. A generalized inverse $\Sigma_q^{\{1\}}(t)$ for the linear transform $\Sigma_q(t)$ is a matrix defined [3] by, for every t,

$$\Sigma_q(t)\Sigma_q^{\{1\}}(t)\Sigma_q(t) = \Sigma_q(t)$$

For example, a generalized inverse, $\Sigma_q^{\{1\}}(t)$, for $\Sigma_q(t)$ can be obtained from the time-varying singular value factorization of $\Sigma(t)$ or from its QR factorization [7], [5]. The solution set of (15) can then be expressed a

$$\begin{aligned} M_q(t) &= \left[\; m_{C,0}(t) \quad m_{l,0}(t) \quad \cdots \quad m_{C,q-1}(t) \right. \\ &\qquad \left. m_{l,q-1}(t) \quad m_{C,q}(t) \;\right], \\ &= \mathcal{D}^q(l(t))\Sigma_q^{\{1\}}(t) \\ &\qquad + W(t)(I_n - \Sigma_q(t)\Sigma_q^{\{1\}}(t)), \end{aligned}$$

where $W(t)$ is an arbitrary matrix of adapted dimensions.

When, for every t, $\operatorname{rank}(\Sigma_q(t)) = \operatorname{rank}\left(\overline{\Sigma}_q(t)\right) = (m+1)\,q + m$, the solution set is reduced to the unique element $\mathcal{D}^q(l(t))\Sigma_q^{\{1\}}(t)$. Let us notice that this element is nondependent on the choice of $\Sigma_q^{\{1\}}(t)$. Otherwise, when

$$\operatorname{rank}(\Sigma_q(t)) = \operatorname{rank}\left(\overline{\Sigma}_q(t)\right) < (m+1)\,q + m$$

then the dimension of the spanned space vector by is

$$r = (m+1)\,q + m - \operatorname{rank}(\Sigma_q(t)).$$

In this last case there are r vectors, $\omega_1(t)$, …, $\omega_r(t)$,

$$\operatorname{span}\{\omega_1(t), \ldots, \omega_r(t)\} = I\left\{W(t)(I_n - \Sigma_q(t)\Sigma_q^{\{1\}}(t))\right\},$$

which can be used to stabilize the designed q-order observer. Namely, to obtain a matrix $F(t)$ which is an Hurwitz matrix.

This step is a very important one but, for shortness sake, cannot be expressed here in a general way. Only, we use the example in the next section to give a flavour of the procedure. The possibility for getting an Hurwitz matrix lays on detecting an uniformly observable and using eigenvalues assignement techniques ([20], [12], [4]), we can yield uniform exponential stability at any desired rate[6], [17] for the observation error system $\dot{\eta}(t) = F(t)\eta(t)$.

This procedure is an extension of the procedure we proposed in [16] to get a minimum functional observer for a time-varying linear system.

IV. Example

Let us consider the observable single-output system (2) with

$$A(t) = \begin{bmatrix} 0 & 0 & 0 & -a_1(t) \\ 1 & 0 & 0 & -a_2(t) \\ 0 & 1 & 0 & -a_3(t) \\ 0 & 0 & 1 & -a_4(t) \end{bmatrix}, \; B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \\ b_4(t) \end{bmatrix},$$

$$C(t) = C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

where $n > 2$. Let us consider three cases for the single linear functional (1) :

- case 1 :
$$l(t) = \begin{bmatrix} 0 & 0 & l(t) & 0 \end{bmatrix}; \tag{16}$$
- case 2 :
$$l(t) = \begin{bmatrix} 0 & l(t) & 0 & 0 \end{bmatrix}; \tag{17}$$
- case 3 :
$$l(t) = \begin{bmatrix} l(t) & 0 & 0 & 0 \end{bmatrix}, \tag{18}$$

where in every case $l(t)$ is a given differentiable function. Our purpose is to find conditions for the existence of asymptotic observers of $v(t) = l(t)x(t)$ by applying, in every case, the proposed observer design detailed in the previous parts.

A. Case 1

We have

$$\mathcal{D}^1(C(t)) = \begin{bmatrix} 0 & 0 & 1 & -a_4(t) \end{bmatrix},$$

thus

$$\Sigma_1(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & l(t) & 0 \\ 0 & 0 & 1 & -a_4(t) \end{bmatrix}$$

and

$$\mathcal{D}^1(l(t)) = \begin{bmatrix} 0 & l(t) & \dot{l}(t) & -a_3(t)l(t) \end{bmatrix}.$$

It is obvious that except at time where $l(t) = 0$, $\text{rank}\Sigma_1(t) < \text{rank}\overline{\Sigma}_1(t)$. Consequently, a one-order observer cannot exist.

To look for a second-order let us calculate :

$$\mathcal{D}^2(C(t)) = \begin{bmatrix} 0 & 1 & -a_4(t) & a_4^2(t) - a_3(t) - \dot{a_4}(t) \end{bmatrix}$$

which leads to :

$$\Sigma_2(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & l(t) & 0 \\ 0 & 0 & 1 & -a_4(t) \\ 0 & l(t) & \dot{l}(t) & -a_3(t)l(t) \\ 0 & 1 & -a_4(t) & a_4^2(t) - a_3(t) - \dot{a_4}(t) \end{bmatrix}.$$

As the first component of $\mathcal{D}^2(l(t))$ is $l(t)$, it is obvious that, as we have not get a one-order observer, we cannot obtain a second-order single functional observer for this system.

The only solution is to design a time-varying Cumming-Gopinath reduced third-order observer. This design is always possible due to observability hypothsesis.

B. Case 2

As the rows $\mathcal{D}^i(C(t))$ are independent of the observed single functional we have

$$\Sigma_1(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & l(t) & 0 & 0 \\ 0 & 0 & 1 & -a_4(t) \end{bmatrix}$$

and

$$\mathcal{D}^1(l(t)) = \begin{bmatrix} l(t) & \dot{l}(t) & 0 & -a_2(t)l(t) \end{bmatrix}.$$

Obviously, $\text{rank}\Sigma_1(t) < \text{rank}\overline{\Sigma}_1(t)$ and a one-order observer cannot be designed.

We calculate

$$\Sigma_2(t) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & l(t) & 0 & 0 \\ 0 & 0 & 1 & -a_4(t) \\ l(t) & \dot{l}(t) & 0 & -a_2(t)l(t) \\ 0 & 1 & -a_4(t) & a_4^2(t) - a_3(t) - \dot{a_4}(t) \end{bmatrix},$$

and

$$\mathcal{D}^2(l(t)) = \begin{bmatrix} 2\dot{l}(t) & \ddot{l}(t) & -a_2(t)l(t) & -\alpha_{2,0}(t)l(t) - \alpha_{2,1}(t)\dot{l}(t) \end{bmatrix}$$

where :

$$\begin{aligned} \alpha_{2,0}(t) &= a_1(t) + a_2(t)a_4(t) + \dot{a_2}(t), \\ \alpha_{2,1}(t) &= 2a_2(t). \end{aligned}$$

As, $\text{rank}\Sigma_2(t) = 4$ we conclude that a second-order observer may exist. Let us suppose that, for every t, $l(t)$ does not vanish. In this case, we can see that

$$\Sigma_2(t) = \begin{bmatrix} \Pi_2(t) \\ \Delta_2(t) \end{bmatrix}$$

where $\Delta_2(t)$ is the last row of $\Sigma_2(t)$ and $\text{rank}(\Pi_2(t)) = 4$. Due to full row rank of $\Pi_2(t)$, on the one hand, there exists a unic vector $\Lambda_2(t)$ such that $\Delta_2(t) = \Lambda_2(t)\Pi_2(t)$. Thus, we have

$$\Lambda_2(t) = \Delta_2(t)\Pi_2^{\{1\}}(t),$$

where $\Pi_2^{\{1\}}(t)$ is a generalized inverse of $\Pi_2(t)$. We can remark that $\Pi_2(t)$ is nonsingular thus, we have

$$\Pi_2^{\{1\}}(t) = \Pi_2^{-1}(t).$$

Moreover,

$$\Pi_2(t) = P_2\widetilde{\Pi}_2(t)$$

where P_2 is the permutation matrix

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and,

$$\widetilde{\Pi}_2(t)=\begin{bmatrix} l(t) & \dot{l}(t) & 0 & -a_2(t)l(t)\\ 0 & l(t) & 0 & 0\\ 0 & 0 & 1 & -a_4(t)\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Thus,

$$\widetilde{\Pi}_2^{-1}(t)=\begin{bmatrix} \frac{1}{l(t)} & -\frac{\dot{l}(t)}{l(t)^2} & 0 & a_2(t)\\ 0 & l(t) & 0 & 0\\ 0 & 0 & 1 & a_4(t)\\ 0 & 0 & 0 & 1\end{bmatrix}$$

and we obtain

$$\Pi_2^{-1}(t)=\widetilde{\Pi}_2^{-1}(t)P_2=\begin{bmatrix} a_2(t) & -\frac{\dot{l}(t)}{l(t)^2} & 0 & \frac{1}{l(t)}\\ 0 & l(t) & 0 & 0\\ a_4(t) & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}.$$

On the other hand, there exists a unic vector $\Phi_2(t)$ such that $\mathcal{D}^2(l(t))=\Phi_2(t)\Pi_2(t)$. Thus, we have

$$\Phi_2(t)=\mathcal{D}^2(l(t))\Pi_2^{\{1\}}(t).$$

In order to solve the linear equation $\mathcal{D}^2(l(t))=M_2(t)\Sigma_2(t)$, we get

$$\Phi_2(t)\Pi_2(t)=M_2^1(t)\Pi_2(t)+m_{C,2}(t)\Lambda_2(t)\Pi_2(t)$$

where $M_2^1(t)=\begin{bmatrix} m_{C,0}(t) & m_{l,0}(t) & m_{C,1}(t) & m_{l,1}(t)\end{bmatrix}$. We deduce

$$M_2^1(t)=\Phi_2(t)-m_{C,2}(t)\Lambda_2(t)$$

and, $m_{C,2}(t)$ appears as a free scalar function which parametrizes the solution $M_2(t)$ as

$$M_2(t)=\begin{bmatrix}\Phi_2(t) & 0\end{bmatrix}-m_{C,2}(t)\begin{bmatrix}\Lambda_2(t) & -1\end{bmatrix}.$$

Following the definition of $F(t)$ in (14), namely,

$$F(t)=\begin{bmatrix} 0 & \mu_{l,0}(t)\\ 1 & \mu_{l,1}(t)\end{bmatrix}$$

with

$$\begin{aligned}\mu_{l,0}(t) &= m_{l,0}(t)-\dot{m_{l,1}}(t),\\ &= \Phi_{2,2}(t)-m_{C,2}(t)\Lambda_{2,2}(t)\\ &\quad -\dot{\Phi_{2,4}}(t)+\dot{m_{C,2}}(t)\Lambda_{2,4}(t)+m_{C,2}(t)\dot{\Lambda_{2,4}}(t),\\ \mu_{l,1}(t) &= \Phi_{2,4}(t)+m_{C,2}(t)\Lambda_{2,4}(t).\end{aligned}$$

The purpose is now to find a scalar function $m_{C,2}(t)$ such that $F(t)$ is a Hurwitz matrix. Let us suppose that $\Lambda_{2,4}(t)$ does not vanish then we get from the second relationship

$$m_{C,2}(t)=\frac{\mu_{l,1}(t)-\Phi_{2,4}(t)}{\Lambda_{2,4}(t)}$$

and, the first equation indicates the constraint to ensure the solvability with respect to $m_{C,2}(t)$. Namely, when $\mu_{l,0}(t)$ and $\mu_{l,1}(t)$ are chosen to ensure that $F(t)$ is an Hurwitz matrix, they are to fulfill the differential equation

$$\mu_{l,0}(t)=\Phi_{2,2}(t)-\frac{\mu_{l,1}(t)-\Phi_{2,4}(t)}{\Lambda_{2,4}(t)}\Lambda_{2,2}(t)-2\dot{\Phi_{2,4}}(t)+\dot{\mu_{l,1}}(t).$$

As we only solve the stable observer problem, depending of the known functions $\Phi_{2,j}(t)$ and $\Lambda_{2,j}(t)$, most of the time we can adapt the scalar functions $\mu_{l,0}(t)$ and $\mu_{l,1}(t)$ to obtain $m_{C,2}(t)$ to yield uniform exponential stability at any desired rate for the observation error system.

In order to illustrate this later point let us consider now $l(t)\equiv 1$. In this case we obtain

$$\Pi_2^{-1}(t)=\widetilde{\Pi}_2^{-1}(t)P_2=\begin{bmatrix} a_2(t) & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ a_4(t) & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

thus, with $\sigma(t)=a_4^2(t)-a_3(t)-\dot{a_4}(t)$,

$$\begin{aligned}\Lambda_2(t) &= \begin{bmatrix} 0 & 1 & -a_4(t) & \sigma(t)\end{bmatrix}\Pi_2^{-1}(t),\\ &= \begin{bmatrix} -a_3(t)-\dot{a_4}(t) & 1 & -a_4(t) & 0\end{bmatrix}\end{aligned}$$

which leads to $\Lambda_{2,2}(t)=1$ and, $\Lambda_{2,4}(t)=0$. Consequently, the definitions for $\mu_{l,0}(t)$ and $\mu_{l,1}(t)$ are reduced to

$$\begin{aligned}\mu_{l,0}(t) &= \Phi_{2,2}(t)-m_{C,2}(t)-\dot{\Phi_{2,4}}(t),\\ \mu_{l,1}(t) &= \Phi_{2,4}(t).\end{aligned}$$

Let us calculate now $\Phi_{2,2}(t)$ and $\Phi_{2,4}(t)$. We get

$$\mathcal{D}^2(l(t))=\begin{bmatrix} 0 & 0 & -a_2(t) & -\alpha_{2,0}(t)\end{bmatrix}$$

with $\alpha_{2,0}(t)=a_1(t)+a_2(t)a_4(t)+\dot{a_2}(t)$. Thus,

$$\Phi_2(t)=\begin{bmatrix} -\alpha_{2,0}(t)-a_2(t)a_4(t) & 0 & -a_2(t) & 0\end{bmatrix}$$

from which we deduce $\Phi_{2,2}(t)=\Phi_{2,4}(t)=0$. Thus

$$F(t)=\begin{bmatrix} 0 & -m_{C,2}(t)\\ 1 & 0\end{bmatrix}$$

which leads to the scalar observation error $\ddot{e}(t)=-m_{C,2}(t)e(t)$. Let us chose 2 scalar functions $\lambda(t)$ and $\varphi(t)$ such that $e(t)=\varphi(t)\exp(-\lambda(t))$ and, $\lim_{t\to\infty}e(t)=0$. Then we must choose

$$\begin{aligned}m_{C,2}(t) &= -\frac{1}{\varphi(t)}\Big(\ddot{\varphi}(t)-2\dot{\lambda}(t)\dot{\varphi}(t)\\ &\quad -\ddot{\lambda}(t)\varphi(t)+\dot{\lambda}^2(t)\varphi(t)\Big)\end{aligned}$$

to obtain a second-order asymptotic observer for $v(t)=x_1(t)$ with an uniform exponential stability at any desired rate. The precise design of this observer is let to the reader as an exercise.

V. Conclusion

In the time-varying case, we have proposed a procedure to design a single functional linear observer. Some specific features of our algorithm can be underlined : the first step uses derivatives of the single functional to be observed; the second step uses realization of an input-output differential relationship; for stabilization it uses two unique matrix factorizations based on linearly independent rows of a time-varying matrix; the proceduredo not require the determination of $T(t)$. This last point overcomes the determination of the solution of a Sylvester differential equation. With respect to other procedures [18] our design method is carried out without needing to solve a differential Sylvester equation. Moreover, the proposed algorithm points out whether we can fix at any desired rate the convergence of the observation error. In addition, if there exists a Lyapunov transform $P(t)$ [6] such that $F(t) = P(t)\Phi P(t)^{-1} + \dot{P}(t)P(t)^{-1}$ where Φ is a constant Hurwitz matrix, this can be performed by means of eigenvalues of Φ. This standpoint has already been used in [15] to design a minimal order single functional stable observer for linear time-invariant systems and the proposed design can be considered as a nontrivial extension of this result to the time-varying case. Our next development will consider the unknown input single functional observer design.

VI. Appendix

Let us see how $T(t)$ can be determined for the second-order single functional observer (14). In this case, on the one hand we have with respect to the Luenberger observer (3)

$$G(t) = \begin{bmatrix} \beta_0(t) \\ \beta_1(t) \end{bmatrix}$$

with

$$\begin{aligned} \beta_0(t) &= \big(-m_{C,1}(t)\mathcal{D}^0(C(t)) - m_{l,1}(t)\mathcal{D}^0(l(t)) \\ &\quad -m_{C,2}(t)\mathcal{D}^1(C(t)) + \dot{m}_{C,2}(t)\mathcal{D}^0(C(t)) + \mathcal{D}^1(l(t))\big) \\ &\quad \times B(t), \\ \beta_1(t) &= \left(-m_{C,2}(t)\mathcal{D}^0(C(t)) + \mathcal{D}^0(l(t))\right) B(t). \end{aligned}$$

On the other hand $G(t) = T(t)B(t)$. Thus we can propose that

$$T(t) = \begin{bmatrix} \gamma_0(t) \\ \gamma_1(t) \end{bmatrix} \qquad (19)$$

with

$$\begin{aligned} \gamma_0(t) &= -m_{C,1}(t)\mathcal{D}^0(C(t)) - m_{l,1}(t)\mathcal{D}^0(l(t)) \\ &\quad -m_{C,2}(t)\mathcal{D}^1(C(t)) + \dot{m}_{C,2}(t)\mathcal{D}^0(C(t)) + \mathcal{D}^1(l(t)), \\ \gamma_1(t) &= -m_{C,2}(t)\mathcal{D}^0(C(t)) + \mathcal{D}^0(l(t)). \end{aligned}$$

The last step consists in verifying that this expression verifies the differential Sylvester equation (4). After some calculations it is done and it can be concluded that (19) is the searched matrix. Let us remind that $T(t)$ gives the relation ship between $z(t)$ and $x(t)$ with

$$z(t) = T(t)x(t).$$

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