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Necessary and sufficient conditions for large contractions in fixed point theory

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Abstract. Many problems in integral and differential equations involve an equation in which there is almost a contraction mapping. Through some type of transformation we arrive at an operator of the form $H(x) = x - f(x)$. The paper contains two main parts. First we offer several transformations which yield that operator. We then offer necessary and sufficient conditions to ensure that the operator is a large contraction. These operators yield unique fixed points. A partial answer to a question raised in [D. R. Smart, Fixed point theorems, *Cambridge University Press*, Cambridge, 1980] is given. The last section contains examples and applications.

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1 Introduction

This paper is concerned with the following problem. Suppose that $(S, \|\cdot\|)$ is a complete metric space of continuous scalar functions $\phi : [0, T] \to \Re$ with the supremum distance function $\|\cdot\|$. Give necessary and sufficient conditions on f to ensure that

$$H(x) = x - f(x)$$

is a **large contraction** mapping S into itself.

We begin with a brief sketch of the definitions, relations to contractions, and two basic theorems about large contractions. It will place matters in context to have before us the definition of a contraction operator and the contraction mapping principle.

Definition 1.1. Let (M, ρ) be a metric space and $P : M \to M$. P is said to be a *contraction* if there exists an $\alpha < 1$ so that $\psi, \phi \in M$ implies that $\rho(P\psi, P\phi) \leq \alpha\rho(\psi, \phi)$.

The basic fixed point theorem then tells us that if (M, ρ) is a complete metric space, then P will have a unique fixed point in M.

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Frequently we encounter problems in which $\rho(P\psi, P\phi) < \rho(\psi, \phi)$ for $\psi, \phi \in M$ with $\psi \neq \phi$ and the question is raised, can we ask a bit more and still get a fixed point? We gave one answer to that question in [5] and in more detail in [7, pp. 22–23] in terms of large contractions. We showed that $H(x) = x - x^3$ is a large contraction. The references which we give in the next 10 pages show that this property advanced aspects of linear theory to cubic theory. In this paper that property is extended to $H(x) = x - f(x)$ for large classes of f, thereby again advancing aspects of linear theory to those functions f.

Moreover, in the next section we give life to these definitions and theorems by studying

$$x'(t) = -x^3(t) + \int_0^t (t-s)^{-1/2} g(s, x(s)) ds$$

showing how $H(x) = x - x^3$ is generated satisfying the definition of a large contraction and then how main parts of the following two theorems can be satisfied.

Definition 1.2. Let (M, ρ) be a metric space and $P : M \to M$. P is said to be a *large contraction* if $\rho(P\phi, P\psi) < \rho(\phi, \psi)$ for $\phi, \psi \in M$, with $\phi \neq \psi$, and if for each $\epsilon > 0$ there exists a $\delta < 1$ such that $\{\phi, \psi \in M, \rho(\phi, \psi) \geq \epsilon\}$ implies that $\rho(P\phi, P\psi) \leq \delta\rho(\phi, \psi)$.

Theorem 1.3. *Let (M, ρ) be a complete metric space and P a large contraction. Suppose there is an $x \in M$ and an $L > 0$, such that $\rho(x, P^n x) \leq L$ for all $n \geq 1$. Then P has a unique fixed point in M.*

Theorem 1.4. *Let (S, ρ) be a Banach space and M a bounded, closed, convex nonempty subset of S. Suppose that $F, P : M \to M$ and that*

$$x, y \in M \implies Fx + Py \in M,$$

F is continuous and FM is contained in a compact subset of M, P is a large contraction. Then there is a $y \in M$ with $Fy + Py = y$.

We now preview two results which follow easily from others cited in Sections 6 and 7. The first one allows us to sort through possibilities quickly, while the second one examines the ones which meet the necessary condition to see if they satisfy a sufficient condition.

Theorem 1.5. *Let $H(x) = x - f(x)$ and let $H : [a, b] \to \Re$. A necessary condition for H to be a large contraction on $[a, b]$ is that $f(a)f(b) \leq 0$.*

Theorem 1.6. *A necessary and sufficient condition for $H(x) = x - f(x)$ to be a large contraction on $[a, b]$ is that $H : [a, b] \to [a, b]$ and that f satisfies the following condition. For $x, y \in I = [a, b]$ then*

$$0 < \frac{f(x) - f(y)}{x - y} < 2, \quad x, y \in I, x \neq y. \tag{A}$$

The proof of Theorem 1.5 is cited in the beginning of Section 7 while the proof of Theorem 1.6 is cited after Theorem 6.7 in Section 6.

After the aforementioned example, the first order of business is to show that there are large collections of differential and integral equations arising in applied mathematics in which the operator H is found.

Then comes the major part of this paper in establishing necessary and sufficient conditions to ensure that H is a large contraction.

Finally, we give a number of examples using the properties of H.

Before we leave this introduction, we would mention the source of the name, large contraction.

We seek a contraction constant for $x - x^3$ from the derivative $1 - 3x^2$ and note that the contraction constant increases as x tends to zero, ending up at 1 when x reaches zero. As 1 is too large, we call such situations a large contraction.

2 Bringing $H(x)$ into the equation

Our first example will center on a class of integral equations which has been widely studied in the literature and features some of the most prominent problems, including heat equations. In the process we will sketch the requirements needed to yield a fixed point for

$$x'(t) = -x^3(t) + \int_0^t (t-s)^{-1/2} g(s, x(s))ds, \quad x(0) = x_0. \tag{2.1}$$

We will present the main parts of a complete problem so that the direction of the remaining paper will be established.

This equation is simply a vehicle to lead us to these concepts and is not offered as a fundamental application. Those are found in the last section.

First, as one might expect, $H(x)$ does not appear in the opening equation. To bring $H(x)$ into the equation and write (2.1) as an integral equation defining a natural fixed point mapping, we write

$$x'(t) = -x(t) + [x(t) - x^3(t)] + \int_0^t (t-s)^{-1/2} g(s, x(s))ds, \quad x(0) = x_0.$$

Now we have $H(x) = x - x^3$ and it is readily shown that if

$$M = \left\{ \phi : [0, \infty) \to \Re : \|\phi\| \le \sqrt{3}/3 \right\},$$

then $H : M \to M$. It was shown in [5] and [7] that H is a large contraction, although we continue to discuss that in this paper.

Next, we want to write this as an integral equation. There are several reasons for avoiding a direct integration and they will show as we proceed. Write

$$x'(t) = -x(t) + [x(t) - x^3(t)] - \int_0^t (t-s)^{-1/2} g(s, x(s))ds, \quad x(0) = x_0$$

yielding $H(x)$ in the square bracket. To get the integral equation, use the variation of parameters formula treating the last two terms as a forcing function

$$x' = -x + q(t), \quad x(0) = x_0,$$

to obtain

$$x(t) = x_0 e^{-t} + \int_0^t e^{-(t-s)} q(s) ds$$

and finally

$$x(t) = x_0 e^{-t} + \int_0^t e^{-(t-s)} [x(s) - x^3(s)] ds + \int_0^t \int_0^u e^{-(u-s)} g(s, x(s)) ds du.$$

There is much discussion in [12], [3], and [2] concerning the conclusion that expressions such as

$$\int_0^t e^{t-s} H(x(s)) ds$$

are large contractions when H is.

It is shown in both [9] and [10] that when g is continuous the last term is a compact map. Actually, the same argument here shows that the first term is also a compact map. So either by

Schauder's theorem or Theorem 1.4 we have a fixed point of the natural mapping whenever we can find a closed bounded convex non-empty set M with our mapping sending M into M and H being a large contraction.

This concludes our discussion of (2.1). Now we focus only on finding H.

The implicit function theorem. While it does not involve large contractions, there is a marvelous classical example of finding H which should be mentioned here. We suppose that we have a function of two variables $G(t,x)$ and we would like to solve the equation $G(t,x(t)) = 0$ for a function $x(t)$. Long ago some bright investigator recognized that it was just set up for fixed point theory. We define a mapping by the equation

$$(P\phi)(t) = \phi(t) - G(t,\phi(t)).$$

We define a complete metric space (M,ρ) of continuous functions on some interval $[a,b]$ with the sup-norm and, under a variety of conditions (see Smart [15, pp. 5–6]) we obtain a fixed point by contraction mappings. Immediately we have

$$\phi(t) = \phi(t) - G(t,\phi(t))$$

or

$$G(t,\phi(t)) = 0$$

on the entire interval. The point here is that we have found $H(t,x) = x - G(t,x)$. It is, of course, possible to consider $G(\phi(t))$.

In (2.1) we have used $H(x) = x - x^3$ which was shown in the original paper to be a large contraction. But many others have been found and appear in the literature. We would mention in particular Eloe, Jonnalagadda, and Raffoul [11], who have advanced the x^3 to x^5, $f(x^5)$, and even

$$H(x) = x - x^{2n+1}, \quad n \in 1,\dots,70,$$

showing that all are large contractions.

In view of these examples, especially the ones with scant connection between the kernel and the data, it is certainly reasonable to expect that our kernel be at least neutral with respect to fixed point methods. And such a kernel does exit. That class of kernels is typified by $(t-s)^{q-1}$, $0 < q < 1$, but many others are included. That class is discussed in depth by Miller [14, p. 209] with consequences on pp. 212–213 and Gripenberg [13]. They are defined as follows:

(A1) $A(t) \in C(0,\infty) \cap L^1(0,1)$.

(A2) $A(t)$ is positive and non-increasing for $t > 0$.

(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

In those references it is shown that when A has an infinite integral then the resolvent equation is

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds, \tag{2.2}$$

and that

$$0 < R(t) \le A(t), \quad \int_0^\infty R(t)dt = 1. \tag{2.3}$$

When the integral of A is finite, then the integral of R is less than one. Notice that if J is a positive constant, then $JA(t)$ still satisfies (A1)–(A3).

In a sequence of papers we showed the advantages of transforming equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds, \quad t \geq 0, \tag{2.4}$$

using a variation of parameters formula of Miller [14, pp. 191–192] into

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds, \tag{2.5}$$

with

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds. \tag{2.6}$$

Here are the steps. Starting with (2.4) and $a(t)$ continuous on $[0,\infty)$ while A satisfies (A1)–(A3) we have

$$\begin{aligned} x(t) &= a(t) - \int_0^t A(t-s)[Jx(s) - Jx(s) + f(s,x(s))ds] \\ &= a(t) - \int_0^t JA(t-s)x(s)ds + \int_0^t JA(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \end{aligned}$$

The linear part is

$$z(t) = a(t) - \int_0^t JA(t-s)z(s)ds \tag{2.7}$$

and the resolvent equation is

$$R(t) = JA(t) - \int_0^t JA(t-s)R(s)ds \tag{2.8}$$

so that by the linear variation-of-parameters formula we have

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds \tag{2.9}$$

and the non-linear variation of parameters formula then yields (2.4)

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds.$$

The transformation from (2.4) to (2.5) was first given in [8] for a Caputo equation in which case there are few difficulties. Further discussion of the transformation is found in [4].

We cannot over emphasize the role of J which seems to have entered simply at our pleasure. Our work centers around making H a contraction locally. The trivial case is

$$H(x) = x - \frac{6x}{J}.$$

Clearly taking $J = 12$ gives $H(x) = x/2$, a perfect contraction. Even so, all is not well without the transformation. For if we started (2.4) having $f(t,x) = x/2$ and $A(t) = t^{-1/2}$ which is the heat problem, for example, the integral of $A(t-s)$ is so large it would destroy the contraction. We see that (2.3) saves it.

Remark 2.1. To emphasize what has happened here, everything was against a contraction but it was all rescued by the transformation. The kernel, A, was simply too large. The function f had no contraction properties. In the end it was the insertion of $x(s)$ into the integrand by the transformation which was then slightly modified by f and J that made the contraction. Then (2.3) made R so small that a contraction in the integrand became a contraction of the entire integral.

3 A universal derivative transformation

In this section we are concerned with a very general integro-differential equation which we will transform into an equation with $H(x)$. The technique is clear if we simply consider the elementary differential equation

$$\frac{dx}{dt} = -f(t, x(t)), \quad x(0) \in \Re, \tag{3.1}$$

and start with some $J > 0$ to subtract and add $Jx(t)$ followed by multiplication by e^{Jt} yielding

$$\begin{aligned}
[x'(t) = & -Jx(t) + Jx(t) - f(t, x(t))]e^{Jt} \\
(x' + Jx)e^{Jt} &= (Jx - f(t, x))e^{Jt} \\
(xe^{Jt})' &= Je^{Jt}\left[x - \frac{f(t, x)}{J}\right] \\
x(t)e^{Jt} &= x(0) + \int_0^t Je^{Js}\left[x(s) - \frac{f(s, x(s))}{J}\right]ds \\
x(t) &= e^{-Jt}x(0) + \int_0^t Je^{-J(t-s)}\left[x(s) - \frac{f(s, x(s))}{J}\right]ds \\
x(t) &= x(0)\left[1 - \int_0^t Je^{-Js}ds\right] + \int_0^t Je^{-J(t-s)}\left[x(s) - \frac{f(s, x(s))}{J}\right]ds
\end{aligned}$$

since

$$x(0)\left[1 - \int_0^t Je^{-Js}ds\right] = x(0)[1 + e^{-Jt} - 1] = x(0)e^{-Jt}.$$

Again, we have arrived at H.

Proposition 3.1. *Let R satisfy (2.3) and*

$$x(t) = a(s) - \int_0^t R(t-s)a(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds.$$

Let $(B, \|\cdot\|)$ denote the Banach space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the sup-norm. If $P : B \to B$ is defined by $\phi \in B$ implies that

$$(P\phi)(t) = \phi(t) - \frac{f(t, \phi(t))}{J}$$

is a contraction with constant α, so is $Q : B \to B$ defined by $\phi \in B$ implies that

$$(Q\phi)(t) = a(t) - \int_0^t R(t-s)a(s)ds + \int_0^t R(t-s)\left[\phi(s) - \frac{f(s, \phi(s))}{J}\right]ds.$$

Proof. If $\phi, \psi \in B$ then

$$\begin{aligned}
|(Q\phi)(t) - (Q\psi)(t)| &\le \int_0^t R(t-s)\left|\left[\phi(s) - \frac{f(s, \phi(s))}{J}\right] - \left[\psi(s) - \frac{f(s, \psi(s))}{J}\right]\right|ds \\
&\le \alpha \int_0^t R(t-s)\|\phi - \psi\|ds \le \alpha\|\phi - \psi\|
\end{aligned}$$

as required. □

Proposition 3.2. *Let $(B, \|\cdot\|)$ be the Banach space as before and suppose there is a $D > 0$ and $J > 0$ so that*

$$\|\phi\| \leq D \implies \left|\phi(t) - \frac{f(t, \phi(t))}{J}\right| \leq D.$$

Let R satisfy (2.2) with $a(t) = x(0)$ and let P be defined by $\phi \in B \implies$

$$(P\phi)(t) = D[1 - \int_0^t R(s)ds] + \int_0^t R(t-s)\left[\phi(s) - \frac{f(s, \phi(s))}{J}\right]ds.$$

Then $\|\phi\| \leq D \implies |(P\phi)(t)| \leq D$.

Proof. If $\|\phi\| \leq D$ then

$$|(P\phi)(t)| \leq D\left[1 - \int_0^t (s)ds\right] + \int_0^t R(t-s)Dds = D$$

as required. □

4 A special integro-differential equation transformation

We now come to the most common passage to $H(x)$ that is seen in the literature. It is the case in which we almost have a linear term which could be used in the variation of parameters formula to transform our equation into an integral equation for a fixed point mapping. Suppose that we have, for example, a general function of t, say $G(t)$ which could include some integrals of the unknown function x which we write as

$$x'(t) = -x^3 + G(t).$$

Subtract and add x to obtain

$$x'(t) = -x(t) + [x(t) - x^3(t)] + G(t).$$

Apply the variation of parameters formula to obtain

$$x(t) = x(0)e^{-t} + \int_0^t e^{-(t-u)}[(x(u) - x^3(u)) + G(u)]du.$$

The function $H(x) = x - x^3$ is again captured.

5 Delay, Caputo, and higher order equations: references their applications

There is a detailed example in [7, pp. 191–199] concerning

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t))$$

which is very instructive concerning existence, uniqueness, boundedness, limit sets, and other properties. It is found in a less instructive context in the journal article [6]. The idea again is to write

$$x'(t) = -a(t)x(t) + a(t)[x(t) - x^3(t)] + b(t)x^3(t - r(t))$$

and apply the variation of parameters formula.

An example in [9, pp. 318–321] is given concerning stability and asymptotic stability of the fractional differential equation of Caputo type

$$^{c}D^{q}x = -x^{3}$$

which inverts as

$$x(t) = x(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\{x(s) - [x(s) - x^3(s)]\}ds.$$

In much the same way Ardjouni and Djoudi [2] study

$$\frac{d}{dt}x(t) = -a(t)x^3(t) + \frac{d}{dt}Q(t, x(g(t))) + G\big(t, x^3(t), x^3(g(t))\big)$$

with $x(t+T) = x(t)$ in search for periodic solutions.

E. Essel and E. Yankson [12] study

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)h(x(t)) = \frac{d}{dt}g(t, x(t-r(t))) + f(t, x(t), x(t-r(t))),$$

using large contractions and obtain a positive periodic solution. Evidently, it is an equation of considerable interest because Ardjouni and Djoudi [3] study the same equation obtaining a periodic solution. The latter authors have long publication lists found in the mathematical reviews in which they use large contractions. Both papers rely on a sufficient condition found in [1] to conclude that H is a large contraction.

Forty more papers often using large contractions can be found in MathSciNet by looking at the citations of [5].

6 Large contractions

In this section we present nasc for large contractions acting on proper subspaces of the space $\mathcal{S} := (S_K, \|\cdot\|)$ of continuous functions defined on a closed interval K of the real line, equipped with the usual sup-norm $\|\cdot\|$. To be more specific, we are interested in large contractions coming from real functions defined on a closed interval $[a, b]$ and viewed as operators acting on a related normed function space.

Given a real function $H : [a, b] \to \Re$ we consider the subspace $\mathcal{M} \subset \mathcal{S}$ defined by $\mathcal{M} := (M, \|\cdot\|)$ where

$$M := \{x \in S_K : a \le x(t) \le b, t \in K\},$$

and the operator $\mathcal{H} : \mathcal{M} \to \mathcal{S}$ defined by $x \in \mathcal{M}$ is mapped to

$$\mathcal{H}(x)(t) := H(x(t)), \quad t \in K. \tag{H}$$

A first observation is that $\mathcal{M}$ is a complete metric space with the metric induced by the sup-norm.

Theorem 6.1. *Let $a < b \in \Re$, K be a closed interval of $\Re$, and $\mathcal{S} := (S_K, \|\cdot\|)$ be the space of continuous functions on K equipped with the usual sup-norm. Then the subspace $\mathcal{M}$ of $\mathcal{S}$ is a complete metric space.*

Proof. It suffices to prove that $\mathcal{M}$ is a closed subset of the complete metric space $(S_K, \|\cdot\|)$. Indeed, if (x_n) is a sequence of elements in $\mathcal{M}$ converging (in the sup-norm) to $x \in S_K$, then for a given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$, $n \geq n_0$. Then for $t \in K$ we have

$$a - \varepsilon < a - \|x_n - x\| \leq x(t) = x_n(t) + x(t) - x_n(t) \leq b + \|x_n - x\| < b + \varepsilon$$

or

$$a - \varepsilon < x(t) < b + \varepsilon, \quad t \in K.$$

As $\varepsilon > 0$ is arbitrary, from the last inequality we see that

$$a \leq x(t) \leq b, t \in K,$$

so $x \in M$. It follows that $\mathcal{M}$ is a closed subset of S_K. □

By a close look at the Definition 1.2, one may see that an operator $\mathcal{H} : \mathcal{M} \to \mathcal{S}$ is a large contraction on $\mathcal{M}$ if and only if it satisfies

$$\mathcal{H}(\mathcal{M}) \subseteq \mathcal{M}, \tag{H0}$$

$$\|\mathcal{H}(x) - \mathcal{H}(y)\| < \|x - y\|, \quad x, y \in \mathcal{M}, x \neq y, \tag{H1}$$

and

$$\begin{cases} \text{for any } \varepsilon > 0 \text{ there exists a } \delta_\varepsilon \in (0,1) \text{ with} \\ x, y \in \mathcal{M} : \|x - y\| \geq \varepsilon \Longrightarrow \|\mathcal{H}(x) - \mathcal{H}(y)\| \leq \delta_\varepsilon \|x - y\|. \end{cases} \tag{H2}$$

In an attempt to pass from a real function $H : [a, b] \to \Re$ to a large contraction $\mathcal{H} : \mathcal{M} \to \mathcal{S}$ we are interested in posing conditions on H which yield that $\mathcal{H}$ satisfies (H0)–(H2). To this end, in the two lemmas below we show that if H satisfies

$$H([a, b]) \subseteq [a, b] := I, \tag{h0}$$

$$|H(x) - H(y)| < |x - y|, \quad x, y \in I, x \neq y, \tag{h1}$$

and

$$\begin{cases} \text{for any } \varepsilon > 0 \text{ there exists a } \delta_\varepsilon \in (0,1) \text{ with} \\ x, y \in I : |x - y| \geq \varepsilon \Longrightarrow |H(x) - H(y)| \leq \delta_e |x - y|, \end{cases} \tag{h2}$$

then the corresponding mapping $\mathcal{H}$ satisfies (H0)–(H2), and vice versa.

Lemma 6.2. *Let $H : [a, b] \to [a, b]$ be a real function and $\mathcal{H}$ be the operator defined by (H). Then $H([a, b]) \subseteq [a, b]$ implies $\mathcal{H}(\mathcal{M}) \subset \mathcal{M}$ and vice versa.*

Proof. To see that $H([a, b]) \subseteq [a, b]$ implies that $\mathcal{H}(\mathcal{M}) \subset \mathcal{M}$ we note that for $f \in \mathcal{M}$ and $t \in K$ we have $f(t) \in [a, b]$, so

$$\mathcal{H}(f)(t) = H(f(t)) \subset [a, b], \quad t \in K.$$

Conversely, if $\mathcal{H}(\mathcal{M}) \subset \mathcal{M}$ then for $x \in [a, b]$, the constant function $x(t) = x$, $t \in K$ belongs to $\mathcal{M}$, thus $\mathcal{H}(x) \in \mathcal{M}$ and

$$a \leq \mathcal{H}(f)(t) = H(f(t)) = H(x) \leq b, \quad t \in K.$$ □

Lemma 6.3. *Let $H : [a,b] \to \Re$ be a real function and $\mathcal{H} : \mathcal{M} \longrightarrow \mathcal{C}$ be the operator defined by (H). Then $\mathcal{H}$ satisfies (H1)–(H2) if and only if H satisfies (h1)–(h2).*

Proof. Assume that H satisfies (h1)–(h2). To verify (H1) we let $x, y \in \mathcal{M}$ and, due to continuity, we consider a $t_1 \in K$ with $|Hx(t_1) - Hy(t_1)| = \|Hx - Hy\|$. Then by (h1) we have

$$\|Hx - Hy\| = |Hx(t_1) - Hy(t_1)| < |x(t_1) - y(t_1)| \leq \|x - y\|,$$

so (H1) holds. Next, we consider an (arbitrary) $\varepsilon > 0$. By (h2), for the positive number $\frac{\varepsilon}{2}$ there exists a $\delta_\varepsilon^1 > 0$ such that

$$x, y \in I : |x - y| \geq \frac{\varepsilon}{2} \Longrightarrow |H(x) - H(y)| \leq \delta_\varepsilon^1 |x - y|.$$

Let $x, y \in \mathcal{M}$ with $\|x - y\| \geq \varepsilon$. Then for $t \in K$, we distinguish between the following two cases: either (i) $|x(t) - y(t)| < \frac{\varepsilon}{2}$, or, (ii) $|x(t) - y(t)| \geq \frac{\varepsilon}{2}$.

In case (i), by (h1) we have

$$|(Hx)(t) - (Hy)(t)| < |x(t) - y(t)| \leq \frac{\varepsilon}{2} \leq \frac{1}{2}\|x - y\|,$$

while in case (ii), by (h2) we take

$$|(Hx)(t) - (Hy)(t)| \leq \delta_\varepsilon^1 |x(t) - y(t)| \leq \delta_\varepsilon^1 \|x - y\|,$$

so we always have

$$|(Hx)(t) - (Hy)(t)| \leq \max\left\{\frac{1}{2}, \delta_\varepsilon^1\right\} \|x - y\|, \quad t \in K,$$

implying that

$$x, y \in \mathcal{M} : \|x - y\| \geq \varepsilon \Longrightarrow \|Hx - Hy\| \leq \delta_\varepsilon \|x - y\|,$$

so (H2) holds with $\delta_\varepsilon := \max\left\{\frac{1}{2}, \delta_\varepsilon^1\right\} \in (0,1)$. In conclusion, we see that if the function $H : [a,b] \to \Re$ satisfies (h1) and (h2) on the interval $[a,b]$, then the corresponding mapping $\mathcal{H}$ satisfies (H1) and (H2) on the function space $\mathcal{M}$.

Now we assume that (H1) and (H2) hold true and consider an arbitrary $\varepsilon > 0$ and the corresponding $\delta_\varepsilon \in (0,1)$ resulting from (H2). Then for $x, y \in I$, the constant functions $x(t) = x, y(t) = y$, $t \in K$ belong to $\mathcal{M}$ so (H1) implies

$$\begin{aligned} |H(x) - H(y)| &= |H(x(t)) - H(y(t))| \\ &= \|(\mathcal{H}x)(t) - (\mathcal{H}y)(t)\| < \|x - y\| = |x - y| \end{aligned}$$

that is (h1) holds true. Finally, from (H2) we have for $|x - y| \geq \varepsilon$

$$|H(x) - H(y)| = \|\mathcal{H}x - \mathcal{H}y\| < \delta_\varepsilon \|x - y\| = \delta_\varepsilon |x - y|,$$

and (h2) is also satisfied. □

In view of the above lemmas we see that if $H : [a,b] \to \Re$ is a real function and $\mathcal{H} : \mathcal{M} \to \mathcal{C}$ is the operator defined by (H), then $\mathcal{H}$ is a large contraction on $\mathcal{M}$ if and only if H satisfies (h0)–(h2). In other words,

Corollary 6.4. *The operator $\mathcal{H}$ is a large contraction in $\mathcal{M}$ if and only if the function H is a large contraction in $C([a,b], |\cdot|)$.*

We now proceed to presenting necessary and sufficient conditions (nasc) for the operator $\mathcal{H}$ defined by (H) to be a large contraction. For convenience, we focus on the case where $f(t,x) = f(x)$, thus, we will assume that

$$H(x) = x - f(x), \quad x \in [a,b]. \tag{f}$$

Thanks to Corollary 6.4, all we have to do is to provide conditions yielding that the real function $H : [a,b] \to [a,b]$ satisfies (h1)–(h2). To this direction, the next result establishes nasc so that the function H satisfies (h1).

Proposition 6.5. *Let f and H be as in (f). Then H satisfies (h1) if and only if*

$$0 < \frac{f(x) - f(y)}{x - y} < 2, \; x, y \in I, \, x \neq y. \tag{A}$$

Proof. Assume, first, that H satisfies (h1).

Then for $x, y \in I$ with $x \neq y$ we have

$$|x - y - [f(x) - f(y)]| = |H(x) - H(y)| < |x - y|$$

implying that

$$\left| 1 - \frac{f(x) - f(y)}{x - y} \right| < 1,$$

and so

$$0 < \frac{f(x) - f(y)}{x - y} < 2, \; x, y \in I, \, x \neq y,$$

that is (A) holds true.

Conversely, assume that (A) is true. Then for any $x, y \in I$ with $x \neq y$ we have

$$-1 < \frac{f(x) - f(y)}{x - y} - 1 < 1$$

$$\left| 1 - \frac{f(x) - f(y)}{x - y} \right| < 1$$

$$|x - y - [f(x) - f(y)]| < |x - y|$$

and so

$$|H(x) - H(y)| < |x - y|, \quad x, y \in I, \, x \neq y,$$

that is (h1) is satisfied. □

The next result comes as a little surprise: though one might expect nasc so that H satisfy (H2), it turns out that there is no such need.

Proposition 6.6. *Let f and H be as in (f). If H satisfies (h1), then H satisfies (h2).*

Proof. Let H satisfy (h1). Due to Proposition 6.5 we may assume that (A) is true. For an arbitrary $\varepsilon \in (0, b - a]$ consider the set

$$S_\varepsilon := \{(x,h) : a \leq x \leq b - h, \varepsilon \leq h \leq b - x\}$$

which is the closed triangle area in the $h - x$ plane included by the lines

$$h = \varepsilon, \quad x = a \quad \text{and } x + h = b,$$

and define the function $g_\varepsilon : S_\varepsilon \to \Re$ with

$$g_\varepsilon(x,h) := \frac{f(x+h)-f(x)}{h}, \quad (x,h) \in S_\varepsilon.$$

We have that g_ε is continuous on the compact set S_ε while it is strictly positive on S_ε [due to the left-hand-side of (A)], so by continuity the function g_ε attains a positive minimum m_e on S_ε, i.e.,

$$0 < m_\varepsilon := \min_{(x,h)\in S_\varepsilon} g_\varepsilon(x,h) \le \frac{f(x+h)-f(x)}{h} \quad \text{for all } (x,h) \in S_\varepsilon\,,$$

also, by the right-hand-side of (A), a maximum $M_\varepsilon \in (0,2)$ on S_ε. Thus, for any $(x,h) \in S_\varepsilon$ we have

$$2 > M_\varepsilon \ge \frac{f(x+h)-f(x)}{h} \ge m_\varepsilon > 0$$

or

$$-1 < 1 - M_\varepsilon \le 1 - \frac{f(x+h)-f(x)}{h} \le 1 - m_\varepsilon < 1. \tag{6.1}$$

Since $0 < m_\varepsilon \le M_\varepsilon < 2$ implies that

$$-1 < 1 - M_\varepsilon \le 1 - m_\varepsilon < 1 \Longrightarrow |1-M_\varepsilon|\,, |1-m_\varepsilon| \in (0,1)\,,$$

from (6.1) we have

$$\left|1 - \frac{f(x+h)-f(x)}{h}\right| \le \delta_e := \max\{|1-m_\varepsilon|\,, |1-M_\varepsilon|\}\,, \quad \text{for } (x,h) \in S_\varepsilon.$$

Then for $x, y \in I : |x-y| \ge \varepsilon$, say $y - x = h \ge \varepsilon$, by the last inequality we find

$$\begin{aligned} |H(x) - H(y)| &= |x - y - [f(x) - f(y)]| \\ &= |x-y| \left|1 - \frac{f(x)-f(y)}{x-y}\right| \\ &= |x-y| \left|1 - \frac{f(x+h)-f(x)}{h}\right| \\ &\le \delta_e\,|x-y|\,, \end{aligned}$$

i.e., (h2) is satisfied with $\delta_e := \max\{|1-m_\varepsilon|\,, |1-M_\varepsilon|\} \in (0,1)$. □

From Proposition 6.6 we see that if H satisfies (h1), then H satisfies both (h1) and (h2). Combining Proposition 6.5 and Proposition 6.6 we have the following result.

Theorem 6.7. *Let f, H be as in (f) with $H : [a,b] \to [a,b]$. Then $\mathcal{H}$ defined by (H) is a large contraction on $\mathcal{M}$:*

(i) if and only if f satisfies (A).

(ii) if and only if H satisfies (h1).

Proof. Clearly, by Proposition 6.5 it suffices that we deal only with (ii). Firstly, if $\mathcal{H}$ is a large contraction on $\mathcal{M}$, then by Corollary 6.4 we have that H is a large contraction in $C([a,b], |\cdot|)$ and so, by definition, we see that H satisfies (h1).

Conversely, assume that H satisfies (h1). Then by Proposition 6.6 we have that H satisfies (h2). Moreover, from as $H([a,b]) \subseteq [a,b]$, by definition we have that H is a large contraction in $C([a,b], |\cdot|)$. In turn, Corollary 6.4 yields that $\mathcal{H}$ is a large contraction in $\mathcal{M}$. □

Proof of Theorem 1.6. If H is a large contraction on $[a, b]$ then, by definition we must have $H([a, b]) \subseteq [a, b]$ as well as that H satisfy (h1), so (A) follows from Proposition 6.5. The converse follows from Theorem 6.7. □

In view of Theorem 6.1, and Theorem 6.7 we have the following.

Theorem 6.8. *Let $a, b \in \Re$, $H : I = [a, b] \to [a, b]$, K be a closed interval of $\Re$, and consider the set*

$$M := \{x \in C(K, \Re) : a \leq x(t) \leq b, t \in K\}.$$

Then $\mathcal{H} : \mathcal{M} := (M, \|\cdot\|) \to \mathcal{C}$ defined by $x \to H(x(t)), t \in K$ is a large contraction on $\mathcal{M}$ if and only if H satisfies (h1) on I, [or, if and only if the function $f(x) = x - H(x)$ satisfies (A) on I].

For the next result we note that Theorem 1.3 holds true (with $n = 1$) in the special case of a bounded metric space (M, ρ), which is exactly the case of our normed space $\mathcal{M}$, so a combination of Theorem 1.3 with Theorem 6.8 immediately leads to the next one.

Theorem 6.9. *Let $H, I, \mathcal{H}$ and $\mathcal{M}$ be as in Theorem 6.8. If the function H satisfies (h1) on I [or, equivalently, if the function $f(x) = H(x) - x$ satisfies (A)], then the mapping $\mathcal{H}$ has a unique fixed point in $\mathcal{M}$.*

Theorem 6.10. *Let $H, I, \mathcal{H}$ and $\mathcal{M}$ be as above, $g \in C(K, \Re)$ and $\mathcal{P} : \mathcal{M} \longrightarrow \mathcal{C}$ be given by*

$$(\mathcal{P}x)(t) := g(t) + (\mathcal{H}x)(t), \quad t \in K.$$

If $\mathcal{P}(\mathcal{M}) \subseteq \mathcal{M}$ and $\mathcal{H}$ satisfies (H1)–(H2) [or, H satisfies (h1)], then $\mathcal{P}$ is a large contraction in $\mathcal{M}$ having a unique fixed point in $\mathcal{M}$.

Proof. For $x, y \in \mathcal{M}$ we have for $t \in K$

$$(\mathcal{P}x)(t) - (\mathcal{P}y)(t) = (\mathcal{H}x)(t) - (\mathcal{H}y)(t),$$

so if $\mathcal{H}$ satisfies (H1)–(H2) then so does $\mathcal{P}$, and hence $\mathcal{P}$ is a large contraction on $\mathcal{M}$. If H satisfies (h1), then by Theorem 6.7 $\mathcal{P}$ is a large contraction. □

At this point we would like to point out that the results above may show the way to a partial answer to a question raised in [15], p. 38, concerning the existence of fixed points of contractive mappings in metric spaces.

Definition 6.11. Let (M, ρ) be a complete metric space. A mapping $\mathcal{P} : (M, \rho) \to (M, \rho)$ is called *contractive* if

$$\rho(\mathcal{P}(x), \mathcal{P}(y)) < \rho(x, y), \quad x, y \in M, \quad x \neq y.$$

Clearly, a mapping $\mathcal{P}$ is contractive on (M, ρ) if $\mathcal{P}(M) \subseteq M$ and $\mathcal{P}$ satisfies (h1). Recall that, by this terminology, one may say that the real function $H : [a, b] \to [a, b]$ satisfying (h1) is contractive in the space $C([a, b], \Re)$ equipped with the usual absolute value.

What we have proved so far is that a mapping $\mathcal{H}$ on $\mathcal{M}$ which is defined through a real function H by (H), turns out to be a large contraction having a unique fixed point in this complete normed space $\mathcal{M}$, provided that the real function h is contractive in $C([a, b], |\cdot|)$. In other words, if we start with a contractive real function h defined on a closed interval, then the mapping $\mathcal{H}$ defined by the specific way of (H) on the space $\mathcal{M}$ is not only contractive, but, also, a large contraction, thus having a unique fixed point in $\mathcal{M}$. To summarize, starting with a real function H which is contractive in $C([a, b], |\cdot|)$ we pass to a large contraction $\mathcal{H}$

in the space $\mathcal{M} := (M, \|\cdot\|)$ where $M := \{x \in C(K, \Re) : a \leq x(t) \leq b, t \in K\}$ with K a closed interval of $\Re$, and we conclude that $\mathcal{H}$ has a unique fixed point in $\mathcal{M}$. It is worth noticing here that the set K has no essential role and there are no conditions posed except that it is a closed interval. As it may arbitrarily be chosen, the results obtained may refer to the whole real line. We should note that this situation may change when the mapping $\mathcal{H}$ is not of the type considered so far (i.e., coming directly from H). We discuss such a case at the end of the next section.

A natural question may then arise: what could be said about existence of fixed points of an arbitrary contractive function in $\mathcal{M}$? It is known ([5]) that a contractive mapping has a fixed point when it is a large contraction, and this is what does happen in the work so far with the mapping being of the specific type given in (H) and the metric space being of the specific type of $\mathcal{M}$. Below we are concerned with metric spaces of the type of $\mathcal{M}$ and ask for conditions so that an *arbitrary* contractive mapping defined on $\mathcal{M}$ have a unique fixed point in $\mathcal{M}$. The next Theorem gives an affirmative answer to this question describing a specific class of mappings, and this may be viewed as a partial answer to a question raised in [15], p. 38. The key idea for this result is that a contractive mapping defined on the specific type of complete metric spaces with the property of mapping constant functions to constant functions is, in fact, a large contraction, so it has a unique fixed point. Note that this condition is not a necessity, and this may be verified by the example at the end of the next section.

Theorem 6.12. *If $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ is a contractive mapping which maps constant functions to constant functions, then it has a unique fixed point in $\mathcal{M}$.*

Proof. Let $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ be a contractive mapping that maps constant functions to constant functions. It follows that for any $x \in [a, b]$ the constant function with value $x(t) = x, t \in K$ belongs to $\mathcal{M}$ so the function $h : [a, b] \to \Re$ defined by $x \in [a, b]$ is mapped to

$$h(x) := (\mathcal{H}x)(t) = \mathcal{H}x,$$

is well defined on $[a, b]$. Moreover, as $\mathcal{H}(\mathcal{M}) \subseteq \mathcal{M}$ we have $h([a, b]) \subseteq [a, b]$. Since $\mathcal{H}$ is contractive it satisfies (H1), so we have for $x, y \in [a, b]$

$$|h(x) - h(y)| = \left|\mathcal{H}(x) - \mathcal{H}(y)\right| = \|\mathcal{H}(x) - \mathcal{H}(y)\| < \|x - y\| = |x - y|,$$

i.e., the real function h satisfies (h1). By Theorem 6.9, the mapping $\mathcal{H}$ is a large contraction with a unique fixed point in $\mathcal{M}$. □

We close this section by citing two useful observations concerning large contractions. The first result concerns the composition of two functions satisfying (h1).

Proposition 6.13. *If $H_1, H_2 := [a, b] \to [a, b]$ satisfy (h1), then the composition $\mathcal{H} := \mathcal{H}_2 \circ \mathcal{H}_1$ of the large contractions $\mathcal{H}_1, \mathcal{H}_2$ resulting from H_1, H_2 is also a large contraction.*

Proof. For $x, y \in [a, b]$ with $x \neq y$ we have $H_1(x) \neq H_1(y)$ and

$$\begin{aligned}
\left|\frac{H(x) - H(y)}{x - y}\right| &= \left|\frac{H_2(H_1(x)) - H_2(H_1(y))}{x - y}\right| \\
&= \left|\frac{H_2(H_1(x)) - H_2(H_1(y))}{H_1(x) - H_1(y)}\right| \left|\frac{H_1(x) - H_1(y)}{x - y}\right| \\
&< 1.
\end{aligned}$$

□

7 Remarks and applications

We start this section with citing two remarks concerning condition (A). Then we present some very simple examples of real functions that are large contractions on closed intervals of $\Re$, yet some useful observations concerning differentiable and odd real functions. Finally, we are concerned with application of our results to integral equations. Notation and definitions are as in the preceding sections.

Remark 7.1. As easily verified by (A) and in view of Theorem 6.8, *in order that* $\mathcal{H}$ *be a large contraction it is necessary that the function* f *be continuous and strictly increasing on* I. Furthermore, $H(I) \subset I$ implies

$$a \le a - f(a), b - f(b) \le b,$$

and

$$-(b-a) \le f(a) \le 0 \le f(b) \le b-a,$$

and so,

$$H : [a,b] \to [a,b] \text{ is a large contraction } \Longrightarrow f(a) f(b) \le 0\,.$$

Proof of Theorem 1.5. It follows immediately from Remark 7.1. □

Remark 7.2. Since a continuous function which is $1-1$ on an interval I it is strictly monotone on I, in order to verify that $\frac{f(x)-f(y)}{x-y} > 0$ it suffices to know that f is continuous, $1-1$ and $f(a) < f(b)$.

In the examples below we focus on verifying that H satisfies (h1) [or, equivalently, f satisfies (A)]. In view of Theorems 6.7–6.9, this yields that the operator $\mathcal{H}$ defines a large contraction in the corresponding space $\mathcal{M}$ and so it has a has a unique fixed point in $\mathcal{M}$.

Example 7.3. Let $H : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \Re$ with

$$H(x) = \begin{cases} x^2 + x, & -\frac{1}{2} \le x \le 0, \\ x^2, & 0 < x \le \frac{1}{2}. \end{cases}$$

Set $I := \left[-\frac{1}{2}, \frac{1}{2}\right]$ and note that H is increasing with $H\left(-\frac{1}{2}\right) = -\frac{1}{4}$, $H\left(\frac{1}{2}\right) = \frac{1}{4}$ so $H(I) \subset I$. By direct calculations we may easily verify that (h1) holds true for $x, y \in \left[-\frac{1}{2}, 0\right]$ and $x, y \in \left[0, \frac{1}{2}\right]$, while for $-\frac{1}{2} \le x < 0 < y \le \frac{1}{2}$ we have

$$0 < \frac{y^2 + (|x| - x^2)}{y-x} = \frac{y^2 - x^2 - x}{y-x} = \frac{H(y) - H(x)}{y-x} < \frac{y-x}{y-x} = 1.$$

so (h1) is satisfied for all $x, y \in I, x \neq y$.

Note that

a) H is not a contraction on I since $H'(0-) = 1 = H'\left(\frac{1}{2}\right)$,

b) H is not odd on I,

c) H is not differentiable at $0 \in I$.

We now would like to illustrate a comment given at the end of the second paragraph after the Definition 6.11. The reader may well see that the set K is not present in the calculations above. It can be an arbitrary closed interval of the real line $\Re$, and for such a choice, the space $\mathcal{M}$ consists of the set

$$M := \{x \in C(K, \Re) : a \le x(t) \le b, t \in K\},$$

with the operator $\mathcal{H} : \mathcal{M} \to \mathcal{C}$ be defined by $x \in \mathcal{M}$ is mapped to $(\mathcal{H}x)(t) := H(x(t)), t \in K$. Our result states that there exists a unique function $x \in M$ with $H(x(t)) = x(t), t \in K$. As K is arbitrary, this x, (being unique) may be extended to the whole real line.

Now we would like to apply the results of the previous section to two interesting types of functions, namely to differentiable functions, and, to odd functions.

Clearly, when f is differentiable on I then so is H, and vice versa. For such a case we have the following lemma.

Lemma 7.4. *Assume that H, f are differentiable on I. Then:*

(i) Condition (A) is equivalent to

$$0 \le f'(x) \le 2, \quad \text{with} \quad f(x), f(x) - 2x : 1-1, \quad x \in I, \tag{A$_d$}$$

(ii) Condition (h1) is equivalent to

$$|H'(x)| \le 1 \quad \text{with} \quad H(x) \pm x : 1-1, \quad x \in I. \tag{h$_d$}$$

Proof. We only prove (ii), the proof of (i) following immediately from (ii).

Let H be differentiable on I. Assume, first, that (h1) holds. Then letting $y \to x$ in (h1) we have that $|H'(x)| \le 1$, yet from (h1) we take for $x, y \in I$, $y < x$,

$$-1 < \frac{H(x) - H(y)}{x - y} < 1$$
$$y - x < H(x) - H(y) < x - y$$

from which we have

$$H(y) + y < H(x) + x \quad \text{and} \quad H(x) - x < H(y) - y$$

so both $H(x) \pm x$ are $1-1$ on I and (h$_d$) holds true.

Conversely, if (h$_d$) holds then by the mean value theorem we have for $x, y \in I$, $x \ne y$,

$$\left|\frac{H(x) - H(y)}{x - y}\right| \le 1,$$

Now if there exist $x, y \in I$, $x \ne y$ with $|\frac{H(x)-H(y)}{x-y}| = 1$ then either $\frac{H(x)-H(y)}{x-y} = 1$ or $\frac{H(x)-H(y)}{x-y} = -1$. In the first case we have that $H(x) - x = H(y) - y$ with $x \ne y$ so $H(x) - x$ is not $1-1$ on I, while in the second case we get that $H(x) + x$ is not $1-1$ on I, in both cases reaching a contradiction to (h$_d$). It follows that $|\frac{H(x)-H(y)}{x-y}| \ne 1$ and (h$_d$) implies (h1). □

Example 7.5. The function $H : \left[-\frac{1}{2}, 0\right] \to \Re$ with

$$H(x) = x^2 + x,$$

satisfies (h1) on $I := \left[-\frac{1}{2}, 0\right]$. Indeed, we easily find that

$$0 \leq H'(x) = 2x + 1 \leq 1, \quad x \in I$$

that is H is nondecreasing with

$$H\left(-\frac{1}{2}\right) = -\frac{1}{4}, H(0) = 0 \Longrightarrow H(I) \subset I,$$

while $H(x) - x = x^2$ and $H(x) + x = x^2 + 2x$ are $1-1$ on I. Since

$$0 < \frac{H(x) - H(y)}{x - y} = 1 + x + y < 1, \quad x, y \in \left[-\frac{1}{2}, 0\right], \quad x \neq y,$$

we see that H satisfies (h1).

Note that H does not define a contraction on I since $H'(0) = 1$.

From Lemma 7.4 and Theorem 6.8 we have the following corollary.

Corollary 7.6. *Let $a, b \in \Re$, $H : I = [a, b] \to [a, b]$ be differentiable, S_K be the space of continuous functions on a closed interval $K \subset \Re$ and set $\mathcal{M} := (M, \|\cdot\|)$ for the (complete) metric space with*

$$M := \{x \in S_K : a \leq x(t) \leq b, t \in K\},$$

equipped with the metric induced by the usual sup-norm. Then $\mathcal{H} : \mathcal{M} \to \mathcal{M}$ defined by $(\mathcal{H}x)(t) = H(x(t)), t \in K$ is a large contraction on $\mathcal{M}$ if and only if H satisfies (h_d) on I [equivalently, if and only if f satisfies (A_d) on I].

Remark 7.7. When f' is (strictly) increasing, then to verify that $\frac{f(x)-f(y)}{x-y} < 2$ it suffices to show that $f'(b) \leq 2$, since for $x, y \in I$ with $x \neq y$ we have

$$\frac{f(x) - f(y)}{x - y} = f'\left(\xi_{x,y}\right) < f'(b) \leq 2.$$

Similarly, to verify that $0 < \frac{f(x)-f(y)}{x-y}$ it suffices to show that $0 \leq f'(0)$.

Example 7.8. Let $f : [0, 1] \to \Re$ be given by

$$f(x) := \frac{1}{e}\left(e^{x^2} - 1\right), \quad x \in [0, 1] := I.$$

We have

$$f'(x) = \frac{1}{e} e^{x^2} 2x \geq 0, \quad x \in [0, 1]$$

so f is increasing on I, thus

$$f(I) = f([0, 1]) = [f(0), f(1)] = \left[0, 1 - \frac{1}{e}\right] \subset I.$$

As both functions $2x$ and e^{x^2} are nonnegative and strictly increasing on $[0, 1]$, the same is true for f'. Thus

$$0 \leq f'(0) \leq f'(x) \leq f'(1) = 2, \quad x \in I.$$

By Remark 7.7 it follows that f satisfies (A) so from Theorem 6.7 we see that the function $H := x - \frac{1}{e}\left(e^{x^2} - 1\right)$, $x \in [0, 1]$ is a large contraction on I. Note that H is not a contraction on I since $H'(x) = 1 - \frac{1}{e} e^{x^2} 2x$ and $H'(0) = 1$.

Next, we consider the case where f is an odd function. Then so is H, and vice versa. In order to show that the operator $\mathcal{H}$ is a large contraction, it suffices to verify that f satisfies (A) only on $[0,k]$. Indeed, if f satisfies (A) for $x,y \in [0,k]$ with $x \neq y$, then clearly it does so for $x,y \in [-k,0]$, $x \neq y$. To prove that f satisfies (A) for $k \leq x < 0 < y \leq k$ f is odd and continuous we have that $f(0) = 0$ so for any $x \in (0,k]$ we have

$$0 < \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} < 2 \Longrightarrow f(x) < 2x.$$

Now for $-k \leq x < 0 \leq y \leq k$, with $x \neq y$ we have $-x > 0$ and so

$$\begin{aligned} 0 < \frac{f(x) - f(y)}{x - y} &= \frac{-f(-x) - f(y)}{x - y} = \frac{-[f(-x) + f(y)]}{-(-x) - y} \\ &= \frac{f(-x) + f(y)}{(-x) + y} < \frac{2(-x) + 2f(y)}{(-x) + y} = 2, \end{aligned}$$

which completes the proof of our claim. By the above discussion and in view of Theorem 6.7, we have the following corollary.

Corollary 7.9. *Let $H : [-k,k] \to [-k,k]$ be an odd function and $f(x) = H(x) - x$.*

(i) $\mathcal{H}$ is a large contraction if and only if H satisfies (h1) [resp., f satisfies (A)] on $[0,k]$ (equivalently, on $[-k,0]$).

(ii) When f is differentiable on $[0,k]$ (equivalently, on $[-k,0]$), then $\mathcal{H}$ is a large contraction if and only if f satisfies (A_d) [resp., H satisfies (h_d)] on $[0,k]$ (equivalently, on $[-k,0]$).

Example 7.10. The function $H : I = \left[-\frac{1}{2}, \frac{1}{2}\right] \to \Re$ with

$$H(x) = \begin{cases} x^2 + x, & -\frac{1}{2} \leq x \leq 0 \\ x - x^2, & 0 < x \leq \frac{1}{2}, \end{cases}$$

is an odd function with $H\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$, which, in view of Example 7.5 satisfies (h1) on $\left[-\frac{1}{2}, 0\right]$. By Corollary 7.9 we have that $\mathcal{H}$ is a large contraction on $\mathcal{M}$.

The above example is a special case of the next more general one.

Example 7.11. The functions of the type

$$H(x) := x - |x|^{a-1} x, \ \ a > 1$$

define large contractions on symmetric, properly small neighborhoods of zero.

Proof. Here we have

$$f(x) = |x|^{a-1} x, \quad x \in \Re \ (a > 1),$$

and f is odd on sets $I = [-k,k]$ for any $k \in (0,1)$. Note that f is differentiable with

$$f'(x) = a|x|^{a-1}, \quad x \in \Re.$$

since, by definition it holds $f'(0) = 0$ while for $x \neq 0$ we have $(|x|)' = \operatorname{sgn} x$ so

$$\begin{aligned} \left(|x|^{a-1} x\right)' &= |x|^{a-1} + x(a-1)|x|^{a-2} \operatorname{sgn} x \\ &= |x|^{a-1} + (a-1)|x|^{a-2}|x| \\ &= |x|^{a-1}(1 + a - 1) \\ &= a|x|^{a-1}. \end{aligned}$$

Next, it is not difficult to see that both functions $f \pm x = (|x|^{a-1} \pm 1)x$ are $1-1$ on $[0,k]$. As f' is increasing on $[0,k]$ we have that (A_d) holds true, so in view of Corollary 7.9 we may consider only $x \geq 0$ and see that H is a large contraction if and only if

$$f'(x) = ax^{a-1} \leq 2 \Longrightarrow x \leq \left(\frac{2}{a}\right)^{\frac{1}{a-1}}.$$

Thus, in order that $H(I) \subset I$ it suffices to ask is that $|H(k)| \leq k$, i.e.,

$$-k \leq k - k^a \leq k.$$

Noting that we always have $H(k) \leq k$ all we have to require is that

$$k^a \leq 2k \Longrightarrow 0 < k \leq 2^{\frac{1}{a-1}},$$

so we conclude that H is a large contraction on any set $[-k,k]$ with

$$k \leq k_0 := \min\left\{\left(\frac{2}{a}\right)^{\frac{1}{a-1}}, 2^{\frac{1}{a-1}}\right\}, \quad (a > 1). \qquad \square$$

In particular, for $a = 2n+1, n \in \mathbb{N}$ we have

$$k_0 := \min\left\{\left(\frac{2}{a}\right)^{\frac{1}{a-1}}, 2^{\frac{1}{a-1}}\right\} = \min\left\{\left(\sqrt[2n]{\frac{2}{2n+1}}\right), \sqrt[2n]{2}\right\} = \sqrt[2n]{\frac{2}{2n+1}}$$

so the function $x - x^{2n+1}, n \in \mathbb{N}$ is a large contraction on the set $\left[-\sqrt[2n]{\frac{2}{2n+1}}, \sqrt[2n]{\frac{2}{2n+1}}\right]$ for any $n \in \mathbb{N}$ (see, also, examples in [1,11]).

Remark 7.12. If f is a twice differentiable odd function with $0 \leq f'(x) \leq 2$, $x \in [0,k]$ and such that $f''(x) > 0$, $x \in (0,k]$, then $H := x - f(x)$ is a large contraction on I.

Example 7.13. The (odd) function

$$H(x) := x - \frac{1}{3}\sin^3 x, \quad x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

defines a large contraction since

$$0 \leq f'(x) = [x - H(x)]' = \left[\frac{1}{3}\sin^3 x\right]' = \sin^2 x \cos x \leq 2, \quad x \in \left[0, \frac{\pi}{4}\right],$$

while

$$\begin{aligned} f''(x) &= \left[\sin^2 x \cos x\right]' = 2\sin x \cos^2 x - \sin^3 x = \sin x\left[2\cos^2 x - \sin^2 x\right] \\ &= \sin x \cos^2 x\left[2 - \tan^2 x\right] > 0, \quad x \in \left(0, \frac{\pi}{4}\right], \end{aligned}$$

and the result follows from 7.12.

Some simple observations may be useful in proving local existence results for highly non-linear functions. For example, if g, f are odd functions with $0 \leq f'(x), g'(x) \leq 1$ and $f''(x), g''(x) > 0$, $x > 0$, then $H(x) := x - g(f(x))$ defines a large contraction provided that there exists some $r \in (0,k]$ such that $H([-r,r]) \subseteq [-r,r]$. Indeed, we have that $g(f(x))$ is odd and

$$[g(f(x))]' = g'(f(x)) f'(x) \in [0,1],$$

yet $f''(x) > 0$ for $0 < x \leq k$, thus f' is strictly increasing and so $f'(x) > 0$ for $x > 0$. It follows that

$$[g(f(x))]'' = g''(f(x)) [f'(x)]^2 + g'(f(x)) f''(x) > 0, \quad x \neq 0,$$

and the result holds true by Remark 7.12.

For example, taking $f(x) = (e^{x^2} - 1) \operatorname{sgn} x$ and $g(x) = x^3$ one may verify that the conditions above are satisfied in a properly small neighborhood of zero. Moreover, for the (odd) function

$$H(x) := x - \left(e^{x^6} - 1\right) \operatorname{sgn} x$$

we have that $H(x) < x$ for $x > 0$, and, (by $H'(0) = 1$ and continuity of H') $-1 < H'(x)$, $x \in [0,r]$ for some properly small $r > 0$. It follows that $H([0,r]) \subseteq [-r,r]$, so $H([-r,r]) \subseteq [-r,r]$ and we may conclude that H defines a large contraction near zero.

Clearly, an odd function $H : [-k,k] \to \Re$ which is a large contraction on $[-k,k]$, has a unique fixed point in $[-k,k]$ and, due to continuity, this point must be 0. In such a case it is not difficult to see that the same is true for the mapping $\mathcal{H}$ since $H(0) = 0$ implies

$$\mathcal{H}(\mathbf{0})(t) = H(\mathbf{0}(t)) = H(0) = 0 = \mathbf{0}(t),$$

with $\mathbf{0} : K \to \Re$ being the zero function.

Proposition 7.14. *If $H : [-k,k] \to \Re$ is an odd function which is a large contraction on $I := [-k,k]$, then the unique fixed point of $\mathcal{H}$ in $\mathcal{M}$ is the zero function.*

Finally, we are interested in showing how our results in this paper may be employed in order to obtain existence and uniqueness of solutions to nonlinear integral equations. To this direction, we consider the equation

$$x(t) = g(t) + \int_0^t Q(t,s) H(x(s)) ds, \quad t \geq 0, \tag{7.1}$$

let $K := [0,T]$ with $T > 0$ being arbitrary and assume that H is contractive.

Lemma 7.15. *Let $H : [a,b] \longrightarrow [a,b]$ satisfy (h1) and the kernel Q is such that*

$$\int_0^t Q(t,s) ds \leq 1, \quad t \in [0,T]. \tag{7.2}$$

Then the operator $\widetilde{\mathcal{H}} : \mathcal{M} \longrightarrow \mathcal{C}$ defined by

$$\left(\widetilde{\mathcal{H}}x\right)(t) := g(t) + \int_0^t Q(t,s) (Hx)(s) \, ds, \quad t \in [0,T],$$

satisfies (H1)–(H2).

Proof. We have for $t \in [0, T]$

$$\left|\left(\widetilde{\mathcal{H}}x\right)(t) - \left(\widetilde{\mathcal{H}}y\right)(t)\right| \le \int_0^t |Q(t,s)|\,|(Hx)(s) - (Hy)(s)|\,ds$$
$$< \int_0^t |Q(t,s)|\,|x(s) - y(s)|\,ds$$
$$\le \int_0^t |Q(t,s)|\,ds\,\|x - y\|,$$

i.e.,

$$\left|\left(\widetilde{\mathcal{H}}x\right)(t) - \left(\widetilde{\mathcal{H}}y\right)(t)\right| < \int_0^t |Q(t,s)|\,ds\,\|x - y\|, \quad t \in [0, T].$$

By continuity of the functions on the compact set $[0, T]$ and the condition (7.2), from the last relation we take

$$\left\|\widetilde{\mathcal{H}}x - \widetilde{\mathcal{H}}y\right\| < \|x - y\|,$$

so $\mathcal{H}$ satisfies (H1).

Now, in view of (h2), for an arbitrary $\varepsilon > 0$ consider a $\delta_\varepsilon > 0$ yielded by Lemma 6.3, thus for $x, y \in \mathcal{M}$ with $\|x - y\| > \varepsilon$ we have

$$\|\mathcal{H}x - \mathcal{H}y\| < \delta_\varepsilon \|x - y\|.$$

Then for $t \in [0, T]$ we have

$$\left|\left(\widetilde{\mathcal{H}}x\right)(t) - \left(\widetilde{\mathcal{H}}y\right)(t)\right| \le \int_0^t |Q(t,s)|\,ds\,\|x - y\|$$
$$< \delta_\varepsilon \|x - y\| \int_0^t |Q(t,s)|\,ds$$
$$\le \delta_\varepsilon \|x - y\|.$$

Employing once more continuity of functions on the compact set $[0, T]$, from the last relation we take

$$\|\mathcal{H}x - \mathcal{H}y\| < \delta_\varepsilon \|x - y\|,$$

which verifies (H2) and completes the proof. □

An interesting example satisfying condition (7.2) is the fractional type kernel $Q(t,s) = \frac{1}{\Gamma(p)\Gamma(q)}(t-s)^{q-1}s^{p-1}$, $p, q \in (0,1), p + q = 1$, a non-constant kernel with integral 1 for all $t > 0$ having singularities at $t = s$ and $s = 0$.

In view of Theorem 6.10 and Lemma 7.15 we have the following existence result for the integral equation (7.1).

Proposition 7.16. *Assume that $H : [a, b] \longrightarrow [a, b]$ satisfies (h1), condition (7.2) holds true and let $\widetilde{\mathcal{H}}, \mathcal{M}$ be as in Lemma 7.15. If, in addition, there exists $\mathcal{M}_0 \subseteq \mathcal{M}$ with $\widetilde{\mathcal{H}}(\mathcal{M}_0) \subseteq \mathcal{M}_0$, then $\widetilde{\mathcal{H}}$ has a unique fixed point in $\mathcal{M}_0$ and the integral equation (7.1) has a unique solution on $[0, T]$.*

Here are a few things worth mentioning. The first one is that the operator $\widetilde{\mathcal{H}}$ again comes from the real function H but not directly, so the situation here is not exactly the same as in the previous section, and this is the reason that we need Lemma 7.15 to proceed. The function H is involved in the integral, and this may assign an essential role of the set $K = [0, T]$: T may not be arbitrarily chosen as the behavior of the integral may destroy (7.2) or even $\widetilde{\mathcal{H}}(\mathcal{M}_0) \subseteq \mathcal{M}_0$. Note, also, that, even when $g = 0$, in general $\widetilde{\mathcal{H}}$ does not map constant functions to constant ones so the condition in Theorem 6.12 is not a necessary one.

To illustrate Proposition 7.16, we recall our starting equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s, x(s))ds, \quad t \geq 0, \tag{7.3}$$

which we write as

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds,$$

with J arbitrarily chosen,

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds,$$

and

$$\int_0^t R(s)ds \leq 1, \quad t \geq 0,$$

by the non-linear variation of parameters formula. Clearly, the kernel R satisfies (7.2). To keep things as simple as can be, we assume that $a \in C([0, \infty), [0, D])$ is positive and decreasing and $f(t, x) \equiv f(x)$ is strictly increasing with $f(0) = 0$ and L-Lipschitz on $[-D, D]$. Choose $J = L + 1$, set $\widetilde{f}(x) = \frac{f(x)}{J}$ and $H(x) = x - \widetilde{f}(x)$. Then equation (7.3) takes the form

$$x(t) := z(t) + \int_0^t R(t-s)H(x(s))\,ds, \quad t \geq 0.$$

Note that for $x, y \in [-D.D]$ we have

$$0 < \frac{\widetilde{f}(x) - \widetilde{f}(y)}{x - y} = \frac{1}{J}\frac{f(x) - f(y)}{x - y} \leq \frac{L}{L+1} < 1,$$

so $\widetilde{f}$ satisfies (A), thus H satisfies (h1). Now for $T > 0$ arbitrary, consider the operator $\mathcal{P} : \mathcal{M} := C([0, T], [-D, D]) \longrightarrow C([0, T], \Re)$ given by

$$\mathcal{P}(x)(t) := z(t) + \int_0^t R(t-s)Hx(s)ds, \quad t \in [0, T].$$

Observing that $f(0) = 0$ and f is L-Lipschitz implies $0 \leq \frac{f(x(s))}{Jx(s)} < 1$, for $x \in [-D, D]$ we have

$$|H(x)| = \left|x - \widetilde{f}(x)\right| = |x|\left|1 - \frac{f(x)}{J}\right| \leq D,$$

i.e., $H([-D, D]) \subseteq [-D, D]$.

Also, for $x, y \in C([0,T], [-D,D])$ we have for $t \in [0,T]$

$$\begin{aligned} |\mathcal{P}(x)(t)| &\le \left| a(t) - \int_0^t R(t-s)a(s)ds \right| + \int_0^t R(t-s) \left| x(s) - \frac{f(x(s))}{J} \right| ds \\ &\le a(t) \left[1 - \int_0^t R(t-s)ds \right] + \int_0^t R(t-s) |x(s)| \left| 1 - \frac{f(x(s))}{Jx(s)} \right| ds \\ &\le D \left[1 - \int_0^t R(t-s)ds \right] + D \int_0^t R(t-s)ds \\ &= D, \end{aligned}$$

so $\mathcal{P}(\mathcal{M}) \subset \mathcal{M}$ with $M := \{x \in C([0,T]) : |x(t)| \le D\}$.

In sum, $H : [-D,D] \to [-D,D]$ satisfies (h1), R satisfies (7.2) and $\mathcal{P}(\mathcal{M}) \subset \mathcal{M}$, so, by Proposition 7.16 we have that equation (7.3) has a unique fixed point in $\mathcal{M}$. As $T > 0$ is arbitrary, we conclude that equation (7.3) has a unique solution defined for all $t \ge 0$. Note that the Lipschitz constant may be unbounded on the half-line.

References

[1] M. Adivar, M. N. Islam, Y. N. Raffoul, Separate contraction and existence of periodic solutions in totally nonlinear delay differential equations, *Hacet. J. Math. Stat.* **41**(2012), No. 1, 1–13. MR2976906

[2] A. Ardjouni, A. Djoudi, Existence and positivity of solutions for a totally nonlinear neutral periodic differential equation, *Miskolc Math. Notes*, **14**(2013), No. 3, 757–768. https://doi.org/10.18514/mmn.2013.742; MR3153963

[3] A. Ardjouni, A. Djoudi, Periodic solutions for a second order nonlinear neutral functional differential equation with variable delay, *Matematiche (Catania)* **69**(2014), 103–115. MR3274099

[4] L. C. Becker, T. A. Burton, I. K. Purnaras, An inversion of a fractional differential equation and fixed points, *Nonlinear Dyn. Syst. Theory* **15**(2015), No. 3, 242–271. MR3410724

[5] T. A. Burton, Integral equations, implicit functions, and fixed points, *Proc. Amer. Math. Soc.* **124**(1996), No. 8, 2383–2390. https://doi.org/10.1090/S0002-9939-96-03533-2; MR1346965

[6] T. A. Burton, Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem, *Nonlinear Stud.* **9**(2002), 181–190. MR1898587

[7] T. A. Burton, *Stability by fixed point theory for functional differential equations*, Dover, Mineola, N.Y., 2006. MR2281958

[8] T. A. Burton, Fractional differential equations and Lyapunov functionals, *Nonlinear Anal.* **74**(2011), No. 16, 5648–5662. https://doi.org/10.1016/j.na.2011.05.050; MR2819307

[9] T. A. Burton, B. Zhang, Fixed points and fractional differential equations: examples, *Fixed Point Theory* **14**(2013), No. 2, 313–326. MR3137175

[10] D. Dwiggins, Fixed point theory and integral equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math Anal.* **23**(2016), No. 1, 47–57. MR3453290

[11] P. Eloe, J. M. Jonnalagadda, Y. N. Raffoul, The large contraction principle and existence of periodic solutions for infinite delay Volterra difference equations, *Turkish J. Math.* **43**(2019), 1988–1999. MR3992665

[12] E. Essel, E. Yankson, On the existence of positive periodic solutions for totally nonlinear neutral differential equations of the second-order with functional delay, *Opuscula Math.* **34**(2014), No. 3, 469–481. https://doi.org/10.7494/OpMath.2014.34.3.469; MR3239077

[13] G. Gripenberg, On positive, nonincreasing resolvents of Volterra equations, *J. Differential Equations* **30**(1978), 380–390. https://doi.org/10.1016/0022-0396(78)90007-4; MR521860

[14] R. K. Miller, *Nonlinear Volterra integral equations*, Benjamin, Menlo Park, CA, 1971. MR0511193

[15] D. R. Smart, *Fixed point theorems*, Cambridge University Press, Cambridge, 1980.