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Existence of solution for two classes of Schrödinger equations in $\mathbb{R}^N$ with magnetic field and zero mass

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Abstract. In this paper, we consider the existence of a nontrivial solution for the following Schrödinger equations with a magnetic potential A

$$-\Delta_A u = K(x) f(|u|^2)u, \quad \text{in } \mathbb{R}^N$$

where $N \geqslant 3$, K is a nonnegative function verifying two kinds of conditions and f is continuous with subcritical growth. We discuss the above equation with K asymptotically periodic and $K \in L^r$.

Keywords: Schrödinger equation, magnetic field, zero mass, periodic condition, asymptotically periodic condition.

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1 Introduction

In this paper, we consider the existence of a nontrivial solution for the following equation

$$-\Delta_A u = K(x) f(|u|^2)u, \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

where $N \geqslant 3$, $K : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative function and $f : \mathbb{R} \to \mathbb{R}$ is continuous with subcritical growth.

Problem (1.1) is motivated by the following nonlinear Schrödinger equation

$$\left(\frac{h}{i}\nabla - A(x)\right)^2 \psi = K(x) f(|\psi|^2)\psi,$$

where $N \geqslant 3$, h is the Planck constant and A is a magnetic potential of a given magnetic field $B = \operatorname{curl} A$, and the nonlinear term f is a nonlinear coupling and K is nonnegative. The function $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a magnetic potential and the Schrödinger operator is defined by

$$-\Delta_A \psi = -\Delta\psi + |A|^2\psi - 2iA\nabla\psi - i\psi \operatorname{div} A, \quad \text{in } \mathbb{R}^N.$$

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This class of problem with the nonlinearity f verifying the condition $f'(0) = 0$ is known as zero mass.

In recent years, much attention has been paid to the nonlinear Schrödinger equations, we may refer to [6, 13, 23, 25–29]. In particular, we notice that the existence of solutions for the problems with zero mass and without magnetic field, namely, $A \equiv 0$ and $f'(0) = 0$. In [5], Alves and Souto investigated the following problem

$$-\Delta u = K(x)f(u), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where f is a continuous function with quasicritical growth and K is nonnegative function. Using the variational method and some technical lemmas, the authors gave the existence of positive solution for problem (1.2).

In [20], Li, Li and Shi considered a nonlinear Kirchhoff type problem

$$-\Big(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2\Big)\Delta u = K(x)f(u), \qquad x \in \mathbb{R}^N,$$

where $N \geqslant 3$, a is a positive constant, $\lambda \geqslant 0$ is a parameter and K is a potential function. The authors used a priori estimate and a Pohozaev type identity in the case with constant coefficient nonlinearity. And in the problem with the variable-coefficient, a cut-off functional and Pohozaev type identity were used to find Palais–Smale sequences.

In [1], Alves studied a quasilinear equation given by

$$-\Delta u + V(x)u - k\Delta(u^2)u = K(x)f(u), \qquad x \in \mathbb{R}^N,$$

where $N \geqslant 1$, $k \in R$, $V : \mathbb{R}^N \to \mathbb{R}$ is the potential, and $f : \mathbb{R} \to \mathbb{R}$ and $K : \mathbb{R}^N \to \mathbb{R}$ are continuous. The variational methods were used to establish a Berestycki–Lions type result. For further results about the elliptic equations with zero mass, we may refer to [4,7,8,19,24].

Inspired by [1,5,20], we would like to consider Schrödinger equations in $\mathbb{R}^N$ with magnetic field and zero mass.

Due to the appearance of the magnetic field, the problem cannot be changed into a pure real-valued problem, hence we should deal with a complex-valued directly, which causes more new difficulties in employing the methods and some estimates. Thus there are a few results for the Schrödinger equations with magnetic field than ones for that without the magnetic field. In [18], Ji and Yin showed the existence of nontrivial solutions for the following Schrödinger equation

$$-\Delta_A u + V(x)u = f(|u|^2)u, \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 3$, f has subcritical growth, and the potential V is nonnegative. The solution is obtained by the variational method combined with penalization technique of del Pino and Felmer [17] and Moser iteration.

In [15], Chabrowski and Szulkin discussed the semilinear Schrödinger equation

$$-\Delta_A u + V(x)u = Q(x)|u|^{2^*-2}u, \qquad u \in H^1_{A,V^+}(\mathbb{R}^N),$$

where V changes sign. The authors considered the problem by a min-max type argument based on a topological linking. For the more results involving the magnetic Schrödinger equations, we see [2,3,9,11,12,16,25] and the references therein.

In this paper, we consider problem (1.1) with the different function K. First of all, we assume the potential A verifying

(A) $A \in L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$.

In the first case, we propose the following assumptions for function K:

(K1) there exist $k_0 > 0$ such that

$$K(x) \geqslant k_0, \quad \text{for } \forall x \in \mathbb{R}^N,$$

(K2) there exist a positive continuous periodic function $K_p : \mathbb{R}^N \to \mathbb{R}$

$$K_p(x+y) = K_p(x), \qquad \forall x \in \mathbb{R}^N \text{ and } \forall y \in \mathbb{Z}^N,$$

such that

$$|K(x) - K_p(x)| \to 0 \quad \text{as } |x| \to +\infty.$$

(K3) K_P is defined in (K2) such that

$$K(x) \geqslant K_p(x), \qquad \forall x \in \mathbb{R}^N.$$

In addition, we assume that function f satisfies:

($f1$) there holds

$$\lim_{t\to 0^+} \frac{f(t)}{t^{\frac{2^*-2}{2}}} = \lim_{t\to+\infty} \frac{f(t)}{t^{\frac{2^*-2}{2}}} = 0,$$

where $2^* = \frac{2N}{N-2}$ and $N \geqslant 3$.

($f2$) function F is defined by $F(t) = \int_0^t f(s)ds$, and

$$\frac{F(t)}{t} \to \infty \quad \text{as } t \to +\infty,$$

($f3$) function $H(t) = tf(t) - F(t)$ is increasing in t and $H(0) = 0$.

Now we are in a position to state the first result.

Theorem 1.1. *Assume that (A), (K1)–(K3) and ($f1$)–($f3$) hold. Then, problem* (1.1) *has a nontrivial solution.*

In the second case, we involve that K is positive almost everywhere:

(K4) the Lebesgue measure of $\{x \in \mathbb{R}^N : K(x) \leqslant 0\}$ is zero.

Then, we state the second result as follows.

Theorem 1.2. *Assume that $K \in L^\infty(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, for some $r \geqslant 1$, satisfies (K4), and (A), ($f1$)–($f3$) hold. Then, problem* (1.1) *has a ground state solution.*

Remark 1.3. In fact, we consider the second case under a weaker condition than $K \in L^r(\mathbb{R}^N)$. We only require to suppose that for all $R > 0$ and any sequence of Borel sets $\{E_n\}$ of $\mathbb{R}^N$ such that $|E_n| \leqslant R$, for every n, we have

$$\lim_{R\to+\infty} \int_{E_n \cap B_R^c(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N}. \tag{1.3}$$

The paper is organized as follows. In the next section, we state the functional setting and give some preliminary lemmas. In Section 3, when K verifies the periodic condition, we study problem (1.1) and establish the existence of a ground state solution. In Section 4, we give the existence of a nontrivial solution for asymptotically periodic problem, proving Theorem 1.1. In the last section we consider problem (1.1) with condition (K4) and we prove Theorem 1.2.

2 Preliminaries

In this section, we outline the variational framework for problem (1.1) and give some preliminary lemmas. We write

$$\Delta_A u := (\nabla + iA)^2 u$$

and

$$\nabla_A u := (\nabla + iA)u.$$

Let $N \geqslant 3$ and $2^* = 2N/(N-2)$. We denote $D_A^{1,2}(\mathbb{R}^N)$ the Hilbert space with the scalar product

$$\langle u, v\rangle_A = \mathrm{Re}\int_{\mathbb{R}^N} (\nabla u + iA(x)u)\overline{(\nabla v + iA(x)v)}dx,$$

and the norm induced by the product $\langle \cdot, \cdot \rangle_A$ is

$$\begin{aligned}\|u\|_A &= \Big(\int_{\mathbb{R}^N} |\nabla_A u|^2 dx\Big)^{\frac{1}{2}}\\ &= \Big(\int_{\mathbb{R}^N} |\nabla u + iA(x)u|^2 dx\Big)^{\frac{1}{2}}\\ &= \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |A(x)|^2|u|^2)dx - 2\,\mathrm{Re}\int_{\mathbb{R}^N} iA(x)u\nabla u dx\Big)^{\frac{1}{2}},\end{aligned}$$

and $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $D_A^{1,2}(\mathbb{R}^N)$ with respect to the norm $\|u\|_A$. It is easy to know that

$$D_A^{1,2}(\mathbb{R}^N) := \Big\{u \in L^{2^*}(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C})\Big\}.$$

Furthermore, the following diamagnetic inequality (see [21, Theorem 7.21]) will be used frequently:

$$\big|\nabla_A u(x)\big| \geqslant \big|\nabla|u(x)|\big|, \quad \text{for } \forall u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}), \tag{2.1}$$

and it implies that if $u(x) \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$, the fact that $|u(x)| \in D^{1,2}(\mathbb{R}^N, \mathbb{R})$ will holds. Therefore, by Sobolev embedding $\int_{\mathbb{R}^N} \big|\nabla|u|\big|^2 dx \geqslant S\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{\frac{2}{2^*}}$, the embedding $D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{2^*}(\mathbb{R}^N, \mathbb{C})$ is continuous for $N \geqslant 3$.

3 A periodic problem

In the section, we will discuss the existence of a ground state solution for the following equation

$$\begin{cases} -\Delta_A u = K_p(x) f(|u|^2)u, & \text{in } \mathbb{R}^N,\\ u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}), \end{cases} \tag{3.1}$$

where $K_p : \mathbb{R}^N \to \mathbb{R}$ is a continuous function verifying the following hypotheses

(K5) for all $x \in \mathbb{R}^N$ and $y \in \mathbb{Z}^N$,

$$K_p(x+y) = K_p(x),$$

(K6) there is a positive constant $k_1 \geqslant 0$ such that

$$K_p(x) \geqslant k_1, \qquad \forall x \in \mathbb{R}^N.$$

In this section, the main result is the following.

Theorem 3.1. *Assume that (A), (K5)–(K6) and (f1)–(f3) hold. Then, problem (3.1) has a nontrivial solution.*

We denote by $I : D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ the energy functional for the problem (3.1), which is defined by

$$I(u) = \frac{1}{2}\|u\|_A^2 - \frac{1}{2}\int_{\mathbb{R}^N} K_p(x)F(|u|^2)dx, \tag{3.2}$$

with derivative, for $\forall u, v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$,

$$I'(u)v = \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u\overline{\nabla_A v}dx - \mathrm{Re}\int_{\mathbb{R}^N} K_p(x)f(|u|^2)u\bar{v}dx. \tag{3.3}$$

The weak solution for (3.1) are the critical points of I. furthermore, we can use (f1)–(f3) to check that functional I satisfies the geometry of the mountain pass. There is a sequence $(u_n) \subset D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that

$$I(u_n) \to c \tag{3.4}$$

and

$$\Big(1 + \|u_n\|_A\Big)\|I'(u_n)\| \to 0, \tag{3.5}$$

where c is the mountain pass level given by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\big(\gamma(t)\big)$$

with

$$\Gamma = \Big\{\gamma \in C\big([0,1], D_A^{1,2}(\mathbb{R}^N, \mathbb{C})\big) : \gamma(0) = 0 \text{ and } I\big(\gamma(1)\big) \leqslant 0\Big\}.$$

This sequence is called as Cerami sequence for I at level c, see [14].

Notice that from (f3) one obtains $H(s) \geqslant 0$ for every $s \in \mathbb{R}$. Then, we have the next estimates: by (f1), for $\forall\varepsilon > 0$, there exist a $\tau = \tau(\varepsilon)$ and $c_\varepsilon > 0$ such that

$$\big|s^2 f(s^2)\big| \leqslant \varepsilon|s|^{2^*} + c_\varepsilon|s|^p\chi_{\{|s|\geqslant\tau\}}(s) \tag{3.6}$$

and, by (f3),

$$\big|F(s^2)\big| \leqslant \varepsilon|s|^{2^*} + c_\varepsilon|s|^p\chi_{\{|s|\geqslant\tau\}}(s) \tag{3.7}$$

where χ is the characteristic function to the set $T = \{t \in \mathbb{R}^N : |t| \geqslant \tau\}$.

In the proof of Theorem 3.1, we announce a lemma which resembles a classical result in [22].

Lemma 3.2. *Let (u_n) be a bounded sequence in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. Then either*

(i) there are $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\displaystyle\int_{B_R(y_n)} |u_n|^2 \geqslant \eta$, for all n,

or

(ii) $\displaystyle\int_{\mathbb{R}^N} |\hat{u}_n|^q \to 0,$ where $\hat{u}_n = u_n\chi_{\{|s|\geqslant\tau\}}$, $\forall q \in (2, 2^*)$ and $\tau > 0$.

Proof. If (i) does not happen, going if necessary to a subsequence, we have

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}}\int_{B_R(y)} |u_n|^2 = 0.$$

Let $\psi : \mathbb{C} \to \mathbb{R}$ be a smooth function such that

$$0 \leqslant \psi(s) \leqslant 1, \quad \psi(s) = 0 \quad \text{for } |s| < \frac{\tau}{2} \quad \text{and} \quad \psi(s) = 1 \quad \text{for } |s| \geqslant \tau,$$

it is easy to check that the sequence $\tilde{u}_n = \psi(u_n)u_n$ belongs to $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ and satisfies

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |\tilde{u}_n|^2 = 0.$$

Hence, by [22],

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\tilde{u}_n|^p = 0, \qquad \forall q \in (2, 2^*),$$

from where it follows that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\hat{u}_n|^p = 0, \qquad \forall q \in (2, 2^*) \text{ and } \tau > 0,$$

finishing the proof. □

The next lemma is used to prove that the Cerami sequence is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.

Lemma 3.3. *There is a positive constant $M > 0$ such that $I(tu_n) \leqslant M$ for every $t \in [0, 1]$ and $n \in \mathbb{N}$.*

Proof. Let $t_n \in [0, 1]$ be such that $I(t_n u_n) = \max_{t\geqslant 0} I(tu_n)$. If either $t_n = 0$ or $t_n = 1$, we are done. Thereby, we can assume that $t_n \in (0, 1)$, and so $I'(t_n u_n)t_n u_n = 0$. From this

$$2I(t_n u_n) = 2I(t_n u_n) - I'(t_n u_n)t_n u_n = \int_{\mathbb{R}^N} K_p(x) H(|t_n u_n|^2).$$

Once that K_p is positive, it follows that $(f3)$

$$2I(t_n u_n) \leqslant \int_{\mathbb{R}^N} K_p(x) H(|u_n|^2) = 2I(u_n) - I'(u_n)u_n = 2I(u_n) + o_n(1).$$

Since $(I(u_n))$ converges to c, so $I(tu_n)$ is bounded. □

Lemma 3.4. *The sequence (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.*

Proof. Suppose by contradiction that $\|u\|_A \to \infty$ and set $w_n = \frac{u_n}{\|u_n\|_A}$. Since $\|w_n\|_A = 1$, there exists $w \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that $w_n \rightharpoonup w$ in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. Next, we will show that $w = 0$. First of all, notice that

$$o_n(1) + 1 = \int_{\mathbb{R}^N} \frac{K_p(x) F(|u_n|^2)}{\|u_n\|_A^2} = \int_{\mathbb{R}^N} \frac{K_p(x) F(|u_n|^2)}{|u_n|^2} |w_n|^2.$$

By $(f2)$, for each $M > 0$, there is $\xi > 0$ such that

$$\frac{F(s^2)}{s^2} \geqslant M, \quad \text{for } |s| \geqslant \xi,$$

hence

$$o_n(1) + 1 \geqslant \int_{\Omega\cap\{|u_n|\geqslant\xi\}} \frac{K_p(x) F(|u_n|^2)}{|u_n|^2} |w_n|^2 \geqslant M k_1 \int_{\Omega\cap\{|u_n|\geqslant\xi\}} |w_n|^2,$$

where $\Omega = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. By Fatou's Lemma

$$1 \geqslant M k_1 \int_{\Omega} |w|^2 dx.$$

Therefore $|\Omega| = 0$, showing that $w = 0$.

Notice that for each $C > 0$, one has $\frac{C}{\|u_n\|_A} \in [0,1]$ for n sufficiently large. Thus

$$I(t_n u_n) \geqslant I\Big(\frac{C}{\|u\|_A} u_n\Big) = I(Cw_n) = \frac{C^2}{2} - \frac{1}{2}\int_{\mathbb{R}^N} K_p(x) F\big(C^2|w_n|^2\big).$$

We claim that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K_p(x) F\big(C^2|w_n|^2\big) = 0. \tag{3.8}$$

We postpone for minutes the proof of (3.8). But if it were true, we would get

$$\lim_{n\to+\infty} I(t_n u_n) \geqslant \frac{C^2}{2}, \quad \text{for every } C > 0,$$

which is a contradiction with Lemma 3.3, since $(I(t_n u_n)) \leqslant M$.

We prove (3.8) by using Lemma 3.2, which gives two alternatives: either

$$\int_{B_R(y_n)} |w_n|^2 \geqslant \eta \quad \text{for some } \eta > 0 \text{ and } (y_n) \in \mathbb{Z}^N,$$

or

$$\int_{\mathbb{R}^N} |\hat{w}_n|^p dx \to 0, \quad \text{where } \hat{w}_n = w_n \chi_{\{|u_n| \geqslant \tau\}},\ p \in (2, 2^*) \text{ and } \tau > 0.$$

By showing the boundedness of (u_n), we will prove that the first alternative does not hold. If the first alternative occurs, we define $\tilde{u}_n = u_n(x + y_n)$ and $\tilde{w}_n = \frac{\tilde{u}_n}{\|u_n\|_A}$. These two sequences satisfy

$$I(\tilde{u}_n) \to c, \qquad \Big(1 + \|\tilde{u}_n\|_A\Big)\|I'(\tilde{u}_n)\| \to 0 \quad \text{and} \quad \tilde{w}_n \rightharpoonup \tilde{w} \neq 0,$$

which is a contraction compared to what we have written in the beginning of this proof. Hence, the second alternative holds and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\hat{w}_n|^p dx = 0.$$

Then

$$\big|K_p(x) F\big(C^2|w_n|^2\big)\big| \leqslant \|K_p\|_\infty \big|F\big(C^2|w_n|^2\big)\big| \leqslant \|K_p\|_\infty \Big[\varepsilon C^{2^*}|w_n|^{2^*} + c_\varepsilon C^p |w_n|^p \chi_{\{C|w_n| \geqslant \delta\}}\Big],$$

from where it follows

$$\big|K_p(x) F\big(C^2|w_n|^2\big)\big| \leqslant \|K_p\|_\infty [\varepsilon C^{2^*}|w_n|^{2^*} + c_\varepsilon C^p |w_n|^p].$$

Consequently

$$\int_{\mathbb{R}^N} \big|K_p(x) F\big(C^2|w_n|^2\big)\big| dx \leqslant \|K_p\|_\infty \Big[\varepsilon C^{2^*} \int_{\mathbb{R}^N} |w_n|^{2^*} dx + c_\varepsilon C^p \int_{\mathbb{R}^N} |w_n|^p dx\Big],$$

showing that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \big|K_p(x) F\big(C^2|w_n|^2\big)\big| dx = 0,$$

and the proof is finished. $\square$

Proof of Theorem 3.1. Since (u_n) is bounded, by applying Lemma 3.2, we have two alternatives, either

$$(i) \qquad \text{there are } R,\eta > 0 \text{ and } (y_n) \subset \mathbb{R}^N \text{ such that } \int_{B_R(y_n)} |u_n|^2 \geqslant \eta, \text{ for all } n,$$

or

$$(ii) \qquad \int_{\mathbb{R}^N} |\hat{u}_n|^q \to 0, \quad \text{where } \hat{u}_n = u_n \chi_{\{|s|\geqslant\tau\}},\ q \in (2, 2^*) \text{ and } \tau > 0.$$

Notice that (ii) does not occur. Otherwise, the inequality

$$\int_{\mathbb{R}^N} \left|K_p(x) f(|u_n|^2)|u_n|^2\right| \leqslant \|K_p\|_\infty \left[\varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*} + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^p\right]$$

leads to

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \left|K_p(x) f(|u_n|^2)|u_n|^2\right| = 0,$$

and so

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K_p(x) f(|u_n|^2)|u_n|^2 = 0.$$

The fact that $I'(u_n)u_n = o_n(1)$ imply that $\|u_n\|_A \to 0$, constituting a contradiction. Since alternative (i) is true and K_p is periodic, the sequence $\tilde{u}_n(x) = u_n(x + y_n)$ is a Cerami sequence for I at level c, namely,

$$I(\tilde{u}_n) \to c, \qquad \left(1 + \|\tilde{u}_n\|_A\right)\|I'(u_n)\| \to 0 \quad \text{and} \quad \tilde{u}_n \rightharpoonup \tilde{u} \text{ in } D_A^{1,2}(\mathbb{R}^N, \mathbb{C}).$$

A direct computation indicates that $I'(\tilde{u}) = 0$, and $\tilde{u}$ is a nontrivial weak solution for problem (3.1). Then, we will prove that $\tilde{u}$ is a ground state solution for (3.1).we will check that $I(\tilde{u})$ accords with the mountain pass level. By Fatou's Lemma,

$$2c = \liminf_{n\to+\infty} 2I(\tilde{u}_n) = \liminf_{n\to+\infty}\left(2I(\tilde{u}_n) - I'(\tilde{u}_n)\tilde{u}_n\right) = \liminf_{n\to+\infty}\int_{\mathbb{R}^N} K_p(x)H(|\tilde{u}_n|^2) \geqslant \int_{\mathbb{R}^N} K_p(x)H(|\tilde{u}|^2).$$

Since

$$2I(\tilde{u}) = 2I(\tilde{u}) - I'(\tilde{u})\tilde{u} = \int_{\mathbb{R}^N} K_p(x)H(|\tilde{u}|^2)dx,$$

we can conclude that $I(\tilde{u}) \leqslant c$. But then, the condition $(f3)$ leads to

$$c = \inf\left\{I(u) : u \in D_A^{1,2}(\mathbb{R}^N)\backslash\{0\} \text{ and } I'(u)u = 0\right\}.$$

It follows that $I'(\tilde{u}) \geqslant c$, and so $I'(\tilde{u}) = c$. □

4 The proof of Theorem 1.1

In the section, we will discuss the existence of a nontrivial solution for problem (1.1), thus showing Theorem 1.1. Therefore, we need to prove Lemmas 4.1 and 4.2 below. Hence, we will presume that the condition (A), (K1)–(K3) and $(f1)$–$(f3)$ hold.

We recall that $u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ is a weak solution of problem (1.1), if

$$\operatorname{Re}\int_{\mathbb{R}^N} \nabla_A u \overline{\nabla_A v} dx = \operatorname{Re}\int_{\mathbb{R}^N} K(x)F(|u|^2)u\bar{v}dx,$$

for all $v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.

The Energy functional associated to (1.1) is

$$J(u) = \frac{1}{2}\|u\|_A^2 - \frac{1}{2}\int_{\mathbb{R}^N} K(x)F(|u|^2)dx, \qquad \forall u \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}) \tag{4.1}$$

with derivative

$$J'(u)v = \mathrm{Re}\int_{\mathbb{R}^N} \nabla_A u \overline{\nabla_A v} dx - \mathrm{Re}\int_{\mathbb{R}^N} K(x)f(|u|^2)u\bar{v}dx, \qquad \forall u, v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C}). \tag{4.2}$$

As in the proof of the periodic case, one observes that J satisfying the geometry of the mountain pass. Therefore, there is a sequence $(v_n) \subset D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ verifying

$$J(v_n) \to d \quad \text{and} \quad \Big(1 + \|v_n\|_A\Big)\|J'(v_n)\| \to 0, \tag{4.3}$$

where d denotes the mountain pass level correlative of J.

Since $I(u) = c$, by property (K3), one obtains $d \leqslant c$. With loss of generality, we can assume that $K \not\equiv K_p$, consequently

$$d \leqslant \max_{t \geqslant 0} J(tu) = J(t_0 u) < I(t_0 u) \leqslant I(u) = c. \tag{4.4}$$

Lemma 4.1. *The sequence (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.*

Proof. Let $t_n \in [0,1]$ be such that $J(t_n v_n) = \max_{t \geqslant 0} J(t v_n)$. If either $t_n = 0$ or $t_n = 1$, we are done. Thereby, we can assume $t_n \in (0,1)$, and so $J'(t_n v_n)t_n v_n = 0$. From this

$$2J(t_n v_n) = 2J(t_n v_n) - J'(t_n v_n)t_n v_n = \int_{\mathbb{R}^N} K(x)H\big(t_n^2|v_n|^2\big).$$

Since K is a nonnegative function, from $(f3)$,

$$2J(t_n v_n) \leqslant \int_{\mathbb{R}^N} K(x)H\big(|v_n|^2\big) = 2J(v_n) - J'(v_n)v_n = 2J(v_n) + o_n(1).$$

Since $(J(v_n))$ is convergent, so it is bounded.

Suppose by contradiction that $\|v_n\|_A \to \infty$. Proving as in Lemma 3.4, the sequence $w_n = \frac{v_n}{\|v_n\|_A}$ weakly converges to 0 in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. Since $\|w_n\|_A = 1$, by applying Lemma 3.2, we have two alternatives, either

(i) there are $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\displaystyle\int_{B_R(y_n)} |w_n|^2 \geqslant \eta$, for all n,

or

(ii) $\displaystyle\int_{\mathbb{R}^N} |\hat{w}_n|^q \to 0$, where $\hat{w}_n = w_n \chi_{\{|s| \geqslant \tau\}}$, $\forall q \in (2, 2^*)$ and $\tau > 0$.

If that (i) occurred, we could define the functions $\tilde{v}_n(x) = v_n(x + y_n)$ and $\tilde{w}_n(x) = \frac{\tilde{v}_n(x)}{\|(v)_n\|_A}$. These two sequences satisfy

$$J(\tilde{v}_n) \to d, \qquad \Big(1 + \|\tilde{v}_n\|_A\Big)\|J'(\tilde{v}_n)\| \to 0 \quad \text{and} \quad \tilde{w}_n \rightharpoonup \tilde{w} \neq 0,$$

which contradicts $w_n \rightharpoonup 0$.

Suppose that (ii) is true. As in the proof of Lemma 3.4

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} K(x)F\big(C^2|w_n|^2\big)=0 \tag{4.5}$$

for each $C>0$, and one has $\frac{C}{\|v_n\|_A}\in[0,1]$ for n sufficiently large. There is a constant $M>0$ such that $J(tv_n)\leqslant M$ for every $t\in[0,1]$ and $n\in\mathbb{N}$. Thus

$$J(t_nv_n)\geqslant J\Big(\frac{C}{\|v_n\|_A}v_n\Big)=J(Cw_n)=\frac{C^2}{2}-\frac{1}{2}\int_{\mathbb{R}^N}K(x)F\big(C^2|w_n|^2\big).$$

By (4.5), one would get

$$\lim_{n\to+\infty}J(t_nv_n)\geqslant\frac{C^2}{2},\quad\text{for every } C>0,$$

which constitutes a contradiction, since $\big(J(t_nv_n)\big)$ is bounded. Consequently, the sequence (v_n) is bounded. □

From the preceding lemma, since the Hilbert space $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ is reflexive, there exists $v\in D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$ and a subsequence of (v_n), still denoted by (v_n), such that $v_n\rightharpoonup v$ in $D_A^{1,2}(\mathbb{R}^N,\mathbb{C})$.

Lemma 4.2. *The weak limit v of (v_n) is nontrivial.*

Proof. Suppose by contradiction that $v\equiv 0$. Since

$$\int_{B_R}|K(x)-K_p(x)|\big|F(|v_n|^2)\big|dx\leqslant\varepsilon\int_{B_R}|K(x)-K_p(x)||v_n|^{2^*}dx+\int_{B_R}|K(x)-K_p(x)||v_n|^p dx,$$

as consequence of $v\equiv 0$, it follows that

$$\int_{B_R}|K(x)-K_p(x)|\big|F(|v_n|^2)\big|dx\to 0\quad\text{as } n\to+\infty. \tag{4.6}$$

On the other hand, from (K2), given $\epsilon>0$ there exists $R=R(\epsilon)$ such that

$$\big|K(x)-K_p(x)\big|<\epsilon,\quad\text{for all } |x|>R.$$

Thus

$$\int_{B_R^c}|K(x)-K_p(x)|\big|F(|v_n|^2)\big|dx\leqslant\epsilon M \tag{4.7}$$

where

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}\big|F(|v_n|^2)\big|dx=M.$$

From (4.6) and (4.7)

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|K(x)-K_p(x)|\big|F(|v_n|^2)\big|dx=0, \tag{4.8}$$

and

$$|J(v_n)-I(v_n)|\to 0\quad\text{as } n\to+\infty.$$

A similar argument shows that

$$|J'(v_n)v_n-I'(v_n)v_n|\to 0\quad\text{as } n\to+\infty.$$

Consequently,

$$I(v_n) = d + o_n(1) \quad \text{and} \quad I'(v_n)v_n = o_n(1). \tag{4.9}$$

Let s_n be positive number verifying

$$I'(s_n v_n)v_n = 0. \tag{4.10}$$

We claim that (s_n) converges to 1 as $n \to +\infty$. We begin proving that

$$\limsup_{n\to+\infty} s_n \leqslant 1. \tag{4.11}$$

Suppose by contradiction that, going if necessary to a subsequence, $s_n \geqslant 1 + \delta$ for all $n \in \mathbb{N}$, for some $\delta > 0$. From (4.9),

$$\|v_n\|_A^2 = \int_{\mathbb{R}^N} K_p(x) f(|v_n|^2)|v_n|^2 dx + o_n(1).$$

On the other hand, from (4.10),

$$s_n\|v_n\|_A^2 = \int_{\mathbb{R}^N} K_p(x) f(s_n^2|v_n|^2) s_n|v_n|^2 dx.$$

Consequently

$$\int_{\mathbb{R}^N} K_p(x) \left[f\left(s_n^2|v_n|^2\right) - f\left(|v_n|^2\right)\right] |v_n|^2 dx = o_n(1),$$

and from $(f3)$ combined with (K1)–(K3),

$$\int_{\mathbb{R}^N} \left[f\left(s_n^2|v_n|^2\right) - f\left(|v_n|^2\right)\right] |v_n|^2 dx = o_n(1). \tag{4.12}$$

Since (v_n) is bounded, by Lemma 3.2 again, we have two alternatives, either

(i) there are $R, \eta > 0$ and $(y_n) \subset \mathbb{R}^N$ such that $\displaystyle\int_{B_R(y_n)} |v_n|^2 \geqslant \eta$, for all n,

or

(ii) $\displaystyle\int_{\mathbb{R}^N} |\hat{v}_n|^q \to 0$, where $\hat{v}_n = v_n \chi_{\{|s| \geqslant \tau\}}$, $\forall q \in (2, 2*)$ and $\tau > 0$.

In case (ii), we derive

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} f\left(|v_n|^2\right)|v_n|^2 dx = 0,$$

which implies $v_n \to 0$ in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ that is impossible.

Let (y_n) be given by (i), and define $\tilde{v}_n(x) = v_n(x + y_n)$. Since

$$\int_{B_R(0)} |\tilde{v}_n|^2 dx \geqslant \eta > 0,$$

there exists $\tilde{v} \neq 0$ in n $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ such that (v_n) is weakly convergent to $\tilde{v}$ in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. From (4.12) and $(f3)$, Fatou's Lemma yields,

$$0 < \int_{\mathbb{R}^N} \left[f\left((1+\delta)^2|\tilde{v}_n|^2\right) - f\left(|\tilde{v}_n|^2\right)\right] |\tilde{v}_n|^2 dx = 0,$$

which is impossible. Hence

$$\limsup_{n\to+\infty} s_n \leqslant 1.$$

From this, (s_n) is bounded. Without loss of generality, we can assume that

$$\lim_{n\to+\infty} s_n = s_0 \leqslant 1.$$

If $s_0 < 1$, we have that $s_n < 1$ for n large enough. Hence, by Fatou's Lemma

$$0 < \int_{\mathbb{R}^N} \left[f\left(|\tilde{v}_n|^2\right) - f\left(s_0^2|\tilde{v}_n|^2\right) \right] |\tilde{v}_n|^2 dx = 0, \quad \text{when } s_0 > 0,$$

and

$$0 < \int_{\mathbb{R}^N} f\left(|\tilde{v}_n|^2\right) |\tilde{v}_n|^2 dx = 0, \quad \text{when } s_0 = 0,$$

which are impossible. Therefore,

$$\lim_{n\to+\infty} s_n = 1. \tag{4.13}$$

As a consequence of (4.13),

$$\int_{\mathbb{R}^N} K_p(x) F\left(s_n^2|v_n|^2\right) dx - \int_{\mathbb{R}^N} K_p(x) F\left(|v_n|^2\right) dx = o_n(1)$$

and

$$\left(s_n^2 - 1\right) \|v_n\|_A^2 = o_n(1),$$

leading to

$$I(s_n v_n) - I(v_n) = o_n(1).$$

Then, by(4.9)

$$c \leqslant I(s_n v_n) = I(v_n) + o_n(1) = d + o_n(1).$$

Taking $n \to +\infty$, we find $c \leqslant d$, which obtain a contradiction, because, by (4.4), $d < c$. This contradiction comes from the assumption that $v \equiv 0$. □

5 The proof of Theorem 1.2

In this section, we mean to prove Theorem 1.2. As the proof in the preceding section, we can prove that the functional I satisfies the geometry of the mountain pass and there is a Cerami sequence $(u_n) \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$ satisfying (3.4) and (3.5). Finally, we have proved Lemma 3.3. In order to check that (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$, we should show that the (3.8) holds and proceed as in the proof of Lemma 3.4.

Let Ω, ξ, w, M be defined as in the proof of Lemma 3.4. Notice that $\left|\Omega\right| = 0$, since

$$o_n(1) + 1 \geqslant \int_{\Omega \cap \{|u_n| \geqslant \xi\}} \frac{K(x) F(|u_n|^2)}{|u_n|^2} |w_n|^2$$

implies that

$$1 \geqslant M \int_{\Omega} K(x) |w|^2,$$

and from (K4), we have $w = 0$.

Let us prove the limit (3.8). From $(f1)$, for each $\varepsilon > 0$, we have $\delta > 0$ and $C_\varepsilon > 0$ such that

$$\left|s^2 f(s^2)\right| \leqslant \varepsilon|s|^{2^*} + C_\varepsilon \chi_{\{|s|\geqslant\delta\}}, \quad \text{for all } s \in \mathbb{R}^N, \tag{5.1}$$

and

$$\left|F(s^2)\right| \leqslant \varepsilon|s|^{2^*} + C_\varepsilon \chi_{\{|s|\geqslant\delta\}}, \quad \text{for all } s \in \mathbb{R}^N. \tag{5.2}$$

By Sobolev embedding and (2.1), there exists $\hat{S} > 0$ such that

$$\int_{\mathbb{R}^N} |v|^{2^*} dx \leqslant \hat{S}\Big(\int_{\mathbb{R}^N} |\nabla_A v|^2 dx\Big)^{\frac{2^*}{2}},$$

for all $v \in D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. Observe that $\Delta_n = \{x \in \mathbb{R}^N : |Cw_n(x)| \geqslant \delta\}$ is such that

$$\int_{\Delta_n} |w_n|^{2^*} \leqslant \hat{S}.$$

This implies, besides (5.2), that

$$\int_{|x|\geqslant R} K(x)F\big(|Cw_n|^2\big)dx \leqslant \varepsilon C^{2^*}\|K\|_\infty \int_{B_R^c(0)} |w_n|^{2^*} dx + C_\varepsilon \int_{B_R^c(0)\cap\Delta_n} K(x)dx,$$

and from (1.3)

$$\lim_{R\to+\infty} \int_{|x|\geqslant R} K(x)F\big(|Cw_n|^2\big)dx \leqslant \varepsilon\hat{S}C^{2^*}\|K\|_\infty, \quad \text{uniformly in } n.$$

On the other hand, for any $R > 0$, from $(f1)$ and Strauss' compactness lemma (see [10])

$$\lim_{n\to+\infty} \int_{|x|\leqslant R} K(x)F\big(|Cw_n|^2\big)dx = 0,$$

which shows that (3.8) holds and (u_n) is bounded in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$.

To prove Theorem 1.2, it is important to show that (u_n) converges in $D_A^{1,2}(\mathbb{R}^N, \mathbb{C})$. In this way we can see that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} K(x)f(|u_n|^2)|u_n|^2 dx = \int_{\mathbb{R}^N} K(x)f(|u|^2)|u|^2 dx. \tag{5.3}$$

To verify (5.3), consider $E_n = \{x \in \mathbb{R}^N : |u_n(x)| \geqslant \delta\}$ which satisfies $\sup_{n\in\mathbb{N}} |E_n| < \infty$. From (5.1)

$$\int_{|x|\geqslant R} K(x)f(|u_n|^2)|u_n|^2 dx \leqslant \varepsilon\|K\|_\infty \int_{B_R^c(0)} |u_n|^{2^*} dx + C_\varepsilon \int_{B_R^c(0)\cap E_n} K(x)dx$$

and from (1.3)

$$\limsup_{R\to+\infty} \int_{|x|\geqslant R} K(x)f(|u_n|^2)|u_n|^2 dx \leqslant \varepsilon\hat{S}\|K\|_\infty, \quad \text{uniformly in } n.$$

Again, from $(f1)$ and Strauss' compactness lemma

$$\lim_{n\to+\infty} \int_{|x|\leqslant R} K(x)f(|u_n|^2)|u_n|^2 dx = \int_{|x|\leqslant R} K(x)f(|u|^2)|u|^2 dx,$$

for all $r > 0$ fixed, and it shows that (5.3) holds. Since $I'(u_n)u_n \to 0$, (5.3) implies that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |\nabla_A u_n|^2 dx = \int_{\mathbb{R}^N} K(x)f(|u|^2)|u|^2 dx = \int_{\mathbb{R}^N} |\nabla_A u|^2 dx$$

finishing the proof of Theorem 1.2.

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