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On a class of difference equations involving a linear map with two dimensional kernel

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Abstract. We establish necessary and sufficient conditions for the existence of periodic solutions to second-order nonlinear difference equations of the form $\Delta^2 x_i + \lambda x_i + \Delta f(x_i) = e_i$, $i \in \mathbb{N}$, and for a simpler equation with difference-free nonlinearity.

The linear part of the equation has two-dimensional kernel.

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1 Introduction

The problem of finding periodic solutions for discrete semilinear systems has been studied in recent years by many authors, with emphasis in a variety of features and with recourse to several techniques. Among the extensive literature on this kind of problems, let us mention a selection of papers (see also their references) which display also a variety of methods used: Lyapunov–Schmidt reduction, Brouwer fixed point theorem [1, 11, 12], minimax methods, critical point theory, Morse theory [3, 8, 10, 13, 15], upper and lower solutions [2, 4, 5]. See also [14] for the analysis of linear eigenvalue theory.

If one considers, in particular, second order scalar difference equations, it turns out that an interesting feature of periodic problems is that they provide resonance models that may involve a linear operator whose kernel has dimension one or two. Both settings have been considered in some of the above mentioned articles. An illustration of peculiarities of such problems can found in [11].

Our purpose in this paper is to study a problem where, on one hand, we have to deal with a two-dimensional kernel and, on the other hand, the nonlinear part involves first order differences. Our motivation goes back to the paper of A. C. Lazer [9], where the existence of 2π-periodic solutions to the resonant problem

$$u'' + u + \big(F(u)\big)' = e(t) \tag{1.1}$$

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is studied. Here e is continuous, 2π-periodic, and F is C^1. Necessary and sufficient conditions for existence are found, in terms of the size of the projection of e onto the kernel of the linear part: namely, if $a \sin t + b \cos t$ appears in the Fourier series of e, then the condition for existence is found to be

$$\pi \sqrt{a^2 + b^2} < 2\big(F(\infty) - F(-\infty)\big). \tag{1.2}$$

We propose to consider the difference equation whose structure is reminiscent of (1.1). Specifically, we want to give criteria for the existence of N-periodic solutions to the second-order nonlinear difference equation

$$\Delta^2 x_i + \lambda x_i + \Delta f(x_i) = e_i, \qquad i \in \mathbb{N}, \tag{1.3}$$

where, considering the jump $h = \frac{2\pi}{N}$, we define the difference operators as

$$\Delta^2 x_i = \frac{1}{h^2}(x_{i+1} - 2x_i + x_{i-1})$$

and

$$\Delta f(x_i) = \frac{1}{h}\big(f(x_i) - f(x_{i-1})\big).$$

In addition, $f : \mathbb{R} \to \mathbb{R}$ is a given function, $\lambda = \frac{N^2}{\pi^2} \sin^2 \frac{\pi}{N}$ is the smallest positive eigenvalue of $-\Delta^2$ with N-periodic conditions (which approaches 1 as N grows larger) and $e = (e_i)$ is a N-periodic vector.

Therefore, the underlying linear operator in our discrete system has in fact two-dimensional kernel; on the other hand the nonlinear term contains first order differences. However, because it appears as a by-product of the method, we deal also with the (simpler) version in which the nonlinearity is difference-free

$$\Delta^2 x_i + \lambda x_i + f(x_i) = e_i, \qquad i \in \mathbb{N}. \tag{1.4}$$

It is our purpose to relate the existence of periodic solutions to (1.3) – or (1.4) – to some relationship between f, e and the kernel of the linear operator $\Delta^2 + \lambda$ acting on N-periodic vectors.

We shall proceed by rephrasing the Poincaré–Miranda theorem in appropriate form, so that it can be used to recover results that correspond to those given by Lazer in [9]. Our necessary or sufficient conditions for existence are a little more complicated than those in [9] because the discretization does not allow a sharp statement; they are close to the conditions in [9] when N is large, but it will be seen that we need to introduce "correcting terms" in the corresponding inequalities.

Since N-periodic sequences can be identified with vectors in $\mathbb{R}^N$, we henceforth identify the elements of $\mathbb{R}^N$ with such sequences, that may be indexed in $\mathbb{Z}$. It will be convenient to consider the following norm and the associated inner product in N-dimensional space:

$$\|x\| = \sqrt{h \sum_{i=1}^{N} x_i^2}\,.$$

It is easy to see that the kernel of the operator $\Delta^2 + \lambda$ is 2-dimensional and is spanned by $\underline{s}$ and $\underline{c}$, with

$$s_j = \sin\left(\frac{2\pi j}{N}\right) \quad \text{and} \quad c_j = \cos\left(\frac{2\pi j}{N}\right).$$

With the previous definition in mind, we have that $\underline{s}$ and $\underline{c}$ are orthogonal and $\|\underline{s}\|^2 = \|\underline{c}\|^2 = \pi$.

Another useful observation is that the linear operator Δ^2 acting on periodic vectors is symmetric. That is, we can write it in matrix form as the $N \times N$ symmetric matrix

$$\frac{N^2}{4\pi^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2 \end{bmatrix}.$$

Hence, setting

$$\mathcal{A} = \Delta^2 + \lambda,$$

we have

$$\sum_{i=1}^{N}(\Delta^2 a_i + \lambda a_i)b_i = (\mathcal{A}a)\cdot b = a \cdot (\mathcal{A}b) = \sum_{i=1}^{N} a_i(\Delta^2 b_i + \lambda b_i).$$

From this, it also follows that the kernel and the image of the operator $\mathcal{A}$ are orthogonal ($\mathrm{Im}(\mathcal{A}) = \mathrm{Ker}(\mathcal{A})^{\perp}$) and any $x \in \mathbb{R}^N$ can be written uniquely as $x = \alpha\underline{s} + \beta\underline{c} + w$, for some $\alpha, \beta \in \mathbb{R}$ and $w \in M := \mathrm{Im}(\mathcal{A})$.

As already stated, we think of e and the solution x as N-periodic vectors, which are identified with elements of $\mathbb{R}^N$. We consider the orthogonal projection of e on $\mathrm{Ker}(\mathcal{A})$, denoted by

$$A\underline{s} + B\underline{c}$$

meaning that

$$A = \frac{h}{\pi}\sum_{i=1}^{N} e_i \sin\left(\frac{2\pi i}{N}\right), \qquad B = \frac{h}{\pi}\sum_{i=1}^{N} e_i \cos\left(\frac{2\pi i}{N}\right). \tag{1.5}$$

We also set

$$f(-\infty) = \lim_{t\to-\infty} f(t), \qquad f(\infty) = \lim_{t\to+\infty} f(t)$$

and

$$m = \sup_{t\in\mathbb{R}} |f(t)|. \tag{1.6}$$

Before stating the main results, further notation must be introduced. For $\theta \in \mathbb{R}$ consider the N-periodic vector $\sigma_j = \sigma_j(\theta) = \sin\big(\theta + \frac{2\pi j}{N}\big)$. Let $x^+ = \max\{x, 0\}$. We introduce the numbers α_N, β_N by

$$\alpha_N = \min_{\theta\in\mathbb{R}} h\sum_{j=1}^{N}\sigma_j^+, \qquad \beta_N = \max_{\theta\in\mathbb{R}} h\sum_{j=1}^{N}\sigma_j^+ \tag{1.7}$$

and we also set

$$\alpha'_N := 2\cos\frac{\pi}{N}\,\cos\frac{2\pi}{N}. \tag{1.8}$$

It is easily seen that the sequences α_N, β_N and α'_N have limit 2 as $N \to \infty$.

In order to simplify the statements and proofs, *we shall take N to be a multiple of* 4. This assumption will not appear in the statements.

Theorem 1.1. *Let $\{e_i\}_{i\in\mathbb{N}}$ be N-periodic and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(\infty)$ and $f(-\infty)$ are finite. Then with the notation of* (1.5), (1.6) *and* (1.8)*:*

(i) Suppose that $\forall x\in\mathbb{R}$, $f(-\infty)<f(x)<f(\infty)$. Then if the equation (1.3) *has a N-periodic solution, the condition*

$$\pi\sqrt{A^2+B^2}<2\big(f(\infty)-f(-\infty)\big)$$

is satisfied.

(ii) Assume that

$$\pi\sqrt{A^2+B^2}+4\,m\sin\frac{\pi}{N}<\alpha'_N\big(f(\infty)-f(-\infty)\big). \tag{1.9}$$

Then equation (1.3) *has a N-periodic solution.*

Theorem 1.2. *Let $\{e_i\}_{i\in\mathbb{N}}$ be N-periodic and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(\infty)$ and $f(-\infty)$ are finite. With the notation of* (1.5), (1.6) *and* (1.7)*:*

(i) Suppose that $\forall x\in\mathbb{R}$, $f(-\infty)<f(x)<f(\infty)$. Then if the equation (1.4) *has a N-periodic solution, the condition*

$$\pi\sqrt{A^2+B^2}<\beta_N\big(f(\infty)-f(-\infty)\big) \tag{1.10}$$

holds.

(ii) Assume that

$$\pi\sqrt{A^2+B^2}+8\,m\pi^2/N^2<\alpha_N\big(f(\infty)-f(-\infty)\big). \tag{1.11}$$

Then equation (1.4) *has a N-periodic solution.*

Remark 1.3. In the above conditions (1.9), (1.10), (1.11), we must use the approximations α_N, β_N, α'_N, rather than the constant 2 (the integral of $\sin^+$ over a period) that appears in [9]. Moreover, we add "correcting terms" that behave as $O(1/N)$ and $O(1/N^2)$, respectively, and are not needed when one deals with a differential equation. Our conditions make sense for large values of N.

2 Auxiliary results

We shall use the following elementary formula for "summing by parts".

Lemma 2.1. *Let a_i and b_i be two N-periodic vectors. Setting $\Delta a_i=a_i-a_{i-1}$ we have:*

$$\sum_{i=1}^{N}\Delta a_i b_i=-\sum_{i=1}^{N}a_i\,\Delta b_{i+1}.$$

Let us recall the Poincaré–Miranda's theorem, stated as follows.

Theorem 2.2. *Let $L_i>0$, $i=1,\dots,N$, $\Omega=\big\{x\in\mathbb{R}^N:\ |x_i|\le L_i,\ i=1,\dots,N\big\}$ and $f:\Omega\to\mathbb{R}^N$ be continuous satisfying:*

$$f_i\big(x_1,x_2,\dots,x_{i-1},-L_i,x_{i+1},\dots,x_N\big)\ge 0\quad \text{for } 1\le i\le N,$$
$$f_i\big(x_1,x_2,\dots,x_{i-1},+L_i,x_{i+1},\dots,x_N\big)\le 0\quad \text{for } 1\le i\le N.$$

Then, $f(x)=0$ has a solution in Ω.

We need slight variations of this statement, where the vector field is defined on a product of intervals with a ball. Although such versions may be related to the approach of [7], we include simple proofs for completeness.

In what follows we shall denote by γ the orthogonal projection of $\mathbb{R}^N = \mathbb{R}^{N-2} \times \mathbb{R}^2$ onto the second factor $\mathbb{R}^2$.

Proposition 2.3. *Let L_i $(i=1,\dots,N)$ and R be positive numbers. Let $\Omega = \{x \in \mathbb{R}^N : |x_i| \le L_i,$ $i = 1,\dots,N-2,\ x_{N-1}^2 + x_N^2 \le R^2\} = \prod_{i=1}^{N-2}[-L_i, L_i] \times \overline{B_R} \subseteq \mathbb{R}^{N-2} \times \mathbb{R}^2$ and $f : \Omega \to \mathbb{R}^N$ be a continuous function satisfying:*

$$f_i(x_1, x_2, \dots, x_{i-1}, -L_i, x_{i+1}, \dots, x_N) < 0 \quad \text{for } 1 \le i \le N-2,$$
$$f_i(x_1, x_2, \dots, x_{i-1}, +L_i, x_{i+1}, \dots, x_N) > 0 \quad \text{for } 1 \le i \le N-2$$

and

$$\forall x \in \prod_{i=1}^{N-2}[-L_i, L_i] \times \partial\overline{B_R}, \qquad f(x) \cdot \gamma x > 0.$$

Then there exists $x^* \in \Omega$ such that $f(x^*) = 0$.

Proof. We use a standard compactness argument to show that there exists $\varepsilon > 0$ such that the mapping $x \mapsto x - \varepsilon f(x)$ maps Ω into Ω. The conclusion follows from Brouwer's fixed point theorem. In fact, if the claim is not true, we find $\epsilon_n \downarrow 0$ and $x_n \in \Omega$ such that $x_n - \varepsilon_n f(x_n) \notin \Omega$. Then, considering subsequences if necessary, either there exists $i \in \{1, \dots, n-2\}$ such that, say

$$x_{ni} - \epsilon_n f_i(x_n) > L_i$$

or

$$\|\gamma x_n - \varepsilon_n \gamma f(x_n)\| > R^2.$$

We may suppose that $x_n \to x$. In the first case we obtain $x_i \ge L_i$, that is, $x_i = L_i$, and then, by the continuity of f and the assumption on f_i, the first inequality gives a contradiction for large n. In the second case, setting $M = \max_{z\in\Omega} \|f(z)\|$, we have

$$\|\gamma x_n\|^2 - 2\epsilon_n \gamma x_n \cdot \gamma f(x_n) + M^2 \epsilon_n^2 > R^2.$$

The previous argument then gives $\|\gamma x\| = R$ and, since by the assumptions $\lim_{n\to\infty} \gamma x_n \cdot \gamma f(x_n) > 0$, again a contradiction for large n is obtained. □

Proposition 2.3 is a very natural generalization of Poincaré–Miranda's theorem, as the dot product condition gives a reasonable notion of the vector field "to point outside" of the domain. Finally, we state a last version of the result, with a variation of the dot product condition.

Proposition 2.4. *Let Ω be as in the preceding proposition and $f : \Omega \to \mathbb{R}^N$ be a continuous function satisfying:*

$$f_i(x_1, x_2, \dots, x_{i-1}, -L_i, x_{i+1}, \dots, x_N) < 0 \quad \text{for } 1 \le i \le N-2,$$
$$f_i(x_1, x_2, \dots, x_{i-1}, +L_i, x_{i+1}, \dots, x_N) > 0 \quad \text{for } 1 \le i \le N-2$$

and

$$\forall x \in \left(\prod_{i=1}^{N-2}[-L_i, L_i]\right) \times \partial\overline{B_R}, \qquad f(x) \cdot \rho(\gamma x) > 0,$$

where ρ denotes a rotation of angle $\frac{\pi}{2}$ in the plane $\mathbb{R}^2$.

Then there exists $x^* \in \Omega$ such that $f(x^*) = 0$.

Proof. Define $g : \Omega \to \mathbb{R}^N$ by $g(x) = f\big(x - \gamma(x), \rho^{-1}(\gamma(x))\big)$. Then g satisfies the conditions of the previous proposition. The conclusion follows. □

Now let $Q, P : \mathbb{R}^N \to \mathbb{R}^N$ be the orthogonal projections onto $\mathrm{Ker}(\mathcal{A})$ and $M = \mathrm{Ker}(\mathcal{A})^{\perp}$, respectively. Let $K : M \to M$ be defined by

$$K = \left(\mathcal{A}_{|M}\right)^{-1}.$$

We now write problem (1.3) in operator form as

$$\mathcal{A}x + \mathcal{G}(x) = e$$

where $\mathcal{G} : \mathbb{R}^N \to \mathbb{R}^N$ is the nonlinear map whose i-th component is $\frac{1}{h}\big(f(x_i) - f(x_{i-1})\big)$.

Using the orthogonal decomposition $x = u + v$, with $u \in \mathrm{Ker}(\mathcal{A})$ and $v \in M$, we obtain

$$\mathcal{A}x + \mathcal{G}(x) = e \iff \mathcal{A}v + \mathcal{G}(u + v) = e$$

or equivalently

$$v - K\big(-P\mathcal{G}(u+v) + Pe\big) = 0, \qquad Q\mathcal{G}(u+v) - Qe = 0. \tag{2.1}$$

We can then define $V : M \times \mathrm{Ker}(\mathcal{A}) \to M \times \mathrm{Ker}(\mathcal{A})$ by:

$$V(v,u) = \Big(v - K\big(-P\mathcal{G}(u+v) + Pe\big),\, Q\mathcal{G}(u+v) - Qe\Big),$$

and conclude that:

Proposition 2.5. *The periodic problem* (1.3) *has a solution if and only if there is a solution to* $V(v,u) = 0$.

3 Proof of Theorem 1.2

We start with some simple remarks and notation. Recall the meaning of the expression

$$\sigma_i = \sigma_i(t) = \sin\left(t + \frac{2\pi i}{N}\right)$$

and set

$$S^+ = \big\{i : \sigma_i > 0,\ i = 1, \dots, N\big\}, \qquad S^- = \big\{i : \sigma_i < 0,\ i = 1, \dots, N\big\}.$$

Since N is even, there is at most an index $i* \in S^+$ such that $0 < \sigma_{i*} < \sin\frac{\pi}{N}$. In such case, there exists a (unique) $j* \in S^-$ with $|\sigma_{j*}| = \sigma_{i*} < \sin\frac{\pi}{N}$. In fact it is easy to see that, assuming without loss of generality that $-\frac{2\pi}{N} < t \le 0$, we have $i* = 1$ or $i* = \frac{N}{2}$. Let us then define

$$S^+* = S^+ \setminus i*, \qquad S^-* = S^+ \setminus j*\,.$$

Otherwise, if $\sigma_i \ge \sin\frac{\pi}{N}$ for all $i \in S^+$, put

$$S^+* = S^+, \qquad S^-* = S^-.$$

We are now ready to present the proof for the case of a difference-free nonlinearity. The abstract approach is very similar to the one described above, where we replace $\mathcal{G}$ with $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$ which is defined component-wise as $\mathcal{F}_i(x) = f(x_i)$. Hence we consider the operator problem

$$\mathcal{A}x + \mathcal{F}(x) = e.$$

As before, finding a periodic solution to (1.2) is equivalent to solving

$$W(v,u) := \Big(v - K\big(-P\mathcal{F}(u+v) + Pe\big),\, Q\mathcal{F}(u+v) - Qe\Big) = 0.$$

Proof. (i) Let x be a solution of (1.4) and consider the orthogonal splitting of e,

$$e = A\underline{s} + B\underline{c} + w,$$

where A, $B \in \mathbb{R}$ and $w \in M$. The inner product of equation (1.4) with $z = A\underline{s} + B\underline{c}$ yields

$$\mathcal{F}(x)\cdot z = e\cdot z = \|z\|^2 = \pi(A^2 + B^2).$$

On the other hand

$$\mathcal{F}(x)\cdot z = h\sum_{i=1}^{N} f(x_i)\, z_i$$

and there exists $\varphi \in \mathbb{R}$ such that $z_i = \sqrt{A^2+B^2}\sin\big(\varphi + \frac{2\pi i}{N}\big)$. Hence summing separately over the sets of indices where the z_i are positive and where the z_i are negative and using the definition of β_N and the assumption of (i) we obtain

$$\pi(A^2+B^2) < \beta_N\sqrt{A^2+B^2}\,\big(f(\infty) - f(-\infty)\big).$$

(ii) We have to prove that $W(v,u) = 0$ has a solution, using the analogue of Proposition 2.5. Suppose that (1.11) holds.

First, we want to show that there exists an $L > 0$ such that

($*$) If $v_i = L$, then $W_i(v,u) > 0$ (respectively if $v_i = -L$, then $W_i(v,u) < 0$), for $1 \le i \le N-2$. Here of course the v_i are coordinates with respect to some basis of M.

To this purpose it suffices to prove that $K\big(-P\mathcal{F}(u+v) + Pe\big)$ is bounded.

Since K is linear there is a constant C such that:

$$\|Kx\| \le C\|x\|, \qquad \forall x \in \mathbb{R}^N.$$

Since f is bounded, so is $\mathcal{F}$ and we have

$$\|-\mathcal{F}(u+v) + e\| \le C^* \quad \text{for some } C^* \in \mathbb{R}.$$

Since P is an orthogonal projection, it follows then that

$$\begin{aligned}\big\|K\big(-P\mathcal{F}(u+v)+e\big)\big\| &\le C\big\|P\big(-\mathcal{F}(u+v)+e\big)\big\|\\ &\le CC^*.\end{aligned}$$

Therefore we can pick up a positive number L with the property ($*$).

Now fix ε such that

$$\pi\sqrt{A^2+B^2} + 8\,m\pi^2/N^2 < \alpha_N\big(f(\infty) - f(-\infty) - 2\,\varepsilon\big).$$

Consider a ball in $\mathrm{Ker}(\mathcal{A})$ with radius R. Let u be on the boundary of the ball, with $u = \alpha \underline{s} + \beta \underline{c}$. There exists $t \in \mathbb{R}$ so that we can write

$$u = \sqrt{\alpha^2 + \beta^2}\, \sigma, \qquad \sigma_i = \sin\left(\frac{2\pi i}{N} + t\right).$$

In particular $R = \sqrt{\pi(\alpha^2+\beta^2)}$. Let $v \in M$ with $|v_i| \le L$. Then, with the notation introduced in the beginning of this section

$$\begin{aligned}
Q\big(\mathcal{F}(u+v) - e\big)\cdot u \;&=\; \mathcal{F}(u+v)\cdot u - e \cdot u \\
&\ge\; h \sum_{i=1}^{N} f(u_i+v_i)u_i \;-\; \pi\,\sqrt{A^2+B^2}\,\sqrt{\alpha^2+\beta^2} \\
&=\; h \sum_{i \in S^{+*}} f\left(\frac{R}{\sqrt{\pi}}\,\sigma_i + v_i\right)\frac{R}{\sqrt{\pi}}\,\sigma_i \;+\; hf\left(\frac{R}{\sqrt{\pi}}\,\sigma_{i*} + v_{i*}\right)\frac{R}{\sqrt{\pi}}\,\sigma_{i*} \\
&\quad +\, h \sum_{i \in S^{-*}} f\left(\frac{R}{\sqrt{\pi}}\,\sigma_i + v_i\right)\frac{R}{\sqrt{\pi}}\,\sigma_i \;+\; hf\left(\frac{R}{\sqrt{\pi}}\,\sigma_{j*} + v_{j*}\right)\frac{R}{\sqrt{\pi}}\,\sigma_{j*} \\
&\quad -\; \pi\,\sqrt{A^2+B^2}\,\sqrt{\alpha^2+\beta^2}
\end{aligned}$$

where the summands that contain $hf\big(\frac{R}{\sqrt{\pi}}\,\sigma_{i*} + v_{i*}\big)$ and $hf\big(\frac{R}{\sqrt{\pi}}\sigma_{j*} + v_{j*}\big)$ appear only if $i*$ and $j*$ exist.

Let R be so large that

$$\frac{R}{\sqrt{\pi}} \sin\frac{\pi}{N} - L > T$$

where T is such that

$$f(x) > f(+\infty) - \varepsilon \quad \forall x \ge T, \qquad f(x) < f(-\infty) + \varepsilon \quad \forall x \le -T.$$

Hence, using symmetry, in any case the above expression is greater than

$$\begin{aligned}
&\frac{R}{\sqrt{\pi}}\left(\big(f(+\infty)-\varepsilon\big)\,h \sum_{i\in S^{+*}} \sigma_i - \big(f(-\infty)+\varepsilon\big)\,h \sum_{i \in S^{-*}} |\sigma_i| - 2\,hm\frac{\pi}{N} - \pi\sqrt{A^2+B^2}\right) \ge \\
&\qquad \ge\; \frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - f(-\infty) - 2\,\varepsilon\big)\,h \sum_{i\in S^{+*}} \sigma_i - 4\,m\frac{\pi^2}{N^2} - \pi\sqrt{A^2+B^2}\right) \\
&\qquad \ge\; \frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - f(-\infty) - 2\,\varepsilon\big)\left(\alpha_N - h\frac{\pi}{N}\right) - 4\,m\frac{\pi^2}{N^2} - \pi\sqrt{A^2+B^2}\right) \\
&\qquad \ge\; \frac{R}{\sqrt{\pi}}\left(\big(f(+\infty) - f(-\infty) - 2\,\varepsilon\big)\,\alpha_N - 8\,m\frac{\pi^2}{N^2} - \pi\sqrt{A^2+B^2}\right) \;>\; 0.
\end{aligned}$$

By Proposition 2.3, it follows that there is a solution to $W(v,u) = 0$ and, consequently, a solution to the periodic problem (1.2). $\square$

4 Proof of the main result

First we list some elementary facts to be used in the sequel.

Lemma 4.1. *If $\sigma_i(t) > \sin\frac{\pi}{N}$, then $\sigma_{i+1}(t+\frac{\pi}{2}) < \sigma_i(t+\frac{\pi}{2})$. If $0 \le \sigma_k(t) \le \sin\frac{\pi}{N}$ then*

$$\left|\sigma_{k+1}\left(t+\frac{\pi}{2}\right) - \sigma_k\left(t+\frac{\pi}{2}\right)\right| \le 2\sin\frac{\pi}{N}.$$

Proof. It suffices to remark that $\sigma_{i+1}(t+\frac{\pi}{2}) - \sigma_i(t+\frac{\pi}{2}) = -2\sin\frac{\pi}{N}\sin(\frac{2\pi i}{N}+\frac{\pi}{N}+t)$. □

Lemma 4.2. $\displaystyle\sum_{i=1}^{N}\left(\sigma_{i+1}(t)-\sigma_i(t)\right)^+ \le 2.$

Lemma 4.3. $\displaystyle\sum_{i\in S^{+}*}\left(\sigma_{i+1}\left(t+\frac{\pi}{2}\right)-\sigma_i\left(t+\frac{\pi}{2}\right)\right)^- \ge 2\cos\frac{2\pi}{N}\cos\frac{\pi}{N}.$

Proof. Suppose first that $i*$ exists, and to fix ideas $i* = 1$. Then we may take $S^+* = \{2,\dots,N/2\}$ and $-\frac{2\pi}{N} < t < -\frac{\pi}{N}$ (so that in fact $0 < \frac{2\pi}{N}+t < \frac{\pi}{N}$). Then, writing $N = 4p$ and using the elementary formula for $\sin x - \sin y$,

$$\begin{aligned}
\sum_{i\in S^{+}*}\left(\sigma_{i+1}\left(t+\frac{\pi}{2}\right)-\sigma_i\left(t+\frac{\pi}{2}\right)\right)^- &= \sum_{i=2}^{N/2}\left[\sin\left(\frac{2\pi(i+1+p)}{N}+t\right)-\sin\left(\frac{2\pi(i+p)}{N}+t\right)\right]^- \\
&= \sin\left(\frac{2\pi(2+p)}{N}+t\right)-\sin\left(\frac{2\pi(\frac{N}{2}+p+1)}{N}+t\right) \\
&= \sin\left(\frac{4\pi+2\pi p}{N}+t\right)-\sin\left(\frac{N\pi+2\pi+2\pi p}{N}+t\right) \\
&= 2\cos\left(\frac{3\pi}{N}+t\right)\cos\frac{\pi}{N}.
\end{aligned}$$

Since $\frac{3\pi}{N}+t \in [\frac{\pi}{N},\frac{2\pi}{N}]$, the inequality follows.

Now suppose that $S^+* = S^+$. Then either $S^+* = \{1,\dots,N/2\}$ with $t = -\frac{\pi}{N}$ or $S^+* = \{1,\dots,N/2-1\}$ with $t=0$. In the first case the sum is $2-2(1-\cos\frac{\pi}{N}) = 2\cos\frac{\pi}{N}$. In the second case the sum is equal to $2-(1-\cos\frac{2\pi}{N}) = 1+\cos\frac{2\pi}{N}$. In both cases the result is greater than $2\cos\frac{2\pi}{N}\cos\frac{\pi}{N}$. □

Remark 4.4. The fact that N is a multiple of 4 yields a simple formulation and proof of the above lemma.

We now prove Theorem 1.1.

Proof. (i) Let x be a solution to (1.1) and consider again the orthogonal splitting of e,

$$e = A\underline{s} + B\underline{c} + w,$$

where $A, B \in \mathbb{R}$ and $w \in M$. The inner product of equation (1.3) with $z = A\underline{s}+B\underline{c}$ yields

$$\mathcal{G}(x)\cdot z = e\cdot z = \|z\|^2 = \pi(A^2+B^2).$$

On the other hand, by Lemma 2.1,

$$\mathcal{G}(x)\cdot z=-h\sum_{i=1}^{N}\frac{f(x_i)\,(z_{i+1}-z_i)}{h}.$$

There exists $\varphi\in\mathbb{R}$ such that $z_i=\sqrt{A^2+B^2}\sin\big(\varphi+\frac{2\pi i}{N}\big)$. Hence splitting the sum into

$$-\sum_{i=1}^{N}f(x_i)\,(z_{i+1}-z_i)^{+}+\sum_{i=1}^{N}f(x_i)\,(z_{i+1}-z_i)^{-}$$

and using the assumptions and Lemma 4.2 we obtain

$$\pi(A^2+B^2)<2\sqrt{A^2+B^2}\,\big(f(\infty)-f(-\infty)\big).$$

(*ii*) By Proposition 2.5, we only need to prove that $V(v,u)=0$ has a solution, which we do using Proposition 2.4. Suppose that (1.9) holds.

First, we want to show that there exists an L such that if $v_i=L$, then $V_i(v,u)>0$ (respectively if $v_i=-L$, then $V_i(v,u)<0$), for $1\le i\le N-2$. It suffices then to prove that $K\big(-P\mathcal{G}(u+v)+Pe\big)$ is bounded, and this is done the same way as given in the proof of Theorem 2.2 (note that $\mathcal{G}$ is bounded as well).

Let $\varepsilon>0$ be such that

$$\big(f(+\infty)-f(-\infty)-2\,\varepsilon\big)\,\alpha'_N-4\,m\sin\frac{\pi}{N}-\pi\sqrt{A^2+B^2}\;>\;0$$

and fix $T>0$ such that

$$f(x)>f(+\infty)-\varepsilon\quad\forall x\ge T,\qquad f(x)<f(-\infty)+\varepsilon\quad\forall x\le -T.$$

Consider now a ball in $\mathrm{Ker}(\Delta^2+\lambda)$ with radius R so that $\frac{R}{\sqrt{\pi}}\sin\frac{\pi}{N}-L>T$. Let u be on the boundary of the ball, with $u=\alpha\underline{s}+\beta\underline{c}$, meaning that $R=\sqrt{\pi(\alpha^2+\beta^2)}$. Consider the rotation ρ of angle $\pi/2$ in this two-dimensional subspace, given by

$$\rho(u)=-\beta\underline{s}+\alpha\underline{c}.$$

It is easily seen that, if $u_i=\frac{R}{\sqrt{\pi}}\sin\big(\frac{2\pi i}{N}+t\big)$, then $\rho(u)_i=\frac{R}{\sqrt{\pi}}\sin\big(\frac{2\pi i}{N}+t+\frac{\pi}{2}\big)$. Then we compute, with $|v_i|\le L$:

$$\begin{aligned}Q\big(\mathcal{G}(u+v)-e\big)\cdot\rho(u)&=\mathcal{G}(u+v)\cdot\rho(u)-e\cdot\rho(u)\\&\ge h\sum_{i=1}^{N}\Delta f(u_i+v_i)\,\rho(u)_i\;-\;\pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2}\\&=-\sum_{i=1}^{N}f(u_i+v_i)\,\big(\rho(u)_{i+1}-\rho(u)_i\big)-\pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2}.\end{aligned}$$

Noticing that the σ_i and the differences $\rho(u)_{i+1}-\rho(u)_i$ have opposite signs (as they lie in sine graphs misaligned by a translation of $\frac{\pi}{2}$) we may write

$$\begin{aligned}&-\sum_{i=1}^{N}f(u_i+v_i)\,\big(\rho(u)_{i+1}-\rho(u)_i\big)\\&\quad=\sum_{i\in S^+}f\Big(\frac{R}{\sqrt{\pi}}\sigma_i+v_i\Big)\big(\rho(u)_{i+1}-\rho(u)_i\big)^{-}-\sum_{i\in S^-}f\Big(\frac{R}{\sqrt{\pi}}\sigma_i+v_i\Big)\big(\rho(u)_{i+1}-\rho(u)_i\big)^{+}.\end{aligned}$$

Hence

$$
\begin{aligned}
&Q\big(\mathcal{G}(u+v)-e\big)\cdot\rho(u)\\
&\qquad\geq \sum_{i\in S^{+*}} f\Big(\frac{R}{\sqrt{\pi}}\,\sigma_i+v_i\Big)\big(\rho(u)_{i+1}-\rho(u)_i\big)^- - \sum_{i\in S^{-*}} f\Big(\frac{R}{\sqrt{\pi}}\,\sigma_i+v_i\Big)\big(\rho(u)_{i+1}-\rho(u)_i\big)^+\\
&\qquad\quad - m\big(\rho(u)_{i*+1}-\rho(u)_{i*}\big)^- - m\big(\rho(u)_{j*+1}-\rho(u)_{j*}\big)^+ - \pi\sqrt{A^2+B^2}\sqrt{\alpha^2+\beta^2}\,.
\end{aligned}
$$

By Lemmas 4.1 and 4.3 and the definition of α'_N we obtain

$$
Q\big(\mathcal{G}(u+v)-e\big)\cdot\rho(u)\;\geq\;\frac{R}{\sqrt{\pi}}\Big(\big(f(+\infty)-f(-\infty)-2\,\varepsilon\big)\alpha'_N-4\,m\sin\frac{\pi}{N}-\pi\sqrt{A^2+B^2}\Big)>0.
$$

We then conclude that there exists a solution to $V(v,u)=0$ and therefore there exists a periodic solution to (1.3). □

A final remark is in order. The estimates for L and R obtained in the proof of Theorem 1.1 depend on N. However under natural assumptions we can show that norms of the solutions are kept below some constant. This is so because there exist a priori bounds for the solutions of (1.3) which do not depend on N. To see this, suppose that $e=e_N$ is defined for all N and that

$$
E:=\sup_N\|e_N\|<\infty.
$$

Keeping the notation introduced in section 2, consider a solution $x=v+u$. Let us decompose v into

$$
v=(c,c,\ldots,c)+w
$$

where $c\in\mathbb{R}$ and w is orthogonal to $(1,1,\ldots,1)$ (and, of course, to $\underline{s}$ and $\underline{c}$ as well). The inner product of (1.3) with $(c,c,\ldots,c)$ yields

$$
\lambda|c|\leq E.
$$

The next step consists in proving that w is bounded. In fact the inner product of (1.3) with w gives

$$
-\frac{1}{h}\sum_{i=1}^{N}\big(w_{i+1}w_i-2\,w_i^2+w_{i-1}w_i\big)\;=\;\lambda\|w\|^2+\sum_{i=1}^{N}f(u_i+v_i)\,(w_{i+1}-w_i)-e\cdot w+2\pi\,\lambda c\|w\|.
$$

Hence

$$
\frac{N}{2\pi}\sum_{i=1}^{N}(w_{i+1}-w_i)^2\;\leq\;\lambda\|w\|^2+C\|w\|+m\sqrt{N\sum_{i=1}^{N}(w_{i+1}-w_i)^2}
$$

where C is a constant independent of N. Recall that $\lambda=\lambda_N$ stays close to 1 for large N. Now we claim that for all w orthogonal to $(1,1,\ldots,1)$, $\underline{s}$ and $\underline{c}$ we have

$$
\sum_{i=1}^{N}(w_{i+1}-w_i)^2\;\geq\;4\,\sin^2\frac{2\pi}{N}\sum_{i=1}^{N}w_i^2. \tag{4.1}
$$

Combining this with the previous inequality we conclude that the quantity

$$
\sum_{i=1}^{N}|w_{i+1}-w_i|
$$

is bounded independently of N and therefore (using the fact that w has components with both signs) it follows that there is a constant L such that, for all N,

$$|w_i| \leq L, \qquad \forall i = 1, \ldots, N.$$

Finally we consider the boundedness of the component u. Assume in addition that there exists $\delta > 0$ such that

$$\pi\sqrt{A^2 + B^2} + \delta < 2\big(f(\infty) - f(-\infty)\big)$$

for all sufficiently large N (recall that $A = A_N$ and $B = B_N$ although we omit the subscript). If the components of u are $u_i = R\sin\big(t + \frac{2\pi i}{N}\big)$, we consider $\tilde{u}$ with $\tilde{u}_i = R\sin\big(t + \frac{\pi}{2} + \frac{2\pi i}{N}\big)$. The inner product of the second equation in (2.1) with $\tilde{u}$ gives

$$\sum_{i=1}^{N} f(u_i + v_i)\,(\tilde{u}_{i+1} - \tilde{u}_i) = Q\,e \cdot \tilde{u}$$

or equivalently

$$\sum_{i=1}^{N} f\left(R\sin\left(t + \frac{2\pi i}{N}\right) + v_i\right) 2\,\sin\frac{\pi}{N}\,\sin\left(\frac{2\pi i}{N} + \frac{\pi}{N} + t\right) = h\sum_{i=1}^{N} e_i \sin\left(t + \frac{\pi}{2} + \frac{2\pi i}{N}\right),$$

which implies

$$\xi_N\,\frac{2\pi}{N}\,\sum_{i=1}^{N} f\left(R\,\sin\left(t + \frac{2\pi i}{N}\right) + v_i\right)\sin\left(\frac{2\pi i}{N} + \frac{\pi}{N} + t\right) \leq \pi\sqrt{A^2 + B^2}$$

where $\xi_N \to 1$ as $N \to \infty$. Given the boundedness of the v_i it is not difficult to see that, for all large N and R sufficiently large, the left-hand side becomes arbitrarily close to $2\big(f(\infty) - f(-\infty)\big)$, a contradiction with the assumption.

For completeness, we provide a

Proof of (4.1). We compute the minimum of the quadratic form $\sum_{i=1}^{N}(w_{i+1} - w_i)^2$ in the unit sphere (for the standard norm of $\mathbb{R}^N$) of the subspace M' consisting of vectors orthogonal to $(1, 1, \ldots, 1)$, $\underline{s}$ and $\underline{c}$. Since in the unit sphere

$$\sum_{i=1}^{N}(w_{i+1} - w_i)^2 = 2 - 2\sum_{i=1}^{N}(w_{i+1}w_i)$$

we have only to compute the maximum of $2\sum_{i=1}^{N}(w_{i+1}w_i)$ in the sphere. Now the matrix of this quadratic form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

is symmetric and circulant, hence it shares the same eigenvectors of the matrix for Δ^2. By elementary properties of circulant matrices (see e.g. [6]), the eigenvalues corresponding to eigenvectors in M' are the numbers $2\cos\frac{j\pi}{N}$, $j = 4, \ldots, \frac{N}{2} - 1$. The greatest of them is $2\cos\frac{4\pi}{N} = 2 - 4\sin^2\frac{2\pi}{N}$. This completes the proof. □

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