Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **85**, 1–23; https://doi.org/10.14232/ejqtde.2019.1.85 www.math.u-szeged.hu/ejqtde/

Existence and asymptotics of traveling wave fronts for a coupled nonlocal diffusion and difference system with delay

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Received 27 April 2019, appeared 19 November 2019
Communicated by Tibor Krisztin

Abstract. In this paper, we consider a general study of a recent proposed hematopoietic stem cells model. This model is a combination of nonlocal diffusion equation and difference equation with delay. We deal with the properties of traveling waves for this system such as the existence and asymptotic behavior. By using the Schauder's fixed point theorem combined with the method based on the construction of upper and lower solutions, we obtain the existence of traveling wave fronts for a speed $c > c^\star$. The case $c = c^\star$ is studied by using a limit argument. We prove also that $c^\star$ is the critical value. We finally prove that the nonlocality increases the minimal wave speed.

Keywords: traveling wave front, nonlocal diffusion systems with delay, difference equation, monostable equation.

2010 Mathematics Subject Classification: 35C07, 35K08, 35K57.

1 Introduction

Propagation and invasion phenomena are often analyzed through the study of traveling wave. A traveling wave is a solution of special form and it can be seen as an invariant function with respect to spatial translation describing processes. Many researchers have used such solutions to model the dynamics of biological invasions and the spread of population (see [25, 31] and references therein). The theory of these solutions has been widely developed for the reaction-diffusion equations and there has been some success studied for establishing the existence of traveling wave for the reaction-diffusion equations with or without delayed local or nonlocal nonlinearity (see [1, 5, 7, 11, 13, 15, 17, 23–26, 30, 34, 35] and references therein). On the other hand, there have been studies about traveling waves for nonlocal diffusion systems where the diffusion is described by integral, see [6, 8–10, 20, 21, 32, 33, 36].

Recently in [1], a new model (based on the model of Mackey [18]) describing hematopoiesis was presented and discussed. This model is the following coupled reaction-diffusion and

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difference system (see also [16] for a particular case)

$$\begin{cases} \dfrac{\partial N(t,x)}{\partial t} = D\dfrac{\partial^2 N(t,x)}{\partial x^2} - (\delta(N(t,x)) + \beta(N(t,x)))N(t,x) \\ \qquad\qquad + 2(1-K)e^{-\gamma r}\displaystyle\int_{-\infty}^{+\infty} \Gamma(r, x-y)u(t-r,y)dy, \\ u(t,x) = \beta(N(t,x))N(t,x) + 2Ke^{-\gamma r}\displaystyle\int_{-\infty}^{+\infty} \Gamma(r,x-y)u(t-r,y)dy. \end{cases} \tag{1.1}$$

This system (1.1) describes mature-immature blood cells interaction. The blood cell population is split into two compartments of mature and immature cells. Each compartment represents the cells in resting phase and proliferating phase, respectively. In this system, N represents the density of resting cells and u the density of new active or proliferating cells (see also [2–4,18]). As mentioned, a special case of the above system is treated in [16] which corresponds to $K = 0$. The one-dimensional domain was taken. The positive coefficients D and d represent the diffusion rates in the quiescent and proliferating phase, respectively. The delay $r > 0$ describes the duration of the active phase and $\gamma > 0$ a programed cell death rate. The terms $2(1-K)e^{-\gamma r}$ and $2Ke^{-\gamma r}$, for $0 \leq K < 1$, describe the part of divided cells (coefficient 2 represents the division) that enter the quiescent and proliferating phase, respectively. The nonlinearities are given by $(\delta(x) + \beta(x))x$ and $\beta(x)x$ where δ is a natural death rate and β is the rate of flux between the both phases (see [1] for more interpretations of parameters).

The system (1.1) shows the non-local effect that is caused by cells diffusing (with a rate d) during proliferating phase where Γ denote the Green's function given by

$$\Gamma(t,x) = \frac{1}{2\sqrt{d\pi t}} \exp\left(-\frac{x^2}{4dt}\right), \quad t > 0,\ x \in \mathbb{R}.$$

We note that Γ satisfies

$$\int_{-\infty}^{+\infty} \Gamma(t,x)dx = 1, \quad t > 0. \tag{1.2}$$

In the case where $K = 0$, the system (1.1) is equivalent to the following one dimensional scalar delayed reaction-diffusion equation

$$\begin{aligned} \frac{\partial N}{\partial t}(t,x) &= D\frac{\partial^2 N(t,x)}{\partial x^2} - (\delta(N(t,x)) + \beta(N(t,x)))N(t,x) \\ &\quad + 2e^{-\gamma r}\int_{-\infty}^{+\infty} \Gamma(r,x-y)\beta(N(t-r,y))N(t-r,y)\mathrm{d}y. \end{aligned} \tag{1.3}$$

When $d \mapsto 0^+$ and using the heat kernel property, the nonlocal term in (1.3) becomes the following expression

$$\beta(N(t-r,x))N(t-r,x) = \lim_{d\mapsto 0^+}\int_{-\infty}^{+\infty} \Gamma(r,x-y)\beta(N(t-r,y))N(t-r,y)\mathrm{d}y.$$

This case corresponds to a problem with a local nonlinearity. Moreover, if $r = 0$ then the system (1.3) is reduced to

$$\frac{\partial N}{\partial t}(t,x) = D\frac{\partial^2 N(t,x)}{\partial x^2} - \delta(N(t,x))N(t,x) + \beta(N(t,x))N(t,x).$$

Considering $\delta(x) = x$ and $\beta(x) = 1$, we get the classical Fisher–KPP equation [12,14,22,25].

In [1], the authors considered a hematopoietic dynamics model that took into account spatial diffusion of cells where the Laplacian operator $\Delta := \partial^2/\partial x^2$ is local. This suggests that the influence is caused by the neighborhood variations. As in many areas, when the density of the considered population is not small, such as the dynamics of cells, the local diffusion is not sufficiently accurate (see, [19]). Moreover, it is emphasized that the nonlocal operator has some properties of the Laplacian one and is reduced in some cases to it (see, [21]). In this work, we deal with the case of nonlocal diffusion which means that we replace the local Laplacian operator by the following convolution nonlocal diffusion

$$(h * v)(t,x) - v(t,x) := \int_{-\infty}^{+\infty} h(x-y)[v(t,y) - v(t,x)]dy,$$

with $h : \mathbb{R} \to \mathbb{R}$ is a nonnegative function satisfying

$$h(x) = h(-x) \quad \text{for } x \in \mathbb{R}, \qquad \int_{-\infty}^{+\infty} h(x)dx = 1,$$

and

$$\int_{-\infty}^{+\infty} h(x)e^{-\lambda x}dx < +\infty, \quad \text{for any } \lambda > 0.$$

We shall focus on the following coupled nonlocal diffusion and difference system with delay, for $t > 0$ and $x \in \mathbb{R}$,

$$\begin{cases} \dfrac{\partial N(t,x)}{\partial t} = D[(h * N)(t,x) - N(t,x)] - (\delta(N(t,x)) + \beta(N(t,x)))N(t,x) \\ \qquad\qquad + 2(1-K)e^{-\gamma r} \displaystyle\int_{-\infty}^{+\infty} \Gamma(r, x-y)u(t-r,y)\mathrm{d}y, \\ u(t,x) = \beta(N(t,x))N(t,x) + 2Ke^{-\gamma r} \displaystyle\int_{-\infty}^{+\infty} \Gamma(r,x-y)u(t-r,y)\mathrm{d}y. \end{cases} \tag{1.4}$$

Our purpose is to prove the existence of fronts of the above system.

This paper is organized as follows. In the next section, we start by some preliminaries about the solution of the system. Section 3 is devoted to the proof of the existence of traveling wave fronts when the speed is greater or equal to a threshold denoted $c^\star$. We also prove the nonexistence when the speed is less than $c^\star$. We proceed by giving a result about the monotonicity of the critical speed wave with respect to the diffusion parameters. Section 4 is devoted to the discussion.

2 Preliminaries

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|_X$ and $X^+ := \{\phi \in X \colon \phi(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$. The space X is a Banach lattice under the partial ordering induced by the closed cone X^+.

We set

$$(T(t)\omega)(x) := \int_{\mathbb{R}} \Gamma(t, x-y)\omega(y)dy, \quad t > 0,\ x \in \mathbb{R}.$$

Then, we get (see for instance [29]) an analytic semigroup $T(t) \colon X \to X$ such that $T(t)X^+ \subset X^+$, for all $t \geq 0$.

Throughout this paper, we make the following hypotheses on the functions β and δ.

The function $N \mapsto \beta(N)$ is continuously differentiable on $\mathbb{R}$ and decreasing on $\mathbb{R}^+$ with $\lim_{N \to +\infty} \beta(N) = 0$. (2.1)

The function $N \mapsto \delta(N)$ is continuously differentiable on $\mathbb{R}$ and increasing on $\mathbb{R}^+$. (2.2)

In [1], the authors studied mainly the existence of traveling wave fronts by using the monotone iteration technique coupled with the sub- and super-solutions method developed in [30]. For the system (1.4), we shall use the same technique based on the construction of upper and lower solutions. Recall that a traveling wave of (1.4) is a solution of special form

$$(N(t,x), u(t,x)) = (\phi(x+ct), \psi(x+ct)),$$

where $\phi, \psi \in C^1(\mathbb{R}, \mathbb{R}^+)$ and $c > 0$ is a constant corresponding to the wave speed (see [19, 25, 29]). We consider the functions $f,\ g : \mathbb{R}^+ \to \mathbb{R}^+$, defined by

$$f(s) = (\delta(s) + \beta(s))\, s \quad \text{and} \quad g(s) = \beta(s)s, \quad s \in \mathbb{R}^+.$$

We set $z = x + ct$ and substitute $(N(t)(x), u(t)(x))$ with $(\phi(z), \psi(z))$ into (1.4). We obtain the corresponding wave system

$$\begin{cases} c\phi'(z) = D[(h * \phi)(z) - \phi(z)] - f(\phi(z)) + 2(1-K)e^{-\gamma r}\,(T(r)\psi)\,(z - cr), \\ \psi(z) = g(\phi(z)) + 2Ke^{-\gamma r}\,(T(r)\psi)\,(z - cr). \end{cases} \tag{2.3}$$

The following proposition ensures existence of positive constant steady state under additional conditions. Recall that $0 \le K < 1$, we suppose the following necessary condition,

$$2Ke^{-\gamma r} < 1. \tag{2.4}$$

Proposition 2.1. *Assume that $\delta(0) > 0$. If*

$$\frac{\delta(0) + (1 - 2e^{-\gamma r})\beta(0)}{2e^{-\gamma r}\delta(0)} < K < \frac{1}{2e^{-\gamma r}}, \tag{2.5}$$

then (1.4) has two distinct steady states: $(0,0)$ and $(N^\star, u^\star)$. If (2.5) does not hold, then $(0,0)$ is the only equilibrium of (1.4).

It is shown in [1] that there exists $c^\star > 0$ such that the system (2.3), with local diffusion or (1.1), has a monotone solution (ϕ, ψ) defined on $\mathbb{R}$, for each $c \ge c^\star$, subject to the following asymptotic boundary condition

$$\phi(-\infty) = \psi(-\infty) = 0, \quad \phi(+\infty) = N^\star \quad \text{and} \quad \psi(+\infty) = u^\star, \tag{2.6}$$

where $(N^\star, u^\star)$ is the only constant positive equilibrium of (1.1). In this case, the corresponding solution $(N(t,x), u(t,x)) = (\phi(x+ct), \psi(x+ct))$ is called a traveling wave front with wave speed $c > 0$ of (1.1). Such result needs the following assumptions.

$$\text{The function } N \mapsto g(N) := \beta(N)N \text{ is increasing on } [0, N^\star], \tag{2.7}$$

$$\beta(N) + \delta(N) \ge \beta(0) + \delta(0), \text{ for all } N \in [0, N^\star]. \tag{2.8}$$

The main result of this paper is given in the following theorem where we show, under the same conditions as in [1], that there exists a minimal wave speed (of course other than that in [1]) for the existence of fronts for (1.4) (system with nonlocal diffusion).

Theorem 2.2. *Assume that (2.1), (2.2), (2.4), (2.5), (2.7) and (2.8) hold. Then, there exists $c^\star > 0$ such that for every $c \ge c^\star$, (1.4) has a traveling wave front which connects $(0,0)$ to the positive equilibrium $(N^\star, u^\star)$. Let $c \in (0, c^\star)$. Then, there is no traveling front of (1.4).*

3 Existence of traveling wave fronts

In this section, we study the existence of traveling wave solutions of system (1.4). This is treated mainly by the Schauder's fixed point theorem with the notion of upper and lower solutions and Laplace transform. Let $A\colon X \to X$ be the bounded linear operator defined by

$$(A\psi)(z) = 2Ke^{-\gamma r}\,(T(r)\psi)\,(z-cr), \quad z \in \mathbb{R}. \tag{3.1}$$

A direct computation leads to $|A|_{\mathcal{L}(X)} = 2Ke^{-\gamma r} < 1$. Then, the operator A is a contraction. Thereby, ψ can be calculated explicitly according to ϕ by considering the inverse of the operator $\mathrm{Id} - A$. In fact, if we put, for $z \in \mathbb{R}$, $k \in \mathbb{N}$,

$$\xi_k(z) = \frac{(2Ke^{-\gamma r})^k}{2(kd\pi r)^{1/2}} \exp\left(-\frac{(z-kcr)^2}{4kdr}\right), \tag{3.2}$$

we have, for $\varphi \in X$,

$$(\mathrm{Id} - A)^{-1}(\varphi) = \sum_{k=0}^{+\infty} A^k\varphi = \xi * \varphi,$$

where

$$\xi(z) = \sum_{k=0}^{+\infty}(2Ke^{-\gamma r})^k\Gamma(kr, z-kcr) = \sum_{k=0}^{+\infty}\xi_k(z), \quad z \in \mathbb{R}. \tag{3.3}$$

The function ξ_k, $k \in \mathbb{N}$ satisfies

$$\int_{-\infty}^{+\infty}\xi_k(y)dy = (2Ke^{-\gamma r})^k.$$

The system (2.3) becomes an uncoupled system

$$\begin{cases} c\phi'(z) = D[(h*\phi)(z) - \phi(z)] - f(\phi(z)) + 2(1-K)e^{-\gamma r}\,(T(r)\psi)\,(z-cr), \\ \psi(z) = (\xi * g(\phi))(z), \end{cases} \tag{3.4}$$

with ξ given by (3.3). We can then write (3.4) as a single differential equation

$$c\phi'(z) = D[(h*\phi)(z) - \phi(z)] - f(\phi(z)) + 2(1-K)e^{-\gamma r}\,[T(r)\,(\xi * g(\phi))]\,(z-cr). \tag{3.5}$$

It is clear that if (ϕ, ψ) is a monotonic solution of (2.3)–(2.6), then ϕ is a monotone solution of (3.5) and

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = N^\star. \tag{3.6}$$

Under (2.4) and (2.7) we prove easily that even if ϕ is a monotone solution of (3.5)-(3.6), then $(\phi, \xi * g(\phi))$ is a monotone solution of (2.3)–(2.6). Hence, we only need to consider the solutions of (3.5) subject to boundary condition (3.6).

Our objective is to show the existence of traveling wave front solutions for the coupled nonlocal diffusion and difference system (1.4). To this end, we use the method based on the notion of an upper and a lower solutions combined with Schauder's fixed point theorem [21, 27].

Let

$$C_{[0,N^\star]}(\mathbb{R},\mathbb{R}) = \left\{\phi \in C(\mathbb{R},\mathbb{R}) :\ 0 \le \phi(z) \le N^\star,\ z \in \mathbb{R}\right\}.$$

Define the operator $H : C_{[0,N^\star]}(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ by

$$H(\phi)(z) = D(h*\phi)(z) + (\mu - D)\phi(z) - f(\phi(z)) + 2(1-K)e^{-\gamma r}\,[T(r)\,(\xi * g(\phi))]\,(z-cr),$$

where $\mu > D + \max_{s\in[0,N^\star]} f'(s)$ is a constant. Next, we show that H satisfies the condition given in the following lemma.

Lemma 3.1. *Assume that* (2.1), (2.2), (2.4), (2.5) *and* (2.7) *hold. Then, H satisfies the following property*

$$H(\phi_1)(z) - H(\phi_2)(z) \geq 0,$$

for all $\phi_1, \phi_2 \in X^+$ such that $0 \leq \phi_2(z) \leq \phi_1(z) \leq N^\star$, for all $z \in \mathbb{R}$.

The proof of this lemma is easy to establish, so we omit the details here.

Now, it is easy to remark that (3.5) is equivalent to the following simplified equation,

$$c\phi'(z) = -\mu\phi(z) + H(\phi)(z). \tag{3.7}$$

Define the operator $F : C_{[0,N^\star]}(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ by

$$F(\phi)(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\mu}{c}(z-s)} H(\phi)(s)ds.$$

We can easily see that the operator F is well defined satisfying (3.7) and the existence of solutions for (3.5) is changed into investigating the existence of a fixed point of operator F. The following remark is a key to show the existence of such fixed point.

Remark 3.2. Lemma 3.1 implies that either H and also F are monotone for ϕ. Moreover, we can deduce that $H(\phi)(z)$ and $F(\phi)(z)$ are both nondecreasing in $z \in \mathbb{R}$ with the assumption that $\phi \in C_{[0,N^\star]}(\mathbb{R}, \mathbb{R})$ is nondecreasing in $z \in \mathbb{R}$.

For $0 < \nu < \frac{\mu}{c}$, define

$$B_\nu(\mathbb{R}, \mathbb{R}) = \left\{\phi \in C(\mathbb{R}, \mathbb{R}) : \ \sup_{z\in\mathbb{R}} |\phi(z)|e^{-\nu|z|} < +\infty\right\}.$$

and the exponential decay norm

$$|\phi|_\nu = \sup_{z\in\mathbb{R}} |\phi(z)|e^{-\nu|z|}, \quad \text{for} \quad \phi \in B_\nu(\mathbb{R}, \mathbb{R}).$$

It is easy to check that $(B_\nu(\mathbb{R}, \mathbb{R}), |\cdot|_\nu)$ is a Banach space.

Now, we define the meaning of an upper and lower solutions of (3.5).

Definition 3.3. A continuous function $\overline{\phi} \in C_{[0,N^*]}(\mathbb{R}, \mathbb{R})$ is called an upper solution of (3.5) if $\overline{\phi}'$ exists almost everywhere (a.e.) and satisfy

$$\begin{aligned} c\overline{\phi}'(z) \geq{}& D[(h * \overline{\phi})(z) - \overline{\phi}(z)] - f(\overline{\phi}(z)) \\ &+ 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\overline{\phi})\right)\right](z - cr), \quad \text{a.e. in } \mathbb{R}. \end{aligned}$$

A lower solution $\underline{\phi}$ of (3.5) is defined in a similar way but it satisfies the above differential inequality in reversed order.

Existence of traveling wave front solutions need to find suitable upper $\overline{\phi}$ and lower $\underline{\phi}$ solutions of (3.5). For this purpose, we consider the transcendental characteristic function for the linearized problem of (3.5) near the zero solution. Let

$$\lambda^+(c) = \frac{c}{2d}\left(1 + \sqrt{1 + \frac{4d}{rc^2}\ln\left(\frac{e^{\gamma r}}{2K}\right)}\right).$$

For $\lambda \in (0, \lambda^+(c))$, we have

$$1 - 2Ke^{-\gamma r}e^{dr\lambda^2 - cr\lambda} > 0, \tag{3.8}$$

and define

$$\Delta_c(\lambda) = -D\left[\int_{-\infty}^{+\infty} h(y)e^{-\lambda y}dy - 1\right] + c\lambda + \delta(0) + \beta(0) - \frac{2(1-K)\beta(0)e^{-\gamma r}e^{dr\lambda^2 - cr\lambda}}{1-2Ke^{-\gamma r}e^{dr\lambda^2-cr\lambda}}. \quad (3.9)$$

We have, for $\lambda \in (0, \lambda^+(c))$,

$$1 - 2Ke^{-\gamma r}e^{dr\lambda^2 - cr\lambda} > 0 \quad \text{and} \quad \lim_{\lambda \to \lambda^+(c)} \Delta_c(\lambda) = -\infty.$$

It is not difficult to see that

$$\Delta_c(0) = \delta(0) + \beta(0) - \frac{2(1-K)\beta(0)e^{-\gamma r}}{1-2Ke^{-\gamma r}} < 0.$$

Furthermore, the second derivative of the function $\lambda \mapsto \Delta_c(\lambda)$ satisfies, for all $\lambda \in [0, \lambda^+(c))$,

$$\frac{\partial^2}{\partial \lambda^2}\Delta_c(\lambda) < 0.$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}c}\left[\Delta_c\left(\lambda\right)\right] > 0 \quad \text{and} \quad \lim_{c \to +\infty} \Delta_c\left(\lambda\right) = +\infty.$$

We conclude that there exists a unique $c^\star > 0$ such that

$$\Delta_{c^\star}(\lambda^\star(c^\star)) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda}\Delta_{c^\star}(\lambda)\big|_{\lambda = \lambda^\star(c^\star)} = 0.$$

According to the above arguments, we have the following result.

Lemma 3.4. *Assume that (2.4) and (2.5) hold. Then, there exists a unique $c^\star > 0$ and for each $c > 0$ there exists a unique $\lambda^\star(c)$ such that*

1. *if $c = c^\star$, $\Delta_{c^\star}(\lambda^\star(c^\star)) = \frac{\partial}{\partial \lambda}\Delta_{c^\star}(\lambda)\big|_{\lambda=\lambda^\star(c^\star)} = 0$,*
2. *if $c > c^\star$, there exist two real roots, $\lambda_1(c)$ and $\lambda_2(c)$, of the equation $\Delta_c(\lambda) = 0$ such that $0 < \lambda_1(c) < \lambda_2(c) < \lambda^+(c)$ and $\Delta_c(\lambda) > 0$ for all $\lambda \in (\lambda_1(c), \lambda_2(c))$,*
3. *if $0 < c < c^\star$, $\Delta_c(\lambda) < 0$ for all $\lambda \in (0, \lambda^+(c))$.*

Next, we fix $c > c^\star$ and we put $\lambda_1 := \lambda_1(c)$, $\lambda_2 := \lambda_2(c)$. We put

$$\begin{cases} \kappa_1(\lambda) = 1 - 2Ke^{-\gamma r}e^{dr\lambda^2 - cr\lambda}, \\ \kappa_2(\lambda) = 2(1-K)\beta(0)e^{-\gamma r}e^{dr\lambda^2 - cr\lambda}. \end{cases}$$

We need the following lemma.

Lemma 3.5. *For $z \in \mathbb{R}$ and $\lambda \in (0, \lambda^+(c))$, we have the following equality*

$$T(r)\left(\xi * e^{\lambda \cdot}\right)(z - cr) = \frac{\kappa_2(\lambda)}{2(1-K)e^{-\gamma r}g'(0)\kappa_1(\lambda)}e^{\lambda z}.$$

Proof. We start by computing the following quantity

$$\begin{aligned}\left(\xi * e^{\lambda\cdot}\right)(z-cr) &= \int_{-\infty}^{+\infty} \xi(z-cr-y)e^{\lambda y}dy,\\ &= e^{-\lambda cr}\sum_{k=0}^{+\infty}(2Ke^{-\gamma r})^k \int_{-\infty}^{+\infty} \Gamma\left(kr, z-y-kcr\right)e^{\lambda y}dy,\\ &= e^{-\lambda cr}e^{\lambda z}\sum_{k=0}^{+\infty}(2Ke^{-\gamma r})^k e^{drk\lambda^2-crk\lambda},\\ &= \frac{e^{-\lambda cr}}{1-2Ke^{-\gamma r}e^{dr\lambda^2-cr\lambda}}e^{\lambda z}.\end{aligned}$$

Then,

$$\begin{aligned}T(r)\left(\xi * e^{\lambda\cdot}\right)(z-cr) &= \frac{e^{-\lambda cr}}{1-2Ke^{-\gamma r}e^{dr\lambda^2-cr\lambda}}\int_{-\infty}^{+\infty}\Gamma\left(r, z-y\right)e^{\lambda y}dy,\\ &= \frac{e^{\lambda^2 dr-\lambda cr}}{1-2Ke^{-\gamma r}e^{dr\lambda^2-cr\lambda}}e^{\lambda z}.\end{aligned}$$

The proof is completed. □

We prove the existence of a continuous upper and lower solutions of (3.5).

Lemma 3.6. *Assume that (2.1), (2.2), (2.4), (2.5), (2.7) and (2.8) hold. Let $c > c^\star$ be fixed, with $c^\star$ given in Lemma 3.4, and $N^\star$ be the positive steady state. We put $\lambda_1 := \lambda_1(c)$, $\lambda_2 := \lambda_2(c)$, with $\lambda_1(c)$ and $\lambda_2(c)$ defined in Lemma 3.4. Then, The function $\overline{\phi}\colon \mathbb{R} \to \mathbb{R}^+$ defined by $\overline{\phi}(z) = \min\{N^\star, e^{\lambda_1 z}\}$ is an upper solution of (3.5).*

Proof. As $\lambda_1 > 0$, for $z_1 = \frac{1}{\lambda_1}\ln(N^\star)$, we have

$$\overline{\phi}(z) = \begin{cases} N^\star, & z \geq z_1,\\ e^{\lambda_1 z}, & z < z_1.\end{cases} \tag{3.10}$$

Suppose that $z \in [z_1, +\infty)$. Then, $\overline{\phi}(z) = N^\star$, $\overline{\phi}'(z) = \overline{\phi}''(z) = 0$ and as g is an increasing function on $[0, N^\star]$, we have

$$\left[T(r)\left(\xi * g(\overline{\phi})\right)\right](z-cr) \leq \left[T(r)\left(\xi * g(N^\star)\right)\right](z-cr) = \frac{g(N^\star)}{1-2Ke^{-\gamma r}}.$$

Then, we obtain

$$\begin{aligned}&c\overline{\phi}'(z) - D[(h*\overline{\phi})(z) - \overline{\phi}(z)] + f(\overline{\phi}(z)) - 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\overline{\phi})\right)\right](z-cr)\\ &\geq f(N^\star) - 2(1-K)e^{-\gamma r}\left(\frac{g(N^\star)}{1-2Ke^{-\gamma r}}\right) = 0.\end{aligned}$$

Suppose that $z \in (-\infty, z_1)$. Then, $\overline{\phi}(z) = e^{\lambda_1 z}$. Consequently,

$$c\overline{\phi}'(z) - D[(h*\overline{\phi})(z) - \overline{\phi}(z)] \geq \left(c\lambda_1 - D\left[\int_{-\infty}^{+\infty} h(y)e^{-\lambda_1 y}dy - 1\right]\right)e^{\lambda_1 z}$$

and due to (2.8)

$$f(\overline{\phi}(z)) \geq f'(0)\overline{\phi}(z) = f'(0)e^{\lambda_1 z}.$$

Furthermore, we have

$$
\begin{aligned}
\left[T(r)\left(\xi * g(\overline{\phi})\right)\right](z-cr) &\leq g'(0)\left[T(r)\left(\xi * \overline{\phi}\right)\right](z-cr) \\
&\leq g'(0)\left[T(r)\left(\xi * e^{\lambda_1 \cdot}\right)\right](z-cr) \\
&= \frac{g'(0)e^{dr\lambda_1^2+(z-cr)\lambda_1}}{1-2Ke^{-\gamma r}e^{dr\lambda_1^2-cr\lambda_1}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&c\overline{\phi}'(z) - D[(h*\overline{\phi})(z) - \overline{\phi}(z)] + f(\overline{\phi}(z)) \\
&\qquad - 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\overline{\phi})\right)\right](z-cr) \geq \Delta_c(\lambda_1)e^{\lambda_1 z} = 0.
\end{aligned}
$$

□

Lemma 3.7. *Assume that the hypotheses of Lemma 3.6 hold. Then, the function $\underline{\phi}\colon \mathbb{R} \to \mathbb{R}^+$ defined by $\underline{\phi}(z) = \max\{0, e^{\lambda_1 z} - Me^{\omega\lambda_1 z}\}$, with $\omega \in (1, \min\{2, \lambda_2/\lambda_1\})$ and $M > 1$ large enough, is a lower solution of (3.5). Moreover, $\underline{\phi}(z) \leq \overline{\phi}(z)$, for all $z \in \mathbb{R}$.*

Proof. Let $\nu \in (\omega - 1, \min\{2, \lambda_2/\lambda_1\} - 1)$. It is clear that $0 < \nu < 1$. Recall that $g(N) \leq g'(0)N$ and $f(N) \geq f'(0)N$ for $[0, N^\star]$. Under the assumption δ and β are C^1-function, there exists $\overline{\alpha} > 0$ such that, for $u \in [0, N^\star]$,

$$
\delta(u) + \beta(u) - (\delta(0) + \beta(0)) \leq \overline{\alpha}u^\nu \quad \text{and} \quad \beta(0) - \beta(u) \leq \overline{\alpha}u^\nu. \tag{3.11}
$$

We will construct a lower solution $\underline{\phi}$ of the form

$$
\underline{\phi}(z) = \begin{cases} e^{\lambda_1 z} - Me^{\omega\lambda_1 z}, & z < z_2, \\ 0, & z \geq z_2, \end{cases}
$$

with

$$
z_2 := \frac{1}{(\omega-1)\lambda_1}\ln\left(\frac{1}{M}\right),
$$

and $M > 1$ is a constant. Then, $z_2 < 0$. First, remark that to get $\underline{\phi} \leq \overline{\phi}$, it suffices to choose $M > (N^\star)^{1-\omega}$.

Let $z \in [z_2, +\infty)$. Then, $\underline{\phi}(z) = 0$. Thus,

$$
\begin{aligned}
&c\underline{\phi}'(z) - D[(h*\underline{\phi})(z) - \underline{\phi}(z)] + f(\underline{\phi}(z)) - 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\underline{\phi})\right)\right](z-cr) \\
&\quad = -D(h*\underline{\phi})(z) - 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\underline{\phi})\right)\right](z-cr).
\end{aligned}
$$

The function $\underline{\phi}$ is nonnegative on $\mathbb{R}$. We conclude that, for all $z \in [z_2, +\infty)$,

$$
c\underline{\phi}'(z) - D[(h*\underline{\phi})(z) - \underline{\phi}(z)] + f(\underline{\phi}(z)) - 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\underline{\phi})\right)\right](z-cr) \leq 0.
$$

Let $z \in (-\infty, z_2)$. Then, $\underline{\phi}(z) = e^{\lambda_1 z} - Me^{\omega\lambda_1 z}$. We have

$$
\begin{aligned}
c\underline{\phi}'(z) - D[(h*\underline{\phi})(z) - \underline{\phi}(z)] &\leq c\lambda_1 e^{\lambda_1 z} - cM\omega\lambda_1 e^{\omega\lambda_1 z} + De^{\lambda_1 z} - DMe^{\omega\lambda_1 z}. \\
&\quad - D\int_{-\infty}^{+\infty} h(y)[e^{\lambda_1(z-y)} - Me^{\omega\lambda_1(z-y)}]dy.
\end{aligned}
$$

Thanks to $\Delta_c(\lambda_1) = 0$, we obtain

$$c\underline{\phi}'(z) - D[(h * \underline{\phi})(z) - \underline{\phi}(z)] \leq \frac{\kappa_2(\lambda_1)}{\kappa_1(\lambda_1)} e^{\lambda_1 z} - f'(0)e^{\lambda_1 z} - \Delta_c(\omega\lambda_1) M e^{\omega\lambda_1 z} - \frac{\kappa_2(\omega\lambda_1)}{\kappa_1(\omega\lambda_1)} M e^{\omega\lambda_1 z} + f'(0) M e^{\omega\lambda_1 z}.$$

We know that $\omega \in (1, \min\{2, \lambda_2/\lambda_1\})$, then

$$\Delta_c(\omega\lambda_1) > 0 \quad \text{and} \quad 1 - 2Ke^{-\gamma r} e^{dr\omega^2\lambda_1^2 - cr\omega\lambda_1} > 0.$$

By Lemma 3.5, we get

$$\frac{\kappa_2(\lambda_1)}{\kappa_1(\lambda_1)} e^{\lambda_1 z} - \frac{\kappa_2(\omega\lambda_1)}{\kappa_1(\omega\lambda_1)} M e^{\omega\lambda_1 z} \leq 2(1-K)e^{-\gamma r} g'(0) T(r) \left(\xi * \underline{\phi}\right)(z - cr).$$

We conclude that

$$\begin{aligned} &c\underline{\phi}'(z) - D[(h * \underline{\phi})(z) - \underline{\phi}(z)] + f(\underline{\phi}(z)) \\ &\quad \leq -\Delta_c(\omega\lambda_1) M e^{\omega\lambda_1 z} + \overline{\alpha}\underline{\phi}^{(\nu+1)}(z) + 2(1-K)e^{-\gamma r} g'(0) T(r) \left(\xi * \underline{\phi}\right)(z - cr). \end{aligned}$$

It is not difficult to see that

$$\underline{\phi}^{(\nu+1)}(s) \leq e^{(\nu+1)\lambda_1 s}, \quad \text{for all } s \in \mathbb{R}.$$

Then,

$$\begin{aligned} &c\underline{\phi}'(z) - D[(h * \underline{\phi})(z) - \underline{\phi}(z)] + f(\underline{\phi}(z)) - 2(1-K)e^{-\gamma r} \left[T(r) \left(\xi * g(\underline{\phi})\right)\right](z - cr) \\ &\quad \leq -\Delta_c(\omega\lambda_1) M e^{\omega\lambda_1 z} + \overline{\alpha} e^{(\nu+1)\lambda_1 z} + 2(1-K)e^{-\gamma r} \overline{\alpha} T(r) \left(\xi * e^{(\nu+1)\lambda_1 \cdot}\right)(z - cr). \end{aligned}$$

Using lemma 3.5, we have

$$2(1-K)e^{-\gamma r} \overline{\alpha} T(r) (\xi * e^{(\nu+1)\lambda_1 \cdot})(z - cr) = \frac{\overline{\alpha}\kappa_2((\nu+1)\lambda_1)}{g'(0)\kappa_1((\nu+1)\lambda_1)} e^{(\nu+1)\lambda_1 z}.$$

So,

$$\begin{aligned} &c\underline{\phi}'(z) - D[(h * \underline{\phi})(z) - \underline{\phi}(z)] + f(\underline{\phi}(z)) - 2(1-K)e^{-\gamma r} \left[T(r) \left(\xi * g(\underline{\phi})\right)\right](z - cr) \\ &\quad \leq e^{\omega\lambda_1 z} \left[-\Delta_c(\omega\lambda_1) M + \overline{\alpha} e^{(\nu+1-\omega)\lambda_1 z} \left(1 + \frac{\kappa_2((\nu+1)\lambda_1)}{g'(0)\kappa_1((\nu+1)\lambda_1)}\right)\right]. \end{aligned}$$

Recall that $\nu + 1 - \omega > 0$, which implies that $e^{(\nu+1-\omega)\lambda_1 z} < 1$, for all $z < z_2$. Then, for $z < z_2$,

$$\begin{aligned} &c\underline{\phi}'(z) - D[(h * \underline{\phi})(z) - \underline{\phi}(z)] + f(\underline{\phi}(z)) - 2(1-K)e^{-\gamma r} \left[T(r) \left(\xi * g(\underline{\phi})\right)\right](z - cr) \\ &\quad \leq e^{\omega\lambda_1 z} \left[-\Delta_c(\omega\lambda_1) M + \overline{\alpha} \left(1 + \frac{\kappa_2((\nu+1)\lambda_1)}{g'(0)\kappa_1((\nu+1)\lambda_1)}\right)\right]. \end{aligned}$$

Finally, we can choose

$$M > \max\left\{1, (N^\star)^{1-\omega}, \widetilde{C}\left[\Delta_c(\omega\lambda_1)\right]^{-1}\right\},$$

with

$$\widetilde{C} := \overline{\alpha} \left(1 + \frac{\kappa_2((\nu+1)\lambda_1)}{g'(0)\kappa_1((\nu+1)\lambda_1)}\right).$$

The proof is completed. □

We define the profile set of traveling wave fronts as

$$\Theta = \left\{ \begin{array}{ll} & \phi \in C_{[0,N^\star]}(\mathbb{R},\mathbb{R}) : \\ (i) & \phi(z) \text{ is nondecreasing on } \mathbb{R}, \\ (ii) & \underline{\phi}(z) \le \phi(z) \le \overline{\phi}(z), \text{ for all } z \in \mathbb{R}. \end{array} \right\}$$

Next, we state two lemmas that ensure the existence of a fixed points of operator F. For the set Θ, it is easy to see that the following lemma holds.

Lemma 3.8. *The set Θ is nonempty, bounded, closed and convex subset of $B_\nu(\mathbb{R},\mathbb{R})$ with respect to the norm $|\cdot|_\nu$.*

The proof of the following result is similar to the proof of the corresponding results in [21, 27].

Lemma 3.9. *Assume (2.1), (2.2), (2.4), (2.5) and (2.7) hold. Then, $F(\Theta) \subset \Theta$ and $F : \Theta \to \Theta$ is continuous and compact with respect to the norm $|\cdot|_\nu$.*

In conclusion, we get the following theorem that state the existence result.

Theorem 3.10. *Assume that (2.1), (2.2), (2.4), (2.5), (2.7) and (2.8) hold. Then, for every $c > c^\star$, (1.4) has a traveling wave front which connects $(0,0)$ to the positive equilibrium $(N^\star, u^\star)$.*

Proof. From Lemmas 3.6, 3.7, 3.8 and 3.9, for $c > c^\star$, we obtain the existence of a fixed point ϕ of F belonging to Θ, that is,

$$\phi(z) = \frac{1}{c}\int_{-\infty}^{z} e^{-\frac{\mu}{c}(z-s)} H(\phi)(s)ds. \tag{3.12}$$

On the other band, we have that $\underline{\phi}(z) \le \phi(z) \le \overline{\phi}(z)$ for all $z \in \mathbb{R}$, which implies that $\lim_{z\to-\infty}\phi(z) = 0$. Moreover, $\phi(z)$ is a nondecreasing function bounded above by $N^\star$. Then, there exists N_0 such that $\lim_{z\to+\infty}\phi(z) = N_0 \le N^\star$. Recall that $0 \le \underline{\phi}(z) \not\equiv 0$ for all $z \in \mathbb{R}$. This implies that $N_0 \in (0, N^\star]$. Using L'Hospital's rule for (3.12), we obtain

$$\begin{aligned} N_0 = \lim_{z\to+\infty}\phi(z) &= \lim_{z\to+\infty}\frac{1}{\mu}H(\phi)(z), \\ &= \frac{1}{\mu}\left[\mu N_0 - f(N_0) + \frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}g(N_0)\right]. \end{aligned}$$

We deduce that $(1-2Ke^{-\gamma r})f(N_0) = 2(1-K)e^{-\gamma r}g(N_0)$. Since we have the uniqueness of the positive steady state, we conclude that $N_0 = N^\star$. As a consequence, we get the existence of traveling wave front satisfying (3.6). The proof is complete. □

The following theorem concerns the result for the critical velocity.

Theorem 3.11. *Assume that (2.1), (2.2), (2.4), (2.5), (2.7) and (2.8) hold and $c = c^*$. Then, Equation (1.4) has a traveling wave front which connects $(0,0)$ to the positive equilibrium $(N^\star, u^\star)$.*

Proof. Let $c = c^\star$. We use some ideas developed in [24, 34, 35]. We consider a sequence $(c_m)_{m\ge1} \subseteq (c^\star, +\infty)$ such that $\lim_{m\to+\infty} c_m = c^\star$. For instance, we can choose $c_m = c^\star + 1/m$. It follows from Theorem 3.10 that for $c = c_m > c^\star$, (1.4) has a solution in Θ. We denote by ϕ_m

this solution. Without loss of generality, we may assume that $\phi_m(0) = N^\star/2$. Furthermore, ϕ_m is given by

$$\phi_m(z) = \frac{1}{c_m}\int_{-\infty}^{z} e^{-\frac{\mu}{c_m}(z-s)} H(\phi_m)(s)ds.$$

We can verify the boundedness of ϕ_m' on $\mathbb{R}$ by differentiating the above equality with respect to z. It follows that ϕ_m is uniformly bounded and equicontinuous sequences of functions on $\mathbb{R}$. By Ascoli's theorem there exists a subsequence of $(c_m)_{m\geq 1}$ (for simplicity, we preserve the same sequence $(c_m)_{m\geq 1}$), such that $\lim_{m\to+\infty} c_m = c^\star$ and $\phi_m(z)$ converge uniformly on every bounded interval. Then, they converge pointwise on $\mathbb{R}$ to $\phi(z)$. By using Lebesgue's dominated convergence theorem, we get

$$\phi(z) = \frac{1}{c^\star}\int_{-\infty}^{z} e^{-\frac{\mu}{c^\star}(z-s)} H(\phi)(s)ds.$$

Then, ϕ is a solution of (3.5) with $c = c^\star$. It is not difficult to see that ϕ is nondecreasing on $\mathbb{R}$ and satisfying $\phi(0) = N^\star/2$ and $0 \leq \phi(z) \leq N^\star$ for all $z \in \mathbb{R}$. Then, $\lim_{z\to-\infty}\phi(z)$ and $\lim_{z\to+\infty}\phi(z)$ exist. Obviously, $\lim_{z\to-\infty}\phi(z) = 0$ and $\lim_{z\to+\infty}\phi(z) = N^\star$. As a consequence, we have that for $c = c^\star$, (3.5) has a solution in Θ. The proof is complete. □

In the next results, we give some properties of the traveling wave fronts of (3.5).

Lemma 3.12. *Let ϕ be a traveling wave front of* (3.5) *connecting* 0 *to the positive equilibrium $N^\star$ and let $z \in \mathbb{R}$. Then, we have*

1. $\displaystyle\int_{-\infty}^{z} |(1-2Ke^{-\gamma r})\,[T(r)\,(\xi * \phi)]\,(y-cr) - \phi(y)|dy < +\infty,$
2. $\displaystyle\varphi(z) := \int_{-\infty}^{z}\phi(y)\mathrm{d}y < +\infty,$
3. $\displaystyle\int_{-\infty}^{z}[T(r)\,(\xi * \phi)]\,(y-cr)dy = ((\Gamma * \xi) * \varphi)\,(z-cr),$
4. $\displaystyle\int_{-\infty}^{z} |(1-2Ke^{-\gamma r})\,((\Gamma * \xi) * \varphi)\,(y-cr) - \varphi(y)|dy < +\infty.$

Proof. (1) The definition of convolution product implies, for $t < z$,

$$\int_t^z (1-2Ke^{-\gamma r})\,[T(r)\,(\xi*\phi)]\,(y-cr) - \phi(y)dy$$
$$= \int_t^z (1-2Ke^{-\gamma r})\int_{-\infty}^{+\infty}(\Gamma*\xi)\,(l)\phi(y-cr-l)dl - \phi(y)dy.$$

We can check that $(1-2Ke^{-\gamma r})\int_{-\infty}^{+\infty}(\Gamma*\xi)\,(l)dl = 1$. Then, we can write the following equality

$$\int_t^z (1-2Ke^{-\gamma r})\,[T(r)\,(\xi*\phi)]\,(y-cr) - \phi(y)dy$$
$$= (1-2Ke^{-\gamma r})\int_t^z\int_{-\infty}^{+\infty}(\Gamma*\xi)\,(l)\,[\phi(y-cr-l)-\phi(y)]\,dldy.$$

Moreover, we have

$$\phi(y-cr-l)-\phi(y) = -\int_{y-cr-l}^{y}\phi'(x)\mathrm{d}x = -(l+cr)\int_0^1\phi'(y-\eta(l+cr))d\eta.$$

Then, we get

$$\int_t^z (1-2Ke^{-\gamma r}) \left[T(r)\left(\xi * \phi\right)\right](y-cr) - \phi(y)dy$$
$$- (1-2Ke^{-\gamma r}) \int_t^z \int_{-\infty}^{+\infty} (l+cr) \left[\int_0^1 \phi'(y-\eta(l+cr))d\eta\right] (\Gamma * \xi)(l) dl dy.$$

Fubini's theorem with the dominated convergence theorem implies

$$\int_{-\infty}^z (1-2Ke^{-\gamma r}) \left[T(r)\left(\xi * \phi\right)\right](y-cr) - \phi(y)dy$$
$$= -(1-2Ke^{-\gamma r}) \int_{-\infty}^{+\infty} (l+cr) \int_0^1 \left[\lim_{t\to-\infty} \int_t^z \phi'(y-\eta(l+cr))dy\right] d\eta\, (\Gamma * \xi)(l) dl.$$

By using the fact that $\lim_{x\to-\infty}\phi(x)=0$, we get

$$\lim_{t\to-\infty} \int_t^z \phi'(y-\eta(l+cr))\mathrm{d}y = \phi(z-\eta(l+cr)).$$

This yields

$$\int_{-\infty}^z (1-2Ke^{-\gamma r}) \left[T(r)\left(\xi * \phi\right)\right](y-cr) - \phi(y)dy$$
$$= -(1-2Ke^{-\gamma r}) \int_{-\infty}^{+\infty} (l+cr) \int_0^1 \phi(z-\eta(l+cr))d\eta\, (\Gamma * \xi)(l) dl.$$

The function $(l,z) \in \mathbb{R}\times\mathbb{R} \longmapsto \int_0^1 \phi(z-\eta(l+cr))d\eta$ is bounded. We have also that $\int_{-\infty}^{+\infty} |l \ \ (\Gamma * \xi)(l)| dl < +\infty$. Then, for $z\in\mathbb{R}$,

$$\int_{-\infty}^z |(1-2Ke^{-\gamma r}) \left[T(r)\left(\xi * \phi\right)\right](y-cr) - \phi(y)| dy < +\infty.$$

(2) The function ϕ is positive and satisfies

$$c\phi'(z) = D[(h*\phi)(z)-\phi(z)] - f(\phi(z)) + 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\phi)\right)\right](z-cr), \tag{3.13}$$

with $\phi(-\infty)=0, \quad \phi(+\infty)=N^\star$. The continuity of β and δ implies that

$$\lim_{y\to-\infty}\beta(\phi(y)) = \beta(0) \quad \text{and} \quad \lim_{y\to-\infty}\delta(\phi(y))+\beta(\phi(y)) = \delta(0)+\beta(0).$$

Then, for $\varepsilon>0$ small enough ($\varepsilon<\beta(0)$), there exists $\overline{y}_\varepsilon<0$ such that, for all $y<\overline{y}_\varepsilon$, we have

$$\begin{cases} \beta(0)-\varepsilon < \beta(\phi(y)) \le \beta(0)+\varepsilon, \\ \delta(0)+\beta(0)-\varepsilon \le \delta(\phi(y))+\beta(\phi(y)) < \delta(0)+\beta(0)+\varepsilon. \end{cases}$$

Then, (3.13) implies, for $y<\overline{y}_\varepsilon$,

$$\begin{aligned} c\phi'(y) \ge{}& D[(h*\phi)(z)-\phi(z)] - (\varepsilon+\delta(0)+\beta(0))\phi(y) \\ &+ 2(1-K)e^{-\gamma r}(\beta(0)-\varepsilon)\left[T(r)\left(\xi * \phi\right)\right](z-cr). \end{aligned} \tag{3.14}$$

Let us denote by

$$J_\varepsilon = \frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}(\beta(0)-\varepsilon) - (\varepsilon+\delta(0)+\beta(0)), \quad \text{with } 0<\varepsilon<\beta(0).$$

We rewrite the inequality (3.14), for $y < \overline{y}_\varepsilon$, in the form

$$\begin{aligned}J_\varepsilon\phi(y) &\le c\phi'(y) - D[(h*\phi)(z) - \phi(z)]\\ &\quad - \frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}(\beta(0)-\varepsilon)[(1-2Ke^{-\gamma r})\left[T(r)\left(\xi*\phi\right)\right](z-cr) - \phi(y)].\end{aligned}$$

On the other hand, the condition for the existence of positive equilibrium can be written as

$$\frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}\beta(0) - (\delta(0)+\beta(0)) > 0.$$

Then, we can choose $\varepsilon \in (0, \beta(0))$ small enough such that

$$\varepsilon\left(\frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}} + 1\right) < \frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}\beta(0) - (\delta(0)+\beta(0)).$$

Hence, J_ε is positive. As a consequence, for $z < \overline{y}_\varepsilon$,

$$\begin{aligned}0 \le J_\varepsilon \int_{-\infty}^{z}\phi(y)dy &\le c\phi(z) - D\int_{-\infty}^{z}[(h*\phi)(y) - \phi(y)]dy\\ - \frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}(\beta(0)-\varepsilon)&\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left[T(r)\left(\xi*\phi\right)\right](y-cr) - \phi(y)dy < +\infty.\end{aligned}\tag{3.15}$$

Then, for all $z \in \mathbb{R}$, $0 \le \displaystyle\int_{-\infty}^{z}\phi(y)\mathrm{d}y < +\infty$.
(3) By Fubini's theorem, we can check that, for $t < z$,

$$\int_t^z \left[T(r)\left(\xi*\phi\right)\right](y-cr)dy = \int_{-\infty}^{+\infty}\left(\Gamma*\xi\right)(z-cr-y)\int_t^y \phi(s)dsdy.$$

By using the dominated convergence theorem, we obtain

$$\begin{aligned}\int_{-\infty}^{z}\left[T(r)\left(\xi*\phi\right)\right](y-cr)dy &= \int_{-\infty}^{+\infty}\left(\Gamma*\xi\right)(x)\left[\lim_{t\to-\infty}\int_t^z \phi(y-cr-x)dy\right]dx,\\ &= \int_{-\infty}^{+\infty}\left(\Gamma*\xi\right)(x)\int_{-\infty}^{z-cr-x}\phi(s)dsdx,\\ &= \left(\left(\Gamma*\xi\right)*\varphi\right)(z-cr).\end{aligned}$$

(4) By the same techniques as in the proof of (1), we have

$$\begin{aligned}&\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left(\left(\Gamma*\xi\right)*\varphi\right)(y-cr) - \varphi(y)dy\\ &\quad = (1-2Ke^{-\gamma r})\int_{-\infty}^{z}\int_{-\infty}^{+\infty}\left(\Gamma*\xi\right)(l)\left[\varphi(y-cr-l) - \varphi(y)\right]dldy.\end{aligned}$$

Then, we get

$$\begin{aligned}&\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left(\left(\Gamma*\xi\right)*\varphi\right)(y-cr) - \varphi(y)dy\\ &\quad = -(1-2Ke^{-\gamma r})\int_{-\infty}^{z}\int_{-\infty}^{+\infty}(l+cr)\left[\int_0^1 \varphi'(y-\eta(l+cr))d\eta\right]\left(\Gamma*\xi\right)(l)dldy.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left((\Gamma*\xi)*\varphi\right)(y-cr)-\varphi(y)dy\\&\quad=-(1-2Ke^{-\gamma r})\int_{-\infty}^{+\infty}(l+cr)\left[\int_0^1\varphi(z-\eta(l+cr))d\eta\right](\Gamma*\xi)(l)dl.\end{aligned}\tag{3.16}$$

We have proved

$$\int_{-\infty}^{z}|(1-2Ke^{-\gamma r})\left((\Gamma*\xi)*\varphi\right)(y-cr)-\varphi(y)|dy<+\infty.$$

□

Now, let us consider ϕ solution of (3.5) satisfying (3.6). The next proposition establish the asymptotic behavior of the profile $\phi(z)$ when $z\to-\infty$.

Proposition 3.13. *There exists a positive constant $\mu_0<\lambda^+(c)$ such that $\phi(z)=O(e^{\mu_0 z})$ as $z\to-\infty$. Moreover,*

$$\sup_{z\in\mathbb{R}}\left[e^{-\mu_0 z}\phi(z)\right]<+\infty.\tag{3.17}$$

Proof. From the Lemma 3.12 and the inequality (3.15), we have, for $y<\overline{y}_\varepsilon$

$$\begin{aligned}0\le J_\varepsilon\varphi(y)&\le c\phi(y)-D\int_{-\infty}^{y}[(h*\phi)(s)-\phi(s)]ds\\&-\frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}(\beta(0)-\varepsilon)\left[(1-2Ke^{-\gamma r})\left((\Gamma*\xi)*\varphi\right)(y-cr)-\varphi(y)\right]<+\infty.\end{aligned}\tag{3.18}$$

By integrating the both sides of (3.18), from $-\infty$ to $z\le\overline{y}_\varepsilon$, we obtain

$$\begin{aligned}J_\varepsilon\int_{-\infty}^{z}\varphi(y)\mathrm{d}y&\le c\varphi(z)-D\int_{-\infty}^{z}\int_{-\infty}^{y}[(h*\phi)(s)-\phi(s)]dsdy\\&\quad-\frac{2(1-K)e^{-\gamma r}}{1-2Ke^{-\gamma r}}(\beta(0)-\varepsilon)\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left((\Gamma*\xi)*\varphi\right)(y-cr)-\varphi(y)dy.\end{aligned}$$

Thanks to (3.16), we have

$$\begin{aligned}&-\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left((\Gamma*\xi)*\varphi\right)(y-cr)-\varphi(y)dy\\&\quad=(1-2Ke^{-\gamma r})\int_{-\infty}^{+\infty}(l+cr)\left[\int_0^1\varphi(z-\eta(l+cr))d\eta\right](\Gamma*\xi)(l)dl.\end{aligned}$$

The function $\eta\in[0,1]\mapsto(s+cr)\varphi(z-\eta(s+cr))$ is decreasing. Then, we obtain

$$\begin{aligned}&-\int_{-\infty}^{z}(1-2Ke^{-\gamma r})\left((\Gamma*\xi)*\varphi\right)(y-cr)-\varphi(y)dy\\&\quad\le(1-2Ke^{-\gamma r})\varphi(z)\int_{-\infty}^{+\infty}(l+cr)\left(\Gamma*\xi\right)(l)dl.\end{aligned}$$

Moreover, from the fact that $\phi'(z)\ge0$ for $z\in\mathbb{R}$, we have

$$\begin{aligned}\int_{-\infty}^{z}\int_{-\infty}^{y}[(h*\phi)(s)-\phi(s)]dsdy&=\int_{-\infty}^{z}(h*\varphi)(y)-\varphi(y)]dy,\\&=\int_{-\infty}^{z}\int_{-\infty}^{+\infty}h(t)[\varphi(y-t)-\varphi(y)]dtdy,\\&=\int_{-\infty}^{z}\int_{0}^{+\infty}h(t)[\varphi(y+t)+\varphi(y-t)-2\varphi(y)]dtdy,\\&=\int_{-\infty}^{z}\int_{0}^{+\infty}h(t)\int_{y-t}^{y}[\phi(t+s)-\phi(s)]\,dsdtdy,\\&\ge0.\end{aligned}$$

Then,

$$J_\varepsilon \int_{-\infty}^{z} \varphi(y)\mathrm{d}y \le c\varphi(z) + 2(1-K)e^{-\gamma r}(\beta(0)-\varepsilon)\int_{-\infty}^{+\infty} (l+cr)\left(\Gamma * \xi\right)(l)dl\varphi(z).$$

We denote by

$$L_\varepsilon = 2(1-K)e^{-\gamma r}(\beta(0)-\varepsilon)\int_{-\infty}^{+\infty} (l+cr)\left(\Gamma * \xi\right)(l)dl.$$

Hence, for all $z \le \overline{y}_\varepsilon$,

$$J_\varepsilon \int_{-\infty}^{z} \varphi(y)\mathrm{d}y \le (c+L_\varepsilon)\varphi(z). \tag{3.19}$$

This implies that, for all $z \le \overline{y}_\varepsilon$,

$$J_\varepsilon \int_{-\infty}^{0} \varphi(z+y)\mathrm{d}y \le (c+L_\varepsilon)\varphi(z).$$

Then, for all $z \le \overline{y}_\varepsilon$ and all $\eta > 0$, we have

$$J_\varepsilon \int_{-\eta}^{0} \varphi(z+y)\mathrm{d}y \le (c+L_\varepsilon)\varphi(z).$$

By integration by parts, we get for all $z \le \overline{y}_\varepsilon$ and all $\eta > 0$,

$$J_\varepsilon \eta\varphi(z-\eta) \le J_\varepsilon \left[\eta\varphi(z-\eta) + \int_0^\eta y\phi(z-y)\mathrm{d}y\right] \le (c+L_\varepsilon)\varphi(z).$$

We choose $\eta_0 > 0$ large enough such that

$$\theta_0 := \frac{c+L_\varepsilon}{J_\varepsilon \eta_0} \in (0,1).$$

Then, for all $z \le \overline{y}_\varepsilon$,

$$\varphi(z-\eta_0) \le \theta_0 \varphi(z).$$

We put

$$j(x) = e^{-\mu_0 x}\varphi(x), \quad \text{for } x \in \mathbb{R},$$

with

$$\mu_0 = \frac{1}{\eta_0}\ln\left(\frac{1}{\theta_0}\right) = \frac{1}{\eta_0}\ln\left(\frac{J_\varepsilon \eta_0}{c+L_\varepsilon}\right).$$

We know that

$$\lim_{\eta_0 \to +\infty} \frac{1}{\eta_0}\ln\left(\frac{J_\varepsilon \eta_0}{c+L_\varepsilon}\right) = 0.$$

Then, we can choose $\eta_0 > 0$ large enough such that $\mu_0 < \lambda^+(c)$. On the one hand, we have $\varphi(x) = \int_{-\infty}^{x} \phi(y)\mathrm{d}y \le \int_{-\infty}^{0} \phi(y)\mathrm{d}y + N^\star x$, for $x \ge 0$. Then, $\lim_{x\to+\infty} j(x) = 0$. On the other hand, we have for $z \le \overline{y}_\varepsilon$

$$j(z-\eta_0) = e^{-\mu_0(z-\eta_0)}\varphi(z-\eta_0) \le \theta_0 e^{\mu_0\eta_0} e^{-\mu_0 z}\varphi(z).$$

Recall that $\theta_0 e^{\mu_0\eta_0} = 1$. Then, $j(z-\eta_0) \le j(z)$, for all $z \le \overline{y}_\varepsilon$. Consequently, there exists $j_0 > 0$ such that

$$j(z) \le j_0, \quad \text{for all } z \in \mathbb{R}.$$

Thus,

$$\varphi(z) \leq j_0 e^{\mu_0 z}, \quad \text{for all } z \in \mathbb{R}.$$

Thanks to (3.19), we have $\phi(z) \leq \dfrac{c + L_\varepsilon}{d_1}\varphi(z)$. Then, we conclude that there exists $q_0 > 0$ such that

$$\phi(z) \leq q_0 e^{\mu_0 z}, \quad \text{for all } z \in \mathbb{R}.$$

This means that $\varphi(z) = O(e^{\mu_0 z})$ as $z \to -\infty$. The same conclusion could be obtained from the equation of the wave for $\phi(z)$ and we get $\phi(z) = O(e^{\mu_0 z})$ as $z \to -\infty$. Moreover, since $z \mapsto \phi(z)$ and $z \mapsto e^{-\mu_0 z}$ are bounded on $(0, +\infty)$, we obtain

$$\sup_{z\in\mathbb{R}} \left[e^{-\mu_0 z}\phi(z)\right] < +\infty. \qquad \square$$

Remark 3.14. Proposition 3.13 implies that the Laplace transform $\int_{-\infty}^{+\infty} e^{-\lambda z}\phi(z)\mathrm{d}z$ is well defined for all $\lambda \in \mathbb{C}$ such that $0 < \mathrm{Re}(\lambda) < \mu_0$.

Now, we consider the case of non-existence of wave.

Theorem 3.15. *Assume that* (2.1), (2.2), (2.4), (2.5), (2.7) *and* (2.8) *hold and that f and g are twice differentiable on $[0, N^\star]$. For $c \in (0, c^\star)$, there exists no non-trivial traveling front of the equation* (1.4).

Proof. We prove this theorem by contradiction. Let $c \in (0, c^\star)$ and assume that there exists a non-trivial traveling wave front ϕ of the following equation

$$c\phi'(z) = D[(h * \phi)(z) - \phi(z)] - f(\phi(z)) + 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\phi)\right)\right](z - cr), \tag{3.20}$$

with

$$\phi(-\infty) = 0, \quad \phi(+\infty) = N^\star.$$

The remark 3.14 implies that the two sided Laplace transform on $\mathbb{R}$ of ϕ, for all $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}(\lambda) < \mu_0$, is well defined. We define

$$\mathcal{L}(\lambda)(\phi) = \int_{\mathbb{R}} e^{-\lambda z}\phi(z)dz.$$

We know from [28] (page 58) and since $\phi > 0$ that $\lambda \mapsto \mathcal{L}(\lambda)(\phi)$ is analytic for $0 < \mathrm{Re}(\lambda) < \sigma$ with σ is a singularity of $\mathcal{L}(\lambda)(\phi)$. From (3.20), we have

$$\begin{aligned}
&-c\phi'(z) + D[(h*\phi)(z) - \phi(z)] - f'(0)\phi(z) + 2(1-K)e^{-\gamma r}g'(0)\left[T(r)\left(\xi * \phi\right)\right](z - cr) \\
&\quad = -f'(0)\phi(z) + 2(1-K)e^{-\gamma r}g'(0)\left[T(r)\left(\xi * \phi\right)\right](z - cr) + f(\phi(z)) \\
&\qquad - 2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\phi)\right)\right](z - cr).
\end{aligned} \tag{3.21}$$

Let $\lambda \in \mathbb{C}$ such that $0 < \mathrm{Re}(\lambda) < \mu_0 < \lambda^+(c)$. Fubini's theorem implies the following identity

$$\int_{\mathbb{R}} e^{-\lambda z}\left[T(r)\left(\xi * \phi\right)\right](z - cr)dz = \int_{\mathbb{R}} \phi(z)\left[T(r)\left(\xi * e^{\lambda \cdot}\right)\right](-z - cr)dz. \tag{3.22}$$

We have the following expression

$$\int_{\mathbb{R}} e^{-\lambda z}\left[T(r)\left(\xi * \phi\right)\right](z - cr)dz = \frac{(1 - 2Ke^{-\gamma\tau})e^{dr\lambda^2 - cr\lambda}}{1 - 2Ke^{-\gamma r}e^{dr\lambda^2 - cr\lambda}}\mathcal{L}(\lambda)(\phi).$$

By applying the two sided Laplace transform to the equation (3.21), we obtain, by means of (3.22),

$$
\begin{aligned}
-\Delta_c(\lambda)\mathcal{L}(\lambda)(\phi) = \int_{\mathbb{R}} e^{-\lambda z} \big[& f(\phi(z)) - f'(0)\phi(z) \\
& +2(1-K)e^{-\gamma r} g'(0) \left[T(r)\left(\xi * \phi\right)\right](z-cr) \\
& -2(1-K)e^{-\gamma r} \left[T(r)\left(\xi * g(\phi)\right)\right](z-cr) \big]\, dz.
\end{aligned} \tag{3.23}
$$

Recall that $f(0) = g(0) = 0$. Since,

$$\lim_{z \to -\infty} \phi(z) = 0,$$

then, when $z \to -\infty$, Taylor's theorem implies that

$$f(\phi(z)) - f'(0)\phi(z) = O(\phi^2(z)),$$

and

$$
\begin{aligned}
& g'(0) \left[T(r)\left(\xi * \phi\right)\right](z-cr) - \left[T(r)\left(\xi * g(\phi)\right)\right](z-cr) \\
& \quad = \left[T(r)\left(\xi * \left[g'(0)\phi - g(\phi)\right]\right)\right](z-cr) = O(\left[T(r)\left(\xi * \phi^2\right)\right](z-cr)).
\end{aligned}
$$

We add the both aboves equality and we obtain, when $z \to -\infty$,

$$
\begin{aligned}
& f(\phi(z)) - f'(0)\phi(z) + 2(1-K)e^{-\gamma r} g'(0) \left[T(r)\left(\xi * \phi\right)\right](z-cr) \\
& \quad - 2(1-K)e^{-\gamma r} \left[T(r)\left(\xi * g(\phi)\right)\right](z-cr) = O(\phi^2(z) + \left[T(r)\left(\xi * \phi^2\right)\right](z-cr)).
\end{aligned}
$$

Since,

$$\lambda \mapsto \mathcal{L}(\lambda)(\phi),$$

is well defined for $0 < \mathrm{Re}(\lambda) < \mu_0$, then the above equality implies that the right hand side of the equation (3.23)

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-\lambda z} \big[f(\phi(z)) - f'(0)\phi(z) + 2(1-K)e^{-\gamma r} g'(0) \left[T(r)\left(\xi * \phi\right)\right](z-cr) & \\
-2(1-K)e^{-\gamma r} \left[T(r)\left(\xi * g(\phi)\right)\right](z-cr) \big]\, dz, &
\end{aligned}
$$

is well defined for all λ with $0 < \mathrm{Re}(\lambda) < 2\mu_0$. We have by hypothesis that $0 < c < c^\star$. So, $\Delta_c(\lambda)$ has no real root in the interval $(0, \lambda^+(c))$. This leads to the following observation

$$\lambda \mapsto \mathcal{L}(\lambda)(\phi),$$

is well defined for $0 < \mathrm{Re}(\lambda) < \lambda^+(c)$ because $\mathcal{L}(\lambda)$ in the equality (3.23) has no singularity for $0 < \mathrm{Re}(\lambda) < \lambda^+(c)$. In this way, we can arrive at a contradiction with the existence of traveling wave from for $0 < c < c^\star$. In, fact

$$\lim_{\substack{\lambda \to \lambda^+(c) \\ <}} \Delta_c(\lambda) = -\infty.$$

Then, there exists $A > 0$ such that for $\lambda \in \mathbb{R}$ with $A < \lambda < \lambda^+(c)$, we have

$$
\begin{aligned}
& \Delta_c(\lambda)\phi(z) + f(\phi(z)) - f'(0)\phi(z) + 2(1-K)e^{-\gamma r} g'(0) \left[T(r)\left(\xi * \phi\right)\right](z-cr) \\
& \qquad - 2(1-K)e^{-\gamma r} \left[T(r)\left(\xi * g(\phi)\right)\right](z-cr) < 0.
\end{aligned}
$$

Therefore, if we multiply this inequality by $e^{-\lambda z}$ and integrate it, we obtain

$$\int_{\mathbb{R}} e^{-\lambda z}\left[\Delta_c(\lambda)\phi(z) + f(\phi(z)) - f'(0)\phi(z) + 2(1-K)e^{-\gamma r}g'(0)\left[T(r)\left(\xi * \phi\right)\right](z-cr)\right.$$
$$\left. -2(1-K)e^{-\gamma r}\left[T(r)\left(\xi * g(\phi)\right)\right](z-cr)\right] dz < 0.$$

It is impossible with the fact that the equation (3.23) holds. The conclusion is that the equation (1.4) has no fronts connecting the zero equilibrium and the positive equilibrium $N^\star$. □

Let us now give some information about the monotonicity of $c^\star$ with respect to D and d.

Proposition 3.16. *Assume that* (2.4) *and* (2.5) *hold. Let $c^\star > 0$ the minimal wave speed. Then, $c^\star$ is nondecreasing with respect to D and d.*

Proof. Let check the influence of D on the minimal speed $c^\star$. We consider $c^\star := c^\star(D)$ as a function of D. Using the implicit function theorem and Lemma 3.4, we have

$$\Delta_{c^\star(D)}(D, \lambda^\star(D)) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda}\Delta_{c^\star(D)}(D, \lambda(D))\big|_{\lambda(D)=\lambda^\star(D)} = 0.$$

The first equation implies that

$$\frac{\partial}{\partial D}\Delta_{c^\star(D)}(D, \lambda^\star(D)) + \frac{d}{dD}\lambda^\star(D)\frac{\partial}{\partial \lambda}\Delta_{c^\star(D)}(D, \lambda(D))\big|_{\lambda(D)=\lambda^\star(D)}$$
$$+ \frac{d}{dD}c^\star(D)\frac{\partial}{\partial c}\Delta_{c(D)}(D, \lambda^\star(D))\big|_{c(D)=c^\star(D)} = 0.$$

Then,

$$\frac{\partial}{\partial D}\Delta_{c^\star(D)}(D, \lambda^\star(D)) + \frac{d}{dD}c^\star(D)\frac{\partial}{\partial c}\Delta_{c(D)}(D, \lambda^\star(D))\big|_{c(D)=c^\star(D)} = 0.$$

Therefore,

$$\frac{d}{dD}c^\star(D) = \frac{\left[\int_{-\infty}^{+\infty} h(y)e^{-\lambda y}dy - 1\right]}{\lambda^\star(1+\eta)} > 0,$$

with

$$\eta = \frac{2r(1-K)e^{-\gamma r}\beta(0)e^{dr\lambda^{\star 2} - \lambda^\star c^\star r}}{(1-2Ke^{-\gamma r}e^{dr\lambda^{\star 2} - \lambda^\star c^\star r})^2} \geq 0.$$

As above, we can proceed for d. We consider $c^\star := c^\star(d)$ as a function of d. We obtain

$$\frac{d}{dd}c^\star(d) = \frac{2\lambda^\star r(1-K)e^{-\gamma r}\beta(0)e^{dr\lambda^{\star 2} - \lambda^\star c^\star r}}{(1+\eta)(1-2Ke^{-\gamma r}e^{dr\lambda^{\star 2} - \lambda^\star c^\star r})^2} \geq 0.$$ □

We finish by the following result.

Theorem 3.17. *Assume that* (2.1), (2.2), (2.4), (2.5) *and* (2.8) *hold. Let (N, u) be a solution of* (1.4). *If the initial condition (N_0, u_0) is such that $0 \leq N_0(x) < N^\star$, $0 \leq u_0(\theta, x) < u^\star$, for $x \in \mathbb{R}$ and $\theta \in [-r, 0]$, and (N_0, u_0) is null for x outside a bounded interval, then*

$$\lim_{t \to +\infty} \sup_{|x| \geq ct}\left[N(t,x) + u(t,x)\right] = 0, \quad \textit{for } c > c^\star.$$

Proof. The idea of the proof is adapted from [35]. Let $c > c^\star$. We choose $0 < \lambda < \lambda^+(c)$ such that $\Delta_c(\lambda) > 0$. We put $\overline{N}(t,x) = Me^{\lambda(ct-zx)}$ and $\overline{u}(t,x) = M\beta(0)e^{\lambda(ct-zx)}/(1-2Ke^{-\gamma r}e^{dr\lambda^2-cr\lambda})$, for some $M > 0$ and $z = \pm 1$. It is easy to check that

$$\overline{u}(t,x) = \beta(0)\overline{N}(t,x) + 2Ke^{-\gamma r}\int_{-\infty}^{+\infty}\Gamma(r,x-y)\overline{u}(t-r,y)dy.$$

On the other hand, we have

$$\begin{aligned}\frac{\partial \overline{N}(t,x)}{\partial t} &- D[(h*\overline{N})(t,x) - \overline{N}(t,x)] + (\delta(0)+\beta(0))\overline{N}(t,x)\\ &- 2(1-K)e^{-\gamma r}\int_{-\infty}^{+\infty}\Gamma(r,x-y)\overline{u}(t-r,y)dy\\ &\geq \overline{N}(t,x)\Delta_c(\lambda) > 0.\end{aligned}$$

So, $(\overline{N},\overline{u})$ is an upper solution of the linear system

$$\begin{cases}\dfrac{\partial N(t,x)}{\partial t} = D[(h*N)(t,x) - N(t,x)] - (\delta(0)+\beta(0))N(t,x)\\ \qquad\qquad + 2(1-K)e^{-\gamma r}\displaystyle\int_{-\infty}^{+\infty}\Gamma(r,x-y)u(t-r,y)dy,\\ u(t,x) = \beta(0)N(t,x) + 2Ke^{-\gamma r}\displaystyle\int_{-\infty}^{+\infty}\Gamma(r,x-y)u(t-r,y)dy.\end{cases} \tag{3.24}$$

We fix $\tilde{c} \in (c^\star, c)$. From Lemma 3.4, we ensure that there exists $0 < \tilde{\lambda} < \lambda^+(c)$ such that $\Delta_{\tilde{c}}(\tilde{\lambda}) > 0$. Using the fact that (N_0, u_0) has a compact support, we can choose M large enough such that, for $z = \pm 1$ and $(\theta, x) \in [-r, 0] \times \mathbb{R}$,

$$N_0(x) \leq Me^{-\tilde{\lambda}zx} \quad \text{and} \quad u_0(\theta,x) \leq \frac{M\beta(0)e^{\tilde{\lambda}(\tilde{c}\theta - zx)}}{1-2Ke^{-\gamma r}e^{dr\tilde{\lambda}^2-\tilde{c}r\tilde{\lambda}}}.$$

Furthermore, the solution (N,u) of the system (1.4) satisfies

$$\begin{cases}\dfrac{\partial N(t,x)}{\partial t} \leq D[(h*N)(t,x) - N(t,x)] - (\delta(0)+\beta(0))N(t,x)\\ \qquad\qquad + 2(1-K)e^{-\gamma r}\displaystyle\int_{-\infty}^{+\infty}\Gamma(r,x-y)u(t-r,y)dy,\\ u(t,x) \leq \beta(0)N(t,x) + 2Ke^{-\gamma r}\displaystyle\int_{-\infty}^{+\infty}\Gamma(r,x-y)u(t-r,y)dy.\end{cases}$$

Then, (N,u) is a lower solution of (3.24). By the comparison principle, we have, for all $(t,x) \in [0,+\infty) \times \mathbb{R}$,

$$N(t,x) \leq Me^{\tilde{\lambda}(\tilde{c}t-zx)} \quad \text{and} \quad u(t,x) \leq \frac{M\beta(0)e^{\tilde{\lambda}(\tilde{c}t-zx)}}{1-2Ke^{-\gamma r}e^{dr\tilde{\lambda}^2-\tilde{c}r\tilde{\lambda}}}.$$

We put, for $x \neq 0$, $z = x/|x|$ and we get

$$N(t,x) \leq Me^{\tilde{\lambda}(\tilde{c}t-|x|)} \quad \text{and} \quad u(t,x) \leq \frac{M\beta(0)e^{\tilde{\lambda}(\tilde{c}t-|x|)}}{1-2Ke^{-\gamma r}e^{dr\tilde{\lambda}^2-\tilde{c}r\tilde{\lambda}}}.$$

This proves the result. □

4 Summary

In this paper, we proposed and analyzed a new mathematical model describing a cell population dynamics. The model is a coupled nonlocal diffusion and difference system, which is a generalization of the model studied in [1] to a model with nonlocal diffusion. Our interest was to deal with the properties of traveling waves of such system. The main difficulty was to combine two different theories: the nonlocal diffusion equations and the difference equations with delay theories. The problem of existence of traveling wave fronts was treated by using the Schauder's fixed point theorem with the method based on the construction of upper and lower solutions. In conclusion, it was clarified that there exists a critical threshold $c^\star$ for which the existence of traveling wave fronts is guaranteed for a speed $c \geq c^\star$ and that no monotone wave exists when $c < c^\star$. Moreover, we established an asymptotic behavior of the profile of the wave near $-\infty$. We proved also that the nonlocality, through the diffusions rates, can increase the minimal wave speed. In the forthcoming works, we will continue to analyze the influence of the parameters on the wave velocity. Future work will include also the stability of wave either for the local or nonlocal diffusion.

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