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Traveling waves for a diffusive SIR-B epidemic model with multiple transmission pathways

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Abstract. In this work, we consider a diffusive SIR-B epidemic model with multiple transmission pathways and saturating incidence rates. We first present the explicit formula of the basic reproduction number $\mathcal{R}_0$. Then we show that if $\mathcal{R}_0 > 1$, there exists a constant $c^* > 0$ such that the system admits traveling wave solutions connecting the disease-free equilibrium and endemic equilibrium with speed c if and only if $c \geq c^*$. Since the system does not admit the comparison principle, we appeal to the standard Schauder's fixed point theorem to prove the existence of traveling waves. Moreover, a suitable Lyapunov function is constructed to prove the upward convergence of traveling waves.

Keywords: reaction-diffusion equations, saturation incidence rates, upper-lower solutions, traveling waves, minimum wave speed.

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1 Introduction

There have been intensive studies about the existence of traveling waves and the minimum wave speed for various epidemic models, which are of great importance for the prediction and control of infectious diseases. For the transmission of communicable diseases, most of epidemic models are proposed based on the classic susceptible-infected-recovered (SIR) epidemic model [13], whose basic assumption is that the disease is only transmitted directly by human-to-human contacts. This assumption is reasonable for many viral diseases (e.g., measles, influenza). However, in addition to direct human-to-human transmission pathway, cholera and many other waterborne diseases are mainly transmitted by indirect environment-to-human contacts via ingestion of contaminated water or food [5,8]. As a consequence, mathematical modeling and dynamical analysis for infectious diseases with multiple transmission pathways have attracted much attention of researchers. We refer to [6,8,17,18,22,25] for ordinary differential equations (ODE), and [26,27,33] for diffusive PDE models with multiple transmission pathways.

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Based on the basic SIR model, Codeço proposes the following SIR-B epidemic model to describe the transmission of cholera [3], which includes a fourth compartment for bacterial concentration in water

$$\begin{cases} \dfrac{dS}{dt} = \mu(N_0 - S) - \beta f(B)S, \\ \dfrac{dI}{dt} = \beta f(B)S - (\sigma + \mu)I, \\ \dfrac{dB}{dt} = eI - (\mu_B - \pi_B)B, \\ \dfrac{dR}{dt} = \sigma I - \mu R, \end{cases} \tag{1.1}$$

where $\beta f(B)S$ is the incidence function for indirect environment-to-human transmission. For more generalizations of Codeço's model, we refer readers to recent works [8, 21]. It is clear that Codeço's model ignores the direct human-to-human transmission pathway, which also plays an important role in the transmission of waterborne diseases [4].

In this work, we intend to study the following diffusive SIR-B epidemic model, which describes the transmission of waterborne disease with direct and indirect transmissions

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_1 \dfrac{\partial^2 S}{\partial x^2} + \mu_H(N_0 - S) - \beta_1 S f_1(I) - \beta_2 S f_2(B), \\ \dfrac{\partial I}{\partial t} = d_2 \dfrac{\partial^2 I}{\partial x^2} + \beta_1 S f_1(I) + \beta_2 S f_2(B) - (\sigma + \mu_H)I, \\ \dfrac{\partial B}{\partial t} = d_3 \dfrac{\partial^2 B}{\partial x^2} + \eta I - (\mu_B - \pi_B)B, \\ \dfrac{\partial R}{\partial t} = d_4 \dfrac{\partial^2 R}{\partial x^2} + \sigma I - \mu_H R. \end{cases} \tag{1.2}$$

Note that model (1.2) is an extension of Codeço's model. The variables $S(x,t)$, $I(x,t)$, $R(x,t)$, and $B(x,t)$ represent, respectively, the density of susceptible, infected, recovered individuals, and the bacterial concentration in contaminated environment at location $x \in (-\infty, \infty)$ and time $t \in [0, \infty)$; the constant N_0 is the total population size at time $t = 0$; $d_i > 0, i = 1,2,3,4$ is the diffusion coefficient for populations; μ_H is the natural birth/death rate of humans; σ is the recovery rate of populations; $\mu_B > \pi_B$ are loss and growth rates of the bacteria; β_1, β_2 are the contact rate of the individual with the infectious and the contaminated environment, respectively; η is the contribution rate of each infectious individual to the population of bacteria; $\beta_1 S f_1(I)$, $\beta_2 S f_2(B)$ are density-dependence incidence functions for direct and indirect transmissions. For more details of the biological background of (1.2), we refer to [3,5,21,25,26] and references therein.

To model the spread of an infectious disease with multiple transmission pathways, one of crucial issues is how to model the incidence rates of the disease, which depend on both the population behavior and the infectivity of the disease. Bilinear incidence rates $\beta_1 SI$ and/or $\beta_2 SB$ have been frequently used, see for example [3, 17, 21, 22, 27]. However, nonlinearity in the incidence rates has been observed in the transmission of many diseases. For example, based on the careful study of the cholera epidemic spread in Bari in 1973, Capasso and Serio [1] introduced a nonlinear saturated incidence rate $\frac{SI}{1+aI}$, $a > 0$, into epidemic models. The saturation incidence rate is more realistic than the bilinear, which takes into account the saturation phenomena in reality. Therefore, we will focus on the saturation incidence rates, and hereafter we assume

$$f_1(I) = \frac{I}{1 + aI}, \qquad f_2(B) = \frac{B}{K_B + B},$$

where constant $K_B > 0$ is half saturation concentration of the bacteria in the contaminated environment [3]. Another crucial issue is how to model the relative magnitude of the direct contact rate to the indirect transmission rate. According to [5, 8], the waterborne disease is mainly transmitted by indirect environment-to-human contacts, and clean water provision may reduce, even stop, the disease transmission. Hence, in this work, we assume the direct contact rate $\beta_1 < (\sigma + \mu_H)/N_0$ relatively small such that the epidemic may not happen in the absence of indirect waterborne transmission.

So far, many results have been done on the threshold dynamics of SIR-B models with respect to the so-called basic reproduction number $\mathcal{R}_0$. For example, see [3, 21, 25] for ODE systems, and recent works [26, 27, 35] for diffusive SIR-B models. However, due to the complexity of the model, little work is to study the existence of traveling waves for the diffusive SIR-B model with multiple transmission pathways. In the absence of bacterium ($B(x,t) \equiv 0$), model (1.2) is the standard SIR epidemic model. For the existence of traveling wave solutions for SIR models with standard or saturation incidence rates, we refer to [2, 7, 24, 28, 31]. Using the Schauder's fixed point theorem, the authors in [7, 31] proved the existence of traveling waves for diffusive SIR models with time delay and saturation incidence rates. In the case of $\beta_1 = 0$ in (1.2), from a mathematical point of view, the system is essentially a diffusive SEIR epidemic model. Based on Schauder's fixed point theorem and Laplace transform, the authors of recent works [20,32] establish the existence and nonexistence of traveling waves for diffusive SEIR models with standard and saturation incidence rates, respectively.

However, in the case of $\beta_i \neq 0, i = 1, 2$, system (1.2) becomes more complicated, and it is essentially different with SIR or SEIR model. As far as we know, the existence of traveling waves and the minimum wave speed of (1.2) has not been studied in literatures. Since the solution semiflow associated with (1.2) does not admit the comparison principle, the powerful theory [14,15] for monotone dynamical systems cannot be applied. To overcome the difficulty due to the lack of monotonicity, we appeal to the standard Schauder's fixed point theorem (see e.g. [11, 16]) for an equivalent non-monotone solution operator, where upper-lower solutions are constructed for the verification of a suitable invariant convex set for the solution operator.

Note that the spatially homogeneous system of (1.2) is given by the following ODE system:

$$\begin{cases} \dfrac{dS}{dt} = \mu_H(N_0 - S) - \beta_1 S f_1(I) - \beta_2 S f_2(B), \\ \dfrac{dI}{dt} = \beta_1 S f_1(I) + \beta_2 S f_2(B) - (\sigma + \mu_H) I, \\ \dfrac{dB}{dt} = \eta I - (\mu_B - \pi_B) B, \\ \dfrac{dR}{dt} = \sigma I - \mu_H R. \end{cases} \tag{1.3}$$

Using the linearization of (1.3) at disease-free equilibrium $(N_0, 0, 0, 0)$ and the next-generation matrix theory given in [23], we can verify that the basic reproduction number $\mathcal{R}_0$ of (1.3) is given by

$$\mathcal{R}_0 = \frac{\beta_1 N_0}{\sigma + \mu_H} + \frac{\beta_2 N_0 \eta}{(\mu_B - \pi_B) K_B (\sigma + \mu_H)} =: \mathcal{R}_0^I + \mathcal{R}_0^B,$$

where

$$\mathcal{R}_0^I = \frac{\beta_1 N_0}{\sigma + \mu_H}$$

is the basic reproduction number induced by direct human-to-human transmission ($\beta_2 = 0$),

and
$$\mathcal{R}_0^B = \frac{\beta_2 N_0 \eta}{(\mu_B - \pi_B)K_B(\sigma + \mu_H)}$$
is the basic reproduction number induced by the indirect environment-to-human transmission ($\beta_1 = 0$). Then by similar arguments as given in [25], we have the following threshold dynamics for (1.3) with respect to $\mathcal{R}_0$.

Proposition 1.1. *If $\mathcal{R}_0 < 1$, then system (1.3) admits only one non-negative disease-free equilibrium (DFE), which is globally asymptotically stable. If $\mathcal{R}_0 > 1$, then the DFE becomes unstable, and system (1.3) has a unique endemic equilibrium, which is locally asymptotically stable.*

The main purpose of this work is to investigate the existence of traveling waves connecting the DFE and endemic equilibrium. Hence, we further assume $\mathcal{R}_0 > 1$ in the remainder of this paper. Our study is mainly motivated by recent works [7,31,32], where the Schauder's fixed point theorem is applied to determine the existence of traveling waves and the minimal wave speed for SIR/SEIR model with saturation incidence rate. Note that our model (1.2) is more complex than the standard SIR/SEIR model, hence the construction of upper-lower solutions is different with that given in [7,31,32]. For the construction idea of such vector-value upper-lower solutions, we refer to [29,32] and other related works.

The rest of this paper is organized as follows. In Section 2, by solving an eigenvalue problem, we establish the existence of the critical value c^*. Then we construct and verify a pair of upper and lower solutions for the associated wave equations. In Section 3, we first construct a closed and convex set, in which we apply the Schauder's fixed point theorem to an equivalent non-monotone solution operator to obtain the existence of traveling waves with $c > c^*$. Moreover, a suitable Lyapunov function is constructed to prove the upward convergence of traveling waves. The existence of traveling waves with $c = c^*$ also obtained by a limiting argument. Finally, a short conclusion and discussion finishes this paper.

2 Upper and lower solutions

In this section, we first determine the existence of critical value c^* by solving an eigenvalue problem. Then we construct and verify a pair of upper and lower solutions for wave equations with $c > c^*$, which is used to construct a closed and convex set for the Schauder's fixed point theorem.

Note that the variable $R(x,t)$ in (1.2) does not appear in the first three equations. Thus, it suffices to consider the closed subsystem for variables S, I, and B. Let $(S, I, B) = (u_1, u_2, u_3)$ for the simplicity of notations, Then we have
$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + \mu_H(N_0 - u_1) - \beta_1 u_1 f_1(u_2) - \beta_2 u_1 f_2(u_3), \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + \beta_1 u_1 f_1(u_2) + \beta_2 u_1 f_2(u_3) - (\sigma + \mu_H) u_2, \\ \dfrac{\partial u_3}{\partial t} = d_3 \dfrac{\partial^2 u_3}{\partial x^2} + \eta u_2 - (\mu_B - \pi_B) u_3. \end{cases} \tag{2.1}$$
From proposition 1.1, we know that, if $\mathcal{R}_0 > 1$, system (2.1) admits a DFE $E_0 := (N_0, 0, 0)$, and a unique endemic equilibrium $E^* = (u_1^*, u_2^*, u_3^*)$. With the assumption of $\mathcal{R}_0 > 1$, we are interested in the existence of monostable traveling waves connecting the disease-free equilibrium E_0 and the endemic equilibrium E^*, which describe the propagation of the disease from an initial disease-free steady state to the endemic steady state.

Definition 2.1. A traveling wave solution of (2.1) connecting E_0 to E^* with speed $c > 0$ is a nonnegative solution of (2.1) with the following form

$$(u_1(x,t), u_2(x,t), u_3(x,t)) = (U_1(s), U_2(s), U_3(s)) := U(s), \qquad s = x + ct,$$

and satisfies

$$U(-\infty) = E_0,\ U(+\infty) = E^*. \tag{2.2}$$

A constant $c^* > 0$ is called the minimum wave speed if system (2.1) admits a traveling wave solution with speed c if and only if $c \geq c^*$.

Substituting the wave profile $U(s)$ defined above to system (2.1), we get the following second order wave equations:

$$\begin{cases} cU_1' = d_1 U_1'' + \mu_H (N_0 - U_1) - \beta_1 U_1 f_1(U_2) - \beta_2 U_1 f_2(U_3), \\ cU_2' = d_2 U_2'' + \beta_1 U_1 f_1(U_2) + \beta_2 U_1 f_2(U_3) - (\sigma + \mu_H) U_2, \\ cU_3' = d_3 U_3'' + \eta U_2 - (\mu_B - \pi_B) U_3, \end{cases} \tag{2.3}$$

where $'$ denotes the derivative with respect to variable s. Then the existence of traveling waves of (2.1) is equivalent to the existence of the nonnegative solutions U of (2.3) with condition (2.2). For the simplicity of notations, we define

$$\begin{aligned} G_1(u_1, u_2, u_3) &:= \mu_H (N_0 - u_1) - \beta_1 u_1 f_1(u_2) - \beta_2 u_1 f_2(u_3), \\ G_2(u_1, u_2, u_3) &:= \beta_1 u_1 f_1(u_2) + \beta_2 u_1 f_2(u_3) - (\sigma + \mu_H) u_2, \\ G_3(u_1, u_2, u_3) &:= \eta u_2 - (\mu_B - \pi_B) u_3. \end{aligned}$$

2.1 Eigenvalues problem

Linearizing system (2.3) at $E_0 = (N_0, 0, 0)$, we get the following linearization:

$$\begin{cases} c\phi_1'(s) = d_1 \phi_1''(s) - \mu_H \phi_1(s) - \beta_1 N_0 \phi_2(s) - \dfrac{\beta_2 N_0}{K_B} \phi_3(s), \\ c\phi_2'(s) = d_2 \phi_2''(s) + \beta_1 N_0 \phi_2(s) + \dfrac{\beta_2 N_0}{K_B} \phi_3(s) - (\sigma + \mu_H) \phi_2(s), \\ c\phi_3'(s) = d_3 \phi_3''(s) + \eta \phi_2(s) - (\mu_B - \pi_B) \phi_3(s). \end{cases} \tag{2.4}$$

Note that the last two equations of (2.4) are closed. Plugging $(\phi_2, \phi_3) = e^{\lambda s}(\kappa_2, \kappa_3)$ into the last two equations of (2.4), we get the following eigenvalue problem

$$A_\lambda \mathcal{K} = 0,$$

where

$$A_\lambda = \begin{bmatrix} p_2(\lambda) & \beta_2 N_0 / K_B \\ \eta & p_3(\lambda) \end{bmatrix}, \qquad \mathcal{K} = \begin{bmatrix} \kappa_2 \\ \kappa_3 \end{bmatrix},$$

$$p_2(\lambda) = d_2 \lambda^2 - c\lambda - (\sigma + \mu_H - \beta_1 N_0), \qquad p_3(\lambda) = d_3 \lambda^2 - c\lambda - (\mu_B - \pi_B).$$

Letting $C(\lambda) := \det(A_\lambda) = 0$ be the characteristic equation, then we get the following equation:

$$P_c(\lambda) - \beta_2 N_0 \eta / K_B = 0, \tag{2.5}$$

where $P_c(\lambda) = p_2(\lambda) p_3(\lambda)$. Now we need to consider the roots of the following fourth order polynomial equation

$$P_c(\lambda) = \beta_2 N_0 \eta / K_B. \tag{2.6}$$

Lemma 2.2. *Let $\mathcal{R}_0 > 1$, then there exists a positive constant c^* such that the following statements are valid:*

(i) If $c > c^*$, (2.6) has four distinct real roots, where one is negative, and the others are positive. Let λ_1 be the smallest positive one, then for $\epsilon > 0$ small enough, we have

$$p_2(\lambda_1 + \epsilon) < 0, \qquad p_3(\lambda_1 + \epsilon) < 0, \qquad P_c(\lambda_1 + \epsilon) > \beta_2 N_0 \eta / K_B.$$

(ii) If $c = c^*$, (2.6) has one negative and two positive real roots, where the smaller positive one is repeated.

(iii) If $0 < c < c^*$, (2.6) has two distinct real roots, and two conjugate complex roots.

Proof. Since $\mathcal{R}_0 > 1$, we have $P_c(0) = (\sigma + \mu_H - \beta_1 N_0)(\mu_B - \pi_B) < \beta_2 N_0 \eta / K_B$, which implies that zero is not a root of (2.6) for any $c > 0$. Denote the roots of $P_c(\lambda) = 0$ to be

$$\lambda_2^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_2(\sigma + \mu_H - \beta_1 N_0)}}{2d_2}, \qquad \lambda_3^{\pm} = \frac{c \pm \sqrt{c^2 + 4d_3(\mu_B - \pi_B)}}{2d_3},$$

then they are all real. Setting

$$\lambda_m^{\pm} = \min\{\lambda_2^{\pm}, \lambda_3^{\pm}\}, \qquad \lambda_M^{\pm} = \max\{\lambda_2^{\pm}, \lambda_3^{\pm}\},$$

then we have $\lambda_M^- < 0 < \lambda_m^+$. Since $P_c(\pm\infty) = +\infty$, the mean value theorem implies that (2.6) has one negative root in $(-\infty, \lambda_m^-)$, and one positive root in (λ_M^+, ∞). Moreover, we can verify that $P_c(\lambda) < 0 < \eta\beta_2 N_0 / K_B$ for all $\lambda \in (\lambda_m^-, \lambda_M^-) \cup (\lambda_m^+, \lambda_M^+)$. Now we consider the interval $I = (\lambda_M^-, \lambda_m^+)$, in which $P_c(\lambda) > 0$ and $p_i(\lambda) < 0, i = 2, 3$. It is easy to observe λ_M^- is strictly decreasing and λ_m^+ is increasing with respect to c. For fixed $\lambda \in I$, we have

$$\frac{dP_c(\lambda)}{dc} = -\lambda(p_2(\lambda) + p_3(\lambda)),$$

which implies that $P_c(\lambda)$ is strictly increasing with respect to c for fixed $\lambda \in (0, \lambda_m^+)$, and it is decreasing for $\lambda \in (\lambda_M^-, 0)$. Moreover, for $c = 0$, we can verify

$$\max_{\lambda \in I} P_0(\lambda) = P_0(0) = (\sigma + \mu_H - \beta_1 N_0)(\mu_B - \pi_B) < \beta_2 N_0 \eta / K_B,$$

then the monotonicity of $P_c(\lambda)$ with respect to c implies that (2.6) has no real root in $(\lambda_M^-, 0)$ for any $c > 0$. Denoting λ_m^+ to be λ_{m0}^+ if $c = 0$, then for any $\lambda \in (0, \lambda_{m0}^+)$, we have

$$\lim_{c \to +\infty} P_c(\lambda) = +\infty > \beta_2 N_0 \eta / K_B.$$

Then the monotonicity of $P_c(\lambda)$ with respect to c for $\lambda \in I$ implies that there exists a constant $c^* > 0$ such that (2.6) has two positive real roots in I for $c > c^*$, no real root in I for $0 < c < c^*$, and there is a positive repeated root in I for $c = c^*$, which implies statements (i)–(iii) hold. □

To ensure the existence of positive solution U of (2.3) satisfying condition (2.2), it is necessary to ask the eigenvalues of (2.5) are all real. Otherwise, a spiral solution near E_0 will destroy the positivity of the state variable $U_i, i = 1, 2, 3$. Then Lemma 2.2 (c) implies that system (2.1) does not admit traveling wave solution for $0 < c < c^*$. For the nonexistence of traveling waves for $0 < c < c^*$, a similar argument as given in [32, Theorem 3.3] also could be applied. Now we mainly focus on the existence of traveling waves of (2.1) for $c \geq c^*$.

2.2 Construction of upper and lower solutions

To prove the existence of traveling waves for $c > c^*$, we need to construct suitable vector-value upper and lower solutions for (2.3). Let λ_1 be given in Lemma 2.2 and $\kappa := -\eta/p_3(\lambda_1) > 0$. Then we define the following continuous functions:

$$\begin{aligned}
&\overline{U}_1(s) = N_0, && \underline{U}_1(s) = \max\{N_0 - \sigma_1 e^{\alpha s}, 0\},\\
&\overline{U}_2(s) = \min\{e^{\lambda_1 s}, N^*\}, && \underline{U}_2(s) = \max\{e^{\lambda_1 s}(1 - \sigma_2 e^{\epsilon s}), 0\},\\
&\overline{U}_3(s) = \min\{\kappa e^{\lambda_1 s}, \omega N^*\}, && \underline{U}_3(s) = \max\{\kappa e^{\lambda_1 s}(1 - \sigma_3 e^{\epsilon s}), 0\},
\end{aligned}$$

where positive constants $N^*, \alpha, \omega, \epsilon, \sigma_i, i = 1, 2, 3$ will be determined. The we have the following lemmas.

Lemma 2.3. *There exists a positive constant N^* (> 1, large enough) such that the functions $\overline{U_2}$ and $\overline{U_3}$ satisfies inequalities*

$$c\overline{U}_2' \geq d_2\overline{U}_2'' + G_2(N_0, \overline{U}_2, \overline{U}_3), \quad \text{for } s \neq \bar{s}_2, \tag{2.7}$$

$$c\overline{U}_3' \geq d_3\overline{U}_3'' + G_3(N_0, \overline{U}_2, \overline{U}_3), \quad \text{for } s \neq \bar{s}_3, \tag{2.8}$$

where $\bar{s}_2 = \frac{\ln N^*}{\lambda_1}$ *and* $\bar{s}_3 = \frac{\ln \frac{\omega N^*}{\kappa}}{\lambda_1}$.

Proof. We may assume $\bar{s}_2 \leq \bar{s}_3$. The case of $\bar{s}_2 > \bar{s}_3$ is similar. If $s < \bar{s}_2$, then $\overline{U}_2(x) = e^{\lambda_1 x}$, $\overline{U}_3(x) = \kappa e^{\lambda_1 x}$ and

$$\begin{aligned}
&d_2\overline{U}_2'' - c\overline{U}_2' + G_2(N_0, \overline{U}_2, \overline{U}_3)\\
&\quad = d_2\lambda_1^2 e^{\lambda_1 x} - c\lambda_1 e^{\lambda_1 x} + \beta_1 N_0 f_1(\overline{U}_2) + \beta_2 f_2(\overline{U}_3) N_0 - (\sigma + \mu_H)\overline{U}_2\\
&\quad \leq e^{\lambda_1 x}\left[d_2\lambda_1^2 - c\lambda_1 + \beta_1 N_0 - (\sigma + \mu_H) + \beta_2 N_0 \frac{\kappa}{K_B}\right] + \beta_2 N_0 f_2(\overline{U}_3) - \beta_2 N_0 \frac{\kappa}{K_B} e^{\lambda_1 x}\\
&\quad = \beta_2 N_0 f_2(\overline{U}_3) - \beta_2 N_0 \frac{\kappa}{K_B} e^{\lambda_1 x} \leq 0.
\end{aligned} \tag{2.9}$$

Similar to (2.9), it can be concluded that

$$\begin{aligned}
d_3\overline{U}_3'' - c\overline{U}_3' + G_3(N_0, \overline{U}_2, \overline{U}_3) &= d_3\kappa\lambda_1^2 e^{\lambda_1 x} - c\kappa\lambda_1 e^{\lambda_1 x} + \eta e^{\lambda_1 x} - (\mu_B - \pi_B)\kappa e^{\lambda_1 x}\\
&= [d_3\lambda_1^2 - c\lambda_1 + \frac{\eta}{\kappa} - (\mu_B - \pi_B)]\kappa e^{\lambda_1 x} = 0.
\end{aligned}$$

If $s > \bar{s}_3$, then $\overline{U}_2 = N^*, \overline{U}_3 = \omega N^*$, and

$$\begin{aligned}
d_2\overline{U}_2'' - c\overline{U}_2' + G_2(N_0, \overline{U}_2, \overline{U}_3) &= \beta_1 N_0 \frac{N^*}{1 + aN^*} + \beta_2 N_0 \frac{\omega N^*}{K_B + \omega N^*} - (\sigma + \mu_H)N^*\\
&\leq \beta_1 N_0/a + \beta_2 N_0 - (\sigma + \mu_H)N^* \leq 0,
\end{aligned} \tag{2.10}$$

where

$$N^* > \frac{\beta_2 N_0 + \beta_1 N_0/a}{\sigma + \mu_H}.$$

It is easy to see that

$$d_3\overline{U}_3'' - c\overline{U}_3' + G_3(N_0, \overline{U}_2, \overline{U}_3) = \eta N^* - (\mu_B - \pi_B)\omega N^* = 0,$$

where $\omega = \frac{\eta}{\mu_B - \pi_B} > 0$.

If $\bar{s}_2 \leq s \leq \bar{s}_3$, it can be similarly shown that (2.7) and (2.8) hold. This completes the proof. □

Lemma 2.4. *For*

$$0<\alpha<\frac{1}{2}\min\left\{\frac{c}{d_1},\lambda_1\right\},\qquad \sigma_1>\max\left\{N_0,\frac{\beta_2 N_0\frac{\kappa}{K_B}+\beta_1 N_0}{(c-d_1\alpha)\alpha}\right\},$$

the function $\underline{U}_1(s)$ satisfies inequality

$$c\underline{U}_1'\le d_1\underline{U}_1''+G_1(\underline{U}_1,\overline{U}_2,\overline{U}_3)$$

for any $s\neq s_1:=\frac{1}{\alpha}\ln\frac{N_0}{\sigma_1}$.

Proof. Without loss of generality, we may assume σ_1 is large enough such that $s_1<0$ and $s_1<\bar{s}_2\le\bar{s}_3$. If $s>s_1$, then $\underline{U}_1(s)=0$, the inequality holds.
If $s<s_1$, then $\underline{U}_1(s)=N_0-\sigma_1 e^{\alpha s}$, $\overline{U}_2(s)=e^{\lambda_1 s}$, $\overline{U}_3(s)=\kappa e^{\lambda_1 s}$. Hence we have

$$\begin{aligned}
&d_1\underline{U}_1''-c\underline{U}_1'+G_1(\underline{U}_1,\overline{U}_2,\overline{U}_3)\\
&\quad=-d_1\sigma_1\alpha^2e^{\alpha s}+c\sigma_1\alpha e^{\alpha s}+\mu_H\sigma_1e^{\alpha s}-\beta_1(N_0-\sigma e^{\alpha s})f_1(\overline{U}_2)-\beta_2(N_0-\sigma e^{\alpha s})f_2(\overline{U}_3)\\
&\quad\ge(c-d_1\alpha)\sigma_1\alpha e^{\alpha s}-\beta_1N_0f_1(\overline{U}_2)-\beta_2N_0f_2(\overline{U}_3)\\
&\quad\ge\left[(c-d_1\alpha)\sigma_1\alpha-[\beta_1N_0+\frac{\beta_2N_0\kappa}{K_B}]e^{(\lambda_1-\alpha)s}\right]e^{\alpha s}\ge0,
\end{aligned}\tag{2.11}$$

where $(\lambda_1-\alpha)s<0$ and $\sigma_1>\max\left\{N_0,\frac{\beta_2N_0\frac{\kappa}{K_B}+\beta_1N_0}{(c-d_1\alpha)\alpha}\right\}$. This completes the proof. □

Lemma 2.5. *There exist positive constants ϵ (small enough), σ_2 and σ_3 (large enough) such that functions $\underline{U}_2(s)$ and $\underline{U}_3(s)$ satisfy inequalities*

$$c\underline{U}_2'\le d_2\underline{U}_2''+G_2(\underline{U}_1,\underline{U}_2,\underline{U}_3),\quad \text{for } s\neq\underline{s}_2:=-\frac{1}{\epsilon}\ln\sigma_2,\tag{2.12}$$

$$c\underline{U}_3'\le d_3\underline{U}_3''+G_3(\underline{U}_1,\underline{U}_2,\underline{U}_3),\quad \text{for } s\neq\underline{s}_3:=-\frac{1}{\epsilon}\ln\sigma_3.\tag{2.13}$$

Proof. Without loss of generality we suppose $\underline{s}_3<\underline{s}_2$, which implies $\sigma_2<\sigma_3$. It is clear that (2.12) holds for $s>\underline{s}_2$ and (2.13) holds for $s>\underline{s}_3$. Since

$$\lim_{s\to-\infty}\underline{U}_1(s)=N_0,\qquad \lim_{s\to-\infty}\underline{U}_i(s)=0,\qquad \lim_{\epsilon\to0^+,\ \sigma_i\to+\infty}\underline{s}_i=-\infty,\qquad i=2,3,$$

we can set ϵ small enough and σ_2,σ_3 large enough such that $\underline{s}_2<0$ and $\underline{s}_2<s_1$. In the remainder of this proof we assume $s\le\underline{s}_2$, which implies

$$\underline{U}_1(s)=N_0-\sigma_1e^{\alpha s},\qquad \underline{U}_2(s)=e^{\lambda_1 s}(1-\sigma_2e^{\epsilon s}),\qquad \underline{U}_3(s)\ge\kappa e^{\lambda_1 s}(1-\sigma_3e^{\epsilon s})=:\underline{\underline{U}}_3,$$

where $\underline{U}_3=\underline{\underline{U}}_3$ if and only if $s\le\underline{s}_3$. By Taylor's theorem we have

$$\begin{aligned}
G_2(\underline{U}_1,\underline{U}_2,\underline{\underline{U}}_3)&=\beta_1N_0\underline{U}_2+\beta_1\frac{(\underline{U}_1-N_0)\underline{U}_2}{(1+a\theta_2\underline{U}_2)^2}-\frac{a\beta_1\theta_2\underline{U}_1(\underline{U}_2)^2}{(1+\theta_2a\underline{U}_2)^3}-(\sigma+\mu_H)\underline{U}_2\\
&\quad+\beta_2\frac{N_0\underline{\underline{U}}_3}{K_B}+\beta_2(\underline{U}_1-N_0)\underline{\underline{U}}_3\frac{K_B}{(K_B+\theta_1\underline{\underline{U}}_3)^2}-\frac{\beta_2\theta_1K_B\underline{U}_1\underline{\underline{U}}_3^{\,2}}{(K_B+\theta_1\underline{\underline{U}}_3)^3},
\end{aligned}\tag{2.14}$$

where $0 < \theta_i < 1,\ i = 1, 2$. Therefore

$$
\begin{aligned}
& e^{-\lambda_1 s}[d_2\underline{U}_2'' - c\underline{U}_2' + G_2(\underline{U}_1, \underline{U}_2, \underline{U}_3)] \\
& \quad \geq e^{-\lambda_1 s}[d_2\underline{U}_2'' - c\underline{U}_2' + G_2(\underline{U}_1, \underline{U}_2, \underline{U}_3)] \\
& \quad \geq d_2\lambda_1^2 - c\lambda_1 + \beta_1 N_0 - (\sigma + \mu_H) + \beta_2 N_0 \frac{\kappa}{K_B} - [d_2(\lambda_1 + \epsilon)^2 - c(\lambda_1 + \epsilon) \\
& \qquad - (\sigma + \mu_H) + \beta_1 N_0]\sigma_2 e^{\epsilon s} - \frac{\beta_2 N_0 \kappa \sigma_3}{K_B} e^{\epsilon s} - R_1(s)\sigma_1 e^{\alpha s} - R_2(s) e^{\lambda_1 s} \\
& \quad = -[d_2(\lambda_1 + \epsilon)^2 - c(\lambda_1 + \epsilon) - (\sigma + \mu_H) + \beta_1 N_0]\sigma_2 e^{\epsilon s} - \frac{\beta_2 N_0 \kappa \sigma_3}{K_B} e^{\epsilon s} \\
& \qquad - R_1(s)\sigma_1 e^{\alpha s} - R_2(s) e^{\lambda_1 s} \\
& \quad = -p_2(\lambda_1 + \epsilon)\sigma_2 e^{\epsilon s} - \frac{\beta_2 N_0}{K_B}\kappa\sigma_3 e^{\epsilon s} - R_1(s)\sigma_1 e^{\alpha s} - R_2(s) e^{\lambda_1 s},
\end{aligned} \tag{2.15}
$$

where

$$
\begin{aligned}
R_1(s) &= \kappa(1 - \sigma_3 e^{\epsilon s}) \frac{\beta_2 K_B}{(K_B + \theta_1 \underline{U}_3)^2} + \frac{\beta_1(1 - \sigma_2 e^{\epsilon s})}{(1 + a\theta_2 \underline{U}_2)^2}, \\
R_2(s) &= \kappa^2(1 - \sigma_3 e^{\epsilon s})^2 \frac{\beta_2 K_B \theta_1 N_0}{(K_B + \theta_1 \underline{U}_3)^3} + \frac{a\beta_1\theta_2 N_0(1 - \sigma_2 e^{\epsilon s})^2}{(1 + a\theta_2 \underline{U}_2)^3}.
\end{aligned}
$$

For $s < \underline{s}_2$, it is easy to show that

$$
0 \leq 1 - \sigma_2 e^{\epsilon s} \leq 1, \qquad 1 - \frac{\sigma_3}{\sigma_2} \leq 1 - \sigma_3 e^{\epsilon s} \leq 1. \tag{2.16}
$$

Similar to (2.15), we have

$$
e^{-\lambda_1 s}[d_3\underline{U}_3'' - c\underline{U}_3' + G_3(\underline{U}_1, \underline{U}_2, \underline{U}_3)] = -\eta\sigma_2 e^{\epsilon s} - P_3(\lambda_1 + \epsilon)\kappa\sigma_3 e^{\epsilon s} \tag{2.17}
$$

for all $s < \underline{s}_3$. Consider the following inequalities

$$
\begin{cases}
p_2(\lambda_1 + \epsilon)x_2 + \dfrac{\beta_2 N_0}{K_B} x_3 < 0, \\
\eta x_2 + p_3(\lambda_1 + \epsilon)x_3 < 0.
\end{cases} \tag{2.18}
$$

Remember that $p_i(\lambda_1 + \epsilon) < 0, i = 2, 3$, and $P_c(\lambda_1 + \epsilon) - \frac{\beta_2 N_0 \eta}{K_B} > 0$. Then [9, Lemma 3.2] implies that there exist positive constant $x_i, i = 1, 2$ such that inequalities (2.18) hold and satisfy

$$
\lim_{\epsilon \to 0} \frac{x_3}{x_2} = \kappa.
$$

Note that for fixed x_i, it is easy to check $\zeta x_i, i = 2, 3$, still satisfy (2.18) for any positive constant $\zeta > \max\{x_2, \kappa/x_3\}$. Setting

$$
\sigma_2 := \zeta x_2, \qquad \sigma_3 := \frac{\zeta x_3}{\kappa},
$$

then for small ϵ, we have

$$
-1 < 1 - \frac{\sigma_3}{\sigma_2} \leq 1 - \sigma_3 e^{\epsilon s} \leq 1.
$$

This and (2.16) imply that there exists a positive constant M_1 such that

$$
|R_1(s)| \leq M_1, \qquad |R_2(s)| \leq M_1 \tag{2.19}
$$

for all $s < \underline{s}_2$. It follows from (2.15) that

$$\begin{aligned}
& e^{-\lambda_1 s}\left[d_2\underline{U}_2'' - c\underline{U}_2' + G_2(\underline{U}_1, \underline{U}_2, \underline{U}_3)\right] \\
& \geq \left[-p_2(\lambda_1+\epsilon)\sigma_2 - \frac{\beta_2 N_0}{K_B}\kappa\sigma_3\right] e^{\epsilon s} - R_1(s)\sigma_1 e^{\alpha s} - R_2(s)e^{\lambda_1 s} \\
& = [\zeta L_\epsilon - \sigma_1 R_1(s)e^{(\alpha-\epsilon)s} - R_2(s)e^{(\lambda_1-\epsilon)s}]e^{\epsilon s} \\
& \geq (\zeta L_\epsilon - \sigma_1 M_1 - M_1)e^{\epsilon s} > 0,
\end{aligned}$$

where

$$L_\epsilon = -P_2(\lambda_1+\epsilon)x_2 - \frac{\beta_2 N_0}{K_B}x_3 > 0, \qquad \zeta > \frac{(\sigma_1+1)M_1}{L_\epsilon}, \qquad \epsilon < \min\{\alpha, \lambda_1\}.$$

Here we use inequalities in (2.18) and the fact $s < \underline{s}_2 < 0$. The inequality (2.13) can be proved similarly. □

3 Existence of traveling waves

Using the upper and lower solutions determined in above lemmas, we define the set

$$\Gamma = \{(U_1(\cdot), U_2(\cdot), U_3(\cdot)) \in C(\mathbb{R}, \mathbb{R}^3) : \underline{U}_i(s) \leq U_i(s) \leq \overline{U}_i(s), i = 1,2,3, s \in \mathbb{R}\}.$$

To apply Schauder's fixed point theorem on Γ, we rewrite equations (2.3) as follows

$$\begin{cases} -d_1U_1'' + cU_1' + \gamma_1 U_1 = H_1[U(\cdot)](s), \\ -d_2U_2'' + cU_2' + \gamma_2 U_2 = H_2[U(\cdot)](s), \\ -d_3U_3'' + cU_3' + \gamma_3 U_3 = H_3[U(\cdot)](s), \end{cases} \tag{3.1}$$

where

$$\begin{aligned}
U(s) &= (U_1(s), U_2(s), U_3(s)), \\
H_1[U(\cdot)](s) &= \gamma_1 U_1(s) + G_1(U(s)), \\
H_2[U(\cdot)](s) &= \gamma_2 U_2(s) + G_2(U(s)), \\
H_3[U(\cdot)](s) &= \gamma_3 U_3(s) + G_3(U(s)),
\end{aligned}$$

and positive constant $\gamma_i, i = 1,2,3$ is large enough such that each $H_i[U(\cdot)](s)$, $i = 1,2,3$, is monotone increasing with respect to $U_i(\cdot)$. Actually, by the definition of functions $G_i, i = 1,2,3$, it is enough to choose

$$\gamma_1 > \sup_{u\in\Gamma_0}\left\{-\frac{\partial G_1(u)}{\partial u_1}\right\} + \beta_1, \qquad \gamma_2 > \sup_{u\in\Gamma_0}\left\{-\frac{\partial G_2(u)}{\partial u_2}\right\}, \qquad \gamma_3 > \sup_{u\in\Gamma_0}\left\{-\frac{\partial G_3(u)}{\partial u_3}\right\}$$

where

$$\Gamma_0 := \{(u_1, u_2, u_3) : 0 < u_1 \leq N_0, 0 < u_2 \leq N^*, 0 < u_3 \leq \kappa N^*\}.$$

Let $\Lambda_{i1} < 0 < \Lambda_{i2}$, $i = 1,2,3$ be the roots of

$$d_i\Lambda^2 - c\Lambda - \gamma_i = 0,$$

and define the operator $F = (F_1, F_2, F_3) : \Gamma \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$F_j[U(\cdot)](s) = \frac{1}{d_j\Lambda_j}\left[\int_{-\infty}^{s} e^{\Lambda_{j1}(s-t)}H_j[U(\cdot)](t)dt + \int_{s}^{\infty} e^{\Lambda_{j2}(s-t)}H_j[U(\cdot)](t)dt\right], \tag{3.2}$$

where $\Lambda_j := \Lambda_{j2} - \Lambda_{j1} > 0, i = 1,2,3$. Then we can check that a fixed point of operator F in Γ is a nonnegative and bounded solution of (3.1). Therefore, to prove the existence of traveling waves, it is enough to prove the existence of fixed points of operator F. Then we have the following lemmas.

Lemma 3.1. *The operator F maps Γ into Γ, i.e. $F(\Gamma) \subset \Gamma$.*

Proof. Let $U(\cdot) \in \Gamma$, that is, $\underline{U}_i(s) \le U_i(s) \le \overline{U}_i(s)$ for any $s \in \mathbb{R}$, $i = 1,2,3$. Then it suffices to prove

$$\underline{U}_i(s) \le F[U_i(s)] \le \overline{U}_i(s)$$

for any $s \in \mathbb{R}$, $i = 1,2,3$. Note that $H_i[U(\cdot)]$ is increasing with respect to $U_i(\cdot)$, and hence $H_i[U(\cdot)] \ge 0, i = 1,2,3$ for $s \in \mathbb{R}$.

If $s \ge \underline{s}_2$, we have $\underline{U}_2(s) = 0$ and $F_2[U(\cdot)](s) > 0 = \underline{U}_2(s)$ due to $H_2[U(\cdot)](s) \ge 0$ and $H_2[U(\cdot)](s) \not\equiv 0$. Now suppose $s \le \underline{s}_2$, then we have

$$\begin{aligned} -d_2\underline{U}_2'' + c\underline{U}_2' + \gamma_2\underline{U}_2 &\le \gamma_2\underline{U}_2 + G_2(\underline{U}_1, \underline{U}_2, \underline{U}_3) \\ &\le \gamma_2 U_2 + G_2(U_1, U_2, U_3) \\ &= H_2[U(\cdot)](s). \end{aligned}$$

Hence,

$$\begin{aligned} F_2[U(\cdot)](s) &= \frac{1}{d_2\Lambda_2}\left[\int_{-\infty}^{s} e^{\Lambda_{21}(s-t)} H_2[U(\cdot)](t)dt + \int_{s}^{\infty} e^{\Lambda_{22}(s-t)} H_2[U(\cdot)](t)dt\right] \\ &\ge \frac{1}{d_2\Lambda_2}\int_{-\infty}^{s} e^{\Lambda_{21}(s-t)}\left[-d_2\underline{U}_2''(t) + c\underline{U}_2'(t) + \gamma_2\underline{U}_2(t)\right]dt \\ &\quad + \frac{1}{d_2\Lambda_2}\int_{s}^{\underline{s}_2} e^{\Lambda_{22}(s-t)}\left[-d_2\underline{U}_2''(t) + c\underline{U}_2'(t) + \gamma_2\underline{U}_2(t)\right]dt \\ &\quad + \frac{1}{d_2\Lambda_2}\int_{\underline{s}_2}^{\infty} e^{\Lambda_{22}(s-t)}\left[-d_2\underline{U}_2''(t) + c\underline{U}_2'(t) + \gamma_2\underline{U}_2(t)\right]dt \\ &= \underline{U}_2(s) + \frac{1}{\Lambda_2}e^{\Lambda_{22}(s-\underline{s}_2)}[\underline{U}_2'(\underline{s}_2+0) - \underline{U}_2'(\underline{s}_2-0)] \\ &> \underline{U}_2(s) \ge 0, \end{aligned} \tag{3.3}$$

where the second inequality holds because of $\underline{U}_2'(\underline{s}_2+0) = 0$ and $\underline{U}_2'(\underline{s}_2-0) < 0$. Therefore

$$F_2[U(\cdot)](s) \ge \underline{U}_2(s)$$

for any $s \in \mathbb{R}$. Other cases can be proved similarly. □

Choosing the positive number $\mu < \min\{-\Lambda_{21}, \Lambda_{21}\}$, then we define the functional space

$$B_\mu(\mathbb{R}, \mathbb{R}^3) := \{\Phi(s) = (\phi_1(s), \phi_2(s), \phi_3(s)) \in C(\mathbb{R}, \mathbb{R}^3) : \|\Phi(s)\| < \infty\}$$

with the norm

$$\|\Phi(s)\| := \max\left\{\sup_{s\in\mathbb{R}}|\phi_1(s)|e^{-\mu|s|}, \sup_{s\in\mathbb{R}}|\phi_2(s)|e^{-\mu|s|}, \sup_{s\in\mathbb{R}}|\phi_3(s)|e^{-\mu|s|}\right\}.$$

It is easy to see that Γ is a closed and convex subset of $B_\mu(\mathbb{R}, \mathbb{R}^3)$.

Lemma 3.2. *The operator $F : \Gamma \to \Gamma$ is continuous with respect to the norm $\|\cdot\|$.*

Proof. Assume $\phi(s), \psi(s) \in \Gamma$ with $\phi(s) \not\equiv \psi(s)$, where

$$\phi(s) = (\phi_1(s), \phi_2(s), \phi_3(s)), \qquad \psi(s) = (\psi_1(s), \psi_2(s), \psi_3(s)).$$

Note that

$$\begin{aligned}
&\left|\phi_1(s)\frac{\phi_2(s)}{1+a\phi_2(s)} - \psi_1(s)\frac{\psi_2(s)}{1+a\psi_2(s)}\right| \\
&\quad = \left|\phi_1(s)\frac{\phi_2(s)}{1+a\phi_2(s)} - \psi_1(s)\frac{\phi_2(s)}{1+a\phi_2(s)} + \psi_1(s)\frac{\phi_2(s)}{1+a\phi_2(s)} - \psi_1(s)\frac{\psi_2(s)}{1+a\psi_2(s)}\right| \\
&\quad \le N^*|\phi_1(s)-\psi_1(s)| + N_0|\phi_2(s)-\psi_2(s)|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\phi_1(s)\frac{\phi_3(s)}{K_B+\phi_3(s)} - \psi_1(s)\frac{\psi_3(s)}{K_B+\psi_3(s)}\right| \\
&\quad = \left|\phi_1(s)\frac{\phi_3(s)}{K_B+\phi_3(s)} - \psi_1(s)\frac{\phi_3(s)}{K_B+\phi_3(s)} + \psi_1(s)\frac{\phi_3(s)}{K_B+\phi_3(s)} - \psi_1(s)\frac{\psi_3(s)}{K_B+\psi_3(s)}\right| \\
&\quad \le |\phi_1(s)-\psi_1(s)| + \frac{N_0}{K_B}|\phi_3(s)-\psi_3(s)|.
\end{aligned}$$

It follows from the definitions of Γ and G that

$$|H_1[\phi(\cdot)](s) - H_1[\psi(\cdot)](s)|e^{-\mu|s|} \le M_2\|\phi(\cdot)-\psi(\cdot)\|,$$

where $M_2 = \beta_2(N_0/K_B+1) + \beta_1(N_0+N^*) + \mu + \gamma_1$. Further more, we have

$$\begin{aligned}
&|F_1[\phi(\cdot)](s) - F_1[\psi(\cdot)](s)|e^{-\mu|s|} \\
&\quad \le \frac{e^{-\mu|s|}}{d_1\Lambda_1}\left[\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)+\mu|t|}|H_1[\phi(\cdot)](t) - H_1[\psi(\cdot)](t)|e^{-\mu|t|}dt\right. \\
&\qquad\qquad \left. + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)+\mu|t|}|H_1[\phi(\cdot)](t) - H_1[\psi(\cdot)](t)|e^{-\mu|t|}dt\right] \\
&\quad \le \frac{M_2e^{-\mu|s|}}{d_1\Lambda_1}\left[\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)+\mu|t|}dt + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)+\mu|t|}dt\right]\|\phi(\cdot)-\psi(\cdot)\|.
\end{aligned}$$

Now we assume $\mu < \min\{-\Lambda_{11}, \Lambda_{12}\}$. If $s<0$, we have

$$\begin{aligned}
&|F_1[\phi(\cdot)](s) - F_1[\psi(\cdot)](s)|e^{-\mu|s|} \\
&\quad \le \frac{M_2e^{\mu s}}{d_1\Lambda_1}\left[e^{\Lambda_{11}s}\int_{-\infty}^{s} e^{-(\Lambda_{11}+\mu)t}dt + e^{\Lambda_{12}s}\int_{s}^{0} e^{-(\Lambda_{12}+\mu)t}dt + e^{\Lambda_{12}s}\int_{0}^{\infty} e^{(\mu-\Lambda_{12})t}dt\right]\|\phi(\cdot)-\psi(\cdot)\| \\
&\quad = \frac{M_2}{d_1\Lambda_1}\left[\frac{-1}{\Lambda_{11}+\mu} + \frac{1-e^{(\Lambda_{12}+\mu)s}}{\Lambda_{12}+\mu} + \frac{e^{(\Lambda_{12}+\mu)s}}{\Lambda_{12}-\mu}\right]\|\phi(\cdot)-\psi(\cdot)\| \\
&\quad \le \frac{M_2}{d_1\Lambda_1}\left(\frac{-1}{\Lambda_{11}+\mu} + \frac{1}{\Lambda_{12}+\mu} + \frac{1}{\Lambda_{12}-\mu}\right)\|\phi(\cdot)-\psi(\cdot)\|.
\end{aligned}$$

If $s \geq 0$, we have

$$\begin{aligned}
&|F_1[\phi(\cdot)](s) - F_1[\psi(\cdot)](s)|e^{-\mu|s|} \\
&\leq \frac{M_2 e^{-\mu s}}{d_1\Lambda_1}\left[e^{\Lambda_{11}s}\int_{-\infty}^{s} e^{-(\Lambda_{11}+\mu)t}dt + e^{\Lambda_{12}s}\int_{s}^{0} e^{-(\Lambda_{12}-\mu)t}dt + e^{\Lambda_{12}s}\int_{0}^{\infty} e^{(\mu-\Lambda_{12})t}dt\right]\|\phi(\cdot)-\psi(\cdot)\| \\
&= \frac{M_2}{d_1\Lambda_1}\left[\frac{-e^{(\Lambda_{11}-\mu)s}}{\Lambda_{11}+\mu} + \frac{1-e^{(\Lambda_{11}-\mu)s}}{\mu-\Lambda_{11}} + \frac{1}{\Lambda_{12}-\mu}\right]\|\phi(\cdot)-\psi(\cdot)\| \\
&\leq \frac{M_2}{d_1\Lambda_1}\left(\frac{-1}{\Lambda_{11}+\mu} + \frac{1}{\mu-\Lambda_{11}} + \frac{1}{\Lambda_{12}-\mu}\right)\|\phi(\cdot)-\psi(\cdot)\|.
\end{aligned}$$

In conclusion, we have shown that

$$\|F_1[\phi(\cdot)](s) - F_1[\psi(\cdot)](s)\| \leq M_3\|\phi(\cdot)-\psi(\cdot)\|,$$

where

$$M_3 = \frac{M_2}{d_1\Lambda_1}\max\left\{\frac{-1}{\Lambda_{11}+\mu} + \frac{1}{\Lambda_{12}+\mu} + \frac{1}{\Lambda_{12}-\mu}, \frac{-1}{\Lambda_{11}+\mu} + \frac{1}{\mu-\Lambda_{11}} + \frac{1}{\Lambda_{12}-\mu}\right\}.$$

Therefore, we know that $F_1 : \Gamma \to C(\mathbb{R},\mathbb{R})$ is continuous with respect to the norm $\|\cdot\|$. Similarly, it can be proved that $F_i : \Gamma \to C(\mathbb{R},\mathbb{R})$, $i = 2,3$ is continuous with respect to the norm $\|\cdot\|$ as well. □

Lemma 3.3. *The operator $F : \Gamma \to \Gamma$ is compact with respect to the norm $\|\cdot\|$.*

Proof. Since Γ is a closed subset of $B_\mu(\mathbb{R},\mathbb{R}^3)$ and $F(\Gamma) \subset \Gamma$, it suffices to prove that $F : \Gamma \to B_\mu(\mathbb{R},\mathbb{R}^3)$ is compact with respect to the norm $\|\cdot\|$.

For any $\Phi = (U_1(s), U_2(s), U_3(s)) \in \Gamma$, there exists a positive constant M_4 such that

$$|H_i[\Phi](s)| = |\gamma_i U_i(s) + G_i(U_1(s), U_2(s), U_3(s))| \leq M_4, \qquad i = 1,2,3,$$

for all $s \in \mathbb{R}$. Consequently, we have

$$\begin{aligned}
\left|\frac{d}{ds}F_1[\Phi](s)\right| &= \frac{1}{d_1\Lambda_1}\left[\Lambda_{11}\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)}H_1[\Phi(\cdot)](t)dt + \Lambda_{12}\int_{s}^{+\infty} e^{\Lambda_{12}(s-t)}H_1[\Phi(\cdot)](t)dt\right] \\
&\leq \frac{M_4}{d_1\Lambda_1}\left[|\Lambda_{11}|\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)}dt + \Lambda_{12}\int_{s}^{+\infty} e^{\Lambda_{12}(s-t)}dt\right] \\
&= \frac{2M_4}{d_1\Lambda_1}
\end{aligned} \tag{3.4}$$

which implies

$$\left\|\frac{d}{ds}F_1[\Phi](\cdot)\right\| \leq \frac{2M_4}{d_1\Lambda_1}.$$

Similarly, we can show that $\|\frac{d}{ds}F_i[\Phi](\cdot)\| \leq \frac{2M_4}{d_i\Lambda_i}$, $i = 2,3$. On the other hand, there exists a constant M_5 such that

$$|F_i[\Phi](s)| \leq M_5, \qquad \forall\ \Phi \in \Gamma,\ \forall s \in \mathbb{R}, \qquad i = 1,2,3.$$

Hence, for any $\varepsilon > 0$, letting $N \in \mathbb{N}^+$ such that

$$\sum_{i=1}^{3}|F_i[\Phi](s)|e^{-\mu|s|} < 3M_5 e^{-\mu N} < \varepsilon, \qquad \forall\,|s| > N. \tag{3.5}$$

Then by Arzelà–Ascoli theorem, we can choose finite elements in $F(\Gamma)$ such that there exists a finite $\varepsilon-$net of $F(\Gamma)(s)$ in the sense of supremum norm if we restrict them on $[-N,N]$, which is also a finite $\varepsilon-$net of $F(\Gamma)(s)(s\in\mathbb{R})$ in the space $B_\mu(\mathbb{R},\mathbb{R}^3)$. This implies that F is compact with respect to the norm $\|\cdot\|$ in $B_\mu(\mathbb{R},\mathbb{R}^3)$. □

Now we are in the position to state the main result of this section.

Theorem 3.4. *Let $\mathcal{R}_0>1$, then for any $c\ge c^*$, system (2.3) admits a positive solution $U(s)=(U_1(s),U_2(s),U_3(s))\in\Gamma$ satisfying condition (2.2). That is, system (2.1) has traveling wave solutions connecting E_0 to E^* with minimum wave speed c^*.*

Proof. Using the Schauder's fixed point theorem, it follows from above lemmas that there exists a nonnegative $U(s)\in\Gamma$ satisfying $U(s)=F(U(s))$. Then $U(s)$ is a nonnegative solution of (2.3). Moreover, the inequality (3.3) implies that $U_2(s)>0$ for all $s\in\mathbb{R}$. Similarly, we can prove $U_i(s)>0, i=1,3$. That is, $U(s)$ is a positive solution of (2.3). Furthermore, from the construction of upper-lower solutions, it is easy to check that $U(-\infty)=E_0$. Since the determined wave solutions may not be monotone, it is not easy to observe the upward convergence of wave profiles. In order to prove $U(+\infty)=E^*$, we appeal to the Lyapunov function method proposed in [19] (also see [7,34]) to establish the upward convergence of traveling waves.

Denote $V_1(s):=U_1'(s), V_2(s):=U_2'(s), V_3(s):=U_3'(s)$, then $U(s)$ also satisfies the following ODE system:

$$\begin{cases} U_1'(s)=V_1(s),\\ d_1V_1'(s)=cV_1(s)-G_1(U_1,U_2,U_3),\\ U_2'(s)=V_2(s),\\ d_2V_2'(s)=cV_2(s)-G_2(U_1,U_2,U_3),\\ U_3'(s)=V_3(s),\\ d_3V_3'(s)=cV_3(s)-G_3(U_1,U_2,U_3). \end{cases}$$

Define a Lyapunov function as the following

$$\begin{aligned} L(s)&:=L_1(s)+L_2(s)+\frac{\beta_2u_1^*f_2(u_3^*)}{\eta u_2^*}L_3(s),\\ L_i(s)&:=cU_i(s)-d_iV_i(s)+\frac{u_i^*d_iV_i(s)}{U_i(s)}-cu_i^*\ln U_i(s), \qquad i=1,2,3. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{dL_i(s)}{ds}&=\Big(cV_i(s)-d_iV_i'(s)\Big)\frac{U_i(s)-u_i^*}{U_i(s)}-\frac{u_i^*d_iV_i^2(s)}{(U_i(s))^2}\\ &=:\mathcal{J}_{i1}-\mathcal{J}_{i2} \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_{i1}&=\Big(cV_i(s)-d_iV_i'(s)\Big)\frac{U_i(s)-u_i^*}{U_i(s)}=G_i(U_1,U_2,U_3)\frac{U_i(s)-u_i^*}{U_i(s)},\\ \mathcal{J}_{i2}&=\frac{u_i^*d_iV_i^2(s)}{(U_i(s))^2}\ge 0. \end{aligned}$$

Therefore, $\frac{dL(s)}{ds} = \mathcal{J}_1 - \mathcal{J}_2$ with

$$\mathcal{J}_1 := \mathcal{J}_{11} + \mathcal{J}_{21} + \frac{\beta_2 u_1^* f_2(u_3^*)}{\eta u_2^*}\mathcal{J}_{31}, \qquad \mathcal{J}_2 := \mathcal{J}_{12} + \mathcal{J}_{22} + \frac{\beta_2 u_1^* f_2(u_3^*)}{\eta u_2^*}\mathcal{J}_{32} \geq 0.$$

Then simple calculations yield that

$$\begin{aligned}\mathcal{J}_1 = \mu_H(u_1^* - U_1)(1 - \frac{u_1^*}{U_1}) + \beta_1 u_1^* f_1(u_2^*)\left[2 - \frac{u_1^*}{U_1} + \frac{U_2(1+au_2^*)}{u_2^*(1+aU_2)} - \frac{U_2}{u_2^*} - \frac{U_1(1+au_2^*)}{u_1^*(1+aU_2)}\right] \\ + \beta_2 u_1^* f_2(u_3^*)\left[3 - \frac{u_1^*}{U_1} + \frac{U_3(K_B+u_3^*)}{u_3^*(K_B+U_3)} - \frac{U_1 u_2^* U_3(K_B+u_3^*)}{u_1^* U_2 u_3^*(K_B+U_3)} - \frac{U_3}{u_3^*} - \frac{U_2 u_3^*}{u_2^* U_3}\right].\end{aligned}$$

Note that

$$\begin{aligned}&2 - \frac{u_1^*}{U_1} + \frac{U_2(1+au_2^*)}{u_2^*(1+aU_2)} - \frac{U_2}{u_2^*} - \frac{U_1(1+au_2^*)}{u_1^*(1+aU_2)} \\ &= \left[\frac{U_2(1+au_2^*)}{u_2^*(1+aU_2)} - 1\right]\left[1 - \frac{1+aU_2}{1+au_2^*}\right] + 3 - \frac{u_1^*}{U_1} - \frac{U_1(1+au_2^*)}{u_1^*(1+aU_2)} - \frac{1+aU_2}{1+au_2^*} \\ &\leq -\ln\frac{u_1^*}{U_1} - \ln\frac{U_1(1+au_2^*)}{u_1^*(1+aU_2)} - \ln\frac{1+aU_2}{1+au_2^*} = 0,\end{aligned}$$

where we use the fact that $1 - x \leq -\ln x$ for all $x > 0$ and the equality holds if and only if $x = 1$. Similarly, we can show that

$$\left[3 - \frac{u_1^*}{U_1} + \frac{U_3(K_B+u_3^*)}{u_3^*(K_B+U_3)} - \frac{U_1 u_2^* U_3(K_B+u_3^*)}{u_1^* U_2 u_3^*(K_B+U_3)} - \frac{U_3}{u_3^*} - \frac{U_2 u_3^*}{u_2^* U_3}\right] \leq 0,$$

and the equality holds if and only if $U_1 = u_1^*, U_2 = u_2^*, U_3 = u_3^*$. In conclusion, we have $\mathcal{J}_1 \leq 0$ and $\mathcal{J}_1 = 0$ if and only if $U(s) = E_1$, which implies that $\frac{dL_i(s)}{ds} \leq 0$ and that $\frac{dL_i(s)}{ds} = 0$ if and only if

$$\Big(U_1(s), V_1(s), U_2(s), V_2(s), U_3(s), V_3(s)\Big) = (u_1^*, 0, u_2^*, 0, u_3^*, 0).$$

Then the Lyapunov–LaSalle's invariance principle implies $U(+\infty) = E^*$.

In the case of $c = c^*$, we use a limiting arguments as used in [9,10] to prove the existence of traveling waves of (2.1). Choosing a sequence $c_n \in (c^*, c^* + 1]$ such that $c_n \to c^*$ as $n \to \infty$, then system (2.1) admits a traveling wave solution

$$U_n(s) := \Big(U_{1,n}(z), U_{2,n}(z), U_{3,n}(z)\Big)$$

connecting E_0 to E^* with speed c_n. Note that U_n satisfies (3.1) and the integral equations (3.2). Then we can check that the sequences $\{U_n(s)\}$, $\{U_n'(s)\}$, and $\{U_n''(s)\}$ are uniformly bounded and equi-continuous on $\mathbb{R}$. Then the Arzelà–Ascoli theorem implies that there exists a subsequence of $\{c_n\}$, denoted again by $\{c_n\}$ for simplicity, and function $U_* := (U_{1*}, U_{2*}, U_{3*})$ such that

$$c_n \to c^*, \qquad U_n \to U_*, \qquad U_n' \to U_*{}', \qquad U_n'' \to U_*{}'',$$

uniformly as $n \to \infty$ on any bounded and closed interval of $\mathbb{R}$, and hence pointwise on $\mathbb{R}$. As $n \to \infty$, then we have

$$\begin{cases} -d_1 U_{1*}'' + c^* U_{1*}' + \gamma_1 U_{1*} = H_1[U_*(\cdot)](s), \\ -d_2 U_{2*}'' + c^* U_{2*}' + \gamma_2 U_{2*} = H_2[U_*(\cdot)](s), \\ -d_3 U_{3*}'' + c^* U_{3*}' + \gamma_3 U_{3*} = H_3[U_*(\cdot)](s). \end{cases}$$

Moreover, by the strong maximum principle, we can further verify that $U_{1*}(s) > 0, U_{2*}(s) > 0, U_{3*}(s) > 0$ for any $s \in \mathbb{R}$. Thus, U_* is a positive traveling wave solution of (2.1) with speed $c = c^*$. Then the same Lyapunov function argument as used above for $c > c^*$ implies that U_* satisfies condition (2.2). We complete the proof. □

4 Conclusion and discussion

In this work, we establish the existence of traveling waves for a reaction-diffusion SIR-B epidemic model with direct and indirect transmission pathways. The model is an extension of the standard SIR epidemic model to describe the transmission of cholera or other waterborne diseases. Based on the mechanism of the transmission of waterborne diseases, we assume the incidence rates are nonlinear and saturated. Since we are interested in the existence of traveling wave solutions connecting the disease-free equilibrium and endemic equilibrium, we assume the basic reproduction number $\mathcal{R}_0 > 1$, which is a threshold value for the existence of endemic equilibrium. With given assumptions, we show that the system admits traveling waves connecting the DFE and endemic equilibrium with speed c if and only if $c \geq c^*$.

Since the solution semiflow associated with our system is non-monotone, we cannot use the general theory for monotone dynamical systems to determine the existence of traveling waves. By constructing suitable upper-lower solutions, we apply the Schauder's fixed point theorem to obtain the existence of traveling waves for $c > c^*$, then a limiting argument is used to determine the existence of critical waves for $c = c^*$. Note that the determined wave solutions may not be monotone, then it is tricky to determine the asymptotic states of waves. In this work, a suitable Lyapunov function is employed to get the upward convergence of wave solutions. Here we should emphasize that the skills we used here could work for more general incidence functions with monotonicity and boundedness properties.

It is well known that the minimal wave speed may coincide with the asymptotic spreading speed for many monotone dynamical systems [15]. Although we have no way to prove this fact is true for our system, it can be expected that the minimal wave speed c^* could approximate the invasion speed of the disease. For the numerical calculation of c^*, we refer readers to a similar procedure as given in [35]. We can verify that c^* is the minimum wave speed of (2.1) if and only if c^* is the unique positive real root of a quartic equation with respect to c^2. Then the numerical value of c^* can be calculated by the "roots" command in Matlab software for given model parameters.

Finally, we should point out that the saturation incidence rates used in this work are monotone and bounded. However, as mentioned in [30], the incidence rates may be non-monotone due to the "psychological" effect. For example, the non-monotone incidence rates $\frac{SI}{1+aI^2}$ is proposed by Xiao and Ruan in [30], which is increasing when I is small and decreasing when I is large. In such case, the construction and verification of vector-value upper-lower solutions become much more difficult and challenging. We will consider this interesting topic in our follow-up works.

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