Recharging Rational Number Understanding

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Abstract

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In 1978, only 24% of 8th grade students in the United States correctly answered whether 12/13+7/8 was closest to 1, 2, 19, or 21 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). In 2014, only 27% of 8th grade students selected the correct answer to the same problem, despite the ensuing forty years of effort to improve students' conceptual understanding (Lortie-Forgues, Tian, & Siegler, 2015). This is troubling, given that 5th grade students' fraction knowledge predicts mathematics achievement in secondary school (Siegler et al, 2012) and that achievement in math is linked to greater life outcomes (Murnane, Willett, & Levy, 1995). General rational number knowledge (fractions, decimals, percentages) has proven problematic for both children and adults in the U.S. (Siegler & Lortie-Forgues, 2017). Though there is debate about which type of rational number instruction should occur first, it seems it would be beneficial to use an integrated approach to numerical development consisting of all rational numbers (Siegler, Thompson, & Schneider, 2011). Despite numerous studies on specific types of rational numbers, there is limited information about how students translate one rational number notation to another (Tian & Siegler, 2018).

The present study seeks to investigate middle school students' understanding of the relations among fraction, decimal, and percent notations and the influence of a daily, brief numerical magnitude translation intervention on fraction arithmetic estimation. Specifically, it explores the benefits of *Simultaneous presentation* of fraction, decimal, and percent equivalencies on number lines versus *Sequential presentation* of fractions, decimals, and

percentages on number lines. It further explores whether rational number review using either Simultaneous or Sequential representation of numerical magnitude is more beneficial for improving fraction arithmetic estimation than *Rote practice with fraction arithmetic*. Finally, it seeks to make a scholarly contribution to the field in an attempt to understand students' conceptions of the relations among fractions, decimals, and percentages as predictors of estimation ability.

Chapter 1 outlines the background that motivates this dissertation and the theories of numerical development that provide the framework for this dissertation. In particular, many middle school students exhibit difficulties connecting magnitude and space with rational numbers, resulting in implausible errors (e.g., 12/13+7/8=1, 19, or 21, 87% of 10>10, 6+0.32=0.38). An integrated approach to numerical development suggests students' difficulty in rational number understanding stems from how students incorporate rational numbers into their numerical development (Siegler, Thompson, & Schneider, 2011). In this view, students must make accommodations in their whole number schemes when encountering fractions, such that they appropriately incorporate fractions into their mental number line. Thus, Chapter 1 highlights number line interventions that have proven helpful for improving understanding of fractions, decimals, and percentages.

In Chapter 2, I hypothesize that current instructional practices leave middle school students with limited understanding of the relations among rational numbers and promote *impulsive calculation,* the act of taking action with digits without considering the magnitudes before or after calculation. Students who *impulsively calculate* are more likely to render implausible answers on problems such as estimating 12/13+7/8 as they do not think about the magnitudes (12/13 is about equal to one and 7/8 is about equal to one) before deciding on a calculation strategy, and they do not stop to judge the reasonableness of an answer relative to an estimate after performing the calculation. I hypothesize that *impulsive calculation* likely stems from separate, sequential instructional approaches that do not provide students with the appropriate desirable difficulties (Bjork & Bjork, 2011) to solidify their understanding of individual notations and their relations.

Additionally, in Chapter 2, I hypothesize that many middle school students are unable to view equivalent rational numbers as being equivalent. This hypothesis is based on the documented tendency of many students to focus on the operational rather than relational view of equivalence (McNeil et al., 2006). In other words, students typically focus on the equal sign as signal to perform an operation and provide an answer (e.g., 3+4=7) rather than the equal sign as a relational indicator (e.g., 3+4=2+5). Moreover, this hypothesis is based on the documented whole number bias exhibited by over a quarter of students in 8th grade, such that students perceived equivalent fractions with larger parts as larger than those with smaller parts (Braithwaite & Siegler, 2018b). If middle school students are unable to perceive equivalent values within the same notation as equivalent in size, it seems probable that they might also struggle perceiving equivalent rational numbers as equivalent across notations. This is especially true in light of evidence that many teachers often do not use equal signs to describe equivalent values expressed as fractions, decimals, and percentages (Muzheve & Capraro, 2012). Chapter 2 underscores the importance of highlighting the connections among notations by discussing the pivotal role of notation connections in prior research (Moss & Case, 1999) and the benefit of interleaved practice in math (Rohrer & Taylor, 2007). Finally, I propose a plan for improving students' understanding of rational numbers through linking notations with number line instruction, as an integrated theory of numerical development (Siegler et al, 2011) suggests that all rational numbers are incorporated into one's mental number line.

Chapter 3 details two experiments that yielded empirical evidence consistent with the hypotheses that students do not perceive equivalent rational numbers as equivalent in size and that this lack of *integrated number sense* influences estimation ability. The findings identify a discrepancy in performance in magnitude comparison across different rational number notations, in which students were more accurate when presented with problems where percentages were *larger* than fractions and decimals than when they were presented with problems where percentages were *smaller* than fractions and decimals. Superficially, this finding of a *percentages-are-larger bias* suggests students have a bias towards perceiving percentages as larger than fractions and decimals; however, it appears this interpretation is not true on all tasks. If students always perceive percentages as larger than fractions and decimals, then their placement of percentages on the number line should be larger than the equivalent fractions or decimals. However, this was not the case. The experiments revealed that students' number line estimation was most accurate for percentages rather than the equivalent fraction and decimal values, demonstrating that students who are influenced by the *percentages-are-larger bias* are most likely not integrating understanding of fractions, decimals, and percentages on a single mental number line. Furthermore, empirical evidence provided support for the theory of *impulsive calculation* defined earlier, such that many students perform worse when presented with distracting information ("lures") meant to elicit the use of flawed calculation strategies than in situations without such lures. Importantly, *integrated number sense*, as measured by the composite score of all cross-notation magnitude comparison trials, was shown to be an important predictor of estimation ability in the presence of distracting information on number lines and fraction arithmetic estimation tasks, often above and beyond number line estimation ability and general math ability.

The experiments reported in Chapter 3 also evaluated whether *Simultaneous*, integrated instruction of all notations improved integration of rational number notations more than *Sequential* instruction of the three notations or a control condition with *Rote practice* in fraction arithmetic. The experiments also evaluated whether the instructional condition influenced fraction arithmetic estimation ability. The findings supported the hypothesis that a *Simultaneous* approach to reviewing rational numbers provides greater benefit for improving integrated number sense, as measured by more improvement in the composite score of magnitude comparison across notations. However, there was no difference among conditions in fraction arithmetic estimation ability at posttest. The experiments point to potential areas for improvement in future work, which are described subsequently.

Chapter 4 attempts to explore further students' understanding of the relations among notations. For this analysis, a number of data sources were examined, including student performance on assessments, interview data, analysis of student work, and classroom observations. Three themes emerged: (1) students are employing a flawed translation strategy, where students concatenate digits from the numerator and denominator to translate the fraction to a decimal such that a/b=0.ab (e.g., 3/5=0.35). (2) percentages can serve as a useful tool for students to judge magnitude, and (3) students equate math with calculation rather than estimation (e.g., in response to being asked to estimate addition of fractions answers, a student responded, "I can't do math, right?"). Moreover, case studies investigated the differential effect of condition (Simultaneous, Sequential, or Control) on students' strategy use. The findings suggest that the Simultaneous approach facilitated a more developed schema for magnitude, which is crucial given that a student's degree of mathematical understanding is determined by the strength and accuracy of connections among related concepts (Hiebert & Carpenter, 1992).

Chapter 5 concludes the dissertation by discussing the contributions of this work, avenues for future research, and educational implications. Ultimately, this dissertation advances the field of numerical cognition in three important ways: (1) by documenting a newly discovered bias of middle school students perceiving *percentages as larger than fractions and decimals* in magnitude comparisons across notations and positing that a lack of integrating notations on the same mental number line is a likely mechanism for this bias; (2) by demonstrating that students exhibit *impulsive calculation*, as measured by the difference in performance between situations where students are presented with distracting information ("lures") meant to elicit the use of flawed calculation strategies and situations that do not involve lures; and (3) by finding that *integrated number sense,* as measured by the composite score for magnitude comparison across notations, is a unique predictor of estimation ability, often above and beyond general mathematical ability and number line estimation. In particular, students with higher integrated number sense are more than twice as likely to correctly answer the aforementioned 12/13+7/8 estimation problem than their peers with the same number line estimation ability and general math ability. This finding suggests that integrated number sense is an important inhibitor for *impulsive calculation*, above estimation ability for individual fractions and a general standardized test of math achievement. Finally, this dissertation advances the field of mathematics education by suggesting instruction that connects equivalent values with varied notations might provide superior benefits over a sequential approach to teaching rational numbers. At a minimum, this dissertation suggests that more careful attention must be paid to relating rational number notations. Future work might examine the origins of *impulsive* *calculation* and the observed *percentages-are-larger bias.* Future research might also examine whether *integrated number sense* is predictive of estimation ability beyond general number sense within notations. From these investigations, it might be possible to design a more impactful intervention to improve rational number outcomes.

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Dedication

This dissertation is dedicated to my loving husband, Brad, and our wonderful children, Lincoln, Anastasia, and Calvin. You never cease to amaze me with your unfailing support. I love you all very much.

Chapter 1: Overview

1.1 The Problem

Many students are unable to see the implausibility of results such as 12/13+7/8=19/21 (Gelman, 1991; Stafylidou & Vosniadou, 2004; Hecht & Vagi, 2010; Ni & Zhou, 2005). Students using simple estimation strategies should determine that the sum of two fractions that are each close to one (e.g., 12/13 + 7/8) yields a result of approximately two, not less than one. Errors on simple estimation tasks are problematic and commonplace. In 1978, only 24% of 8^{th} grade students in the United States correctly answered whether 12/13+7/8 was closest to 1, 2, 19, or 21 (Carpenter et al., 1980). In 2014, only 27% of 8^{th} grade students selected the correct answer to the same problem despite the ensuing forty years of effort to improve students' conceptual understanding (Lortie-Forgues et al., 2015).

Students make similar implausible errors with decimals and percentages. For example, 43% of 5^{th} grade students aligned the rightmost digit of addends to calculate 6 + 0.32= 0.38 (Hiebert & Wearne, 1985). Again, simple estimation dictates that adding six and a number less than one would give a result that was slightly more than six, not less than one. Moreover, in a more recent study, the issue with decimal point alignment accounted for about half of the errors with decimal addition and subtraction for Australian middle school students (Lai & Murray, 2014). Though the decimal point alignment problem diminishes with age, it persists at least into the high school years (Hiebert & Wearne, 1986).

While less is known about the understanding of percentages (Tian & Siegler, 2018), estimations of percent cause difficulty for many students. Only 45% of 7^{th} and 8^{th} grade students correctly answered that *87% of 10* was less than 10 (Gay & Aichele, 1997). Additionally, only

69% of 11th grade students and 37% of 7th grade students indicated that *76% of 20* would be less than 20 in a multiple-choice question (Kouba et al., 1988).

The struggles described in the preceding paragraphs reflect a lack of number sense. They are especially troubling because rational number understanding is linked to greater math outcomes. In particular, one's ability to compute fractions is related to advanced mathematical outcomes even after controlling for several other cognitive abilities (Siegler, Duncan, Davis-Kean, Duckworth, Claessens, Engel, Susperreguy, & Chen, 2012). A study of 6th and 8th graders demonstrated that fraction magnitude representations are important for understanding mathematics achievement scores, apart from fraction arithmetic fluency, when examining fraction magnitude knowledge, number line estimation, fraction arithmetic proficiency, and school mathematics achievement (Siegler et al., 2011). Additionally, computational estimation skills relate to math performance (Hanson & Hogan, 2000). Because math achievement is linked to greater life outcomes (Murnane, Willett, & Levy, 1995), and because rational numbers are essential for the workplace (Handel, 2016), this lack of number sense warrants attention.

On a fundamental level, the errors described point to a disconnect between magnitude and space, as students struggle with understanding the direction of effects for operations with rational numbers. Specifically, many children struggle allocating attention to the magnitude of rational numbers, resulting in estimations with implausible results. As such, it is vital to understand how children incorporate conceptions of rational numbers in their numerical development and whether they can build a mental number line that includes all rational numbers.

1.2 Theories of Numerical Development

Numerical development is a complex cognitive process. Young human infants possess crude quantitative estimation ability (Dehaene, 2011; Xu & Arriaga, 2007); exact representations of number are thought to be developed from an infant's crude estimation ability as a consequence of culture (Piazza, Pica, Izard, Spelke, & Dehaene, 2013), with language playing a primary role in number development. While an intuitive sense of number is present at birth, children must gradually acquire a connection between number words and numerosity (Wynn, 1992). During the first few years of life, children gradually understand "one" represents a quantity corresponding to one object, two represents two objects, and three represents three objects. Once a child understands four objects, they comprehend that any number refers to the quantity of the set. Thus, it seems children construct an understanding of whole numbers in succession, as if they are gradually building a mental number line.

Consistent with this theory, mathematical cognition research suggests that whole number quantities are represented on a mental number line (Dehaene, 2011). Classic evidence for the mental number line hypothesis includes the spatial numerical association of response codes (SNARC) effect (Dehaene, Bossini, & Giraux, 1993) and the distance effect (Moyer & Landauer, 1967). The SNARC effect suggests small numbers are associated with the left side of space and larger numbers are associated with the right side of space. The result that individuals are quicker at indicating a small number with their left hand and a large number with their right hand than vice versa provides evidence of the SNARC effect (Dehaene et al., 1993). This link between number and space contributes to the hypothesis that people represent numerosity on a mental number line (Dehaene et al., 1993). Similarly, there exists a numerical distance effect, such that people are quicker and more accurate at comparing the magnitude of quantities when the ratio between the two numbers is larger (Moyer & Landauer, 1967). The numerical distance effect is present in the processing of both symbolic and non-symbolic quantities, as demonstrated by numerous behavioral studies.

If a mental number line must be constructed for whole numbers, it seems logical that children must also construct a mental number line that includes all rational numbers. However, the process of constructing a mental number line that includes fractions, decimals, and percentages is not as straightforward as whole numbers. Research has indicated that preschool children as young as three years old can calculate sums of fractions in a nonverbal task in a similar way to calculating nonverbal whole number sums (Mix, Levine, & Huttenlocher, 1999) and that students have rich knowledge of fractions outside of formalized school (Mack, 1990); however, it is not entirely clear how children incorporate rational numbers into their overall numerical development.

One theory of numerical development suggests that an understanding of whole numbers interferes with learning of fractions, decimals, and percentages. For example, Hartnett and Gelman (1999) argue that fraction learning is hindered by an understanding of counting numbers, such that each number has a unique successor. They claim that because understanding of fractions is not consistent with counting principles, fractional representations are often misinterpreted in young children and that new conceptual structures need to be fostered (Hartnett & Gelman, 1999). Additionally, other researchers insist that there is a whole number bias which interferes with learning of fractions (Ni & Zhou, 2005; Stafylidou & Vosniadou, 2006). According to Ni and Zhou (2005), "the whole number bias thus refers to a robust tendency to use the single-unit counting scheme to interpret instructional data on fractions" (p. 28). This view of numerical development privileges whole numbers and suggests that the learning of whole numbers hinders the development of fraction understanding. For example, students might misapply knowledge of whole numbers by stating that 1/4 is larger when comparing fractions such as 1/3 and 1/4 because 4 is larger than 3, adding across numerators/denominators such as computing 1/2 + 1/3 = 2/5, counting non-congruent parts in a shape and naming it as a specific fraction, and failing to conceive of numbers as being between 0 and 1 (Ni & Zhou, 2005). Another important area to consider in this theory of whole number bias is the relation between language and mathematical notation. Ni and Zhou (2005) argue that children's difficulty with fraction symbols is "not merely a matter of not mastering the notations but it has more to do with the internal processes of conceptual restructuring" (Ni & Zhou, 2005, p. 47). In other words, in trying to conceptualize manipulation of various representations of fractions, children must begin to restructure their cognitive system by conceptualizing fractions as continuous quantities.

On the other hand, a competing theory of numerical development suggests that there exists an integrated approach to acquisition of the concept of number across all rational numbers. For example, Case and Okamoto (1996) proposed that the mental number line possibly includes whole numbers and rational numbers. In their view, children develop a counting schema with a motor routine and verbal tags. Children then map this routine onto conceptual categories and finally they are able to apply a newly "integrated structure, recursively, to the new numerical symbols they have acquired" (Case et al., 1996, p. 57). Additionally, Steffe (2001) proposed a *reorganization hypothesis*, suggesting that whole numbers do not hinder children's understanding of fractions but that children must make accommodations in their whole number schemes when they encounter fractions. Similarly, Siegler and colleagues (2011) posit that the mental number line model has proven useful for understanding children's concept of whole number but that this can also extend to understanding of fractions. They further suggest that an "integrated theory promises to broaden and deepen our understanding of numerical development" (Siegler et al., 2011, p. 292). As opposed to "whole number bias" claims which stipulate that whole numbers interfere with learning about fractions (Ni & Zhou, 2005), Siegler and colleagues (2011) argue that difficulty in understanding fractions "stems from drawing inaccurate analogies to whole numbers rather than from drawing analogies between whole numbers and fractions per se" (p. 291). Properties of whole numbers such as having "unique successors, can be represented by a single symbol, are countable, never decrease with multiplication, never increase with division, and so on" do not apply to fractions (Siegler, Fazio, Bailey, & Zhou, 2013, p. 13). Therefore, it is beneficial to encourage children to draw correct analogies to whole numbers by teaching them that like whole numbers, "fractions can express a proportion of another number (3/5:1 :: 60:100 :: 60% of 100) or that fractions, like whole numbers, can provide absolute measures of quantity (6 in. = ½ foot = 1/6 yard)" (Siegler et al., 2011, p. 291). Moreover, a critical assumption of the integrated theory of numerical development is that fraction magnitude understanding is vital for overall mathematics learning; therefore, rational numbers play a central role in numerical development, and theories that diminish their central role are "unnecessarily truncated" (Siegler et al., 2013, p. 13). Thus, "generating a mature understanding of rational numbers requires understanding both the one property that all rational numbers share-- that they have magnitudes that can be located and ordered on number lines— and understanding that other properties that unite whole numbers do not unite rational numbers" (Torbeyns, Schneider, Xin, & Siegler, 2015, p. 3).

Results from magnitude comparison across notations lend support to this theory of integrated numerical development. Hurst and Cordes (2016) demonstrated that adults' performance reveals ratio effects across notations (i.e., decimal compared to fractions, decimal compared to whole numbers, and whole numbers compared to fractions), which extended a line of previous research demonstrating significant response time ratio effects within notations (e.g., Moyer & Landauer, 1967; Meert, Gregoire, & Noel, 2010; DeWolf, Grounds, Bassok, &

Holyoak, 2014; Schneider & Siegler, 2010). Moreover, Hurst and Cordes (2016) found no evidence of biases, such as whole numbers being judged as larger than fractions, as response times, were similar whether the larger value was in fraction or whole number form. Hurst and Cordes (2016) argued that these results provided compelling support for an integrated sense of number, where all notations are represented on an integrated continuum. Moreover, eye-tracking data revealed longer fixation on more difficult trials, where the ratio between values being compared was smaller and thus more difficult. Taken together, behavioral and eye-tracking data on adult performance with comparison across notations (whole numbers, fractions, and decimals) suggested that adults represent these values on an integrated continuum, independent of notation. Given Siegler and colleagues' (2011) theory of integrated numerical development, it seems likely that the mental number line encompasses all rational numbers, including fractions, decimals, and percentages. However, to my knowledge, there have been no studies examining magnitude comparison across fractions, decimals, and percentages, and very little is known about individuals' understanding of percentages (Tian & Siegler, 2018).

1.3 Difficulties with Rational Numbers

Many children across different countries struggle with rational numbers, yet they are universally crucial for mathematics achievement (Torbeyns et al., 2014). Given that depth of understanding is associated with connections among related concepts (Hiebert & Carpenter, 1992), it is essential to consider students' difficulties with fractions and related forms (decimals and percentages) given the pivotal role of fraction magnitude knowledge in overall mathematics achievement.

Siegler and Lortie-Forgues (2017) distinguish between two main sources of difficulty with rational numbers: inherent and culturally contingent sources of difficulty. According to

Siegler and Lortie-Forgues (2017), inherent sources of difficulty are universal and would be present regardless of the educational system, whereas, culturally contingent sources of difficulty are ones that involve instruction or learners' prior knowledge.

Inherent sources of difficulty include understanding that there is an infinite number of other numbers between any two rational numbers, there is an infinite number of ways to express any rational number (e.g., 3/5, 6/10, 36/60), and that longer trains of digits for whole numbers suggests a larger number but this is not the case for decimals (e.g., 0.123 versus 0.5) (Nesher & Peled, 1986; Resnick et al., 1989). Beyond the longer-train-of-digits-whole-number misconception, many children have difficulty with the role of 0 (Durkin & Rittle-Johnson, 2015). For example, students ignore the value of 0 to the right of the decimal point and treat the next non-zero digit as being in the tenths place (e.g., .08 would be treated as 0.8). As in the case of whole numbers, students might reason that putting a zero at the rightmost end of a train of decimal digits makes the magnitude larger (e.g., that .430 is larger than .43). Furthermore, the close relation between fractions and decimals might promote the fraction misconception with decimals (Resnick et al., 1989). Students exhibiting the fraction misconception inappropriately import knowledge of fractions to decimal magnitude judgment, where students might reason that a decimal number that has a digit in the thousandths place is smaller than a decimal number that has digits in the tenths place (e.g., 0.893 is less than .4 because thousandths are less than tenths). Additionally, relations between rational and whole number arithmetic are complex. For example, adding/subtracting fractions requires that the denominator remain unchanged provided there is a common denominator as in 3/5+4/5=7/5, but multiplication requires that multiplication is applied independently to the numerators and denominators as in 3/5*4/5=12/25. These complex relations often cause procedural problems such as adding numerators and denominators.

Moreover, the variety of interpretations for rational numbers themselves can cause difficulty (Behr, Lesh, Post, & Silver, 1983). According to Kieren (1976), rational numbers can be interpreted as different subconstructs: part-to-whole comparison, decimal, ratio, indicated division (quotient), operator, and measure of continuous quantities. Similarly, the essential feature of percentages in daily life centers around understanding the quantitative relationship between a part and a whole, expressed by the equation percentage = part/whole, which can be written as part= percentage x whole or whole= part/percentage, can be problematic due to the relational nature of equivalency (McNeil et al, 2006). Moreover, while much less is known about the understanding of percentages, we might be taking for granted that students understand the absolute magnitude of percentages. Ginsburg, Gal, and Schuh, (1995) discussed how adult learners who could justify their use of 100% as representing a whole were more likely to answer questions related to this quantitative relationship. Students also displayed more difficulty with computations when they could not justify that percent means part out of 100 (Lembke & Reys, 1994).

Culturally contingent sources of difficulty include teacher knowledge, textbooks, and language (Siegler & Lortie-Forgues, 2017). For example, teachers in the US and Canada show strikingly weak conceptual understanding of rational number multiplication and division, and this is problematic because lack of general knowledge is related to poor understanding of how to teach these topics (Depaepe et al., 2015). Moreover, math textbook content has been shown to influence student learning, but students are often receiving less examples of more difficult problems (Braithwaite, Pyke, & Siegler, 2017). For example, Korean textbooks present far more fraction division than fraction multiplication problems, as compared to US textbooks, which did the opposite (Son & Senk, 2010). Beyond types of problems, there is an overemphasis on instruction involving the rote procedure of inverting-and-multiplying in US textbooks, with little focus on understanding the meaning of division of fractions (Son & Senk, 2010; Ma, 1999). Language might also play a role in sources of difficulty, as teaching US children English versions of Korean fractional expressions which highlight the relational meaning of fractions was associated with increased performance (Paik & Mix, 2003). On that note, teachers often use inappropriate and mathematically inaccurate language when teaching about translation between rational numbers that could actually instill misconceptions or encourage a treatment of fractions as whole numbers (e.g., “north” or “nanny” to describe the numerator or “getting rid of the decimal” to write a decimal as a percent) (Muzheve & Capraro, 2012). Misconceptions might also arise from an emphasis on the part-whole interpretations of fractions (Ni & Zhou, 2005). While the approach of teaching children that 1/8 can be represented as one part of a pizza cut into 8 pieces has value due to its concreteness, this instructional approach does not encourage children to think about the fraction as 1/8 of the distance between 0 and 1 on the number line (Moseley, Okamoto, & Ishida, 2007). As a result, many elementary and middle school students hold misconceptions about fractions, such as not understanding that there are numbers between 0 and 1 (Ni, 2001; Ni & Zhou, 2005). Similar, misconceptions are observed with decimals (Resnick et al., 1989; Durkin & Rittle-Johnson, 2015).

This trend in US schools is strikingly different from instructional approaches in Japan and China that emphasize number line representations of fractions to a greater extent than in the US (Moseley et al., 2007). Furthermore, given the evidence of an integrated theory of numerical development for whole numbers and fractions (Siegler et al., 2011), it is not surprising that these countries also exhibit better understanding of fractions (Moseley et al., 2007). Additionally, pre-segmented visual models can interfere with students’ abilities to understand fraction magnitude by eliciting counting-based strategies rather than encouraging the use of an intuitive sense of number (Boyer, Levine, & Huttenlocher, 2008). Finally, instruction in the realm of percentages can "make the students' concepts of percent less intuitive and more rule-driven, actually narrowing rather than expanding the strategies and the computational methods students use when working with percents" (Lembke & Reys, 1994, p. 256).

1.4 Efforts to Improve Rational Number Understanding

Efforts to improve rational number understanding have demonstrated that the measurement (or continuous) approach is more beneficial than a part-whole (or discretized) approach for fractions. Utilizing number lines as a visual model for rational numbers has proven especially useful. An intervention study aimed at improving at-risk fourth graders' understanding of fractions explored two different approaches to instructional intervention: a measurement interpretation of fractions and a part-whole interpretation of fractions (Fuchs et al., 2013). The fourth grade students in this study were identified as at risk based on whole number calculation skills because they ranked below the 35^{th} percentile prior to the intervention. This study involved a 12-week program where the intervention focused on a measurement interpretation of fractions, which consisted of "representing, comparing, ordering, and placing fractions on a 0 to 1 number line" (Fuchs et al., 2013, p. 687). On the other hand, the instruction of the control condition focused on calculation procedures and "part-whole understanding by using shaded regions and other manipulatives related to the area model" (Fuchs et al., 2013, p. 687). Post-assessments of fraction performance revealed that at-risk students in the intervention group performed better than their peers in the control condition. Furthermore, students' understanding of fractions as measure mediated the effects of the intervention suggesting that a measurement interpretation of fractions is critical to developing students' fraction knowledge. Finally, the study demonstrated that a focus on fractions as measure narrowed the gap between at-risk and low-risk students; whereas, the control condition did not have the same effect in narrowing this gap. Many other studies have demonstrated the benefit of number lines to improve rational number understanding (e.g., Fuchs et al., 2014; Psycharis, Latsi, Kynigos, & others, 2007; Davydov & Tsvetkovich, 1991; Fazio, Kennedy, & Siegler, 2016). Furthermore, studies comparing visual models have demonstrated an advantage of number lines over circular models or no model (Hamdan & Gunderson, 2017). Finally, instruction using the number line has been shown to improve students' abilities performing operations with fractions (Sidney, Thompson, & Rivera, 2019).

The benefit of the number line approach to improving number sense is not limited to fractions. Rittle-Johnson, Siegler, and Alibali (2001) demonstrated the benefit of utilizing number lines for improving both procedural and conceptual aspects of decimal knowledge. Moreover, number lines are a useful tool to counteract common misconceptions about decimals (Durkin & Rittle-Johnson, 2012); however, it should be noted that as whole number misconceptions diminished (e.g., longer decimal train signifies larger magnitude), a new misconception sometimes replaced the old one. Specifically, as students began to pay attention to the fractional component of the decimals, they noticed that the hundredth place value signified a smaller fractional part than the tenths place for example and this would lead them to incorrectly judge .84 as less than .3 because hundredths are smaller parts than tenths. Thus, it is important to monitor misconceptions and utilize number lines to guide students away from these flawed ideas that sometimes arise from interventions.

While there is limited information about percentages, it seems plausible that number lines might also be a useful tool for this rational number notation, given its success with fractions and decimals. Moreover, a novel curriculum utilizing percentages as an entry into learning about rational numbers and an emphasis on a measurement approach underscored the importance of number lines across notations (Moss & Case, 1999). I will discuss more about this pivotal study subsequently. However, it is important to note that all notations can be placed on the number line and equivalent values can be expressed in any form: fraction, decimal, and percentage. Ultimately, it appears that number lines provide a powerful tool for eliciting understanding of magnitude for rational numbers.

Chapter 2: Solidifying an Integrated Sense of Number

The following sections discuss my theory that current instructional practices leave middle school students with limited understanding of the relations among rational numbers and promote *impulsive calculation,* the act of taking action with digits without considering the magnitudes before or after calculation. Students who *impulsively calculate* are more likely to render implausible answers on problems such as estimating 12/13+7/8 as they do not think about the magnitudes (12/13 is about equal to one and 7/8 is about equal to one) before deciding on a calculation strategy, and they do not stop to judge the reasonableness of an answer with an estimate after performing the calculation. I hypothesize that *impulsive calculation* likely stems from separate, sequential instructional approaches to instruction with different rational number notations that do not provide students with the appropriate desirable difficulties (Bjork & Bjork, 2011) needed to solidify their understanding of individual notations and their relations.

Additionally, this chapter describes the inability of many middle school students to view equivalent rational numbers as being equivalent in size. This idea is based on the documented tendency of many students to focus on the operational rather than relational view of equivalence (McNeil et al., 2006). In other words, students typically focus on the equal sign as signal to perform an operation and provide an answer (e.g., 3+4=7) rather than the equal sign as a relational indicator (e.g., 3+4=2+5). Moreover, this idea is based on the documented whole number bias exhibited by over a quarter of students in 8^{th} grade, such that students perceived equivalent fractions with larger parts as larger than those with smaller parts (Braithwaite & Siegler, 2018b). If middle school students are unable to perceive equivalent values within the same notation as equivalent in size, it seems probable that they might also struggle perceiving equivalent rational numbers as equivalent across notations. This is especially true in light of evidence that many teachers often do not use equal signs to describe equivalent fraction, decimal, and percent values (Muzheve & Capraro, 2012). Thus, Chapter 2 underscores the importance of highlighting the connections among notations by discussing the pivotal role of notation connections in prior research (Moss & Case, 1999) and the benefit of interleaved practice in math (Rohrer & Taylor, 2007). Finally, I propose a plan for improving students' understanding of rational numbers through linking notations with number line instruction, as the integrated theory of numerical development (Siegler et al, 2011) suggests that all rational numbers are incorporated into one's mental number line.

2.1 Potential Reasons for a Lack of an Integrated Sense of Number

Both children and adults exhibit poor understanding of rational numbers despite clear evidence in favor of a number line approach to rational number instruction (Carpenter et al., 1980; Siegler & Lortie-Forgues, 2017). Moreover, on the 2017 NAEP, only 27% of 8th grade students were correct in identifying point A, B, and the midpoint between the two points for the figure below (Figure 1). Obviously, a disconnect between magnitude and space still exists despite educational research demonstrating the importance of a number line approach.

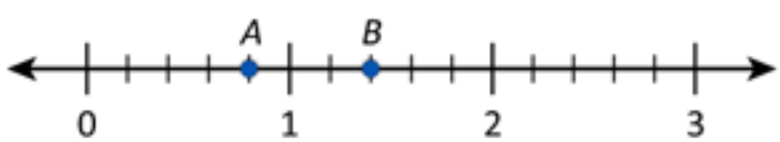


Figure 1: Assessment item from the NAEP (2017) for 8th grade students in math.

Yet, if many students struggle at understanding rational numbers, why do the vast majority of children across the United States pass math class? Perhaps, the very nature of how rational numbers are taught in isolation yields evidence of performance versus learning (Soderstrom & Bjork, 2015). In other words, instruction may alter student performance, producing “temporary fluctuations in behavior or knowledge that can be observed and measured during or immediately after the acquisition process” (Soderstrom & Bjork, 2015, p. 176). However, these fluctuations in behavior or knowledge may not be those that reflect learning, or “the relatively permanent changes […] that support long-term retention and transfer” (Soderstrom & Bjork, 2015, p. 176). Typical rational number instruction covers fractions, decimals, and percentages in sequence (Common Core State Standards Initiative, 2010). Therefore, immediately after a unit on each topic students might exhibit correct performance on each topic but not conceptual understanding that is the type that supports long-term retention. Thus, students might exhibit understanding of fractions that often does not emphasize fraction magnitude but rather a part-whole interpretation of fractions (Moseley et al, 2007). Students might also demonstrate a basic understanding of decimals but likely rife with misconceptions (Resnick et al., 1989). Finally, students may be able to perform operations with percentages but exhibit a problematic understanding of percentages (Gay & Aichele, 1997). Ultimately, students may be able to perform operations with rational numbers without learning the individual notations themselves, much less their interconnections (Vamvakoussi & Vosniadou, 2010). Moreover, the translation between the notations is likely taught via rote memorization rather than as a meaningful cognitive activity (Wang & Siegler, 2013).

Therefore, when students are presented with a task that presents misleading information, such as partitions that do not match the location of the sought after midpoint in Figure 1, student performance is easily manipulated by the type of assessment. Similarly, in Siegler and Thompson (2014), the relationship between mathematics achievement and fraction knowledge is lowered when children are met with potentially distracting partitions on number lines. The finding that student performance is easily manipulated with potentially distracting information provides evidence that students likely do not have a good sense of magnitude. In other words, immediately after a chapter on fraction addition, students will perform the operation reasonably well. Still, after some time and, perhaps after being taught several other operations that might cause confusion, students exhibit behaviors that contradict reason, such as estimating that adding 12/13+7/8= 19 or 21 or 19/21. In this case, students may exhibit incremental changes in performance but not the relatively permanent changes involved in learning, which are necessary for long-term retention and transfer (Soderstrom & Bjork, 2015). Thus, students are passed to the next grade, where teachers of older grades gripe about students still not understanding fractions after it has been taught since 3rd and 4th grade (Hoffer et al., 2007).

While researchers debate about which notation is the best to initiate instruction to improve conceptual understanding (see Tian & Siegler, 2018 for a review), perhaps we are debating the wrong issue. Because procedural and conceptual knowledge have been shown to develop iteratively (Rittle-Johnson et al, 2001), translation between notations and conceptual understanding of each individual notation are likely to develop iteratively. In this vein, Moss & Case (1999) demonstrated that an experimental curriculum aimed at highlighting connections among the notations brought deeper conceptual understanding of rational numbers. In this experiment, the fourth-grade students in the treatment condition received intensive training on understanding continuous quantity, measurement, and equivalence among different representations (fractions, decimals, percentages) of rational numbers. The authors believe that deeper understanding of fractions was achieved by attempting “to move the children beyond the understanding of any single form of rational number representation toward a deeper understanding of the rational number system as a whole” (Moss & Case, 1999, p. 142). This echoes other work, which suggests that depth of understanding is indicated by the strength of connections among related concepts (Hiebert & Carpenter, 1992. Furthermore, Moss and Case (1999) argue that starting with percentages and decimals before fractions enabled students to build on whole number knowledge in conceptualizing the rational number system in an intuitive way. Moreover, Wang and Siegler (2013) demonstrated that improving fraction magnitude knowledge can increase fraction understanding and notation translation. Thus, highlighting the shared magnitude of the notations can bring about better conceptual understanding of the individual notations and the translation process. On that note, teachers need to ensure that they are using an equal sign to denote the equivalent relationships between fractions, decimals, and percentages, as this can drastically affect students' use of equal signs and perhaps their understanding of equivalence between the notations (Muzheve & Capraro, 2012). Ultimately, it seems that instruction that highlights the equivalence of rational numbers in different notations, rather than instruction in which the notations are presented separately and sequentially, maximizes benefits for students.

Moreover, sequential instructional approaches do not afford students with desirable difficulties that provide an opportunity to fully integrate their conceptions of rational number. Research on desirable difficulties in education draws upon research from motor learning; decades of research on motor learning have revealed that varied and mixed practice are better for overall performance, rather than blocked or fixed practice (for a review see Bjork & Bjork, 2011). For example, Kerr & Booth (1978) demonstrated that children who practiced tossing a bean bag into a target from varied positions (2 feet and 4 feet away) performed better when tested at 3 feet away than children who practiced at a fixed position of 3 feet away (the exact distance they were tested on). Similar results were observed when children practiced tossing bean bags of different weights in an intermixed order rather than blocked by weight (Carson &

Wiegand, 1979). Drawing parallels to the field of motor learning, research on desirable difficulties in the realm of higher cognitive learning suggests that there is benefit of blocking topics and massed practice in the short term but retention is substantially worse (Bjork & Bjork, 2011; Siegler & Stern, 1998); and yet, that is precisely the approach that is typical in this sequential approach to rational numbers. Moreover, Rohrer and Taylor (2007) demonstrated the benefits of spaced practice and interleaved topics in the domain of mathematics. One of the most striking findings from this study is that the students in the interleaved condition, where topics were mixed, appeared to do worse during the practice session than the students of the blocked condition that focused on one topic alone. However, when the students were tested later, the students in the mixed condition performed better than those in the blocked condition. Possibly, this same phenomenon is occurring after children complete separate units of instruction on rational numbers through a sequential approach. It might appear that students perform sufficiently on the individual notations but when they are tested at a later time they perform poorly. By contrast, students perform better when instruction highlights the connections between the notations, especially on tasks that provide students with misleading information (Moss & Case, 1999). Moreover, while rational numbers are revisited later in the curriculum, perhaps the instruction is not targeted at enhancing magnitude representation through varying the notation (i.e., students do not have ample opportunity to explore the absolute magnitude of 4/5 as 80% and as 0.8).

Perhaps, this sequential approach to rational number instruction is not so dissimilar from the bean bag tossing example from the motor learning research studies mentioned earlier. In other words, practice at understanding the magnitude of fractions by themselves is a great start, but it might not be enough to provide the variable practice required for retention and transfer (e.g., students might not utilize magnitude understanding to evaluate the implausibility of estimates such as 12/13+7/8=19/21). In line with this reasoning, Goode, Geraci, and Roediger (2008) demonstrated that practicing variations of a task might lead to better transfer than repeated practice of the same task in regards to anagram solutions. In this study, there were three conditions: same, varied, and different. The participants in the same condition practiced the same word three times and were later tested on that word (e.g., they practiced solving LDOOF three times and were tested on LDOOF). Those in the different condition practiced the same word three times and were later tested on a different word (e.g., they practiced solving DOLOF three times and were tested on LDOOF). The varied practice condition practiced three different versions of the word and was tested on a different word (e.g., they practiced solving DOLOF, FOLOD, OOFLD and were tested on LDOOF). Similar to the Rohrer & Taylor (2007) study, individuals in the repeated practice conditions (i.e., same and different conditions) appeared to be quicker at solving the anagrams during the three practice sessions but, at immediate post-testing, those in the varied condition solved a significantly greater proportion of anagrams than those in either of the repeated practice conditions. Goode and colleagues (2008) posit that the benefit of variable practice over repeated practice might be interpreted through the framework of schema theory (Schmidt, 1975) and elaborative processing (Battig, 1979; Shea & Zimny, 1983). In other words, varying the anagram allows an individual to generate a lexical schema for solving anagrams, and greater contextual interference during learning provides a better opportunity for elaborative processing, which leads to improved transfer.

Relatedly, because current instructional approaches do not vary the notation of individual magnitudes, students are noticing the wrong aspects about the values. Thus, students make inaccurate hypotheses about the size of rational numbers and they have limited resources for checking their hypotheses. By and large, students have not developed an appropriate schema by which they can interpret the magnitude of rational numbers; instead, they are paying attention to aspects of rational numbers that are superficial, rather than reflecting deep conceptual understanding of the structural pattern. Evidence of noticing the wrong aspects of rational numbers is apparent in misconceptions about fractions (e.g., Stafylidou & Vosniadou, 2006), decimals (e.g., Durkin & Rittle-Johnson, 2012), and percentages (e.g., Gay & Aichele, 1997). The critical feature of fractions, decimals, and percentages is their magnitude. Research has demonstrated that individuals possess an intuitive sense of approximate fractional magnitude (Fazio, Bailey, Thompson, & Siegler, 2014; Matthews & Chesney, 2015). However, the process for determining the symbol to magnitude correspondence does not seem so straightforward.

Children tend to focus on the superficial aspects of rational numbers, rather than making meaning of the magnitudes. Research has shown that experts are able to focus on structural features of problems, whereas novices focus on superficial features (Chi, Feltovich, & Glaser, 1981). Similarly, children are novices in the domain of rational numbers, focusing on superficial features of rational numbers in an attempt to make meaning of the magnitude or, perhaps because of how they were taught, they simply are not trying to make meaning of the magnitude. In other words, students see a fraction and they immediately think, “What do I do with this?” rather than “How big is this number?” Thus, children tend to think of rational numbers, not as quantities but as entities that need to be acted upon.

Impulsive Calculation

This tendency to take action with digits without considering the magnitudes before or after calculation is what I refer to as *impulsive calculation*. Students who *impulsively calculate* are more likely to render implausible answers on problems such as estimating 12/13+7/8 as they do not think about the magnitudes (12/13 is about equal to one and 7/8 is about equal to one) before deciding on a calculation strategy, and they do not stop to judge the reasonableness of an answer with an estimate after performing the calculation. I hypothesize that at least one source of *impulsive calculation* is the separate, sequential instructional approaches in rational number instruction. These instructional approaches do not provide students with the appropriate desirable difficulties (Bjork & Bjork, 2011) to solidify their understanding of individual notations and their relations. Moreover, as I'll discuss subsequently, impulsive calculation likely stems from instruction that has not provided students with ample opportunity to make inferences about the patterns observed with the notations themselves.

Specifically, I argue that this tendency to *impulsively calculate* rather than map symbols onto magnitudes cannot arise solely from inherent difficulties with rational numbers. In other words, young children have a great deal of informal understanding of rational numbers (Mack, 1990) that they could extend, but do not extend it to symbolic operations with the same problems (Mack, 1995). For example, a student was able to discuss that 1/8 of a pizza and another 1/8 of a pizza was the same as 2/8 of a pizza but yet when it came to the symbolic 1/8+1/8, the child said the sum was 2/16 because she imagined it being 1/8 of one pizza and 1/8 of another pizza, thus 2 of the 16 parts (Mack, 1995). The problem here is that she is focusing on the parts of the symbols (i.e., 1 of 8 parts and another 1 of 8 parts is the same as 2 of 16 parts) and not understanding that the implicit whole is one. Thus, it should be interpreted as 1/8 of a whole and 1/8 of a whole is the same as 2/8 of a whole, which reflects multiplicative rather than additive thinking (Lamon, 1999). Thus, the understanding of the multiplicative relation is essential, but this understanding may be lost when children work with symbols because of the way rational numbers are taught in isolation.

Though they may have rich intuitive understandings about fractions, children may not have enough opportunity to utilize inductive reasoning to make inferences about the patterns observed with symbols for rational numbers (e.g., n/6 is going to be smaller than n/5). A study of inductive reasoning with function finding for college students demonstrated that successful and unsuccessful problem solvers did not differ in the patterns that they observed but that "successful participants do not merely compute quantities; they *analyze* them" (Haverty, Koedinger, Klahr, & Alibali, 2000, p. 262). Moreover, successful problem solvers integrate pattern finding and hypothesis generation through translating a pattern into symbols (Haverty et al., 2000). Taking it a step further in line with what the authors suggest, perhaps early number sense is so intricately related to advanced mathematical outcomes because of this intricate link between noticing patterns and translation. Thus, an inductive understanding of fractions requires switching back and forth between finding a pattern with the digits to generating a hypothesis about magnitude. Specifically, understanding of magnitude involves "examination, modification, or manipulation of numerical instances for the purpose of understanding the quantity in question" (Haverty et al., 2000, p. 259).

In other words, instead of *impulsively calculating* when students encounter the symbol 27/30, they should be reasoning that this number is close to 1. They might also pursue the idea that they can transform this number into a value that might make it easier for them to evaluate the magnitude more effectively. Putting the fraction in lowest terms might help them see that 9/10 is the same as .9 or 90%, which is in line with their original hypothesis that 27/30 is close to 1. Though students might initially falter when first encountering rational number symbols by focusing on the componential parts, this gradually diminishes from 4^{th} to 8^{th} grade, suggesting that perhaps related experiences encountering decimals, percentages, ratios, rates, proportions, and rational number arithmetic are useful in helping students map magnitudes onto symbols (Braithwaite & Siegler, 2018b). Thus, appropriately interpreting the symbols of rational numbers involves mapping between non-symbolic intuitive understanding of these numbers and symbols; importantly, one is aided in this effort through the act of translation. In other words, individuals are able to check their hypothesis about the size of a particular value by weighing it against a translation of the value to another form. Fluid understanding of the connections among rational number notations equips individuals with tools to better analyze their ideas about magnitude.

In sum, despite growing evidence of the importance of number lines, there exists a disconnect between magnitude and space, as students still struggle understanding the location of values on number lines (e.g., 2017 NAEP) and evaluating the direction of effects in regards to rational numbers (e.g., 12/13+7/8 cannot equal 19/21). Current instructional approaches emphasize the fraction, decimal, percentage sequence (Common Core State Standards Initiative); yet, instruction that highlights equivalency among the notations has been shown to provide greater benefit (Moss & Case, 1999) because conceptual understanding and translation procedures are likely to develop iteratively (Rittle-Johnson et al., 2001). Moreover, this sequential approach provides blocked rather than interleaved practice, which research has shown allows for better performance in the short-term but does not result in improved long-term learning (Rohrer & Taylor, 2007). Relatedly, focusing on one notation individually does not provide sufficient varied practice for accessing magnitude representations. Varied practice has been shown to be more effective than repeated practice in producing retention and transfer in other high-cognitive demand learning activities (Goode et al, 2008). Finally, students often notice the wrong aspects of individual notations, as evidenced by their numerous pervasive misconceptions (Resnick et al., 1989: Durkin & Rittle-Johnson, 2015; Stafylidou & Vosniadou, 2006; Gay & Aichele, 1997). Successful problem solvers translate observed patterns into symbols (Haverty et. al, 2000) and better magnitude knowledge is related to translation (Wang & Siegler, 2013). Thus, students need to be equipped with a method for translating a pattern they observe in one notation into a numeric symbol that helps them generate a hypothesis about magnitude.

2.2 A Theoretical Instructional Plan

These theories provide a case for implementing instruction that focuses on the relation between magnitude and space for all rational numbers to fully integrate understanding of rational number. Drawing on the literature on desirable difficulties, the instruction will involve a form of interleaved and varied practice, where the instruction will target improving rational number understanding over several weeks and will vary the symbolic notation daily. The fundamental aspects of instruction will include number lines, will recognize the difficulties that students have with concepts of equivalence, will carefully relate individual notations, and will use percentages as a strategy for linking rational number notations.

I propose that number lines are a powerful tool to explicitly draw the connections among the magnitudes of equivalent fractions, decimals, and percentages. I argue that instruction should guide students to attend to the most important aspect of rational numbers and help them use interconnections among the rational number notations as a tool for monitoring their own understanding of magnitude. In other words, students should notice that their translation does not align with their original intuition when they compare their estimate for a fraction and its translation on the number line, such as when students look at 4/5 and then translate the fraction as .45 or 45%. The most concrete way for students to notice the connection among rational number notations is through the number line (Moss & Case, 1999). The connection between what children are guided to notice and how they learn has been demonstrated with whole numbers (McNeil & Alibali, 2005) and fractions (Kellman et al, 2008). As discussed previously, number lines have proven quite useful in improving magnitude representation for fractions and decimals (e.g., Fuchs et al., 2013; Durkin & Rittle-Johnson, 2015). Moreover, highlighting the connection among notations has proven beneficial in bringing about better understanding of the rational number system as a whole (Moss & Case, 1999). Thus, simultaneously displaying the magnitudes of fractions, decimals, and percentages on number lines would likely bring about deeper understanding of the relation among the notations, as students notice that each notation occupies the same position on the number line relative to the endpoints.

This understanding that equivalent values written in different notations occupy the same position on the number line is not something that can be taken for granted. This is especially true given that Braithwaite and Siegler (2018b) found that at least a quarter of 8^{th} grade students estimated equivalent fractions with larger components as larger in size. It is likely that students will struggle understanding that equivalent values written in different notations occupy the same position on the number line, if students struggle with understanding that equivalent values written in the *same* notation are equivalent in size. Moss (2005) raised this issue when she wrote, "textbooks typically treat the notation system as something that is obvious and transparent and can simply be given by a definition at a lesson's outset" (p. 319). Indeed, studies of textbooks have revealed differences in the treatment of the concept of equivalence as operational rather than relational view (McNeil et al., 2006). In other words, students typically focus on the equal sign as signal to perform an operation and provide an answer (e.g., 3+4=7) rather than the equal sign as a relational indicator (e.g., 3+4=2+5). This is problematic considering that algebra often involves the relational understanding of equivalence (e.g., 5x-2=3x+4). Textbooks emphasize the operational rather than the relational view and student performance mirrors the textbooks' treatment of equivalence (Li et al, 2008). Specifically, Chinese students that receive textbook input that stresses the relational nature of the equal sign perform better than US students whose instruction emphasizes the operational view of the equal sign (though we cannot take for granted other variables, such as culture). Even still, we are seeing that many students tend to only think about the equal sign in terms of performing an operation rather than the relational view of equivalence possibly due to textbook treatment of these concepts. Importantly, modifications to curriculum that emphasize the relational view results in better understanding of the concept of equivalence (McNeil, Fyfe, & Dunwiddie, 2015). Moreover, teachers do not use equal signs when expressing relationships between equivalent fractions, decimals, and percentages (Muzheve & Capraro, 2012). Even with an emphasis towards using multiple representations within the domain of fractions (e.g., fraction bars, number lines, and pie charts), it still is not entirely clear to students exactly how these representations are related to each other (Murray et al., 2015). This is troubling because ultimately we are seeing that children may not understand the relation between fractions, decimals, and percentages.

Therefore, instruction should be aimed at promoting connections to foster depth of understanding. Because understanding involves incorporation of concepts into an internal network, degree of understanding is determined by the strength and accuracy of connections among related concepts (Hiebert & Carpenter, 1992). Students that have a superficial understanding of the relation among notations will not have the robust rational number understanding that educators desire. It is simply not enough to say that fractions, decimals, and percentages are related and expect that students will be able to 'conceptually transcode' among these different notations (Berch, 2017). For example, students that had weak or limited understanding of place value were unable to conceptually transcode their understanding of place value when the task became misleading (Miura & Okamoto, 1989). In this study, children were given thirteen cubes, asked to place four in each of the three cups, and leave the remaining cube outside the cups. When shown an index card with the number '13' printed on it, children that had weak understanding of place value explained that the one cube represented the '1' digit and the three cups represented the '3' digit. Similarly, students in Siegler and Thompson (2014) exhibited worse performance in the condition where they had to place a fraction on a number line that was partitioned and labeled with tenths over the condition where the number line did not have such partitions. Thus, superficial understanding on a numerical processing level can result in problems with conceptual transcoding among representations. Students need explicit instruction that carefully relates the notations especially because "virtually no time is spent in relating the various representations- decimals, fractions, percentages- to each other" (Moss, 2005, p. 320).Furthermore, teachers often do not use equal signs with equivalent rational numbers, which may implicitly suggest the values are not equivalent (Muzheve & Capraro, 2012). Ultimately, fluid understanding of the connections among fractions, decimals and percentages could lead to deeper understanding and superior performance (Moss & Case, 1999).

Consistent with Moss and Case (1999), I suggest that percentages are central to helping students notice the relationships among the notations and integrate conceptions of magnitude. As such, students will be encouraged to draw upon their intuitions about percentages to inform their estimates of magnitude of decimals and fractions and use this knowledge to monitor their translation activity and placement of values on the number line. As compared to fractions and decimals, percentages are typically used to express relations between a part and a whole rather than absolute magnitudes. For example, Tian (2018) noted that textbooks often use addition and subtraction of fractions and decimals (3/4+2/3 and .75 +.67) but rarely include problems involving addition of percentages (75% + 67%). However, just because percentages are not thought of in daily contexts as absolute magnitudes, does not mean that it is not useful to explore them as such. Thus, while percentages are typically described colloquially as a relation between part and whole, we can actually think about them as existing on a number line. Anchoring concepts of percent to a number line appears almost trivial because it essentially transforms the space between 0 and 1, to match a number line with whole numbers from 0 to 100, except the labels have a percent sign next to them. So, a number line of percentages essentially becomes a number line that taps whole number (or decimal) knowledge. One might argue that this could potentially confuse students by encouraging them to draw inappropriate connections between whole numbers and rational numbers, but the opposite seems more likely to be true. As discussed previously, students who were introduced to rational numbers through exploration of percentages first and who then examined how percentages relate to fractions and decimals exhibited greater rational number understanding than those who followed the typical fractions first sequence (Moss & Case, 1999). Moreover, Siegler and colleagues (2011) found that a common effective strategy to estimate a fraction is to translate "the fraction being estimated into a percentage of the distance between the two endpoints and then to use the percentage as if it were a whole number on a 0-100 number line. […] Improvements in number line estimation accuracy between 6th and 8th grade seem partially attributable to the 8th graders, but not the 6th graders having been taught about percentages" (p. 291). Finally, research suggests that humans have intuitive access to non-symbolic ratio magnitudes and this might support symbolic knowledge (Matthews & Chesney, 2015). Clearly, making the connection between percentages and the other rational number notations seems both accessible and intuitive for students.

Chapter 3: Implementing and Evaluating the Instructional Plan

3.1 Research Questions

I seek to understand the current state of middle school students' understanding of the relations among rational number notations and whether it is possible to help students integrate these conceptions of individual notations (Siegler et al, 2011) through daily, brief targeted instruction. Additionally, I seek to understand whether integrated number sense will support students in inhibiting *impulsive calculation* by helping them focus on magnitude before and after calculating. In particular, I seek to answer these questions: What are students' perceptions about the relations among rational number notations? What is the effect of an integrated understanding of rational numbers on students' ability to estimate in the presence of distracting information? What effect does Simultaneous versus Sequential instruction of notations have on solidifying rational number understanding?

3.2 Hypotheses

1) Integrated number sense adds explanatory power to mathematical outcomes:

a) Middle school students do not perceive equivalent rational numbers as equivalent in size.

b) Individual differences in integrated number sense predict students' estimation ability in the presence of distraction.

2) ***Number line instruction improves integrated number sense:***

a) Rational number review with number lines results in better outcomes than review without number lines.

b) Simultaneous review of notations will improve outcomes, especially tasks that measure integrated number sense, more than Sequential review of notations.

3.3 Overview of the Project

Introduction

I conducted two experiments to investigate these questions. Experiment 1 was a pilot (n=43 students) to test instructional and assessment materials and determine whether it would be worthwhile to continue the investigation with a larger sample. Experiment 2 was based on Experiment 1 with three notable differences: (1) there were small modifications to the assessments and instruction based on learning gained from Experiment 1, (2) the sample size was substantially larger (n=264 students), and (3) the students' teachers led the daily instruction as opposed to the lead researcher who led all instruction in Experiment 1. I selected 7th and 8th grade students for experiment participation for two reasons: (1) percentages are typically not a focus of mathematics education until 7th grade (Common Core, 2019), and (2) the experiments' intervention was designed as a review of notations rather than teaching new material.

Testing of the first hypothesis involved examination of pretest performance. Testing of the second hypothesis involved examination of improvement on a number of measures from pretest to posttest, with instruction taking place between assessments as designated by each class's assigned condition. The following sections will provide an overview of (1) assessment tasks, (2) the instructional conditions, and (3) the rationale for the methodology of testing each hypothesis. After the overview, there are detailed sections for each Experiment covering the following: (1) the method and design of the experiment, (2) the rationale and procedure of the tasks, (3) the design of and procedures for the intervention, and (4) the analysis of the results.

Overview of Assessment Tasks:

Five assessment tasks were administered before and after the intervention. Two tasks assessed understanding of individual fractions: number line estimation with endpoints 0-1 and number line estimation with endpoints 0-5. Two tasks assessed students' understanding of relations among notations: magnitude comparison across notations and comparison of student performance placing equivalent fractions, decimals, and percentages on a decile number line (a line partitioned in tenths). One task assessed fraction arithmetic estimation.

The tasks did not have substantial differences between Experiments 1 and 2 except for the fraction arithmetic estimation task, which differed due to an issue with data collection. These differences between the two fraction arithmetic tasks will be described in Experiment 2. Other tasks had minor changes for Experiment 2 (e.g., different fractions, decimals, and percentages presented in the problems) due to learning from Experiment 1. These adjustments are also explained in Experiment 2.

In addition to the tasks, relevant standardized testing data and demographic information was obtained from the school districts. These data were combined with the pretest and posttest results to build a coherent picture of the results of Experiments 1 and 2.

Overview of Instructional Conditions:

The instructional conditions and the instruction were designed to investigate Hypothesis 2: number line instruction improves integrated number sense. Three instructional conditions were developed: the Simultaneous condition, the Sequential condition, and the Control condition (Experiment 2 only). Specifically, this dissertation explores the benefits of *Simultaneous presentation* of fraction, decimal, and percent equivalencies on number lines versus *Sequential presentation* of fractions, decimals, and percentages on number lines. In Experiment 2, it further explores whether rational number review using either Simultaneous or Sequential representation of numerical magnitude is more beneficial for improving fraction arithmetic estimation than the Control condition, which involves *Rote practice with fraction arithmetic*. The instruction was intended to be a brief 5-minute warm-up review activity at the beginning of class spread out over three weeks (15 lessons in total).

The Rationale and Methodology for Testing Each Hypothesis

Hypothesis 1: Integrated number sense adds explanatory power to mathematical outcomes

The Integrated Theory of Numerical Development (Siegler et al, 2011) demonstrates that students' understanding of fraction magnitudes is an essential part of numerical development. Siegler, Thompson, and Schneider (2011) further suggest that future research might explore students' understanding of the relations among fractions, decimals, and percentages. However, to my knowledge, no research has explicitly examined the understanding of the relations among these notations or the role that understanding of the relations among notations plays in mathematical outcomes.

I theorized that integrated number sense, characterized by understanding of the relations among notations, would add unique explanatory power to understanding individual differences in mathematical outcomes. This theory was derived from research indicating the importance of fraction magnitude representation in numerical development (Siegler et al, 2011) and research suggesting that depth of understanding involves making connections among related concepts (Hiebert & Carpenter, 1992). Thus, testing of Hypothesis 1 involves investigating whether students' understanding of the relations among fractions, decimals, and percentages is predictive of math achievement on standardized tests, beyond the predictive power of fraction magnitude representations as found by Siegler, Thompson, and Schneider (2011). Specifically, the current study tested whether integrated number sense, as measured by the composite score of magnitude comparison across notations, adds unique explanatory power to the model explaining variance in math achievement tests. Furthermore, the dissertation explored whether students struggle perceiving equivalent rational numbers as equivalent in size (Hypothesis 1a) and whether individual differences in integrated number sense predicts students' estimation ability in the presence of distracting information (Hypothesis 1b).

Hypothesis 1a: Students do not perceive equivalent rational numbers as equivalent in size.

Theoretically, students who do not have an understanding about how the different notations (fractions, decimals, and percentages) are related to one another should not perceive equivalent rational numbers as equivalent in size. As discussed previously, this hypothesis was based on the finding that a substantial number of middle school students did not perceive equivalent fractions as equivalent in size (Braithwaite & Siegler, 2018b). Hence, it was probable that they might also struggle perceiving equivalent rational numbers as equivalent across notations. Thus, it was predicted that students would differ in accuracy when placing equivalent fraction, decimal, and percentages on the number line (e.g., 1/19, .052, 5%). Specifically, I expected PAE for number line estimation accuracy of equivalent values to be best for percentages, followed by decimals, and worst for fractions. This ordering of performance accuracy was based on the idea that percentages can be seen as most closely related to whole numbers, followed by decimals, and then fractions. Additionally, the aforementioned decile number line was used rather than the typical 0-1 number line because I wanted to see whether the potentially distracting partitions would encourage students to abandon attention to magnitude. For example, students might place 5% at the midpoint because the 5/10 label distracts them.

For similar reasons, it was also predicted that students would differ in their accuracy for magnitude comparison across notations, such that they would likely be more accurate on trials where percentages were larger if they were following a heuristic (e.g., percentages are larger). As discussed previously, magnitude comparison has been utilized widely as a measure of magnitude representation but there have been no studies to date that have examined magnitude comparison across fractions, decimals, and percentages. Studies examining magnitude comparison across fractions, decimals, and whole numbers have suggested magnitudes are represented on an integrated continuum for adults (Ganor-Stern, 2013; Hurst & Cordes, 2016) and children (Hurst & Cordes, 2018). Therefore, I planned to examine accuracies across the six different categories of comparison trials (Percent>Fraction, Fraction>Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction).

Hypothesis 1b: Individual differences in integrated number sense predict students' estimation ability in the presence of distraction.

Theoretically, students with integrated number sense would likely not exhibit *impulsive calculation* and students without integrated number sense are more likely to *impulsively calculate,* especially in situations that might elicit flawed calculation strategies. Earlier, I defined *impulsive calculation* as a tendency to take action with digits without considering the magnitudes before or after calculation. I reasoned that placing fractions on the decile number line could create a distracting situation where students without integrated number sense would estimate worse in favor of doing something with the digits in the fractions (i.e., students would exhibit impulsive calculation). This phenomenon was observed in Sielger & Thompson (2014), which found that number line estimation performance was significantly worse in the decile condition than a typical 0-1 number line estimation condition. Siegler & Thompson (2014) suggested the task might measure magnitude knowledge and the ability to inhibit potentially distracting landmarks.

This suggestion led to the development of hypothesis 1b, in which I theorize that integrated number sense helps students estimate better, even in the presence of distracting information. In other words, students who were distracted on the decile number line task did not have a solid sense of magnitude and the type of assessment easily manipulated their performance. To test this hypothesis, students placed fractions on both the 0-1 number line and the decile number line. Students were expected to perform significantly worse on the decile number line than the 0-1 number line, which would replicate findings of Siegler & Thompson (2014). Importantly, I hypothesized that integrated number sense, as measured by magnitude comparison across notations, would predict performance on the decile number line task (with the partitions serving as distracting information). In other words, individual differences in magnitude comparison across notations would predict fraction decile number line estimation, even after controlling for unlabeled 0-1 number line estimation and math achievement scores.

Students were also expected to perform worse on estimating fraction arithmetic than their performance on estimating individual fractions (0-1 number line estimation). This result would replicate the finding of Braithwaite, Tian, & Siegler (2018), who found that students' PAE was substantially worse for estimating the sums of fractions than individual fractions. Importantly, I predicted that integrated number sense, as measured by magnitude comparison across notations, would predict performance on fraction arithmetic estimation, even after controlling for unlabeled 0-1 number line estimation and math achievement test scores.

I chose not to examine the fraction arithmetic estimation part of this hypothesis in Experiment 1, given an issue with data collection to be described later. However, analysis of common student errors in Experiment 1 helped inform the multiple-choice design of the fraction arithmetic estimation task in Experiment 2. The new design of the fraction arithmetic estimation task in Experiment 2 had the added benefit of constraining student answer choices in an important way. Answer choices involved lure trials and no lure trials to test student estimation abilities both in and without the presence of distracting information (i.e., one of the three answer choices included a lure answer such as adding across numerators and denominators or answer choices did not have an obvious lure). The structure of this design allowed me to examine whether there would be a difference in performance between lure and no lure trials, parallel to the difference between decile (labeled) and 0-1 number line (unlabeled) estimation. I theorized that integrated number sense would predict how students perform on these lure fraction addition estimation trials, above their standardized test scores, performance on no lure trials, and 0-1 number line estimation performance.

Finally, I also sought to investigate predictors of whether students would be correct with the infamous 12/13+7/8 problem (whether the sum is closest to 1, 2, 19, or 21?) (Carpenter et al, 1980). Therefore, the 12/13 + 7/8 question was included in Experiment 2 and logistic regression was conducted to analyze it separately from the other tasks. Ultimately, I theorized that integrated number sense, as measured by magnitude comparison across notations, would be a significant predictor for accuracy with the 12/13 + 7/8 problem above 0-1 number line estimation and general math ability.

Hypothesis 2: Number line Instruction improves integrated number sense

a) Rational number review with number lines results in better outcomes than review without number lines.

b) Simultaneous review of notations will improve outcomes, especially tasks that measure integrated number sense, more than Sequential review of notations.

I predicted that improvement from pretest to posttest would result in

Simultaneous>Sequential>Control for all five assessment measures: (1) number line estimation on lines with endpoints 0 and 1, (2) number line estimation with endpoints 0 and 5, (3) magnitude comparison across notations, (4) performance placing equivalent fractions, decimals, and percentages on a decile number line, and (5) fraction arithmetic estimation. In other words, students in the Simultaneous condition would make greater improvement over the Sequential and Control condition on all the above named measures. Moreover, the Sequential condition would make greater improvement over the Control condition on these same measures.

The hypothesis that both number line experimental conditions (Simultaneous and Sequential) would make greater improvement over the Control conditions was based on the documented benefits of number line interventions in improving rational number magnitude representations (e.g., Fuchs et al, 2013; Fuchs et al., 2014; Psycharis, Latsi, Kynigos, & others, 2007; Davydov & Tsvetkovich, 1991; Fazio, Kennedy, & Siegler, 2016; Sidney, Thompson, & Rivera, 2019; Rittle-Johnson, Siegler, & Alibali, 2001). The hypothesis that Simultaneous rather than Sequential review of notations will provide more integrated number sense was based on prior research that underscored the importance of highlighting the connections among notations (Moss & Case, 1999), the benefit of interleaved practice for review in math (Rohrer & Taylor, 2007; Rohrer et al, 2019), and the integrated theory of numerical development (Siegler et al, 2011), which suggests that all rational numbers are incorporated into one's mental number line.

3.4 Experiment 1

Experiment 1 was conducted to explore students' conceptions of the relations among rational numbers and examine how students respond to instruction. Specifically, Experiment 1 was conducted to pilot-test instructional and assessment materials and determine whether to continue the investigation with a larger sample, which was done in Experiment 2.

3.4.1 Method

Participants

Participants were 43 middle school students from a private school located in a middle-class neighborhood in northern New Jersey. There were 22 8th grade students and 21 7th grade students. There were 24 boys and 19 girls. The experiment also included 27 6th grade students who were excluded from these analyses for two reasons: (1) there was only one class of 6th grade students, which would have resulted in an unbalanced experiment with far more students in one condition than the other as it wasn't practical to assign half of the students to each condition, and (2) there was a concern that the 6th grade students did not have thorough instruction on percentages, meaning the instruction would be initial learning instead of review (the focus of this study). Since classes were grouped by ability, an attempt was made to assign classes to a condition that resulted in each condition having a mix of high and low achieving students. Two classes were assigned to the Simultaneous condition (N=21) and two classes to the Sequential condition (N=22). Testing was done via Qualtrics in a quiet classroom setting.

Tasks

Number line estimation

Rationale: Number line estimation tasks using lines with 0-1 and 0-5 endpoints are widely used as a measure of individual fraction estimation ability (e.g., Siegler et al, 2011; Siegler & Pyke, 2013; Siegler & Thompson, 2014; Braithwaite, Tian, & Siegler, 2018). The fraction magnitude knowledge assessed by number line estimation tasks has been shown to be predictive of advanced mathematics outcomes (Siegler et al, 2012). Thus, I used number line estimation tasks to measure students' magnitude knowledge of individual fractions and as a predictor for explaining estimation ability. This was important for determining whether integrated number sense is more important than general fraction magnitude representation in predicting math outcomes (Hypothesis 1).

Procedure: Adapted from the number line estimation task (Siegler & Pyke, 2013), students completed the number line task on a computer via Qualtrics. The students were presented number lines. For the 0 to 1 number line task, each number line had 0 at the left end, 1 at the right, and the fraction to be estimated above the line. Students responded by moving the slider to the position on the line that they thought corresponded to the number being estimated and then clicked the computer's track pad. After completing each problem, a new number line with a different fraction appeared, and the students repeated the process. To acquaint students with the slider procedure, the researcher first asked them to locate the practice fractions ½ and ¼. The instructor then asked if anyone needed clarification on the activity. Clarification involved explaining the directions individually to any students that asked a question. Feedback on accuracy was not provided. After the practice trials, students estimated the positions of 10 fractions: 1/19, 2/13, 1/5, 1/3, 3/7, 7/12, 5/8, 3/4, 7/8, and 13/14. Two fractions were drawn from each fifth of the number line. Here as on all experimental tasks, presentation order of items was random, and no feedback was provided.

The 0 to 5 number line task was identical, except the right endpoint was labeled 5, and the practice fraction was 7/2. Again, after the practice question, the instructor asked if any students needed clarification on the activity. Feedback on accuracy was not provided. Then, the students were randomly presented with 10 fractions to be estimated on the 0-5 number line task: 1/5, 7/8, 11/7, 9/5, 13/6, 7/3, 13/4, 10/3, 9/2, and 19/4.

Decile Number Line Estimation (i.e., a number line labeled with tenths)

Rationale: I chose to use a decile number line instead of the typical 0-1 number line because I intentionally wanted to examine whether students attended to the digits in the fractions or maintained focus on magnitude when placing fractions on the decile number line. As part of my theory on *impulsive calculation*, I suggested that students often take action with digits and ignore the magnitudes of values. Research shows students struggle with placing values on a number line in the presence of visually distracting information. As such, a decile number line, a line partitioned and labeled by tenths, was used to examine student performance when placing fractions and their equivalent decimal and percent values on a decile number line as compared to a regular number line. Students in Siegler and Thompson (2014) were highly distracted when tasked with placing a fraction on the decile number line, as evidenced by worse performance. Similarly, Moss & Case (1999) discussed the importance of having students estimate magnitudes of values with a visually misleading task because even Piaget believed children needed to be presented with misleading tasks or else the assessment would just measure their ability to parrot instruction. Additionally, as discussed previously, students with limited or weak place value understanding demonstrated difficulty with basic numerical processing when presented with a visually misleading task and asked to describe the meaning of the digits in the number thirteen (Miura and Okamoto, 1989). Thus, the decile number line task was used to capture issues with students' rational number processing given that a superficial understanding of numerical processing level can result in problems with conceptual transcoding among representations. Moreover, I theorized in Hypothesis 1 that students, who have integrated number sense, have a better developed schema for magnitude and, therefore, would not be distracted by the visually misleading partitions.

Furthermore, students placing equivalent fraction, decimal, and percent values on the same number line enables examining students' understanding of the relations among notations without testing their procedural ability to translate between notations. In other words, students may not understand magnitude representations among notational forms but might get a correct answer without understanding magnitude by seamlessly executing a process for translating between fractions and decimals such as long division or an equivalent fraction strategy (e.g., 4/5=80/100=0.80). Braithwaite and Siegler (2018) employed a similar methodology when they asked students to place equivalent fractions on number lines and found that fractions with larger componential parts would yield larger estimates for younger students (e.g., 16/20 would be judged as larger than 4/5). Therefore, the decile number line task seeks to assess whether there are differences in accuracy based on notation. Any differences observed in performance across notations would lend support to Hypothesis 1 about the lack of integrated number sense being reflected by perceiving equivalent values as not equivalent in size.

Procedure: Students placed 8 fractions and their equivalent decimal and percent on a decile number line (e.g., 1/19, 0.052, and 5%). Adapted from Siegler & Thompson (2014), students completed the decile number line estimation task on a computer via Qualtrics. Students were presented number lines, each with 0 at the left end and 1 at the right, with the line partitioned and labeled by tenths. The fraction, decimal, and percent to be estimated appeared above the line for the 0-1 number line task. Students responded by clicking on a location on the line that they thought corresponded to the number being estimated. Then, a new number line with a different value appeared, and the process repeated. The fraction values were specifically chosen to be distracting because the partitioning might promote encoding strategies that do not encourage attention to the magnitude of the value (e.g., a line partitioned into tenths does not indicate where 6/17 should be in a simple way other than if it is translated to a decimal and then related to the tenths markers). The 8 fraction trials were matched with equivalent decimals and percentages for a total of 24 trials. (See Appendix A for all assessment items)

Fraction Arithmetic Estimation

Rationale: Based on the integrated theory of numerical development (Siegler et al, 2011), I theorized that individuals who understand the relation among rational numbers would be less likely to make egregious errors with fraction arithmetic estimation (e.g., 1/3+1/3=2/6), which is in line with the first hypothesis. Moreover, it was also theorized that students who received number line training through a Simultaneous rather than Sequential approach would make greater improvement on fraction arithmetic estimation at posttest, which is in line with the second hypothesis. So, a measure was used that directly assessed students' fraction arithmetic estimation.

Procedure: Students were presented 20 problems, 10 for fraction addition and 10 for fraction subtraction, one at a time on a computer screen. Students were instructed not to compute the exact answer but to generate the nearest number to the answer, whether it be a fraction, a decimal, a percentage or a whole number. They were given 20 seconds to answer the question before the program automatically moved onto the next problem. They input their answer by clicking on a box and typing their response. (See Appendix A for all assessment items)

Magnitude Comparison Across Notations

Rationale: Magnitude Comparison within notation has been utilized widely in the field as a measure of magnitude representation for fractions, decimals, and whole numbers (DeWolf et al., 2014; Meert et al., 2010; Moyer & Landauer, 1967; Schneider & Siegler, 2010). More recently, magnitude comparison across fractions, decimals, and whole numbers in both adults (Ganor-Stern, 2013; Hurst & Cordes, 2016) and children (Hurst & Cordes, 2018) has provided compelling evidence that all notations are represented on an integrated continuum (Siegler et al, 2011). Thus, I sought to assess middle school students' magnitude comparison ability across rational number notations (fractions, decimals, and percentages) as a measure of integrated magnitude representation. I chose to use percentages rather than whole numbers because little is known about percentages (Tian & Siegler, 2018) and understanding of percentage is an important linking representation between rational numbers (Moss & Case, 1999). Importantly, I reasoned that integrated number sense is likely to be higher if there is understanding of fractions and its interrelated concepts: decimals and percentages (Hiebert & Carpenter, 1992). Thus, I theorized in the first hypothesis, that students who represent magnitude along an integrated continuum would be able to select the larger value independent of notation and without any evidence of a heuristic (e.g., percent is larger than fraction). Furthermore, I planned to use overall performance accuracy as a measure of integrated magnitude representation similar to how Hurst and Cordes (2018) operationalized the composite score of comparisons across fractions, decimals, and whole numbers as rational number magnitude ability.

Procedure: Students were presented with 24 comparison problems across rational number notations with identical or nearly identical digits (e.g., compare 4/5 versus 45%) to assess their integration of notations. I planned to examine how students performed across six category types (Fraction>Percent, Percent>Fraction, Decimal>Percent, Percent>Decimal, Fraction>Decimal, Decimal>Fraction) to determine whether students view rational numbers as equivalent in size. Theoretically, if students perceived rational numbers as equivalent in size there should be no difference in performance between related trials (e.g., performance should be about the same if the percent is larger or smaller value when compared to fractions). Following analysis of pretest results, there was a concern that there might be some confounding factor with the selection of values for given trials. Thus, a revision was made to include 18 trials with identical digits as before and 18 trials were matched for magnitude across all notations with small, medium, and large differences between compared values (e.g., compare .40 versus 25%, 2/5 versus .25, .4 versus 1/4, etc.). (See Appendix A for all assessment items)

Standardized math achievement tests

Rationale: It is common practice in the field to use standardized tests of math achievement as a necessary control for math general knowledge and as an outcome variable. For example, I wanted to examine whether other predictors explain estimation ability above general math knowledge.

Procedure: I obtained the children's percentile rank for the mathematics section of the TerraNova, a standardized test administered to private school students in New Jersey and private and public schools in other states around the country. The test was given toward the end of the students' previous grade level, about a year before the study began. These tests served as measures of students' overall mathematical ability.

Student Demographic Information

Rationale: It is common practice in the field to use relevant demographic information as control a control in certain analyses.

Procedure: The school provided relevant demographic information about students' gender, disability status, and English Language Learning (ELL) status. This information was de-identified and utilized as necessary controls and to examine any trends in the data across conditions.

General Procedure

Each student completed a pretest on the computer with the primary researcher during one math period. Most students completed the assessment in approximately 30 minutes. Then, students completed 15 daily warm-up activities as designated by their condition spread out over a little more than 3 weeks though not entirely successive days for each class due to school functions. The principle researcher delivered the instructions for the warm-up activity at the start of class and the regular classroom teacher taught the remainder of the class period. After completing 15 lessons of warm-up activities according to condition, students took a posttest that included half of the same items as the pretest and half new items (see Appendix A for all assessment items).

General Overview for Lessons 1-15

Students in the Simultaneous condition focused on equivalent fraction, decimal, and percent values throughout all lessons and students in the Sequential condition focused on specific notations by week (fractions for Lessons 1-5, decimals for Lessons 6-10, and percentages for Lessons 11-15). The Simultaneous condition received general review of fractions, decimals, and percentages in Lesson 1. The Sequential condition received general review of fractions in Lesson 1, general review of decimals in Lesson 6, and general review of percentages in Lesson 11.

Students were provided feedback on their number line placement by receiving a "rational number ruler" to check the accuracy of the placement of the value. Lessons were approximately 5 minutes, though slightly longer during initial activities.

Part 1. Lesson 1-3 of the intervention involved:

(1) General review of rational number concepts by condition with a scripted PowerPoint presentation

(2) Opportunity to Estimate the Amount that was shaded in an area model (Simultaneous used a battery image and Sequential used a continuously shaded rectangle)

(3) Practice partitioning the area model to determine precisely how much of the battery was shaded with a centimeter ruler

(4) Students plotted this value on the number line (Simultaneous plotted the value on a fraction, decimal, percent number line and the Sequential condition plotted the value on a fraction number line)

Part 2. Lessons 4-15 of the intervention did not present students with an area model. Instead, the students were presented with a value and then estimated its magnitude by shading a small area model to represent it and plotting the magnitude on the number line.

Detailed Description of Instructional Interventions

Part 1. Use of An Area Model in Conjunction with Number lines

The first three days of the intervention involved connecting area models to the number line. Day 1 of the intervention involved review of rational number concepts by linking an area model to the number line. Day 2 and Day 3 also included exploration of how an area model connected to number lines. The primary difference between the two conditions was that the

Simultaneous condition focused on highlighting the connection among the notations during review and the Sequential condition focused on individual notations separately and Sequentially over the course of three weeks. In other words, the Sequential condition focused solely on fractions during the first week, decimals during the second week, and percentages during the third week. No attempt was made to connect understanding among the notations in the Sequential condition; whereas, the primary goal of the Simultaneous condition was to link understanding among the notations. To make the connection among notations more salient for the Simultaneous condition, instructional materials used an image of a battery power indicator to encourage students to naturally draw connections between percent and fractions. This was similar to how Moss and Case (1999) tried to highlight the connection between percentages and fractions with a halving/splitting method for determining how full the beakers were. On the other hand, the Sequential condition utilized a simple rectangle (Figure 2). Finally, it is important to note, that the Simultaneous condition received review of all rational number concepts on the same day and reinforced throughout the entire intervention. Whereas, the Sequential condition received the review of rational number concepts in a Sequential order and reinforced only during the week of that notation being emphasized. In other words, the Sequential condition received review of the fraction concepts in Lesson 1 and reinforced fraction concepts throughout Lessons 1-5, they received review of the decimal concepts in Lesson 6 and reinforced decimal concepts throughout Lessons 6-10, and a review of the percent concepts on Day 11 and reinforced percent concepts throughout Lessons 11-15. Moreover, students in the Sequential condition were encouraged to estimate with using only the designated notation of focus during each week, though many students in the Sequential condition gave percent estimates at times. Students were redirected to focus only on the notation of the week if they spoke about reasoning with a different notation during a week designated as a particular notation (e.g., spoke about percent during the fraction week).

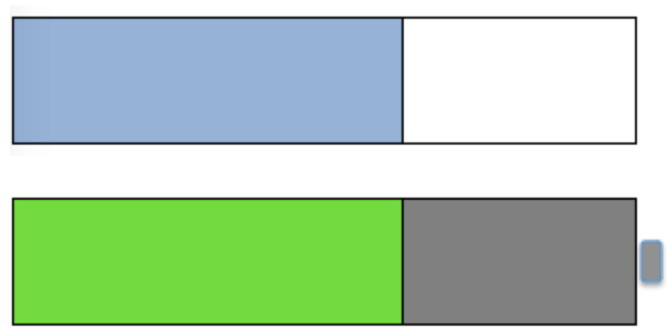


Figure 2: Area model used in instruction by condition. Sequential condition (top) and Simultaneous condition (bottom).

During these first three warm-up activities, students focused on estimating the amount of the figure that was shaded in by generating a number. Then, they used a centimeter ruler to partition the shape into equal parts to try to generate an exact number to represent the part that was shaded in. Students could not always generate an exact number, though the process of trying to determine a number was likely beneficial given research indicating the value of invention versus tell-and-practice (Schwartz, Chase, Oppezzo, & Chin, 2011). Immediately after trying to generate a value that could represent the shaded part of the image, students were instructed to work on placing a fraction on a number line marked with endpoints 0 and 1. In addition to placing a fraction on a number line, students in the Simultaneous condition also had to generate/place the equivalent decimal on a number line marked with endpoints 0 and 1. The decimal line was partitioned into ten parts and labeled by 0.1. Students in the Simultaneous condition also had to generate/place the equivalent percent on a number line with endpoints 0 and 100%, partitioned into ten parts and labeled by 10%. All students were encouraged not only to label the value on the number line, but also to shade above the number line to represent the magnitude of that value. The purpose of shading above the number line was to remind them that the value conveys an absolute magnitude and a physical location on the number line (Figure 3). Students in both conditions were encouraged to notice the position of the value relative to the endpoints, which involved noticing that the values were in equivalent locations across the fraction, decimal, and percent number lines in the Simultaneous condition only.

Figure 3: Number line displayed in instructional presentation for 7/20 for both Simultaneous and Sequential Condition during Week 1.

Finally, all students were provided with a "rational number ruler" to check the accuracy of their estimate. In other words, students were provided with a strip of paper that displayed a number line that was partitioned and labeled according to the value that was being estimated (Figure 4). It is important to note that the Sequential condition only received a "fraction ruler" during the week focused on fractions, a "decimal ruler" during the week focused on decimals, and a "percent ruler" during the week focused on percentages. Because the Simultaneous condition always was presented with partitioned decimal and percent number lines in their student activity book, the Simultaneous condition was only given a "fraction ruler."

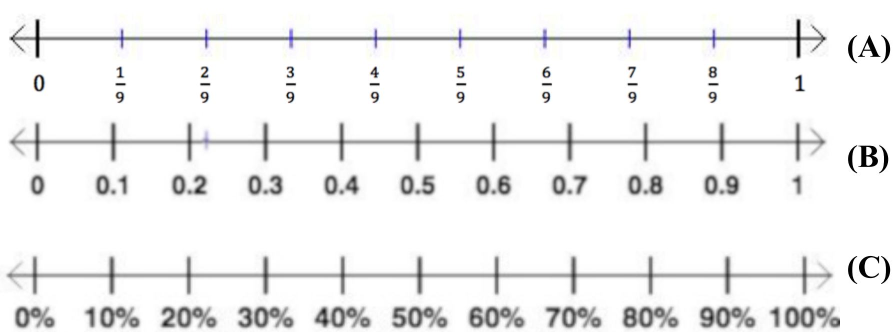


Figure 4: Images of "Rational Number Rulers."
(A) "fraction ruler" (B) "decimal ruler" (C) "percent ruler."

Part 2.

During Lessons 4-15 of the intervention, students practiced estimating a value and then placing it on the number line. The procedure for estimating a value was to either shade in a rectangle (Sequential condition) or battery power icon (Simultaneous condition) to represent the approximate size of the value. Instruction always encouraged students to think about known values to guide their estimates. Specifically, the instruction encouraged cross-notation thinking in the Simultaneous condition (e.g., 2/9 can be thought of as about 20% because I know that 2/10 is 20% and I know 2/9 is going to be less than 50% because 4.5 is half of 9) and same notation thinking in the Sequential condition (e.g., 2/9 can be thought of as 2/10 and I know 2/9 is less than half because 4 1/2 is half of 9). Once students estimated the size of the value, they moved onto the number line activity. The students in the Simultaneous condition had to place the equivalent fraction, decimal, and percent on the corresponding number line, and the principle researcher highlighted understanding of percentages as critical for helping them make a better estimate of the fraction. On the other hand, students in the Sequential condition just placed a fraction on the number line (Lessons 1-5), just a decimal on the number line (Lessons 6-10), or just a percent on the number line (Lessons 11-15). Students in the Simultaneous condition were provided with a “fraction ruler” to check their estimates and students in the Sequential condition were provided with either a “fraction ruler,” “decimal ruler,” or “percent ruler” to check the accuracy of their estimates, according to what topic was being emphasized that week (e.g., Fractions covered Lessons 1-5, Decimals covered Lessons 6-10, and Percentages covered Lessons 11-15). The reason students in the Simultaneous condition did not also receive a “decimal ruler” or “percent ruler” was because students in the Simultaneous condition always had a decimal number line labeled with deciles and a percent number line partitioned and labeled by 10% below the fraction number line in their activity book (Figure 5). An overview of the values involved in each lesson across condition is listed in Appendix B and an example of the difference in student activity book appears below (Figure 5). See Appendix B for more examples of student activity book pages across several lessons. Finally, all students were always asked to consider and write a response for the question: "What did you do to help you figure out a good answer?"

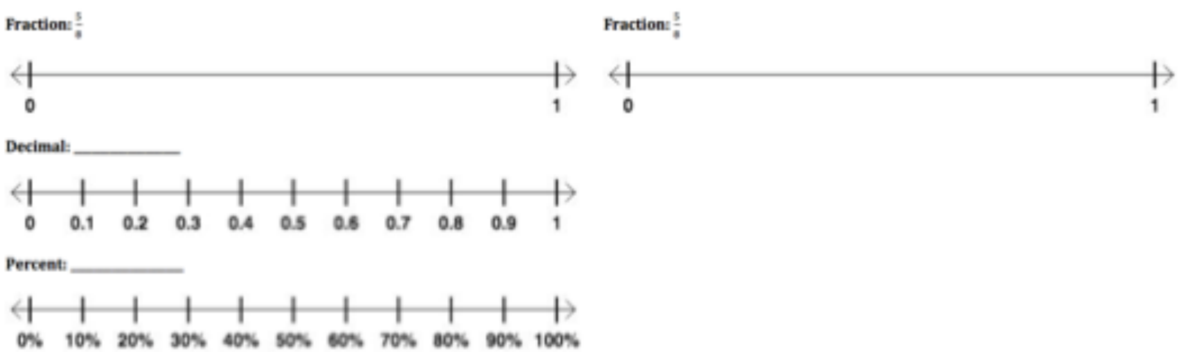


Figure 5: Comparison of student activity book by condition: Simultaneous (left) and Sequential condition (right). Note: 5/8 activity page was shrunk to be compared adjacently.

In sum, students in the Simultaneous and Sequential condition received practically identical instruction as a brief warm-up activity at the beginning of class. The primary difference was that the Simultaneous condition received review of each notation on the first day of instruction; whereas, the Sequential condition received review of each notation separately and sequentially (e.g., received review of fractions on day 1, review of decimals on day 6, and review of percentages on day 11). The Simultaneous condition provided three number lines daily (a fraction, a decimal, and a percent number line) and encouraged students to leverage understanding of equivalency among notations to be precise with placing equivalent values on the appropriate lines. The Sequential condition encouraged students to focus on one notation per week as they worked on being precise with placing values on a fraction number line for the first week, a decimal number line for the second week, and a percent number line for the third week.

Analyses

Number line estimation task performance was measured using percent absolute error (PAE), defined as |Student's Answer–Correct Answer|/Numerical Range. For example, a participant estimating 3/5 on a 0-1 number line at the location corresponding to 0.65 would result in a PAE of 0.5 (|0.65–0.6|/1 = .05). Therefore, lower PAE indicates higher accuracy.

Magnitude comparison accuracy was measured using percent correct. Performance in across notation comparison in the six categories (Percent>Fraction, Fraction> Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction) was analyzed to detect trends in overall perception of the size of individual notations as it was hypothesized that students did not think about equivalent rational numbers as being equivalent in size.

Additionally, performance across the different notations on the decile number line task was compared to determine whether there were differences in placing equivalent fraction, decimal, and percent values on the number line. If student performance was different across notations, this would provide further support of the hypothesis of rational numbers not being perceived as equivalent in size.

Due to an issue with the testing format (described later), the open response answers for the arithmetic estimation task were examined and categorized based on the type of strategy the student employed to estimate the fraction addition and subtraction problems. Thus, I decided not to quantify student performance on the fraction arithmetic estimation task in the present study but to use student errors to inform the subsequent experiment.

Because classes were grouped by ability, an attempt was made to match classes during assignment to condition based on ability level. A number of one-way ANOVAs were conducted to determine whether there were any differences across conditions in demographics (e.g., grade, general ability, gender) or relevant measures. There were no differences with the exception that Percent Decile Number Line estimation was significantly worse in the Simultaneous condition at pretest ($p=.022$). Moreover, because the data included students nested within classes, multi-level analysis was considered. However, because the sample was insufficient with only two classes per condition (Maas & Hox, 2005), I proceeded by using students' scores as the unit of analysis.

Ultimately, paired *t*-tests were used to compare performance on the pretest and posttest within each condition, and change scores (i.e., difference in performance from pretest to posttest) were submitted to *ANCOVA* with condition as between-subjects factor and pretest score as covariate.

3.4.2 Results

Table 1 shows students' performance on all tasks at pretest and post-test in the Simultaneous and Sequential conditions.

Table 1: Mean (standard deviation) performance on assessment tasks by condition and test time. Note: PAE denotes percent absolute error.

	Sequential Condition		Simultaneous Condition	
Task	Pretest	Posttest	Pretest	Posttest
0-1 Number line (No Partitions) PAE	.13 (*.15*)	.072 (*.08*)	.17 (*.15*)	.07 (*.05*)
0-5 Number line (No Partitions) PAE	.24 (*.13*)	.21 (*.11*)	.23 (*.12*)	.24 (*.14*)
Fraction Decile Number line (0-1 line partitioned and labeled by tenths) PAE	.151 (*.14*)	.114 (*.10*)	.20 (*.20*)	.11 *(.09*)
Decimal Decile Number line (0-1 line partitioned and labeled by tenths) PAE	.18 *(.16)*	.12 *(.14)*	.21 *(.15)*	.09 *(.15)*
Percent Decile Number line (0-1 line partitioned and labeled by tenths) PAE	.03 *(.05)*	.01 *(.02)*	.09 *(.11)*	.01 *(.02)*
Integrated Magnitude Comparison (Across Notations) % Correct	.69 *(.15)*	.83 *(.11)*	.76 *(.15)*	.81 *(.16)*

Number line Estimation (No Partitions, 0 to 1 endpoints)

PAE on this task improved (i.e., decreased) in both the Sequential condition, t(21)=2.118, p=.046, *d*= .45 and the Simultaneous condition, t(20)=3.38, p=.003 d=.74. When change scores were submitted to ANCOVA with pretest scores as a covariate, there was no significant difference in improvement by condition (p>.05). This result indicates that, while each condition improved, the improvement was not substantially greater in one condition over the other.

Number line Estimation (No Partitions, 0 to 5 endpoints)

PAE on this task did not improve in either the simultaneous or sequential conditions ($p>.05$).

Fraction Decile Number line (Partitioned and labeled by tenths, 0 to 1 endpoints)

PAE improvement on this task did not reach statistical significance in the Sequential condition $t(21)=1.669$, $p=.110$. However, PAE did improve (i.e., decrease) in the Simultaneous condition, $t(20)=2.489$, $p=.022$, $d=.54$. When change scores were submitted to ANCOVA with pretest scores as a covariate, there was no significant difference in improvement by condition ($p>.05$). Thus, improvement was not substantially greater in one condition over the other.

Decimal Decile Number line (Partitioned and labeled by tenths, 0 to 1 endpoints)

PAE improvement on this task did not reach statistical significance in the Sequential condition $t(21)=1.654$, $p=.113$. However, PAE did improve (i.e., decrease) in the Simultaneous condition, $t(20)=2.543$, $p=.019$, $d=.55$. When change scores were submitted to ANCOVA with pretest scores as a covariate, there was no significant difference in improvement by condition ($p>.05$). Thus, improvement was not substantially greater in one condition over the other.

Percent Decile Number line (Partitioned and labeled by tenths, 0 to 1 endpoints)

PAE improvement on this task did not reach statistical significance in the Sequential condition $t(21)=1.70$ $p=.104$. However, PAE did improve (i.e., decrease) in the Simultaneous condition, $t(20)=3.276$, $p=.004$, $d=.71$. It should be noted that this finding should be interpreted with caution, as there was a significant difference in pretest scores, such that the average scores were worse in the Simultaneous condition than the Sequential condition. That being said, when change scores were submitted to ANCOVA with pretest scores as a covariate, there was no significant difference in improvement by condition ($p>.05$). Thus, improvement was not substantially greater in one condition over the other.

Relation among Notations on Number line Performance

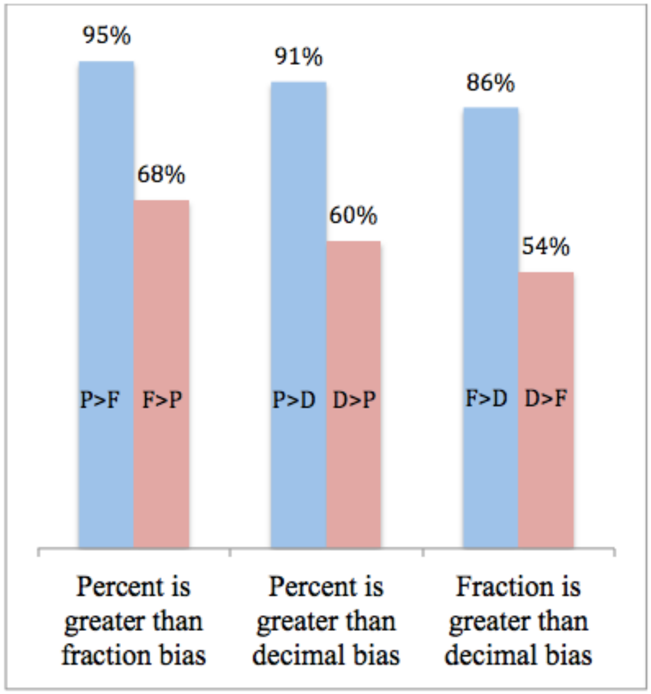
The relative performance across decile number line estimation for the different notations (fractions, decimals, percentages) was analyzed using a repeated-measures analysis of variance (ANOVA) on PAE. There was a main effect of notation $F(2, 84) = 12.190$, $p < .001$, $\eta^2 = .225$, suggesting that students performed best on the percent decile number line estimation task ($M_{percent}=.06$) compared to the fraction and decimal decile number line estimation. However, Bonferroni test for multiple comparisons demonstrated there was no difference in performance ($p>.05$) between fraction decile number line estimation ($M_{fraction}=.18$) and decimal decile number line estimation ($M_{decimal}=.20$).

Magnitude Comparison Across Notations

Analyses were conducted for magnitude comparisons across all notation comparison problems as well as looking at specific notation comparison types (Percent>Fraction, Fraction> Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction).

Across all Notation Comparison Problems: Percent correct in magnitude comparison across notations improved in the Sequential condition $t(21)=-5.997$, $p<.001$, $d= 1.278$; the students in the Simultaneous condition made marginal improvement $t(20)=-1.982$, $p=.061$, d=.43. When change scores were submitted to ANCOVA with pretest scores as a covariate, there was a marginal significant difference in improvement favoring the Sequential condition ($p=.08$).

Specific Notation Comparisons: To test the hypothesis that students do not think about equivalent rational numbers as being equivalent in size, I examined accuracy across trials to determine whether students perceive one notation as larger than another. Figure 6 displays students' performance across the six categories of comparisons at pretest: Percent>Fraction, Fraction> Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction. At pretest, accuracy was 95% for items where percent is larger than the fraction, as compared to 68% accurate for items where percent is smaller than fractions. Moreover, accuracy was 91% for items where the percent was larger than the decimal, as compared to 60% accurate for items where the decimal was smaller than the percent. Students were 86% accurate when the fraction was greater than the decimal and 54% accurate when the decimal was greater than the fraction. Paired t-tests were conducted to determine whether the difference between these categories of comparison were significant. Results demonstrated that middle school students have a bias towards perceiving percentages as larger than fractions/decimals and fractions as larger than decimals, as evidenced by statistically significant differences between mean scores for items where Percent>Fraction and Fraction>Percent ($t(42)= 5.287$, $p<.001$), Percent>Decimal and Decimal>Percent ($t(42)= 4.937$, $p<.001$) and Fraction>Decimal and Decimal>Fraction ($t(42)=6.750$, $p<.001$).



(A) (B) (C)

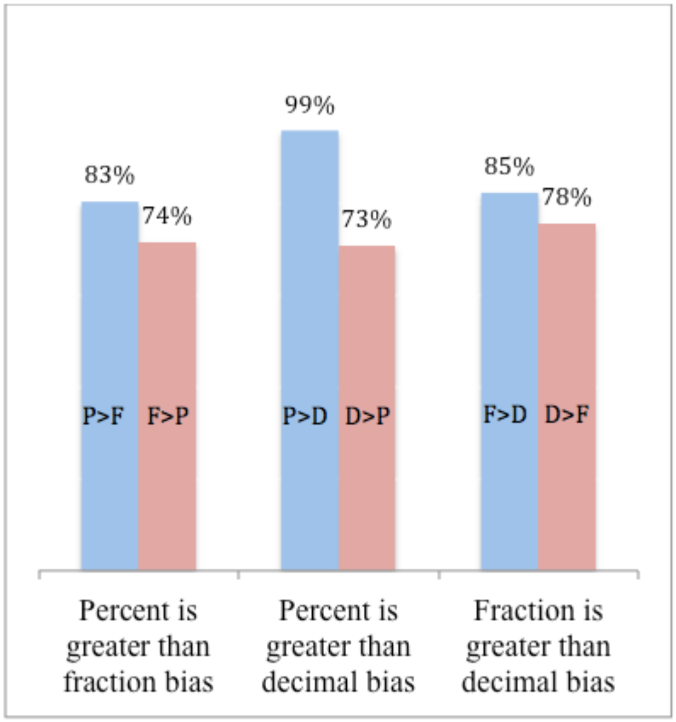
Figure 6: Experiment 1 pretest accuracy for magnitude comparison across notations. The chart is segmented by students' inferred biases of which notation is greater. The data includes all 7th and 8th grade students. The comparison types include: (A) Percent-to-Fraction Comparisons, (B) Percent-to-Decimal Comparisons, and (C) Fraction-to-Decimal Comparisons.

Given the results, there was a concern that the *percent is greater bias* result was influenced by a confounding factor (e.g., the ratio between compared values was not equivalent across comparison types). As such, the posttest was modified to include more trials (24 at pretest and 36 at posttest). Furthermore, the posttest included half of the trials with comparisons across notations with identical digits (e.g., 3/5 versus 35%) and half with trials that were matched for magnitude across all notations with small, medium, and large differences between compared values (e.g., compare .40 versus 25%, 2/5 versus .25, .4 versus 1/4, etc.). The pretest only included identical digits and not trials that matched for magnitude across notations. The inclusion of these trials would help to control for absolute magnitude of values. Furthermore, I decided that a subsequent experiment would also include these modifications as part of the pretest design.

However, caution should be used in interpreting any pretest to posttest changes in the current experiment because the task was altered substantially in the aforementioned way from pretest to posttest.

Figure 7 displays children's performance across the six categories of comparisons at posttest: Percent>Fraction, Fraction> Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction. At posttest, accuracy was 83% for items where percent is larger than the fraction, as compared to 74% accurate for items where percent is smaller than fractions. Moreover, accuracy was 99% for items where the percent was larger than the decimal, as compared to 73% accurate for items where the decimal was smaller than the percent. Students were 85% accurate when the fraction was greater than the decimal and 78% accurate when the decimal was greater than the fraction. Paired t-tests were conducted to determine whether the difference between these categories of comparison were significant.

Results demonstrated there was a marginally significant difference between P>F and F>P, such that middle school students have less of a bias towards perceiving percentages as larger than fractions in the posttest as a result of the intervention ($t(42)= 1.933$, $p=.06$). Additionally, there was a marginally significant difference between F>D and D>F, such that middle school students have less of a bias towards perceiving fractions as larger than decimals in the posttest as a result of the intervention ($t(42)=1.833$, $p=.07$). However, the bias of perceiving percentages as larger than decimals still held as there was a difference between P>D and D>P, such that students are more accurate when the percent is larger than the decimal ($t(42)=5.56$, $p<.001$). It is important to note that caution should be used in interpreting these changes from pretest to posttest due to the increased number of magnitude comparison trials and the modifications to the design of the measure after the pretest to control for magnitude across all notations. The Magnitude Comparison Across Notations task would reflect these changes in both pretest and posttest for the subsequent experiment.



(A) (B) (C)

Figure 7: Experiment 1 posttest accuracy for magnitude comparison across notations. The chart is segmented by students' inferred biases of which notation is greater. The data includes all 7th and 8th grade students. The comparison types include: (A) Percent-to-Fraction Comparisons, (B) Percent-to-Decimal Comparisons, and (C) Fraction-to-Decimal Comparisons.

The students in the Simultaneous and the Sequential conditions followed a similar pattern of results at pretest – greater accuracy when the percent was larger than the fraction/decimal and greater accuracy when the fraction was larger than the decimal. One-way ANOVAs yielded that there was no difference in these biases by condition at pretest ($p>.05$ for all biases). However, the pattern of results was slightly different at posttest by condition for fraction-to-percent and fraction-to-decimal comparisons. Figure 8 displays side-by-side comparisons for Simultaneous and Sequential posttest results. For the fraction-to-decimal comparisons, there is no difference in performance whether one is larger than the other for the Simultaneous condition ($t(20)=.309$, p=.761). There is a difference for the Sequential condition (t(21)=2.153, p=.043), such that accuracy is higher when the fraction is larger than the decimal. For the fraction-to-percent comparisons, there is a difference in performance whether one is larger than the other for the students in the Simultaneous condition (t(20)=2.216, p=.038), such that they perform better when the percent is larger. There is no Percent>Fraction bias for the students in the Sequential condition (t(21)=.130, p=.898).

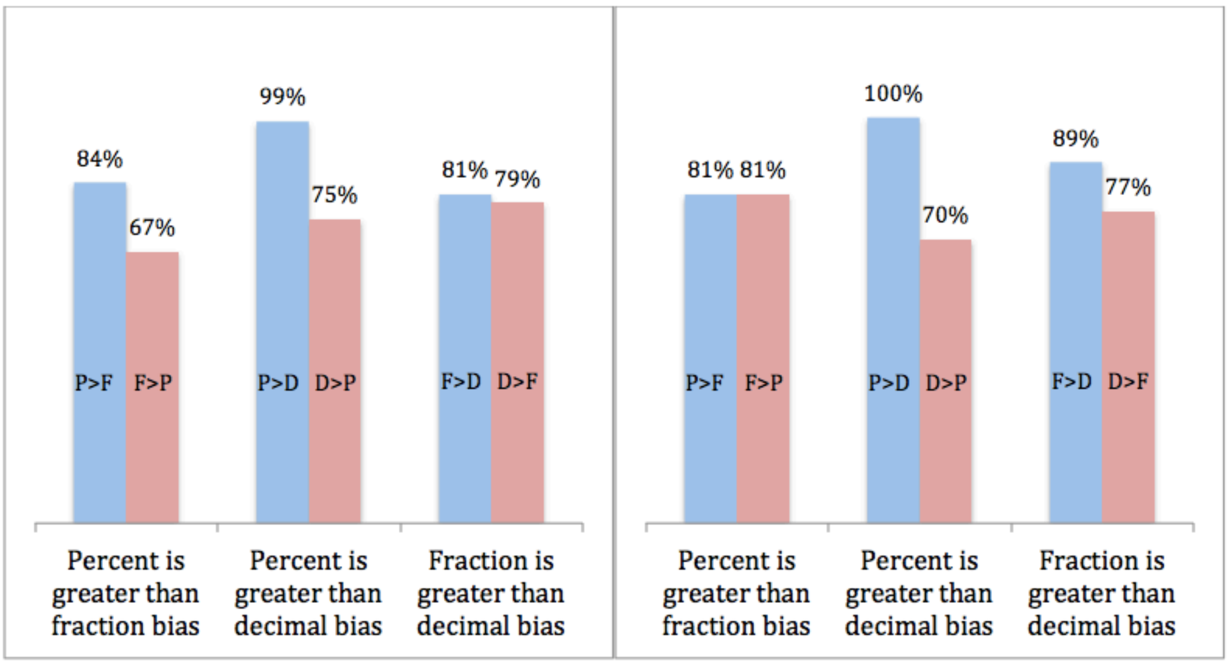


Figure 8: Experiment 1 posttest accuracy for magnitude comparison across notations by condition: Simultaneous (left) and Sequential (right).

To better understand how these biases affected individual students, a 'Bias Score' was generated for each student on each bias (Percent greater than fraction, Percent greater than decimal, and Fraction greater than decimal). The Bias Score was defined as the difference in average performance between items congruent with the bias and items incongruent with the bias (e.g., performance on items where the percent was larger than the fraction minus performance on items where the fraction was larger than the percent). In other words, if a student was 90% accurate on items where the percent was larger than the fraction and 60% accurate on items where the fraction was larger than the percent, the student would receive a bias score of 30% for the Percent>Fraction bias.

Examination of Bias Score by condition at posttest (Figure 8) suggests that the Percent>Fraction bias is weakest in the Sequential condition at posttest. The mean difference in percentage points between P>F and F>P is 0 for students in the Sequential condition, as compared to 17 for students in the Simultaneous condition. When posttest Bias scores were submitted to ANCOVA with condition as between subject effect and controlling for pretest Bias scores, gender, and standardized achievement test, the effect of condition was significant $F(1, 36) = 7.964$, p =.008, $\eta^2 = .181$, suggesting that the bias is considerably weaker in the Sequential condition at posttest.

Examination of Bias Score by condition at posttest (Figure 8) suggests that the Percent>Decimal bias is weakest for students in the Simultaneous condition at posttest. The mean difference in percentage points between P>D and D>P is 24 for students in the Simultaneous condition, as compared to 30 in the Sequential condition. When posttest Bias scores were submitted to ANCOVA with condition as between subject effect and controlling for pretest Bias scores, gender, and standardized achievement test, the effect of condition was not significant $F(1, 36) = .71$, p =.405, $\eta^2 = .019$, suggesting that there is no difference in the strength of the bias at posttest between conditions.

Examination of Bias Score by condition at posttest (Figure 8) suggests that the Fraction>Decimal bias is weakest in for students in the Simultaneous condition at posttest. The mean difference in percentage points between F>D and D>F is 2 for students in the Simultaneous condition, as compared to 12 for students in the Sequential condition. When posttest Bias scores were submitted to ANCOVA with condition as between subject effect and controlling for pretest Bias scores, gender, and standardized achievement test, the effect of condition was not significant $F(1, 36) = 1.069$, $p = .308$, $\eta^2 = .029$, suggesting that there is no difference in the strength of the bias at posttest between conditions.

Arithmetic Estimation Strategies

The analysis of arithmetic estimation strategies was exploratory in nature rather than quantifiable for two reasons: (1) the format of the fraction arithmetic estimation task was open response via Qualtrics, resulting in difficulties for some students due to the unfamiliar format (e.g., many students were unsure of how to type a fraction value using a keyboard), and (2) it was a timed task, resulting often in incomplete answers. Therefore, the analysis focused on understanding the strategies students were using to estimate fraction addition and subtraction solutions by examining their responses. Since the instructors and principal researcher did not directly ask students about the strategies that they employed, the strategies students were employing cannot be determined with certainty. Although, there were many examples of clearly defined strategies that many students employed based on their responses. I decided to move ahead with the analysis given the systematic errors many students made in obtaining incorrect responses and because research has shown value in examining student work for understanding students' mathematical activity (Kazemi & Franke, 2004).

The most common and easily identifiable systematic-error strategies found in student estimation responses are the Across Strategy and the Hybrid Across Strategy. The other strategy that was unequivocally employed was the Calculate the Exact Answer Strategy, which does not reflect an error but rather a failure to provide an estimation. The Across Strategy involves treating numerators and denominators as whole numbers and adding/subtracting across numerators or denominators (e.g., 1/3+1/3=2/6). The Hybrid Across Strategy involves finding a common denominator and then adding/subtracting across numerators (e.g., 1/3+1/3=2/9). The Calculate Exact Answer Strategy involves providing an exact answer for the problem with a common denominator (e.g., 8/10-1/9=62/90). Some students were accurate calculating the answer exactly in this manner but obviously their answers do not reveal whether they can estimate answers for fraction arithmetic.

Table 2 displays the different types of systematic-error strategies students used to estimate the answer to fraction addition and subtraction problems. Also, there were some other strategies that were not used consistently throughout the task. These strategies are more difficult to define and cannot be verified without individual student reports on which strategies they used. As such, these inconsistent strategies are labeled in Table 2 with an asterisk due to being speculative in nature.

The improvement from pretest to posttest will not be quantified due to difficulties characterizing the exact strategies each student used without individual student reports. However, it appears that the systematic-error strategies used were less frequent at posttest and that some students who used an inappropriate strategy at pretest (e.g., the Across Strategy) moved to more appropriate estimation strategies at posttest. This is just speculation and may warrant further investigation in future studies. Ultimately, the arithmetic estimation strategies task provided evidence of common flawed estimation strategies that students may systematically be using. The learning from Experiment 1 informed the design of a more appropriate fraction arithmetic estimation task for Experiment 2 discussed in the next section.

Table 2: Inferred Fraction Arithmetic Estimation Strategies at pretest in Experiment 1.

Strategy	Definition	Example
Across Strategy (inappropriate)	Student adds across numerators and denominators	$\frac{5}{8}+\frac{1}{6}=\frac{6}{14}$
Hybrid Across Strategy (inappropriate)	Find a common denominator and then add across the numerators	$\frac{5}{8}+\frac{1}{6}=\frac{6}{48}$
Calculate Exact Answer (often with egregious errors)	Find a common denominator and calculate mentally	$\frac{8}{10}-\frac{1}{9}=\frac{62}{90}$
*First fraction Estimation Strategy	Estimate the size of first fraction and add/subtract the numerator of the second fraction (often without regard for the size of the second fraction)	$\frac{5}{8}+\frac{2}{3}=\frac{7}{8}$
*Common Denominator Estimation Strategy	Find a common denominator and make an educated estimate about what the numerator would be	$\frac{4}{10}+\frac{2}{8}=\frac{50}{80}$
*Percent Strategy	Translate the fractions to percentages and estimate the answer as a percent	$\frac{5}{6}-\frac{2}{4}=40\%$
*Decimal Strategy	Translate the fractions to decimal and estimate the answer as a decimal	$\frac{8}{10}+\frac{1}{9}=.9$
*Fraction Strategy	Transform the fractions into easier numbers to work with to arrive at a fraction solution that does not involve a common denominator, typically a canonical or benchmark fraction answer	$\frac{4}{9}-\frac{1}{5}=\frac{1}{4}$

*Speculative strategy: it is quite difficult to know for sure without student report

Individual differences in predicting estimation ability in the presence of distracting information

The decile number line estimation task measures a student's ability to attend to magnitude and estimate a fraction, decimal, or percent's location on a number line in the presence of distracting information (i.e., decile partitions). This analysis seeks to test Hypothesis 1, integrated number sense, by examining individual differences in estimation ability in the presence of potentially distracting information.

Students performed worse on the decile number line task (PAE=.18) as compared to the unlabeled 0-1 number line task (PAE=.15), though the task was not significantly more distracting overall – a paired t-test demonstrated no difference in performance on number line estimation for labeled versus unlabeled number line PAE ($t(42)=.997$, $p=.324$). However, considering the documented potential for distraction (Siegler & Thompson, 2014), the PAE for placing fractions on the decile line was operationalized as how students perform on estimation in the presence of distraction. Thus, hierarchical linear regression (Table 3) was used to determine whether each additional variable explains additional variance. The following three variables were added to the model sequentially to determine whether they added more explanatory power to performance in placing fractions on the decile number line:

1. *Achievement score:* The Achievement score, which is operationalized as the students' percentile rank on the TerraNova standardized test of math achievement, was added to the model first to account for any predictive value of general math ability. The Achievement score was a significant predictor of performance on the decile number line task (Step 1, $p<.001$).
2. *0-1 Number Line PAE:* 0-1 Number line PAE (on an unlabeled 0 to 1 number line) was entered because it was hypothesized to be most closely related to how students

would perform on the Decile Number line task, given that how students perform on a 0-1 number line should be the same as a 0-1 number line partitioned into tenths. However, the R^2 change was not significant and both variables (unlabeled 0-1 number line estimation and Achievement) were non-significant (Step 2, $p>.05$ for both).

3. *Cross-Notation Comparison:* Cross-Notation Comparison, magnitude comparison across notations, ability was entered last. Cross-Notation Comparison ability was calculated as a composite of the scores (i.e., average percentage correct) on comparisons that required students to compare between distinct notations (i.e., Fraction versus Decimal, Decimal versus Percent, and Percent versus Fraction). This variable was added to the model because it was theorized that integrated number sense would help students persevere in attending to magnitude in the presence of distraction. When it is added to the model, Cross-Notation Comparison is a significant predictor (Step 3, $p<.001$), and it adds 11% of explanatory power to the model predicting Decile Number line PAE. The fit of the final model is significant $F(3,40)=4.152$ $p=.012$, suggesting that when controlling for Achievement and Unlabeled 0-1 Number line estimation, Cross-Notation Comparison is the only significant predictor of decile number line PAE.

The hierarchical linear regression analysis suggests that higher performance on the Cross-Notation Comparison task is associated with better performance on the decile number line task when controlling for general math ability and unlabeled number line performance. This finding is consistent with the first hypothesis about integrated number sense, as indicated by performance on the magnitude comparison task, helps students fight their way through the distracting number line information.

Table 3: Hierarchical Linear Regression Analysis Predicting Fraction Decile Number Line PAE for Experiment 1.

	b (*unstandardized*)	*SE b*	*β* (*standardized*)
Step 1			
Constant	.315	.062	
Achievement	-.002	.001	-.328*
Step 2			
Constant	.240	.085	
Achievement	-.001	.001	.210
Number line 0 to 1 PAE	.223	.178	.222
Step 3			
Constant	.485	.133	
Achievement	.000	.001	-.045
Number line 0 to 1 PAE	.089	.178	.089
Cross-Notation Comparison	-.403	.174	-.421*

Note. R^2=.11 For Step 1*; ΔR^2=.035 for Step 2, ΔR^2=.11 for Step 3*
*p<.05, **p<.01, ***p<.001

3.4.3 Discussion

Summary

The results of this pilot study served to provide some evidence in support of the hypotheses for a few measures and set the stage for a larger experiment. Below is a summary of the results as they relate to each hypothesis followed by a short discussion of each of the results. An important caveat in interpreting the findings is that the nature of this experiment was exploratory, with the primary purpose being to pilot-test assessment and instructional materials. Some critical limitations of this initial experiment include the lack of a control group and assignment to conditions because students were grouped according to ability level. While an attempt was made to balance conditions based on ability, there was a difference by condition on one pretest measure (i.e., students in the Simultaneous condition performed significantly worse at pretest on percent number line estimation). This difference between conditions could be problematic because it may reflect lack of a thorough understanding of percentage prior to instruction for the Simultaneous condition. Thus, any observed differences might have more to do with lack of thorough understanding at pretest than differences across instructional conditions. Finally, there was a substantial modification to the Magnitude Comparison Across Notations task between the pretest and the posttest, such that only 5 of the 36 posttest trials had appeared on the pretest. The modification to the Magnitude Comparison Across Notations task was made after pretest results yielded newly discovered biases. The modifications were made to control more closely for any potential confounding factors and to pilot-test the assessment with the intent of using the task for a subsequent study. Thus, caution must be exercised when interpreting improvement from pretest to posttest, since it is possible that the pretest may have been unbalanced and measuring different abilities than the posttest.

Hypothesis 1: Integrated Number Sense

The analysis indicates that understanding of the relations among notations may be an important aspect of numerical development beyond fraction magnitude representation alone. As part of the integrated theory of numerical development, Siegler, Thompson, and Schneider (2011) posited that fraction magnitude representation is central to numerical development. However, I argue that fraction magnitude representation alone may not tell the whole story when determining why many students make implausible errors with fraction arithmetic estimation (e.g., 12/13+7/8=19/21). This theory is supported by research documenting that students perform worse on estimating sums of fractions than on estimating fractions individually (Braithwaite, Tian, & Siegler, 2018). Therefore, some students might be able to reason about magnitudes of individual fractions when the task explicitly asks them to do so. However, many students do not apply this knowledge. Understanding of fractions and its related forms (decimals and percentages) is likely a better indicator of magnitude representation because depth of understanding is characterized by strength of connections among related concepts (Hiebert & Carpenter, 1992). Moreover, a focus on analysis of quantity distinguishes successful problem solvers (Haverty et al, 2000). Therefore, students with integrated number sense, or an understanding of how fractions, decimals, and percentages are related to one another, are better equipped with tools to evaluate magnitude. Indeed, it appears that integrated number sense provides more explanatory power than fraction magnitude representations alone in how students deal with situations that could potentially cause them to lose focus on magnitude.

a) Students do not perceive equivalent rational numbers as equivalent in size.

i. Many students do not view equivalent rational numbers as equivalent in size, as evidenced by worse performance when the percentage is the smaller than the fraction/decimal and when fractions are smaller than decimals.

ii. The bias towards perceiving percentages as larger than fractions/decimals is not evident when students place equivalent values written in the three notations on the number line (e.g., if students always thought percentages were larger, then they would place the percent value as larger than the fractions/decimals). However, students demonstrated a high degree of accuracy with placing percentages on the number line and less accuracy placing fractions and decimals on the number line.

b) *Individual differences in integrated number sense predict estimation ability in the presence of distraction.*

i. A decile number line was selected as a distracting task based on documented worse performance on this task than on an unlabeled 0-1 number line (Siegler and Thompson, 2014). Integrated number sense, as operationalized as the composite score on Magnitude Comparison Across Notations, was the only significant predictor of the accuracy of placing fractions on the decile number line, when controlling for 0-1 number line estimation and standardized math scores.

ii. Due to the aforementioned issues with the fraction arithmetic estimation task in Experiment 1, I did not have another measure of how students perform in the presence of distraction. Thus, it remains to be seen whether integrated number sense will also help students inhibit implausible errors (e.g., 12/13+7/8=19/21).

Hypothesis 2: Improving Integrated Number Sense

Based on the assumption that integrated number sense was important, I sought to try to improve integrated number sense through two different number line interventions. I reasoned that both would help improve outcomes but that the Simultaneous Condition would result in greater improvement over the Sequential Condition. The results did not support the hypothesis overall but did on some measures. However, more research needs to be done because it is unclear whether any differences can be attributed to effects of the intervention, since the experimental design involved an attempt to assign classes to condition by ability rather than random assignment to condition. Summary of results from Experiment 1 that were relevant to Hypothesis 2 are outlined below. The section starts with a simplified chart of the measures and whether students in each condition made Significant, Not Significant (NS), or Marginally Significant

(MS) improvement from pretest to posttest. Finally, the third column details whether the Simultaneous or the Sequential Condition led to greater improvement in students' performance.

Table 4: Simplified summary of findings comparing improvement within and between conditions for Experiment 1.

Task	Sequential Improvement	Simultaneous Improvement	Significant Improvement over the other condition
0-1 Number line (No Partitions) PAE	Significant	Significant	NS
0-5 Number line (No Partitions) PAE	NS	NS	NS
Fraction Decile Number line (0-1 line partitioned and labeled by tenths) PAE	NS	Significant	NS
Decimal Decile Number line (0-1 line partitioned and labeled by tenths) PAE	NS	Significant	NS
Percent Decile Number line (0-1 line partitioned and labeled by tenths) PAE	NS	Significant	NS
Integrated Magnitude Comparison (Across Notations) % Correct	Significant	Marginally Significant (p=.06)	Marginally Significant in favor of Sequential over Simultaneous (p=.08)

a) Review of rational number notations with number lines improves outcomes

    i. 0-1 number line estimation improved for both conditions (though I did not have a control condition in Experiment 1; thus, it is difficult to know whether it is number line review per se that drove any improvements).

b) The Simultaneous versus Sequential review of notation is better for math outcomes

    i. Consistent with the hypothesis, students in the Simultaneous condition made significant improvement from pretest to posttest on all decile number line tasks, where the students in the Sequential condition did not make significant improvement. This suggests that students in the Simultaneous condition are able to estimate better in the presence of distracting information (i.e., decile partitions). However, this improvement is not substantially greater. Moreover, it is important to note that there was a significant difference by condition at pretest in percent number line estimation, such that Simultaneous students performed worse. Thus, Simultaneous students may have had greater opportunity for gain on this measure.

    ii. Contrary to the hypothesis, the students in the Sequential condition made significant improvement and students in the Simultaneous condition only made marginally significant improvement on the Magnitude comparison across notation task ($p=.06$). Additionally, the students in the Sequential condition made marginally greater improvement in Magnitude Comparison across notations ($p=.08$). However, an important caveat is that the posttest task was changed substantially such that only 5 of the 36 posttest comparison trials had appeared on the pretest. Moreover, when the modifications are excluded from the analysis, there is no significant difference in improvement by condition controlling for pretest scores (p=.477). Additionally, it is important to note again that students in the Simultaneous condition were significantly worse at pretest on percent number line estimation. Thus, it is highly likely that students in the Simultaneous condition did not have a thorough understanding of percent prior to instruction.

Consistent with the Hypothesis 1a that students do not perceive rational numbers as equivalent in size, middle school students demonstrated a bias towards perceiving percentages as larger than fractions and decimals at pretest. In other words, students were more accurate when the percentages were larger than fractions and decimals than when the percentages were smaller than fractions and decimals. Additionally, they demonstrated a bias towards perceiving fractions as larger than decimals in this sample at pretest. At posttest, the percent is larger than decimal bias persisted, though the other two biases weakened. The fact that these biases are still somewhat present in the data following three weeks of number line training suggest that these skewed perceptions might be a real phenomenon. Though this finding is consistent with research suggesting that students have difficulties with concepts of equivalence in general (McNeil et al, 2006), there is no documented evidence of these biases in the literature. Thus, it is important to test whether any of these biases will replicate in another sample.

Also, though I did not ask students to translate values directly, I utilized equivalent fraction, decimal, and percentage values for the decile number line task to determine whether there might be a mismatch between students' perception of size for equivalent rational numbers (Hypothesis 1a). Consistent with the hypothesis that students do not perceive rational numbers as equivalent in size, is the difference in PAE on the decile number line task where students placed equivalent fraction, decimal, and percent values on a line with endpoints 0-1 that was partitioned and labeled by tenths ($PAE_{percent}$=.06, $PAE_{fraction}$=.18, $PAE_{decimal}$=.20). In other words, students were better at placing percentages on the decile number line than placing their equivalent fractions and decimals on the same number line. There was no difference in performance in placing fractions and decimals on the decile number line. The finding that students were highly accurate with placing percentages on the number line and not as accurate in placing their equivalent decimals/fractions provides additional support that students do not think about equivalent rational numbers as being equivalent in size. In particular, it suggests that students' representations of magnitude for percent are likely most transferrable to new contexts (Moss & Case, 1999), which is one of the essential features of the instructional approach to rational number review in the Simultaneous condition in the current study.

Moreover, consistent with the first hypothesis about the importance of integrated rational number sense (Siegler et al, 2011) in predicting estimation ability, Cross-Notation Comparison, was found to predict fraction estimation ability in the presence of distraction on the decile number line. I argued that estimation involves both attending to magnitude of individual values and attending to magnitude of combining two values to perform an arithmetic operation. Moreover, I posited that the decile number line task is particularly informative of how students perform in the presence of distraction because the tenths partitions do not aid in placing a value on the number line in any meaningful way unless the fractional value is translated to a decimal (Siegler & Thompson, 2014). The finding that performance was (not-significantly) worse on the fraction decile number line task than on the unlabeled number line task is consistent with the finding that students with weaker knowledge of place value perform worse in the presence of distracting information (Miura & Okamoto, 1989). The result that integrated rational number sense added unique explanatory power above and beyond general math ability and number line estimation ability suggests that students who have an integrated sense of rational number are better able to persevere in attending to magnitude in potentially distracting situations.

Furthermore, consistent with the second hypothesis about the superiority of the Simultaneous condition, students in the Simultaneous condition made significant improvement on all decile number line tasks (including fraction, decimal, and percent trials); whereas, the students in the Sequential condition did not make significant improvement in these tasks. An important caveat is that students in the Simultaneous condition were worse at pretest on the percent decile number line task. Thus, it is possible that the students in the Simultaneous condition may have had greater opportunity for gains on this task. Since there was not a control condition, it is possible that simply another three weeks of school contributed to these differences in performance however.

Several results were inconsistent with the hypotheses. For example, there was no difference in improvement by condition for any of the measures, when change scores were submitted to ANCOVA with pretest scores as a covariate. Students in the Sequential condition made marginally significantly more improvement on Magnitude Comparison across notations than the students in the Simultaneous condition, which is also inconsistent with the hypothesis. Though, it is important to note that caution should be used in interpreting any changes from pretest to posttest with the magnitude comparison across notations task. Specifically, I increased the number of trials at posttest from 24 to 36 trials. I also adapted the task to control for magnitude across all notations and to better control for the ratios between compared values. Ultimately, the posttest trials only included 5 out of 36 trials that had appeared on the pretest. Furthermore, when the 31 modified items are excluded from the analysis, there was no significant difference in improvement by condition controlling for pretest scores ($p=.477$).

Moreover, the results about reduction of cross-notation bias were slightly mixed. After number line training, the *Percent is larger than fraction bias* and the *Fraction is larger than decimal bias* weakened though the conditions likely made separate contributions to the weakening of these biases. In other words, the Simultaneous condition facilitated a weakening of the bias in the fraction-to-decimal comparisons (though it was not statistically significant), whereas, the Sequential condition facilitated a weakening of the bias in the percent-to-fraction comparisons (which was statistically significant).

Relatedly, an important point to remember when considering differences between the Simultaneous and Sequential condition in this study is that classes were grouped according to ability level. Though an attempt was made to match ability levels when assigning classes to condition, it is difficult to know whether differences in condition were due to condition alone or whether there might be some other unknown factor at play related to the sample. For example, students in the Simultaneous condition performed worse on the percent decile number line task than the students in the Sequential condition at pretest. Thus, any differences observed in magnitude comparisons involving percent may have to do with less general knowledge about percentages in that condition.

I had hoped to examine the effects of condition on fraction arithmetic estimation ability and examine individual differences in predicting this type of estimation. However, because of the aforementioned issues with typing responses during the timed activity via Qualtrics, I opted to pursue an exploratory rather than quantitative investigation of fraction arithmetic estimation with this data set, focusing on strategies students employed. Given that I did not ask students to report the specific strategies that they used, I analyzed student answers for potential evidence of specific strategies. Three common types of answers that appeared repeatedly in student responses that indicated what strategies were being employed were: Adding Across, where students added across numerators and denominators (e.g., 1/2+1/3=2/5); Hybrid Across, where students found a common denominator and then added the numerators (1/2+1/3=2/6); and calculating exactly, where students calculated the answer exactly with a common denominator (e.g., 1/2+1/3=5/6). There were some other strategies that I was able to classify, but since I did not have student reports of the strategies they employed, I could not verify that these were the strategies that were actually being employed and thus, I listed them as speculation. Perhaps, future research might explore some of these strategies I speculated that students were employing but that is beyond the focus of this dissertation. Despite the issue with the open response format of this fraction arithmetic task, it seemed that true estimation ability was severely lacking and students exhibited *impulsive calculation* rather than estimation (e.g., using a flawed calculation strategy or calculating exactly). However, an open question was still whether students that could calculate the answers exactly could actually estimate. In other words, students that calculated the exact answers could have done so procedurally without thinking about the magnitude of the individual values. Despite Common Core's (2019) emphasis on students being able to judge the reasonableness of fraction arithmetic answers, it appeared that virtually no students knew how to estimate because they either used a flawed strategy or they calculated the exact answer. As far as I could tell, there was only one student that I classified as a true estimator at pretest. Therefore, it seems largely unclear whether students could actually estimate rather than *impulsively calculate*.

With the small sample sizes in each condition and the fact that an attempt was made to balance conditions with high and low achieving students due to classes being grouped by ability levels, it is unclear whether instructional conditions were the primary driving force in any differences observed. Moreover, it is important to note that caution should be used in interpreting any changes from pretest to posttest with the magnitude comparison across notations task because of the difference in number of items on the two tests. I also adapted the task to control for magnitude across all notations and to better control for the ratios between compared values. Ultimately, the posttest trials only included 5 out of 36 trials that had appeared on the pretest. Still, this initial study provided some evidence that review of rational numbers through daily number line training was valuable for improving students' abilities to estimate in the presence of distraction. Additionally, the current study yielded some novel findings about students' understanding of the relation among rational numbers. Therefore, I performed a second study with a larger sample size and other methodological improvements.

3.5 Experiment 2

Experiment 1 suggested further exploration was warranted, particularly of students' understanding of the relations among notations and whether instruction can improve this understanding. Experiment 2 had several goals: (1) to test whether results obtained in Experiment 1 replicate with a significantly larger sample that is randomly assigned by classes to condition, (2) to modify the pretest, posttest and instruction based on the limitations of Experiment 1 (e.g., instruction was done by the lead researcher with a small sample size), and (3) to add a Control condition where students underwent rote practice of addition and subtraction of fractions without the use of number lines as a warm-up. Students in the Control condition used the same fraction values from the exercises completed by students in the Simultaneous and Sequential conditions.

Experiment 2 was designed to account for potential issues in Experiment 1. The Control condition was used to rule out the possibility that simply another three weeks of math instruction was a contributing factor to any observed gains. The students' regular classroom instructors were used to alleviate concerns over the effect I may have had on the outcomes. For example, as a former educator and researcher in the field, I may possess greater theoretical and practical understanding that might affect the teaching moves (Empson & Jacobs, 2008) that were implemented during instruction in Experiment 1. It is also possible that perhaps the novelty of a new instructor could be driving any effects.

The use of the students' regular classroom instructors had the added benefit of determining whether an educator who is not a researcher in the field can implement the instructional intervention effectively. Moreover, care was taken to ensure fidelity to instructional conditions, including professional development before the intervention and ongoing daily support. Furthermore, overhead slides with scripted text were provided to ensure that students in each condition were exposed to the appropriate instruction and content. Finally, teachers were unaware of the hypotheses involved in this experiment.

The content of the Simultaneous and Sequential conditions in Experiment 2 was nearly identical to Experiment 1, except for one important change: the use of area models in student work and classroom presentations. In Experiment 1, the area model was an image of a battery power indicator for the Simultaneous condition and the area model was a plain rectangle with continuous shading for the Sequential condition. There was a concern that the contextual nature of a battery power indicator rather than an image of a shaded rectangle introduced experimental variation with contextualization versus without it (Cox & Griggs, 1982; Pollard & Evans, 1987). Thus, any differences observed between the Simultaneous and Sequential conditions in Experiment 1 may have had less to do with the timing and order of the presentation of materials and more to do with the contextual scenario of a battery power indicator versus a simple rectangular area model. Based on this concern, the determination was made to use the image of the battery power indicator in both the Simultaneous and Sequential conditions.

Modifications were made to the format of the arithmetic estimation task to better determine whether students can evaluate the reasonableness of an estimate for fraction arithmetic by providing students with three choices rather than an open response format. The open-response format in Experiment 1 was problematic because typing fraction responses on a computer was unfamiliar to them . Moreover, the timed nature seemed to encourage students to adopt a strategy (e.g., attempt to calculate exactly, add numerators and denominators, etc.) and maintain the strategy throughout the task. In other words, it was not always clear if students could evaluate the reasonableness of an answer and perhaps the timed nature of the task did not allow them to utilize a particular strategy, then evaluate the estimate, and finally type the result. In providing multiple-choice answers, I sought to determine whether students could judge the reasonableness of given answers This choice has practical significance because Common Core standards (2019) suggest using “benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2". Moreover, I utilized the multiple choice nature of this task to determine whether having an answer that employed a flawed strategy that elicited *impulsive calculation* resulted in worse performance than when answer choices did not elicit *impulsive calculation*. Answer choices that might elicit impulsive calculation involved an estimate with a flawed approach such as adding across numerators and denominators (e.g., answers like 19/21 for problems like 12/13+7/8 versus round benchmark numbers such as 1/3,1/2, 1, etc.). All of these changes will be described in length in the Method section that follows.

3.5.1 Method

Participants

Participants were 264 middle school students across 19 classrooms, taught by 5 classroom teachers recruited from a middle-income public school district in northern New Jersey. For assignment to condition, stratified random assignment was conducted such that each teacher's 3 general education classes were randomly assigned to one of three conditions (Simultaneous, Sequential, and Control). Therefore, this design could partially control for teacher effect since each teacher taught each of the conditions. Teachers also taught an inclusion class, where there were several students with special needs and the possible inclusion of other struggling students without special needs. Each inclusion class was randomly assigned to Simultaneous, Sequential, or Control. Thus, there were 2 Simultaneous, 2 Sequential, and 1 Control inclusion class. Finally, in discussing with the school district, we opted not to include the advanced classes in the study because these classes were on an accelerated track, where they were covering 9th and 10th grade content material rather than middle school content. There was a concern that the instructional content of the intervention would not be beneficial to students and would detract from their accelerated schedule. Relatedly, the school district also opted to exclude the self-contained special education classes, where students have more significant cognitive and behavioral problems. Administrators were concerned that the content would not meet the specific needs of the students in these classes. Finally, one classroom teacher opted not to participate in the study. So, in total there were 7 Simultaneous classes, 7 Sequential classes, and 5 control classes, where consent/assent was received from all but 2 students. Due to attrition with some students not having taken the posttest, the final sample size was 252 (93 students in the Simultaneous condition, 85 in the Sequential condition, and 74 students in the Control condition). Testing was done via Qualtrics in a quiet classroom setting. Instruction occurred in students' regular math classrooms by the regular math teacher.

Tasks

Number line estimation

No Modifications from Experiment 1: The format and content of this task was identical to Experiment 1, where students were tasked with placing fractions on a number line from 0 to 1 and placing fractions on a number line from 0 to 5.

Decile Number Line Estimation (i.e., 0-1 line labeled with tenths)

Modifications and Rationale: The format of this task was identical to Experiment 1, where students were tasked with placing fractions, decimals, and percentages on a 0-1 number line partitioned and labeled by tenths. The only difference with this task was a slight variation in the fractions, decimals, and percentages that were estimated on the number line. In Experiment 1, students were more accurate with placing fractions such as 2/7 on the decile number line than other fractions such as 6/17. I hypothesized that this observed difference had to do with a flawed fraction to decimal translation strategy, where students were concatenating some digits from fractions such that a/b=0.ab and using this decimal value to place the fraction on the decile number line. For example, 2/7 when translated to decimal form using this flawed translation strategy (a/b=0.ab) is quite close to .27 (versus the actual answer of .286). Many of the students placed the fraction 2/7 precisely at 0.27 and PAE for this item was a lot lower than that for other items. On the other hand, 6/17 when translated to decimal form (0.35) is quite far away from .67, which was where most of the students placed the fraction 6/17 on the decile line.

Based on these observations from Experiment 1, I selected fractional values whose decimal value is quite far from the hypothesized translation error (e.g., the numerical distance between 6/17 translated through the flawed approach (.67) and the actual decimal value (.35) is .32). Thus, I purposely selected the fractions in this task to yield a high PAE for individual items if they were using this hypothesized flawed translation strategy (Table 5). In this way, the task design ensured that students could not get close to the right answer using the a/b=0.ab flawed approach. In other words, if a student placed 2/7 at 0.27 because they were using the a/b=0.ab strategy, then their accuracy would actually be quite high for that trial even though they were using a flawed approach because their PAE would be .02 for that trial.

Table 5: Values Selected for Fraction Decile Number Line Trials, hypothesized flawed translation, and resulting PAE if hypothesized flawed translation is employed.

Fraction Decile Number Line Trial	Hypothesized Flawed Translation	Resulting PAE if hypothesized flawed translation is used
1/19	0.19	0.14
9/20	0.92	0.47
9/17	0.97	0.44
6/17	0.67	0.32
5/6	0.56	0.27
8/14	0.84	0.27
9/15	0.95	0.35
4/5	0.45	0.35

Fraction Addition Estimation

Modifications and Rationale: Instead of the open response format from Experiment 1, students were presented with 24 multiple choice fraction addition estimation problems. Through this methodology, I constrained students' strategy use by not allowing them to calculate the exact answer because none of the answer choices included the exact answer. This methodology had the benefit of assessing whether students were able to judge the reasonableness of an estimate for fraction arithmetic, as outlined by Common Core Standards (2019). Furthermore, I was able to determine whether students would gravitate towards answers that employed flawed calculation strategies when answer choices with such "lures" were present or not. An example of a fraction "lure" is "What is the best estimate for 1/5+1/2: 2/7, 1/3, or ¾?" In this example, 2/7 is a lure because it is the result of employing the flawed Across Strategy from Experiment 1, where students add across the numerators and denominators. An example of a fraction "no lure" item is "What is the best estimate for 2/10+2/4: 1/5, 1/3 or 2/3?" In this example, the answer choices involve answer choices that are more like benchmark fractions and do not include components that are the sums or products of the individual digits of the addends, such as 4/14 in this case. I reasoned that if students were less accurate when trials contained "lures" than when trials did not contain "lures," this would provide evidence of what I call *impulsive calculation*. I defined *impulsive calculation* as taking action with the digits without thinking about the values. Thus, I predicted that students would be more accurate on trials that contained "no lures" than trials that contained "lures," because it would not allow them to *impulsively calculate*. Moreover, I reasoned that students would be more likely to select the "lure" than the other wrong answer on the "lure trials."

Procedure: Across the 24 trials, there were 12 with "lure" responses and 12 with "no lure responses." The 12 problems with "lure" responses included 3 with "across lures," 3 with "hybrid lures," 3 with "decimal lures," and 3 with "percent lure" responses. The 12 problems with "no lure" responses included 6 problems with fraction responses that consist of "round number" fractions that do not elicit lures, 3 problems with decimal responses that do not elicit lures, and 3 problems with percent responses that do not elicit lures (e.g., a decimal lure for

7/8+2/3 is .9 because the answer involves adding the numerators and placing a decimal next to the sum). Students were instructed not to compute the exact answer but to select the best estimate for the given problems. They were given 20 seconds to answer the question before the program automatically moved onto the next problem. They recorded their estimate by clicking on the multiple choice item response that they thought best estimated the sum. Additionally, I also included the 12/13+7/8 estimation problem with answer choices 1,2,19,21 (Carpenter et al, 1980) to be analyzed separately to determine how aligned performance of this population is to past performance and determine whether there are any individual differences that predict ability to answer the estimation problem correctly (See Appendix A for all assessment items).

Magnitude Comparison across Notations

Modifications and Rationale: The format of the Magnitude Comparison Across Notations task was identical to the posttest of Experiment 1, such that students were presented with 36 comparison problems across rational number notations. As a reminder, the Experiment 1 posttest was slightly different from the Experiment 1 pretest because I wanted to control for confounding factors. Therefore, there were 18 magnitude comparison trials that required comparison of values with identical or nearly identical digits (e.g., compare 4/5 versus 45%) and 18 trials that were matched for magnitude across all notations between compared values (e.g., compare .40 versus 25%, 2/5 versus .25, .4 versus 1/4, etc.). The problems were carefully selected so that there were equal numbers of items in each of 6 categories: Fraction>Percent, Percent>Fraction, Decimal>Percent, Percent>Decimal, Fraction>Decimal, and Decimal>Fraction. The posttest for Experiment 2 contained half of the same comparison items and half novel items (See Appendix A for all assessment items).

Standardized math achievement tests

Modifications and rationale: The only difference from Experiment 1 is the type of standardized math test (Experiment 1 used TerraNova and Experiment 2 used PARCC) because these were the schools' test of choice for standardized testing. Thus, in Experiment 2, students' scores from the mathematics section of the PARCC, the standardized test typically administered to public school students in New Jersey, were obtained from the school. The test was given toward the end of the students' previous grade level, about a year before the study began. Scores on the PARCC range from 650-850. These test scores served as measures of students' overall mathematical ability.

Student Demographic Information

No Modifications from Experiment 1: Relevant demographic information about students' gender, disability status, and English Language Learning (ELL) status were collected from the district along with the standardized test scores. This information was de-identified and utilized as necessary controls and to examine any trends in the data across conditions.

General Procedure

Prior to the start of the study, the teachers received professional development on how to implement the instruction for all conditions, including a demo lesson with students that were not included in the study. To ensure that teachers did not carry over strategies from one condition to another, scripted lessons with overhead slides were provided and teachers were directed to maintain fidelity to the conditions with on-going daily support provided by the principal researcher.

Classes completed a pretest on the computer with the primary researcher and their regular classroom teacher during one math period. Most students completed the assessment in approximately 30 minutes. Then, spread out over a little more than 3 weeks, largely though not entirely on successive days (the non-successive days being due to school functions), students completed the15 warm-up activities as designated by their condition. The regular classroom teachers taught all brief warm-up activity lessons. After students completed the 15 lessons of warm-up activities according to condition, they took a post assessment that included half of the same items as the pretest and half new items.

Student work was collected and classes were observed randomly throughout the three weeks to ensure fidelity of instruction to the conditions. Student work was organized in a booklet and was collected and redistributed daily. At the end of the three weeks, the researcher collected the booklets to be coded for analysis (Examples of Student Workbook Pages in Appendix B). Feedback was provided daily by giving the correct answers to the whole class, together with explanations of how the answer was arrived at and why it was correct. Each instructional condition will be described in more detail subsequently. Following the completion of the intervention, the students completed the post-assessment on a computer during a single 30-minute session. The post-assessment included the same number line and fraction arithmetic estimation tasks with half familiar problems and half novel problems (See Appendix A for all assessment items). The researcher also randomly selected students from each condition for a clinical interview about their rational number understanding within 1-2 weeks of the post-assessment. The clinical interview involved items from the Rational Number Test utilized by Moss & Case (1999), questions regarding number line estimation strategies, and a question on fraction arithmetic estimation (Appendix A). Analysis of the interviews in conjunction with other data sources will be discussed in Chapter 4.

Instructional Procedures

Each day for three weeks, the students encountered a quick instructional activity at the beginning of each class, which varied according to their condition: Simultaneous, Sequential, or Control condition.

For the Simultaneous and Sequential conditions, the instructional procedures in Experiment 2 were identical to those in Experiment 1 except for the use of area models in student work and classroom presentations. In Experiment 1, the area model was an image of a battery power indicator for the Simultaneous condition and the area model was a plain rectangle with continuous shading for the Sequential condition. A concern was that the contextual nature of a battery power indicator rather than an image of a continuously shaded rectangle introduced too much experimental variation. In other words, perhaps the differences that were observed between the Simultaneous and Sequential conditions had more to do with a contextual scenario of a battery power indicator rather than a simple rectangular area model and less to do with the instructional materials. Based on this concern, I opted to utilize the image of the battery power indicator in both the Simultaneous and Sequential conditions. Thus, overhead slides and student activity books included images of battery power indicators for any partitioning or estimation activities with area models in both the Simultaneous and Sequential condition.

Students in the Control condition engaged in rote review of fraction addition/subtraction utilizing the same values from the other two conditions over the course of three weeks. The students in the Control condition engaged in activities for the same amount of time as the

Simultaneous and Sequential conditions, approximately five minutes per day, for 15 classes in total. This activity was chosen because number line training was hypothesized to be more important than mere practice with fraction arithmetic for improving fraction addition estimation. To maintain consistency across conditions, teachers were also provided with scripted review of fraction addition/subtraction and daily review lessons accompanying overhead slides. Importantly, discussion about estimation or use of different notations was excluded from any warm-up activities in the control condition. The Control condition focused on practice of fraction addition/subtraction procedures alone. The reason for this was to test the hypothesis that the number line conditions, specifically the Simultaneous condition, would be more beneficial in improving student estimation ability with fraction arithmetic at posttest over rote practice with fraction arithmetic alone. Typically in school, children are not provided with an opportunity to practice estimation of fraction arithmetic but they are often provided with additional rote practice with fraction arithmetic. I hypothesized that practice alone with fraction arithmetic was not enough to improve their estimation abilities for adding fractions. Therefore, the hypothesis was tested by determining whether the intervention aimed at improving number sense was more beneficial for improving fraction arithmetic estimation than rote practice with fraction arithmetic.

Analyses

Performance on all number line estimation tasks (endpoints 0-1, 0-5, and decile number line) was measured using percent absolute error (PAE), defined as |Participant's Answer–Correct Answer|/Numerical Range. For example, if a participant was asked to estimate 3/5 on a number line marked with endpoints 0 and 1 and marked the location corresponding to 0.65, PAE for that trial would be |0.65–0.6|/1 = .05, where lower PAE indicates higher accuracy.

Magnitude comparison accuracy was scored as percent correct. Performance across notation comparison in the six categories (Percent>Fraction, Fraction> Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction) was analyzed to detect trends in overall perception of the size of individual notations to explore whether the results about biases found in Experiment 1 are replicated in Experiment 2.

Given that the data included students nested within classes, the intraclass correlation (ICC) value was computed based on the unconditional mean model (i.e., no predictor). The ICC results of the outcome variables were small on measures ($p<0.05$), indicating that the proportion of between classroom variance is small compared to the total variance. Furthermore, preliminary analysis found no significant differences on the outcomes of interest between classrooms and other student demographics (e.g., gender). With the non-significant results and an insufficient number of classrooms (N=19, split among three conditions) for multi-level analysis (Maas & Hox, 2005), I used students' scores as the unit of analysis.

Thus, paired *t*-tests were used to compare performance on the pretest and posttest within each condition, and change scores (i.e., difference in performance from pretest to posttest) were submitted to *ANCOVA* with condition as between-subjects factors and pretest score as a covariate.

3.5.2 Results

Table 6 shows students' performance on all tasks at pretest and posttest in the Simultaneous, Sequential, and Control conditions.

Table 6: Mean (*standard deviation*) performance on assessment tasks by condition and test time. Note: PAE denotes percent absolute error.

	Sequential Condition		Simultaneous Condition		Control Condition	
Task	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
0-1 Number line (No Partitions) PAE	.08*(.07)*	.07*(.05)*	.10 *(.10)*	.07 *(.07)*	.09 *(.09)*	.09*(.09)*
0-5 Number line (No Partitions) PAE	.18*(.10)*	.18*(.08)*	.21 *(.11)*	.21*(.11)*	.18*(.10)*	.18*(.10)*
Fraction Decile Number line (0-1 line partitioned and labeled by tenths) PAE	.21*(.12)*	.15*(.10)*	.21*(.13)*	.16*(.12)*	.21*(.13)*	.19*(.12)*
Decimal Decile Number line (0-1 line partitioned and labeled by tenths) PAE	.14*(.14)*	.07*(.09)*	.12*(.14)*	.06*(.09)*	.12(*.13)*	.09*(.12)*
Percent Decile Number line (0-1 line partitioned and labeled by tenths) PAE	.05*(.08)*	.04*(.04)*	.05 *(.06)*	.04 *(.05)*	.05*(.07)*	.06*(.09)*
Magnitude Comparison (Across Notations) % Correct	.78*(.17)*	.82*(.14)*	.79*(.18)*	.85*(.14)*	.78*(.17)*	.81*(.16)*

Number line Estimation (No Partitions, 0 to 1 endpoints)

Paired t-tests within condition demonstrated that PAE on this task improved (i.e., decreased) in both the Sequential condition, t(84)=2.158, p=.034, *d*=.22 and the Simultaneous condition, t(92)=3.395, p<.001 d=.35. PAE on this task did not improve in the Control condition p>.6. When difference scores were submitted to ANCOVA with condition as between subject effect and pretest score as a covariate, the effect of condition was significant $F(2, 249) = 3.93$, p =.021, $\eta^2 = .031$, and Bonferroni adjustments for multiple comparisons demonstrated that improvement was considerably greater in the Simultaneous than the Control condition (p=.017). There was not a difference between the Sequential and Control condition (p=.225) or between the Simultaneous and Sequential condition (p=.973).

Number line Estimation (No Partitions, 0 to 5 endpoints)

PAE on this task did not improve (i.e., decrease) in any of the conditions. When difference scores were submitted to ANCOVA with condition as between subject effect and pretest score as a covariate, there was no significant difference in improvement by condition (p>.05).

Fraction Decile Number line (Partitioned and labeled by tenths, 0 to 1 endpoints)

Paired t-tests within condition demonstrated that PAE on this task improved (i.e., decreased) in both the Sequential condition, t(84)=5.027, p<.001, *d*=.53 and the Simultaneous condition, t(92)=5.178, p<.001 d=.54. PAE on this task marginally improved in the Control condition p=.063. When change scores were submitted to ANCOVA with pretest scores as a covariate, there was no significant difference in improvement by condition (p>.05).

Decimal Decile Number line (Partitioned and labeled by tenths, 0 to 1 endpoints)

Paired t-tests within condition demonstrated that PAE on this task improved (i.e., decreased) in the Sequential condition, t(84)=4.476, p<.001, *d*=.5, the Simultaneous condition, t(92)=4.255, p<.001 d=.44, and the Control condition t(74)=2.01, p=.048, d=.23. When change scores were submitted to ANCOVA with pretest scores as a covariate, there was no significant difference in improvement by condition ($p>.05$).

Percent Decile Number line (Partitioned and labeled by tenths, 0 to 1 endpoints)

Paired t-tests within condition demonstrated that PAE on this task marginally improved (i.e., decreased) in the Simultaneous condition, $t(92)=1.899$, $p=.061$, $d=.2$. PAE on this task did not improve in the Sequential or Control condition. When change scores were submitted to ANCOVA with pretest scores as a covariate, the effect of condition was significant $F(2, 249) = 3.301$, $p=.038$, $\eta^2 = .026$, and Bonferroni adjustments for multiple comparisons demonstrated that improvement was marginally greater in the Simultaneous than the Control condition ($p=.058$). There was not a difference between the Sequential and Control condition ($p=.1$) and the Simultaneous and Sequential condition ($p=1.0$).

Relation among Notations on Number line Performance

In addition to investigating how students performed on individual number line measures, I also investigated the relative performance across number line estimation for the different notations (Fraction Decile Number line, Decimal Decile Number line, Percent Decile Number line) using a repeated-measures analysis of variance (ANOVA) on PAE. There was a main effect of notation (reporting the Huynh-Feldt correction for a violation of sphericity), $F(1.95, 515.29) = 187$, $p<.001$, $\eta^2 = .415$, suggesting that performance was best for the percent decile number line estimation ($M_{percent}=.05$), next best for the decimal decile number line estimation ($M_{decimal}=.13$), and worst for the fraction decile number line estimation ($M_{fraction}=.21$).

Average performance for each notation (fraction, decimal, and percent) was calculated to examine whether performance improved from pretest to posttest by condition. Thus, I generated a variable called average decile performance, which averaged PAE across the fraction, decimal, and percent decile number line tasks. When change scores were submitted to ANCOVA with average pretest scores as a covariate, there was a significant difference in improvement by condition $F(2, 249) = 6.86$, $p = .001$, $\eta^2 = .052$, and Bonferroni adjustments for multiple comparisons demonstrated that improvement in the Simultaneous (p= .004) and Sequential condition (p=.003) were significantly greater than the Control condition. There was no difference between the Simultaneous and Sequential condition on improvement (p=1.0).

Magnitude Comparison Across Notations

Analyses were conducted for magnitude comparison across all notation comparison problems as well as looking at specific notation comparison types (Percent>Fraction, Fraction>Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction).

Across All Notations Comparisons: Paired t-tests within condition demonstrated that overall percent accuracy on Magnitude Comparison across Notations improved in the Sequential condition, t(84)=-2.901, p=.005, *d*=.31, the Simultaneous condition, t(92)=-4.475, p<.001 d=.46, and the Control condition t(73)=-2.978, p=.004, d=.35. When difference scores were submitted to ANCOVA with condition as a between subject effect and pretest score as a covariate, the effect of condition was significant suggesting greater differential improvement according to condition $F(2, 251) = 3.109$, $p = .046$, $\eta^2 = .024$. Bonferroni adjustments for multiple comparisons demonstrated that improvement was considerably greater in the Simultaneous condition than the Sequential (p=.03) and Control conditions (p=.04).

Specific Notations Comparisons: To test the hypothesis that students do not think about equivalent rational numbers as being equivalent in size, I examined accuracy across trials to determine whether students perceive one notation as larger than another as in Experiment 1. Figure 9 displays students' performance across the six categories of comparisons at pretest: Percent>Fraction, Fraction> Percent, Percent>Decimal, Decimal>Percent, Fraction>Decimal, Decimal>Fraction. This analysis yielded a similar pattern of results to Experiment 1 except for the lack of a difference between fraction>decimal and decimal>fraction performance.

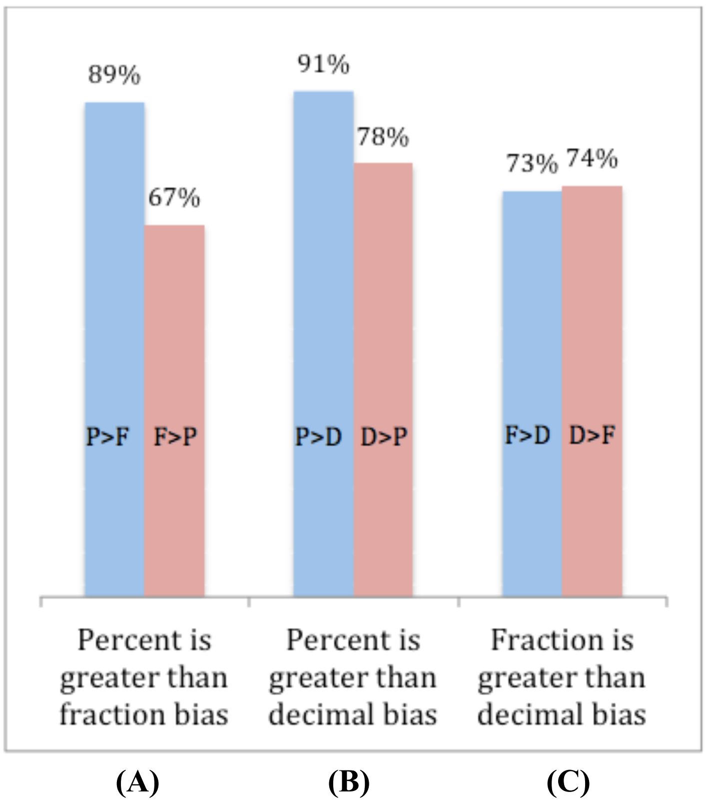


Figure 9: Experiment 2 pretest accuracy for magnitude comparison across notations. The chart is segmented by students' inferred biases of which notation is greater. The data includes all 7th and 8th grade students. The comparison types include: (A) Percent-to-Fraction Comparisons (B) Percent-to-Decimal Comparisons (C) Fraction-to-Decimal Comparisons.

At pretest, overall accuracy was 89% for items where percent is larger than the fraction, as compared to 67% accurate for items where percent is smaller than fractions. Moreover, accuracy was 91% for items where the percent was larger than the decimal, as compared to 78%

accurate for items where the decimal was smaller than the percent. Students were 74% accurate when the decimal was greater than the fraction and 73% accurate when the fraction was greater than the decimal. Paired t-tests were conducted to determine whether the difference between these categories of comparison were significant. Results replicated the finding that middle school students (N=264) have a bias towards perceiving percentages as larger than fractions/decimals Specifically, paired t-tests revealed differences between mean scores for items where Percent>Fraction and Fraction>Percent (t(263)= -11.227, p<.001) and Percent>Decimal and Decimal>Percent (t(263)= -6.864, p<.001). However, unlike in Experiment 1, the bias towards perceiving fractions as larger than decimals did not exist in this sample, as there was not a significant difference between mean scores for Fraction>Decimal as compared to Decimal>Fraction (t(263)=-.686, p=.493).

Since these biases have not been documented previously in the literature, I carefully examined the fraction, decimal, and percent values themselves to determine whether there could be some other confounding variable to explain these results. For example, I found that the mean value of percentage across all trials (M=0.405) was slightly less than the mean value of fractions/decimals across these same trials (M=0.407). This slight difference does not explain why students perceived percentages to be larger because in actuality they were slightly smaller on average. Furthermore, the ratios between values across trials comparing fractions/decimals to percentages was on average slightly larger when the fraction or decimal was the correct answer (M=2.99) than when the percent was the correct answer (M=2.41). Decades of research on magnitude comparison suggest that it is easier for individuals to select the larger value as the ratio between two values increases. In other words, it is easier to select the larger value for 40 vs. 20 than 21 vs. 20 because the ratio between the first two values is 2 and the ratio between the second two values is 1.05. Thus, it theoretically should have been easier for students to select the fraction/decimal as the larger value in comparisons to percentages because the ratio between the values was larger on average when the correct answer was a fraction/decimal compared to a percent. For example, the ratio between 52% vs. 2/5 and the ratio between 5/6 vs. 65% is about the same (1.3 and 1.28 respectively), suggesting that these trials are of approximately equal difficulty level. Yet, there was a difference in performance when the percent was the larger value (89.4%) than when the fraction was the larger value (50.2%). Even on trials where the ratio between values compared was identical because it used identical magnitudes but in different notations, there were discrepancies in performance. For example, accuracy was very high for the comparison of 40% vs. 1/4 (M=91%) as compared to quite low performance for the comparison of equivalent values where the larger value is written in fraction form 2/5 vs. 25% (M=63%). Additionally, it could be that what was being measured is actually just general magnitude comparison ability rather than cross notation ability (i.e., if a student knows that 1/10 is a number close to 0, than the probability is high that the other number is likely larger). If this is the case, then the magnitude comparison task measured how students could relate at least one value to the endpoints of 0 and 1. Thus, performance would be better if at least one value was close to an endpoint. However, I examined performance on comparison tasks where one of the values was greater than or equal to .8 or less than or equal to .2. Performance was slightly worse when one of the values contained a value within .2 of the endpoints (M=77.06%) as compared to trials with no values within .2 of the endpoints (M=78.2%). Thus, we can likely exclude the explanation that students were using knowledge of extreme values and guessing with a high degree of accuracy that the other value was smaller or larger. In sum, we can likely exclude a number of explanations that might explain the results.

To determine whether condition had an effect on weakening these biases, I calculated a 'Bias score' for each student, which was the difference between related categories (i.e., the difference between mean accuracy for Percent>Fraction and Fraction>Percent items and the difference between mean accuracy for Percent>Decimal and Decimal>Percent items). Theoretically, the difference between scores for items involving Fraction>Percent and Percent>Fraction should be about 0 because students should perform equally well on items whether the notation is larger or smaller than another notation. For example, students should perform equally well on items where the larger value is a percent (e.g., 2/5 vs. 52%) and where the larger value is a fraction (e.g., 25% vs. 2/5). Inspection of the bar graph for pretest scores (Figure 9) demonstrates that the difference between Fraction>Decimal and Decimal>Fraction is about 0, and indeed there is no statistically significant difference between mean scores for these items as discussed previously. Thus, I will not focus on Bias scores in fraction-to-decimal comparisons because there doesn't appear to be a bias. However, there is a great discrepancy between scores on items where the percent is greater than the fraction/decimal and, as discussed previously. At pretest, all conditions followed the same general pattern of perceiving percent as greater than fraction and percent as greater than decimal. One-way ANOVAs were conducted to determine whether there was a difference in 'Bias scores' by condition at pretest, but this was not the case ($p>.05$ for both biases).

Examination of Percentage is Greater than Fraction Bias by condition at posttest (Figure 10) suggests that the bias was weakest for students in the Simultaneous condition at posttest because the mean difference in percentage correct between P>F and F>P is 17, as compared to 20 in the Sequential condition, and 24 in the Control condition. When posttest Percentages are Larger than Fraction (P>F) Bias scores were submitted to ANCOVA with condition as between subject effect and controlling for pretest bias, gender, and standardized achievement test the effect of condition was marginally significant $F(2, 231) = 2.714$, $p = .068$, $\eta^2 = .023$, suggesting that P>F bias is weakest in the Simultaneous condition at posttest.

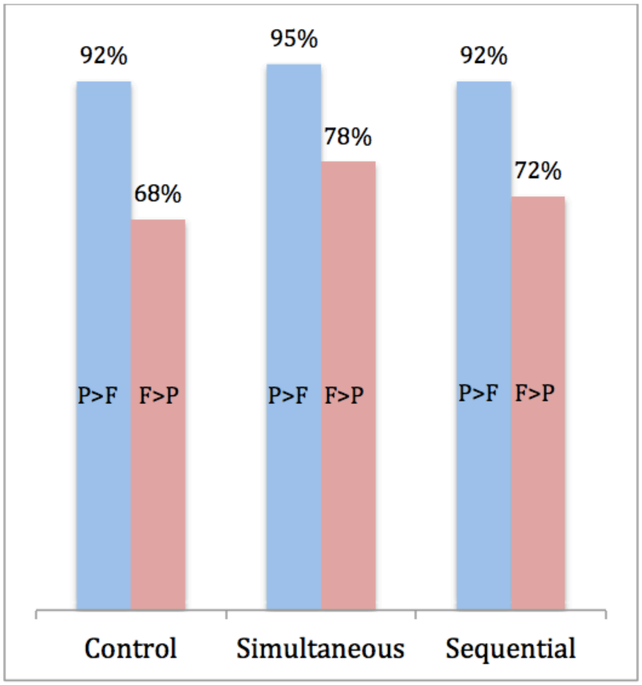


Figure 10: Experiment 2 posttest accuracy for the Percent is Greater than Fraction Bias (percent-to-fraction comparisons) by condition.

Examination of Percent is greater than Decimal Bias by condition (Figure 11) suggests that the bias is weakest for students in the Control condition at posttest. The mean difference in percentage points between P>F and F>P for students in the Control condition is 15, as compared to 18 for the Simultaneous condition, and 17 for the Control condition. When posttest Percent greater than Decimal (P>D) Bias scores were submitted to ANCOVA with condition as between subject effect and controlling for pretest bias, gender, and standardized achievement test $F(2, 228) = .526$, $p = .592$, $\eta^2 = .005$, the effect of condition was not significant suggesting that these differences among conditions in P>D bias are not significant.

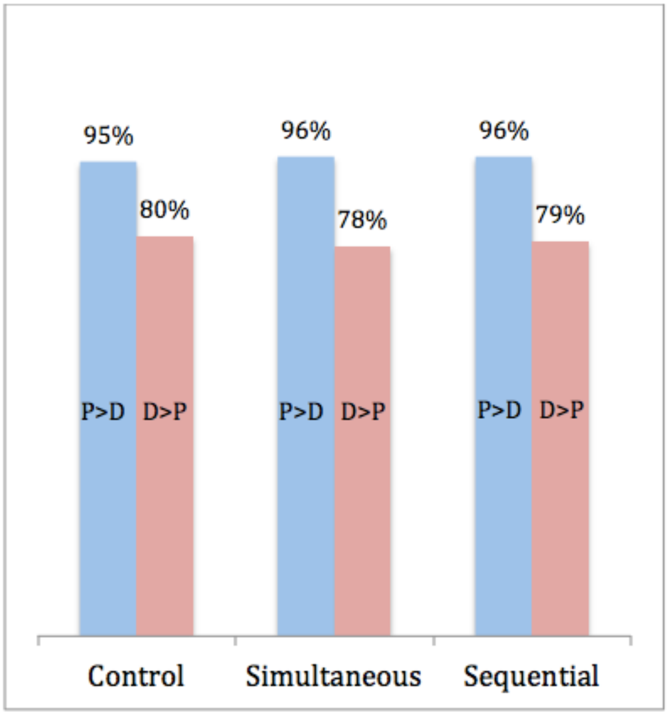


Figure 11: Experiment 2 posttest accuracy for the Percent is Greater than Decimal Bias (percent-to-decimal comparisons) by condition.

Finally, the effect of condition on improving Fraction>Percent items for students of high and low ability was examined, since there was a marginal difference by condition in the Percent is greater than Fraction Bias but not the Percent is greater than Decimal Bias. Theoretically, students of high ability would likely be at ceiling and not show very much improvement, whereas students with low ability would likely make greater improvement. Students of high ability were defined as those above the mean at pretest for Fraction>Percent items and students of low ability were those below the mean at pretest for Fraction>Percent items. Fraction>Percent performance was chosen as a discriminating measure since most students did much worse on these items than on the Percent>Fraction items. Further analysis of the Fraction>Percent items enabled a focus on an area that had the greatest potential for improvement. The low performing group improved more than the high-performing group ($F(1,246)=51.986$, $p<.001$) and students in the Simultaneous condition improved more than those in the other conditions overall controlling for pretest ability (F(2,246)=5.041, p=.007). Figure 12 displays profile plots by ability and condition. For high performers, the condition had no effect (F(2,246)=.055, p= .947). For low performers, the condition predicted greater improvement (F(2,246)= 8.100, p<.001). Specifically, pairwise comparisons with Bonferroni adjustment for multiple comparisons demonstrated that the low performing students in the Simultaneous condition improved more than the low students in the Sequential (p=.004) and Control conditions (p=.001). This finding suggests that the Simultaneous condition was the most influential for improving low performers' accuracy with fraction>percent items at posttest. For these low performing students, mean improvement in Fraction>Percent was .30 (SD=.34) p<.001, d=.88 for the Simultaneous condition, .13 (SD=.29), p=.007, d=.42 for the Sequential condition, and .10(SD=.24) p=.009, d=.42 for the Control condition.

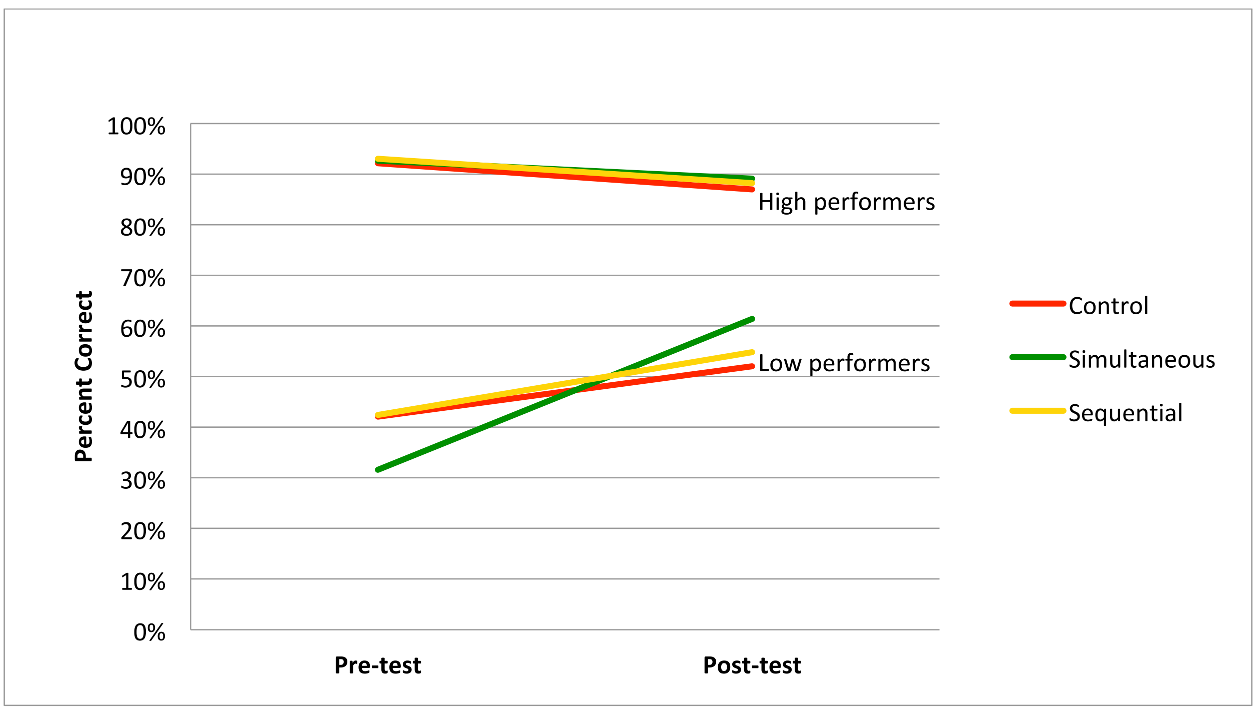


Figure 12: Experiment 2 profile plot of performance on items when the Fraction is Greater than Percent in Magnitude Comparison by ability (low versus high performers), condition, and test time.

This is an important finding as improved fraction>percent performance indicates that the percent is greater than fraction bias is less influential at posttest for children who performed poorly at pretest. Additionally, performance on fraction>percent items at pretest is one of the most closely related measures to performance on the fraction addition estimation lure items (.378). This correlation between fraction>percent items and fraction addition estimation is exceeded only by standardized tests of achievement (.388) and Magnitude Comparison Across Notations (.407), which encompasses F>Percent because it is a composite score across all comparison types. Table 7 displays correlations for variables of interest. Finally, it is important to note that the Fraction> Percent mean for these low performers improved from .32 to .61, which was just slightly below the pretest mean for all students. Thus, the Simultaneous Condition brought the lowest performing students up to the level of the average students after the intervention.

Table 7: Correlations for variables of interest at pretest for Experiment 2.

Measure	1	2	3	4	5	6	7
1. F>P Items Accuracy	-						
2. Fraction Addition Lure Estimation	.378**	-					
3. PARCC scores	.468**	.388**	-				
4. Cross-Notation Comparison Accuracy	.810**	.407**	.562**	-			
5. 0-1 Number line PAE	-.490**	-.274**	-.559**	-.547**	-		
6. Fraction Decile Number line PAE	-.558**	-.360**	-.476**	-.620**	.438**	-	
7. Decimal Decile Number line PAE	-.389**	-.195**	-.400**	-.482**	.322**	.383**	-
8. Percent Decile Number line PAE	-.333**	-.264**	-.260**	-.413**	.235**	.287**	.285**

**. Correlation is significant at the .01 level

Fraction Addition Estimation

All students did quite poor on fraction addition estimation at pretest with accuracy just slightly greater than chance given three multiple-choice options (M=34%, SD=15%). At posttest, performance was about the same (M= 33%, SD=18%) and there were no significant differences by condition. Thus, contrary to the hypothesis, condition had no effect on fraction arithmetic estimation.

However, the fraction addition estimation task was also utilized to measure whether students exhibit *impulsive calculation*. Theoretically, students should do worse on items where there was a fraction lure than items without a lure because the trials elicited calculation without thinking about magnitude. Lure trials involved an answer choice with a flawed strategy (e.g., what is the best estimate for 1/5+1/2: 2/7, 1/3, or ¾?, where 2/7 is an "across lure"). No lure trials involved all answer choices with round benchmark fractions (e.g., What is the best estimate for 2/10+2/4: 1/5, 1/3 or 2/3?). Results revealed that there was a difference in performance at pretest between items where fraction addition estimation had a fraction lure and when there was no fraction lure. In particular, students were 41% accurate on "No Lure" items as compared to 34% accurate on "Lure" items, $t(261)=-3.687$, $p<.001$. Furthermore, wrong answers were statistically significantly more likely to be a lure than the other wrong answer $t(255)=6.136$, $p<.001$. This lends some support to the theory about students exhibiting *impulsive calculation*.

Individual differences in predicting estimation ability

In addition to the improvements on the measures described earlier, I also wanted to examine individual differences in predicting estimation ability, particularly in the presence of potentially distracting information. In this analysis, I focused on estimation ability in the presence of distraction at both the level of an individual value (decile number line estimation) and at the level of combination of values (fraction addition estimation with lure answer choices and a special focus on the infamous fraction addition estimation problem 12/13+7/8 from Carpenter et al, 1980).

Individual Values- Based on the findings from Siegler and Thompson (2014) that demonstrated weaker performance for number line estimation on lines that were partitioned and labeled by tenths, I theorized that the decile number line could create a potentially distracting situation. Students that have weak understanding of the holistic value of a fraction might focus on the numerator or the denominator rather than focus on the magnitude of the fraction. In particular, the first hypothesis suggested that integrated rational number sense would help students persevere in attending to magnitude during situations that could potentially be distracting such as number lines that were partitioned and labeled by tenths.

Indeed, performance was significantly worse on the decile number line task (PAE=.21) as compared to the unlabeled 0-1 number line task(PAE=.09) ($t(263)=15.738$, $p<.001$), suggesting the task was sufficiently distracting. Hierarchical linear regression (Table 8) was used to determine whether each additional variable explains additional variance in predicting fraction decile number line estimation. The following three variables were added to the model to determine whether they added more explanatory power to performance in placing fractions on the decile number line:

1. *PARCC Score*: The PARCC score, the standardized test of math achievement, was added to the model first to account for any predictive value of general math ability. PARCC score was a significant predictor of performance on the decile number line task (Step 1, $p<.001$).
2. *0-1 Number Line Estimation*: Number line PAE (on an unlabeled 0 to 1 number line) was entered next because it was hypothesized to be most closely related to how students would perform on the Decile Number line task. Theoretically, how students perform on a number line with endpoints 0 and 1 should be the same as a

number line with endpoints 0 and 1 that is labeled by tenths. Both Unlabeled Number line estimation and PARCC scores were predictive of Decile Number line PAE (Step 2, $p<.001$ for both).

3. *Cross-Notation Comparison*: Cross-Notation Magnitude Comparison was entered last. Cross-Notation Comparison was calculated as a composite of the scores (i.e., average percentage correct) on comparisons that required students to compare between distinct notations (i.e., Fraction versus Decimal, Decimal versus Percent, and Percent versus Fraction). When it was added to the model, Cross-Notation Comparison was a significant predictor (Step 3, $p<.001$) and it added 14% of explanatory power to the model predicting Decile Number line PAE.

Importantly, when controlling for PARCC and Cross-Notation Comparison Ability, Unlabeled Number line estimation was no longer a significant predictor of ($p>.05$) decile number line PAE. Examination of the standardized beta coefficients in Table 8 suggests that Cross-Notation Comparison was the most important predictor of Decile Number Line PAE in the final model, above and beyond standardized test of achievement and unlabeled number line PAE. This suggests that higher performance on the Cross-Notation Comparison task was associated with better performance on the decile number line task, controlling for general math ability and unlabeled number line performance. This seems consistent with the hypothesis of better skill at integrating the three notations, as indicated by performance on the Cross-Notation Comparison task, helps students fight their way through the distracting number line information.

Table 8: Hierarchical Linear Regression Analysis predicting Fraction Decile Number Line PAE for Experiment 2.

	b (unstandardized)	*SE b*	*β* (standardized)
Step 1			
Constant	2.631	.286	
PARCC	-.003	.000	-.476***
Step 2			
Constant	1.881	.341	
PARCC	-.002	.000	-.335***
Number line 0 to 1 PAE	.369	.097	.252***
Step 3			
Constant	1.270	.318	
PARCC	-.001	.000	-.155*
Number line 0 to 1 PAE	.134	.092	.092
Cross-Notation Comparison	-.362	.048	-.480***

Note. R^2=.226 For Step 1***; ΔR^2=.044 for Step 2***, ΔR^2=.140 for Step 3***
*p<.05, **p<.01, ***p<.001

Combination of values- Similar to examining what predicts ability to persevere in attending to magnitude of fractions at the individual level, I also wanted to examine what predicts ability to attend to magnitude of more than one fraction in situations that might be misleading. Thus, I explored what predicted students' ability to estimate addition of fractions with lure item choices. For example, a fraction addition estimation task with lure responses would ask students to select the best estimate for 1/5+1/2 from these choices: 2/7, 1/3, or ¾. The answer choice 2/7 would be the "lure" response in the aforementioned problem because it encourages students to calculate across numerators and denominators rather than focus on magnitude. Thus, hierarchical linear regression was conducted to investigate individual predictors of fraction lure arithmetic estimation (Table 9). The following four variables were added to the model to determine whether they added more explanatory power to the variance in performance in estimating fraction arithmetic sums, where 1 of the 3 answer choices had a "lure" choice:

1. *PARCC score:* The PARCC score, the standardized test of math achievement, was added to the model first to account for any predictive value of general math ability. PARCC score was a significant predictor of performance on the decile number line task (Step 1, $p<.001$).
2. *No Lure Estimation Performance:* Next, how students performed on fraction arithmetic estimation was added because, theoretically, how students performed on fraction arithmetic estimation without lures should most closely predict how students perform on fraction arithmetic estimation problems with lures. No lure trials involved simple benchmark fraction choices like 1/3, 3/8, or 3/4. By contrast, lure trials involved an answer choice with a "lure" such as 2/7 for 1/2+1/5, where the lure is derived from a calculation involving a flawed calculation strategy. Thus, it was entered into the model after the standardized measure of achievement (PARCC scores). Performance on "No Lure" fraction estimation significantly predicted performance on "Lure" estimation trials (Step 2, $p<.001$).
3. *0-1 Number Line Estimation*: Next, Number line estimation ability (0 to 1) was added to the model, as it has been shown to be an indicator of advanced mathematics outcomes (Siegler et al., 2012) and individual fraction estimation ability theoretically should predict how students estimate sums of fractions. However, Number line estimation ability did not add more explanatory power to the model (Step 3, $p>.05$) and it was not a significant predictor ($p>.05$).
4. *Cross-Notation Comparison*: Cross-Notation Comparison Ability was included last as a measure of integrated number sense. Cross-Notation Comparison Ability is the composite score for all magnitude comparison across notation trials. I hypothesized that integrated

number sense allows students to persevere in attending to magnitude, especially in situations that may be distracting (Hypothesis 1). This analysis demonstrated that Cross-Notation Comparison is a unique predictor of fraction addition estimation ability in the presence of distracting answer choices (Step 4, p<.001).

Importantly, the standardized coefficient beta for Cross-Notation Comparison (.294) suggests that Cross-Notation Comparison is the most important predictor in estimation ability, even above general mathematical ability (PARCC scores). When accounting for other variables, Number Line Estimation PAE (performance on 0-1 unmarked number lines) is no longer a significant predictor of estimation ability.

Table 9: Hierarchical Linear Regression Analysis predicting Fraction Addition Estimation for Experiment 2.

	b *(unstandardized)*	*SE b*	*β* (standardized)
Step 1			
Constant	-3.597	.614	
PARCC	.005	.001	.382***
Step 2			
Constant	-3.344	.603	
PARCC	.005	.001	.348***
Fraction No Lure	.248	.068	.214***
Step 3			
Constant	-2.930	.734	
PARCC	.004	.001	310***
Fraction No Lure	.243	.068	.209***
Number line 0 to 1 PAE	-.204	.206	-.070
Step 4			
Constant	-2.170	.734	
PARCC	.003	.001	.200**
Fraction No Lure	.237	.066	.205***
Number line 0 to 1 PAE	.081	.211	.027
Cross-Notation Comparison	.447	.109	.294***

Note. R^2=.146 For Step 1***; ΔR^2=.044 for Step 2***, ΔR^2=.003 for Step 3, ΔR^2=.053 for Step 4*** *p<.05, **p<.01, ***p<.001

Additionally, I explored separately whether there might be individual differences in performance that can predict whether students are correct or incorrect with the 12/13+7/8 estimation problem (Carpenter et al, 1980). Since this problem has been studied repeatedly since it was asked in 1978 and there has been little improvement (Lortie-Forgues et al., 2015), I wanted to test whether the theory that integrated number sense (as measured by Cross-Notation Comparison) predicts accuracy with this problem. In this analysis, I controlled for other important measures of mathematical ability such as PARCC scores (standardized test of math achievement) and Number line estimation PAE (on 0 to 1 unmarked number lines). I reasoned that general math ability should be important for estimation skills and magnitude representations for individual values should be predictive of estimating sums of two fractions.

At pretest, students were asked to select the best estimate for 12/13+7/8: 1, 2, 19, or 21. Of the 210 students that answered the question, 33% selected the correct answer 2, 63% chose 19 or 21, and 4% chose 1 as their answer choice. It is important to note that the original problem 12/13+7/8 (Carpenter et al, 1980) was posed to 8th grade students. However, since the performance of the 8th grade students alone (N=135) was not different from the overall performance (M=33% correct), the following analysis includes all 7th and 8th grade students. Table 10 displays mean values for other predictors by problem accuracy. As discussed previously, PARCC scores can range in values from 650-850, PAE refers to the percent absolute error, and Cross-Notation (CN) Comparison ability is scored as mean percentage correct across all comparison trials. According to Table 10, students that were right had a mean PARCC score of 758.5, 0-1 Number line PAE of .1, and Cross-Notation (CN) Comparison score of 88% accuracy; students that were wrong had a mean PARCC score of 748.3, PAE of .15, and Cross-Notation (CN) Comparison score of 75% accuracy.

Table 10: Descriptive statistics by estimate accuracy (right or wrong) for the 12/13+7/8 estimation problem.

Estimate	Total Sample (N)	PARCC (SD)	0-1 PAE (SD)	CN-Comparison (SD)
Right	70	758.5 (17.3)	.10 (.06)	.88 (.13)
Wrong	140	748. 3 (19.4)	.15 (.09)	.75 (.17)
Summary	210	751.7 (19.3)	.14 (.09)	.79 (.17)

Logistic regression was applied with accuracy for the 12/13+7/8 estimation problem as the dependent variable (coded 0=wrong and 1=right). Predictors included PARCC score (standardized test of achievement), Number line Estimation PAE (on 0 to 1 unmarked number lines), and Cross-Notation Comparison (composite score for magnitude comparison trials across fractions, decimals, and percentages). According to the model (Table 11), the log odds of a student being correct with selecting an appropriate estimate was positively correlated with Cross-Notation Comparison (p=.001) but not with the other predictors. In other words, the higher the Cross-Notation Comparison score, the more likely that the student would select the right answer controlling for other measures. The odds of selecting the right estimate for the problem were 2.024 (=$e^{0.705}$; Table 11) times greater for higher scores of Cross-Notation Comparison.

The inferential goodness-of-fit test is the Hosmer-Lemeshow (H-L) test yielded a $\chi^2(8)$ of 3.80 and was insignificant (p>.05), suggesting that the model was fit well to the data.

Table 11: Logistic Regression Predicting Accuracy for 12/13+7/8 Estimation Problem

Predictor	β	SE β	Wald's χ^2	*df*	*p*	e^{β}
Constant	-.856	.170	25.335	1	<.001	.425
zPARCC	.132	.206	.413	1	.520	1.141
zPAE (0-1 line)	-.225	.266	.716	1	.397	.799
zComparison	.705	.220	10.230	1	.001	2.024
Test			χ^2	*df*	*p*	
Overall model evaluation						
Likelihood ratio test			29.381	3	<.001	
Goodness-of-fit test						
Hosmer & Lemeshow			3.80	8	.875	

Thus, we can infer that integration of notations as measured by the cross-notation comparison score is an important predictor for fraction addition estimation, above typical measures of math ability such as standardized tests of achievement or even fraction number line estimation. In other words, students that are of equal ability levels as measured by a standardized test of achievement and fraction number line estimation but perform better on selecting the larger of two values across notations are over two times more likely to select the appropriate estimate for this fraction addition estimation problem.

3.5.3 Discussion

Summary

Number line estimation tasks (0-1,0-5, and a decile number line), magnitude comparison across notations, and fraction addition estimation were used to (a) better characterize how middle school students process magnitude information for fractions, decimals, and percentages and (b) investigate how magnitude understanding across distinct notations may be differentially implicated in the relation between rational number ability and estimation. Two approaches to review of rational numbers were evaluated to determine whether there might be superior benefits of one instructional approach for review of rational numbers and whether one aids in improving estimation ability. The motivation for evaluating these two approaches was based on a theory of integrated numerical development (Siegler et al, 2011). Furthermore, I theorized that current approaches to instruction on fractions, decimals, and percentages leave students with limited understanding of the relation among notations (Moss & Case, 1999). Thus, classes of middle school students were randomly assigned to three different conditions (Simultaneous, Sequential, or Control) to determine whether there might be superior benefits of one instructional approach in solidifying or "recharging" rational number understanding.

In the following section, I summarize the hypotheses and the results that provide evidence in support or against the hypotheses. Finally, the section concludes with a brief discussion of the findings.

Hypothesis 1: Integrated Number Sense

The analysis of Experiment 2 also indicated that understanding of the relations among notations is another important aspect of numerical development, apart from fraction magnitude representations alone. In the integrated theory of numerical development, Siegler, Thompson, and Schneider (2011) posited that fraction magnitude representation is central to numerical development. However, I argue that fraction magnitude representation alone does not tell the whole story when determining why many students make implausible errors with fraction arithmetic estimation (e.g., 12/13+7/8=19/21). Instead, it appears that integrated number sense, an understanding of how fractions, decimals, and percentages are related to one another, provides more explanatory power in how students attend to magnitude in situations that could be potentially distracting situations. Beyond distracting situations, the correlation between the standardized measure of math achievement and integrated number sense is strong (r=.562, p<.001), similar to the strength of its correlation with fraction estimation on a 0-1 number line (r=-.559, p<.001). Therefore, integrated number sense as measured by magnitude comparison across notations is an important skill worthy of attention. In particular, the data provided evidence that there is a lack of integrated number sense in middle school students. Moreover, individual differences in integrated number sense explain variance in estimation ability.

a) *Students do not perceive equivalent rational numbers as equivalent in size.*

    i. Many students do not view equivalent rational numbers as equivalent in size, as evidenced by worse performance when a percentage is the smaller value compared to fractions/decimals

    ii. Students do not exhibit the bias towards perceiving percentages as larger than fractions/decimals in all situations. Specifically, students did not estimate percentages as larger than their equivalent fractions and decimals. In fact, their decile number line performance with equivalent values across the three notations demonstrated greatest accuracy with percentages, followed by decimals, and then fractions.

b) *Individual differences in integrated number sense predict students' estimation ability in the presence of distraction*

    i. Performance data within subjects provided evidence of *impulsive calculation* in the presence of distracting information, such that performance was higher in the non-distracting situations. In other words, students performed better on the 0-1 number line estimation task as compared to fraction decile number line estimation (with potentially distracting partitions). Students also performed better on fraction arithmetic estimation when answer choices had no lures as compared to fraction arithmetic estimation accuracy when one answer choice was a lure (e.g., adding across numerators and denominators). Moreover, wrong answer choices were more likely to be the lure choice than the other wrong answer.

ii. Integrated number sense, as operationalized as the composite score on Magnitude Comparison Across Notations, was the most important predictor of placing fractions on the decile number line, above 0-1 number line estimation and standardized math scores.

iii. Integrated number sense was the most important predictor of fraction arithmetic estimation with lure answer choices, even when controlling for their performance with trials that did not include lures, their standardized test of math achievement, and their 0-1 number line estimation ability. Moreover, 0-1 number line estimation ability was not a significant predictor in the model.

iv. Logistic regression suggested that integrated number sense predicted higher accuracy with the 12/13+7/8 estimation problem, when math achievement test scores and 0-1 number line estimation ability are held constant.

Hypothesis 2: Improving Integrated Number Sense

Based on findings about the importance of integrated number sense from Experiment 1, the current study tested a Simultaneous approach to number line instruction to determine if it resulted in greater outcomes over the Control condition. I reasoned that a Simultaneous and a Sequential number line intervention would improve outcomes over the Control condition but that the Simultaneous Condition would result in greater improvement for students over the Sequential Condition. Performance of students in the Simultaneous condition was not unequivocally better than the other conditions. However, the results leaned in support of the hypothesis in an important way, suggesting that the intervention modestly improved integrated number sense. The findings point to avenues for future research and improved educational interventions. Table 12 provides a simplified summary of whether students in each condition made Significant, Not Significant (NS), or Marginally Significant (MS) improvement from pretest to posttest. Finally, the third column details whether Simultaneous (Sim.), Sequential (Seq.), or Control (Con.) condition made substantially greater improvement over the other conditions.

Table 12: Simplified summary of findings comparing improvements between conditions for Experiment 2

Task	Sequential Improvement	Simultaneous Improvement	Control Improvement	Significant Improvement over other conditions
0-1 Number line (No Partitions) PAE	Significant	Significant	NS	Significant for Sim. over Con.
0-5 Number line (No Partitions) PAE	NS	NS	NS	NS
Fraction Decile Number line (0-1 line partitioned and labeled by tenths) PAE	Significant	Significant	Marginally Significant ($p=.06$)	NS
Decimal Decile Number line (0-1 line partitioned and labeled by tenths) PAE	Significant	Significant	Significant	NS
Percent Decile Number line (0-1 line partitioned and labeled by tenths) PAE	NS	Marginally Significant ($p=.06$)	NS	Marginally Significant for Sim. over Con.
Average Decile Performance across the 3 Notations	Significant	Significant	NS	Significant for Sim. over Con. Seq. over Con.
Integrated Magnitude Comparison (Across Notations) % Correct	Significant	Significant	Significant	Significant for Sim. over Seq. & Con.

a) Number line instruction will be beneficial over the Control instruction

i. Supporting the hypothesis, improvement in students' average performance across all fraction, decimal, and percent decile number line estimation was considerably greater in the Simultaneous and Sequential condition over the Control condition.

ii. In some cases, the students in the Simultaneous condition made greater improvement than the Control condition; however, in these same cases, the students in the Sequential condition did not make greater improvement over the Control. The measures where Simultaneous improvement (but not Sequential) was greater than the Control were: 0-1 number line estimation and Percent Decile number line estimation (marginally significant improvement).

iii. Contrary to the hypothesis, there was no difference on 0-5 number line estimation, the individual fraction and decimal decile number line estimation, or fraction arithmetic estimation at posttest

b) The Simultaneous review will improve outcomes over Sequential review of notations

i. The students in the Simultaneous condition made the greatest improvement in Magnitude Comparison Across Notations from pretest to posttest above the Sequential and Control conditions. Moreover, the lowest performing students in the Simultaneous condition made substantially greater improvement than the low performing students in the other conditions.

ii. Aforementioned cases where students in the Simultaneous condition made greater improvement over Control, but the students in the Sequential did not make improvement over control also provided support of the hypothesis that the

Simultaneous is preferred over the Sequential approach (0-1 number line estimation and Percent Decile number line estimation)

iii. Contrary to the hypothesis, the Simultaneous condition was not greater than the Sequential and Control condition on the other measures (0-5 number line estimation, fraction arithmetic estimation, fraction/decimal decile number line estimation).

Consistent with the hypothesis that students do not perceive rational numbers as equivalent in size, Experiment 2 replicated the finding from Experiment 1 that middle school students demonstrate a bias towards perceiving percentages as larger than fractions and decimals. In other words, students were more accurate when the percentages were larger than fractions and decimals than when the percentages were smaller than fractions and decimals. Also, consistent with the hypothesis that students do not perceive rational numbers as equivalent in size, is the difference in PAE on the decile number line task for equivalent values ($M_{percent}$=.05, $M_{decimal}$=.13,$M_{fraction}$=.21). For this task, students placed equivalent fraction, decimal, and percent values on a line with endpoints 0-1 that was partitioned and labeled by tenths. For example, students were most accurate placing 35% on the decile number line, followed by .35, and finally 6/17. Moreover, Experiment 2 posttest results demonstrated that students do not perceive equivalent rational numbers as equivalent in size, as many students still demonstrated a bias towards perceiving Percentages as larger than Fractions and Decimals.

Unlike Experiment 1, there was no bias towards perceiving fractions as larger than decimals in this sample at pretest. It is possible that this discrepancy was due to the sample in Experiment 1 being less proficient with decimals, as this fraction>decimal bias was non-existent at posttest following review of rational numbers in Experiment 1. This explanation is likely given that performance on the decimal decile number line estimation task and fractional decile number line estimation task was about the same in Experiment 1. This finding is different from the finding in Experiment 2, which demonstrated that it was easier for students to place decimals than fractions on a decile number line. As discussed previously, there are many varied misconceptions that students hold about decimals (Nesher & Peled, 1986; Resnick et al, 1989, Durkin & Rittle-Johnson, 2015). It seems likely that many students in Experiment 1 might have been basing their magnitude comparison across notations choices primarily on decimal misconceptions such as interpreting shorter train as smaller number (e.g., students might have been interpreting .8 as .08). This shorter-train misconception would lead them to select the fraction as the larger value on trials that included single-digit decimals. Another possibility is that the format of the pretest in Experiment 1 only included identical or nearly identical digits and did not include items that match across equivalent values as in Experiment 2 (e.g., 2/5 vs. 25%, 40% vs. ¼, 2/5 vs. .25, etc.). The lack of this control could have had an impact on skewing the data in this way. As discussed earlier, there was no difference in performance by condition at posttest in Experiment 1 when these modifications were excluded from the analysis.

Consistent with the hypothesis about the superiority of the Simultaneous review of rational numbers, students in the Simultaneous condition made greater improvement than the Sequential and Control condition in Magnitude Comparison across notations. In particular, the low performing students made substantial gains in Fraction>Percent items over the Sequential and Control conditions with an effect size of Cohen's d=.88. This is consistent with other number line training interventions that have been particularly helpful for low-performing students (e.g., Fuchs et al, 2013). Additionally, students in the Simultaneous condition made significantly greater improvement on 0-1 number line estimation and marginally greater improvement on Percent Decile Number line estimation than the students in the Control condition. Moreover, the difference between Sequential and Control condition for these measures was not significant. However, the hypothesis did not hold for all of the measures, as student improvement in the Simultaneous condition was not significantly greater than the students in other conditions in all cases. Additionally, there was no improvement in fraction arithmetic estimation by condition, as overall estimation ability was about at chance.

Finally, consistent with the hypothesis about the importance of integrated number sense, an examination of individual differences in predicting estimation ability demonstrated that integrated number sense was a significant predictor of estimation ability in the presence of distraction. Integrated number sense was operationalized as the composite score for magnitude comparison across notations. The analysis focused on estimation ability in the presence of distraction at both (1) the level of an individual value (decile number line estimation) and (2) at the level of combination of values (fraction addition estimation with lure answer choices and a special focus on the 12/13+7/8 fraction addition estimation problem from Carpenter et al, 1980). The results of the current study demonstrate that performance was worse on the decile number line than the unlabeled number line, which is consistent with the results of Siegler and Thompson (2014). Similarly, the results demonstrate that performance was worse on the fraction addition estimation task where trials contained lure responses than trials that did not contain lure responses (e.g., those that contained "lure choices" such as 4/6 for an estimate of the sum of 2/3+2/3 because it is derived from adding the numerators and denominators). Moreover, wrong answers were most likely to be lure answer choices than the other wrong answer. Thus, as I expected, performance was worse on tasks that could be potentially distracting lending support to the theory about students exhibiting *impulsive calculation.* At both the individual level (decile number line) and the combination of values level (fraction arithmetic with lure choices), accuracy with magnitude comparison across notations added unique explanatory power to the variance in estimation ability. In particular, holding general math ability and number line estimation ability constant, the log odds of selecting the correct estimate for the 12/13+7/8 problem were more than twice as likely for higher levels of magnitude comparison across notations. An important caveat is that students may have been using some form of a compensatory strategy in this magnitude comparison task that avoided actually comparing across notations (e.g., locating a value as being close to 0 or 1 and then guessing with a high degree of accuracy that the other value was either smaller or larger). However, it is unlikely this was the case because performance was about equal whether the comparisons contained an extreme value or not. Thus, it is possible though perhaps unlikely, that it is mere comparison ability rather than Cross-Notation Comparison ability per se that is yielding the predictive explanatory power in estimation. Future research might include both within notation and across notation comparisons to tease apart better the contributions of general comparison ability versus cross-notational comparison ability. Even still, the finding that magnitude comparison across notations is predictive of estimation ability is especially important when considering that the students in the Simultaneous condition made the greatest gains in magnitude comparison across notation. The next chapter examines Experiment 2 data from multiple sources, including interview data.

Chapter 4: Analysis of Qualitative Data in Conjunction with Quantitative Data

Given that little is known about students' understanding of the relation among fractions, decimals, and percentages (Tian & Siegler, 2018), in this section, I attempt to investigate students' understanding of the relations among these notations through a grounded theory approach (Glaser, 1992; Glaser & Strauss, 1967; Strauss, 1987). Grounded theory is particularly well suited to inductively build theories from data in areas lacking a substantial body of literature. Thus, I am drawing upon grounded theory (Glaser, 1992; Glaser & Strauss, 1967; Strauss, 1987), in conjunction with analysis of student work, notes from classroom observations, quantitative data, and 23 student interviews, to posit theories about students' understanding of estimation and the relations among rational number notations.

4.1 Method

Participants and Setting

Forty 7^{th} and 8^{th} grade students were randomly selected for interviews after the completion of the Experiment 2 posttest. Interview consent forms were sent home with students who agreed to participate. Several students declined to participate; for these students, another student was randomly selected in their place. Interviews were only conducted after receiving parental consent and ensuring proper assent through the appropriate Institutional Review Board protocol. Because the posttest was administered during the two weeks prior to the end of the school year, there was a limited amount of time in which to conduct interviews; however, I was able to conduct interviews with all students who returned a consent form and agreed to participate (N=23). The sample of interview participants contained a mix of students from the Control condition (N=7), the Simultaneous condition (N=9), and the Sequential condition (N=7).

While student selection was random, the final sample was not a random sample given that several students did not agree to participate and 17 others agreed to participate but did not return their consent forms. Ultimately, I tried to maintain random selection of interview candidates by randomly selecting additional students after others declined. The purpose of this was to ensure that the sample of participants was as representative as possible, so that the findings could potentially transfer to new contexts.

The interviews were conducted either in a quiet classroom or library. The interviews were not recorded due to concerns from the school district about audio or video recording. All students were provided with a packet of problems containing ample space on which to document their mathematical activity; these packets were utilized as part of the analysis in lieu of recordings. The principal researcher also took detailed notes and recorded all student responses in a de-identified digital spreadsheet.

Interview Protocol

After ensuring receipt of parental consent, assent was obtained as per the Institutional Review Board protocol. The contents of the interview included three parts: (1) questions adapted from Moss & Case's (1999) assessment items involving fractions, decimals, and percentages; (2) discussion of number line estimation strategies; and (3) fraction arithmetic estimation strategies (see Appendix A for full interview protocol). The principal researcher read each question to the students; students recorded their work for each question in the problem packet and explained their thinking about the strategies used to solve the problems. As part of the interview, students also placed fractions on a 0-1 number line and a 0-1 decile number line via Qualtrics and were asked to explain their strategies. Finally, they discussed how they would estimate 12/13+7/8. The questions were decided in advance, but because the interview was semi-structured, I asked some follow-up questions to better understand what students were thinking. These follow-up questions varied by participant. In total, there were 23 interviews; each lasted between 20-30 minutes.

Data Analysis Methods

This investigation used a grounded theory approach. Grounded theory involves breaking down the data and creatively conceptualizing it in a new way, constantly comparing data to other forms of data until theories emerge (Glaser, 1992; Glaser & Strauss, 1967; Strauss, 1987). These theories are gradually refined as new pieces of data are examined. Thus, upon the completion of the interviews, interview and quantitative data were compiled for these 23 students and trends were examined. In many cases, the principal researcher did not know the student's assigned condition during the interview, but the principal investigator was not completely unaware of each student's condition assignment. That being said, the first stage of the analysis attempted to ignore condition assignment. A constant comparative method was used while (1) looking at each piece of data for general trends in measures and student explanations and (2) looking across student performance in comparison to their explanations. In particular, the method involved (a) generating categories and comparing incidents to these specific categories; (b) synthesizing categories and their properties; (c) contextualizing the theory; and (d) writing the theory (Glaser & Strauss, 1967, p. 105). Thus, it was important to stay very close to the data initially, questioning and comparing whether pieces of information are consistent with categories; then once all individual data elements were analyzed, theories were able to move from low-level abstract theory to more high-level theory (Strauss & Corbin, 1998). Furthermore, the large interviewee sample size (N=23) and diverse types of data sources (e.g., interview data, quantitative data, classroom observations, student work) allowed triangulation of theory and perspective (Glaser, 1992).

Finally, an attempt was made to understand the effects of condition (Simultaneous, Sequential, Control) in facilitating any changes from pretest to posttest interview for the decile number line task, given its importance understanding numerical processing in the presence of distraction (Hypothesis 1a). However, given that project constraints did not allow for interviews at pretest, the strategies students were using at pretest are unknown, and thus, it is difficult to analyze micro-changes occurring in student thinking as a result of condition. That being said, aggregated data has often concealed diversity in patterns of performance in the area of numerical cognition (Braithwaite & Siegler, 2018b; Siegler, 1987, 1989). Thus, case studies were used to examine individual students who were distracted at pretest and interviewed at posttest. This case study analysis consisted of examining the trajectories of students of approximately equal ability level (as measured by their standardized tests of achievement) and combining their pretest to posttest results with the explanations of their thinking from the interviews.

4.2 General Findings

Three theories emerged from the data using a grounded theory approach:

1. Flawed translation strategy: Students use left-to-right whole number and decimal strategies for fractions, resulting in inappropriately concatenating values (a/b = 0.ab)
2. Percent as a tool: Using a percent estimation strategy (e.g., 7/12 is a little more than 50%) for fractions may inhibit students' use of flawed strategies and help them maintain focus on magnitude.
3. Estimation is not valued: Students perceive mathematics as involving calculation rather than estimation for rational numbers.

These theories are important because they shed new light on the focus of inquiry of this dissertation about understanding the relation among notations. In particular, these theories suggest that difficulties with rational numbers may stem from an inability to appropriately evaluate magnitude of fractions based on flawed translations and a lack of belief about the importance of considering their magnitudes. Moreover, it lends support to the hypothesis that instruction aimed at improving integrated number sense might be beneficial for improving mathematics outcomes because estimating a fraction's magnitude as a percent proved particularly useful for students. Perhaps, an even simpler intervention could be to have students estimate all fraction values as percentages and use those rough approximations as a means to evaluate the answers to fraction arithmetic problems. In other words, students might reason that 12/13+7/8 is approximately 90%+90%, which is approximately equal to 180% or 1.8 (the correct answer is actually 1.798). More details will be described in subsequent sections.

Additionally, the case studies provided greater understanding about the effects of condition in improving students' ability to attend to magnitude. Four students of equal ability levels as designated by their math achievement score were analyzed as case studies because their pretest fraction decile number line estimation performance (PAE >.3) suggested that they were using the flawed translation strategy (a/b=0.ab) at pretest. One other student was included in this case study analysis because their standardized scores were borderline "exceeding expectations," yet the student had the worst PAE on the decile number line task. Analyzing these students' quantitative data and comparing it to their interview data provided comprehensive information about how the conditions affected their ability to maintain attending to magnitudes. In particular, it seems that the Simultaneous condition was beneficial for improving the diversity of appropriate strategies that the students had for evaluating magnitude and improvement on fraction arithmetic estimation when there were no "lure" answer choices.

The following sections discuss the three theories that emerged from the data: flawed translation strategy, percent as a tool, and estimation is not valued. The final section highlights case studies to examine the effect of condition in fostering attention to magnitude.

Flawed Translation Strategy involves a/b to 0.ab concatenation of values

In Experiment 1, the majority of students selected the location of .67 for the placement of 6/17 on the 0-1 number line that was partitioned into tenths. Thus, students concatenated a digit from the numerator and the denominator to place a fractional value (a/b) at the location 0.ab on the labeled number line. Experiment 2 found the same pattern of results. Table 13 displays the fractional values presented, the hypothesized student response (i.e., the concatenated wrong answer), the percent that selected the hypothesized wrong answer, the percent that selected the right answer, the mode, and overall accuracy. The hypothesized wrong answer appears as the mode 5 out of 8 times (5 out of 6 times for all values that do not have a denominator that can be easily multiplied to a power of 10).

Table 13: Descriptive Statistics for Fraction Decile Number Line Trials

Decile Number Line Trial	Hypothesized Incorrect Answer	Mode	Percent selected the hypothesized incorrect answer	Percent selected the correct answer
1/19	0.19	.01, .19	8.2%	6.7%
9/20	0.92	0.45	2.7%	21.6%
9/17	0.97	0.97	8.6%	1.5%
6/17	0.67	0.67	14.8%	3.4%
5/6	0.56	0.9	5.7%	1.5%
8/14	0.84	0.84	9.1%	3%
9/15	0.95	0.95	11.9%	10%
4/5	0.45	0.8	9.3%	43.7%

Figure 13 displays the distribution of responses for 6/17 on the decile number line. It appears that the most responses are around .67 (the hypothesized wrong answer) and .35 (the actual magnitude). The pattern of results with frequency of results peaking at the hypothesized wrong answer and to a lesser degree at the actual answer (Figure 13) are similar to the results of the other values students translated inappropriately.

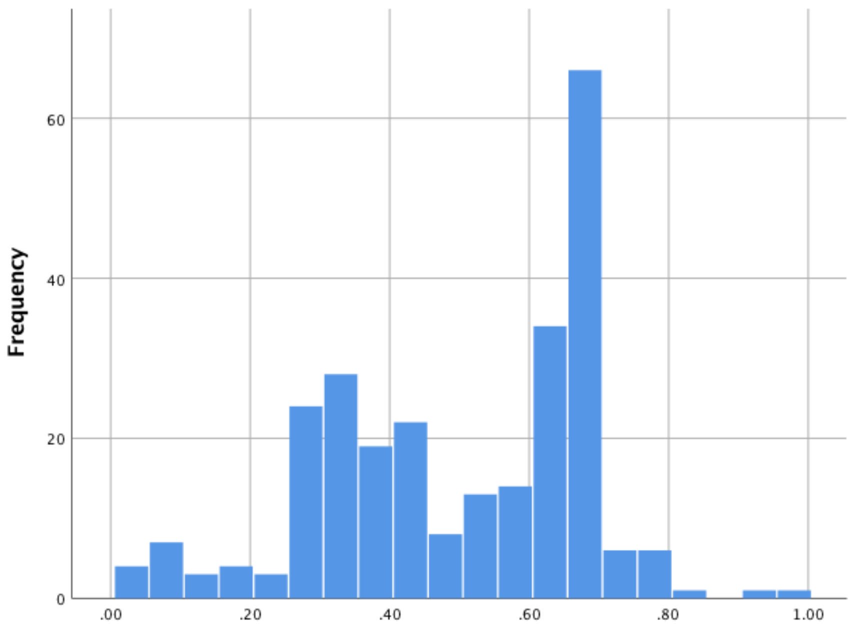


Figure 13: Frequency distribution of responses for placing 6/17 on the Decile Number line. Note: the mode is 0.67, but the actual magnitude of 6/17=0.35

What do students say? Analysis of student explanations during interviews suggested a reliance on interpreting the magnitude through translating from a/b to 0.ab or left-to-right processing of the symbol. For example, when asked to place 6/17 on the number line from 0 to 1 labeled and partitioned by tenths, a student described the reasoning as you must "go to the 6 and then over more for the 17." Similarly, another student placed 3/5 on the number line at .35 and reasoned, "it's in the 3 range and in the middle," suggesting that 3/5 should be placed between .3 and .4 and in the middle of the .3 to .4 range. Even apart from the number line, students, when asked to translate 1/8 to a decimal and percent, described the translation process as "1/8 is small so it is .08 and 8%." In essence, the 1 tells you it is a small number that is near 0 and less than .1 and the 8 tells you where to put it in that range.

Why might the flawed translation strategy exist? Many students' pattern of activity mirrors a similar pattern of activity with other notations when interpreting the magnitude of a value from its written symbol. In whole numbers, the first number indicates the numerical range on a macro level, and the subsequent numbers provide even more fine-grained location information. For example, interpreting the magnitude of the symbol '617' involves understanding that the 6 refers to the six hundreds range (one must locate a range between 600 and 700); the 1 refers to the teens range (one must partition the space between 600 and 700 further to find the 610-620 range); and the 7 refers to the ones (one must partition the space between 610 and 620 even further to find the appropriate location). The processing of magnitude becomes more and more specific as a student interprets the symbol from left to right. Decimals are processed in a similar manner with partitioning getting more fine-grained as the student reads from left to right. To find .617 on the number line, the student would have to locate the .6 to .7 range, then the .61 to .62 range, and then place the .617; granted, decimal processing is not as straightforward for many students given their misconceptions about decimal magnitude (Resnick et al. 1989; Durkin and Rittle-Johnson, 2015). However, students often have little difficulty processing two-digit decimal magnitude using this strategy. Even digital clocks are read "six-seventeen" for 6:17 from left to right. Implicit in the understanding of telling time is that the first number tells that the time is between 6 and 7 o'clock, and the second number provides specific detail as to where it is between 6 and 7 o'clock.

On the other hand, the numerical values in fractions need to be processed in a different fashion than the left-to-right processing of whole numbers and decimals. Students cannot interpret the fraction 6/17 as being between 6 and 7 and then use the 17 to provide specific detail as to where it is between 6 and 7. Yet, in many cases, it is possible to read a fraction (a/b) and concatenate digits from the numerator and denominator to translate it as a decimal (i.e., 0.ab) and obtain a correct or very near correct answer for the actual magnitude of the value (e.g., 3/8 is 0.38). Thus, concatenating digits from a fraction's numerator and the denominator can often provide a "ballpark" fraction to decimal translation. However, the strategy fails in many cases such as 4/5, where translating the value using this flawed strategy to .45 provides a vastly different interpretation of magnitude from the correct one (.8). Thus, the interpretation of magnitude for fraction symbols marks a monumental departure from the interpretation of magnitude for other notational symbols students have encountered. Students' persistence in interpreting the magnitude of fractions through a left-to-right (i.e., a/b=0.ab) approach is consistent with Ganor-Stern's (2013) interpretation that decimals "might be represented more similarly to multi-digit numbers than to unit fractions, as they are similar to multi-digit numbers in their numerical characteristics as well as in their notation, with both numbers written horizontally. Such greater similarity enables an easy mapping of decimal fractions on the same mental number line with whole numbers" (p. 305). And indeed, performance is slower and less accurate for locating fractions on number lines than decimals, such that decimal performance is more similar to integers (Iuculano & Butterworth, 2011).

Presenting students with intentionally misleading situations can reveal information about student understanding of concepts and procedures regarding the interpretation of magnitude. For example, students with limited place value understanding are distracted when visual displays highlight the digits rather than the magnitude (Miura & Okamoto, 1989). In Miura and Okamoto's study, students were given 13 cubes and 3 cups, directed to place 4 cubes in each cup, and shown a card with 13 written on it. Students with weak understanding of place value described the 1 digit as representing 1 cube and the 3 digit as representing the 3 cups filled with 4 cubes rather than the total magnitude of 13 cubes. Similarly, many students in Experiments 1 and 2 demonstrate weaker understanding of fraction magnitude because the a/b to 0.ab fraction to decimal translation does not match the appropriate magnitude (e.g., 3/5 does not equal 0.35). Thus, understanding the magnitude of a written symbol involves the coordination of concepts and procedures governing the interpretation of the notation. While this dissertation begins to shed light on students' translation procedures, particularly the flawed concatenation strategy, more work needs to be done to (1) better understand the situations in which students might use this flawed approach and (2) develop a deeper understanding of how instruction could be designed to minimize the use of flawed translation strategies for fractions.

Percent as a tool

Another discovery that emerged from the data was the finding that using a percent estimation strategy for fractions may have inhibited students' use of flawed strategies and helped them maintain focus on magnitude. In other words, students who thought about the magnitude of a fraction as a percent (e.g. 12/13 is approximately 90%) were better able to maintain focus on magnitude in potentially distracting situations. Similarly, Moss and Case (1999) showed that highlighting the connections among fractions, decimals, and percentages helps students develop a more thorough understanding of magnitude. Students need to understand the various interpretations of fractions and how to interpret the size of a fraction (Behr et al, 1983; Kieran, 1976). Translation between percentages and fractions can be a useful tool for interpreting the magnitude of a fraction (Moss & Case, 1999); however, not much research exists in this area.

Using percent as an estimation tool for attending to magnitude could help students resist the temptation to use a flawed translating strategy for a fraction. For example, students who may use a flawed translation strategy for 4/5 (i.e., would select 0.45 as an equivalent decimal) could reason that 4/5 is greater than 50%, thereby identifying that the decimal equivalent cannot be 0.45 since 45% is less than 50%. Interviewed students (N=23) identified percent as a strategy for number line estimation more often in the Simultaneous condition than the Sequential or Control Condition (p=0.002). In particular, 78% of interviewed students from the Simultaneous condition used a percent strategy at least once, as compared to 14% in the Sequential condition and 0% in the Control condition. The percent strategy appears to have had a stronger effect on students who were distracted by the decile number line task at pretest. For example, one student described "eyeballing" number line estimation by thinking about the fraction as a percent that is larger than 50% and probably closer to 75-85%.

Finally, it is important to note that nearly all students attended well to percent magnitude at pretest even in the presence of distracting information (the decile partitions) on the decile number line. This result even applied to students who performed poorly at pretest on the decile number line task for fractions or decimals. Also, 83% of all students in the Sequential condition (N=85) spontaneously mentioned percent at least once in their student activity book in lessons that focused on fractions or decimals. These findings show that percent may be a more intuitive way to think about rational numbers (Moss & Case 1999) and perhaps could be used as a tool to encourage students to think about the magnitude of rational numbers as numbers rather than concentrating on the part-whole approach to fractions. In other words, students might be taught to estimate with fractions by thinking about them as percentages (e.g., 12/13+7/8 is approximately 90%+90%, which is 180% or 1.8)

Estimation is not valued

The interview data revealed that students do not equate estimation with mathematics. For example, a student responded, “I can’t do math, right?” when asked to estimate addition of fractions instead of performing rote calculations. In other words, many students see math as the calculation of an exact answer alone and that it does not involve critically thinking about an approximate answer. Furthermore, students appeared confused by the request to estimate when adding fractions, asking, “Don’t you have to solve it though?”

Interviews with students yielded a particularly striking interpretation: fractions are not numbers that can be used to estimate. Most students were quite adamant they did not “know how to take a guess” because “you’d have to find the common denominator.” Common Core Standards (2019) recommend that students “use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 < 1/2.” Yet, many students had very little idea that estimation with fractions was even possible, let alone something that they might do to recognize incorrect results when doing their math homework. Given most students were confused at the request to estimate with fractions when explicitly asked to estimate, it is highly unlikely students will estimate on their own to evaluate the reasonableness of their solutions.

Despite a push towards incorporating more measurement approaches to fraction instruction in the US (Common Core Standards, 2019), many students walk away with more emphasis on procedures than concepts. Moreover, student performance on the fraction arithmetic task in Experiments 2 resulted in students generating a correct response only slightly greater than chance. As such, it appears that whatever estimation practices are occurring (if any) in the classroom are not helpful for students. The intervention attempted to encourage more estimation by involving a 5-7 minute warm-up activity once per day over the course of three weeks. The intervention yielded some promising results given the short time frame; however, it has only begun to scratch the surface of understanding how students view the connections between rational number notations and how an intervention might help students employ estimation skills with rational numbers.

4.3 Case Studies

Case studies on specific students were used to examine individual student performance on multiple measures across time and explored whether their interviews could provide additional insight on how the instructional condition affected the posttest outcomes. Specifically, I concentrated the analysis on interviewed students, who performed poorly on the labeled number line task, suggesting they used a flawed strategy (e.g., students who scored above .3 at pretest).

Students with similar standardized test of achievement scores were used so that students with similar overall ability levels could be compared. Therefore, case studies focused on four students of low average ability, designated as “approaching expectations” according to their PARCC scores. Additionally, a student of above average ability level, designated as “met expectations” according to their PARCC score, was examined because their decile number line score was one of the worst at pretest (PAE=0.42), which was surprising given their PARCC score bordered on “exceeding expectations.”

The students focused on in the first part of the analysis are Control-A, Simultaneous-A, Simultaneous-B, and Sequential-A. Table 14 displays their performance on some measures involving distraction (decile number line and lure fraction addition estimation accuracy) and no distraction (0-1 number line estimation and no lure fraction addition estimation).

Table 14: Descriptive statistics by test time for each case study student from Experiment 2.

Student information			Unlabeled 0-1 Line PAE		Fraction Decile 0-1 Line PAE		"No Lure" Fraction addition accuracy		"Lure" fraction addition accuracy	
Students	Gender	PARCC	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest	Pretest	Posttest
Control-A	F	746	.04	.05	.37	.21	.40	.09	.25	.25
Simultaneous-A	F	741	.20	.03	.33	.09	.50	.36	.30	.40
Simultaneous-B	M	730	.39	.49	.35	.37	.17	.58	.25	.25
Sequential-A	F	722	.08	.20	.33	.31	0.0	.25	.36	.33
Simultaneous-C	M	780	.03	.04	.42	.03	.45	.83	.58	.70

Note: The number line estimation tasks are scored in PAE and the Fraction Addition Estimation is scored in percent correct.

Control Student compared to Simultaneous Student that Improved on the Decile Task

Comparing Control-A to Simultaneous-A, both students tested similarly at pretest on the decile number line task. Examination of their actual responses indicated that these students may have been using the a/b=0.ab flawed translation strategy discussed previously (e.g., 5/6=0.56). Additionally, both students were heavily biased towards perceiving percentages as greater than fractions. Accuracies on items for both students were identical (P>F= 67% and F>P=17%). At posttest, both students improved their fraction decile number line PAE score but Simultaneous-A made a greater improvement. Careful examination of their actual responses demonstrated that Control-A mostly maintained the a/b=0.ab flawed strategy but Simultaneous-A did not, suggesting that perhaps the condition helped Simultaneous-A revise their decile number line estimation strategy to a more appropriate one. Indeed, during the number line portion of the interview, Simultaneous-A utilized more diverse number line estimation strategies including a correct translation strategy (not the flawed approach); whereas, Control-A employed limited or flawed number line estimation strategies. Furthermore, Simultaneous-A improved from 17% accuracy on F>P magnitude comparison items to 100% accuracy and from 67% accuracy on P>F items to 100% accuracy. Both students performed better on the no-lure fraction addition estimation items than the lure fraction addition estimation items at pretest. However, both students exhibited no change or worse performance at posttest, with the exception of Simultaneous-A who made a slight improvement on lure fraction addition items (from 30% accuracy to 40% accuracy). Simultaneous-A provided no answer for all of the "Across lure" fraction arithmetic estimation trials. As mentioned previously, students had 20 seconds to respond before the program moved onto the next fraction arithmetic estimation problem. It seems that Simultaneous-A may have run out of time during "Across lure" trials at posttest. Thus, the Simultaneous Condition likely helped Simultaneous-A avoid immediately selecting the lures; although, it is not clear Simultaneous-A would have selected the correct answer if given more time. The 0 to 5 number line task continued to prove problematic for both students, as evidenced by their inappropriate reasoning about the 0 to 5 number line task during interviews.

Simultaneous Student who Did Not Improve on the Decile Number Line Task

While Simultaneous-A showed a fair amount of improvement on the fraction decile number line task, Simultaneous-B scored slightly worse on this task at posttest. The interview responses yielded many procedural explanations such as performing long division, moving over decimal points, and cross simplifying. Simultaneous-B often employed an inappropriate strategy or used the strategy incorrectly (e.g., dividing 5 by 4 instead of 4 by 5 for 4/5). Examining

Simultaneous-B's pretest scores in connection with interview data yielded a more holistic picture of Simultaneous-B's mathematical behaviors. For example, Simultaneous-B's fraction decile number line performance was very poor (PAE=0.35) and unlabeled number line performance was worse (PAE=0.39). By contrast, Simultaneous-B's pretest score for placing decimals on the decile labeled number line was quite good (PAE=0.003). This sharp contrast in performance indicates that Simultaneous-B understands how decimals are related to decimal fractions but does not understand how fractions are related to decimal fractions. Thus, it is likely that Simultaneous-B attempted to use an algorithm such as long division (albeit very poorly) to determine where the fraction value should be placed for both the decile-partitioned and unlabeled number line tasks at pretest and posttest. Even though Simultaneous-B did not improve on the fraction labeled number line items, he improved considerably from pretest to posttest on non-lure fraction estimation items (17% to 58%). The 58% accuracy on non-lure fraction addition estimation items is well above chance and the group mean (33%), which would not have been predicted based on his performance on some of the other measures. However, it is important to note that the fraction arithmetic estimation task is the only timed task. Perhaps this time constraint did not allow him time to apply rules or algorithms, allowing him to focus on magnitude. Perhaps the Simultaneous condition was beneficial in improving his intuition for magnitude when he was not actively trying to apply algorithms.

Sequential Student that was Distracted on the Decile Number Line Task at Pretest

Similar to Control-A and Simultaneous-B, Sequential-A made no practical improvement on the decile number line task and consistently described using flawed procedures for estimating the values of fractions on the number line. For example, when describing her strategy for placing 3/5 on the decile number line, Sequential-A explained that the 3 tells her it goes between 3/10 and 4/10 and the 5 tells her that it goes in the middle of that range. The explanation of her strategy matched her behavior, as she placed the value at .35 (as opposed to the correct location at .6). Her responses on both pretest and posttest are consistent with this a/b=0.ab translation strategy (e.g., placing 2/3 at .23 and 9/15 at .95 on the decile number line). Sequential-A made some improvement on the fraction estimation non-lure items but her performance (25%) was still below chance for a task involving three answer choices.

Simultaneous Student of High Ability who was Distracted at Pretest

Siegler & Thompson (2014) discussed how the correlation between standardized test scores and PAE weakened in the decile number line condition. They further posited that a potential reason for the weakening correlation is that students did not really understand the magnitude of the values. Thus, in addition to focusing on students of similar ability levels, a case study was developed for an above average student (PARC=780) who performed poorly on the decile number line condition. Of the students that were interviewed, Simultaneous-C's performance was one of the worst on the decile number line (PAE=0.42). Given the constraints of Experiment 2, students were not interviewed at pretest. Thus, it can only be inferred what Simultaneous-C's difficulties might have been at pretest. At pretest, he was fairly consistent with constraining his estimate of each fraction to between 0 and 1/10 (e.g., he placed 4/5 at 0.08). Notably, there was one exception where he accurately placed 9/20 at .45, suggesting that he was confused about decimals and/or he did not understand the relation between fractions and decimals. Improvement was 0.39 on the fraction decile number line estimation task, suggesting that it was no longer distracting for him and the decile number line was potentially helpful for him in estimating the magnitude. During the interview, Simultaneous-C discussed a variety of strategies for number line estimation, including frequent use of percentage as an estimation tool.

Additionally, Simultaneous-C's overall fraction addition estimation abilities improved from 52% accuracy at pretest to 77% at posttest. Thus, the intervention was likely fruitful in helping Simultaneous-C solidify the relations among rational number notations, and this solidification of understanding transferred to fraction addition estimation.

How Do Case Studies Fair in Transferring Skills to a New Model?

Finally, since a good deal of weight was placed on number lines in this analysis, the performance of these five students was examined on another spatial model: a circle. As part of the student interviews, some questions from Moss & Case's (1999) interview protocol were included that asked students to shade circles that were already partitioned. One question asked students to shade ¾ of a circle partitioned into eighths, and the other asked students to shade .3 of a circle partitioned into fifths. Of the five students in this analysis, three students, Simultaneous-A and Simultaneous-C and Control-A, were the only students who appropriately shaded ¾ of the circle partitioned into eighths and explained that ¾ is equivalent to 6/8. None of the students accurately shaded .3 of the circle that was partitioned into fifths. However, Simultaneous-A and Simultaneous-C were the only ones to provide an explanation that differed from I shaded "3 because 0.3 has number 3 in it." For example, Simultaneous-A explained that she knew that .3=30/100=30%, but she wasn't sure how to shade the circle to show that so she just shaded 3 parts. Simultaneous-C did long division for .3/5 to find 0.06 and then shaded what she described as .06 of the circle. Thus, Simultaneous-A and Simultaneous-C's knowledge of number lines did not fully transfer to the context of circles, yet their understanding of the relation between fractions/decimals in this new context was likely at an emerging stage.

4.4 Discussion

In conclusion, three theories emerged from the data using a grounded theory approach: students are using a flawed translation strategy (a/b=0.ab), percent is a useful tool for students to evaluate the magnitude of fractions, and students do not value estimation with fractions. These three theories are important because they shed new light on the importance of integrated number sense. In particular, these theories suggest that difficulties with rational numbers may stem from an inability to appropriately evaluate magnitude of fractions based on flawed translations. It is possible that students are learning spurious associations between instances where translations done with the flawed approach yield approximate answers. This may even perpetuate a lack of belief about the importance of considering the magnitudes of fractions because students cannot be sure why these translation strategies may work in some cases and not in others. Moreover, it lends support to the hypothesis that instruction aimed at improving integrated number sense might be beneficial for improving mathematics outcomes because estimating a fraction's magnitude as a percent proved particularly useful for students.

Finally, the case studies suggested that the Simultaneous condition seemed to provide a means for students to triangulate their understanding of magnitude through thinking about a value as a fraction, decimal, and percent. This is a useful way to think about magnitude because understanding involves incorporation of concepts into an internal network, whereby the degree of understanding is determined by the strength and accuracy of connections among related concepts (Hiebert & Carpenter, 1992). Thus, the daily, 5-7 minute intervention over the course of three weeks likely had some influence in deepening the strength of connections among rational number notations, helping students who were easily distracted at pretest. At the very least, it appears that future research in this area is warranted, perhaps including an intervention over a more prolonged period of time.

Chapter 5: General Discussion

In this concluding chapter, I discuss the contributions of this work, avenues for future research, and educational implications. Ultimately, this dissertation advances the field of numerical cognition by expanding upon Siegler, Thompson, and Schneider's (2011) integrated theory of numerical development by demonstrating that (1) students' *integrated number sense,* or understanding of the relations among fractions, decimals, and percentages, accounts for substantial variance in mathematical outcomes beyond that explained by fraction magnitude representations and tests of math achievement and (2) number line-based review of rational numbers can improve students' *integrated number sense*. In particular, the findings related to the hypotheses are the following:

Hypothesis 1: Integrated number sense adds explanatory power to mathematical outcomes

Data from this dissertation support the first hypothesis in three important ways: (1) by documenting a newly discovered bias of middle school students perceiving *percentages as larger than fractions and decimals* in magnitude comparisons across notations and positing that a lack of integrating notations on the same mental number line is a likely mechanism for this bias; (2) by demonstrating that students exhibit *impulsive calculation*, as measured by the difference in performance between situations where students are presented with distracting information ("lures") meant to elicit the use of flawed calculation strategies and situations that do not involve lures; and (3) by finding that *integrated number sense,* as measured by the composite score for magnitude comparison across notations, is a unique predictor of estimation ability, often above and beyond tests of math achievement and number line estimation. In particular, students with higher integrated number sense are more than twice as likely to correctly answer the 12/13+7/8 estimation problem discussed at the beginning of and throughout this dissertation than their peers with the same number line estimation ability and math achievement scores. This finding suggests that *integrated number sense* is the most important inhibitor for *impulsive calculation*, above estimation ability for individual fractions and general math ability.

Hypothesis 2: Number line instruction aimed at improving integrated number sense is beneficial

This dissertation advances the field of mathematics education by suggesting instruction that connects equivalent values with varied notations might provide superior benefits over a separate and sequential approach to reviewing rational numbers. At a minimum, this dissertation suggests that more careful attention must be paid to relating rational number notations.

Limitations and new directions

Future work might examine the origins of *impulsive calculation* and the observed *percentages-are-larger bias.* Future research might also examine whether *integrated number sense* is predictive of estimation ability beyond general number sense within notations. From these investigations, it might be possible to design a more impactful intervention to improve rational number outcomes.

5.1 Introduction

Siegler, Thompson, and Schneider (2011) posited an integrated theory of numerical development, which placed fraction magnitude understanding as central in numerical development. In particular, the study found that differences in magnitude knowledge correlated highly with individual differences in fraction arithmetic ability and with math achievement test scores. The researchers alluded to the idea that fraction magnitude representations may be related to ability to translate among fractions, decimals, and percentages. However, there have been no studies to date that have investigated the relation among fractions, decimals, and percentages. I theorized that integrated number sense would be just as important, if not more important, than individual fraction magnitude representations. This hypothesis was based on research indicating that depth of understanding is characterized by the strength of connections among related concepts (Hiebert & Carpenter, 1992). Therefore, students that have a stronger understanding of fraction magnitude would also have a strong sense of how its varied notations (fractions, decimals, and percentages) are related. In this dissertation, integrated number sense was operationalized as percent accuracy with comparing magnitudes across notations (e.g., 2/5 vs. 25%, 40% vs. .25, 2/5 vs. .25). Consistent with the hypothesis about its importance in math outcomes, integrated number sense was highly correlated with math achievement tests ($r=.562$, $p<.001$), similar to the strength of the correlation between math achievement tests and fraction estimation on a 0-1 number line ($r=-.559$, $p<.001$). Replicating Siegler et al (2011), 0-1 number line estimation was an important predictor in hierarchical regression analyses accounting for 31% of the variance in math achievement scores ($F(1,244)=110.66$, $p<.001$). Importantly, Integrated Number sense added 10% further variance ($F(2,243)=83.27$ $p<.001$). These results suggest that fraction magnitude understanding alone does not explain the whole story in regards to mathematical outcomes; understanding the relations among fractions, decimals, and percentages also plays a crucial role in numerical development. Furthermore, as will be discussed later, integrated number sense is more important than fraction number sense alone for inhibiting implausible errors with fractions.

The following sections will discuss three findings related to the first hypothesis about the importance of integrated number sense in explaining mathematical outcomes. In particular, the sections focus first on problems that arise from lack of integrated number sense: *percentages-are-larger bias* and *impulsive calculation*. The third section will focus on how individual differences in integrated number sense predict students' estimation ability in potentially distracting situations.

Finally, the fourth section of this chapter focuses on findings related to the second hypothesis about improving integrated number sense through number line instruction. Specifically, discussion centers on how the Simultaneous condition offered benefit over the Sequential and Control conditions in improving integrated number sense. This section will conclude with general mathematics education recommendations.

5.2 The Percentages-are-larger bias and the Relation Among Notations

The studies presented in this dissertation are the first to examine magnitude comparisons among fractions, decimals, and percentages. The paragraphs below discuss how the dissertation adds to the field of numerical cognition by documenting a newly discovered bias of middle school students perceiving *percentages as larger than fractions and decimals* in magnitude comparisons across notations. Moreover, I posit that a lack of integrating notations on the same mental number line is a likely mechanism for this bias. In particular, the following sections review the rationale for the hypothesis about students perceiving equivalent rational numbers as equivalent in size based on previous literature and reviews background on prior cross-notation comparison studies. Furthermore, the next sections summarize the finding of the *percentages-are-larger bias* and how possible confounding explanations can be excluded. Moreover, subsequent sections discuss the theory of how lack of integrating notations on the same mental number line is a likely mechanism for this bias. Finally, this section suggests that magnitude comparison across notations could potentially be utilized by the field as a measure of integrated number sense and how future studies should also include within notation comparisons.

I hypothesized that middle school students do not think about rational numbers as being equivalent in size. This hypothesis was based on previous work that suggested students do not understand the fundamental connections among notations in the number system (Moss & Case, 1999). This hypothesis was also based on the documented tendency of many students to focus on the operational rather than relational view of equivalence (McNeil et al., 2006). Furthermore, this hypothesis was based on the documented whole number bias exhibited by over a quarter of students in 8th grade, such that students perceived equivalent fractions with larger parts as larger than those with smaller parts despite years of instruction on equivalent fractions (Braithwaite & Siegler, 2018b). If middle school students are unable to perceive equivalent values within the same notation as equivalent in size, it seems probable that they might also struggle perceiving equivalent rational numbers as equivalent across notations. This is especially true in light of evidence that many teachers often do not use equal signs to describe equivalent fraction, decimal, and percent values (Muzheve & Capraro, 2012). Therefore, I reasoned that middle school students would exhibit difficulty with comparing fractions, decimals, and percentages.

Prior research investigating cross-notation comparison abilities has had important limitations. Previous studies investigated magnitude understanding of fractions and decimals, suggesting decimal notation might be easier for adults when accessing magnitude information (DeWolf, Grounds, Bassok, & Holyoak, 2014; Hurst and Cordes, 2016). Furthermore, research suggests that decimals are represented more similarly to integers than unit fractions (Ganor-Stern, 2013). Other studies examined relative performance across fractions, decimals, and whole number comparisons and how these magnitude representations contribute to pre-algebra ability in students (Hurst & Cordes, 2018). However, most studies utilized decimals with equal number of digits, making the comparisons an almost trivial task because of the potential for participants to ignore the decimal point and treat the values as whole number comparisons rather than decimal comparisons (e.g., 0.12 compared to 0.22 being viewed as 12 compared to 22). Comparing decimals with an equal number of digits may not provide an accurate depiction of students' numerical processing of decimals given students have many misconceptions about decimals such as longer train signifies greater value (e.g., Nesher & Peled, 1986; Resnick et al, 1989; Durkin & Rittle-Johnson, 2015).

There is documented evidence of the importance of fractions for math outcomes (Siegler et al., 2012) and support for a theory of numerical development that integrates all rational numbers (Siegler, Thompson, Schneider, 2011), but little is known about the relations among notations. Yet, it is known that understanding involves the incorporation of concepts into an internal network, whereby the degree of understanding is determined by the strength and accuracy of connections among related concepts (Hiebert & Carpenter, 1992). Therefore, fluid understanding of the connections and relations among fractions, decimals and percentages indicates deeper understanding and superior performance (Moss & Case, 1999). While there is some information about comparison across fractions and decimals (e.g., Hurst & Cordes, 2016), much less is known about comparison across fractions, decimals, and percentages. Before debating which notation is best to learn first (Tian & Siegler, 2018), we must first understand how children conceptualize the relation among all notations.

The studies presented in this dissertation contribute to the literature on students' understanding of the relations among notations by being the first to examine magnitude comparisons across fractions, decimals, and percentages. Ganor-Stern (2013) suggested fractions and decimals are represented along the same mental number line. Ganor-Stern (2013) goes on to posit that the specific notation of fractions might be what distinguishes them from decimals given that decimals are represented more similarly to multi-digit whole numbers than unit fractions. As such, representing magnitudes of fractions requires overcoming the violation of a positive linear relationship between the components of the digits and the holistic magnitude of the value (e.g., 1/9<1/8, but 9>8; whereas, 0.9>0.8).

The studies presented in this dissertation expand on Ganor-Stern's work by asking middle school students to make judgments about the size of values with identical digits written in different notations; the results indicate that many middle school students do not think about equivalent rational numbers as being equivalent in size. Half of the trials compared values with identical or nearly identical digits (e.g., compare 4/5 versus 45%). The other half were matched for magnitude across all notations with small, medium, and large differences between the compared values (e.g., compare .40 versus 25%, 2/5 versus .25, .4 versus 1/4, etc.). Results of Experiments 1 and 2 showed that many middle school students do not think about equivalent rational numbers as being equivalent in size; instead, the experiments showed students have a bias towards perceiving percentages as larger than fractions and decimals. In other words, students were more accurate at deciding when a percentage was greater than a fraction or decimal and less accurate when a percentage was smaller than the fraction or decimal. I call this phenomenon the *percentages-are-larger bias.*

I examined whether there might be a confounding factor that could explain the *percentages-are-larger bias* but concluded that these explanations could likely be excluded. For example, I examined whether students perceived percentages to be larger due to the percentages presented being larger on average across trials; however, the percentages presented were slightly smaller on average. I examined whether the ratios between percent and fraction or decimal might be larger when the percent is larger (making those trials easier to compare) given that research suggests magnitude comparisons are easier involving higher ratios between the values being compared. However, the ratios were similar when the percent is larger than the fraction/decimal and when the percent is smaller than the fraction/decimal. I examined whether students were using a compensatory strategy by locating an "extreme" value as close to either 0 or 1 and guessing with a high degree of probability that the other value is larger or smaller than the extreme value. However, answer accuracy did not improve when one of the values was close to the endpoints. Considering that confounding explanations for these results are unlikely based on the analysis, I will focus on exploring the mechanism by which the *percentages-are-larger bias* occurs.

The *percentages-are-larger bias* demonstrates an error in student noticing consistent with other errors that students make with rational numbers. Evidence of noticing the wrong aspects of rational numbers is apparent in misconceptions about fractions (e.g., Stafylidou & Vosniadou, 2006), decimals (e.g., Durkin & Rittle-Johnson, 2015), and percentages (e.g., Gay & Aichele, 1997). The discrepancy in performance between percent-to-fraction and percent-to-decimal comparisons such that performance is significantly worse if the percent is smaller than the other notation in Experiment 1 and 2 provide cross-notation evidence of errors in student noticing. Perhaps, the explanation of this result is not just that they think percentages are larger than fractions/decimals but that the familiar 0-100 whole number format of the percent activates the whole number bias (Ni & Zhou, 2005), where students use the single-unit counting scheme to interpret rational numbers. Thus, students import knowledge of whole numbers to the comparison between percentages and fractions/decimals, making it more difficult for them to correctly select the percentage as smaller than the fraction or decimal. Colloquially, children think of percent as more similar to whole numbers (Ginsburg, Gal, & Schuh, 1995). For example, students make statements such as “I got a 100 on my test” rather than emphasizing the part-whole relation, “I got all of the problems right, of all of the problems that were asked.” Thus, when tasked with comparing a percent to a fraction, likely a task they have never encountered before given the separate sequential approach that is typical in education, students activate a mental number line schema for whole numbers. Essentially, students seem to have a separate number line for fractions and decimals from their mental number line for whole numbers, or at the very least, imagine all of the fractions/decimals as falling below 1%. Therefore, students may notice that percentages are treated more colloquially as similar to whole numbers but this may lead them to make errors in judgment of absolute magnitude.

The theory that the *percentages-are-larger bias* stems from students drawing connections between percentages and whole numbers is consistent with prior research demonstrating that whole number schemes coopt attention to magnitude. For example, Boyer, Levine, and Huttenlocher (2008) demonstrated that children possess intuitive sense of proportional relationships but student performance is diminished when displays are discretized (an area model has partitions) versus continuous (an area model has no partitions). Boyer and colleagues (2008) suggest that “young children go wrong in reasoning about proportions when the knowledge they have acquired about counting to compare set sizes gets in the way of their intuitive, relative visual comparison, proportional-reasoning processes” (p. 14). This suggests that when activities appear most similar to whole numbers, children will activate a whole number scheme and operate within this whole number scheme. Thus, in the case of Boyer et al (2008), it wasn’t that children did not have an understanding of proportional relationships but that the partitions in the displays elicited an impulse to count rather than a focus on the proportional relationship. Therefore, when students see fraction-to-percent and decimal-to-percent comparisons, it is likely that children are activating whole number mental schemas. Instead of approaching the cross-notation comparison rationally by imagining a number line that includes fractions, decimals, and percentages, students show a preference to select percentages as the larger value because they are viewed as most similar to whole numbers.

Superficially, this finding of a *percentages-are-larger bias* suggests students have a bias towards always perceiving percentages as larger than fractions and decimals; however, it appears that this interpretation is not true on all tasks. An important point to highlight is that students in the current study were not distracted when placing percentages on the decile number line. Theoretically, it could have been distracting for students to view a number line partitioned and labeled by tenths. In other words, the partitioning of the lines could have lead students to place the value 5% at five-tenths because the partitioning encouraged them to count over five tick marks. However, that is not the case as the overall PAE for percentage was quite low (PAE=.05). Thus, students were able to estimate in the presence of distraction because likely they have a schema that suggests that 5% is a small number, close to 0. In fact, only 4% of students had a percent decile number line estimation PAE that was slightly higher than their fraction number line estimation PAE. If students actually perceived percentages as larger than their equivalent fractions and decimals, then we would expect to see that percentages would be placed slightly higher on the number line. In other words, we would expect that students would place 35% higher than 6/17 on the same number line. This is precisely what occurred when Braithwaite and Siegler (2018a) analyzed differences in performance when they asked students to place equivalent fractions on the same number line. They found that students actually placed equivalent fractions with larger componential parts (e.g., 16/20) in a location that was higher than their estimate for the equivalent fraction with smaller componential parts (e.g., 4/5).

So, how do we reconcile these conflicting findings? In this concluding chapter, I argue that perhaps just claiming that students have a bias towards perceiving percentages as larger than fraction/decimals is not sufficient that it must be viewed in context. Students are selecting percentage as larger because they are importing whole number strategies and viewing the percent as more similar to whole numbers. Thus, the mechanism for this bias is a lack of correctly integrating the notational forms on the same mental number line. Students that perform well on this task understand that the implicit whole in all comparisons is one. In other words, comparing 3/5 versus 35% involves understanding that the value being compared is 3/5 of 1 and 35% of 1. While this may seem commonsensical, it is a fundamental understanding of the relation among fractions, decimals, percentages, and whole numbers. To be successful at this task, individuals must constrain their representation of the compared magnitudes as being between 0 and 1, forcing the individual to integrate representations of the notations onto a single number line. In this vein, Siegler et al (2011) stressed the importance of encouraging children to draw correct analogies to whole numbers by teaching them that like whole numbers, "fractions can express a proportion of another number (3/5: 1:: 60: 100:: 60% of 100) or that fractions, like whole numbers, can provide absolute measures of quantity (6 in. = ½ foot = 1/6 yard)" (p. 291). Thus, we need to keep track of what whole we are relating the values to because 20% off versus $5 off sounds like quite a difference if the cost of the item is $100 but the 20% coupon saves less than the $5 coupon if the cost of the item is $15. In the aforementioned example, 20% versus $5 both activate whole number schemes, potentially leading to the erroneous conclusion that 20% off would be a better deal if you did not know the initial cost of the item. Perhaps, this is why advertisers often use percentages to describe discounts, though I am unaware of any particular academic research on the topic.

Similarly, I argue that students who do well on the magnitude comparison task are able to maintain focus on the "of what" or the implicit whole. In comparing 45% and 4/5, 45% may sounds like a lot but if the "of what" is 1 then that isn't very much and 4/5 of 1 is substantially greater. This idea is consistent with Boyer et al (2008)'s claim that their finding of less accuracy in the discrete condition "does not necessarily mean that children are unable to code part-whole relationships, but rather, that parts may be more salient than the wholes" (p. 11). Given that the task of the current study was designed purposely to include identical or nearly identical digits, the task frees up resources that might otherwise be focused on components of the different values. In other words, magnitude comparisons across notations with unique digits (e.g., 3/4 versus 56%) could potentially induce some sort of strategy focusing on one of the components of the values. Other attempts at thwarting participants' use of componential strategies include sequential presentation of values being compared (Ganor-Stern, 2013). Given the advantage of the task design of the current study that utilizes mostly identical digits, it helps exclude explanations that could have something to do with the components and leaves as a primary explanation an assessment of whether they understand the implicit whole as being one. Or taking it a step further, they must understand that the whole could be any other value X, as long as the "of what" is the same. In other words, the larger value will always be 4/5, as long as the "of what" is the same (i.e., the answer will still be 4/5 if the question is which is larger 4/5 of X versus 45% of X). This task has the benefit of the whole being unseen, as the implicit whole in this task is 1. Another variation that could be done is to vary the whole (e.g., the shirt is $50, what is the better deal 3/5 or 35% off). However, I imagine the results would be quite similar or perhaps even more exaggerated given children's difficulties with evaluating direction of effects with fractions, decimals, and percentages (e.g., Gay & Aichele, 1997; Hiebert & Wearne, 1985;

Gelman, 1991). Thus, the magnitude comparison across notations task as I have designed it likely does assess whether students can integrate fractions, decimals, and percentages on the same mental number line.

Students who exhibit the *percentages-are-larger bias* ignore the fact that fractions, decimals, and percentages share a critical feature: their magnitude can be represented on a number line. Research has demonstrated that individuals possess an intuitive sense of approximate rational number magnitude (Fazio, Bailey, Thompson, & Siegler, 2014; Matthews & Chesney, 2015; Boyer et al, 2008). Research has also shown that experts are able to focus on the structural features of problems, whereas novices focus on the superficial features (Chi et al., 1981). In a similar way, children are novices in the domain of rational numbers, focusing on superficial features of the rational numbers in an attempt to make meaning of the magnitude or, perhaps because of how they were taught, they simply are not trying to make meaning of the magnitude. In other words, students see a fraction and they immediately think, "What do I do with this?" rather than "How big is this number?" Or, in the case of what was observed in the cross-notation comparisons, students activate magnitude for familiar 0-100 whole numbers. Therefore, students who exhibit the *percentages-are-larger bias* have not developed the appropriate number line schema that incorporates fractions, decimals, and percentages.

Importantly, the composite score for magnitude comparison across notations yields information about the degree to which students have integrated fractions, decimals, and percentages on the same mental number line. The operationalization of cross-notation comparison ability as a measure of integrated number sense is similar to prior research (Hurst & Cordes, 2018); yet it should be noted that the current work utilized percentages rather than whole numbers in cross-notation comparisons. The magnitude comparison task of the current study has the benefit of examination of fraction magnitude representations and both of its related forms: decimals and percentages. It should also be noted that since I did not assess translation ability directly nor did I include within notation comparisons, we should interpret the claim that rational number comparison across notations is an indicator of integrated number sense with caution. As discussed previously, I examined whether there might be some confounding factor that could explain these results such as using extreme values to guess, comparisons when percent was larger were easier, or percentages were on average larger but these explanations could likely be excluded. Therefore, students who struggled were likely attempting to compare across notations but they did not properly integrate notations on the mental number line. The interpretation that magnitude comparison across notations measures integrated number sense is consistent with several others who theorize that the mental number line possibly includes whole numbers and rational numbers (Case & Okamoto, 1996; Steffe, 2001, Siegler et al., 2011). All of these researchers posit that children must in some way integrate their understanding of whole numbers with other forms of rational numbers, making accommodations to their whole number schemes as they encounter more varied forms of numbers. The current work suggests that magnitude comparison across notations provides a measure for determining integrated number sense, or the degree to which students have integrated notations on a mental number line.

Perhaps, the reason why fraction magnitude knowledge is such a strong predictor of algebra knowledge (Siegler et al, 2012) has something to do with maintaining this relational understanding of the part (fraction) and the implicit whole (1). In other words, half of a 12 inch sandwich is going to be bigger than half of a 6 inch sandwich but if we are talking about half as an absolute measure, the implicit whole is one. Thus, integrating this understanding necessarily integrates understandings of fractions and whole numbers on a mental number line (Siegler et al, 2011). Though the origin of the relation between fraction knowledge and algebra is beyond the scope of this dissertation, my interpretation that integration of rational numbers is related to relational reasoning is consistent with Hurst & Cordes (2018). Results suggested cross-notation comparison ability with fractions, decimals, and whole numbers predicted pre-algebra ability above and beyond grade and rational number arithmetic skill (Hurst & Cordes, 2018). The pre-algebra assessment of Hurst & Cordes (2018) specifically measured students' abilities to relate quantities (e.g., 6-4+3=? + 3). This theory that the ability to relate quantities without computing results is a driving force in algebra has also been described previously (DeWolf, Bassok, & Holyoak, 2015).

In the dissertation, I claim that students tend to think of rational numbers, not as quantities but as entities that need to be acted upon. This tendency to do something with the digits, ignoring the magnitude of the values is what I refer to as *impulsive calculation*. I'll focus more specifically on the concept of *impulsive calculation* in the next section but considering its effect in the context of understanding equivalence has value here. This idea that students *impulsively calculate* rather than relate quantities was shown in student interviews of the current study, when students suggested that estimation involved not being able to "do math." Furthermore, it is reminiscent of Haverty and colleagues' (2000) suggestion that those who are successful at the algebraic function finding task "do not merely compute quantities; they *analyze* them" (p. 262). Thus, students that understand concepts of equivalence among notations are also able to understand equivalence in the form of determining a missing value (e.g., 6-4+3=? + 3) (Hurst & Cordes, 2018). However, the limitation to Hurst and Cordes (2018) as well as the current one, is that they did not specify whether it is specifically the cross-notation comparison ability over general comparison ability within notation that is driving the relationship between cross-notation comparison ability and pre-algebra ability. That being said, the current study had the advantage of using percentages rather than whole numbers and, therefore, requires that participants compare values by relating them to the same implicit whole. Given that performance was worse when the percentages were smaller than the compared fractions/decimals, it is likely that many students did not integrate notations on the same mental number line but instead treated percentages as whole numbers. Again, this provides evidence in support of my claim that magnitude comparison across fractions, decimals, and percentages measures integrated rational number sense.

Future research should include within notation (e.g., Fraction vs. Fraction) and across notation (e.g., Fraction vs. Percent) comparisons for individual participants to understand more specifically children's understanding of the relation among rational numbers. Another potential avenue for research could include determining whether students correctly decide that equivalent values written in different notations are equivalent to one another or whether the percent is larger bias also holds (e.g., students indicate that 2/5 and 40% are equivalent or whether they maintain that 40% is larger because it is a percent). Finally, the Cross-Notation Comparison task as I have designed it has the potential to be utilized in the field as a measure of integrated number sense. Siegler et al (2011) posited an integrated theory of numerical development but did not provide a measure of the degree to which students understand the relation among notations. With future testing, it is possible that this magnitude comparison across notations task could provide a measure to assess the degree to which students have integrated notations.

In conclusion, the current study found that children do not perceive equivalent rational numbers as equivalent in size as evidenced by differing performance placing equivalent fraction, decimal, and percentages on a decile number line. Moreover, there were great discrepancies in performance on magnitude comparisons across notations whether the percent value was smaller or larger than the fraction or decimal, suggesting that the values were not viewed on the same scale. In the next section, I focus on the phenomenon of *impulsive calculation*.

5.3 Impulsive Calculation: A Failure to Focus on Magnitude

The current work demonstrates that students often exhibit *impulsive calculation* with fractions. Earlier, I defined *impulsive calculation,* as the act of taking action with digits without considering the magnitudes before or after calculation. Students who *impulsively calculate* are more likely to render implausible estimates on problems such as estimating 12/13+7/8 as they do not think about the magnitudes (12/13 is about equal to one and 7/8 is about equal to one) before deciding on a calculation strategy, and they do not stop to judge the reasonableness of an answer with an estimate after performing the calculation. The dissertation demonstrated that students exhibit *impulsive calculation*, as measured by the difference in performance between situations where students are presented with distracting information ("lures") meant to elicit the use of flawed calculation strategies and situations that do not involve lures. Specifically, there were two pairs of measures that directly contrasted lure versus no lure performance: (1) student performance on a fraction decile number line was compared to their performance on number lines without these potentially distracting partitions/labels and (2) student performance on fraction arithmetic estimation with and without lure answer choices. Ultimately, the comparison of individual performance on these two pairs of measures in conjunction with interview data suggested that *impulsive calculators* are more prone to making errors on lure problems. Furthermore, this difference in performance on tasks with lures versus without lures raised the possibility of systematic error with processing fraction magnitudes.

Prior research has documented a decrement in performance when situations utilize lures that might elicit whole number counting schemes. Numerous studies have demonstrated that concepts of quantity both discrete and continuous are quite intuitive, even in infants (Xu & Arriaga, 2007; Boyer, Levine, & Huttenlocher, 2008; Mack, 1990). Yet, the problem lies in how children integrate these intuitive understandings with symbols (Mack, 1995). Since children learn to count first, whole numbers schemes are the basis by which they must accommodate all other notations. Thus, when children encounter an unfamiliar or potentially taxing mathematical situation, their first instinct is to draw upon whole number schemes. This was observed in children's tendencies to count rather than focus on magnitude in the discrete but not continuous conditions of Boyer et al (2008). Similarly, worse performance was observed on the decile number line task than a 0-1 number line task in Siegler and Thompson (2014) and in this dissertation. Relatedly, performance was worse on estimating sums of fractions on the number line than performance on estimating individual fractions on number lines (Braithwaite, Tian, & Siegler, 2018). Thus, students are highly distracted in situations that elicit an activation of whole number schemes.

The present study, however, offered new insight into the phenomenon of how students are distracted on the decile number line task; it was not just that students were distracted but that they were distracted in a very specific way, such that they demonstrated a lack of integrating understanding of rational numbers with whole numbers. In placing fractions on a decile number line in the current study, many students interpreted the digits of a fraction in precisely the way that they interpret the magnitude of whole numbers, decimals, and even, digital clocks- horizontally. In other words, they began by processing the numerator first and then the denominator (e.g., the '3' in 3/5 tells you the value should be placed between .3 and .4 and the

‘5’ tells you it is in the middle of the range), when they should be processing the value holistically. Thus, poor performance on the distracting number line task demonstrates a phenomenon that I refer to as *impulsive calculation*. This *impulsive calculation* refers to the act of taking action with digits without considering the magnitudes. Instead of reasoning about 6/17 as being less than half because half of 17 is approximately 8 and 6/17 is clearly smaller than 8/17, students take action with the digits without thinking about magnitudes. So, while they may reason appropriately on an unmarked number line because the task encourages them to reason about magnitude, a task that provides distracting information elicits *impulsive calculation* and diminishes their ability to reason relationally, thus placing the value in an implausible location, greater than half. Specifically, many students were likely performing a translation a/b=0.ab in their *impulsive calculation*. This is different from the interpretation that students get confused in focusing on either the numerator or the denominator but that students who struggle are focusing on *both* but not in the way that is necessary to access magnitude.

The current studies suggest that students are making a systematic error in relating rational numbers. The interviews with students that were distracted by the decile number line are particularly revealing at the numerical processing level because it suggests that many students might actually be processing the magnitude of the fraction symbols through a flawed horizontal approach (e.g., students reasoned that to locate 3/5 on the number line, one should go to the three-tenths range and then it is in the middle of that range, thus placing the value at .35). Even flawed translations that were done separately from the decile number line mirrored this pattern. For example, one student described the translation process as “1/8 is small so it is .08 and 8%” during the random sample of posttest interviews. In essence, the numerator tells you it is a small number that is near 0 and less than .1 and the 8 tells you where to put it in that range. While this reasoning is severely flawed, it is only problematic in some cases. For example, 3/8 when translated to a decimal through this flawed a/b=0.ab approach is 0.38, which is the actual magnitude of the fraction with rounding. Many fraction to decimal translations done with the a/b=0.ab approach yield results somewhat close to the actual magnitude of the value and it is possible that children might interpret these differences as reflective of rounding since many prospective teachers have misconceptions about rounding (Burroughs & Yopp, 2010). So, it is possible that students are implicitly learning a flawed translation strategy, which would not be totally unusual given evidence of children learning spurious associations in mathematics (Braithwaite & Siegler, 2018a). Thus, students might be making systematic errors in relating fractions, decimals, and percentages that sometimes yield approximate magnitudes.

This adds new insight to Nesher and Peled's (1986) observation that when asked "to write 3/4 in decimal form some wrote 3.4, 0.3, or 0.34" (p. 75). It is not just that they are confused at making the coordination between the size of the part and the number of parts in question as suggested by Nesher and Peled (1986) but some students are relying on a left-to-right numerical processing of the digits, as they do with all other numbers. Thus, students are concatenating digits from the numerator and denominator to match this left-to-right processing (a/b=0.ab). In some cases, this flawed translation strategy (a/b=0.ab) actually helps students get in the "ballpark" of the right magnitude (e.g., 3/8=0.38). Indeed, quantitative results demonstrated that the mode for the most frequently selected location for fractions on the decile number line was the hypothesized flawed translation result in 5 out of 6 cases, where the denominator could not be easily multiplied into a power of ten. In other words, I hypothesized that students in Experiment 2 would place the fraction 6/17 at .67 on the decile number line and this was the most frequent response, where 14.8% of students in the sample placed it in that exact location. This finding is consistent with Ganor-Stern's (2013) suggestion that decimals are more similar to multi-digit whole numbers and Moss and Case's (1999) suggestion that percentages are intuitive for linking students' conceptions of whole number magnitude to fractions and decimals. Moreover, it is reminiscent of Ganor-Stern's (2013) suggestion that properly representing magnitude of fractions requires overcoming the violation of "principles of the decimal system, that is, the positive linear relation between the components magnitude and the holistic magnitude of the number" (p. 305). This is different from the interpretation that students get confused in focusing on either the numerator or the denominator but that students who struggle are focusing on *both* but not in the way that is necessary to access magnitude.

Given that I did not ask all students to translate values from fractions to decimals directly, it is important to take caution in interpreting this finding. Future work might ask participants to translate values directly or perhaps simply ask participants to decide whether a translation is close to the value or not (e.g., Is 0.35 a good estimate for 3/5: yes or no?; Is 0.27 a good estimate for 2/7: Yes or no?). Even still, the fact that some students' number line estimation ability is easily manipulated based on the assessment type is suggestive that perhaps their understanding of the magnitude of fractions is actually weaker than originally thought. This idea is consistent with a body of research that differentiates performance versus learning (Soderstorm & Bjork, 2015). A similar phenomenon was observed when young students were introduced to a distracting situation involving place value of whole numbers, when students with superficial knowledge of place value described the digits in the number '13' as representing, '1' cube and '3' cups filled with the remaining cubes rather than the actual quantity of the set (Miura & Case, 1989). In line with this reasoning, Moss & Case (1999) discussed the importance of utilizing visually misleading tasks to assess understanding, as they discussed that even Piaget believed that children needed to be presented with misleading tasks or else the assessment just measures their ability to parrot instruction. Thus, distracting tasks can provide a powerful lens for examining deep rather than superficial understanding.

Beyond the level of an individual fraction, the current study measured this *impulsive calculation* through a multiple choice format for the fraction addition estimation task where half of the trials included one lure choice (e.g., "What is the best estimate for 1/5+1/2: 2/7, 1/3, or ¾?", where 2/7 is an "across lure") and half of the trials did not include a lure choice (i.e., answer choices were specifically chosen not to include potential lures). The direct comparison between student performance on fraction addition estimation with lure fraction choices versus choices without a fraction lure suggests that students can estimate slightly better when they are not allowed to give into *impulsive calculation*. Thus, we are observing a parallel phenomenon in accessing magnitude at the level of individual fractions and accessing magnitude at the level of combination of fractions. The bipartite structure of fractions is very difficult for students to access magnitude information and, particularly, when students must do so in the presence of distraction. These findings are similar to the results of Boyer, Huttenlocher, and Levine (2008) that demonstrated that children's difficulties reflect a tendency to focus on matching the units of the problem and the choice alternatives. Thus, any potential source of distraction can result in students abandoning attention to magnitude in favor of focusing on the digits. Boyer and colleagues (2008) suggest that children may have intuitions about magnitude but then counting gets in the way of their proportional reasoning. This same phenomenon is observed when students add across numerators and denominators to arrive at implausible answers (e.g., 2/3+2/3=4/6); in this case, the *impulsive calculation* outweighs logic.

Impulsive calculation might partially explain why students' estimation ability for sums of fractions was worse than their estimation ability for individual fractions, where over half the trials violated the direction of effects for estimating addition of fractions such that students were no more accurate than if they selected the midpoint in each trial that (Braithwaite, Tian, & Siegler, 2018). It seems likely that the vast majority of students in the Braithwaite and colleagues (2018) study were using a flawed strategy (e.g., adding across numerators/denominators) to estimate the answers to fraction addition problems. This was observed in student answers in Experiment 1 and student explanations of their strategies during interviews in Experiment 2. Students might be able to procedurally represent the magnitude of fractions by themselves on the number line (e.g., 12/13 and 7/8) by partitioning an unlabeled number line for each fraction but *impulsive calculation* takes over when they have to interpret a fraction arithmetic addition problem (e.g., 12/13+7/8) and represent the sum on a number line. In other words, students who impulsively calculate do something with the digits without considering the magnitude of the individual values prior to calculation.

It might be worthwhile to remind students of the implicit whole involved in estimating fraction arithmetic. For example, it might be beneficial to remind students that 1 is the implicit whole (12/13 of 1 + 7/8 of 1= ?). During the random sample of interviews, this discussion was helpful in getting students who impulsively calculated their estimate as 19/21 to reason appropriately. Moreover, it might also be worthwhile to encourage students to estimate the value of a fraction with a percent (e.g. 12/13+7/8 is approximately 90% of 1+90% of 1). Future work might explore whether emphasizing the relational nature of fractions to whole numbers is helpful in inhibiting impulsive calculation.

In sum, *impulsive calculation* involves taking action with digits without thinking about the magnitude of values. Impulsive calculation was observed in discrepancies between pairs of measures that directly contrasted lure versus no lure performance: (1) student performance on a fraction decile number line was compared to their performance on number lines without these potentially distracting partitions/labels and (2) student performance on fraction arithmetic estimation with and without lure answer choices. Ultimately, the comparison of individual performance on these two pairs of measures in conjunction with interview data suggested that *impulsive calculators* are more prone to making errors on lure problems. In the next section, I focus on individual predictors of estimation ability, specifically the importance of this integrated cross-notation ability in inhibiting *impulsive calculation*.

5.4 Integrated Number Sense: Individual Predictor of Estimation Ability

In this dissertation, I hypothesized that integrated number sense enables students to inhibit *impulsive calculation* and maintain focus on magnitude during estimation. The current data suggest that integrated number sense, operationalized as the composite score for magnitude comparison across fractions, decimals, and percentages, was a significant and unique predictor of estimation ability on potentially distracting tasks that might elicit *impulsive calculation* oftentimes over and above their performance on standardized tests of math achievement and general number line estimation skill. This discussion focuses on estimation ability in the presence of distracting aspects that might elicit *impulsive calculation* at both the level of an individual value (decile number line estimation) and at the level of combination of values (fraction addition estimation with lure answer choices and a special focus on the 12/13+7/8 fraction addition estimation problem from Carpenter et al, 1980).

Drawing upon prior research, I hypothesized that a decile number line might elicit *impulsive calculation.* In Siegler and Thompson (2014), students in the decile number line condition performed worse on number line estimation than students in the unlabeled 0-1 number line condition. Therefore, I theorized that the decile number line was creating a potentially distracting situation for students that have little integrated number sense, forcing them to lose focus on the magnitude of the fraction. Results from Experiment 1 demonstrated that fraction estimation was worse in the decile number line than unlabeled 0-1 number line for individual participants, replicating the Siegler and Thompson (2014) finding within subjects. However, it was not significantly worse because many students were actually quite accurate on certain trials and less accurate on other trials. For example, I noticed that many students appeared to be translating fractions to decimals as a/b=0.ab (e.g., 2/7=0.27, which yields a very low PAE because the actual magnitude is 0.285 for that fraction). Therefore, for Experiment 2, I purposely selected fractional values that would yield a particularly high PAE if the students were using this a/b=0.ab flawed strategy to place values on the decile number line (e.g., 5/6=0.56, which yields a high PAE because the actual magnitude is 0.833 for that fraction). I wanted to ensure that students were not able to be somewhat accurate with their number line placement for the wrong reasons (e.g., using the flawed a/b=0.ab would yield a low PAE of 0.02 for the flawed translation 2/7=0.27). In Experiment 2, paired t-tests demonstrated a significant difference between individual PAEs on unlabeled 0-1 number line and the fraction decile number line task, such that performance was worse on average for the fraction decile number line task. Thus, the task elicited *impulsive calculation*, such that students took action with the digits without regard for the magnitude of the values. Specifically, their responses suggested systematic error with translating fractions to decimals as a/b=0.ab.

Importantly, integrated number sense helped students inhibit *impulsive calculation* and persevere in attending to magnitude during the potentially distracting decile number line task. I theorized that students with integrated number sense would be less likely to *impulsively calculate* because their number sense reflects understanding of connections among related concepts (Hiebert & Carpenter, 1992). Indeed, hierarchical linear regression suggested that integrated number sense, operationalized as magnitude comparison across notations, added unique explanatory power to the model predicting performance on the decile number line task, above and beyond number line estimation on an unlabeled line with endpoints 0-1. Moreover, the large absolute value for the standardized beta coefficient suggested that integrated number sense was the most important predictor of how students performed on this distracting task above other predictors in the model. Therefore, integrated number sense is more predictive of whether students will be distracted than would be predicted by their magnitude representations for individual fractions or their standardized measure of math achievement.

Thus, depth of understanding can be assessed by whether misleading information affects performance and the degree to which related concepts are understood. Siegler and Thompson (2014) speculated that conditions that weaken magnitude encoding (e.g., decile number line) are reflective of individual differences in ability to "inhibit distracting landmarks as well as magnitude knowledge" (p. 55). In support of the hypothesis, the current study suggests that integrated number sense enables students to fight their way through distraction and maintain focus on magnitude. This finding is consistent with research on learning versus performance (Soderstorm & Bjork, 2015), such that performance on the decile task might actually provide a clearer depiction of magnitude learning at a numerical processing level. It is possible that number line representation alone does not provide the full picture of students' understanding of fraction magnitude. In a similar way, a student that understands 3+2=5 but does not understand that 5-3=2 has a less strong concept of the relation among magnitudes. This makes sense because depth of understanding is determined by the strength of connections among related concepts (Hiebert & Carpenter, 1992). Therefore, a task that asks students to demonstrate understanding in a scenario that is distracting or unfamiliar is likely to deliver a clearer portrayal of the depth of their understanding.

Integrated number sense provides students with a well-developed schema to call upon in these types of situations that might be distracting. The schema that students are able to call upon is the mental number line that includes all rational numbers. In any type of reasoning, individuals try to access a schema for which they know that might help them in a given situation (Anderson, 1983). Children who do not have a schema or weaker schema for a particular situation are at a disadvantage for solving a problem. Indeed, research has suggested that reasoning is better when children have an appropriate schema to draw upon than when it is decontextualized (Cox & Griggs, 1982; Pollard & Evans, 1987). I argue that integrated rational number sense provides the best schema for interpreting magnitude: the number line. When the number line includes all rational numbers, individuals are able to leverage the intuitive concepts of percent (Moss & Case, 1999; Fazio et al, 2014; Matthews & Chesney, 2015). Using percent as an evaluative tool enables students to maintain focus on magnitude, particularly in situations that might heighten the saliency of the componential parts of fractions. Students that have an integrated schema are able to call upon a well-developed number line that has both a horizontal and vertical dimension. Their understanding of magnitude involves an understanding that there are infinite numbers between numbers (Vamvakoussi & Vosniadou, 2010) and there are an infinite number of ways to describe each spot on the number line (e.g., 4/5=8/10=16/20= 80%=.8=.80=.800, etc.).

In line with this reasoning, we see that integrated number sense again plays a pivotal role in estimation of fraction addition. In particular, the fraction addition estimation task was designed purposely to elicit *impulsive calculation*. A multiple choice format was used for the fraction addition estimation task where half of the trials included one lure choice (e.g., "What is the best estimate for 1/5+1/2: 2/7, 1/3, or ¾?", where 2/7 is an "across lure") and half of the trials did not include a lure choice (i.e., answer choices were specifically chosen not to include potential lures). This task was designed strategically to measure whether students could estimate sums of fractions when they did not have an option to select typical impulsive calculation responses (e.g., adding across numerators and denominators or finding a common denominator and adding numerators). Indeed, students were more accurate when there were no lures than when there were lure answer choices and wrong answer choices were most likely to be lures than just a wrong answer, which provided support for my theory of *impulsive calculation*. Importantly, integrated number sense was a unique predictor of performance on fraction arithmetic estimation with lures controlling for general math ability and their performance on trials without lures. Thus, integrated number sense again plays a critical role in situations that students might typically be inclined to disregard magnitude information.

Notably, fraction number line (0-1) estimation is not a significant predictor of fraction addition estimation, when controlling for general math ability and performance on no lure trials. This is not surprising given students' estimation ability for sums of fractions was worse than their estimation ability for individual fractions, where over half the trials violated the direction of effects for estimating addition of fractions; students were no more accurate than if they selected the midpoint in each trial (Braithwaite, Tian, & Siegler, 2018). It seems likely that the vast majority of students in the Braithwaite and colleagues (2018) study were using a flawed strategy (e.g., adding across numerators/denominators) to estimate the answers to fraction addition problems, which is what I observed in student answers in Experiment 1 and student explanations of their strategies during interviews in Experiment 2. Students might be able to procedurally represent the magnitude of fractions by themselves on the number line (e.g., 12/13 and 7/8) by partitioning an unlabeled number line for each fraction. However, *impulsive calculation* takes over when they have to interpret a fraction arithmetic addition problem (e.g., 12/13+7/8) and represent the sum on a number line. Thus, students who impulsively calculate do something with the digits without considering the magnitude of the individual values and it is this sum that they place on the number line.

This finding that integrated rational number sense predicts fraction arithmetic estimation parallels the results of Hurst and Cordes (2018), which demonstrated the predictive power of magnitude comparison across fractions, decimals, and whole numbers in another domain: pre-algebra. In their experiment, they operationalized pre-algebra ability as understanding of equivalence (e.g., find the number that goes in this box 6-4+3= _ +3, finding the value for c in c+c+4=16). In the tasks that Hurst and Cordes (2018) operationalized as pre-algebra ability, it is essential that individuals (especially those without formal algebra knowledge) can persevere in estimating a guess for the unknowns and checking the accuracy of their guess. These tasks could also be highly distracting for children with no formal algebra knowledge. In order to be successful in the aforementioned pre-algebra task, the students must not only compute quantities, but "*analyze* them" (Haverty et. al, 2000, p. 262). Students with an integrated sense of number are better equipped to do so because they can call upon a well-structured schema: the number line. Ultimately, fluid understanding of the connection among rational number notations equips individuals with tools to better analyze their ideas about magnitude.

Finally, I began the dissertation discussing how only 24% of 8th grade students in the US correctly decided that 12/13+7/8 was closest to 2, where the others selected either 1, 19, or 21 (Carpenter et al, 1980). This finding that the majority of students selected answers that lacked any sort of logic was replicated forty years later with very little improvement (Lortie-Forgues et al, 2015). I theorized that a lack of integrated number sense might explain these results and thus, included the 12/13+7/8 estimation problem to be analyzed separately. When examining predictors of whether students would get the infamous estimation problem right or wrong, logistic regression suggested that integrated number sense, as measured by magnitude comparison across notations, is an important predictor for fraction addition estimation. Integrated number sense was a more important predictor than typical measures of math ability such as standardized tests of achievement and fraction number line estimation. In other words, students that are of equal ability levels as measured by a standardized test of achievement and fraction number line estimation but perform better on selecting the larger of two values across notations are over two times more likely to select the appropriate estimate for a fraction addition estimation problem.

Perhaps, a reason for this could be that a standardized test of math achievement and number line estimation do not accurately assess whether students have an integrated sense of number, such that they understand the relation among fractions, decimals, percentages, and all rational numbers as being a part of the same number line. It could be that students who perform well on the cross-notational magnitude comparison task have internalized the understanding of fractions as numbers rather than entities to be acted upon. On the other hand, it seems that some students can get by in school by memorizing procedures for arithmetic operations and can provide an estimate for a single fraction on a number line task. However, most students are unable to dissociate from a procedure to provide an appropriate estimate for fraction addition. Instead, these students give into *impulsive calculation*, as 94% of students that got the 12/13+7/8 estimation problem wrong in this dissertation selected either 19 or 21, which is the result of adding either the numerators or the denominators together and clearly lacking any sort of reasoning about fraction magnitude. Thus, students that have higher integrated number sense are able to maintain focus on magnitudes of individual values when estimating fraction arithmetic.

Because understanding is determined by the strength and accuracy of connections among related concepts (Hiebert & Carpenter, 1992), students that do well on the magnitude comparison task are able to view the values as sharing the same number line, where the various notations can name the same magnitude in a different way. Taking it a step further, students that can flexibly translate among notations to compare the magnitude of values are probably able to do so when they are asked to estimate the answer to a fraction addition problem. In other words, students might think of 12/13+7/8 as 0.9+0.9 (or 90% of 1+ 90% of 1), which helps them realize the sum is about 2. An important caveat to this interpretation is that we can only make inferences about translation ability from students' performance on magnitude comparison across notations because it did not assess translation ability directly. Additionally, because I did not include within notation comparisons (e.g., fraction versus fraction), it could be that what I have operationalized as integrated number sense is actually just magnitude comparison in general rather than a measure of integration. Future work should examine whether comparison across notations is more predictive of estimation ability than just comparison across individual notations within the same participants. Even still, these findings suggest that magnitude comparison is a more important predictor for estimation ability in the presence of distraction than individual fraction magnitude representation. The next section discusses the differences between conditions and provides general math education recommendations based on the findings in this dissertation.

5.5 Review of Rational Numbers and Educational Implications

While researchers debate about which notation is the best to initiate instruction to promote better conceptual understanding (see Tian & Siegler, 2018 for a review), I argued in this dissertation that perhaps we are debating the wrong issue. Because procedural and conceptual knowledge develop iteratively (Rittle-Johnson et al, 2001), translation between notations and conceptual understanding of each individual notation are likely to develop iteratively. Reviewing rational numbers through a Simultaneous approach was hypothesized to provide more integrated number sense, which would transfer to better estimation ability, than a Sequential approach. Thus, in addition to examining children's understanding of the relation among notations, the current study explored whether it is possible to improve this understanding by testing two experimental conditions (Simultaneous versus Sequential) compared to a Control condition. In the Simultaneous condition, students worked on simultaneously placing a fractional value and its equivalent decimal and percent value on individual fraction, decimal, and percent number lines that were printed one below another in their student activity book. Students were instructed to utilize understanding of other notations to help them be precise in placing values on the number lines and were directed to notice that equivalent values occupy the same space on the number line relative to the endpoints. The Sequential condition received the same review of rational numbers except there was no emphasis on connecting the notations but instead students were directed to focus on placing fractions on a number line for the first week, decimals on a number line for the next week, and percentages on the number line for the last week. Students in the

Control condition practiced addition and subtraction of fractions by rote with the same values utilized in the other two conditions.

The current study demonstrated that a brief number line warm-up at the start of class for three weeks improved children's number line estimates of individual fractions, which replicates improvements in number line estimation of other brief intervention studies (e.g., Fazio et al., 2016). The fact that students in both the Simultaneous and Sequential conditions made significant improvement on fraction number line estimation is important because it might indicate that less time is actually required for improvements in fraction magnitude knowledge. Students in the Sequential condition only worked on fraction magnitudes for approximately 25 minutes over the course of five class sessions. This idea is consistent with other intervention studies that demonstrated gains in fraction magnitude knowledge over short periods of time (e.g., Fazio et al, 2016;). However, it should be noted that the students in the Sequential condition showed significant improvement in their fraction number line estimation two weeks after they concluded their number line training with fractions (i.e., the fraction number line estimation ability of the students in the Sequential condition was essentially measured as a delayed posttest). Perhaps, another explanation for this improvement at delayed testing was that students were still very much involved in number line training over the course of those two weeks but in a different capacity- number line training with decimals and percentages. Importantly, it should be noted that while students in both the Simultaneous and Sequential condition made significant improvement within condition, only the students in the Simultaneous condition made substantially greater improvement on 0-1 number line estimation over students in the Control condition.

Additionally, consistent with the hypothesis about the superiority of the Simultaneous condition, students in the Simultaneous condition made significantly greater improvement than the Sequential and Control condition in Magnitude Comparison across notations. In particular, the low performing students made substantial gains in Fraction>Percent items over the Sequential and Control conditions with a relatively large effect size (Cohen's d=.88). The improvement in magnitude comparison across notations is important for at least two reasons. First, it suggests that helping students notice the connection among fractions, decimals, and percentages could promote an integrated understanding of number (Siegler et al, 2011). Second, because magnitude comparison ability was the largest predictor of estimation ability in the presence of distraction, it seems possible that this type of instruction might offer more lasting benefits for learning (Bjork & Bjrok, 2011). For example, the Simultaneous condition helped students make greater improvement in magnitude representation even in the absence of a spatial model. There is definitely a difference in representing values on a number line versus utilizing a mental model of the number line or perhaps some other strategy to compare the magnitudes of values (see for a review Schneider, Thompson, & Rittle-Johnson, 2018). I posit that the superior Magnitude Comparison ability of students following Simultaneous condition training is evidence of transfer of learning (Bjork & Bjork, 2011), as none of the conditions explicitly practiced comparison of values. Therefore, students may have improved their schema for magnitude by incorporating fractions, decimals, percentages, and whole numbers on the same mental number line. Regardless of the strategy that students may or may not have been using to complete the magnitude comparison task, the results demonstrated that improvement in the magnitude comparison performance from pre- to posttest was considerably greater for students in the Simultaneous condition. Instruction in the Simultaneous condition highlighted the connection among notations. This finding is especially important, given that magnitude comparison was found repeatedly to be an important predictor of estimation ability in these analyses. Though there were no differences at posttest in estimation ability, it seems possible that over time students might improve their estimation abilities as they integrate rational number sense more fully. This theory is consistent with research demonstrating superior conceptual understanding following instruction that highlights the connection among notations (Moss & Case, 1999; Moss, 2005; Kalchman, Moss, & Case, 2001). However, it is also possible that the number line instruction for individual values alone may not be sufficient for improving fraction arithmetic estimation. This idea is consistent with students' difficulties generalizing number line estimation for individual fractions to sums of fractions (Braithwaite, Tian, & Siegler, 2018).

Moreover, to the best of my knowledge, the current study was the first to directly compare a Simultaneous versus Sequential approach to reviewing rational number understanding. In this dissertation, I argue that current educational approaches that emphasize separate and Sequential instruction of fractions, decimals, and percentages do not provide students with an opportunity to fully integrate their conceptions of rational number and that a Simultaneous approach would be preferred. Moss & Case (1999) demonstrated a superior advantage of an experimental curriculum that simultaneously highlighted the connections among notations over a business as usual control. However, their intention was not to directly compare the Simultaneous versus Sequential approach per se but to emphasize the connections among notations. Furthermore, their study examined initial learning rather than review of rational numbers and it was highly constructivist in nature (e.g., students were exploring rational numbers through inventing concepts of percent with containers of water). This kind of constructivist exploration was obviously very valuable, as students outperformed their peers in a business as usual control condition. However, the curriculum required a significant amount of time spent planning and additional considerations involved in classroom management given the resources required.

The goal with this project was to test a brief, low cost warm-up activity aimed at improving rational number understanding that educators could implement daily without additional specialized knowledge or time intensive preparation. For example, Rohrer and colleagues (2019) demonstrated a simple adjustment that could be made in reviewing middle school math concepts: worksheets that applied an interleaved approach were better than worksheets that provided a blocked approach to review. This dissertation directly compared a Simultaneous versus Sequential number line approach to reviewing rational numbers. It was hypothesized that the Simultaneous approach would provide a more solidified understanding of rational number. While both the Simultaneous and Sequential condition individually made greater improvement on measures over the Control condition in most cases, the differences between the Simultaneous and Sequential condition were not significant except in the case of Magnitude Comparison across notations. There could be at least two explanations for this: there is little advantage of one number line condition over the other or it could be that students in the Sequential condition spontaneously linked notations by themselves. For example, analysis of the student activity packets demonstrated that 83% of students in the Sequential condition mentioned percent at least once in their packet in lessons that did not explicitly ask them about percent. It is possible that the image of a battery power icon, where charge is typically measured in percentage, naturally primed students to think about percent. From an educational perspective, it is promising that 83% of students in the Sequential condition made the connection between percentages and fractions/decimals at least once because research suggests that a focus on percentage is an intuitive linking representation among notations (Moss & Case, 1999). From a research perspective, on the other hand, the analysis of the comparison between the Simultaneous and Sequential conditions may have been contaminated by students making these connections. Future research will have to be careful about how to control for this effect or perhaps a more appropriate test of the Simultaneous versus Sequential approach is at the initial stages of learning, where students likely have little formal knowledge of percent.

Given that potentially percent was utilized as a tool to link the notations for students in both the Simultaneous and Sequential conditions (unintentionally for the Sequential condition), it helps reconcile some of the findings of a recent study (Malone et al 2019) with the current one. Malone et al (2019) found that an intervention that integrated fraction and decimal understandings was not more beneficial over just a fraction alone intervention. This appears to stand in opposition to Moss and Case (1999) and the Simultaneous condition of the current study, which integrated notations. However, the common thread between Moss and Case (1999) and the current study that is lacking in the Malone et al (2019) study is the focus on percent as a tool for linking notations. It is very possible that the conflict in findings about integrated instruction are due to the absence of percentage in the Malone et al (2019) study. The absence of percentage in Malone et al (2019) contrasts sharply with its central role in both Moss and Case (1999) and the current one. Case studies and interviews suggested that students used percent as a tool for interpreting magnitude in the current study. Moreover, analysis of student performance at pretest in the current study suggested that performance on fraction>percent items was one of the most closely related measures of performance on fraction addition estimation, exceeded only by standardized test of achievement and the composite magnitude comparison score. Relatedly, low performing students in the Simultaneous condition made substantially greater gains in fraction>percent comparisons than the other conditions. Thus, while fraction-to-decimal comparisons are important, understanding of the relation between fraction and percent appears to be a critical driving force in integrated rational number understanding.

Ultimately, the current study provided evidence that a daily brief warm-up activity to review rational numbers has some benefits and at the very least cannot hurt, especially given research documenting children's difficulties with rational numbers (e.g., Lortie-Forgues et al, 2015; Resnick et al, 1989, Durkin & Rittle-Johnson, 2015, Gay & Aichele, 1997) and the importance of fraction magnitude knowledge in advanced math outcomes (Siegler et al, 2012). That being said, the Simultaneous intervention was not perfect and did not resolve all of the students' difficulties with rational numbers. Also, it is possible that 5 minutes per day over three weeks simply was not enough time to correct children's persistent misunderstandings about rational numbers (Lortie-Forgues et al, 2015), when 4-5 years of previous instruction have essentially confused them. The quantitative and qualitative results suggest that students in the Simultaneous condition may have begun to modify their mental number line to include all rational numbers but perhaps it was only in the initial stages. Also, it is possible that the Simultaneous condition did not introduce enough variability with the notations. In other words, instead of always presenting a fraction that they had to translate into decimal and percent notation and represent all on the number line, the instruction could have presented them with a decimal or a percent that needed to be simultaneously represented in the other notations. This likely would have caused more elaborative processing (Battig, 1979; Shea & Zimny, 1983), which could have lead to greater understanding of the connection among notations. Moreover, instruction could have also incorporated varying notation during estimation of fraction arithmetic (e.g. estimate the answer to 12/13 + 90%). Varying the notation directly during estimation of fraction arithmetic could have provided students with specific direction on how to use their emerging understanding of the relation among rational numbers. Another potential issue with the instruction is that it only involved numbers between 0 and 1, thus constraining children's understanding of the relation among notations across the entire number line. Future research might examine the effects of the Simultaneous condition over several months of warm-up activities and/or more daily practice opportunities perhaps through technology incorporating these instructional changes. At the very least, this study suggests that there is some proof of concept that needs some modifications before further investigation takes place.

General Mathematics Education Recommendations

In addition to evaluation of a Simultaneous versus Sequential approach to review of rational numbers, this dissertation sheds light on several more general educational implications. The finding that students do not think about rational numbers as being equivalent in size suggests that teachers need to be mindful about using equal signs to describe equivalent values and be mindful about the language they use during instruction (Muszheve and Capraro, 2012). For example, teachers should be careful with comments that allude to rote procedural activity such as removing the decimal or getting rid of the percent sign to translate between notations and making the fraction 'bigger' when really describing equivalent fractions with larger components.

Beyond these cautionary words, educators should strive to stimulate discussions on the connections among notations. Educators might engage students in conversations about relations among notations by asking questions such as, "What is the better coupon: 20% off or $5 off?" Teachers could extend this conversation to whether one needs to know the initial cost of the object to know which is more 35% off or 3/5 off. Educators might also engage in conversations about why students think particular notations are used for different purposes in our daily lives.

For example, why do we typically talk about batting averages as decimals (e.g., colloquially, "he bats three-hundred") when we talk about basketball free throw shooting statistics as percentages? (Likely because batting 30% does not sound great and also because the thousandth place provides more discrimination among players' performance). There are lots of questions like these that are ripe fruit for discussion; emphasis should be placed on describing that though we might typically use one notational form, any equivalent number could be used.

Additionally, it is worth explicitly calling attention to the (a/b=0.ab) flawed translation strategy that sometimes works. An educator could begin a discussion by saying that another student at a different school used this procedure and it gives approximately the right answer and challenge students to determine whether it works all of the time. Also, give students number lines that are already partitioned into tenths or thirds or another variation and ask them to locate fractions, decimals, and percentages on it. Engage in discussion about the differences in determining where each notation is on the number line (e.g., the digits of decimals and percentages are processed horizontally, whereas fractions must be processed holistically).

Finally, the striking differences between fraction performance on items that involve distraction as compared to those without distraction suggest that *impulsive calculation* is very real; students tend to think about fractions as entities to be acted upon rather than numbers. Perhaps, a way to combat this impulsivity is to utilize percent as tool. Percentage is an intuitive form of rational number that could help link the notations (Moss & Case, 1999; Matthews & Chesney 2015). Instead of this impulsive calculation when they encounter the symbol 27/30, students should be reasoning that this number should be close to 1 because if this were the amount of battery charge they had on their cellphone it would be almost 100% fully charged. They might also pursue the idea that they can transform this number into a value that might make it easier for them to evaluate the magnitude more effectively. Putting the fraction in lowest terms might help them see that 9/10 is the same as .9 or 90%, which is in line with their original hypothesis that 27/30 is close to 1.

Estimation should be an integral part of mathematics; however, student interviews suggested estimation is not valued for rational numbers despite Common Core recommendations. Moreover, student performance on fraction arithmetic estimation in the current study suggested students' understanding of estimation in practice has very little power. As such, educators should be striving to model estimation with think-aloud procedures and incorporate more number sense discussions connected to number lines into daily instruction. Ultimately, this dissertation suggests that helping students understand the relations among fractions, decimals, and percentages might be key to recharging rational number sense.

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Appendix A

Assessment Items

Experiment 1
Task 1: Fraction Arithmetic Estimation
Part A) Addition Estimation Test ***Directions:*** For this task, you need to estimate the sum of a few fraction addition problems. Do not compute the exact answer. For each problem, generate the nearest number that you can to the sum. It could be a fraction, a decimal, a percentage, or a whole number - whatever you think is closest. The software will automatically move you to the next problem within a few seconds so please estimate and input your answer **quickly.**

Pretest	Posttest
Practice:	**Practice:**
5/10+ 1/7	5/10+ 1/7
4/10 +2/8	4/10 +2/8
Test Items:	**Test Items:**
4/8+1/6	4/8+1/6
5/10+1/6	5/9+1/10
8/10+1/9	8/10+1/9
9/10+1/9	8/10+2/9
¾+ 1/3	¾+ 1/3
3/5+1/2	3/5+1/2
5/6+2/4	5/6+2/4
5/7+2/3	2/3 +1/2
¾+2/3	¾+2/3
¾+4/6	4/5 + 4/7
5/10+1/8	
4/9+1/5	

Part B) Subtraction Estimation Test. ***Directions:*** For this task, you need to estimate the difference of a few fraction subtraction problems. Do not compute the exact answer. For each problem, generate the nearest number that you can to the difference. It could be a fraction, a decimal, a percentage, or a whole number - whatever you think is closest. The software will automatically move you to the next problem within a few seconds so please estimate and input your answer **quickly.**

Pretest	Posttest
5/10-1/7	5/10-1/7
4/10-2/8	4/9-1/5
4/8-1/6	4/8-1/6
5/10-1/6	4/10-1/4
8/10-1/9	8/10-1/9
9/10-1/9	8/10-2/9
¾-1/3	¾-1/3
3/5-1/2	2/3-1/2
5/6-2/4	5/6-2/4
5/7-2/3	3/4-3/5
¾-2/3	
¾-4/6	

Experiment 2 (Revised from Experiment 1)
Task 1: Fraction Arithmetic Estimation
Directions: Select the best estimate for the following problems

Practice Problems

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{1}{2}+\frac{1}{3}$	$\frac{1}{2}$ $\frac{2}{5}$ $\frac{3}{4}$	$\frac{1}{2}+\frac{1}{3}$	$\frac{1}{2}$ $\frac{2}{5}$ $\frac{3}{4}$
$\frac{5}{10}+\frac{1}{4}$	60%, 30%, 100%	$\frac{5}{10}+\frac{1}{4}$	60%, 30%, 100%

Lure: Fraction Across Answer Choices

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{5}{6}+\frac{2}{4}$	$\frac{7}{10}$ $\frac{1}{3}$ $1\frac{1}{4}$	$\frac{5}{6}+\frac{2}{4}$	$\frac{7}{10}$ $\frac{1}{3}$ $1\frac{1}{4}$
$\frac{3}{4}+\frac{1}{10}$	$\frac{4}{14}$ $1\frac{1}{4}$ $\frac{9}{10}$	$\frac{2}{7}+\frac{3}{5}$	$\frac{5}{12}$ $1\frac{1}{4}$ $\frac{9}{10}$
$\frac{1}{5}+\frac{1}{2}$	$\frac{2}{7}$ $\frac{1}{3}$ $\frac{3}{4}$	$\frac{2}{3}+\frac{1}{8}$	$\frac{3}{11}$ $\frac{1}{4}$ $\frac{8}{10}$

Lure: Fraction Hybrid Across Answer Choices

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{3}{5}+\frac{8}{9}$	$\frac{11}{45}$ 2 $1\frac{1}{2}$	$\frac{3}{5}+\frac{8}{9}$	$\frac{11}{45}$ 2 $1\frac{1}{2}$
$\frac{2}{9}+\frac{3}{5}$	$\frac{5}{45}$ $\frac{4}{18}$ $\frac{4}{5}$	$\frac{3}{7}+\frac{2}{5}$	$\frac{5}{35}$ $\frac{2}{4}$ $\frac{8}{10}$
$\frac{3}{4}+\frac{2}{10}$	$\frac{5}{40}$ $1\frac{1}{2}$ 1	$\frac{3}{4}+\frac{2}{10}$	$\frac{5}{40}$ $1\frac{1}{2}$ 1

Lure: Decimal Answer Choices

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{7}{8}+\frac{2}{3}$	.9 2.0 1.5	$\frac{7}{8}+\frac{2}{3}$	.9 2.0 1.5
$\frac{2}{3}+\frac{2}{5}$	.4 1.7 1.0	$\frac{3}{8}+\frac{2}{3}$	.5 1.5 1.0
$\frac{2}{9}+\frac{2}{5}$	.4 .2 .6	$\frac{1}{3}+\frac{4}{10}$	.5 .75 1.0

Lure: Percent Answer Choices

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{5}{6}+\frac{1}{4}$	60% 25% 100%	$\frac{3}{5}+\frac{1}{2}$	47% 75% 100%
$\frac{1}{3}+\frac{3}{5}$	48% 25% 90%	$\frac{1}{3}+\frac{3}{5}$	48% 25% 90%
$\frac{1}{3}+\frac{1}{7}$	20% 100% 50%	$\frac{1}{3}+\frac{1}{7}$	20% 100% 50%

No Lure: Fraction Answer Choices

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{3}{5}+\frac{2}{3}$	$\frac{1}{4}$ $\frac{7}{10}$ $1\frac{1}{4}$	$\frac{3}{5}+\frac{2}{3}$	$\frac{1}{4}$ $\frac{7}{10}$ $1\frac{1}{4}$
$\frac{5}{9}+\frac{1}{3}$	$\frac{1}{2}$ $1\frac{1}{2}$ 1	$\frac{2}{5}+\frac{1}{2}$	$\frac{4}{10}$ $1\frac{1}{2}$ 1
$\frac{3}{8}+\frac{1}{3}$	$\frac{1}{10}$ $\frac{2}{5}$ $\frac{3}{4}$	$\frac{2}{9}+\frac{4}{7}$	$\frac{1}{10}$ $\frac{1}{3}$ $\frac{8}{10}$
$\frac{4}{5}+\frac{2}{3}$	$\frac{1}{2}$ 2 $1\frac{1}{2}$	$\frac{4}{5}+\frac{2}{3}$	$\frac{1}{2}$ 2 $1\frac{1}{2}$
$\frac{2}{10}+\frac{2}{4}$	$\frac{1}{5}$ $\frac{1}{3}$ $\frac{2}{3}$	$\frac{2}{6}+\frac{2}{4}$	$\frac{1}{5}$ $\frac{1}{2}$ $\frac{8}{10}$
$\frac{3}{7}+\frac{5}{9}$	$\frac{3}{4}$ 1 $1\frac{1}{2}$	$\frac{3}{7}+\frac{5}{9}$	$\frac{3}{4}$ 1 $1\frac{1}{2}$

No Lure: Decimal Answer Choices

Pretest Item	Answer Choices	Posttest Item	Answer Choices
$\frac{5}{6}+\frac{3}{4}$	.9 2.0 1.5	$\frac{5}{6}+\frac{3}{4}$	.9 2.0 1.5
$\frac{3}{4}+\frac{3}{10}$	.8 1.5 1.0	$\frac{5}{6}+\frac{2}{10}$	.3 1.5 1.0
$\frac{2}{4}+\frac{1}{9}$	.10 .25 .60	$\frac{2}{7}+\frac{2}{5}$	.5 1.0 .75

No Lure: Percent Answer Choices

Pretest Item	Answer Choices			Posttest Item	Answer Choices		
$\frac{4}{6}+\frac{2}{4}$	25%	75%	100%	$\frac{1}{4}+\frac{6}{7}$	75%	30%	100%
$\frac{4}{5}+\frac{1}{7}$	20%	60%	90%	$\frac{4}{5}+\frac{1}{7}$	20%	60%	90%
$\frac{1}{10}+\frac{1}{3}$	5%	75%	40%	$\frac{1}{10}+\frac{1}{3}$	5%	75%	40%

Experiment 1
Task 2: Mixed Comparison

Directions: For this task, you will compare fractions, decimals, and percentages. Please try your best to select the larger value QUICKLY.

Select the larger value:

Pretest Items

F>D	D>F	F>P	P>F	D>P	P>D
3/5 vs. .35	.97 vs. 7/9	½ vs. 21%	72% vs. 4/7	.51 vs. 15%	42% vs. 24
4/6 vs. .4	.8 vs. 5/8	¾ vs. 43%	85% vs. 5/8	.4 vs. 4%	63% vs. .3
6/7 vs. 6	.9 vs. 4/9	3/5 vs. 35%	97% vs. 7/9	.85 vs. 58%	71% vs. .17
¾ vs. .34	.62 vs. 2/6	2/5 vs. 25%	52% vs. 2/5	.9 vs. 19%	21% vs. .1

Posttest Items

F>D	D>F	F>P	P>F	D>P	P>D
2/5 vs .25	¼ vs .4	2/5 vs 25%	40% vs ¼	.40 vs 25%	40% vs .25
.35 vs 3/5	.6 vs 7/20	35% vs 3/5	7/20 vs 60%	35% vs .6	.35 vs 60%
3/8 vs .08	8/100 vs .38	3/8 vs 8%	38% vs 8/100	.38 vs 8%	.08 vs 38%
.67 vs 6/7	.52 vs 2/5	65% vs 5/6	52% vs 2/5	13% vs .31	42% vs .24
.5 vs 5/6	1/5 vs .51	3/4 vs 43%	1/5 vs 51%	58% vs .85	.47 vs 74%
¾ vs .34	.83 vs 3/8	45% vs 4/5	3/8 vs 83%	.4 vs 4%	51% vs .15

Experiment 2 (Revised from Experiment 1)
Task 2: Mixed Comparison

Directions: For this task, you will compare fractions, decimals, and percentages. Please try your best to select the larger value QUICKLY.

Select the larger value:

Pretest Items

F>D	D>F	F>P	P>F	D>P	P>D
2/5 vs .25	¼ vs .4	2/5 vs 25%	40% vs ¼	.40 vs 25%	40% vs .25
.35 vs 3/5	.6 vs 7/20	35% vs 3/5	7/20 vs 60%	35% vs .6	.35 vs 60%
3/8 vs .08	8/100 vs .38	3/8 vs 8%	38% vs 8/100	.38 vs 8%	.08 vs 38%
.67 vs 6/7	.52 vs 2/5	65% vs 5/6	52% vs 2/5	13% vs .31	42% vs .24
.5 vs 5/6	1/5 vs .51	3/4 vs 43%	1/5 vs 51%	58% vs .85	.47 vs 74%
¾ vs .34	.83 vs 3/8	23% vs 2/3	3/8 vs 83%	.4 vs 4%	51% vs .15

Posttest Items

F>D	D>F	F>P	P>F	D>P	P>D
2/5 vs .25	¼ vs .4	2/5 vs 25%	40% vs ¼	.40 vs 25%	40% vs .25
.45 vs 4/5	.8 vs 9/20	45% vs 4/5	9/20 vs 80%	45% vs .8	.45 vs 80%
3/8 vs .08	8/100 vs .38	3/8 vs 8%	38% vs 8/100	.38 vs 8%	.08 vs 38%
.67 vs 6/7	.52 vs 2/5	65% vs 5/6	52% vs 2/5	2% vs .21	31% vs .13
.4 vs 4/6	3/7 vs .73	3/4 vs 43%	7/9 vs 97%	58% vs .85	.47 vs 74%
2/3 vs .23	.94 vs 4/9	45% vs 4/5	2/7 vs 72%	.4 vs 4%	61% vs .16

EXPERIMENT 1 & 2
TASK 3:
0 to 1 Number Line Estimation:
Directions: For this task, you need to slide the indicator to estimate a fraction on a number line from 0 to 1. Please try to be as accurate as possible.

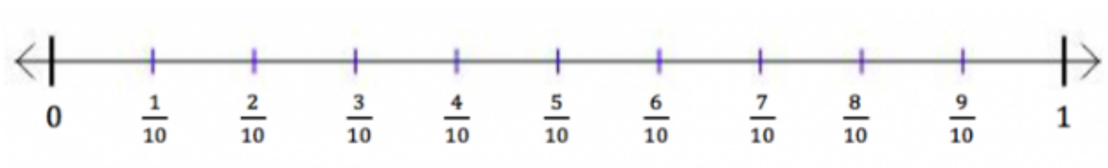
Please slide to estimate the fraction on the number line.
Practice: ½ , ¼
Pretest: 1/19, 2/13, 1/5, 1/3, 3/7, 7/12, 5/8, 3/4, 7/8, and 13/14
Posttest: 1/19, 1/7, 1/5, 3/11, 3/7, 8/14, 5/8, 7/9, 7/8, and 14/15

TASK 4:
0 to 5 Number line Estimation:
Directions: For this task, you need to slide the indicator to estimate a fraction on a number line from 0 to 5. Please try to be as accurate as possible.

Please slide to estimate the fraction on the number line.
Practice: 7/2
Pretest: 1/5, 7/8, 11/7, 9/5, 13/6, 7/3, 13/4, 10/3, 9/2, and 19/4
Posttest: 1/5, 7/8, 13/7, 12/7, 13/6, 8/3, 13/4, 11/3, 9/2, and 18/4

Experiment 1

TASK 5: Distraction Number line Task
Directions: For this task, you will place values on a labeled number line.



Select the approximate location for:

Pretest

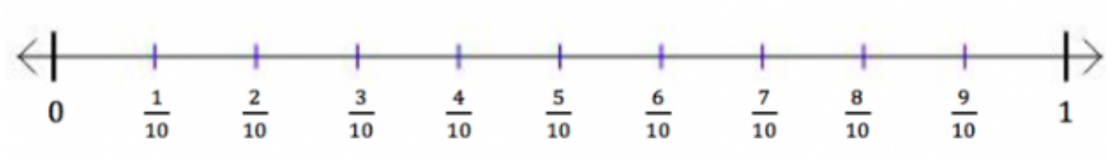
Fraction	Decimal	Percent
1/19	0.052	5%
3/16	0.1875	19%
2/7	0.285	29%
6/17	0.35	35%
4/9	0.444	44%
4/7	0.57	57%
5/8	0.625	63%
11/13	0.8461	85%

Posttest

Fraction	Decimal	Percent
1/19	0.052	5%
3/16	0.1875	19%
6/17	0.35	35%
11/13	0.8461	85%
3/4	.750	75%
5/6	.83	83%
5/18	.277	28%
9/15	.600	60%

Experiment 2 (Revised from Experiment 1)

TASK 5: Distraction Number line Task
Directions: For this task, you will place values on a labeled number line.



Select the approximate location for:

Pretest

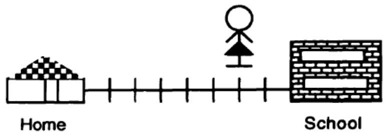
Fraction	Decimal	Percent
1/19	0.052	5%
6/17	0.35	35%
9/20	.450	45%
9/17	.53	53%
8/14	.5714	57%
9/15	.600	60%
4/5	.8	80%
5/6	0.8333	83%

Posttest

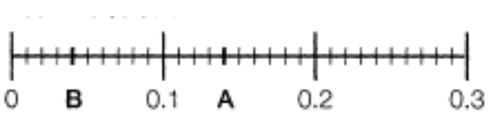
Fraction	Decimal	Percent
1/18	0.056	6%
6/17	0.35	35%
8/20	.4	40%
7/13	.54	54%
8/14	.5714	57%
9/15	.600	60%
2/3	.666	66%
5/6	0.8333	83%

Experiment 2 - Interview Protocol
Part 1: Explain your strategy for solving the problems
(Adapted from Moss & Case,1999)

1) Fifteen blocks spilled out of a bag. This was 75% of the total number of blocks. How many blocks were in the bag to begin with? Explain.
2) What fraction of the distance has Mary traveled from home to school? What is that as a percent? Explain.



3) Can you tell me a number that comes between .3 and .4? Can you tell me a number that comes between 1/3 and ¼? Explain.
4) How much is 1% of four dollars? Explain.
5) Find ¾ of a pizza. (circle partitioned by eighths) Explain.
6) What is 1/8 as a decimal/percent, how do you know? Explain.
7) How much is 2/3 of 6/7? Can you draw a picture to explain how you got the answer? Explain.
8) Look at this number line. What number is marked by the letter A? What number is marked by the letter B? Explain.



9) Could these be the same amount, .06 of a tenth and .6 of a hundredths? Yes or no. Explain.
10) How should you write thirty-five hundredths as a decimal? How should you write seventy-five thousandths as a decimal? Explain.
11) Shade 0.3 of a circle. (circle partitioned by fifths) Explain.
12) How would you write 6% as a decimal? Fraction? Explain.

Part 2: Explain your strategy for placing these on the number line
<u>0 to 1 number line</u>:1/19, 3/7, 7/8
<u>0 to 5 Number line:</u> 1/5**,** 7/3**,** 19/4
<u>0 to 1 decile line:</u> 6/17**,** 3/5**,** 5/6

Part 3:
Estimate: 12/13+7/8

Appendix B

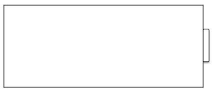
Sample Instructional Materials

Appendix B. Sample Instructional Materials
Overview of Numerical Values for each Condition

	Simultaneous	Sequential	Control
Directions	*Write the fraction as a decimal and percent. Label values on their respective number lines.*	*Label value on the number line.*	*Solve the fraction arithmetic problem. Show your work.*
Lesson Day for Values	1. 4/5 2. 3/10 3. 5/8 4. 2/25 5. 7/20 6. 27/30 7. 4/7 8. 17/19 9. 15/25 10. 2/9 11. 14/20 12. 7/8 13. 34/50 14. 5/6 15. 4/11	1. 4/5 2. 3/10 3. 5/8 4. 2/25 5. 7/20 6. 0.9 7. 0.571 8. 0.8947 9. 0.22222 10. 0.6 11. 70% 12. 68% 13. 87.5% 14. 83.3% 15. 36.4%	1. 4/5 +3/10 2. 5/8+2/25 3. 2/25+7/20 4. 4/5+7/20 5. 3/10+2/25 6. 27/30-4/7 7. 17/19-5/8 8. 15/25-3/10 9. 4/5-15/25 10. 5/8-2/9 11. 7/8-4/5 12. 7/8+5/6 13. 5/6-14/20 14. 14/20-34/50 15. 4/11+5/6
Question In Student Packet	What did you do to help you figure out a good answer?	What did you do to help you figure out a good answer?	What did you do to help you figure out a good answer?

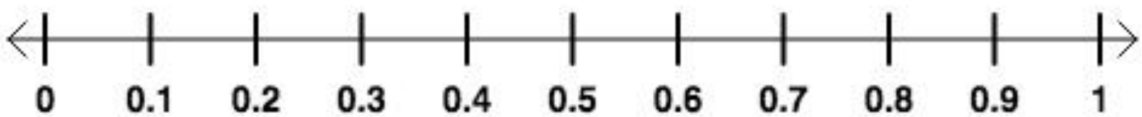
Sample Student Activity - Simultaneous Condition (Typical Throughout Lessons 1-15)
Directions: *Estimate the value by shading in an approximate battery icon for the value. Write the fraction as a decimal and percent. Label values on their respective number lines.*

Estimate: Shade in the battery icon below to show your estimate of the value.

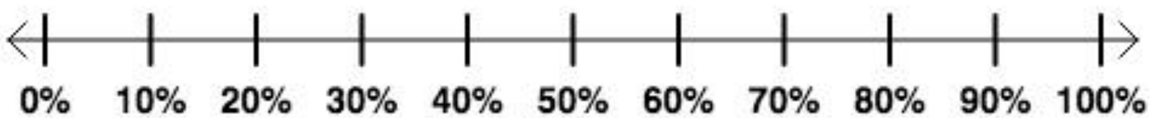


Fraction: $\frac{5}{8}$

Decimal: ___________________



Percent: ____________________



What did you think about to help you figure out a good answer?

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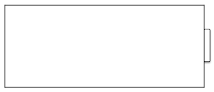
Sample Student Activity - Sequential Condition (Typical of Week 1)
Directions: *Estimate the value by shading in an approximate battery icon for the value. Label value on the number line.*

Estimate: Shade in the battery icon below to show your estimate of the value.



Fraction: $\frac{5}{8}$

What did you think about to help you figure out a good answer?

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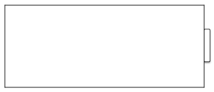
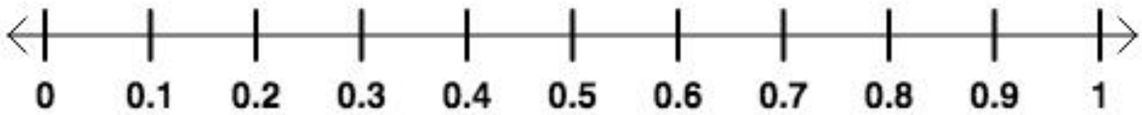
Sample Student Activity - Sequential Condition (Typical of Week 2)
Directions: *Estimate the value by shading in an approximate battery icon for the value. Label value on the number line.*

Estimate: Shade in the battery icon below to show your estimate of the value.



Decimal: 0.571



What did you think about to help you figure out a good answer?

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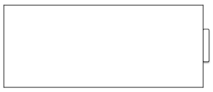
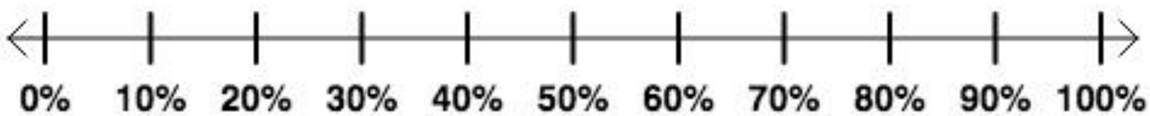
Sample Student Activity - Sequential Condition (Typical of Week 3)
Directions: *Estimate the value by shading in an approximate battery icon for the value. Label value on the number line.*

Estimate: Shade in the battery icon below to show your estimate of the value.



Percent: 68%



What did you think about to help you figure out a good answer?

Sample Student Activity - Control Condition (Typical Throughout Lessons 1-15)
Arithmetic- Day 3
Directions: *Solve the fraction arithmetic problem.*

Problem:

$\frac{2}{25} + \frac{7}{20}$

What did you think about to help you figure out a good answer?

Arithmetic- Day 6
Directions: *Solve the fraction arithmetic problem.*

Problem:

$\frac{27}{30} - \frac{4}{7}$

What did you think about to help you figure out a good answer?
