

## ARTICLE

# Online Ramsey theory for a triangle on $F$ -free graphs

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## Abstract

Given a class  $C$  of graphs and a fixed graph  $H$ , the *online Ramsey game for  $H$  on  $C$*  is a game between two players Builder and Painter as follows: an unbounded set of vertices is given as an initial state, and on each turn Builder introduces a new edge with the constraint that the resulting graph must be in  $C$ , and Painter colors the new edge either red or blue. Builder wins the game if Painter is forced to make a monochromatic copy of  $H$  at some point in the game. Otherwise, Painter can avoid creating a monochromatic copy of  $H$  forever, and we say Painter wins the game. We initiate the study of characterizing the graphs  $F$  such that for a given graph  $H$ , Painter wins the online Ramsey game for  $H$  on  $F$ -free graphs. We characterize all graphs  $F$  such that Painter wins the online Ramsey game for  $C_3$  on the class of  $F$ -free graphs, except when  $F$  is one particular graph. We also show that Painter wins the online Ramsey game for  $C_3$  on the class of  $K_4$ -minor-free graphs, extending a result by Grytczuk, Hałuszczak, and Kierstead [*Electron. J. Combin.* 11 (2004), p. 60].

## KEYWORDS

forbidden subgraph, games on graphs, online Ramsey theory

## 1 | INTRODUCTION

All graphs in this paper are finite. For a host graph  $G$  and a target graph  $H$ , let  $G \rightarrow H$  mean that there exists a monochromatic copy of  $H$  for every (not necessarily proper) 2-edge-coloring of  $G$ . For a graph parameter  $\rho$ , let  $R_\rho(H)$  denote the minimum  $\rho(G)$  where  $G \rightarrow H$ . When  $\rho$  counts the number of vertices in a graph,  $R_\rho(H)$  is the *Ramsey number* of  $H$  and it is often denoted  $R(H)$ . The well-known Ramsey's Theorem [22] from 1930 states that  $R(H)$  is finite for every graph  $H$ .

Burr, Erdős, and Lovász [3] introduced *the chromatic Ramsey number* and *the degree Ramsey number*, which arises when  $\rho$  is the chromatic number and the maximum degree, respectively. Erdős et al. [9] introduced *the size Ramsey number*, denoted  $R_e(H)$ , which arises when  $e(G)$  is the number of edges in a graph  $G$ . We redirect the readers to a thorough survey by Conlon, Fox, and Sudakov [6] for more history regarding these parameters.

Another variant of Ramsey theory is online Ramsey theory, introduced by Beck [2] in 1993. Given a class  $C$  of graphs and a fixed graph  $H$ , an *online Ramsey game for  $H$  on  $C$*  is a game between two players Builder and Painter with the following rules: an unbounded set of vertices is given as an initial state, and on each turn Builder introduces a new edge with the constraint that the resulting graph must be in  $C$ , and Painter colors the new edge either red or blue. Builder wins if Painter is forced to make a monochromatic copy of  $H$  at some point of the game, and we say Builder wins the online Ramsey game for  $H$  on  $C$ . Otherwise, Painter can avoid creating a monochromatic copy of  $H$  forever, and we say Painter wins the online Ramsey game for  $H$  on  $C$ .

If no graph in  $C$  contains  $H$  as a subgraph, then Painter wins the online Ramsey game for  $H$  on  $C$  since a copy of  $H$  cannot be created, let alone a monochromatic one. Therefore it must be that  $H$  is a subgraph of at least one graph in  $C$  for a result to be nontrivial. If  $C$  is the class of graphs with bounded maximum degree, then this is the online version of the degree Ramsey number (see [4,23,24] for results regarding this topic).

This paper concerns the online version of the size Ramsey number. For a graph  $H$ , *the online (size) Ramsey number of  $H$* , denoted  $r(H)$ , is the minimum number of rounds required for Builder to win, assuming that both Builder and Painter play optimally. When there are no restrictions on the graphs Builder can create, it is an easy consequence of Ramsey's theorem [22] that Builder always wins the online Ramsey game for every target graph  $H$ , so  $r(H) \leq R_e(H)$ . For a fixed graph  $H$ , studying the ratio of  $r(H)$  and  $R_e(H)$  was initiated in [2,10,14] and has drawn much attention since then [11–13,20]. There is also a line of research trying to determine some exact online Ramsey numbers [5,7,8,12,20,21]. Additionally, there are some results on the behavior of  $r(H)$  in various random settings [1,15–17,19].

The investigation of online (size) Ramsey theory on specific graph classes was initiated in 2004 by Grytczuk, Hałuszczak, and Kierstead [11]. They studied online Ramsey theory on forests,  $k$ -colorable graphs, outerplanar graphs, and planar graphs. In particular, they conjectured that Builder wins the online Ramsey game for  $H$  on planar graphs if and only if  $H$  is an outerplanar graph. This conjecture was recently disproved by Petříčková [18]. She showed that one direction of the conjecture is true while the other direction is not.

**Proposition 1.1** (Petříčková [18]). *For every outerplanar graph  $H$ , Builder wins the online Ramsey game for  $H$  on planar graphs.*

**Proposition 1.2** (Petříčková [18]). *Builder wins the online Ramsey game for  $K_{2,3}$  on planar graphs.*

In [11], it is shown that Painter wins the online Ramsey game for  $C_3$  on outerplanar graphs, and the graphs containing  $C_3$  as a subgraph are the only known graphs where Painter wins the online Ramsey game on outerplanar graphs. Contrarily, they also demonstrate that Builder wins the online Ramsey game for  $C_3$  on 2-degenerate planar graphs.

**Theorem 1.3** (Grytczuk et al [11]). *Painter wins the online Ramsey game for  $C_3$  on outerplanar graphs.*

**Proposition 1.4** (Grytczuk et al [11]). *Builder wins the online Ramsey game for  $C_3$  on 2-degenerate planar graphs.*

We extend the class of graphs where Painter wins the online Ramsey game for  $C_3$  from outerplanar graphs to  $K_4$ -minor-free graphs. Our proof is a generalization of the proof of Theorem 1.3 in [11].

**Theorem 2.4.** *Painter wins the online Ramsey game for  $C_3$  on  $K_4$ -minor-free graphs.*

We initiate the study of characterizing the graphs  $F$  such that for a given graph  $H$ , Painter wins the online Ramsey game for  $H$  on  $F$ -free graphs. A graph class is  $F$ -free if every graph in the class does not contain  $F$  as a subgraph. We characterize all graphs  $F$  such that Painter wins the online Ramsey game for  $C_3$  on  $F$ -free graphs, except when  $F$  is one special graph. We put the constraint that  $F$  has no isolated vertices because the game is defined to have infinitely many isolated vertices as the initial state. The following theorem is our main result.

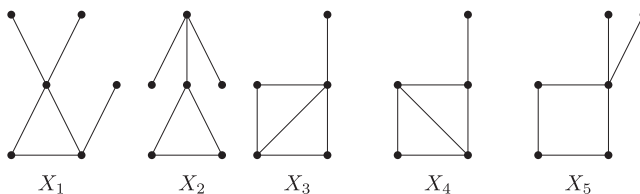
**Theorem 3.1.** *Let  $X_1, \dots, X_5$  be the graphs in Figure 1, and let  $F$  be a graph with no isolated vertices. Given that  $F$  is not isomorphic to  $X_5$ , Painter wins the online Ramsey game for  $C_3$  on  $F$ -free graphs if and only if  $F$  is isomorphic to a subgraph of a graph in  $\{X_1, X_2, X_3, X_4\}$ .*

This paper is organized as follows. In Section 2, we prove Theorem 2.4 and in Section 3, we prove Theorem 3.1. Section 3 is further divided into three sections. Section 3.1 and Section 3.2 deal with the classes of graphs where Builder and Painter wins, respectively. Section 3.3 concludes Section 3.

For an edge  $e$ , we say that “Painter cannot color  $e$ ” if there is a monochromatic copy of  $H$  whether Painter colors  $e$  red or blue; in other words, Builder wins the game no matter what color Painter uses on  $e$ . In particular, we say that “Painter cannot color  $e$  red (blue)” or “Painter must color  $e$  blue (red),” if we already observed that Painter will eventually lose (a monochromatic copy of  $H$  will appear) when Painter colors  $e$  red (blue).

## 2 | THE ONLINE RAMSEY GAME FOR $C_3$ ON $K_4$ -MINOR-FREE GRAPHS

Grytczuk, Hałuszczak, and Kierstead [11] proved that Builder wins the online Ramsey game for  $C_3$  on 2-degenerate planar graphs, but Painter wins the online Ramsey game for  $C_3$  on



**FIGURE 1** The graphs  $X_1, X_2, X_3, X_4, X_5$

outerplanar graphs. We extend the class the graphs on which Painter is known to win the online Ramsey game for  $C_3$ . Since a graph is outerplanar if and only if it does not contain  $K_{2,3}$  and  $K_4$  as a minor, we focus on  $K_{2,3}$ -minor-free graphs and  $K_4$ -minor-free graphs. We show that Painter wins the online Ramsey game for  $C_3$  on  $K_4$ -minor-free graphs, but Builder still wins the online Ramsey game for  $C_3$  on  $K_{2,3}$ -minor-free graphs.

The following proposition shows that Builder wins the online Ramsey game for  $C_3$  on  $K_{2,3}$ -minor-free graphs. Builder will use Strategy 2.1.

**Strategy 2.1.** *Builder draws a copy of  $K_{1,5}$ . Let  $u$  be the vertex of degree 5. By the pigeonhole principle, Painter will color at least three edges with the same color, say  $uv_1, uv_2, uv_3$ . Builder draws the edges  $v_1v_2, v_2v_3$ , and  $v_3v_1$ .*

**Proposition 2.2.** *Builder wins the online Ramsey game for  $C_3$  on  $K_{2,3}$ -minor-free graphs.*

*Proof.* Builder uses Strategy 2.1. Assume  $uv_1, uv_2, uv_3$  are colored red. If Painter colors one of  $v_1v_2, v_2v_3, v_3v_1$  red, then this creates a red  $C_3$ . Therefore Painter must color all of  $v_1v_2, v_2v_3, v_3v_1$  blue, but then this creates a blue  $C_3$  with vertices  $v_1, v_2$ , and  $v_3$ .

The graph resulting from Strategy 2.1 has no  $K_{2,3}$  as a minor. Thus Builder wins the online Ramsey game for  $C_3$  on  $K_{2,3}$ -minor-free graphs. ■

Now, we will prove that Painter wins the online Ramsey game for  $C_3$  on  $K_4$ -minor-free graphs. The key idea of this proof stemmed from the proof of Theorem 1.3 in [11].

Recall that a graph  $G$  contains  $H$  as a minor if there exists a set  $S$  of pairwise disjoint subsets of  $V(G)$  satisfying the following:

- For every vertex  $u$  of  $H$ , there is an element  $S_u \in S$  such that  $G[S_u]$  is connected.
- For every edge  $uv$  of  $H$ , there is an edge between  $S_u$  and  $S_v$ .

We call  $S_u$  the *branch set* of  $u$  in an  $H$ -minor of  $G$  for every vertex  $u$  of  $H$ . When the branch set has one vertex, we also call it a *branch vertex*. For two vertices  $x, y$  in  $G$ , an  $x, y$ -*path* is a path in  $G$  from  $x$  to  $y$ .

**Lemma 2.3.** *Let  $xy$  be an edge of a  $K_4$ -minor-free graph  $G$ , and let  $P$  and  $Q$  be two  $x, y$ -paths in  $G - xy$ . For an integer  $k \geq 3$ , if  $x = v_1, \dots, v_k = y$  are the common vertices of  $P$  and  $Q$ , then these vertices are in the same order on both  $P$  and  $Q$ .*

*Proof.* The claim is trivial when  $k = 3$ , so we may assume  $k > 3$ . By reordering the indices, let  $v_1, \dots, v_k$  be the order of these vertices on  $P$ .

We claim that for  $j > i + 1$ , if there is a  $v_i, v_j$ -path  $R$  in  $G$  that is internally disjoint with  $P$ , then there is no path from  $\{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$  to  $V(P) \setminus \{v_i, v_{i+1}, \dots, v_j\}$  that is internally disjoint with  $P$ . Suppose not. Take an  $a, b$ -path  $P'$  where  $a \in \{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$  and  $b \in V(P) \setminus \{v_i, v_{i+1}, \dots, v_j\}$ . If  $P'$  and  $R$  share a vertex  $z$ , then  $G$  has a  $K_4$ -minor where the branch vertices are  $z, v_i, v_j, a$ . If  $P'$  and  $R$  are vertex disjoint, then  $G$  has a  $K_4$ -minor where the branch vertices are  $a, b, v_i, v_j$ .

Thus, if  $R$  is a subpath of  $Q$ , then  $Q$  can never visit  $v_{i+1}, v_{i+2}, \dots, v_{j-1}$  because otherwise  $Q$  will contain a subpath from  $\{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$  to  $x$  or  $y$ . This is a problem since  $Q$  is an

$x, y$ -path and must go through all of  $v_1, \dots, v_k$ . Therefore, we conclude that  $v_1, \dots, v_k$  are in the same order on both  $P$  and  $Q$ . ■

Given two vertices  $u, v$  on a path  $P$ , let  $P[u, v]$  denote the subpath of  $P$  from  $u$  to  $v$ . For a 2-edge-colored graph  $H$ , let  $f(H)$  denote the number of red edges minus the number of blue edges in  $H$  modulo 3. A 2-edge-colored graph  $H$  is *zero*, *positive*, and *negative* if  $f(H)$  is 0, 1, and 2, respectively. Given a 2-edge-colored graph  $G$ , a zero cycle  $C$  is *good* if there exist two vertices  $\alpha, \beta$  on  $V(C)$  such that an  $\alpha, \beta$ -path on  $C$  is zero and there exists an  $\alpha, \beta$ -path in  $G$  whose internal vertices are disjoint from  $V(C)$ .

**Theorem 2.4.** *Painter wins the online Ramsey game for  $C_3$  on  $K_4$ -minor-free graphs.*

*Proof.* Assume Builder drew the edge  $e = xy$  to the previous graph to obtain the current graph  $G$ , which is 2-edge-colored except for  $e$ . Since the initial graph has no edges, it suffices to show that if  $G - e$  has a 2-edge-coloring such that every zero cycle is good, then this coloring can be extended to  $G$  so that every zero cycle is good. Note that if every zero cycle is good, then there cannot be a monochromatic  $C_3$ , since a monochromatic  $C_3$  is a zero cycle and cannot have a zero path as a subgraph.

Suppose whenever Painter tries to color  $e$  red and blue in  $G$ , there arises a zero cycle  $C^r$  and  $C^b$ , respectively, that is not good. Let  $P^r = C^r - e$  and  $P^b = C^b - e$ . Since  $C^r$  and  $C^b$  are zero cycles,  $P^r$  is negative and  $P^b$  is positive. Let  $x = v_1, v_2, \dots, v_t = y$  be the common vertices of  $P^r$  and  $P^b$ . By Lemma 2.3, they are in the same order on  $P^r$  and  $P^b$ . Without loss of generality, let  $v_1, \dots, v_t$  be the ordering of these vertices on  $P^r$  and  $P^b$ . Note that  $P^r[v_j, v_{j+1}] = P^b[v_j, v_{j+1}]$  might happen for some  $j \in \{1, \dots, t-1\}$ , but there must exist an  $i$  where  $P^r[v_i, v_{i+1}] \neq P^b[v_i, v_{i+1}]$ , since  $P^r$  is negative and  $P^b$  is positive. Fix such an  $i$ , and note that  $P^r[v_i, v_{i+1}]$  and  $P^b[v_i, v_{i+1}]$  are internally disjoint.

We claim that both  $P^r[v_i, v_{i+1}]$  and  $P^b[v_i, v_{i+1}]$  are not zero. Without loss of generality, assume  $P^r[v_i, v_{i+1}]$  was zero. Since  $P^b[v_i, v_{i+1}]$  is a path from  $v_i$  to  $v_{i+1}$  whose internal vertices are disjoint from  $V(C^r)$ , this implies that  $C^r$  is a good cycle, which is a contradiction.

Now we claim that  $P^r[v_i, v_{i+1}]$  and  $P^b[v_i, v_{i+1}]$  are either both positive or both negative. Without loss of generality assume  $P^r[v_i, v_{i+1}]$  is positive and  $P^b[v_i, v_{i+1}]$  is negative. Since the cycle  $D$  formed by  $I_1$  and  $P^b[v_i, v_{i+1}]$  is zero even before Builder drew  $e$ , we know that  $D$  is a good cycle by the induction hypothesis. Therefore, there are two vertices  $\alpha, \beta$  on  $D$  where an  $\alpha, \beta$ -path on  $D$  is zero and  $G - e$  (also,  $G$ ) has an  $\alpha, \beta$ -path whose internal vertices are disjoint from  $V(D)$ . Note that this latter  $\alpha, \beta$ -path cannot share its internal vertices with  $P^r$  and  $P^b$  since this would create a  $K_4$ -minor. If both  $\alpha, \beta$  are on the same  $P^j$  for some  $j \in \{r, b\}$ , then because there are two zero  $\alpha, \beta$ -paths (on  $C^j$ ) and another internally disjoint  $\alpha, \beta$ -path, we can conclude  $C^j$  is good, which is a contradiction. If  $\alpha, \beta$  are on different paths of  $P^r, P^b$ , then  $G$  contains  $K_4$  as a minor where the branch vertices are  $v_i, v_{i+1}, \alpha, \beta$ , which is again a contradiction.

Now we know that  $P^r[v_i, v_{i+1}]$  and  $P^b[v_i, v_{i+1}]$  are both positive or both negative for every  $i \in \{1, \dots, t-1\}$ , which implies that  $P^r$  and  $P^b$  are both positive or both negative, which contradicts that  $P^r$  is negative and  $P^b$  is positive.

Thus, Painter can color  $e$  so that every zero cycle in  $G$  is good, and hence there is no monochromatic  $C_3$  in the coloring Painter produces. ■

We remark that the proof of Theorem 2.4 works for not only  $K_4$ -minor-free graphs, but also  $K_4$ -topological-minor-free graphs.

### 3 | THE ONLINE RAMSEY GAME FOR $C_3$ ON $F$ -FREE GRAPHS

In this section, we attempt to characterize all graphs  $F$  such that Painter wins the online Ramsey game for  $C_3$  on  $F$ -free graphs. We determine the winner of the game in all cases except when  $F$  is the graph  $X_5$ , which is in Figure 1. Recall that we put the constraint that  $F$  has no isolated vertices because the game is defined to have infinitely many isolated vertices as the initial state. Here is our main result.

**Theorem 3.1.** *Let  $X_1, \dots, X_5$  be the graphs in Figure 1. Suppose that  $F$  is a graph with no isolated vertices that is not isomorphic to  $X_5$ . Painter wins the online Ramsey game for  $C_3$  on  $F$ -free graphs if and only if  $F$  is isomorphic to a subgraph of a graph in  $\{X_1, X_2, X_3, X_4\}$ .*

#### 3.1 | When does Builder win the online Ramsey game for $C_3$ on $F$ -free graphs?

In this section, we provide three different classes where Builder wins the online Ramsey game for  $C_3$ . We start by proving Lemma 3.2, which shows that we only need to consider  $F$  to be a subgraph of the graph  $X$ , which is in Figure 2. Then we investigate the classes of (a)  $K_4$ -free graphs, (b)  $K_{1,5}$ -free graphs, and (b)  $Y$ -free graphs where  $Y$  is the graph in Figure 5.

**Lemma 3.2.** *Let  $X$  be the graph in Figure 2. If a graph  $F$  is not isomorphic to a subgraph of  $X$ , then Builder wins the online Ramsey game for  $C_3$  on  $F$ -free graphs.*

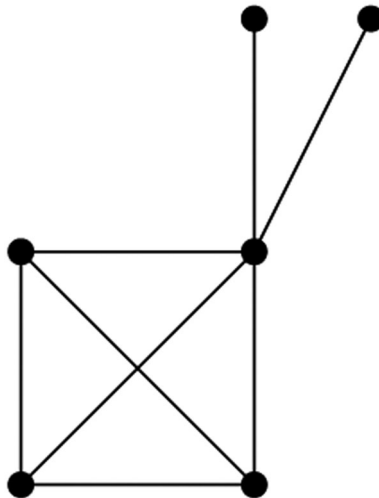
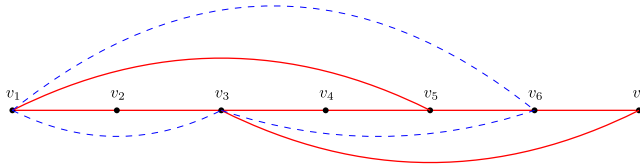
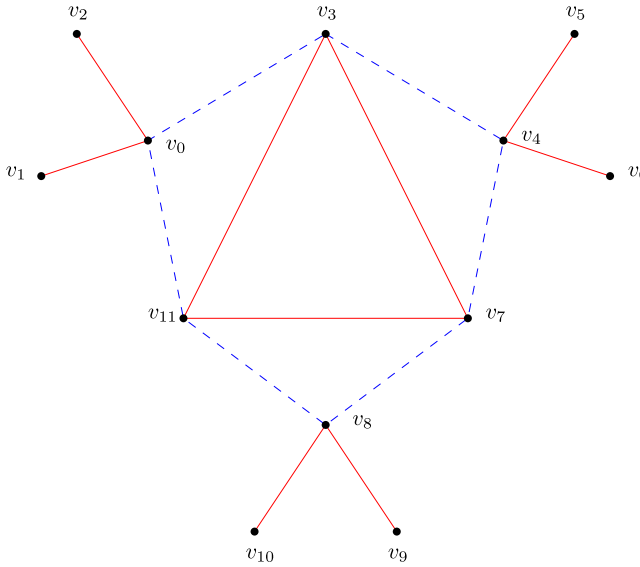


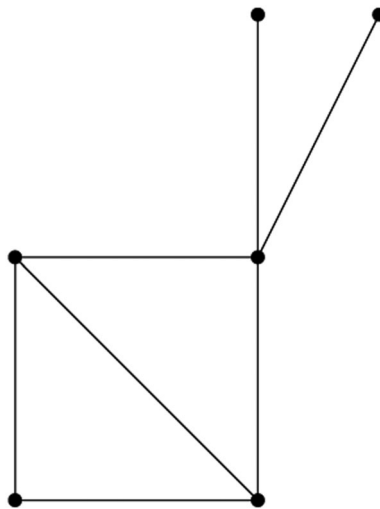
FIGURE 2 The graph  $X$



**FIGURE 3** A strategy for Builder to win the online Ramsey game for  $C_3$  on  $K_4$ -free graphs [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



**FIGURE 4** A strategy for Builder to win the online Ramsey game for  $C_3$  on  $K_{1,5}$ -free graphs [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]



**FIGURE 5** The graph  $Y$

*Proof.* Builder uses Strategy 2.1. Assume  $uv_1, uv_2, uv_3$  are colored red. If Painter colors one of  $v_1v_2, v_2v_3, v_3v_1$  red, then this creates a red  $C_3$ . Therefore Painter must color all of  $v_1v_2, v_2v_3, v_3v_1$  blue, but then this creates a blue  $C_3$  with vertices  $v_1, v_2$ , and  $v_3$ .

There is no  $F$  as a subgraph at every step of the game since the resulting graph is  $X$  and  $F$  is not isomorphic to any of the subgraphs of  $X$ . Hence, Builder wins the online Ramsey game for  $C_3$  on  $F$ -free graphs. ■

The following Proposition 3.3 is a special case of a result in [11], and a more general theorem is proved in [13]. For the sake of completeness, we include a proof of Proposition 3.3.

**Proposition 3.3.** *Builder wins the online Ramsey game for  $C_3$  on  $K_4$ -free graphs.*

*Proof.* We will present a winning strategy for Builder.

Given a forest  $S$ , it is known that Builder wins the online Ramsey game for  $S$  on the class of all forests by [11]. Thus, we may assume that Builder has forced Painter to create a monochromatic path of length 6 while drawing a forest. We label the seven vertices on the path by  $v_1, v_2, \dots, v_7$  and suppose that these vertices on the path are in this order. Without loss of generality, assume the edges of the path are colored red. Note that there might be more edges incident with  $v_i$  for  $i \in \{1, \dots, 7\}$ , but since the whole graph is a forest, it is  $K_4$ -free.

Next, Builder draws  $v_1v_5$  and  $v_3v_7$ . We claim that Painter must color both  $v_1v_5$  and  $v_3v_7$  red. Without loss of generality assume that  $v_1v_5$  is colored blue. Now Builder draws both  $v_1v_3$  and  $v_3v_5$ . Painter must color  $v_1v_3$  blue; otherwise there is a red  $C_3$  with three vertices  $v_1, v_2, v_3$ . Now Painter cannot color  $v_3v_5$ . Therefore, both  $v_1v_5$  and  $v_3v_7$  must be colored red.

Finally, Builder draws three edges  $v_1v_3, v_3v_6$ , and  $v_6v_1$ . If Painter colors any of them red, then a red  $C_3$  is created. Otherwise, Painter colors all of them blue, and this creates a blue  $C_3$  with three vertices  $v_1, v_3$ , and  $v_6$  (see Figure 3).

Four vertices of degree at least 3 appear only in the previous paragraph. It is easy to check that  $K_4$  does not appear as a subgraph in this case, so  $K_4$  does not appear as a subgraph at every step of the game. Hence, Builder wins the online Ramsey game for  $C_3$  on  $K_4$ -free graphs. ■

The following proposition is implied by a result in [4] (see Proposition 4.2). For completeness, we provide a proof here as well.

**Proposition 3.4.** *Builder wins the online Ramsey game for  $C_3$  on  $K_{1,5}$ -free graphs.*

*Proof.* We will present a winning strategy for Builder.

Builder draws five pairwise disjoint induced copies of  $K_{1,3}$ . We claim that Painter must not create a monochromatic copy of  $K_{1,3}$ . Otherwise, without loss of generality, assume that there is a red  $K_{1,3}$ . Now, Builder draws  $K_4$  containing the red  $K_{1,3}$  as a subgraph. If Painter colors any of the newly drawn edges red, then a red  $C_3$  is created. Otherwise, Painter colors all of the newly drawn edges blue, and a blue  $C_3$  is created.

Therefore, since there is no monochromatic copy of  $K_{1,3}$ , we may assume that at least three of the five pairwise disjoint induced copies of  $K_{1,3}$  contain two red edges and one



blue edge; let these copies of  $K_{1,3}$  be  $S_0, S_1, S_2$  where  $V(S_i) = \{v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3}\}$  and  $E(S_i) = \{v_{4i}v_{4i+1}, v_{4i}v_{4i+2}, v_{4i}v_{4i+3}\}$  for  $i \in \{0, 1, 2\}$  while  $v_0v_3, v_4v_7$ , and  $v_8v_{11}$  are blue, and all other edges in  $E(S_0) \cup E(S_1) \cup E(S_2)$  are red.

Next, Builder draws  $v_3v_4, v_7v_8$ , and  $v_{11}v_0$ . We claim that Painter must color all these edges blue. Suppose without loss of generality that Painter colors  $v_3v_4$  red. Then Builder draws  $v_3v_5, v_5v_6$ , and  $v_6v_3$ . If Painter colors any of them red, then a red  $C_3$  is created. If Painter colors all of them blue, then this creates a blue  $C_3$  with vertices  $v_3, v_5$ , and  $v_6$ .

Therefore we may assume that Painter colors  $v_3v_4, v_7v_8$ , and  $v_{11}v_0$  blue. Finally, Builder draws  $v_3v_7, v_7v_{11}$ , and  $v_{11}v_3$ . If Painter colors any of them blue, then a blue  $C_3$  is created. If Painter colors all of them red, then this creates a red  $C_3$  with vertices  $v_3, v_7$ , and  $v_{11}$  (see Figure 4).

It is easy to check that  $K_{1,5}$  does not appear as a subgraph at every step of the game. Hence, Builder wins the online Ramsey game for  $C_3$  on  $K_{1,5}$ -free graphs. ■

**Lemma 3.5.** *Let  $Y$  be the graph in Figure 5. While playing the online Ramsey game for  $C_3$  on  $Y$ -free graphs, Builder can force Painter to create either a monochromatic copy of  $C_3$  or a blue edge  $xy$  with  $\deg(x) = 1$  and  $\deg(y) \leq 2$ .*

*Proof.* This can be proven by letting Builder draw an edge and extend it to a path of length 4. At any moment, if Painter colors any of these edges blue, then that creates the blue edge we seek. Otherwise, we may assume Painter produced a red path of length 4. Let  $P$  be such a red path with vertices  $x_1, x_2, x_3, x_4$ , and  $x_5$  in this order on  $P$ .

Now, Builder draws two edges  $x_2x_6$  and  $x_4x_6$  with a new vertex  $x_6$ . We claim that Painter must color both  $x_2x_6$  and  $x_4x_6$  with the color blue. Without loss of generality, suppose Painter colors  $x_2x_6$  red. Now, Builder draws  $x_1x_3, x_3x_6$ , and  $x_6x_1$ . If Painter colors any of these edges red, then there is a red  $C_3$ . If Painter colors all of these edges blue, then this creates a blue  $C_3$ . Therefore, Painter must color  $x_2x_6$  and  $x_4x_6$  blue.

Finally, Builder draws  $x_2x_4$ . Whenever Painter colors  $x_2x_4$  red or blue, this creates a monochromatic copy of  $C_3$ .

It is easy to check that  $Y$  does not appear as a subgraph at every step of the game. Hence, Builder can force Painter to create either a monochromatic copy of  $C_3$  or a blue edge  $xy$  with  $\deg(x) = 1, \deg(y) \leq 2$ , while playing the online Ramsey game for  $C_3$  on  $Y$ -free graphs. ■

**Proposition 3.6.** *Let  $Y$  be the graph in Figure 5. Builder wins the online Ramsey game for  $C_3$  on  $Y$ -free graphs.*

*Proof.* We will present a winning strategy for Builder.

Builder draws seven pairwise disjoint edges. By the pigeonhole principle, Painter colors at least four edges with the same color. Without loss of generality, assume  $v_1w_1, v_2w_2, v_3w_3$ , and  $v_4w_4$  are red edges.

Next, Builder draws the four edges  $vw_i$  for  $i \in \{1, 2, 3, 4\}$  with a new vertex  $v$ . We claim that Painter must color two of them red and the other two blue. Suppose Painter colors  $vw_1, vw_2$ , and  $vw_3$  red. Now Builder draws  $v_1v_2, v_2v_3$ , and  $v_3v_1$ . If Painter colors any of them red, then a red  $C_3$  is created. If Painter colors all of these edges blue, then this creates a

blue  $C_3$  with vertices  $v_1, v_2$ , and  $v_3$ . Therefore, we may assume that  $vv_1, vw_2$  are red and  $vw_3, vw_4$  are blue.

Next, Builder draws  $w_1w_2$ . Suppose Painter colors  $w_1w_2$  blue. Now, Builder draws  $vw_1$  and  $vw_2$ . If Painter colors any of these edges red, then a red  $C_3$  is created. If Painter colors both  $vw_1$  and  $vw_2$  blue, then a blue  $C_3$  with vertices  $v, w_1$ , and  $w_2$  is created. Therefore we may assume that Painter colors  $w_1w_2$  red.

Now, Builder forces Painter to create a blue edge  $xy$  with  $\deg(x) = 1$  and  $\deg(y) \leq 2$ , which is possible by Lemma 3.5. Next, Builder draws  $xw_1$  and  $xw_2$ . We claim that Painter must color these edges blue. Without loss of generality, suppose  $xw_1$  is colored red. Then Builder draws two more edges  $xv_1$  and  $v_1w_2$ . If Painter colors any of  $xw_2, xv_1$ , and  $v_1w_2$  red, then there is a red  $C_3$ . If Painter colors all of them blue, then this creates a blue  $C_3$  with vertices  $x, v_1$ , and  $w_2$ . Therefore, Painter must color  $xw_1$  and  $xw_2$  blue.

Finally, Builder draws  $yw_1$  and  $yw_2$ . If Painter colors any of them blue, then there is a blue  $C_3$ . If Painter colors all of them red, then this creates a red  $C_3$  with vertices  $y, w_1$  and  $w_2$  (see Figure 6).

It is easy to check that  $Y$  never appears as a subgraph at every step of the game. Hence, Builder wins the online Ramsey game for  $C_3$  on  $Y$ -free graphs.  $\blacksquare$

### 3.2 | When does Painter win the online Ramsey game for $C_3$ on $F$ -free graphs?

In this section, we will prove that Painter wins the online Ramsey game for  $C_3$  on  $F$ -free graphs for various  $F$ . Recall that by Lemma 3.2, we only need to consider  $F$  to be a subgraph of the graph  $X$ , which is in Figure 2. For a fixed  $F$ , it is sufficient to provide a strategy for Painter so that a monochromatic  $C_3$  does not appear forever on  $F$ -free graphs. We will provide three different winning strategies for Painter for three different  $F$ .

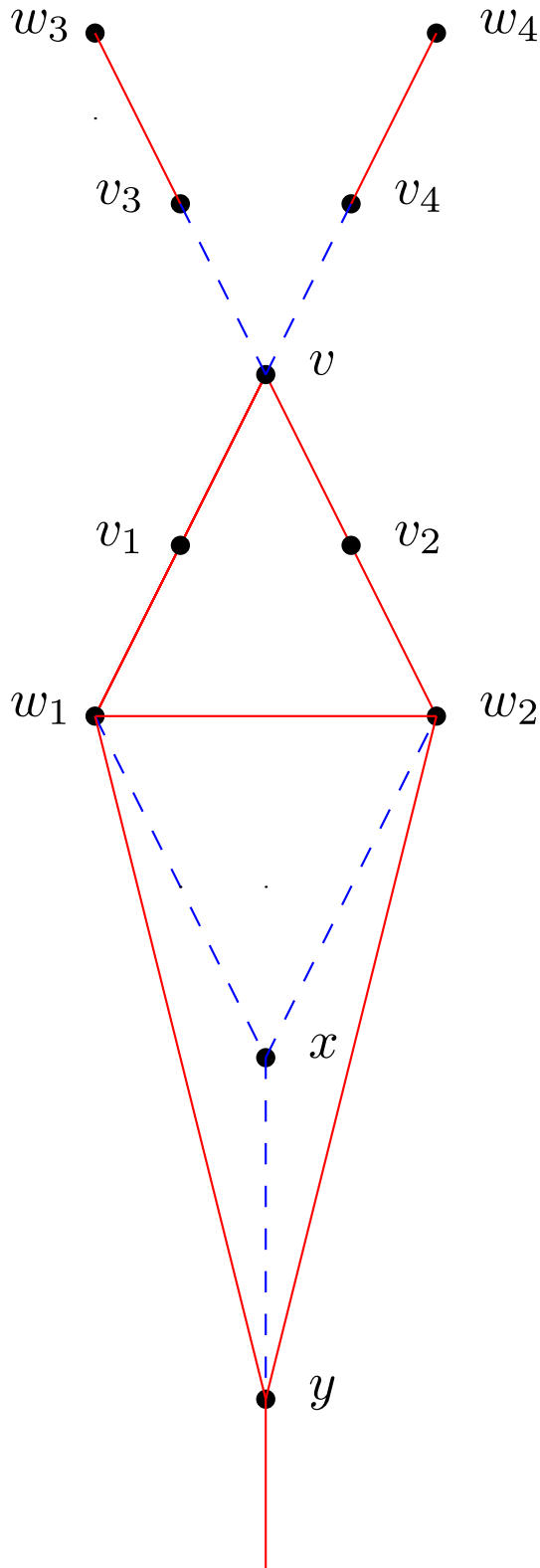
**Strategy 3.7.** *Painter colors each new edge red, unless doing so creates a red  $K_{1,3}$ , a red  $C_3$ , or a red  $C_4$ , in which case the new edge is colored blue.*

**Proposition 3.8.** *Let  $X_1$  be the graph in Figure 1. Painter wins the online Ramsey game for  $C_3$  on  $X_1$ -free graphs.*

*Proof.* Painter will use Strategy 3.7. We claim that Painter can always color the new edge  $e = xy$  with Strategy 3.7. Let  $G$  be the new graph when Builder draws  $e$ . We will use induction on the number of edges. The base case is trivial.

By the induction hypothesis, we may assume that there is no red  $K_{1,3}$ , no red  $C_3$ , no red  $C_4$ , and no blue  $C_3$  in  $G - e$ . The strategy fails when coloring  $e$  blue results in a blue  $C_3$  and coloring  $e$  red results in a red  $K_{1,3}$ , a red  $C_4$ , or a red  $C_3$ . Let  $x, y, z$  be the vertices of the blue  $C_3$  when  $e$  is colored blue. We will prove that if the strategy fails, then  $G$  has  $X_1$  as a subgraph, which is a contradiction, and thus the strategy does not fail. We will divide the cases according to which red subgraph appears when Painter colors  $e$  red.

*Case 1.* Assume a red  $C_3$  is created when Painter colors  $e$  red, and let  $w$  be the third vertex of this red  $C_3$ . Since Painter colored neither  $xz$  nor  $zy$  red, coloring each of  $xz$  and  $yz$  red must have created a red  $C_4$ , a red  $C_3$ , or a red  $K_{1,3}$  in  $G - e$ . We will



**FIGURE 6** A strategy for Builder to win the online Ramsey game for  $C_3$  on  $Y$ -free graphs [Color figure can be viewed at [wileyonlinelibrary.com](http://wileyonlinelibrary.com)]

show that a red  $C_3$  or a red  $C_4$  cannot be created by coloring either  $xz$  or  $yz$  red. Without loss of generality, let us consider  $xz$ .

If coloring  $xz$  red resulted in a red  $C_4$  with vertices  $x, s, t, z$  in cyclic order, then  $t \neq y$  and  $s \neq y$ , since in  $G - e$ , the edge  $yz$  is blue and  $e$  does not exist. We also know that  $t \neq w$ , since otherwise  $G - e$  has a red  $K_{1,3}$  as a subgraph, which is a contradiction to the induction hypothesis. If  $s = w$ , then it must be that  $t = y$  in order for  $G - e$  to not have a red  $K_{1,3}$ , but this contradicts that  $t \neq y$ . This implies that  $s, t \notin \{x, y, z, w\}$ , which means  $G$  has  $X_1$  as a subgraph, which is a contradiction.

If coloring  $xz$  red resulted in a red  $C_3$  with vertices  $x, z, u$ , then  $u \neq w$ , since otherwise  $G - e$  has a red  $K_{1,3}$  as a subgraph, which contradicts the induction hypothesis. This implies that  $u \notin \{x, y, z, w\}$ . Now,  $y$  and  $z$  cannot have neighbors outside of  $\{u, x, y, z, w\}$  since that would create a copy of  $X_1$  in  $G$ . There is no red edge between  $u$  and  $w$  because that would create a red  $C_3$  in  $G - e$ . Since either a red  $yu$  or a red  $zw$  would create a red  $K_{1,3}$ , neither  $y$  nor  $z$  can have more incident red edges, which means  $yz$  could have been colored red, which is a contradiction.

This boils down to the case where both  $xz$  and  $zy$  were colored blue because coloring either one red would create a red  $K_{1,3}$ . Since  $zw$  cannot be a red edge (creates a red  $K_{1,3}$  in  $G - e$ ) and  $z$  cannot have two neighbors outside of  $\{x, y, w\}$  (creates a copy of  $X_1$  in  $G$ ), each of  $x$  and  $y$  has a neighbor  $x'$  and  $y'$ , respectively, such that  $xx'$  and  $yy'$  are red. It cannot be that  $x' = y'$ , since this creates a red  $C_4$  with vertices  $x, w, y, x'$  in  $G - e$ . If  $x' \neq y'$ , then this creates a copy of  $X_1$  in  $G$ . In either case, we obtain a contradiction.

*Case 2.* Assume a red  $K_{1,3}$  is created when Painter colors  $e$  red, and without loss of generality let  $x, y, u, v$  be the vertices of the red  $K_{1,3}$  so that  $xy, ux, xv$  are red edges. Now,  $z$  and  $y$  cannot have neighbors outside of  $\{x, y, z, u, v\}$  since that would create a copy of  $X_1$ . This implies that each of  $z$  and  $y$  cannot have two red edges incident to it, since that would create a red  $C_4$ , with vertices  $z, u, x, v$ . Also,  $uv$  cannot be a red edge since  $G - e$  would have a red  $C_3$ , with vertices  $u, v, x$ . Since  $zy$  was not colored with red, coloring  $zy$  with red must create a red  $K_{1,3}$ , a red  $C_3$ , or a red  $C_4$  in  $G - e$ . The only possible case is when coloring  $zy$  with red creates a red  $C_3$ , which implies that either  $u$  or  $v$  is a vertex of this red  $C_3$ , which implies the existence of a red  $K_{1,3}$  in  $G - e$ , which is a contradiction.

*Case 3.* Assume a red  $C_4$  is created when Painter colors  $e$  red, and let  $xx', x'y', y'y$  be the red edges of this red  $C_4$  other than  $e$ . Now, neither  $x$  nor  $y$  can have a neighbor outside of  $\{x, y, x', y', z\}$  since this would create a copy of  $X_1$  in  $G$ . Also,  $x'$  and  $y'$  cannot have a neighbor  $v \notin \{x, x', y', y\}$  where  $x'v$  and  $y'v$  is red, respectively, since this would create a red copy of  $K_{1,3}$  in  $G - e$ . Since Painter colored neither  $xz$  nor  $yz$  red, coloring each of  $xz$  and  $yz$  red must create a red  $K_{1,3}$ , a red  $C_4$ , or a red  $C_3$ . The only possible case is when there is a red  $K_{1,3}$  centered at  $z$  when Painter colors  $xz$  or  $yz$  red. In particular,  $z$  must have two neighbors  $z', z''$  outside of  $\{x, x', y, y'\}$  where  $zz'$  and  $zz''$  are red edges. Yet, this creates a copy of  $X_1$ , which is a contradiction.

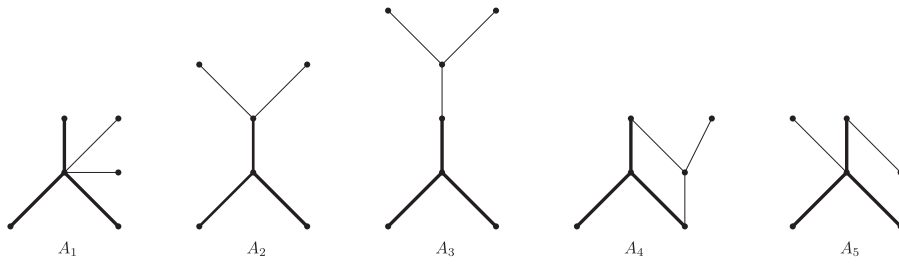


FIGURE 7 The five good  $K_{1,3}$ 's

Therefore, Strategy 3.7 works and thus Painter wins the online Ramsey game for  $C_3$  on  $X_1$ -free graphs. ■

Before starting the proof for the case of  $X_2$ -free graphs, we define some “good subgraphs of a graph. We say a subgraph  $H$  of  $G$  that is isomorphic to either  $K_{1,3}$  or  $C_4$  is good if  $H$  is red, and there exists a subgraph  $I$  of  $G$  where  $H$  is a subgraph of  $I$  in such a way that  $I$  is isomorphic to one of the graphs in Figures 7 and 8, where the thick edges correspond to the edges of  $H$ ; moreover, for  $i \in \{1, \dots, 5\}$ , we say  $H$  is good by property  $A_i$  (or  $B_i$ ) to mean that the corresponding  $I$  is isomorphic to the graph labeled  $A_i$  (or  $B_i$ ) in Figures 7 and 8. We also say  $H$  is good if  $H$  is good because of multiple properties. For example, if  $H$  satisfies the property  $A_1$ , then  $H$  is isomorphic to  $K_{1,3}$  and the vertex of degree 3 of  $G[V(H)]$  has degree at least 5 in  $G$ . We say that a red subgraph  $H$  of  $G$  that is isomorphic to either  $K_{1,3}$  or  $C_4$  is bad if it is not good. Note that if a subgraph  $H$  is bad, then all of its edges are red.

The idea is that we want to forbid  $K_{1,3}$  and  $C_4$  in the graph as much as we can, but we allow copies of  $K_{1,3}$  and  $C_4$  if we can guarantee that there is some structure we can utilize.

**Lemma 3.9.** *Let  $X_2$  be the graph in Figure 1. Let  $G$  be a graph that has a good  $K_{1,3}$  with vertices  $v, v_1, v_2, v_3$  where  $v$  is the vertex of degree 3. If  $v_1v_2, v_2v_3,$  and  $v_3v_1$  are edges in  $G$ , then  $G$  contains  $X_2$  as a subgraph.*

*Proof.* See Figure 9. It is easy to check that  $G$  has  $X_2$  as a subgraph in each case. ■

**Strategy 3.10.** *Painter colors each new edge red, unless doing so creates a red  $C_3$ , a bad  $K_{1,3}$ , or a bad  $C_4$ , in which case the new edge is colored blue.*

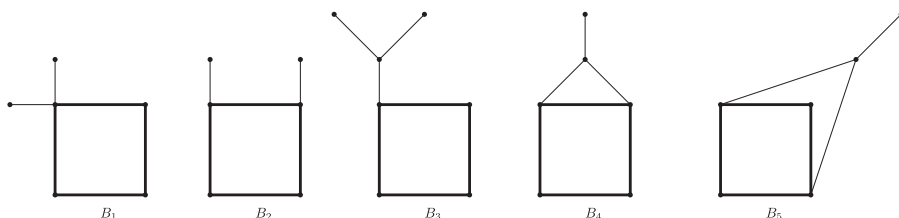


FIGURE 8 The five good  $C_4$ 's

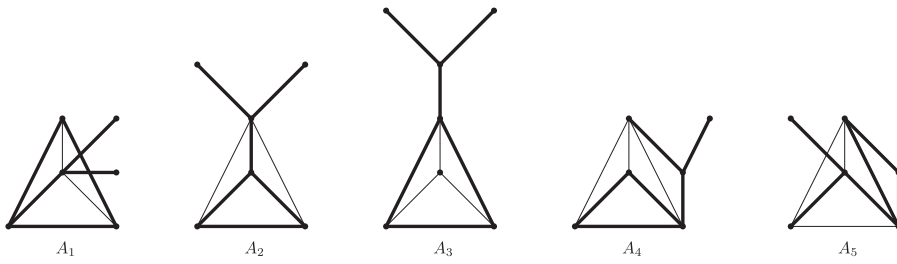


FIGURE 9 Observation for the proof of Lemma 3.9

**Proposition 3.11.** *Let  $X_2$  be the graph in Figure 1. Painter wins the online Ramsey game for  $C_3$  on  $X_2$ -free graphs.*

*Proof.* Painter will use Strategy 3.10. We claim that Painter can always color the new edge  $e = xy$  with Strategy 3.10. Let  $G$  be the new graph when Builder draws  $e$ . We will use induction on the number of edges. The base case is trivial.

By the induction hypothesis, we may assume that none of a red  $C_3$ , a bad  $K_{1,3}$ , or a bad  $C_4$  exists in  $G - e$ . The strategy fails when coloring  $e$  blue results in a blue  $C_3$  and coloring  $e$  red results in a red  $C_3$ , a bad  $K_{1,3}$ , or a bad  $C_4$ . Let  $z$  be the vertex of the blue  $C_3$  so that  $xz$  and  $zy$  are blue. Note that every blue edge has at least two red edges incident with it in  $G$  while Painter uses Strategy 3.10. We will prove that if the strategy fails, then  $G$  has  $X_2$  as a subgraph, which is a contradiction, and thus the strategy does not fail. We will divide the cases according to which red graph appears when Painter colors  $e$  red.

*Case 1.* Assume a red  $C_3$  is created when Painter colors  $e$  red, and let  $w$  be the third vertex of this red  $C_3$ . Since Painter did not color  $xz$  and  $yz$  red, coloring any of  $xz$  and  $yz$  red must have created a red  $C_3$ , a bad  $C_4$ , or a bad  $K_{1,3}$ . By Lemma 3.9, we may assume that there is no red edge between  $z$  and  $w$ . Now, we consider three subcases where coloring  $xz$  red creates one of a red  $C_3$ , a bad  $K_{1,3}$ , or a bad  $C_4$ .

*Subcase 1-1.* Assume that coloring  $xz$  red creates a red  $C_3$  with vertices  $x, z$ , and  $u$ . Since we assumed that there is no red edge between  $z$  and  $w$ , we know that  $u \neq w$ . By Lemma 3.9, we may assume that there is no red edge between  $y$  and  $u$ . Moreover,  $y$  and  $z$  cannot have neighbors outside of  $\{x, y, z, u, w\}$ , since  $G$  cannot have  $X_2$  as a subgraph. However, this is a contradiction because Painter must have colored  $yz$  red (instead of blue) since this does not create any of a bad  $K_{1,3}$ , a bad  $C_4$ , or a red  $C_3$ . Note that although there can be an edge  $uw$  in  $G - e$ , Painter could not color  $uw$  red since this creates a red  $C_3$  in  $G - e$ .

*Subcase 1-2.* Assume that coloring  $xz$  red creates a bad  $C_4$ , say  $R$ , with vertices  $x, u, v$ , and  $z$  in cyclic order. Since there is no red edge between  $z$  and  $w$ , we know that  $v \neq w$ .

Suppose  $u \neq w$ . Note that  $u, v$ , and  $z$  cannot have neighbors outside of  $\{x, y, z, u, v, w\}$  and  $E(G)$  has none of  $vy, vx, vw$ , and  $uz$ , otherwise  $G$  has  $X_2$  as a subgraph. Therefore, there was no red  $C_3$  when Painter colored  $yz$  red.

If there was a bad  $C_4$  when Painter colored  $yz$  red, then the only possible case is when the bad  $C_4$  consists of vertices  $u, v, y$ , and  $z$  since  $u, v$ , and  $z$  have no neighbor outside of  $\{x, y, z, u, v, w\}$ . Note that there is a red  $K_{1,3}$  with vertices  $u, v, x$ , and  $y$ . If  $\{x, y\}$  has no neighbors outside of  $\{x, y, z, u, v\}$ , then this red  $K_{1,3}$  must be bad, which is a contradiction. Therefore, whenever  $xz$  is drawn later than  $yz$  or  $yz$  is drawn later than  $xz$ , the later one must be colored red since the corresponding red  $C_4$  is actually good.

The only remaining reason that Painter colored  $yz$  blue is that there are two red edges incident with  $y$  so that coloring  $yz$  red creates a bad  $K_{1,3}$ , say  $S$ . There are two cases: when there is a red edge between  $y$  and  $u$  so that  $E(S) = \{yz, yu, yw\}$  and when there is no red edge between  $y$  and  $u$  but there is a red edge  $ys$  with a new vertex  $s$  so that  $E(S) = \{yz, ys, yw\}$ .

- When there is a red edge between  $y$  and  $u$  so that  $E(S) = \{yz, yu, yw\}$ .
  - Suppose Builder drew  $xz$  later than  $yz$ . Then  $R$  is good by property  $B_5$ , which is a contradiction.
  - Suppose Builder drew  $yz$  later than  $xz$ . Then  $S$  is good by property  $A_2$ , which is a contradiction.
- When there is no red edge between  $y$  and  $u$  but there is a red edge  $ys$  with a new vertex  $s$  so that  $E(S) = \{yz, ys, yw\}$ .
  - Suppose Builder drew  $xz$  later than  $yz$ . Then  $R$  is good by property  $B_3$ , which is a contradiction.
  - Suppose Builder drew  $yz$  later than  $xz$ . Then  $S$  is good by property  $A_2$ , which is a contradiction.

Note that for both cases,  $xw$  may not be drawn at each step of the game.

Now suppose  $u = w$ . It is easy to check that  $v, w$ , and  $z$  cannot have neighbors outside of  $\{v, w, x, y, z\}$ , since otherwise  $G$  has  $X_2$  as a subgraph. Since a red  $K_{1,3}$  with vertices  $x, y, w, v$  must be good,  $\{x, y\}$  must have at least one neighbor outside of  $\{v, w, x, y, z\}$ . Note that this is only true for those steps of the game in which the red  $K_{1,3}$  has already been drawn.

- Suppose that  $yz$  is drawn later than  $xz$ .
  - Coloring  $yz$  red cannot create a red  $C_3$  since  $z$  cannot have neighbors outside of  $\{v, w, x, y, z\}$ .
  - Coloring  $yz$  red cannot create a bad  $C_4$  since the only possible red  $C_4$  is of vertices  $v, w, y$ , and  $z$ . Since there is a red  $K_{1,3}$ ,  $\{x, y\}$  must have at least one neighbor outside of  $\{v, w, x, y, z\}$  and this implies that the red  $C_4$  is good.
  - Coloring  $yz$  red cannot create a bad  $K_{1,3}$  since  $z$  cannot have neighbors outside of  $\{v, w, x, y, z\}$ . Even if there are two red edges  $ys$  and  $yt$  for vertices  $s$  and  $t$  (one of  $s$  and  $t$  may be equal to  $w$ , but not to  $v$  or  $x$ ), the red  $K_{1,3}$  with vertices  $s, t, y$ , and  $z$  is good by property  $A_2$ .

Note that there are no red edges between  $z$  and  $w$ , between  $v$  and  $y$ , between  $v$  and  $x$ , or between  $x$  and  $y$ .

- Suppose that  $xz$  is drawn later than  $yz$ . Now, there are two cases: when coloring  $yz$  red created a bad  $C_4$  or when coloring  $yz$  red created a bad  $K_{1,3}$ . Note that coloring  $yz$  red cannot create a red  $C_3$ .
  - If coloring  $yz$  red would have created a bad  $C_4$ , then the vertices of this  $C_4$  must be  $v, w, y$ , and  $z$ , since  $v, w$ , and  $z$  cannot have neighbors outside of  $\{v, w, x, y, z\}$  and  $wz$  is not a red edge. Hence,  $yw$  must have been drawn before  $yz$ .
  - If coloring  $yz$  red would have created a bad  $K_{1,3}$ , then there must have been two vertices  $s, t$  such that  $ys$  and  $yt$  are red, and these are drawn earlier than  $yz$ . Whenever  $w \in \{s, t\}$  or not,  $R$  is good, which is a contradiction. Note that there are no red edges between  $v$  and  $x$  or between  $v$  and  $y$ .

*Subcase 1–3.* We may assume that coloring  $xz$  red creates a bad  $K_{1,3}$ , say  $T_1$ . If  $z$  is the center of  $T_1$ , then since there is no red edge between  $z$  and  $w$ ,  $G$  has  $X_2$  as a subgraph, which is a contradiction. Therefore, we may assume that  $x$  is a center of  $T_1$ , and  $xz, xu_1$ , and  $xu_2$  are the three edges of  $T_1$  with new vertices  $u_1$  and  $u_2$ . By symmetry, we may assume that coloring  $yz$  red creates a bad  $K_{1,3}$ , say  $T_2$ , with the center  $y$ . We may also assume that  $yz, yv_1$ , and  $yv_2$  are the three edges of  $T_2$  with new vertices  $v_1$  and  $v_2$ . Note that  $w$  is not necessarily distinct from  $u_1, u_2, v_1, v_2$ .

- If  $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 0$ , then it is easy to check that Painter can color one of  $xz$  and  $yz$  red since  $T_1$  or  $T_2$  must be good by property  $A_3$ , which is a contradiction.
- If  $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 1$ , then it is easy to check that Painter can color one of  $xz$  and  $yz$  red since  $T_1$  or  $T_2$  must be good by property  $A_4$ , which is a contradiction.
- If  $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 2$ , then let  $w_1 = u_1 = v_1$  and  $w_2 = u_2 = v_2$ . We may assume that  $yz$  is drawn later than  $xz$ , by symmetry. Then right before Builder draws  $yz$ , each of  $\{x, y, z, w_1, w_2\}$  cannot have neighbors outside of  $\{x, y, z, w_1, w_2\}$ , since otherwise  $T_2$  becomes good when Painter colors  $yz$  red. However, this is a contradiction since the red  $C_4$  with vertices  $x, w_1, y, w_2$ , is bad in  $G - e$ . This is because a red  $C_4$  in a component of at most five vertices is always bad.

*Case 2.* Assume a bad  $K_{1,3}$ , say  $S$ , is created when Painter colors  $e$  red, and without loss of generality let  $x, y, u$ , and  $v$  be the vertices of  $S$  so that  $ux, xv$  are red edges. Now,  $y$  cannot have neighbors outside of  $\{x, z, u, v\}$  in  $G$  since otherwise  $S$  is good by property  $A_2$  when  $e$  is colored red in  $G$ . If there is a red edge between  $u$  and  $y$  and between  $v$  and  $y$ , then this case is covered by case 1. Therefore, we may assume that there is no red edge between  $u$  and  $y$  and



between  $v$  and  $y$  in  $G$ . Since the blue edge  $yz$  must have two incident red edges in  $G$ , we may assume that the two red edges are incident with  $z$ , say  $zs, zt$ . We can check that  $\{s, t\} = \{u, v\}$ , since otherwise  $S$  is good by property  $A_4$  when  $e$  is colored red in  $G$ . Since  $G - e$  has a red  $C_4$  with vertices  $x, u, z, v$ , say  $R$ , by the induction hypothesis,  $R$  must be good. This implies that the component containing  $R$  must have at least six vertices, thus, one of  $x, y, z, u$ , and  $v$  has a neighbor outside of  $\{x, y, z, u, v\}$ . However, this implies that  $S$  is good when  $e$  is colored red, which is a contradiction.

*Case 3.* Assume a bad  $C_4$ , say  $R$ , is created when Painter colors  $e$  red, and let  $xu, uv, vy$  be the red edges of  $R$ . Now each of  $\{x, y, z, u, v\}$  cannot have neighbors outside of  $\{x, y, z, u, v\}$ , since otherwise  $R$  is good in  $G$ . This also implies that there is no red  $K_{1,3}$  in this component since  $K_{1,3}$  in a component of at most five vertices must be bad. Then the blue edge  $zx$  has no two red edges incident with it in  $G$ , which is a contradiction.

Therefore, Strategy 3.10 works, and thus Painter wins the online Ramsey game for  $C_3$  on  $X_2$ -free graphs.  $\blacksquare$

We present a winning strategy for Painter that can be used for the following proposition covering two cases.

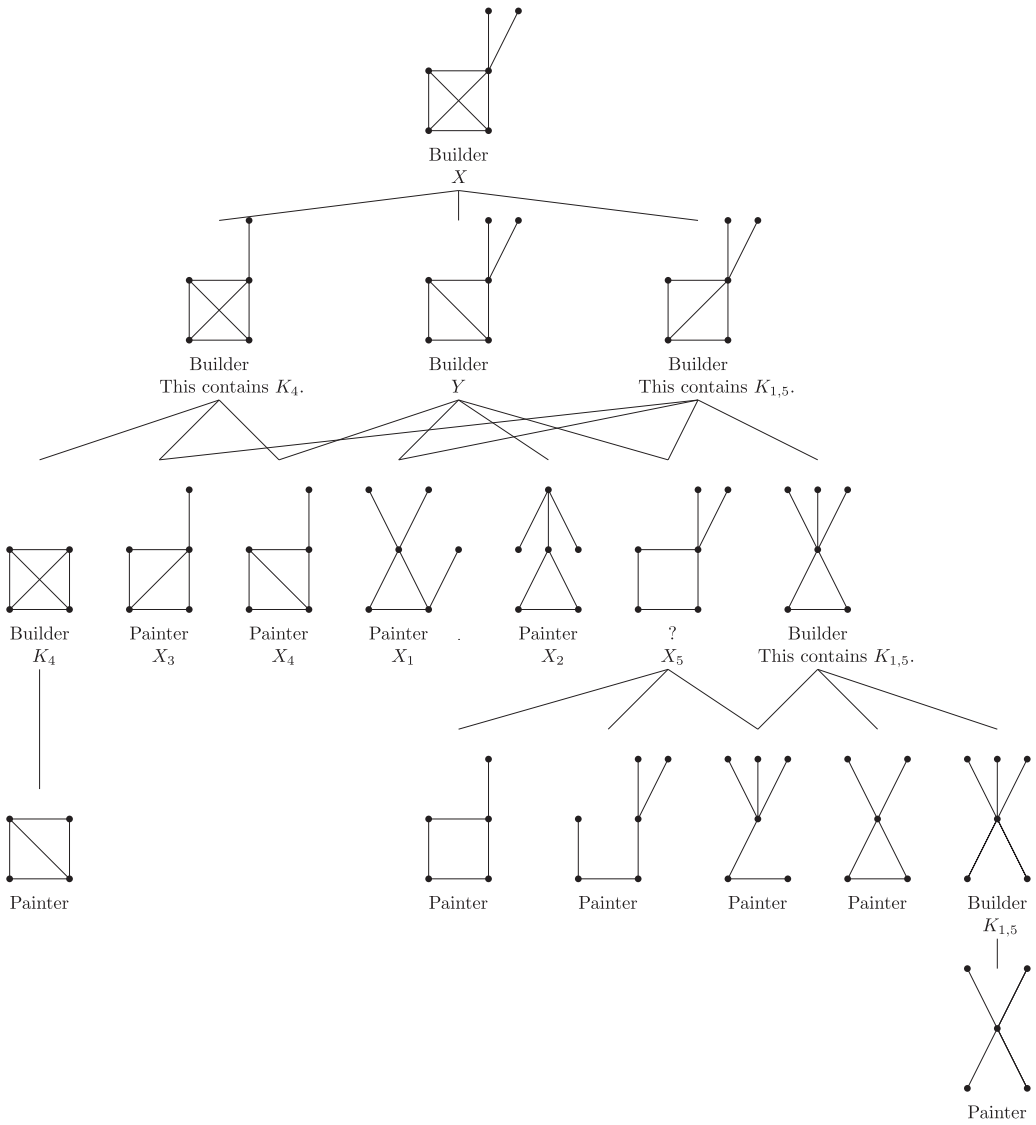
**Strategy 3.12.** *When Builder draws an edge  $e$ , if a blue  $C_3$  is made when Painter colors  $e$  blue or there is no red edge incident to  $e$ , then Painter colors  $e$  red. Otherwise, Painter colors  $e$  blue.*

**Proposition 3.13.** *Let  $X_3$  and  $X_4$  be the graphs in Figure 1. Painter wins the online Ramsey game for  $C_3$  on  $X_3$ -free graphs and Painter wins the online Ramsey game for  $C_3$  on  $X_4$ -free graphs.*

*Proof.* We will prove both statements at the same time. Painter will use Strategy 3.12. We claim that Painter can always color the new edge  $e = xy$  with Strategy 3.12. Let  $G$  be the new graph when Builder draws  $e$ . We will use induction on the number of edges. The base case is trivial.

By the induction hypothesis, we may assume that every blue edge is incident with at least one red edge and that there is no monochromatic  $C_3$  in  $G - e$ . The strategy fails when coloring  $e$  blue and red results in a blue  $C_3$  and a red  $C_3$ , respectively. Let  $z_1, z_2$  be vertices such that  $\{x, z_1, y\}$  and  $\{x, z_2, y\}$  are vertices of the blue  $C_3$  and the red  $C_3$ , respectively. We will prove that if the strategy fails, then  $G$  has both  $X_3$  and  $X_4$  as subgraphs, which is a contradiction, and thus the strategy does not fail in either game.

Without loss of generality, we may assume that Builder has drawn  $xz_2$  later than  $z_2y$ . Consider the graph right after Builder drew  $xz_2$ . Note that  $xz_2$  is incident to a red edge  $z_2y$ . Since Painter uses Strategy 3.12 and Painter colored  $xz_2$  red, there must be a blue  $C_3$  when Painter colors  $xz_2$  blue. Let  $x, z_2, v$  be the vertices of the blue  $C_3$ . Note that  $xv$  and



**FIGURE 10** The lines between graphs imply that the lower graph is a subgraph of the higher graph

$z_2v$  are drawn earlier than  $xz_2$ . If  $v \neq z_1$ , then  $G$  contains both  $X_3$  and  $X_4$  as a subgraph, and thus  $v$  must be the same as  $z_1$ .

Now consider the graph right before Builder drew  $xz_2$ . Since Builder has already drawn  $xz_1$  and Painter colored it blue,  $xz_1$  must have at least one incident red edge in  $G$ . This red edge is incident with either  $x$  or  $z_1$ , but in both cases  $G$  contains both  $X_3$  and  $X_4$  as a subgraph, which is a contradiction, and thus the strategy works.

Therefore, Strategy 3.12 works, and thus Painter wins the online Ramsey game for  $C_3$  on both  $X_3$ -free graphs and  $X_4$ -free graphs. ■

### 3.3 | The final touch

In this section we prove Theorem 3.1. We need two additional lemmas to prove Theorem 3.1.

**Lemma 3.14.** *If Builder wins the online Ramsey game for  $H$  on  $I$ -free graphs for a graph  $I$ , then Builder wins the online Ramsey game for  $H$  on  $J$ -free graphs for every graph  $J$  that has  $I$  as a subgraph.*

*Proof.* Since the set of  $I$ -free graphs is a subset of the set of  $J$ -free graphs, Builder can use the same strategy used in the case of  $J$ -free graphs. ■

**Lemma 3.15.** *If Painter wins the online Ramsey game for  $H$  on  $I$ -free graphs for a graph  $I$ , then Painter wins the online Ramsey game for  $H$  on  $J$ -free graphs for every graph  $J$  that is a subgraph of  $I$ .*

*Proof.* Since the set of  $J$ -free graphs is a subset of the set of  $I$ -free graphs, Painter can use the same strategy used in the case of  $I$ -free graphs. ■

Finally, we prove Theorem 3.1.

*Proof of Theorem 3.1.* By Lemma 3.2, it is enough to consider when  $F$  is a subgraph of  $X$ .

By Propositions 3.8, 3.11, and 3.13, along with Lemma 3.15, Painter wins the online Ramsey game for  $C_3$  on  $F$ -free graphs if  $F$  is isomorphic to a subgraph of a graph in  $\{X_1, X_2, X_3, X_4\}$ . By Propositions 3.3, 3.4, and 3.6, along with Lemma 3.14, Builder wins the online Ramsey game for  $C_3$  on  $F$ -free graphs if  $F$  contains a graph in  $\{X_1, X_2, X_3, X_4\}$  as a proper subgraph.

It is easy to check that all graphs without isolated vertices are covered by the above paragraph except for the graph  $X_5$ . Figure 10 shows subgraphs of  $X$ . Moreover, “Builder” and “Painter” written under some graph in Figure 10 means that Builder and Painter, respectively, wins the online Ramsey game for  $C_3$  on  $F$ -free graphs. ■

We end this section with the only case that is unsolved.

**Question 3.16.** Let  $X_5$  be the graph in Figure 1. Who wins the online Ramsey game for  $C_3$  on  $X_5$ -free graphs?

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