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Non-planar one-loop Parke-Taylor factors in the CHY approach for quadratic propagators

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Abstract: In this work we have studied the Kleiss-Kuijf relations for the recently introduced Parke-Taylor factors at one-loop in the CHY approach, that reproduce quadratic Feynman propagators. By doing this, we were able to identify the non-planar one-loop Parke-Taylor factors. In order to check that, in fact, these new factors can describe non-planar amplitudes, we applied them to the bi-adjoint Φ^3 theory. As a byproduct, we found a new type of graphs that we called *the non-planar CHY-graphs*. These graphs encode all the information for the subleading order at one-loop, and there is not an equivalent of these in the Feynman formalism.

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Contents

1 Introduction

Among the recent developments that apply on-shell methods to the calculation of amplitudes, following Witten's work in 2003 [1], the proposal by Cachazo-He-Yuan (CHY) [2, 3] offers some advantages. The CHY formalism applies to several dimensions and also to a large array of theories [4–7], that go even beyond field theory [8–12]. The formalism is based on the scattering equations (at tree-level)

$$E_a := \sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0, \quad \sigma_{ab} := \sigma_a - \sigma_b, \quad a = 1, 2, \ldots, n, \tag{1.1}$$

with the σ_a's denoting the local coordinates on the moduli space of n-punctured Riemann spheres and $k_a^2 = 0$. To obtain the tree-level S-matrix we have to perform a contour integral localized over solutions of these equations, i.e.

$$\mathcal{A}_n = \int_\Gamma d\mu_n^{\text{tree}}\, \mathcal{I}^{\text{CHY}}_{\text{tree}}(\sigma), \tag{1.2}$$

where the integration measure, $d\mu_n^{\text{tree}}$, is given by

$$d\mu_n^{\text{tree}} = \frac{\prod_{a=1}^n d\sigma_a}{\text{Vol}\,(\text{PSL}(2,\mathbb{C}))} \times \frac{(\sigma_{ij}\sigma_{jk}\sigma_{ki})}{\prod_{b\neq i,j,k}^n E_b}, \tag{1.3}$$

and the contour Γ is defined by the $n-3$ independent scattering equations

$$E_b = 0, \quad b \neq i,j,k\,. \tag{1.4}$$

A different integrand, $\mathcal{I}_{\text{tree}}^{\text{CHY}}$, describes a different theory. In the study of the scattering equations at tree-level and the development of techniques for integration, many approaches have been formulated [4, 13–36]. In particular, the method of integration that we use in the calculations for the present paper was developed by one of the authors in [37], which is called the Λ-algorithm.

The following step for the CHY formalism is going to loop corrections. A prescription that allows to go to higher genus Riemann surfaces was developed in [38–42], which is called the ambitwistor and pure spinor ambitwistor string theory. Another alternative approach employing an elliptic curve was proposed in [43, 44] by one of the authors. Additionally, in [45–47] they took an approach from tree-level, by introducing the forward limit with two additional massive particles that played the role of the loop momenta.

The premise for the previously mentioned prescriptions at one-loop, is that from the CHY formalism a new representation for the Feynman propagators arise, the so called linear propagators, which look like $(2\ell\cdot K+K^2)^{-1}$ [41, 48]. Many interesting developments in one-loop integrands identities (like the Kleiss-Kuijf (KK) identities [49]) and dualities (e.g. the Bern-Carrasco-Johansson (BCJ) color-kinematics duality [50]) have been found from the CHY approach, suported over the tree-level scattering equations with two extra massive particles [51–53].

The linear propagators approach still leaves several open questions. First, there is no direct way to relate some of the results with the ones from traditional field theory, like the BCJ numerators for example. Another question is about direct loop integration, and to see if it is more efficient to compute these new integrals that appear, in comparison with many well known and long time developed integration techniques for the traditional Feynman propagators.

Currently, we are developing a program to obtain the traditional quadratic Feynman propagators, $(\ell+K)^{-2}$, directly from the CHY. The first proposal came by one of the authors in [54] for the Φ^3 scalar theory. Following, two of the current authors in [55] presented a reformulation for the one-loop Parke-Taylor factors that gives quadratic propagators, but it was just made for leading (or planar) contributions. These new Parke-Taylor factors were successfully tested in the massless bi-adjoint Φ^3 theory.

The proposal to obtain the traditional quadratic Feynman propagators from the CHY approach lies in the use of $n+4$ massless scattering equations instead of $n+2$. The extra particles come from splitting the extra two massive loop momenta (ℓ^+,ℓ^-) into four massless ones $((a_1,b_1),(b_2,a_2))$. The splitting was motivated by taking an unitary cut on a n-point Feynman diagram at two-loop (see figure 1), that leave us with a $(n+4)$-point

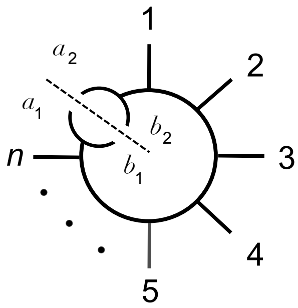

Figure 1. Unitary cut on a two-loop diagram.

tree-level diagram. It was also proved that the four auxiliary loop momenta will always combine in order to give the loop momentum in the forward limit [54–56].

This work was motivated by the tree-level KK relations, which were found for gluons at tree-level in 1988 [49]. Let us remind the algebraic relations that the partial amplitudes satisfy

$$A^{\text{tree}}(\beta_1,\ldots,\beta_s,1,\alpha_1,\ldots,\alpha_r,n) = (-1)^s \sum_{\sigma\in\text{OP}\{\beta\}\{\alpha\}} A^{\text{tree}}(1,\sigma,n), \tag{1.5}$$

where the order preserving product (OP) merges the ordering α^{T} into the ordering β.

An one-loop version of the previous relations was found by Bern-Dixon-Dunbar-Kosower in [57] (so they may be called the BDDK relations instead of the KK relations, since the later are only at tree-level). Their outline for the proof relies on the structure constants, so the result holds for any one-loop gauge theory amplitude where the external particles and the particles circulating around the loop are both in the adjoint representation of SU(N). This relation reads[1]

$$A^{1-\text{loop}}_{n;r}(1,\ldots,r-1\,;\,r,\ldots,n) = (-1)^r \sum_{\sigma\in\text{COP}\{\alpha\}\{\beta\}} A^{1-\text{loop}}_{n;1}(\sigma). \tag{1.6}$$

The amplitudes we have on the left hand side belong to the non-planar sector of the perturbative expansion. These amplitudes are far less understood than the planar ones, starting with the notion of the integrand which is ambiguous. At one-loop order there is no way to draw a Feynman diagram that could lead us to the integrand and we can only make an incursion into them using the previous relations. From another point of view, twistor methods like the ones from [58] apply only to planar diagrams, so the first obstacle to face from this perspective is the lack of duals for non-planar diagrams.

From a physics point of view, 't Hooft's large N limit [59] shows that the planar sector of the amplitudes for a gauge theory with color group U(N) (with fixed 't Hooft coupling) is dominating, so by going beyond and studying these non-planar interactions we can get a better understanding of that limit, where the 't Hooft diagrams have a cylindrical topology.

[1]The cyclic ordering preserving product (COP) merges the cyclic permutations of $\{\alpha\} = \{r-1,\ldots,1\}$ leaving the $\{\beta\} = \{r, r+1,\ldots,n\}$ ordering fixed. This merging of the orderings is quite different from the one at tree-level, and makes the proof for the relation a non-trivial task.

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In twistor string, a non-planar open string amplitude at one-loop (scattering of gluons) has the topology of that of a closed string, so it is a cylinder (an annulus with vertex operators on the two boundaries). This brings the inclusion of conformal graviton exchanges, which may presents itself as a drawback to the formalism.

The idea in the present work is to understand the amplitudes in the non-planar sector from the perspective of the CHY formalism. In this formalism, the ordering in the amplitudes is directly associated with the Parke-Taylor factors, so our main focus will be on those. Since our Parke-Taylor factors for the n-point one-loop amplitude come from those at tree-level with $(n+4)$-point, we will analyse the KK relations in order to extract the BDDK relations for the amplitudes. At tree-level there is already a version of the KK relations for the Parke-Taylor factors in the CHY formalism [52].

One important aspect to realize is that not all of the terms in the $(n+4)$-point KK relations contribute to what may be the n-point BDDK relations for the Parke-Taylor factors. After identifying what contributes to one-loop there is a simple and powerful result, a relation for the partial one-loop Parke-Taylor factors that involves the regular shuffle product for the orderings. The relations for the partial factors can be promoted to relations for the full one-loop Parke-Taylor factors, this process involves taking cyclic permutations and rearranging the partial factors. It is exactly after that promotion that we find the BDDK relations for them.[2] In this way, we provide a new proof for the BDDK relations for the amplitudes from a different approach. Along the path, a more general type of relations appear, and for those we still don't have a direct physical interpretation.

It is known that the BDDK relations connect the non-planar amplitudes with the planar ones, this will allow us to identify what would be the *non-planar one-loop Parke-Taylor factors*, which are the building blocks for such amplitudes. Employing these new objects in the construction of the CHY integrands, will lead to the identification of the non-planar CHY-graphs. This new type of graphs do not have an equivalent in the traditional formalism and encode all the information for the non-planar order in a single expression, they also provide a clear definition of the integrand, at the level of the CYH formalism. The developed non-planar CHY-graphs are applied to the massless Bi-adjoint Φ^3 theory, allowing us to explore the non-planar sector of the theory at one-loop, we will obtain all the integrands which contribute to the amplitude at this quantum correction.

Outline. The present work is organized as follows. In section 2 we review the one-loop Parke-Taylor factors for planar corrections that were proposed in [55], with a few changes in the notation and define the *partial planar one-loop Parke-Taylor factor*, which will be studied in depth in the following sections.

Section 3 is divided in two parts. In the first one we take the previously presented partial Parke-Taylor factors, which are tree-level type ones, and apply the KK relations on them in order to obtain the BDDK relations at one-loop. The development starts by analysing the tree-level relations for $n+4$ particles, from there it becomes clear that not all of the terms contribute to the one-loop case, but a sector from them. With that analysis we

[2]The BDDK relations involve a more elaborated shuffle product for the orderings, the COP product that we mentioned before.

find the simplest one-loop case, from there we will go to the most general relation possible at one-loop, this is possible by the realization of an algebraic relation that includes the shuffle product. The next part is where we find the BDDK relations for the full one-loop Parke-Taylor factors, these being a particular case which resemble the relations for gluon amplitudes. We also find some relations that generalize the BDDK ones.

In section 4 we employ the one-loop non-planar Parke-Taylor factors in the study of one-loop amplitudes for the massless Bi-adjoint Φ^3 theory. Since the CHY integrands for this theory are given by the product of two Parke-Taylor factors, the contributions for the one-loop amplitudes come from different sectors: the planar, the non-planar, and the mixed one (planar; non-planar). The Bi-adjoint Φ^3 theory plays a role in the double copy between supersymmetric Yang-Mills and supergravity [60], also in color-kinematic relations for scalar effective field theories [61]. Therefore, having and understanding the non-planar sector has great value in the development of more relevant theories.

The non-planar CHY-graphs will be introduced in section 5 with some particular worked out examples for the four-point and five-point cases. In section 6 we present the general non-planar CHY-graphs for an arbitrary number of points. Finally, we present our conclusions in section 7. Additionally, in order to make the paper self contained, we provide the detailed calculations of the four-point amplitude from the CHY formalism in appendix A and its exact equivalence with the results obtained from the standard method based on the Feynman rules in appendix B.

Notation. For convenience, in this paper we use the following notation

$$\sigma_{ij} := \sigma_i - \sigma_j, \qquad \omega_{i:j}^{a:b} := \frac{\sigma_{ab}}{\sigma_{ia}\,\sigma_{jb}}. \tag{1.7}$$

Note that $\omega_{i:j}^{a:b}$ is the generalization of the $(1,0)$-forms used in [56] to write the CHY integrands at two-loop. In addition, we define the σ_{ab}'s and $\omega_{i:j}^{a:b}$'s chains as

$$\begin{aligned} (i_1, i_2, \ldots, i_p) &:= \sigma_{i_1 i_2} \cdots \sigma_{i_{p-1} i_p} \sigma_{i_p i_1}, \\ (i_1, i_2, \ldots, i_p)_\omega^{a:b} &:= \omega_{i_1:i_2}^{a:b} \cdots \omega_{i_{p-1}:i_p}^{a:b} \omega_{i_p:i_1}^{a:b} = \omega_{i_1:i_1}^{a:b} \cdots \omega_{i_{p-1}:i_{p-1}}^{a:b} \omega_{i_p:i_p}^{a:b}, \end{aligned} \tag{1.8}$$

To have a graphical description for the CHY integrands on a Riemann sphere (CHY-graphs), it is useful to represent each σ_a puncture as a vertex, the factor $\frac{1}{\sigma_{ab}}$ as a line and the factor σ_{ab} as a dashed line that we call the *anti-line*. Additionally, since we often use the Λ-algorithm[3] [37], then we introduce the color code given in figure 2 and 3 for a mnemonic understanding.

Finally, we introduce the momenta notation

$$k_{a_1,\ldots,a_m} := \sum_{i=1}^{m} k_{a_i} = [a_1, \ldots, a_m], \quad s_{a_1\ldots a_m} := k^2_{a_1,\ldots,a_m}, \quad \tilde{s}_{a_1\ldots a_m} := \sum_{a_i<a_j}^{m} k_{a_i} \cdot k_{a_j}.$$

[3]It is useful to recall that the Λ-algorithm fixes four punctures, three of them by the $\mathrm{PSL}(2,\mathbb{C})$ symmetry and the last one by the scale invariance.

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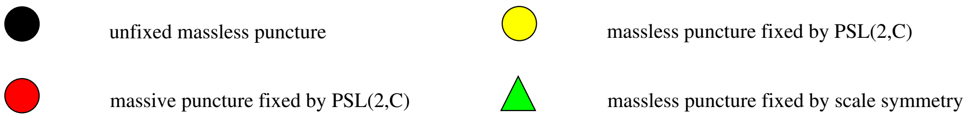

Figure 2. Vertex Color code in the CHY-graphs for the Λ-algorithm.

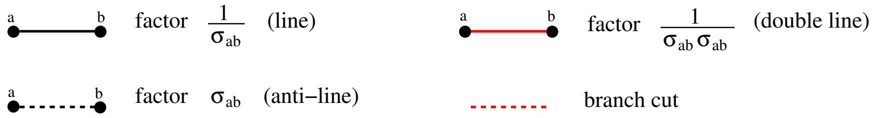

Figure 3. Edges Color code in the CHY-graphs for the Λ-algorithm.

2 Planar Parke-Taylor factors at one-loop

In a previous work, we found a reformulation for the Parke-taylor factors that leads to quadratic Feynman propagators by evaluating only massless scattering equations in the CHY prescription.

Let us remind the expression for the Parke-Taylor factor at tree-level in the CHY approach, it is given by the following

$$\mathrm{PT}[\pi] = \frac{1}{(\pi_1, \pi_2, \ldots, \pi_n)}, \tag{2.1}$$

where "π" is a generic ordering and n is the total number of particles. In addition, it will be useful to define a reduced Parke-Taylor factor, which does not involve all n particles, i.e.

$$\widehat{\mathrm{PT}}[i_1, \ldots, i_p] := \frac{1}{(i_1, \ldots, i_p)}, \tag{2.2}$$

where $p < n$.

As it was shown in [55], the planar one-loop Parke-Taylor factor (on the left-sector, denoted by a-sector) for quadratic propagators is given by[4]

$$\begin{aligned}
\mathrm{PT}^{(1)}_{a_1:a_2}[\pi] &:= \mathrm{PT}[\pi_1, \pi_2, \ldots, \pi_n, a_1, b_1, b_2, a_2] + \mathrm{cyc}(\pi) \\
&= \frac{1}{(a_1, b_1, b_2, a_2)} \sum_{\alpha \in \mathrm{cyc}(\pi)} \frac{1}{\sigma_{\alpha_1\alpha_2}\sigma_{\alpha_2\alpha_3}\cdots\sigma_{\alpha_{n-1}\alpha_n}} \omega^{a_1:a_2}_{\alpha_n:\alpha_1} \\
&= \frac{(a_1, a_2)}{(a_1, b_1, b_2, a_2)} \sum_{\alpha \in \mathrm{cyc}(\pi)} \widehat{\mathrm{PT}}[\alpha_1, \ldots, \alpha_n, a_1, a_2]\,.
\end{aligned} \tag{2.3}$$

[4]As it was explained in [54, 55], in order to compute scattering amplitudes at one-loop we must perform the forward limit, $k_{a_1} = -k_{a_2}$ ($k_{b_1} = -k_{b_2}$). So, from the third line in (2.3), it is clear this definition is totally analog to the one given in [45, 46] by the expression

$$\mathrm{PT}^{(1)}[\pi] := \sum_{\alpha \in \mathrm{cyc}(\pi)} \mathrm{PT}[\alpha_1, \ldots, \alpha_n, +, -],$$

which is just able to reproduce linear propagators.

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It is useful to remember that the Parke-Taylor factor on the right-sector (or b-sector) is defined to be

$$\mathrm{PT}^{(1)}_{b_1:b_2}[\rho] := \frac{(b_1,b_2)}{(a_1,b_1,b_2,a_2)} \sum_{\beta\in \mathrm{cyc}(\rho)} \widehat{\mathrm{PT}}[\beta_1,\ldots,\beta_n,b_1,b_2]. \tag{2.4}$$

Such as we explained in the appendix A in [55], the CHY-integral of the product, $\mathrm{PT}^{(1)}_{a_1:a_2}[\pi]\times \mathrm{PT}^{(1)}_{b_1:b_2}[\rho]$, is always a function of the couples, $(k_{a_1}+k_{b_1})$ and $(k_{a_2}+k_{b_2})$. This means, these type of integrals always produce quadratic propagators at one-loop by the identification, $(k_{a_1}+k_{b_1}) = -(k_{a_2}+k_{b_2}) = \ell$.

Each one of the terms of the sum in (2.3) and (2.4) is called *partial planar one-loop Parke-Taylor factor* which we denote them by $\mathbf{pt}^{(1)}_{a_1:a_2}[i_1,\ldots,i_p]$ ($\mathbf{pt}^{(1)}_{b_1:b_2}[i_1,\ldots,i_p]$), namely

$$\mathbf{pt}^{(1)}_{a_1:a_2}[i_1,\ldots,i_p] := \frac{1}{\sigma_{i_1i_2}\sigma_{i_2i_3}\cdots\sigma_{i_{p-1}i_p}}\,\omega^{a_1:a_2}_{i_p:i_1}, \tag{2.5}$$

in particular we define

$$\mathbf{pt}^{(1)}_{a_1:a_2}[i_1] := \omega^{a_1:a_2}_{i_1:i_1}. \tag{2.6}$$

Thus,

$$\mathrm{PT}^{(1)}_{a_1:a_2}[\pi] = \frac{1}{(a_1,b_1,b_2,a_2)} \sum_{\alpha\in \mathrm{cyc}(\pi)} \mathbf{pt}^{(1)}_{a_1:a_2}[\alpha_1,\ldots,\alpha_n], \tag{2.7}$$

and for the b-sector we must just perform the replacement, $(a_1,a_2)\to(b_1,b_2)$. Therefore, for the rest of this work it is enough to work on the a-sector.

In [55], we found several algebraic manipulations on these Parke-Taylor factors that allowed us to obtain the contributing planar diagrams for the bi-adjoint scalar Φ^3 theory. In order to go beyond the planar case, let us analyse a known relation between planar and non-planar integrands.

3 From KK to BDDK relations and non-planar Parke-Taylor factors at one-loop

In this section we will build an alternative proof for the BDDK relations at one-loop. Since our Parke-Taylor factors come from the $(n+4)$-point ones at tree-level, it makes perfect sense to start analysing the KK relations for those. From the KK relations we will realise than only a sector contributes to the one-loop Parke-Taylor factors, more specifically to the partial one-loop Parke-Taylor factors. By taking the pertinent cyclic permutations we will promote the partial factors to the full Parke-Taylor factors, there the BDDK identity will arise naturally. Along this procedure we will be able to identify a new object, the non-planar Parke-Taylor factors, we also will go beyond and find a generalisation for the BDDK relations.

Let us recall a particular case of the KK relations for the Parke-Taylor factors [49, 52]. At tree-level we have the identity

$$\begin{aligned}\mathrm{PT}[\{\{1\}\sqcup\{2,3,4,\ldots,n-1\},n\}] &:= \mathrm{PT}[1,2,3,4,\ldots,n]+\mathrm{PT}[2,1,3,4,\ldots,n]\\ &\quad+\mathrm{PT}[2,3,1,4,\ldots,n]+\cdots+\mathrm{PT}[2,3,4,\ldots,n-1,1,n]=0,\end{aligned} \tag{3.1}$$

where we have introduced the shuffle product " $\shuffle$ ",[5] just for a simple case of the first entry merging in the next entries up until the one at the (n-1) position, we will generalise this to two lists of arbitrary size and even beyond two lists. This will set the root of our developments towards the case of one-loop. Note that we are making the following abuse in the notation

$$\mathrm{PT}[A_1 + A_2 + \cdots + A_p] \equiv \sum_{i=1}^{p} \mathrm{PT}[A_i]\,, \tag{3.2}$$

where A_i is an ordered list of n different elements.

The relation (3.1) will set the starting point for our analysis.

3.1 KK relations for the partial one-loop Parke-Taylor factors

Translating the relation (3.1) for the partial one-loop Parke-Taylor factors, which from a technical point of view are still at tree-level, it can be written as

$$\begin{aligned}&\mathrm{PT}[1,2,\ldots,n,a_1,b_1,b_2,a_2]+\sum_{i=2}^{n-1}\mathrm{PT}[2,\ldots,i,1,i{+}1,\ldots,n,a_1,b_1,b_2,a_2]+\mathrm{PT}[2,\ldots,n,1,a_1,b_1,b_2,a_2]\\&+\mathrm{PT}[2,\ldots,n,a_1,1,b_1,b_2,a_2]+\mathrm{PT}[2,\ldots,n,a_1,b_1,1,b_2,a_2]=-\mathrm{PT}[2,\ldots,n,a_1,b_1,b_2,1,a_2].\end{aligned} \tag{3.3}$$

Note that in the terms on the second line in (3.3) there is a splitting in the points assigned to the loop momenta. Those terms are perfectly normal at tree-level, but at loop level they would not lead to any quadratic Feynman propagator in the forward limit, in other words, the identification, $(k_{a_1} + k_{b_1}) = -(k_{a_2} + k_{b_2}) = \ell$, is not well defined on them. In any case, we can analyse the terms where the splitting is not present.

Taking the sector of (3.3) where the loop momenta points are not split, we can find a different relation for the partial one-loop Parke-Taylor factors. The new relation is given by

$$\begin{aligned}&\mathrm{PT}[1,2,3,\ldots,n,a_1,b_1,b_2,a_2]+\sum_{i=2}^{n-1}\mathrm{PT}[2,3,\ldots,i,1,i+1,\ldots,n,a_1,b_1,b_2,a_2]\\&+\mathrm{PT}[2,3,\ldots,n,1,a_1,b_1,b_2,a_2]=\omega_{1:1}^{a_1:a_2}\,\widehat{\mathrm{PT}}[2,3,\ldots,n,a_1,b_1,b_2,a_2].\end{aligned} \tag{3.4}$$

Again, with the introduction of the shuffle product, this relation can be written in a compact and more legible way as follows

$$\begin{aligned}\mathbf{pt}^{(1)}_{a_1:a_2}[\{1\}\shuffle\{2,\ldots,n\}]&:=\mathbf{pt}^{(1)}_{a_1:a_2}[1,2,\ldots,n]+\mathbf{pt}^{(1)}_{a_1:a_2}[2,1,\ldots,n]+\cdots+\mathbf{pt}^{(1)}_{a_1:a_2}[2,\ldots,n,1]\\&=\mathbf{pt}^{(1)}_{a_1:a_2}[1]\times\mathbf{pt}^{(1)}_{a_1:a_2}[2,\ldots,n].\end{aligned} \tag{3.5}$$

The result from above can be generalized to lists of arbitrary size. In fact, we have been able to find a more general relation for the *partial planar one-loop Parke-Taylor factors*, which is given by the identity

General KK relations for the partial one-loop Parke-Taylor factors

$$\begin{aligned}&\mathbf{pt}^{(1)}_{a_1:a_2}[\{i_1,i_2,\ldots i_p\}\shuffle\{i_{p+1},\ldots i_q\}\shuffle\cdots\shuffle\{i_m,\ldots,i_n\}]\\&=\mathbf{pt}^{(1)}_{a_1:a_2}[i_1,i_2,\ldots i_p]\times\mathbf{pt}^{(1)}_{a_1:a_2}[i_{p+1},\ldots i_q]\times\cdots\times\mathbf{pt}^{(1)}_{a_1:a_2}[i_m,\ldots,i_n]\end{aligned}$$

[5]It is useful to remember the shuffle product, $\{\alpha_1,\ldots\alpha_p\}\shuffle\{\beta_1,\ldots\beta_q\}$, has a total of $\frac{(p+q)!}{p!\,q!}$ terms.

One particular case is to take all the lists with size 1, a straightforward calculation gives the result

$$\begin{aligned}\mathbf{pt}^{(1)}_{a_1:a_2}[\{1\}\sqcup\{2\}\sqcup\{3\}\sqcup\cdots\sqcup\{n\}] &= \mathbf{pt}^{(1)}_{a_1:a_2}[1]\times\mathbf{pt}^{(1)}_{a_1:a_2}[2]\times\cdots\times\mathbf{pt}^{(1)}_{a_1:a_2}[n] \\ &= \omega^{a_1:a_2}_{1:1}\times\omega^{a_1:a_2}_{2:2}\times\cdots\times\omega^{a_1:a_2}_{n:n} \\ &= \sum_{\alpha\in\mathrm{S}_{n-1}}\frac{1}{\sigma_{\alpha_1\alpha_2}\sigma_{\alpha_2\alpha_3}\cdots\sigma_{\alpha_{n-1}\alpha_n}}\omega^{a_1:a_2}_{\alpha_n:\alpha_1},\end{aligned} \tag{3.6}$$

where $\alpha_1 := 1$ and S_{n-1} is the set of all permutations of $\{2,3,\ldots,n\}$. Let us recall that the expression in (3.6) reproduces the symmetrized *n-gon* at one-loop.

Another particular case is by considering only two lists, taking the canonical ordering we have the following

$$\mathbf{pt}^{(1)}_{a_1:a_2}[\{1,2,\ldots p\}\sqcup\{p+1,\ldots n\}] = \mathbf{pt}^{(1)}_{a_1:a_2}[1,2,\ldots p]\times\mathbf{pt}^{(1)}_{a_1:a_2}[p+1,\ldots n], \tag{3.7}$$

we obtain a similar structure as the BDDK relation between the planar and non-planar amplitudes at one-loop [57, 62], but the shuffle is different, since we are not dealing with full one-loop Parke-Taylor factors yet. We define the *partial non-planar one-loop Parke-Taylor factors* as

$$\begin{aligned}\mathbf{pt}^{(1)}_{a_1:a_2}[i_1,i_2,\ldots,i_p\,|\,i_{p+1},\ldots,n] &:= \mathbf{pt}^{(1)}_{a_1:a_2}[\{i_1,i_2,\ldots i_p\}\sqcup\{i_{p+1},\ldots i_n\}] \\ &= \mathbf{pt}^{(1)}_{a_1:a_2}[i_1,i_2,\ldots,i_p]\times\mathbf{pt}^{(1)}_{a_1:a_2}[i_{p+1},\ldots,i_n].\end{aligned} \tag{3.8}$$

Taking the sum over the corresponding permutations we build the full *non-planar one-loop Parke-Taylor factors*[6] as

$$\begin{aligned}\mathrm{PT}^{(1)}_{a_1:a_2}[\pi_1,\ldots,\pi_p\,|\,\rho_{p+1},\ldots,\rho_n] &:= \frac{1}{(a_1,b_1,b_2,a_2)}\sum_{\substack{\alpha\in\mathrm{cyc}(\pi)\\ \beta\in\mathrm{cyc}(\rho)}}\mathbf{pt}^{(1)}_{a_1:a_2}[\alpha_1,\ldots,\alpha_p\,|\,\beta_{p+1},\ldots,\beta_n] \\ &= \frac{1}{(a_1,b_1,b_2,a_2)}\sum_{\substack{\alpha\in\mathrm{cyc}(\pi)\\ \beta\in\mathrm{cyc}(\rho)}}\mathbf{pt}^{(1)}_{a_1:a_2}[\alpha_1,\ldots,\alpha_p]\times\mathbf{pt}^{(1)}_{a_1:a_2}[\beta_{p+1},\ldots,\beta_n],\end{aligned} \tag{3.9}$$

where $\{\pi_1,\ldots,\pi_p\}$ and $\{\rho_{p+1},\ldots,\rho_n\}$ are two different generic orderings such that $\{\pi_1,\ldots,\pi_p\}\cap\{\rho_{p+1},\ldots,\rho_n\}=\emptyset$ and $\{\pi_1,\ldots,\pi_p\}\cup\{\rho_{p+1},\ldots,\rho_n\}=\{1,2,\ldots,n\}$.

The partial and full non-planar one-loop Parke-Taylor factors defined in (3.8) and (3.9) can be generalized, now for an arbitrary number of lists, in the following way

$$\begin{aligned}&\mathbf{pt}^{(1)}_{a_1:a_2}[i_1,\ldots,i_p\,|\,i_{p+1},\ldots,i_q\,|\,\cdots\,|\,i_m,\ldots,i_n] \\ &\quad := \mathbf{pt}^{(1)}_{a_1:a_2}[\{i_1,\ldots,i_p\}\sqcup\{i_{p+1},\ldots,i_q\}\sqcup\cdots\sqcup\{i_m,\ldots,i_n\}] \\ &\quad = \mathbf{pt}^{(1)}_{a_1:a_2}[i_1,i_2,\ldots i_p]\times\mathbf{pt}^{(1)}_{a_1:a_2}[i_{p+1},\ldots i_q]\times\cdots\times\mathbf{pt}^{(1)}_{a_1:a_2}[i_m,\ldots,i_n],\end{aligned} \tag{3.10}$$

[6]Note that the partial non-planar one-loop Parke-Taylor factors for quadratic propagators can be written as

$$\begin{aligned}&\frac{1}{(a_1,b_1,b_2,a_2)}\,\mathbf{pt}^{(1)}_{a_1:a_2}[1,2,\ldots,p\,|\,n,n-1,\ldots,p+1] \\ &\quad = (-1)^{n-p}\,\frac{(a_1,a_2)}{(a_1,b_1,b_2,a_2)}\times\widehat{\mathrm{PT}}[1,2,\ldots,p,a_1,p+1,\ldots,n-1,n,a_2].\end{aligned}$$

This expression is totally similar to the one given for non-planar linear propagators in [47, 52], where $a_1\to+$ and $a_2\to-$.

and we can write the full version in terms of these like

$$\begin{aligned}&\mathrm{PT}^{(1)}_{a_1:a_2}[\pi_1,\ldots,\pi_p\,|\,\rho_{p+1},\ldots,\rho_q\,|\cdots|\,\gamma_m,\ldots,\gamma_n] \\ &:=\frac{1}{(a_1,b_1,b_2,a_2)}\sum_{\alpha\in\mathrm{cyc}(\pi)}\sum_{\beta\in\mathrm{cyc}(\rho)}\cdots\sum_{\delta\in\mathrm{cyc}(\gamma)}\mathbf{pt}^{(1)}_{a_1:a_2}[\alpha_1,\ldots,\alpha_p\,|\,\beta_{p+1},\ldots,\beta_q\,|\cdots|\,\delta_m,\ldots,\delta_n],\end{aligned} \tag{3.11}$$

where $\{\pi_1,\ldots,\pi_p\}, \{\rho_{p+1},\ldots,\rho_n\},\ldots\{\gamma_m,\ldots,\gamma_n\}$, are different generic orderings such that $\{\pi_1,\ldots,\pi_p\}\cup\{\rho_{p+1},\ldots,\rho_n\}\cup\cdots\cup\{\gamma_{p+1},\ldots,\gamma_n\}=\{1,2,\ldots,n\}$ and they are disjoint, meaning they have no element in common. Although these generalizations are well defined in the CHY side, we still do not have an understanding of what would be their physical meaning. So far we have worked with tree-like expressions so the KK relations still applied, in the following subsection we shall see the one-loop behaviour of the relations.

3.2 The BDDK relations for the one-loop Parke-Taylor factors

In the previous section we found general relations for the partial one-loop Parke-Taylor factors, these are a sector of the KK relations. In the following section our interest lies in finding BDDK relations that involve the full one-loop Parke-Taylor factors defined in (2.3) and (3.9).

To obtain full one-loop Parke-Taylor factors on the right hand side of (3.9) we have to expand the sum and collect the terms related by a cyclic permutation. The result can be arranged again in a sum as follows

$$\mathrm{PT}^{(1)}_{a_1:a_2}[\pi_1,\ldots,\pi_p\,|\,\rho_{p+1},\ldots,\rho_n]=\sum_{\alpha\in\mathrm{cyc}(\pi)\sqcup\rho/\rho_n}\mathrm{PT}^{(1)}_{a_1:a_2}[\alpha_1,\ldots,\alpha_n], \tag{3.12}$$

where $\mathrm{cyc}(\pi)\sqcup\rho/\rho_n\equiv\{\mathrm{cyc}(\pi)\sqcup\{\rho_{p+1},\ldots,\rho_{n-1}\},\rho_n\}$. As an example, for the non-planar ordering $[1,2|3,4,5]$ we have the relation

$$\begin{aligned}\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5]=\;&\mathrm{PT}^{(1)}_{a_1:a_2}[1,2,3,4,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[1,3,2,4,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[1,3,4,2,5]\\&+\mathrm{PT}^{(1)}_{a_1:a_2}[2,1,3,4,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[3,1,2,4,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[3,1,4,2,5]\\&+\mathrm{PT}^{(1)}_{a_1:a_2}[2,3,1,4,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[3,4,1,2,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[2,3,4,1,5]\\&+\mathrm{PT}^{(1)}_{a_1:a_2}[3,2,4,1,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[3,4,2,1,5]+\mathrm{PT}^{(1)}_{a_1:a_2}[3,2,1,4,5].\end{aligned} \tag{3.13}$$

This product for two orderings has been usually denoted in the literature as $\mathrm{COP}\{\pi^{\mathrm{T}}\}\{\rho\}$, where π^{T} is the reversed ordering of π. The proof of this result for gluons subamplitudes can be found in [57], where they approach it from the string theory and field theory sides, here we have obtained it as a consequence of the regular shuffle product that appears at the tree-level KK relations.

Finding the relation between the full one-loop Parke-Taylor allows us to see more clearly the bridge between CHY and the subamplitudes from the traditional field theory approach. Another important point is that a result in [55] shows an expansion of the planar Parke-Taylor factors at one-loop in terms of ω's, which displays an advantage from a computational point of view.

Going to the more general case in (3.11), the same analysis can be performed to find a relation with the full one-loop Parke-Taylor factors. After expanding and collecting the terms we are left with

General BDDK relations for the full one-loop Parke-Taylor factors

$$\mathrm{PT}^{(1)}_{a_1:a_2}[\pi_1,\ldots,\pi_p\,|\,\rho_{p+1},\ldots,\rho_q\,|\,\cdots\,|\,\gamma_m,\ldots,\gamma_n] = \sum_{\alpha\in \mathrm{cyc}(\pi)\sqcup\mathrm{cyc}(\rho)\sqcup\ldots\sqcup\gamma/\gamma_n} \mathrm{PT}^{(1)}_{a_1:a_2}[\alpha_1,\ldots,\alpha_n],$$

where the product of orderings follows the same definition given after (3.12). Here even the simplest non-trivial example (that could be $[1,2|3,4|5,6]$) gives a considerable number of terms to fit on a single page.

4 Bi-adjoint Φ^3 scalar theory

Having found the non-planar Parke-Taylor factors in the previous section, we will apply them to the simplest non-trivial theory we can build from them, the Bi-adjoint Φ^3 scalar theory. These new factors will allow us to dive directly into the non-planar sector of the theory. In the CHY prescription we will have a clearly defined integrand for a non-planar amplitude, something that does not happen in the traditional formalism, and opens a new gate to explore these interactions.

Here we propose a non-planar CHY prescription for the S-matrix at one-loop for the bi-adjoint Φ^3 scalar theory. By construction, with our Parke-Taylor factors we will be able reproduce directly the quadratic propagators, which are the important ones if one wants to study the behaviour of the singularities in the non-planar sector, although the analysis of singularities is beyond the scope of the present work.

4.1 Full bi-adjoint Φ^3 amplitude at one-loop

Along the line of reasoning of [47] and with the partial Parke-Taylor factors defined in section 2, we define the full bi-adjoint Φ^3 amplitude at one-loop with flavor group $\mathrm{U}(N)\times\mathrm{U}(\tilde{N})$ as

$$\begin{aligned}\mathbf{m}_n^{1-\text{loop}} = \int d^D\ell \sum_{\substack{\pi\in S_{n+2}/\mathbb{Z}_{n+2}\\ \rho\in S_{n+2}/\mathbb{Z}_{n+2}}} &\mathrm{Tr}(T^{i_{\pi_1}}T^{i_{\pi_2}}\cdots T^{i_{\pi_n}}T^{i_{\pi_{a_1}}}T^{i_{\pi_{a_2}}})\times\left\{\frac{1}{2^{n+1}}\int d\Omega\, s_{a_1b_1}\right.\\ &\times\int_\Gamma d\mu^t_{n+4}\,\frac{(a_1,a_2)}{(a_1,b_1,b_2,a_2)}\widehat{\mathrm{PT}}[\pi_1,\ldots,\pi_n,\pi_{a_1},\pi_{a_2}]\times\frac{(b_1,b_2)}{(a_1,b_1,b_2,a_2)}\widehat{\mathrm{PT}}[\rho_1,\ldots,\rho_n,\rho_{b_1},\rho_{b_2}]\left.\vphantom{\int}\right\}\\ &\times\mathrm{Tr}(\tilde{T}^{i_{\rho_1}}\tilde{T}^{i_{\rho_2}}\cdots\tilde{T}^{i_{\rho_n}}\tilde{T}^{i_{\rho_{b_1}}}\tilde{T}^{i_{\rho_{b_2}}}),\end{aligned}\tag{4.1}$$

where the measures, $d\Omega$ and $d\mu^{\rm t}_{n+4}$, are given by the expressions[7] [3, 54]

$$d\Omega := d^D(k_{a_1}+k_{b_1})\,\delta^{(D)}(k_{a_1}+k_{b_1}-\ell)\,d^Dk_{a_2}\,d^Dk_{b_2}\,\delta^{(D)}(k_{a_2}+k_{a_1})\,\delta^{(D)}(k_{b_2}+k_{b_1}),\tag{4.2}$$

[7] In this paper we are considering that the D-dimensional momentum space is real, i.e. $k_i\in\mathbb{R}^{D-1,1}$. Therefore, the Dirac delta functions in (4.2) are well defined.

and

$$d\mu^{\rm t}_{n+4} := \frac{\prod_{A=1}^{n+4} d\sigma_A}{\text{Vol (PSL}(2,\mathbb{C}))} \times \frac{(\sigma_{1b_1}\,\sigma_{b_1b_2}\,\sigma_{b_21})}{\prod_{A\neq 1,b_1,b_2}^{n+4} E_A}$$
$$\xrightarrow{\text{fixing PSL}(2,\mathbb{C})} \frac{d\sigma_{a_1}}{E_{a_1}} \times \frac{d\sigma_{a_2}}{E_{a_2}} \times \prod_{i=2}^{n} \frac{d\sigma_i}{E_i} \times (\sigma_{1b_1}\,\sigma_{b_1b_2}\,\sigma_{b_21})^2. \qquad (4.3)$$

The Γ contour is defined by the massless scattering equations

$$E_A := \sum_{\substack{B=1\\ B\neq A}}^{n+4} \frac{k_A\cdot k_B}{\sigma_{AB}} = 0, \qquad A = 1,2,\ldots,n+4, \quad \text{with} \ \sum_{A=1}^{n+4} k_A = 0, \qquad (4.4)$$

where we are making the following identifications

$$n+1 \to a_1,\ \ n+2 \to a_2,\ \ n+3 \to b_1,\ \ n+4 \to b_2\,. \qquad (4.5)$$

Finally, without loss of generality, note that in (4.3) we have fixed $\{\sigma_1, \sigma_{b_1}, \sigma_{b_2}\}$ and $\{E_1, E_{b_1}, E_{b_2}\}$.

It is very important to remark that the proposal given in (4.1) is well defined. After performing the contour integral, $\int_\Gamma d\mu^{\rm t}_{n+4}$, the result obtained has a functional dependence of the loop momenta couples[8] like $(k_{a_1} + k_{b_1})$ and $(k_{a_2} + k_{b_2})$, therefore, the Dirac delta functions in $d\Omega$ can be carried out without any inconvenience.

As it has been argued in [47], since we are looking for the forward limit with the measure $\delta^{(D)}(k_{a_1} + k_{a_2}) \times \delta^{(D)}(k_{b_1} + k_{b_2})$ in (4.2), then it requires that when summing over the U(N) (and U$(\tilde N)$) degrees of freedom of the two internal particles they must be identified. Being more precise, we must introduce the sum

$$\sum_{i_{a_1},i_{a_2}=1}^{N^2} \delta_{i_{a_1}i_{a_2}} \times \sum_{i_{b_1},i_{b_2}=1}^{\tilde N^2} \delta_{i_{b_1}i_{b_2}}, \qquad (4.6)$$

where N^2 ($\tilde N^2$) is the dimension of U(N) (U$(\tilde N)$). Now, by using the identities[9]

$$\sum_{i_{a_1}=1}^{N^2} \text{Tr}(X\,T^{i_{a_1}}\,Y\,T^{i_{a_1}}) = \text{Tr}(X)\,\text{Tr}(Y), \qquad \sum_{i_{a_1}=1}^{N^2} \text{Tr}(X\,Y\,T^{i_{a_1}}\,T^{i_{a_1}}) = N\,\text{Tr}(XY)$$
$$\text{Tr}(T^{m_1}\,T^{m_2}\,\cdots\,T^{m_{p-1}}\,T^{m_p}) = (-1)^p\,\text{Tr}(T^{m_p}\,T^{m_{p-1}}\,\cdots\,T^{m_2}\,T^{m_1}), \qquad (4.7)$$

[8] It is guaranteed by the KK relations in (3.7) and the footnote 6. Since the partial non-planar one-loop Parke-Taylor factors can always be written as a linear combination of the partial planar one-loop Parke-Taylor factors then, as it was explained in the appendix A in [55], this implies the CHY-integral is always a function of the momenta $(k_{a_1} + k_{b_1})$ and $(k_{a_2} + k_{b_2})$.

[9] The second line in (4.7) is known as the reflection identity, which is simple to check for the adjoint representation.

the full amplitude, $\mathbf{m}_n^{1-\text{loop}}$, becomes

$$\begin{aligned}
\mathbf{m}_n^{1-\text{loop}} = 4\times\Bigg\{ & (N\tilde{N}) \sum_{\substack{\pi\in S_n/\mathbb{Z}_n \\ \rho\in S_n/\mathbb{Z}_n}} \text{Tr}(T^{i_{\pi_1}}\cdots T^{i_{\pi_n}})\times m_n^{(1-\text{P;P})}[\pi;\rho]\times \text{Tr}(\tilde{T}^{i_{\rho_1}}\cdots\tilde{T}^{i_{\rho_n}}) \\
& +(N)\sum_{p=1}^{n-1}\sum_{\pi\in S_n/\mathbb{Z}_n}\sum_{\substack{\gamma\in S_p/\mathbb{Z}_p \\ \delta\in S_{n-p}/\mathbb{Z}_{n-p}}} \text{Tr}(T^{i_{\pi_1}}\cdots T^{i_{\pi_n}})\times m_n^{(1-\text{P;NP})}[\pi;\gamma|\delta] \\
& \qquad\qquad\qquad\qquad \times \text{Tr}(\tilde{T}^{i_{\gamma_1}}\cdots\tilde{T}^{i_{\gamma_p}})\times\text{Tr}(\tilde{T}^{i_{\delta_{p+1}}}\cdots\tilde{T}^{i_{\delta_n}}) \\
& +(\tilde{N})\sum_{p=1}^{n-1}\sum_{\substack{\pi\in S_p/\mathbb{Z}_p \\ \rho\in S_{n-p}/\mathbb{Z}_{n-p}}}\sum_{\gamma\in S_n/\mathbb{Z}_n} \text{Tr}(T^{i_{\pi_1}}\cdots T^{i_{\pi_p}})\times\text{Tr}(T^{i_{\rho_{p+1}}}\cdots T^{i_{\rho_n}})\times m_n^{(1-\text{NP;P})}[\pi|\rho;\gamma] \\
& \qquad\qquad\qquad\qquad \times\text{Tr}(\tilde{T}^{i_{\gamma_1}}\cdots\tilde{T}^{i_{\gamma_n}}) \\
& +\sum_{\substack{p=1 \\ q=1}}^{n-1}\sum_{\substack{\pi\in S_p/\mathbb{Z}_p,\gamma\in S_q/\mathbb{Z}_q \\ \rho\in S_{n-p}/\mathbb{Z}_{n-p},\delta\in S_{n-q}/\mathbb{Z}_{n-q}}} \text{Tr}(T^{i_{\pi_1}}\cdots T^{i_{\pi_p}})\times\text{Tr}(T^{i_{\rho_{p+1}}}\cdots T^{i_{\rho_n}})\times m_n^{(1-\text{NP;NP})}[\pi|\rho;\gamma|\delta] \\
& \qquad\qquad\qquad\qquad \times\text{Tr}(\tilde{T}^{i_{\gamma_1}}\cdots\tilde{T}^{i_{\gamma_q}})\times\text{Tr}(\tilde{T}^{i_{\delta_{q+1}}}\cdots\tilde{T}^{i_{\delta_n}})\Bigg\}, \qquad (4.8)
\end{aligned}$$

where P(NP) means planar (non-planar), $m_n^{(1-\text{P;P})}[\pi;\rho]$ is the same amplitude defined in [55] as $\mathfrak{M}_n^{1-\text{loop}}[\pi|\rho]$, and the non-planar contributions, $m_n^{(1-\text{P;NP})}[\pi;\gamma|\delta]$ ($m_n^{(1-\text{NP;P})}[\pi|\rho;\gamma]$) and $m_n^{(1-\text{NP;NP})}[\pi|\rho;\gamma|\delta]$ are given by

$$\begin{aligned}
m_n^{(1-\text{P;NP})}[\pi_1,\ldots,\pi_n\,;\,\gamma_1,\ldots,\gamma_p|\delta_{p+1},\ldots,\delta_n] &:= \frac{1}{2^{n+1}}\int d^D\ell\int d\Omega\times s_{a_1b_1} \\
&\times\int d\mu_{n+4}^{\text{t}}\times\mathbf{I}^{\text{CHY}}_{(1-\text{P;NP})}[\pi_1,\ldots,\pi_n\,;\,\gamma_1,\ldots,\gamma_p|\delta_{p+1},\ldots,\delta_n], \qquad (4.9) \\
m_n^{(1-\text{NP;NP})}[\pi_1,\ldots,\pi_p|\rho_{p+1},\ldots,\rho_n\,;\,\gamma_1,\ldots,\gamma_q|\delta_{q+1},\ldots,\delta_n] &:= \frac{1}{2^{n+1}}\int d^D\ell\int d\Omega\times s_{a_1b_1} \\
&\times\int d\mu_{n+4}^{\text{t}}\times\mathbf{I}^{\text{CHY}}_{(1-\text{NP;NP})}[\pi_1,\ldots,\pi_p|\rho_{p+1},\ldots,\rho_n\,;\,\gamma_1,\ldots,\gamma_q|\delta_{q+1},\ldots,\delta_n], \qquad (4.10)
\end{aligned}$$

with

$$\begin{aligned}
&\mathbf{I}^{\text{CHY}}_{(1-\text{P;NP})}[\pi_1,\ldots,\pi_n\,;\,\gamma_1,\ldots,\gamma_p|\delta_{p+1},\ldots,\delta_n] := \frac{(a_1,a_2)\,(b_1,b_2)}{(a_1,b_1,b_2,a_2)^2} \qquad (4.11) \\
&\times\sum_{\alpha\in\text{cyc}(\pi)}\widehat{\text{PT}}[\alpha_1,\ldots,\alpha_n,a_1,a_2]\times\sum_{\substack{\xi\in\text{cyc}(\gamma)\\ \zeta\in\text{cyc}(\delta)}}\widehat{\text{PT}}[\xi_1,\ldots,\xi_q,b_1,\zeta_{q+1},\ldots,\zeta_n,b_2], \\
&\mathbf{I}^{\text{CHY}}_{(1-\text{NP;NP})}[\pi_1,\ldots,\pi_p|\rho_{p+1},\ldots,\rho_n\,;\,\gamma_1,\ldots,\gamma_q|\delta_{q+1},\ldots,\delta_n] := \frac{(a_1,a_2)\,(b_1,b_2)}{(a_1,b_1,b_2,a_2)^2} \qquad (4.12) \\
&\times\sum_{\substack{\alpha\in\text{cyc}(\pi)\\ \beta\in\text{cyc}(\rho)}}\widehat{\text{PT}}[\alpha_1,\ldots,\alpha_p,a_1,\beta_{p+1},\ldots,\beta_n,a_2]\times\sum_{\substack{\xi\in\text{cyc}(\gamma)\\ \zeta\in\text{cyc}(\delta)}}\widehat{\text{PT}}[\xi_1,\ldots,\xi_q,b_1,\zeta_{q+1},\ldots,\zeta_n,b_2].
\end{aligned}$$

Finally, the amplitude, $m_n^{(1-\text{NP;P})}[\pi|\rho\,;\,\gamma]$, is defined in a similar way to the one given in (4.9).

Since the bi-adjoint theory comes with double trace, we have mixed sectors which are not entirely non-planar. Our focus here will be in the definition of the integrands for such sectors and how they are related to the planar sector.

In the following sections we are going to focus on these non-planar amplitudes, we will give a few examples and define the *non-planar CHY-graphs*.

4.2 Non-planar bi-adjoint Φ^3 amplitudes at one-loop

In the previous section we have defined the non-planar Φ^3 amplitudes at one-loop. In [55], a technology to deal with this type CHY structures and graphs was developed, one of the ideas to remember from there is that by integrating a CHY integrand we can obtain several Feynman integrands and therefore all the information about the possible interactions at a certain order. For the planar case the relation CHY-graphs/Feynman-diagrams was mainly one-to-one, no on the non-planar case we will see that is not the case any more, but again a non-planar CHY diagram contains all the information for the interactions at that level. The new integrands that we have to deal with are $\mathbf{I}^{\rm CHY}_{(1-{\rm P};{\rm NP})}[\pi\,;\,\gamma|\delta]$ and $\mathbf{I}^{\rm CHY}_{(1-{\rm NP};{\rm NP})}[\pi|\rho\,;\,\gamma|\delta]$, so let us see how the look by using the Parke-Taylor factors we found in 3.

From the identity in footnote 6, it is straightforward to see

$$\begin{aligned}
&\mathbf{I}^{\rm CHY}_{(1-{\rm P};{\rm NP})}[\pi_1,\ldots,\pi_n\,;\gamma_1,\ldots,\gamma_p|\delta_{p+1},\ldots,\delta_n]\\
&\quad={\rm PT}^{(1)}_{a_1:a_2}[\pi_1,\ldots,\pi_n]\times{\rm PT}^{(1)}_{b_1:b_2}[\gamma_1,\ldots,\gamma_p\,|\,\delta_n,\delta_{n-1},\ldots,\delta_{p+1}], \qquad (4.13)\\
&\mathbf{I}^{\rm CHY}_{(1-{\rm NP};{\rm NP})}[\pi_1,\ldots,\pi_p|\rho_{p+1},\ldots,\rho_n\,;\gamma_1,\ldots,\gamma_q|\delta_{q+1},\ldots,\delta_n]\\
&\quad={\rm PT}^{(1)}_{a_1:a_2}[\pi_1,\ldots,\pi_p\,|\,\rho_n,\rho_{n-1},\ldots,\rho_{p+1}]\times{\rm PT}^{(1)}_{b_1:b_2}[\gamma_1,\ldots,\gamma_q\,|\,\delta_n,\delta_{n-1},\ldots,\delta_{q+1}]. \qquad (4.14)
\end{aligned}$$

So, for convenience we define:

Definition. *The non-planar partial amplitudes,* $\mathfrak{M}^{(1-{\rm P};{\rm NP})}_n[\pi;\gamma|\delta]$ *and* $\mathfrak{M}^{(1-{\rm NP};{\rm NP})}_n[\pi|\rho;\gamma|\delta]$, *for bi-adjoint* Φ^3 *scalar theory are defined as*

$$\begin{aligned}
&\mathfrak{M}^{(1-{\rm P};{\rm NP})}_n[\pi_1,\ldots,\pi_n\,;\,\gamma_1\ldots,\gamma_p\,|\,\delta_{p+1},\ldots,\delta_n]\\
&\quad:=m^{(1-{\rm P};{\rm NP})}_n[\pi_1,\ldots,\pi_n\ ;\ \gamma_1\ldots,\gamma_p\,|\,\delta_n,\ldots,\delta_{p+1}], \qquad (4.15)\\
&\mathfrak{M}^{(1-{\rm NP};{\rm NP})}_n[\pi_1,\ldots,\pi_p|\rho_{p+1},\ldots,\rho_n\,;\,\gamma_1\ldots,\gamma_q\,|\,\delta_{q+1},\ldots,\delta_n]\\
&\quad:=m^{(1-{\rm NP};{\rm NP})}_n[\pi_1,\ldots,\pi_p|\rho_n,\ldots,\rho_{p+1}\ ;\ \gamma_1\ldots,\gamma_q\,|\,\delta_n,\ldots,\delta_{q+1}]. \qquad (4.16)
\end{aligned}$$

In a similar way we define $\mathfrak{M}^{(1-{\rm NP};{\rm P})}_n[\pi|\rho\,;\,\gamma]$.

In the next section we will present some examples, which can be directly compared with the field theory results.

5 Examples and non-planar CHY-graphs

Let us now translate the integrands we just introduced into their corresponding CHY-graphs in order to get more information from them. In this section we compute some particular examples for lower number of points, $n=4,5$ by using the new proposal given in section 4.2. Moving forward, we will define and analyse the more general structure of the *non-planar CHY-graphs at one-loop* in a simple way, which will be presented in the next section.

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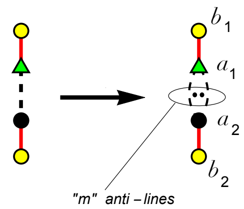
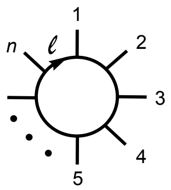
Before giving the examples, it is useful to introduce the *line* notation[10]



$$, \tag{5.1}$$

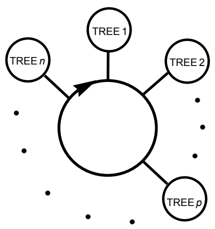
where the number of *anti-lines* "m" is equal to, $m = \#\, lines - 4$. Finally, in order to obtain a more compact notation, we bring in the following definitions

$$\overrightarrow{[1,2,\ldots,n]} :=$$

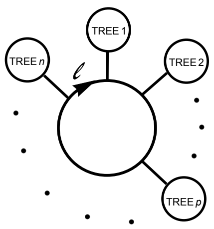


$$= \frac{1}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2\cdots(\ell+k_1+k_2+\cdots+k_{n-1})^2}\,, \tag{5.2}$$

and



$\equiv$



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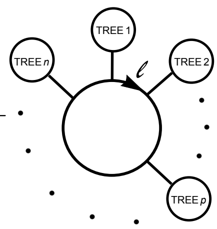
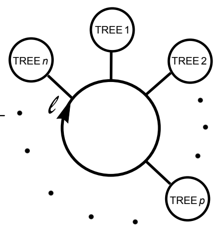


$+\cdots+$



$$= \overrightarrow{[\text{tree}\,1,\text{tree}\,2,\ldots,\text{tree}\,\text{n}]} + \overrightarrow{[\text{tree}\,2,\ldots,\text{tree}\,\text{n},\text{tree}\,1]} + \cdots + \overrightarrow{[\text{tree}\,\text{n},\text{tree}\,1,\ldots,\text{tree}\,(\text{n}-1)]}\,, \tag{5.3}$$

where the arrow over the bracket, $\overrightarrow{[\ldots]}$, means the transit of the loop momentum "ℓ" and "tree i" is a generic Feynman diagram at tree level. For simplicity in the notation we omit the integral, $\int d^D\ell$.

Finally, in order to obtain a correspondence between the CHY-graphs for non-planar bi-adjoint scalar theory and Feynman diagrams at one-loop, we follow the same procedure performed in [55], i.e. we will carry out a power expansion[11] of $\text{PT}^{(1)}_{a_1:a_2}[\cdots]$ in terms of $\omega^{a_1:a_2}_{i:j}$, while for $\text{PT}^{(1)}_{b_1:b_2}[\cdots]$ we will use its original definition.

5.1 Four-point

First of all, we analyse the simplest example, the four-point computation.[12]

Let us consider the NP; P contribution, which is given by the expression

$$\begin{aligned}\mathfrak{M}^{(1-\text{NP};\text{P})}_4[1,2|3,4\,;\,1,2,3,4] &= \frac{1}{2^{4+1}}\int d\Omega \times s_{a_1b_1} \\ &\times \int d\mu^{\text{t}}_{4+4}\;\; \text{PT}^{(1)}_{a_1:a_2}[1,2|3,4] \times \text{PT}^{(1)}_{b_1:b_2}[1,2,3,4].\end{aligned} \tag{5.4}$$

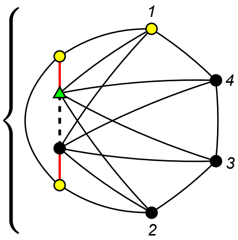
[10] This line is the same *square* defined in [55].

[11] Note that the lowest power of that expansion for the non-planar Parke-Taylor factors is four, it is a consequence of the *Theorem 1* in [55].

[12] In appendix B we carry out the four-point computation using the Feynman rules.

Since we have the identity, $\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4] = \frac{1}{(a_1,b_1,b_2,a_2)} \times \omega^{a_1:a_2}_{1:1}\omega^{a_1:a_2}_{2:2}\omega^{a_1:a_2}_{3:3}\omega^{a_1:a_2}_{4:4}$, which has been shown in [55], it is simple to draw the CHY-graphs

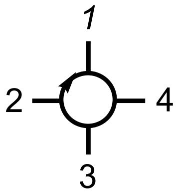
$$\mathfrak{M}^{(1-\mathrm{NP;P})}_4[1,2|3,4;1,2,3,4] = \frac{1}{2^5}\int d\Omega\, s_{a_1b_1}\int d\mu^{\mathrm{t}}_{4+4}\Bigg\{$$



$$+\mathrm{cyc}(1,2,3,4)\Bigg\}. \tag{5.5}$$

These kind of graphs were already computed in [55] and the answer is

$$\mathfrak{M}^{(1-\mathrm{NP;P})}_4[1,2|3,4\,;\,1,2,3,4] = \tag{5.6}$$

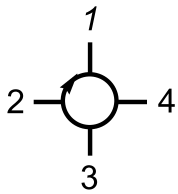


Now, we compute the NP; NP part of the four-point case. Let us consider the amplitude

$$\begin{aligned}
&\mathfrak{M}^{(1-\mathrm{NP;NP})}_4[1,2|3,4;1,2|3,4] \qquad (5.7)\\
&= \frac{1}{2^{4+1}}\int d\Omega\times s_{a_1b_1}\int d\mu^{\mathrm{t}}_{4+4}\,\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4]\times\mathrm{PT}^{(1)}_{b_1:b_2}[1,2|3,4]\\
&= \frac{1}{2^5}\int d\Omega\, s_{a_1b_1}\int d\mu^{\mathrm{t}}_{4+4}\,\frac{1}{(a_1,b_1,b_2,a_2)^2}\,(\omega^{a_1:a_2}_{1:1}\omega^{a_1:a_2}_{2:2}\omega^{a_1:a_2}_{3:3}\omega^{a_1:a_2}_{4:4})\times(\omega^{b_1:b_2}_{1:1}\omega^{b_1:b_2}_{2:2}\omega^{b_1:b_2}_{3:3}\omega^{b_1:b_2}_{4:4}),
\end{aligned}$$

where we have used the same identity as in the above example. This CHY-integral was computed by one of the authors in [54] and the result is

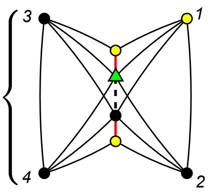
$$\mathfrak{M}^{(1-\mathrm{NP;NP})}_4[1,2|3,4\,;\,1,2|3,4] =$$



$$+\,\mathrm{per}(2,3,4)\,. \tag{5.8}$$

Next, in order to compare this example with the previous one and to obtain more information from it, we use the same procedure as in (5.5). Thus, on the a-sector we apply the identity, $\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4] = (a_1,b_1,b_2,a_2)^{-1}\times\omega^{a_1:a_2}_{1:1}\omega^{a_1:a_2}_{2:2}\omega^{a_1:a_2}_{3:3}\omega^{a_1:a_2}_{4:4}$, but, on the b-sector we use the definition given in (3.9), therefore we have the graph expansion

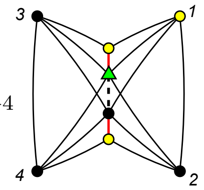
$$\mathfrak{M}^{(1-\mathrm{NP;NP})}_4[1,2\,|\,3,4;1,2\,|\,3,4] = \frac{1}{2^5}\int d\Omega\, s_{a_1b_1}\int d\mu^{\mathrm{t}}_{4+4}\Bigg\{$$

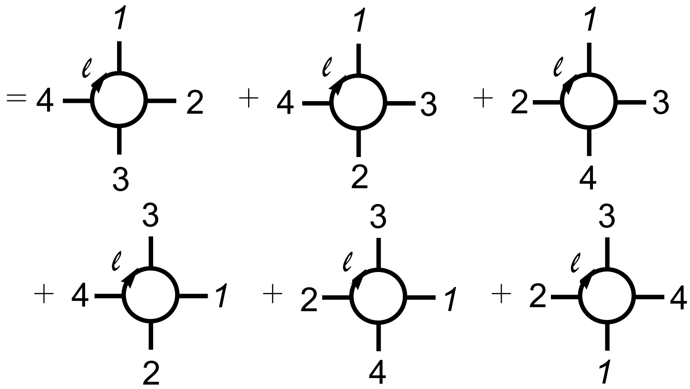


$$\begin{aligned}&+(1\leftrightarrow 2)\\ &+(3\leftrightarrow 4)\\ &+(1\leftrightarrow 2)\times(3\leftrightarrow 4)\end{aligned}\Bigg\}. \tag{5.9}$$

The CHY-graphs obtained above are a new type of graphs, which will be called[13] *the non-planar CHY-graphs (or butterfly graphs)*. Using the Λ-algorithm developed by one of the authors in [37], we can compute this butterfly graph in a simple way, and the result is (see appendix A)

$$\frac{1}{2^5}\int d\Omega \times s_{a_1 b_1}\int d\mu^{\mathrm{t}}_{4+4}$$





$$= \overrightarrow{[1,2,3,4]} + \overrightarrow{[1,3,2,4]} + \overrightarrow{[1,3,4,2]} + \overrightarrow{[3,1,2,4]} + \overrightarrow{[3,1,4,2]} + \overrightarrow{[3,4,1,2]}$$
$$= \overrightarrow{[1,2] \sqcup [3,4]}\,. \tag{5.10}$$

This interesting result is generalized in section 6. Certainly, by summing the four butterfly graphs from (5.9), one obtains the same answer as in (5.8).

Notice that we have not used the Parke-Taylor identities found in section 3.2 in the above examples, for instance

$$\begin{aligned}\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4] = {}& \mathrm{PT}^{(1)}_{a_1:a_2}[1,2,3,4] + \mathrm{PT}^{(1)}_{a_1:a_2}[1,3,2,4] + \mathrm{PT}^{(1)}_{a_1:a_2}[3,1,2,4] \\ & + \mathrm{PT}^{(1)}_{a_1:a_2}[2,1,3,4] + \mathrm{PT}^{(1)}_{a_1:a_2}[2,3,1,4] + \mathrm{PT}^{(1)}_{a_1:a_2}[3,2,1,4]\,.\end{aligned} \tag{5.11}$$

We could do that and then to apply the technology developed in [55]. However, this procedure is longer and tedious, we would also lose the correspondence between the non-planar CHY-graphs and Feynman diagrams at one-loop.

What the previous paragraph is actually telling us, is that we have found a nice way to embed the BDDK identities into a single expression, all this from the relations we found at the level of the Parke-Taylor factors in section 3. From the CHY approach, we can find all the interactions that are involved in the non-planar sector directly by solving the corresponding CHY integrand. Since the 4-point case is so simple and just showed us one type of interaction (the non-planar CHY-graph gave just boxes), let us take a look to a nontrivial example, the 5-point case, where more interactions shall appear.

[13]Note that the graph in (5.9) is like a double copy of the graph in (5.5) without the connection between σ_{b_1} and σ_{b_2}.

5.2 Five-point

In this section we will consider a less trivial example, the five-point case computations. This example is richer than the previous one, here we will see different interactions appearing in the non-planar sector. Diagrams with trees will be attached to the loop will appear now, but those tree diagrams should satisfy certain conditions imposed by the structure of the gauge group, all of this now embedded into the non-planar Parke-Taylor factors (this argument will become clear in the next section). The first contributing integrand we will take into consideration is the mixed non-planar-planar case, it is given by

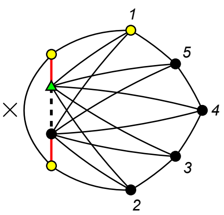
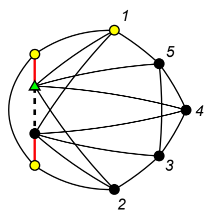
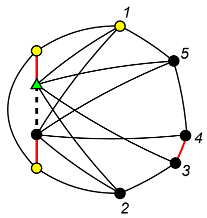
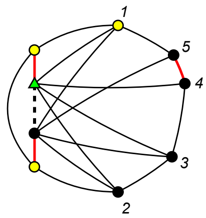
$$\begin{aligned}\mathfrak{M}_5^{(1-\text{NP};\text{P})}[1,2|3,4,5\,;\,1,2,3,4,5] &= \frac{1}{2^{5+1}}\int d\Omega \times s_{a_1 b_1} \\ &\times \int d\mu^{\text{t}}_{5+4}\ \text{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5] \times \text{PT}^{(1)}_{b_1:b_2}[1,2,3,4,5].\end{aligned} \tag{5.12}$$

From the results found in [55], it is straightforward to check the identity

$$\begin{aligned}\text{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5] = &\frac{1}{(a_1,b_1,b_2,a_2)} \times (\omega^{a_1:a_2}_{1:1}\omega^{a_1:a_2}_{2:2}) \times \Big(2\,\omega^{a_1:a_2}_{3:3}\omega^{a_1:a_2}_{4:4}\omega^{a_1:a_2}_{5:5} \\ &+ \frac{\sigma_{34}\,\sigma_{45}\,\omega^{a_1:a_2}_{5:3}}{(3,4,5)}\,\omega^{a_1:a_2}_{4:4} + \frac{\sigma_{45}\,\sigma_{53}\,\omega^{a_1:a_2}_{3:4}}{(3,4,5)}\,\omega^{a_1:a_2}_{5:5} + \frac{\sigma_{53}\,\sigma_{34}\,\omega^{a_1:a_2}_{4:5}}{(3,4,5)}\,\omega^{a_1:a_2}_{3:3}\Big),\end{aligned} \tag{5.13}$$

so, using the definition, $\text{PT}^{(1)}_{b_1:b_2}[1,2,3,4,5] = \frac{1}{(a_1,b_1,b_2,a_2)}\left[\omega^{b_1:b_2}_{1:2}\frac{1}{\sigma_{23}\sigma_{34}\sigma_{45}\sigma_{51}} + \text{cyc}(1,2,3,4,5)\right]$, we obtain the graph expansion

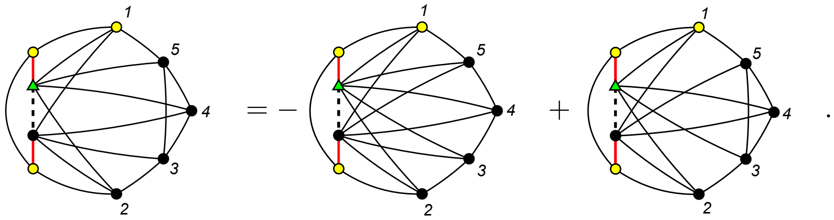
$$\mathfrak{M}_5^{(1-\text{NP};\text{P})}[1,2|3,4,5\,;\,1,2,3,4,5] = \frac{1}{2^6}\int d\Omega \times s_{a_1 b_1} \times \int d\mu^{\text{t}}_{5+4}$$

$\Bigg\{ 2\times$  $+$  $+$  $+$  $+\,\text{cyc}\Bigg\}$, (5.14)

here "cyc" stands for cyclic permutations of the labels, $(1,2,3,4,5)$, but by keeping the connection among them. Note that the graphs in (5.14) are totally similar to the ones obtained in [55], equation (5.3). In fact, the only new graph is the second one, which is a generalization of the graph in *proposition 3* of [55]. Following the same procedure that was applied there, we multiply this graph by the cross-ratio identity, $\mathbb{1} = -\sigma_{53}\,\omega^{a_1:a_2}_{3:5} + \frac{\sigma_{5a_1}\sigma_{3a_2}}{\sigma_{3a_1}\sigma_{5a_2}}$, therefore the second term in (5.13) becomes

$$\frac{\sigma_{34}\,\sigma_{45}\,\omega^{a_1:a_2}_{5:3}}{(3,4,5)}\,\omega^{a_1:a_2}_{4:4} \times \mathbb{1} = -\omega^{a_1:a_2}_{3:3}\,\omega^{a_1:a_2}_{4:4}\,\omega^{a_1:a_2}_{5:5} + \frac{\sigma_{34}\,\sigma_{45}\,\omega^{a_1:a_2}_{3:5}}{(3,4,5)}\,\omega^{a_1:a_2}_{4:4}, \tag{5.15}$$

and the graph turns into

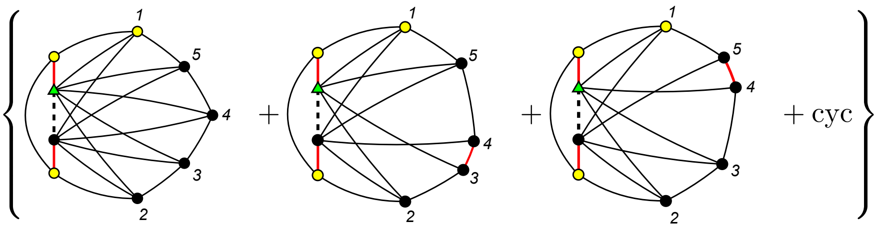


$$(5.16)$$

The second resulting graph is a generalization of the one given in *proposition 2* of [55]. This graph is simple to compute using the Λ-algorithm and it vanishes. So, the $\mathfrak{M}_5^{(1-\mathrm{NP;P})}$ amplitude can now be written as

$$\mathfrak{M}_5^{(1-\mathrm{NP;P})}[1,2|3,4,5\,;\,1,2,3,4,5] = \frac{1}{2^6}\int d\Omega \times s_{a_1 b_1} \times$$

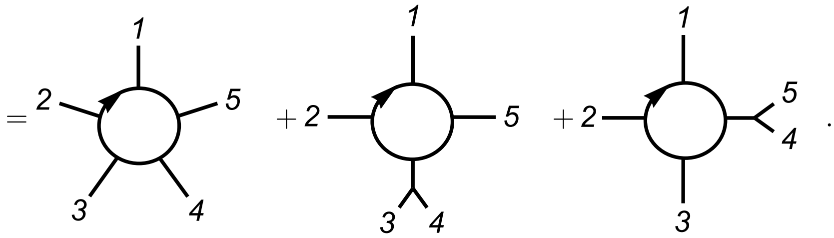
$$\int d\mu^{\mathrm{t}}_{5+4}$$



$$(5.17)$$

where all these graphs were already mapped to Feynman diagrams in [55]. Thus, the final answer is

$$\mathfrak{M}_5^{(1-\mathrm{NP;P})}[1,2|3,4,5\,;\,1,2,3,4,5]$$



$$(5.18)$$

Next, we compute the NP; NP contribution to the five-point case. Let us consider the amplitude

$$\begin{aligned}\mathfrak{M}_5^{(1-\mathrm{NP;NP})}[1,2|3,4,5\,;\,1,2|3,4,5] = \frac{1}{2^{5+1}}\int d\Omega \times s_{a_1 b_1} \\ \times \int d\mu^{\mathrm{t}}_{5+4}\,\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5] \times \mathrm{PT}^{(1)}_{b_1:b_2}[1,2|3,4,5]\,.\end{aligned} \tag{5.19}$$

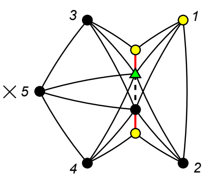
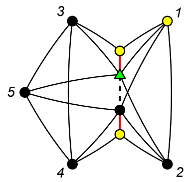
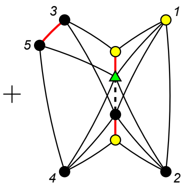
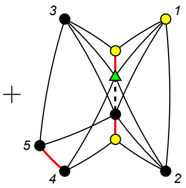
Such as it was done previously, we use the expansion given in (5.13) for $\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5]$

and the original definition for $\mathrm{PT}^{(1)}_{b_1:b_2}[1,2|3,4,5]$ written in (3.9), i.e.

$$
\begin{aligned}
\mathrm{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5] &= \frac{1}{(a_1,b_1,b_2,a_2)} \times \left(\omega^{a_1:a_2}_{1:1}\omega^{a_1:a_2}_{2:2}\right) \qquad (5.20)\\
&\times\left(2\,\omega^{a_1:a_2}_{3:3}\omega^{a_1:a_2}_{4:4}\omega^{a_1:a_2}_{5:5} + \frac{\sigma_{45}\,\sigma_{53}\,\omega^{a_1:a_2}_{3:4}}{(3,4,5)}\,\omega^{a_1:a_2}_{5:5} + \frac{\sigma_{34}\,\sigma_{45}\,\omega^{a_1:a_2}_{5:3}}{(3,4,5)}\,\omega^{a_1:a_2}_{4:4} + \frac{\sigma_{53}\,\sigma_{34}\,\omega^{a_1:a_2}_{4:5}}{(3,4,5)}\,\omega^{a_1:a_2}_{3:3}\right),\\
\mathrm{PT}^{(1)}_{b_1:b_2}[1,2|3,4,5] &= \frac{1}{(a_1,b_1,b_2,a_2)} \times \left(\frac{1}{\sigma_{21}}\,\omega^{b_1:b_2}_{1:2} + \frac{1}{\sigma_{12}}\,\omega^{b_1:b_2}_{2:1}\right)\\
&\times\left(\frac{1}{\sigma_{45}\sigma_{53}}\,\omega^{b_1:b_2}_{3:4} + \frac{1}{\sigma_{53}\sigma_{34}}\,\omega^{b_1:b_2}_{4:5} + \frac{1}{\sigma_{34}\sigma_{45}}\,\omega^{b_1:b_2}_{5:3}\right). \qquad (5.21)
\end{aligned}
$$

Thus, one obtains the graph expansion

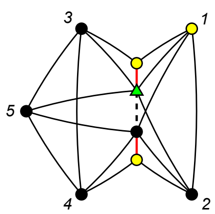
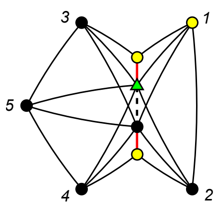
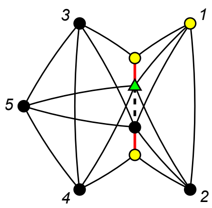
$$
\mathfrak{M}^{(1-\mathrm{NP;NP})}_5[1,2|3,4,5;1,2|3,4,5] = \frac{1}{2^6}\int d\Omega \times s_{a_1b_1} \times \int d\mu^{\mathrm{t}}_{5+4}
$$

$\Bigg\{$ 2 ×  +  +  +  $+(1\leftrightarrow 2)$ $+\mathrm{cyc}(3,4,5)$ $+(1\leftrightarrow 2)\times\mathrm{cyc}(3,4,5)$ $\Bigg\}$. (5.22)

The first, third and fourth CHY-graphs are a generalization of the butterfly graph that was found in the above section, equation (5.9). On the other hand, the second CHY-graph is a combination between the butterfly graph and the graph in *proposition 3* of [55]. So, by multiplying this graph by the cross-ratio identity, $\mathbb{1} = -\sigma_{34}\,\omega^{a_1:a_2}_{4:3} + \frac{\sigma_{3a_1}\sigma_{4a_2}}{\sigma_{4a_1}\sigma_{3a_2}}$, the second term in (5.20) becomes

$$
\frac{\sigma_{45}\,\sigma_{53}\,\omega^{a_1:a_2}_{3:4}}{(3,4,5)}\,\omega^{a_1:a_2}_{5:5} \times \mathbb{1} = -\omega^{a_1:a_2}_{3:3}\,\omega^{a_1:a_2}_{4:4}\,\omega^{a_1:a_2}_{5:5} + \frac{\sigma_{45}\,\sigma_{53}\,\omega^{a_1:a_2}_{4:3}}{(3,4,5)}\,\omega^{a_1:a_2}_{5:5}, \qquad (5.23)
$$

and the graph turns into

 $= -$  $+$  . (5.24)

Like it happened in the previous example, the second resulting graph is a mixing between the butterfly graph and the one given in *proposition 2* of [55]. The result for this graph is

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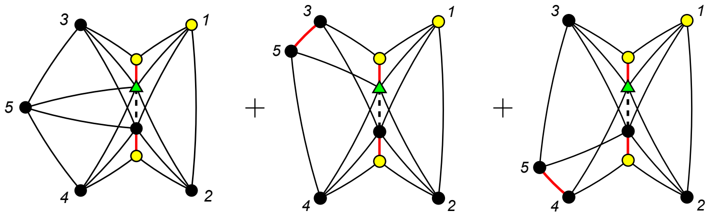
again zero. Therefore, we can now write the $\mathfrak{M}_5^{(1-\mathrm{NP;NP})}$ amplitude as

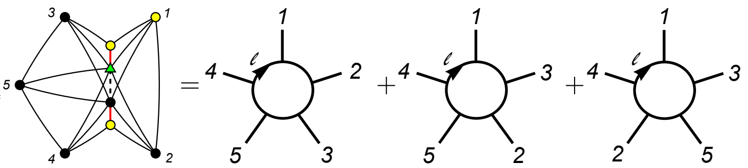
$$\mathfrak{M}_5^{(1-\mathrm{NP;NP})}[1,2|3,4,5\,;\,1,2|3,4,5] = \frac{1}{2^6}\int d\Omega \times s_{a_1 b_1} \times \int d\mu^{\mathrm{t}}_{5+4}$$



$$\left\{ \cdots + (1\leftrightarrow 2) + \mathrm{cyc}(3,4,5) + (1\leftrightarrow 2)\times \mathrm{cyc}(3,4,5) \right\}. \tag{5.25}$$

Certainly, we have been able to rewrite the $\mathfrak{M}_5^{(1-\mathrm{NP;NP})}$ amplitude just in terms of butterfly graphs (non-planar CHY-graphs), in the same way as it was made for $\mathfrak{M}_4^{(1-\mathrm{NP;NP})}$. The computation of these graphs is completely similar to the one performed in (5.10) which leads to

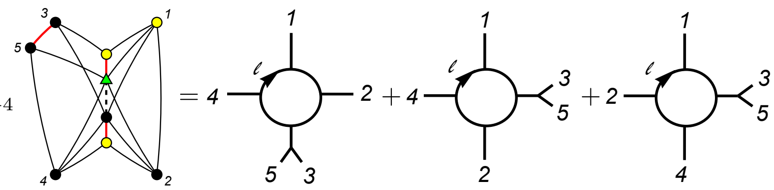
$$\frac{1}{2^6}\int d\Omega\, s_{a_1b_1}\int d\mu^{\mathrm{t}}_{5+4}$$



$$= \cdots + (7\,\text{terms})$$

$$= \overrightarrow{[1,2]\sqcup\!\sqcup[3,5,4]}\,, \tag{5.26}$$

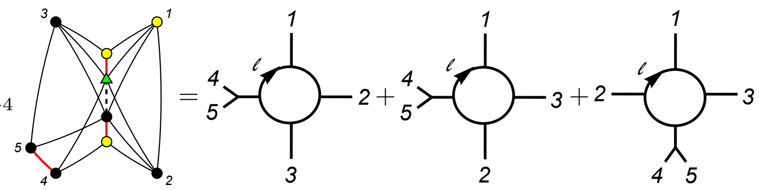
$$\frac{1}{2^6}\int d\Omega\, s_{a_1b_1}\int d\mu^{\mathrm{t}}_{5+4}$$



$$= \cdots + (3\,\text{terms})$$

$$= \overrightarrow{[1,2,\{3,5\},4]} + \overrightarrow{[1,\{3,5\},2,4]} + \overrightarrow{[1,\{3,5\},4,2]} + \cdots = \overrightarrow{[1,2]\sqcup\!\sqcup[\{3,5\},4]}\,, \tag{5.27}$$

$$\frac{1}{2^6}\int d\Omega\, s_{a_1b_1}\int d\mu^{\mathrm{t}}_{5+4}$$



$$= \cdots + (3\,\text{terms})$$

$$= \overrightarrow{[1,2,3,\{5,4\}]} + \overrightarrow{[1,3,2\{5,4\}]} + \overrightarrow{[1,3,\{5,4\},2]} + \cdots = \overrightarrow{[1,2]\sqcup\!\sqcup[3,\{5,4\}]}\,, \tag{5.28}$$

where we have denoted $\{3,5\}$, $\{5,4\}$ and $\{4,3\}$ the tree level sector. Therefore, the final answer is given by

$$\begin{aligned}\mathfrak{M}_5^{(1-\mathrm{NP;NP})}[1,2|3,4,5\,;\,1,2|3,4,5] = {}& \mathbf{S}[1,2|3,5,4] + \mathbf{S}[1,2|5,4,3] + \mathbf{S}[1,2|4,3,5]\\ & \mathbf{S}[2,1|3,5,4] + \mathbf{S}[2,1|5,4,3] + \mathbf{S}[2,1|4,3,5]\,,\end{aligned} \tag{5.29}$$

where we have defined $\mathbf{S}[a_1,a_2|a_3,a_4,a_5]$ as

$$\mathbf{S}[a_1,a_2|a_3,a_4,a_5] \equiv \overrightarrow{[a_1,a_2]\sqcup\!\sqcup[a_3,a_4,a_5]} + \overrightarrow{[a_1,a_2]\sqcup\!\sqcup[\{a_3,a_4\},a_5]} + \overrightarrow{[a_1,a_2]\sqcup\!\sqcup[a_3,\{a_4,a_5\}]}\,. \tag{5.30}$$

It is simple to check the total number of Feynman diagrams in (5.29) is,[14] 60 *pentagons* +72 *boxes* = 132, while in the CHY representation we only have 18 CHY-graphs (equation (5.25)).

Finally, looking at the results obtained in this section, it is interesting to note that the amplitudes, $\mathfrak{M}_4^{(1-\text{NP;P})}[1,2|3,4\,;\,1,2,3,4]$ and $\mathfrak{M}_5^{(1-\text{NP;P})}[1,2|3,4,5\,;\,1,2,3,4,5]$, can be found from the intersection[15]

$$\mathfrak{M}_4^{(1-\text{NP;P})}[1,2|3,4;1,2,3,4]=\mathfrak{M}_4^{(1-\text{NP;NP})}[1,2|3,4;1,2|3,4]\cap\mathfrak{M}_4^{(1-\text{P;P})}[1,2,3,4;1,2,3,4],$$
$$\mathfrak{M}_5^{(1-\text{NP;P})}[1,2|3,4,5;1,2,3,4,5]=\mathfrak{M}_5^{(1-\text{NP;NP})}[1,2|3,4,5;1,2|3,4,5]\cap\mathfrak{M}_5^{(1-\text{P;P})}[1,2,3,4,5;1,2,3,4,5],$$

such as it was done in the planar case, [46, 55]. Although we do not have a formal proof, there are evidences that the previous intersection relation could be applied to higher number of points, therefore we conjecture the following general relation, up to an overall sign

$$\mathfrak{M}_n^{(1-\text{NP;NP})}[\alpha|\beta\,;\,\gamma|\delta]=\mathfrak{M}_n^{(1-\text{NP;NP})}[\alpha|\beta\,;\,\alpha|\beta]\,\cap\,\mathfrak{M}_n^{(1-\text{NP;NP})}[\gamma|\delta\,;\,\gamma|\delta]\,. \tag{5.31}$$

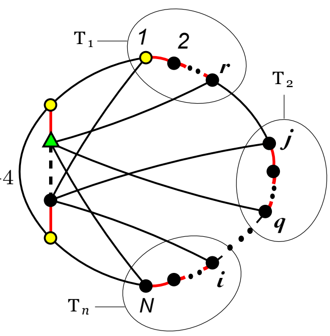
where the ordered lists, $\alpha=\{\alpha_1,\ldots,\alpha_i\}$, $\beta=\{\beta_1,\ldots,\beta_j\}$, $\gamma=\{\gamma_1,\ldots,\gamma_k\}$ and $\delta=\{\delta_1,\ldots,\delta_m\}$, satisfy the conditions, $\alpha\cap\beta=\gamma\cap\delta=\emptyset$ and $\alpha\cup\beta=\gamma\cup\delta=\{1,2,\ldots,n\}$.

In the following section we will generalize the previous results to an arbitrary number of points. Additionally, it is useful to remind that all computations were checked numerically.

6 General non-planar CHY-graphs

We have found a new type of CHY-graphs at one-loop, the *non-planar CHY-graphs* (or *butterfly graphs*). Those type of graphs could be identified as a generalization of the planar case obtained in [55]. In addition, let us recall that when the CHY-integrand associated to a planar CHY-graph is integrated, its result is a sum of Feynman diagrams at one-loop. This fact is summarized by the equality

$$\frac{1}{2^{N+1}}\int d\Omega\times s_{a_1b_1}\times\int d\mu^{\text{t}}_{N+4}$$



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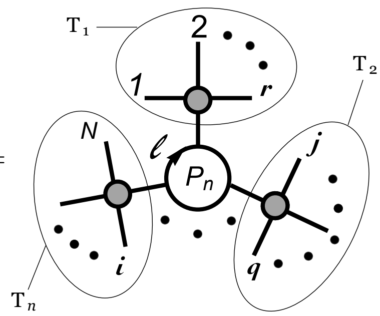


, (6.1)

[14] Under the equivalence relation given by the loop momentum shifting in (5.3), there are 12 nonequivalent *pentagons* and 18 nonequivalent *boxes*. In addition, note that these 12 nonequivalent *pentagons* are in perfect agreement with the expansion given in (3.13) for $\text{PT}^{(1)}_{a_1:a_2}[1,2|3,4,5]$.

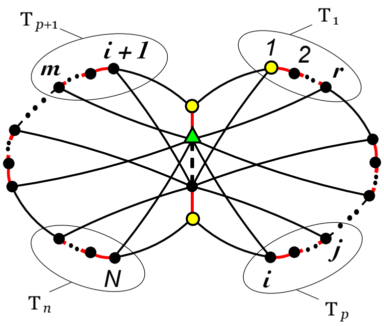
[15] This equality is given up to an overall sign.

where the grey circles mean the sum over all possible trivalent planar diagrams[16] (T_i), the symbol "P_n" in the loop circle denotes a regular polygon of n-edges and "N" is total number of particles.

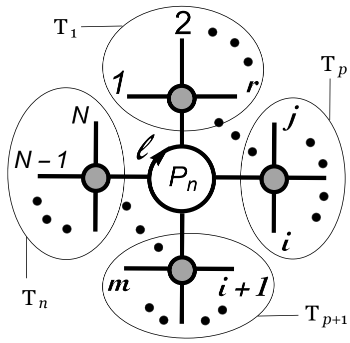
We can make an importan remark here. The tree diagrams attached to the loop satisfy two conditions: 1) they cannot have particles from the two orderings, and 2) one tree cannot contain all the particles belonging to one ordering. This is known from the structure of the gauge group, but here it appears from the Parke-Taylor factors. From the expression (3.9) we can apply an expansion in ω for each individual ordering, like we showed in [55], this will give us the contributing type of diagrams, effectively satisfying the previous conditions.

From the butterfly graphs obtained in (5.9) and (5.25), we generalize the planar CHY-graph in (6.1) to the non-planar case. Additionally, by using the Λ-algorithm, it is straightforward to compute this new kind of graph. Thus, a general butterfly graph and its result in terms of Feynman diagrams is given by the expression

$$\frac{1}{2^{N+1}}\int d\Omega\, s_{a_1b_1}\int d\mu^{\text{t}}_{N+4}$$



$$=$$



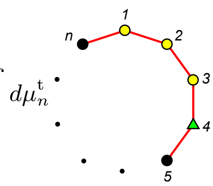
$$+\left\{\left[\binom{n}{p}-1\right]\text{terms}\right\},$$
$$=\overrightarrow{[T_1,\ldots,T_p]\sqcup[T_{p+1},\ldots,T_n]}\,, \tag{6.3}$$

where, as in (6.1), the grey circles mean the sum over all possible trivalent planar diagrams (T_i), "P_n" denotes a regular polygon of n-edges and "N" is the total number of particles.

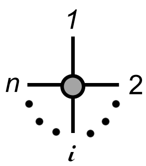
Finally, using the results found in [55] and the CHY-graphs representation obtained for $\mathfrak{M}_4^{1-\text{NP:NP}}[1,2|3,4\,;\,1,2|3,4]$ and $\mathfrak{M}_5^{1-\text{NP:NP}}[1,2|3,4,5\,;\,1,2|3,4,5]$ in (5.9) and (5.25) respectively, we formulate a general expression for $\mathfrak{M}_N^{1-\text{NP;NP}}[\alpha|\beta\,;\,\alpha|\beta]$. To be precise, up

[16] As it is very well known, at tree-level there is a map between the CHY-graphs and Feynman diagrams given by

$$\int d\mu^{\text{t}}_n$$



$$=$$



$$, \tag{6.2}$$

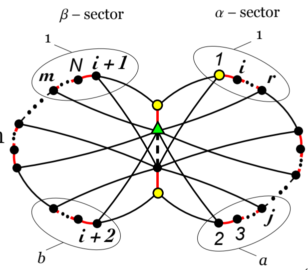
where the grey circle means the sum over all possible trivalent planar diagrams with the ordering $(1,2,\ldots,n)$.

to an overall sign, we propose the following expression

$$\mathfrak{M}_N^{1-\text{NP:NP}}[\alpha|\beta;\alpha|\beta] = \frac{1}{2^{N+1}} \int d\Omega\, s_{a_1 b_1} \int d\mu_{N+4}^{\text{tree}} \left[\sum_{\substack{a=2\\b=2}}^{|\alpha|,|\beta|} \sum_i \text{NPchy}^{a;b}_{(\alpha|\beta)}[[i]] + \text{cyc}(\alpha)\times\text{cyc}(\beta) \right], \tag{6.4}$$

where the ordered lists α and β are given by, $\alpha = \{1, 2, \ldots, p\}$ and $\beta = \{p+1, p+2, \ldots, N\}$, $|\alpha|$ and $|\beta|$ are the lengths of the lists, i.e. $|\alpha| = p$ and $|\beta| = N - p$, and we have defined the set, $\text{NPchy}^{a:b}_{(\alpha|\beta)}$, as

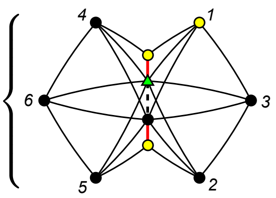
$$\text{NPchy}^{a:b}_{(\alpha|\beta)} := \Big\{ \text{All possible non-planar CHY graphs with the form}$$



$$\Big\}, \tag{6.5}$$

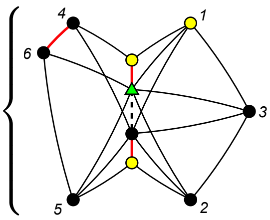
being $\text{NPchy}^{a:b}_{(\alpha|\beta)}[[i]]$ the i-th element in $\text{NPchy}^{a:b}_{(\alpha|\beta)}$. For instance

$$\text{NPchy}^{3:3}_{(1,2,3|4,5,6)} = \Big\{$$

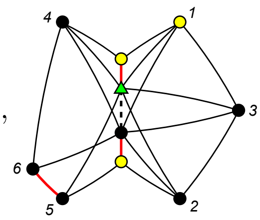


$$\Big\}, \tag{6.6}$$

$$\text{NPchy}^{3:2}_{(1,2,3|4,5,6)} = \Big\{$$



$$,$$



$$\Big\}. \tag{6.7}$$

This is clear that (6.4) is in agreement with (5.9) and (5.25).

We can now summarise the results obtained in this work in the following and final section.

7 Discussions

In this paper we continued the program to obtain one-loop quadratic Feynman propagators directly from CHY prescription, now into the non-planar sector of the amplitudes. Studying the one-loop Parke-Taylor factors presented in [55], we found the well known KK relations for amplitudes in momentum space, now in the context of functions of $\sigma - \textit{variables}$ in CHY formalism. These relations allowed a generalization, and from then on we found the BDDK relations at one-loop level, that leaded us also to obtain a generalization to the *multi-trace one-loop Parke-Taylor factors*, which as a special case have the double-trace one, called the *non-planar one-loop Parke-Taylor factors*.

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The BDDK relations have been traditionally used to obtain the subleading order contributions at one-loop in terms of leading ones. What we have done here, is exploit the BDDK relations and as a consequence of them obtain a remarkable result, the *non-planar CHY-graphs*. There is not an equivalent of these graphs in the traditional formalism and among the advantages they offer we have:

- The BDDK relations are already embedded on them, so we only need to integrate their corresponding integrand to obtain all the contributing interactions in the non-planar sector.
- The subleading order can now be written directly in terms of these graphs, since they encode all the information for this order in a fewer number of graphs, offering a more compact presentation in comparison to Feynman diagrams.
- Easier computation and generalization for higher number of points.
- Using intersections between graphs the amplitudes for mixed orderings can be obtained in a straightforward way.

We applied the non-planar CHY-graphs in the study of the bi-adjoint scalar theory at one-loop. We have found all there is to know, as far as we understand, for this theory at that loop order, at the level of Feynman integrands with on-shell external particles.

There are several directions to move on from the developments we have done in this paper. We are ready to apply all the technology we have developed in the study of the Yang-Mills theory and gravity at one-loop with quadratic Feynman propagators. Since the Parke-Taylor factors enter into the CHY integrand for Yang-Mills like

$$\mathcal{I}^{\rm YM}_{(1-{\rm P})}(1,2,\ldots,n) = {\rm PT}^{(1)}_{a_1:a_2}[1,2,\ldots,n] \times \sum_{\rho\in S_n} \mathbf{n}_{a_2,b_2|\rho_1\cdots\rho_n|b_1,a_1}\, \frac{\mathbf{pt}^{(1)}_{b_1:b_2}[\rho_1,\ldots,\rho_n]}{(a_1,b_1,b_2,a_2)}\,,$$

where $\mathbf{n}_{a_2,b_2|\rho_1\cdots\rho_n|b_1,a_1}$ are the BCJ numerators, which must be found (see [51, 52] for the linear representation), we can apply the developments of this work directly by replacing the planar Parke-Taylor factor by the non-planar one, obtaining the integrand for the non-planar sector of Yang-Mills like

$$\begin{aligned}&\mathcal{I}^{\rm YM}_{(1-{\rm NP})}(1,2,\ldots,i;i+1,\ldots,n)\\ &\quad= {\rm PT}^{(1)}_{a_1:a_2}[1,2,\ldots,i;i+1,\ldots,n] \times \sum_{\rho\in S_n} \mathbf{n}_{a_2,b_2|\rho_1\cdots\rho_n|b_1,a_1}\, \frac{\mathbf{pt}^{(1)}_{b_1:b_2}[\rho_1,\ldots,\rho_n]}{(a_1,b_1,b_2,a_2)}.\end{aligned}$$

This can be done in a straightforward manner, because the Pfaffian, $\sum_{\rho\in S_n} \mathbf{n}_{a_2,b_2|\rho_1\cdots\rho_n|b_1,a_1} \frac{\mathbf{pt}^{(1)}_{b_1:b_2}[\rho_1,\ldots,\rho_n]}{(a_1,b_1,b_2,a_2)}$, does not have anything to do with the ordering.

The BDDK relations we have found here play an important role in a version of the BCJ duality we are currently working on. The study of the BCJ duality have already been done for the case of linear propagators [52, 53], so it will be very interesting comparison to make.

Another important direction, is going to higher number of loops for the Parke-Taylor factors. There, we could study the relations that may exist among them, that is another work in progress. There is not much study done in this part at the moment.

Since we have found a way to encode all the one-loop information in a fewer number of CHY-graphs(integrands), it would be desirable to be able, or understand the viability, of performing the loop integration before the contour integration in the $\sigma - variable$, this could lead to a new and more compact way of calculating loop corrections, and could lead to a better understanding of the singularities in the non-planar sector, a topic not completely understood, except for some particular cases.

Acknowledgments

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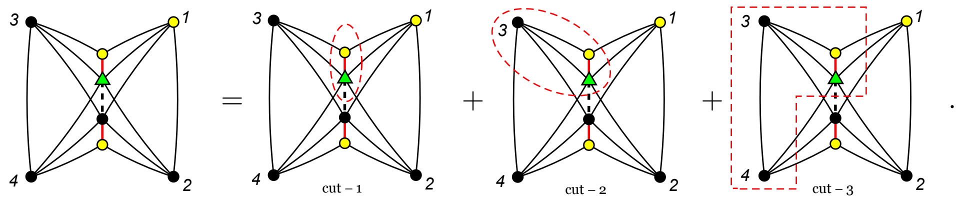
A Four-point non-planar CHY-graph computation

In this appendix we will work out an example using the Λ-algorithm in order to compute the non-planar CHY-graphs. Since that the Λ-algorithm is a graphical method we can omit the integral $\int d\mu^{\rm t}_{4+4}$.

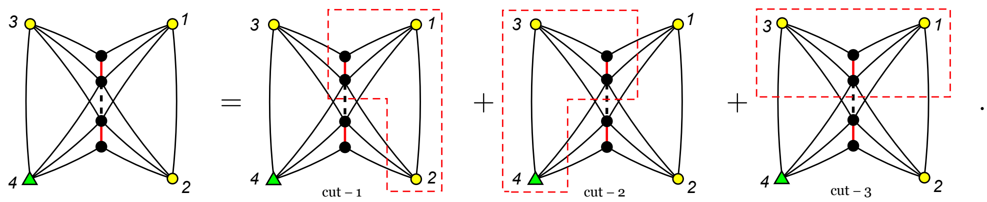
The simplest non-trivial example is the four-point case computation obtained in (5.9). Applying the Λ-rules presented in [37], we identify that there are three non-zero cuts given by



(A.1)

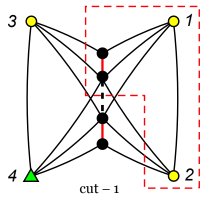
Note that in this paper we have used this gauge fixing for all graphs, which was very useful in the planar case [55]. However, for the non-planar case this gauge is not as efficient. Although it is not evident at first sight, the cuts in (A.1) generate non-trivial CHY-subgraphs with spurious poles. Thus, we are going to choose a new gauge fixing which produces CHY-subgraphs with only physical poles.

Let us fix the punctures $\{\sigma_1, \sigma_2, \sigma_3\}$ by PSL$(2, \mathbb{C})$, and the puncture $\{\sigma_4\}$ by scaling symmetry. Under this gauge fixing, the non-planar graph becomes

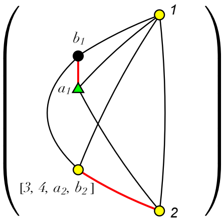


(A.2)

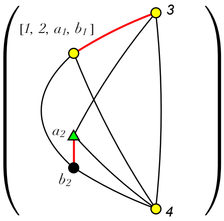
The above cuts are straightforward to be computed and give



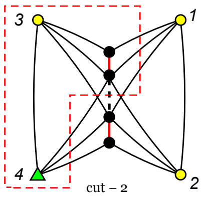
$$= \frac{2}{(k_{a_1}+k_{b_1}+k_1+k_2)^2} \times$$



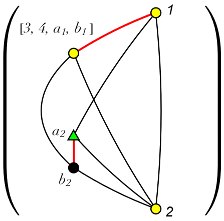
$$\times$$



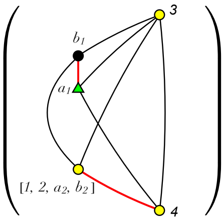
$$, \qquad \text{(A.3)}$$



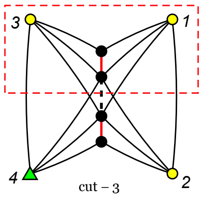
$$= \frac{2}{(k_{a_1}+k_{b_1}+k_3+k_4)^2} \times$$



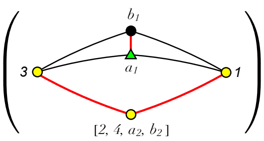
$$\times$$



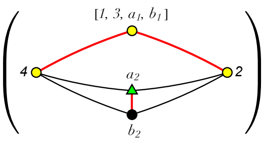
$$, \qquad \text{(A.4)}$$



$$= \frac{2}{(k_{a_1}+k_{b_1}+k_1+k_3)^2} \times$$



$$\times$$



$$. \qquad \text{(A.5)}$$

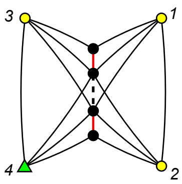
Note that the CHY-subgraphs obtained in (A.3) and (A.4) have the same structure as the one given in (6.1), which was computed in datail in [55]. So, the final answer for cut-1 and cut-2 is

$$\text{cut} - 1 = \frac{2^5}{s_{a_2b_2}\, s_{a_1b_1}\, s_{1a_1b_1}\, s_{12a_1b_1}\, s_{123a_1b_1}},$$
$$\text{cut} - 2 = \frac{2^5}{s_{a_2b_2}\, s_{a_1b_1}\, s_{3a_1b_1}\, s_{34a_1b_1}\, s_{341a_1b_1}}.$$

On the other hand, the subgraphs in (A.5) have the same form as the ones studied in [54]. The computation of these subgraphs is straightforward and the total result of the cut-3 is given by

$$\text{cut} - 3 = \frac{2^5}{s_{a_2b_2}\, s_{a_1b_1}\, s_{13a_1b_1}} \left(\frac{1}{s_{1a_1b_1}} + \frac{1}{s_{3a_1b_1}}\right) \times \left(\frac{1}{s_{132a_1b_1}} + \frac{1}{s_{134a_1b_1}}\right).$$

Adding the cuts, $(\text{cut} - 1) + (\text{cut} - 2) + (\text{cut} - 3)$, we obtain the final result for this four-point non-planar CHY-graph, namely



$$= \frac{2^5}{s_{a_2b_2}\, s_{a_1b_1}} \left(\frac{1}{s_{1a_1b_1}\, s_{12a_1b_1}\, s_{123a_1b_1}} + \frac{1}{s_{1a_1b_1}\, s_{13a_1b_1}\, s_{132a_1b_1}} + \frac{1}{s_{1a_1b_1}\, s_{13a_1b_1}\, s_{134a_1b_1}} \right.$$
$$\left. + \frac{1}{s_{3a_1b_1}\, s_{31a_1b_1}\, s_{312a_1b_1}} + \frac{1}{s_{3a_1b_1}\, s_{31a_1b_1}\, s_{314a_1b_1}} + \frac{1}{s_{3a_1b_1}\, s_{34a_1b_1}\, s_{341a_1b_1}}\right).$$

Therefore, by carrying out the integral, $\frac{1}{2^5}\int d\Omega\, s_{a_1b_1}$, it is trivial to check the previous expression becomes (5.10).

Finally, note that this gauge fixing can be applied to higher number of points.

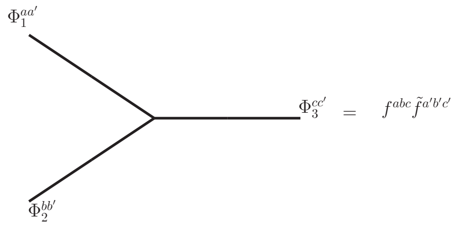


Figure 4. Standard Feynman three-vertex in the bi-adjoint theory.

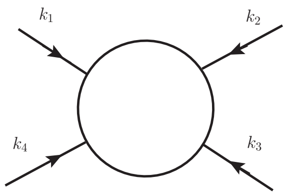


Figure 5. Irreducible contribution to the one-loop four-point amplitude.

B Four-point amplitude from the Feynman rules

In this appendix we calculate the four-point amplitude from the standard Feynman rules to compare with our final results based on the master formulas we obtained in previous sections. We have the following Lagrangian for the bi-adjoint scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial^\mu \Phi^{aa'} \partial_\mu \Phi^{aa'} + \frac{1}{3!} f^{abc} \tilde{f}^{a'b'c'} \Phi^{aa'} \Phi^{bb'} \Phi^{cc'} , \tag{B.1}$$

in which we have the following Lie algebra

$$[T^a, T^b] = i f^{abc} T^c \quad , \quad [\tilde{T}^a, \tilde{T}^b] = i \tilde{f}^{abc} \tilde{T}^c , \tag{B.2}$$

with two sets of generators $\{T^a\}$ and $\{\tilde{T}^a\}$ and their corresponding structure constants, f^{abc} and $\tilde{f}^{abc}$ [32]. In this theory we only have the following three-vertex to be used for constructing tree- and loop-level diagrams, see [4] for some examples in tree-level amplitudes. In this paper we are interested in one-loop amplitudes in bi-adjoint scalar field theory, especially its non-planar contributions. The first one-loop diagram with non-planar counterpart in this theory is the four-point amplitude which will be constructed using the above three-vertex to compare with the CHY results in section 5.1. After sewing four copies of the three-vertex in figure 4 we get the irreducible contribution to the four-point amplitude depicted in figure 5 which leads to the following representation

$$\begin{aligned}
&\mathcal{A}_{\mathrm{Irr}}[k_1, k_2, k_3, k_4] \\
&= \delta^D\left(\sum_{i=1}^{4} k_i\right) \sum_{b,c,d,e=1}^{N^2} f^{ca_1b}\, f^{ba_2d}\, f^{da_3e}\, f^{ea_4c} \sum_{b',c',d',e'=1}^{\tilde{N}^2} \tilde{f}^{c'a'_1b'}\, \tilde{f}^{b'a'_2d'}\, \tilde{f}^{d'a'_3e'}\, \tilde{f}^{e'a'_4c'} \\
&\quad \times \int \frac{d^Dl}{(2\pi)^D} \frac{1}{l^2[l+k_2]^2[l+k_2+k_3]^2[l-k_1]^2} .
\end{aligned} \tag{B.3}$$

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After using identities in (4.7) for the natural ordering of the external scalars in the loop (canonical ordering: 1234) we get

$$\begin{aligned}
&\sum_{b,c,d,e=1}^{N^2} f^{ca_1b}\, f^{ba_2d}\, f^{da_3e}\, f^{ea_4c} \\
&\quad= \sum_{b,c,d,e} \mathrm{tr}(T^c[T^{a_1},T^b])\,\mathrm{tr}(T^b[T^{a_2},T^d])\,\mathrm{tr}(T^d[T^{a_3},T^e])\,\mathrm{tr}(T^e[T^{a_4},T^c]) \\
&\quad= 2N\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + 2\mathrm{tr}(T^{a_1}T^{a_2})\mathrm{tr}(T^{a_3}T^{a_4}) \\
&\qquad + 2\mathrm{tr}(T^{a_1}T^{a_3})\mathrm{tr}(T^{a_2}T^{a_4}) + 2\mathrm{tr}(T^{a_1}T^{a_4})\mathrm{tr}(T^{a_2}T^{a_3})\,,
\end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
\sum_{b',c',d',e'=1}^{\tilde{N}^2} \tilde{f}^{c'a'_1b'}\, \tilde{f}^{b'a'_2d'}\, \tilde{f}^{d'a'_3e'}\, \tilde{f}^{e'a'_4c'} &= 2\tilde{N}\mathrm{tr}(\tilde{T}^{a'_1}\tilde{T}^{a'_2}\tilde{T}^{a'_3}\tilde{T}^{a'_4}) + 2\mathrm{tr}(\tilde{T}^{a'_1}\tilde{T}^{a'_2})\mathrm{tr}(\tilde{T}^{a'_3}\tilde{T}^{a'_4}) \\
&\quad + 2\mathrm{tr}(\tilde{T}^{a'_1}\tilde{T}^{a'_3})\mathrm{tr}(\tilde{T}^{a'_2}\tilde{T}^{a'_4}) + 2\mathrm{tr}(\tilde{T}^{a'_1}\tilde{T}^{a'_4})\mathrm{tr}(\tilde{T}^{a'_2}\tilde{T}^{a'_3})\,,
\end{aligned} \tag{B.5}$$

where N^2 ($\tilde{N}^2$) is the dimension of $\mathrm{U}(N)$ ($\mathrm{U}(\tilde{N})$). Now if we exclude the fully planar part of (B.3) the rest of the amplitude gives

$$\mathcal{A}'_{\mathrm{Irr}}[k_1,k_2,k_3,k_4] = \delta^D\Big(\sum_{i=1}^{4} k_i\Big)\mathbf{T}'^{1234}\int \frac{d^Dl}{(2\pi)^D}\frac{1}{l^2[l+k_2]^2[l+k_2+k_3]^2[l-k_1]^2}\,. \tag{B.6}$$

We define

$$\mathbf{T}'^{1234} = \mathbf{T}'^{1234-\mathrm{P;NP}} + \mathbf{T}'^{1234-\mathrm{NP;P}} + \mathbf{T}'^{1234-\mathrm{NP;NP}}\,, \tag{B.7}$$

in which

$$\begin{aligned}
\mathbf{T}^{1234-\mathrm{P;NP}} = 4N\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\Big\{&\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_2})\mathrm{tr}(\tilde{T}^{a_3}\tilde{T}^{a_4}) + \mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_3})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_4}) \\
&+ \mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_4})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_3})\Big\}\,,
\end{aligned} \tag{B.8}$$

represents the mixed planar-nonplanar (P; NP) part,

$$\begin{aligned}
\mathbf{T}^{1234-\mathrm{NP;P}} = 4\tilde{N}\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_2}\tilde{T}^{a_3}\tilde{T}^{a_4})\Big\{&\mathrm{tr}(T^{a_1}T^{a_2})\mathrm{tr}(T^{a_3}T^{a_4}) + \mathrm{tr}(T^{a_1}T^{a_3})\mathrm{tr}(T^{a_2}T^{a_4}) \\
&+ \mathrm{tr}(T^{a_1}T^{a_4})\mathrm{tr}(T^{a_2}T^{a_3})\Big\}\,,
\end{aligned} \tag{B.9}$$

for the mixed nonplanar-planar (NP; P) part and

$$\begin{aligned}
\mathbf{T}^{1234-\mathrm{NP;NP}} = 4&\Big\{\mathrm{tr}(T^{a_1}T^{a_2})\mathrm{tr}(T^{a_3}T^{a_4})+\mathrm{tr}(T^{a_1}T^{a_3})\mathrm{tr}(T^{a_2}T^{a_4})+\mathrm{tr}(T^{a_1}T^{a_4})\mathrm{tr}(T^{a_2}T^{a_3})\Big\} \\
&\times\Big\{\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_2})\mathrm{tr}(\tilde{T}^{a_3}\tilde{T}^{a_4})+\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_3})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_4})+\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_4})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_3})\Big\}\,,
\end{aligned} \tag{B.10}$$

for the nonplanar-nonplanar (NP; NP) contribution of the color structure. In total we have six inequivalent diagrams for the one-loop four-point amplitude with the following orderings

$$S_4/\mathbb{Z}_4 = \Big\{\{1234\},\{1243\},\{1324\},\{1342\},\{1423\},\{1432\}\Big\}\,, \tag{B.11}$$

which only the first ordering has been shown in (B.3). The full four-point amplitude (irreducible contribution) is the sum of all the orderings in (B.11), which can be written as ($\mathcal{A}'_{\rm Irr}[i,j,k,l] \equiv \mathcal{A}'_{\rm Irr}[k_i,k_j,k_k,k_l]$)

$$\begin{aligned}\mathcal{A}'_{\rm Irr} &= \mathcal{A}'_{\rm Irr}[1,2,3,4] + \mathcal{A}'_{\rm Irr}[1,2,4,3] + \mathcal{A}'_{\rm Irr}[1,3,2,4] \\ &\quad + \mathcal{A}'_{\rm Irr}[1,3,4,3] + \mathcal{A}'_{\rm Irr}[1,4,2,3] + \mathcal{A}'_{\rm Irr}[1,4,3,2]\,.\end{aligned} \tag{B.12}$$

Note that the $\mathbf{T}^{1234-{\rm NP;NP}}$ is invariant under exchanging any two external scalars, which indicates its appearance for all six orderings.

In the following we present the CHY results from (4.8) for this amplitude to compare with the results from the Feynman rules in (B.12). $\mathbf{m}_4^{1-\rm loop}$ in (4.8) has two parts, one for the color decomposition and one for the momentum integral. If we exclude the planar contribution (which has been discussed in [55]) in $\mathbf{m}_4^{1-\rm loop}$ we have:

$$\begin{aligned} I^{(1-{\rm P;NP})} &= 4(N)\sum_{p=1}^{3}\sum_{\pi\in S_4/\mathbb{Z}_4}\sum_{\substack{\gamma\in S_p/\mathbb{Z}_p\\ \delta\in S_{4-p}/\mathbb{Z}_{4-p}}} {\rm Tr}(T^{i_{\pi_1}}\cdots T^{i_{\pi_4}})\times m_4^{(1-{\rm P;NP})}[\pi\,;\gamma|\delta] \\ &\qquad\qquad \times{\rm Tr}(\tilde{T}^{i_{\gamma_1}}\cdots\tilde{T}^{i_{\gamma_p}})\times{\rm Tr}(\tilde{T}^{i_{\delta_{p+1}}}\cdots\tilde{T}^{i_{\delta_4}}) \\ &= 4N{\rm tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\times\Big\{{\rm Tr}(\tilde{T}^{a_1}\tilde{T}^{a_2})\times{\rm Tr}(\tilde{T}^{a_3}\tilde{T}^{a_4})m_4^{(1-{\rm P;NP})}[1,2,3,4;\,1,2|3,4] \\ &\qquad +{\rm Tr}(\tilde{T}^{a_1}\tilde{T}^{a_3})\times{\rm Tr}(\tilde{T}^{a_2}\tilde{T}^{a_4})m_4^{(1-{\rm P;NP})}[1,2,3,4;\,1,3|2,4] \\ &\qquad +{\rm Tr}(\tilde{T}^{a_1}\tilde{T}^{a_4})\times{\rm Tr}(\tilde{T}^{a_2}\tilde{T}^{a_3})m_4^{(1-{\rm P;NP})}[1,2,3,4;\,1,4|2,3]\Big\} \\ &\quad +\text{five more permutations}\,, \end{aligned} \tag{B.13}$$

which coincides with the color structure of (B.6) after replacing $\mathbf{T}'^{1234}$ with $\mathbf{T}'^{1234-{\rm P;NP}}$ and considering other permutations in (B.12). Now the loop integral appearing in (5.6) (after some momentum shifts) corresponds to what we have in (B.6). We also used the fact that

$$\mathfrak{M}_4^{(1-{\rm P;NP})}[1,2,3,4;1,2|3,4] \equiv \mathfrak{M}_4^{(1-{\rm P;NP})}[1,2,3,4;1,3|2,4] \equiv \mathfrak{M}_4^{(1-{\rm P;NP})}[1,2,3,4;1,4|2,3]\,. \tag{B.14}$$

The same discussion holds for the NP; P part.

Now, the most nontrivial part is the NP;NP contribution. For this piece of the amplitude from (4.8) we get

$$
\begin{aligned}
I^{(1-\mathrm{NP};\mathrm{NP})} = 4\sum_{\substack{p=1\\q=1}}^{3} \sum_{\substack{\pi\in S_p/\mathbb{Z}_p,\,\gamma\in S_q/\mathbb{Z}_q\\ \rho\in S_{4-p}/\mathbb{Z}_{4-p},\,\delta\in S_{4-q}/\mathbb{Z}_{4-q}}} & \mathrm{Tr}(T^{i_{\pi_1}}\cdots T^{i_{\pi_p}})\times\mathrm{Tr}(T^{i_{\rho_{p+1}}}\cdots T^{i_{\rho_4}})\times m_4^{(1-\mathrm{NP};\mathrm{NP})}[\pi|\rho;\gamma|\delta] \\
& \times\mathrm{Tr}(\tilde{T}^{i_{\gamma_1}}\cdots\tilde{T}^{i_{\gamma_q}})\times\mathrm{Tr}(\tilde{T}^{i_{\delta_{q+1}}}\cdots\tilde{T}^{i_{\delta_4}}) \\
= 4\mathrm{tr}(T^{a_1}T^{a_2})\mathrm{tr}(T^{a_3}T^{a_4})\Big\{ & m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,2|3,4;1,2|3,4]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_2})\mathrm{tr}(\tilde{T}^{a_3}\tilde{T}^{a_4}) \\
& + m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,2|3,4;13|24]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_3})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_4}) \\
& + m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,2|3,4;14|23]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_4})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_3})\Big\} \\
+4\mathrm{tr}(T^{a_1}T^{a_3})\mathrm{tr}(T^{a_2}T^{a_4})\Big\{ & m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,3|2,4;1,2|3,4]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_2})\mathrm{tr}(\tilde{T}^{a_3}\tilde{T}^{a_4}) \\
& + m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,3|2,4;13|24]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_3})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_4}) \\
& + m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,3|2,4;14|23]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_4})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_3})\Big\} \\
+4\mathrm{tr}(T^{a_1}T^{a_4})\mathrm{tr}(T^{a_2}T^{a_3})\Big\{ & m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,4|2,3;1,2|3,4]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_2})\mathrm{tr}(\tilde{T}^{a_3}\tilde{T}^{a_4}) \\
& + m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,4|2,3;13|24]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_3})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_4}) \\
& + m_4^{(1-\mathrm{NP};\mathrm{NP})}[1,4|2,3;14|23]\mathrm{tr}(\tilde{T}^{a_1}\tilde{T}^{a_4})\mathrm{tr}(\tilde{T}^{a_2}\tilde{T}^{a_3})\Big\}. \quad \text{(B.15)}
\end{aligned}
$$

We have already calculated the momentum integrals appearing here in (5.8), it contains all the six inequivalent momentum integrals we have for the four-point amplitude, which means all $m_4^{(1-\mathrm{NP};\mathrm{NP})}[\cdots]$ in (B.15) are equivalent. After factorizing the momentum integral, (B.15) is exactly what we have for the NP;NP contribution from the standard computation in $\mathbf{T}^{1234-\mathrm{NP};\mathrm{NP}}$ after considering all permutations and momentum integrals in (B.12).

To summarize, in this appendix we have shown the exact equivalence for the non-planar contribution to the four-point amplitude between our formula in the CHY side (4.8) and the one from the standard method based on the Feynman rules for the bi-adjoint scalar field theory in (B.12). The same discussion applies to higher order amplitudes at one-loop level.

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