Rapid Communications

Signatures of many-body localization in steady states of open quantum systems

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Many-body localization (MBL) is a result of the balance between interference-based Anderson localization and many-body interactions in a high-dimensional Fock space. It is expected that dissipation is blurring interference and destroying that balance so that the asymptotic state of a system with an MBL Hamiltonian does not bear localization signatures. This is evidently true in the case of local dephasing which drives any system into an infinite-temperature state. We demonstrate, by using a set of dissipative operators, where each one is acting nontrivially on a pair of neighboring sites (or spins), that an MBL system can be brought into a Hamiltonian-specific steady state. The difference between ergodic and MBL Hamiltonians can be seen in statistics of imbalance, entanglement entropy, and level spacing of the steady-state density operator. By introducing pairwise dissipative operators into an MBL system already exposed to dephasing, these localization signatures can be restored.

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Many-body localization (MBL) is an extension of Anderson localization [1] into the world of many-body systems [2,3]. There is a spectrum of definitions/quantifiers of this multi-faceted phenomenon aimed to highlight peculiar properties of MBL systems, e.g., the absence of conductivity [3] (even in the infinite temperature limit [2]), slow logarithmic growth of the entanglement entropy after an interaction quench [4–7], the existence of an extensive set of local integrals of motion [8], and specific spectral properties of MBL Hamiltonians [9,10]. There is a class of quantifiers which address the properties of a single (eigen)state of an MBL system such as short-range correlations [11], low entanglement entropy [12–14], and large fluctuations of local observables [15].

One of the important questions (especially in the context of recent experiments [16,17]), concerns the impact of the interaction with an environment on the states of MBL systems on large timescales. This question has been addressed recently in a series of papers [18–20], where the action of the environment was modeled with a set of local dephasing operators in the framework of Lindblad master equation. The ultimate fate of the systems is plain: dephasing, however small, grinds *any* system—with or without many-body interactions, governed by an MBL or ergodic Hamiltonian—into an infinite temperature state [21]. The asymptotic state cannot be changed by modifying the system Hamiltonian since the identity (which is the density operator corresponding to the infinite temperature state) commutes with any Hamiltonian.

In this Rapid Communication we show that, by introducing a special (although physically relevant) type of dissipation into a system already subjected to dephasing, we can drive the system into a new asymptotic state which bears detectable signatures of localization (or of its absence, depending on the system Hamiltonian). These signatures can be revealed by analyzing the population imbalance [18–20] (a quantity measured in experiments [16,22]), the operator space entanglement entropy [20,23], and the mean spectrum gap ratio [9] of the steady-state density operator. We do it in two steps. First, we demonstrate that the proposed dissipation, when acting alone, can sculpt an asymptotic state with localization features. Next, we show that these features are robust to the action of dephasing.

Model. We study a conventional MBL model, an open-ended chain of N (an even number) sites occupied by $N/2$ spinless fermions. The fermions are subjected to a random on-site potential $h_l, l = 1,\dots,N$ and interact when located on neighboring sites. The model Hamiltonian has the form

$$H = -J\sum_{l=1}^{N}(c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l) + U\sum_{l=1}^{N} n_l n_{l+1} + \sum_{l=1}^{N} h_l n_l, \tag{1}$$

where $c_l^\dagger$ (c_l) creates (annihilates) a fermion at site l, and $n_l = c_l^\dagger c_l$ is the local particle number operator. Values h_l are drawn from a uniform distribution on the interval $[-h,h]$. For $J = U = 1$ (our choice here) this system undergoes a many-body localization transition when $h > h_{\rm MBL} \backsimeq 3.6$ [11].

The dissipation is captured with a master equation [24],

$$\dot{\varrho}(t) = \mathcal{L}\varrho(t) = \mathcal{L}_H\varrho(t) + \mathcal{L}_{\rm dis}\varrho(t) = -i[H,\varrho(t)] + \sum_{s=1}^{M}\gamma_s\left[A_s\varrho(t)A_s^\dagger - \frac{1}{2}\{A_s^\dagger A_s,\varrho(t)\}\right], \tag{2}$$

where $\varrho(t)$ is the system density operator, and A_s is the jump operator mimicking the sth dissipative channel through which the environment acts on the system with a rate γ_s, and M is the total number of the channels. We are interested in the asymptotic (steady-state) density operator $\varrho_\infty = \lim_{t\to\infty}\varrho(t)$.

For Hermitian dissipative operators (dissipators) $A_s^\dagger = A_s$, we have $\varrho_\infty = \mathbb{1}/L$, where $\mathbb{1}$ is the identity operator in the half-filling subspace and $L = N!/[(N/2)!]^2$ is the number of accessible states. This is the case of local dephasing, $A_l = c_l^\dagger c_l$, $l = 1,...,N$ [25], considered in Refs. [18–20]. On the other hand, formally one could construct a non-Hermitian operator A^i such that $A^i|\phi_i\rangle = 0$, where $|\phi_i\rangle$ is the ith eigenstate of the Hamiltonian H. Then the asymptotic state is $\varrho_\infty = |\phi_i\rangle\langle\phi_i|$ [26]. However, such dissipators are highly nonlocal and too disorder specific to be practically relevant.

We choose dissipative operators which act on a pair of neighboring sites [26],

$$A_l = (c_l^\dagger + c_{l+1}^\dagger)(c_l - c_{l+1}), \quad \forall \gamma_l = \gamma_p. \tag{3}$$

This non-Hermitian operator tries to synchronize the dynamics on two sites by constantly recycling the antisymmetric out-of-phase mode into the symmetric in-phase one. A possible experimental realization of this dissipation, with an array of superconducting microwave resonators, is discussed in Ref. [27].

As we want to consider the situation when both types of dissipation are affecting the system dynamics, the dissipative part of the generator in Eq. (2) includes N dephasing operators $c_l^\dagger c_l$, $l = 1,...,N$, and $N-1$ pairwise dissipators, acting with rates γ_d and γ_p, respectively. The total number of dissipative channels is therefore $M = 2N - 1$.

We start the analysis from the limit $\gamma_d = 0$ and $\gamma_p = 0.1$. By using the Jordan-Wigner transformation, the system, Eqs. (1)–(3), can be mapped onto a model of N spins confined to the manifold $S^z = \sum_{l=1}^N s_l^z = 0$ [28]. This relation allows us to implement the time-evolving block decimation (TEBD) scheme generalized to matrix product operators [29] and propagate the model system with $N > 10$ to its steady state [30]. As the initial state we use $\varrho(0) = |\psi_0\rangle\langle\psi_0|, |\psi_0\rangle = |1010\ldots10\rangle$. For small systems, $N \leqslant 10$, we find asymptotic density operator as a kernel of the Lindblad generator, $\mathcal{L}\varrho_\infty = 0$.

Imbalance. The imbalance is defined as

$$\mathcal{I}(t) = \frac{N_o(t) - N_e(t)}{N/2}, \tag{4}$$

where N_o (N_e) is the number of fermions on odd (even) sites. This characteristic was measured in the recent experiments [16,22].

In the case of dephasing-driven dynamics, the asymptotic imbalance $\mathcal{I} = \lim_{t\to\infty} \mathcal{I}(t)$ is uniformly zero. When the dissipation is non-Hermitian, the steady state is disorder specific and asymptotic imbalance is a real-valued random variable. We sample its probability density function (pdf) $P(\mathcal{I})$ for different system sizes and disorder strengths (see Fig. 1). The imbalance can be considered as a sum of $N/2$ random variables, $\xi_l = n_{2l-1} - n_{2l}$, $l = 1, \ldots, N/2$, where n_s is the occupation of the sth site. Because these variables are correlated, their sums are not subjected to the central limit theorem (CLT) [31]. We check a scaling hypothesis $N^{-\beta_h} P(N^{\beta_h}\mathcal{I}[N])$, with the exponent β_h being a function of disorder ($\beta_h = 1/2$ will correspond to the CLT case). Exponent values can be estimated by calculating the variance of the pdf's for different N and then fitting the obtained dependence with the power law $N^{2\beta_h}$ [see inset in Fig. 1(a)]. We find $\beta_h \simeq 0.55$ for the ergodic regime and $\beta_h \approx 0.8$ for $h = 10,20$. To get some insight, we consider a particular realization $\{n_1, n_2, \ldots, n_N\}$ as a result of uniform sampling from a set of N independent and identically distributed random variables constrained by preservation of the total sum, $\sum_{s=1}^N n_s = N/2$, and condition $\forall n_s < 1$ ("no more than one particle per site") [30]. The result of such sampling for $N = 16$ is in good agreement with the imbalance pdf obtained for the ergodic regime [see dashed line in Fig. 1(a)].

Operator-space entanglement entropy (OSEE). This quantity was introduced by Prosen and Pižorn [23] as an operator generalization of the spatial entanglement entropy (defined for pure states). To calculate the OSEE, one should split the chain into two (equal in our case) parts and calculate the Schmidt decomposition of the density operator, $\varrho = \sum_k \sqrt{\mu_k} C_k \otimes D_k$, where the operators C_k (D_k) act nontrivially on the left (right) half only and form a complete Hilbert-Schmidt basis in the corresponding subspace. The normalized coefficients $\bar{\mu}_k$ define the entropy value $S^\natural = -\sum_k \bar{\mu}_k \log_2 \bar{\mu}_k$. When the state is pure, $S^\natural$ is twice the standard entanglement entropy [32].

The OSEE is a practically relevant characteristic: Small entropy of a state means low complexity of the matrix product

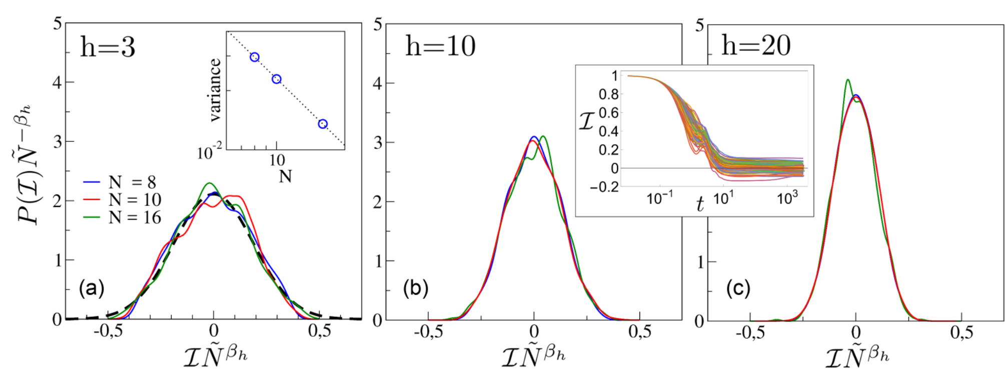

FIG. 1. Probability density function $P(\mathcal{I})$ of the steady-state imbalance $\mathcal{I}$ for different disorder strengths and system sizes. Dashed thick line on panel (a) is the distribution sampled with a constrained random partition for $N = 16$ (see text). Distributions are scaled with $\tilde{N}^{\beta_h}$, where $\tilde{N} = N/8$ and exponent β_h has values 0.55 for $h = 0.3$ (a) and 0.8 for $h = 10,20$ [(b),(c)]. Insets: (a) scaling of the distribution variance with N for $h = 3$ (dashed line is the power law $N^{2\beta_h}$) and (b), (c) the time evolution of the imbalance for 10^2 disorder realizations, $h = 20$ and $N = 32$, obtained with the TEBD propagation (not used for the histograms) [29]. The parameters are $\gamma_p = 0.1$, $U = J = 1$. Number K of realizations is 10^4 ($N = 8$) and 4×10^3 ($N = 10,16$).

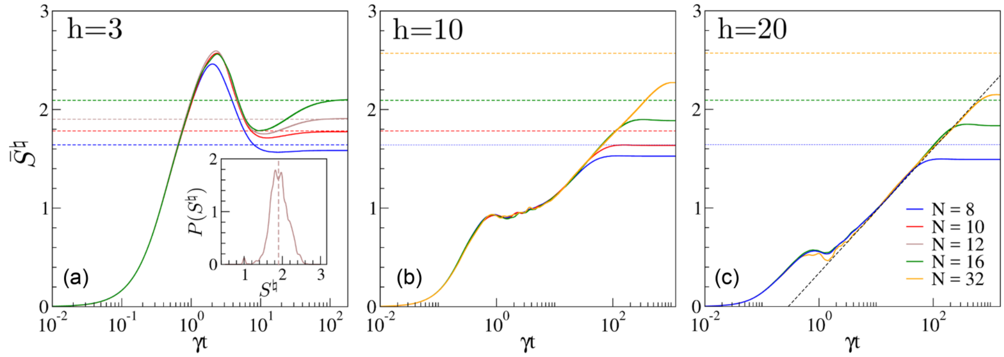

FIG. 2. Averaged operator-space entanglement entropy $\bar{S}^\natural(t)$ of the density operator $\varrho(t)$ as a function of time. Dashed lines are the values of the entropy for the infinite-temperature state. The dotted line on panel (c) is $\frac{1}{5}\log_2(t) + \text{const}$. Inset: The probability density function of the entropy of individual disorder realizations, for $h = 3$ and $N = 12$. The initial condition is $\varrho(0) = |\psi_0\rangle\langle\psi_0|$, $|\psi_0\rangle = |1010\ldots10\rangle$. Other parameters are as in Fig. 1.

operator representation of the corresponding density operator [32]. For example, the entropy is smaller for the infinite temperature state than for a pure state of high entanglement, which is directly opposite in the case of the von Neumann entropy. Therefore, we consider the operator entropy a better choice [20] to characterize ϱ_∞ than the von Neumann entropy [18].

We find that, in the ergodic phase, the averaged (over the disorder) OSEE $\bar{S}^\natural(t)$ saturates to $S^\natural(\mathbb{1})$ [Fig. 2(a)]. This implies an *effective* thermalization of the system: At variance to the case of local dephasing [20], the individual realization entropy values are not all identical to $S^\natural(\mathbb{1})$ but distributed around it [see inset in Fig. 2(a)]. The initial short-time evolution of the entropy follows the Hamiltonian path; it is a linear growth, which in the absence of the dissipation would saturate to the Page value [33], $S^\natural_{\text{Page}} \simeq N - 1$. After time $t \gtrsim \gamma^{-1}$ the contribution of the dissipative part of the generator $\mathcal{L}$ becomes tangible and eventually brings the entropy down to an asymptotic value $S^\natural(\varrho_\infty) \ll S^\natural_{\text{Page}}$.

In the MBL phase, the averaged OSEE saturates to values below $S^\natural(\mathbb{1})$ [see Figs. 2(b) and 2(c)]. This can be explained by generalizing the argument used in Ref. [11] for the Hamiltonian MBL systems. While in the ergodic phase all—even distant—sites (spins) are "tied" by the conservation of the total particle number (total spin), in the MBL phase the correlations are short-ranged and restricted by the localization length. Therefore, the entanglement has to be short-ranged in the MBL phase. It is noteworthy that, similar to the entanglement entropy in the Hamiltonian case [4–7], a relaxation of the OSSE to its asymptotic value is marked by a logarithmic growth, $S^\natural(t) \propto \log(t)$, a feature found before with local dephasing [20].

Ratio of consecutive level spacing for the steady-state density operator. According to the quantum chaos theory [34], Poisson and Wigner-Dyson distributions of the spacing $\delta_j = E_{j+1} - E_j$, where $\{E_j\}$ are eigenvalues of a Hamiltonian sorted in ascending order, correspond to regular (integrable) and chaotic (nonintegrable) quantum systems. Similarly, we can expect Poisson and Wigner-Dyson distributions for MBL and ergodic many-body Hamiltonians, respectively [9,10]. However, these indicators assume the uniform level density which is rarely the case with physical Hamiltonians. To circumvent this problem, Oganesyan and Huse considered the distribution of the ratios $r_j = \min[\lambda_j, \lambda_j^{-1}]$, $\lambda_j = \delta_j/\delta_{j-1}$, which do not depend on the local density of states [9]. It follows that spectral averages of r yield $r_{\text{Poisson}} \simeq 0.386$ for Poisson random variables, $r_{\text{GOE}} \simeq 0.536$ for Gaussian orthogonal (GOE), and $r_{\text{GUE}} \simeq 0.603$ for Gaussian unitary (GUE) ensembles [35].

In another context, Prosen and Žnidarič proposed to quantify the nonequilibrium steady-state density operators in terms of their level spacing distributions [36]. They found that the

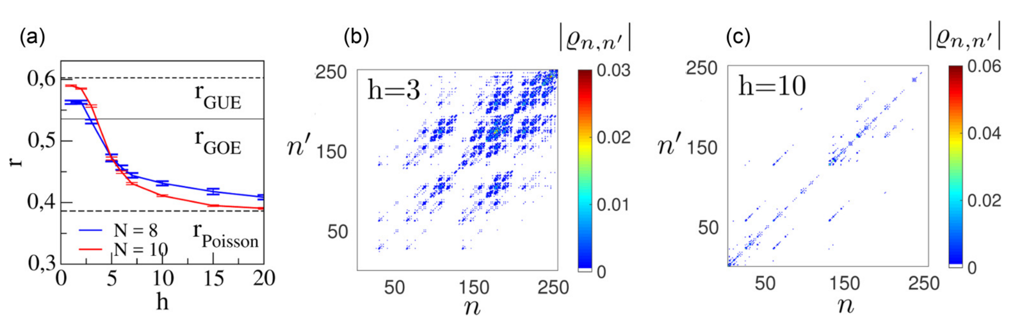

FIG. 3. (a) Averaged ratio of consecutive level spacing r of ϱ_∞ as a function of disorder strength h. The ratio is sampled for chains with $N = 8$ and 10 sites and averaged (for every value of h) over 10^2 disorder realizations. The error bars show the variance of the ratio. (b), (c) Absolute values of the elements of the steady-state density matrix for a single disorder realization and two different values of h. The matrices are expressed in the Fock basis (for the half-filling sector) sorted in the lexicographical order. Only elements with absolute value larger than 10^{-5} are shown. Other parameters are the same as in Fig. 1.

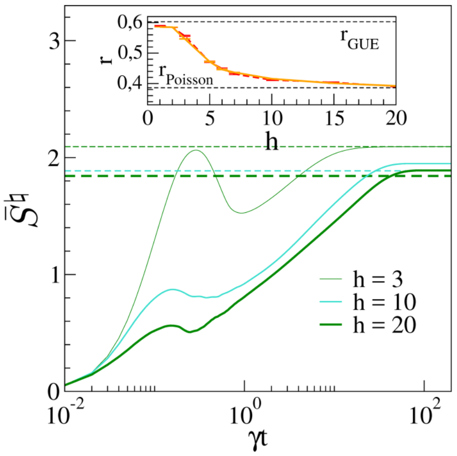

FIG. 4. Time evolution of the averaged operator-space entanglement entropy $\bar{S}^{\natural}(t)$ when both types of dissipation are acting simultaneously. Dashed lines correspond to asymptotic values obtained with the pairwise dissipation (see Fig. 3). The system size is $N = 16$. The inset shows the averaged ratio of consecutive level spacing r of ϱ_∞ ($N = 10$) as a function of disorder strength h for $\gamma_d = 0$ (thick solid line) and $\gamma_d = 0.1$ (thick dashed line). Other parameters are the same as in Fig. 1.

transition from integrability to non-integrability [37] corresponds to the Poisson-to-GUE transition. We follow this idea and find that in the ergodic phase the spectrum of the steady-state density operator yields r values close to r_{GUE}, while in the limit of strong localization it approaches $r_{Poisson}$ [see Fig. 3(a)]. This correspondence improves with increasing N. The structure of the density matrices ϱ_∞ is notably different in the ergodic and strong localization regimes [see Figs. 3(b) and 3(c)]: While in the ergodic phase matrices exhibit a well-developed off-diagonal structure and thus a relatively high purity and interference pattern, in the deep MBL regime they have near diagonal structure, with a few "hot spots" (a similar structure was found before with a dissipative single-particle model [38]).

Combined action of the pairwise dissipation and local dephasing. In this case, the steady-state density operator is the solution of the operator equation $\mathcal{L}_{\mathcal{H}}\varrho_\infty + \mathcal{L}_d\varrho_\infty + \mathcal{L}_p\varrho_\infty = 0$, where two last superoperators are parametrized with rates γ_d and γ_p.

We argue that a steady state obtained with the pairwise dissipation is stable with respect to the action of dephasing. Namely, the quantifiers of the steady-state density operator ϱ_∞ change continuously with the increase of γ_d. This conjecture is based on the notion of stability introduced for many-body dissipative systems with no faster than linear (in time) growth of the support of initially localized operators [39,40]. This seems to be the case for $h = 10, 20$ as has been detected with the OSEE [Figs. 3(b) and 3(c)]. To validate the conjecture, we calculate the OSEE and the mean gap ratio r for $\gamma_d = \gamma_p = 0.1$ (Fig. 4). While the asymptotic operator entropy changes slightly in the localization phase (and remained constant in the ergodic phase), changes of the mean gap ratio values are smaller than the sampling errors (see inset in Fig. 4). Therefore, by introducing the pairwise dissipation into a system already subjected to decoherence, we can restore localization features and distinguish between ergodic and MBL Hamiltonians.

Discussion. We proposed three quantitative identifiers of MBL in open systems. The imbalance statistics is accessible in experiments [16,22] but requires studying systems of different sizes. The operator-space entanglement entropy indicates differences between phases both in the asymptotic limit and during the relaxation toward it. The level spacing of the asymptotic density operator ϱ_∞ bridges MBL and quantum chaos theory [34,36].

There are three factors [and, correspondingly, three terms in the generator $\mathcal{L}$ of the master equation, Eq. (2)] contributing to the formation of the steady-state density operator. The pairwise dissipation tries to build classical and quantum correlations between distant sites and reasonably week local dephasing is not able to wash them out. At the same time, the MBL mechanisms, induced by the Hamiltonian, try to restrict the correlations to the localization length. As a result of the balance between these three factors, an asymptotic state with localization footprints appears.

Future studies could consider the incorporation of the disorder into local rates $\gamma_p(s)$ and, ultimately, a creation of MBL states by dissipative means solely. Disordered pairwise dissipation acquires a relevance in the context of recent experiments with dissipatively coupled exciton-polariton condensate arrays [41].

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