



PUBLISHED FOR SISSA BY SPRINGER

RECEIVED: December 6, 2017

REVISED: April 19, 2018

ACCEPTED: July 8, 2018

PUBLISHED: July 17, 2018

General continuum clockwork

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ABSTRACT: The continuum clockwork is an extra-dimensional set-up to realize certain features of the clockwork mechanism generating exponentially suppressed or hierarchical couplings of light particles. We study the continuum clockwork in a general scheme in which large volume, warped geometry, and localization of zero modes in extra dimension are described by independent parameters. For this, we propose a generalized 5-dimensional linear dilaton model which can realize such set-up as a solution of the model, and examine the KK spectrum and the couplings of zero modes and massive KK modes to boundary-localized operators for the bulk graviton, Abelian gauge bosons and periodic scalar fields. We discuss how those KK spectra and couplings vary as a function of the volume, warping and localization parameters, and highlight the behavior in the parameter region corresponding to the clockwork limit. We discuss also the field range of 4-dimensional axions originating from either 5-dimensional periodic scalar field or the 5-th component of an Abelian gauge field, and comment on the limitations of continuum clockwork compared to the discrete clockwork.

KEYWORDS: Field Theories in Higher Dimensions, Gauge Symmetry, Global Symmetries, Space-Time Symmetries

ARXIV EPRINT: [1711.06228](https://arxiv.org/abs/1711.06228)

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1 Introduction

The clockwork (CW) is a mechanism to generate exponentially suppressed or hierarchical couplings of light particles with N massive states having comparable masses near the threshold scale of the mechanism, where the suppression or hierarchy factor is given by $\mathcal{O}(q^{-N})$ for a CW parameter $q > 1$ [1–3]. Among the many possible implementations of the mechanism [4–23], the CW axions [2, 3] and U(1) gauge bosons [4, 5] are particularly interesting as the key features of the mechanism can be understood in terms of a specific pattern of symmetry breaking of the underlying $N + 1$ (global or local) U(1) symmetries $[\text{U}(1)]^{N+1} = \prod_{i=0}^N \text{U}(1)_i$, which is (explicitly or spontaneously) broken down to a $\text{U}(1)_{\text{CW}}$ subgroup. Furthermore, the key model parameters such as the CW parameter q and the involved axion-instanton couplings and U(1) gauge charges, are required to be integer-valued (in appropriate units) by the compact $[\text{U}(1)]^{N+1}$, so the model has a built-in criterion for natural size of these model parameters. The existence of light axion or U(1) gauge boson having hierarchical couplings can be explained by the unbroken $\text{U}(1)_{\text{CW}}$ generated by the charge operator

$$Q_{\text{CW}} \propto \sum_{i=0}^N \frac{Q_i}{q^i}, \tag{1.1}$$

which has a localized distribution along the linear quiver of the $U(1)$ charges Q_i . As it is a subgroup of the compact $[U(1)]^{N+1}$, the unbroken $U(1)_{CW}$ is yet compact, but has an exponentially large range enhanced by q^N relative to the range of $U(1)_i$, which is achieved by a series of discrete monodromy between the nearest neighbor $U(1)$ symmetries. This allows an exponentially enhanced axion field range or an exponential hierarchy among the quantized charges of the unbroken $U(1)_{CW}$ gauge symmetry, which might have an interesting implication for the fundamental issues such as the weak gravity conjecture [4, 24, 25].

Recently it has been pointed out that the CW mechanism can be implemented in an extra-dimensional setup by taking a limit $N \rightarrow \infty$, while identifying the label i as the coordinate of an extra spacial dimension [5]. Then, depending upon which aspects of the discrete clockwork (DCW) one likes to reproduce, there can be two different approaches to the continuum clockwork (CCW) [5, 26, 27]:¹ one approach based on a specific form of 5D geometry involving an exponentially large volume together with an exponential warp factor [5], which can be obtained as a solution of the 5D linear dilaton model [28], and another approach based on exponentially localized zero mode profile over the 5th dimension in a field basis defined in terms of 5D $U(1)$ or discrete gauge symmetry, which is generated by appropriately tuned bulk and boundary mass parameters [26]. As was briefly discussed in [5, 26, 29] and will be discussed in more detail in this paper, each approach has its own limitations, and as a result can reproduce only certain partial features of the original DCW axions and $U(1)$ gauge bosons. For instance, assuming a 5D discrete shift symmetry which would assure that the involved axion is a periodic scalar, and also taking into account the $U(1)$ gauge symmetry associated with the involved 5D vector field, in CCW either the localized CW symmetry to protect zero mode is not respected by gravitational interactions, or an exponential hierarchy between the zero mode couplings at different boundaries can not be generated. As other limitations relative to DCW, CCW could yield neither an exponentially enhanced trans-Planckian field range of the zero mode axion, nor an exponential hierarchy among the 4D gauge charges, while keeping the quantized nature of gauge charges of an unbroken 4D $U(1)$ gauge symmetry.

Yet the CCW is interesting by itself as it offers a setup to realize some features of the DCW mechanism, while incorporating the well known extra-dimensional solution of the hierarchy problem [28, 30]. Motivated by this, in this paper we study the CCW in a general setup involving large volume, warped geometry, and localized zero mode profile, which are described by independent parameters. Our setup makes it possible to examine how the patterns of low energy couplings (to boundary operators) and Kaluza-Klein (KK) mass spectra behave when the model parameters are varying from the CW limit to other limits such as the large extra dimension limit [31] and the Randall-Sundrum limit [30], from which we can identify the distinctive features of the CCW. We also clarify the limitations of each approach to the CCW to sharpen the differences from the DCW, particularly in connection with the possibility of trans-Planckian axion field range, and also the origin of unbroken $U(1)_{CW}$ which would protect the zero mode axion or $U(1)$ gauge boson.

¹Refs. [5] and [26] are using different criterion about what one would call the CW mechanism, causing a certain amount of confusion. In this paper, we will use a broad notion for CW, accommodating the both criteria adopted in [5] and [26].

The outline of this paper is as follows. In the next section, we first review the DCW with a specific model for clockwork axions and U(1) gauge bosons [2–5], and discuss the possible continuum limits following [5] and [26]. We then introduce a general continuum clockwork which may realize the CCW in more general ground involving large volume, warped geometry, and localized zero mode profile simultaneously. In section 3, we discuss a generalized linear dilaton model yielding the background metric and dilaton profile for general CCW as a solution of the model. We study also specific 5D models for CCW axions and U(1) gauge bosons in section 3, and finally conclude in section 4.

2 Clockwork and its continuum limits

2.1 Discrete clockwork axions and Abelian gauge bosons

In this section, we review the key features of the CW mechanism using an example of discrete clockwork axions [2, 3] and U(1) gauge bosons [4, 5], and discuss the possible continuum limits following [5] and [26]

Generic CW model involves two pieces: (i) a clockwork sector living on a quiver with $N + 1$ sites equipped with asymmetric nearest-neighbor interactions, and (ii) an external sector which couples to the clockwork sector through a specific site in the quiver, e.g. the first and/or the last site. For clockwork axions and U(1) gauge bosons, the CW sector lagrangian is given by

$$\begin{aligned} \mathcal{L}_{CW} = & -\frac{1}{2} \sum_{i=0}^N f^2 \partial_\mu U_i^* \partial^\mu U_i + \frac{1}{2} \sum_{j=0}^{N-1} \frac{1}{2} \Lambda^4 \left(U_{j+1}^q U_j^* + \text{h.c.} \right) \\ & - \sum_{i=0}^N \frac{1}{4g^2} F^{i\mu\nu} F_{\mu\nu}^i - \frac{1}{2} \sum_{j=0}^{N-1} v^2 \left| (\partial_\mu - i\tilde{q}A_\mu^{j+1} + iA_\mu^j) \Sigma_{j+1} \right|^2, \end{aligned} \quad (2.1)$$

where U_i and Σ_i are U(1)-valued fields:

$$U_i = e^{i\phi_i/f}, \quad \Sigma_i = e^{i\omega_i}. \quad (2.2)$$

As usual, here we assume for simplicity that the model parameters, e.g. f, Λ, q , etc., have site-independent values. As for the external sector, we consider the Yang-Mills (YM) gauge fields G_μ^I and the U(1)-charged fermions ψ_I ($I = 0, N$), which couple to the CW sector through the 0-th and N -th sites as follows:²

$$\mathcal{L}_{\text{ext}} = \sum_{I=0,N} \left(-\frac{1}{4g_I^2} G_{\mu\nu}^I G^{I\mu\nu} + \frac{\kappa_I}{32\pi^2} \frac{\phi_I}{f} G_{\mu\nu}^I \tilde{G}^{I\mu\nu} + i\bar{\psi}_I \bar{\sigma}^\mu \partial_\mu \psi_I + Q_I A_\mu^I \bar{\psi}_I \bar{\sigma}_\mu \psi_I \right), \quad (2.3)$$

where $G_{\mu\nu}^I$ and $\tilde{G}_{\mu\nu}^I$ are the YM gauge field strength and its dual, respectively, and we ignore the U(1) gauge anomalies which are not relevant for our discussion.³

²One can consider more general external sector on arbitrary sites with $0 < I < N$, whose feature of interactions with the CW sector can be readily inferred from the subsequent discussions for the 0-th and N -th sites.

³One can easily avoid the U(1) gauge anomalies by introducing additional fermions with opposite gauge charge, without affecting any of our subsequent discussions.

A key component of the model is the clockwork gear composed of asymmetric nearest-neighbor interactions in the CW sector, e.g. the axion potentials and Stückelberg gauge boson mass terms in (2.1), which is responsible for the following (explicit or spontaneous) symmetry breaking:

$$[\mathrm{U}(1)]_{\mathrm{global}}^{N+1} \times [\mathrm{U}(1)]_{\mathrm{local}}^{N+1} \rightarrow [\mathrm{U}(1)_{\mathrm{CW}}]_{\mathrm{global}} \times [\mathrm{U}(1)_{\mathrm{CW}}]_{\mathrm{local}}, \quad (2.4)$$

where $[\mathrm{U}(1)]^{N+1}$ denotes the $N+1$ compact (global or local) $\mathrm{U}(1)$ symmetries of the model:

$$\begin{aligned} [\mathrm{U}(1)]_{\mathrm{global}}^{N+1} : & \quad U_i \rightarrow e^{i\alpha_i} U_i, \\ [\mathrm{U}(1)]_{\mathrm{local}}^{N+1} : & \quad A_\mu^i \rightarrow A_\mu^i + \partial_\mu \beta_i(x), \quad \Sigma_i \rightarrow e^{i(\tilde{q}\beta_i - \beta_{i-1})} \Sigma_i, \quad \psi_I \rightarrow e^{iQ_{iI}\beta_i} \psi_I, \end{aligned} \quad (2.5)$$

and the unbroken $\mathrm{U}(1)_{\mathrm{CW}}$ symmetries are given by

$$\begin{aligned} [\mathrm{U}(1)_{\mathrm{CW}}]_{\mathrm{global}} : & \quad U_i \rightarrow e^{i\alpha/q^i} U_i, \\ [\mathrm{U}(1)_{\mathrm{CW}}]_{\mathrm{local}} : & \quad A_\mu^i \rightarrow A_\mu^i + \frac{1}{\tilde{q}^i} \partial_\mu \beta(x), \quad \Sigma_i \rightarrow \Sigma_i, \quad \psi_I \rightarrow e^{iQ_{iI}\beta/\tilde{q}^i} \psi_I. \end{aligned} \quad (2.6)$$

Note that α_i and β_i are periodic variables with the periodicity 2π , while the periodicity of the unbroken $\mathrm{U}(1)_{\mathrm{CW}}$ symmetries, i.e. the range of α and β , are enlarged to $2\pi q^N$ and $2\pi \tilde{q}^N$, respectively. The axion potential responsible for the explicit symmetry breaking: $[\mathrm{U}(1)]_{\mathrm{global}}^{N+1} \rightarrow [\mathrm{U}(1)_{\mathrm{CW}}]_{\mathrm{global}}$ might be generated by non-perturbative dynamics as discussed in [2], or introduced simply by hand [3], while the Stückelberg mass terms for the spontaneous symmetry breaking: $[\mathrm{U}(1)]_{\mathrm{local}}^{N+1} \rightarrow [\mathrm{U}(1)_{\mathrm{CW}}]_{\mathrm{local}}$ can be achieved by introducing the complex scalar fields σ_j ($j = 1, \dots, N$) which carry the $\mathrm{U}(1)_{j-1} \times \mathrm{U}(1)_j$ gauge charge $Q_{\sigma_j} = (-1, \tilde{q})$, while having the vacuum expectation values $\langle \sigma_j \rangle = v e^{i\omega_j}$. In order for both the effective lagrangian (2.1) and the associated symmetries (2.5) well defined over the full range of the field variables and the symmetry group, the CW parameters q and \tilde{q} , the $\mathrm{U}(1)$ gauge charges Q_0 and Q_N , and the axion-instanton couplings κ_0 and κ_N should be all integer-valued.

The symmetry breaking (2.4) due to the CW gear assures that there exist a massless axion and $\mathrm{U}(1)$ gauge boson associated with the unbroken $\mathrm{U}(1)_{\mathrm{CW}}$, in addition to the N massive gear modes associated with the broken symmetries. For the particular case that all model parameters are site-independent, one can diagonalize the full $(N+1) \times (N+1)$ mass matrices to find the mass eigenstates. This results in⁴

$$\phi_i = \sum_{\ell=0}^N O_{i\ell} \hat{\phi}_\ell, \quad A_\mu^i = \sum_{\ell=0}^N \tilde{O}_{i\ell} \hat{A}_\mu^\ell, \quad (2.7)$$

⁴Even when the model parameters are site-dependent, it is still straightforward to find the explicit form of the unbroken $\mathrm{U}(1)_{\mathrm{CW}}$ symmetries and the associated massless components, which are given by $Q_{\mathrm{CW}} = \sum_i O_{i0} Q_i$ with $O_{i0} \propto 1/\prod_{k=1}^i q_k$ for site-dependent CW parameters q_k ($k = 1, \dots, N$). In such generic situation, one can not get an explicit form of the mass eigenvalues and the mixing matrices for massive modes. However, as long as parameters at different sites are comparable to each other, the qualitative features of the spectrum and mixings are expected to be same as the site-independent case.

where $\hat{\phi}_0, \hat{A}_\mu^0$ denote the canonically normalized massless modes, while $\hat{\phi}_n, \hat{A}_\mu^n$ ($n=1, \dots, N$) stand for the massive gear modes, and the corresponding mixing matrices are given by [3, 5]

$$\begin{aligned} O_{i0} &= \frac{\mathcal{N}_0}{q^i}, & O_{in} &= \mathcal{N}_n \left(q \sin \frac{in\pi}{N+1} - \sin \frac{(i+1)n\pi}{N+1} \right) \quad (n=1, \dots, N) \\ \tilde{O}_{i0} &= gO_{i0}(q \rightarrow \tilde{q}), & \tilde{O}_{in} &= gO_{in}(q \rightarrow \tilde{q}), \end{aligned} \quad (2.8)$$

where

$$\mathcal{N}_0 = \sqrt{\frac{q^2 - 1}{q^2 - q^{-2N}}}, \quad \mathcal{N}_n = \sqrt{\frac{2}{(N+1)\lambda_n}}, \quad \lambda_n = q^2 + 1 - 2q \cos \frac{n\pi}{N+1}. \quad (2.9)$$

One finds also the mass eigenvalues

$$m_{\hat{\phi}_n}^2 = \lambda_n M_{\text{CW}}^2, \quad m_{\hat{A}_n}^2 = \tilde{\lambda}_n \tilde{M}_{\text{CW}}^2 \quad (n=1, \dots, N) \quad (2.10)$$

where

$$M_{\text{CW}} = \frac{\Lambda^2}{f}, \quad \tilde{M}_{\text{CW}} = gv, \quad \tilde{\lambda}_n = \lambda_n(q \rightarrow \tilde{q}). \quad (2.11)$$

Notice that the mixing matrix element O_{k0} in (2.8), which is the component of the massless mode in the k -th site of the clockwork gear, is suppressed by the factor q^{-k} ($= e^{-k \ln q}$), while all the other components of the mixing matrices and λ_n are essentially of order unity. This exponentially localized distribution of O_{i0} along the quiver of the original field variables ϕ_i and A_μ^i is directly responsible for generating small or hierarchical couplings of light particles in the limit when N is moderately large, e.g $N = \mathcal{O}(10)$. More explicitly, one can write down the couplings to the external sector in (2.3) in terms of the mass eigenstates:

$$\begin{aligned} & -\frac{1}{2}(\partial_\mu \hat{\phi}_0)^2 - \frac{1}{2} \sum_n \left((\partial_\mu \hat{\phi}_n)^2 + m_{\hat{\phi}_n}^2 \hat{\phi}_n^2 + \dots \right) \\ & -\frac{1}{4}(\hat{F}_{\mu\nu}^0)^2 + \sum_n \left(-\frac{1}{4}(\hat{F}_{\mu\nu}^n)^2 - m_{\hat{A}_n}^2 (\hat{A}_\mu^n)^2 \right) \\ & + \frac{1}{32\pi^2} \frac{\hat{\phi}_0}{F_0} \left(\kappa_0 G_{\mu\nu}^0 \tilde{G}^{0\mu\nu} + \frac{\kappa_N}{q^N} G_{\mu\nu}^N \tilde{G}^{N\mu\nu} \right) \\ & - \sum_n \frac{1}{32\pi^2} \frac{\hat{\phi}_n}{F_n} \left(\kappa_0 G_{\mu\nu}^0 \tilde{G}^{0\mu\nu} + (-1)^n q \kappa_N G_{\mu\nu}^N \tilde{G}^{N\mu\nu} \right) \\ & + \hat{g}_0 \hat{A}_\mu^0 \left(Q_0 \bar{\psi}_0 \bar{\sigma}_\mu \psi_0 + \frac{Q_N}{q^N} \bar{\psi}_N \bar{\sigma}_\mu \psi_N \right) \\ & - \sum_n \hat{g}_n \hat{A}_\mu^n \left(Q_0 \bar{\psi}_0 \bar{\sigma}_\mu \psi_0 + (-1)^n q Q_N \bar{\psi}_N \bar{\sigma}_\mu \psi_N \right), \end{aligned} \quad (2.12)$$

where we choose the gauge $\Sigma_i = 1$, and

$$\frac{1}{F_0} = \frac{\mathcal{N}_0}{f}, \quad \frac{1}{F_n} = \frac{\mathcal{N}_n \sin \frac{n\pi}{N+1}}{f}, \quad \hat{g}_0 = g\tilde{\mathcal{N}}_0, \quad \hat{g}_n = g\tilde{\mathcal{N}}_n \sin \frac{n\pi}{N+1}. \quad (2.13)$$

Here one can see that the massless modes $\hat{\phi}_0$ and \hat{A}_μ^0 couple to the external sector in the N -th site with an exponentially small coupling suppressed by q^{-N} , while all other couplings are typically of $\mathcal{O}(1/f)$ for axions and $\mathcal{O}(g)$ for gauge bosons, although the couplings of massive modes can be suppressed by the factor $1/\sqrt{N}$ for large N .⁵

The above lagrangian written in terms of the mass eigenstates shows that the couplings of massless modes to the operators at the N -th site are exponentially suppressed by $1/q^N$ compared to the couplings to the similar operators at the 0-th site, if κ_0 and κ_N (Q_0 and Q_N) are comparable to each other, which would be a natural choice for integer-valued $\kappa_{0,N}$ and $Q_{0,N}$. There is another exponential hierarchy of $\mathcal{O}(1/q^N)$ between the massless mode couplings and the massive mode couplings to the operators at the N -th site. Notice that this second hierarchy is independent of the relative size of κ_0 (Q_0) compared to κ_N (Q_N). We will see that this distinction is important when discussing continuum limit of the DCW. Here the suppressed coupling of the zero mode axion to the YM gauge fields at the N -th site is a consequence of the enlarged periodicity of $[\text{U}(1)_{\text{CW}}]_{\text{global}}$ in (2.6), which results in the enlarged field range of the zero mode axion

$$\Delta\hat{\phi}_0 \equiv 2\pi F_{\text{eff}} = 2\pi q^N F_0, \tag{2.14}$$

which is exponentially bigger than the original axion field range $2\pi f$ in (2.2). Another notable feature of the DCW is the spectrum of massive modes. The parameter λ_n in (2.9) for the mass eigenvalues (2.10) satisfies

$$(q-1)^2 < \lambda_n < (q+1)^2, \tag{2.15}$$

while λ_n is increasing in n . Therefore, in the limit $N \gg 1$ the mass eigenvalues m_n in (2.10) become approximately degenerate around the clockwork threshold scale M_{CW} , and their mass gaps behave as $\delta m \sim M_{\text{CW}}/N$, which might have interesting phenomenological implications [27].

Before moving to the discussion of continuum clockwork, we summarize the key features of the discrete CW axions and U(1) gauge bosons.

1. The key model parameters such as the CW parameter q , the axion-instanton couplings κ_I and the U(1) gauge charges Q_I have integer values as required by the underlying compact $[\text{U}(1)]^{N+1}$ symmetry. Therefore the model is equipped with an unambiguous criterion for natural size of the relevant UV parameters, providing a basis for the subsequent discussion of the hierarchical structure of the low energy effective couplings of zero mode axion and zero mode U(1) gauge boson.
2. Zero mode axion and U(1) gauge boson are protected by the unbroken compact $\text{U}(1)_{\text{CW}}$ generated by a charge operator $Q_{\text{CW}} \propto \sum_i Q_i/q^i$ which has a localized distribution along the linear quiver of the charge operators Q_i of $[\text{U}(1)]^{N+1} = \prod_{i=0}^N \text{U}(1)_i$. The localized feature of the unbroken symmetry leads to a multiplicative monodromy

⁵Even for more general external sector located at an arbitrary site I , it can be easily shown that only the massless modes have suppressed couplings to the I -th site by the factor q^{-I} .

structure, enhancing the range of $U(1)_{\text{CW}}$ by $\prod_{j=1}^N q = q^N$. This results in also an exponential hierarchy between the couplings of zero modes at the 0-th and the N -th sites, as well as an exponentially enhanced field range of the zero mode axion.

3. Exponential hierarchy ($\propto q^{-N}$) between the zero mode couplings and the massive mode couplings to the external sector at the N -th site.
4. Approximately degenerate N massive modes around the threshold scale, with $m_n \sim M_{\text{CW}}$ and the small mass splittings $\delta m \sim m_{n+1} - m_n \sim m_n/N$ in the limit $N \gg 1$.

2.2 Continuum limit

Starting from a discrete clockwork model with $N + 1$ sites, its continuum version can be obtained by taking the limit $N \rightarrow \infty$, while identifying the site index i as the coordinate of the 5-th spacial dimension with a finite length $\pi R = N\Delta r$, where Δr is the lattice spacing. In order to keep the hierarchy factors q^N, \tilde{q}^N finite, while having a nonzero mass gap between the zero modes and the massive gear states, one should take also $q, \tilde{q} \rightarrow 1$ with the CW mass scales $M_{\text{CW}} = \Lambda^2/f$ and $\tilde{M}_{\text{CW}} = gv$ approaching to the lattice cutoff scale $1/\Delta r$. More explicitly, the continuum limit takes

$$N \rightarrow \infty, \quad q, \tilde{q} \rightarrow 1, \quad M_{\text{CW}}, \tilde{M}_{\text{CW}} \rightarrow \frac{1}{\Delta r} = \infty, \quad (2.16)$$

with the following parameter combinations keeping a finite nonzero value:

$$\pi R = N\Delta r, \quad \mu = M_{\text{CW}} \ln q, \quad \tilde{\mu} = \tilde{M}_{\text{CW}} \ln \tilde{q}, \quad (2.17)$$

which results in

$$q^N \rightarrow e^{\mu\pi R}, \quad \tilde{q}^N \rightarrow e^{\tilde{\mu}\pi R}. \quad (2.18)$$

To get the continuum limit of the lagrangian densities (2.1) and (2.3), one can make the following substitutions:

$$\sum_i \rightarrow \frac{1}{\Delta r} \int_0^{\pi R} dy, \quad \phi_{i+1} - \phi_i \rightarrow \Delta r \partial_y \phi, \quad (2.19)$$

together with the field and parameter redefinitions:

$$\begin{aligned} M_{\text{CW}} = \frac{\Lambda^2}{f} &\rightarrow \frac{1}{\Delta r}, & \tilde{M}_{\text{CW}} = gv &\rightarrow \frac{1}{\Delta r}, & q - 1 &\rightarrow \mu \Delta r, & \tilde{q} - 1 &\rightarrow \tilde{\mu} \Delta r, \\ \phi_i(x) &\rightarrow \Phi(x, y) \Delta r^{1/2}, & A_\mu^i(x) &\rightarrow A_\mu(x, y), & \omega_i(x) &\rightarrow \Omega(x, y) \Delta r, \\ f &\rightarrow f_5^{3/2} \Delta r^{1/2}, & g^2 &\rightarrow \frac{g_5^2}{\Delta r}, \end{aligned} \quad (2.20)$$

and finally take the limit $\Delta r \rightarrow 0$. One then finds

$$\int dy \left[-\frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{2} (\partial_y \Phi + \mu \Phi)^2 - \frac{1}{4g_5^2} ((F_{\mu\nu})^2 + (\partial_\mu \Omega - \partial_y A_\mu - \tilde{\mu} A_\mu)^2) \right. \\ \left. + \frac{1}{32\pi^2} \frac{\Phi}{f_5^{3/2}} \left(\kappa_0 \delta(y) G_{\mu\nu}^0 \tilde{G}^{0\mu\nu} + \kappa_\pi \delta(y - \pi R) G_{\mu\nu}^\pi \tilde{G}^{\pi\mu\nu} \right) \right. \\ \left. + A_\mu (Q_0 \delta(y) \bar{\psi}_0 \bar{\sigma}^\mu \psi_0 + Q_\pi \delta(y - \pi R) \bar{\psi}_\pi \bar{\sigma}^\mu \psi_\pi) \right], \quad (2.21)$$

where we changed the index for the last site from N to π , e.g. $\kappa_N \rightarrow \kappa_\pi$.

Taking the continuum limit of $[\text{U}(1)]^{N+1}$ in (2.5) and $\text{U}(1)_{\text{CW}}$ in (2.6), we also find the associated symmetry transformations are given by

$$[\text{U}(1)_{5\text{D}}]_{\text{global}} : \quad \delta\Phi = f_5^{3/2} \alpha(y), \quad (2.22) \\ [\text{U}(1)_{5\text{D}}]_{\text{local}} : \quad \delta A_\mu = \partial_\mu \beta(x, y), \quad \delta\Omega = \partial_y \beta + \tilde{\mu} \beta, \quad \delta\psi_{0,\pi} = iQ_{0,\pi} \beta(x, y = 0, \pi R) \psi_{0,\pi},$$

which are explicitly or spontaneously broken down to

$$[\text{U}(1)_{\text{CW}}]_{\text{global}} : \quad \delta\Phi = f_5^{3/2} e^{-\mu y} \alpha_0, \quad (2.23) \\ [\text{U}(1)_{\text{CW}}]_{\text{local}} : \quad \delta A_\mu = e^{-\tilde{\mu} y} \partial_\mu \beta_0(x), \quad \delta\Omega = 0, \quad \delta\psi_0 = iQ_0 \beta_0(x) \psi_0, \quad \delta\psi_\pi = i e^{-\tilde{\mu} \pi R} Q_\pi \beta_0(x) \psi_\pi,$$

where α_0 is a constant, and $\beta_0(x)$ is a function of the 4D spacetime coordinate x^μ .

Here we see an important difference between the original DCW and its continuum limit. In the DCW, one starts with compact $[\text{U}(1)]^{N+1}$ symmetries which are broken down to $\text{U}(1)_{\text{CW}}$, and both the full $\text{U}(1)$ symmetries and the unbroken $\text{U}(1)_{\text{CW}}$ are perfectly compatible with the 4D diffeomorphism and Lorentz symmetries. On the other hand, in the continuum clockwork (CCW) limit, the correspondence (2.18) implies that the CW parameters q, \tilde{q} can not be integer-valued anymore, except for the trivial case of $\mu = \tilde{\mu} = 0$. This raises a question if the $\text{U}(1)$ symmetries (2.22) and (2.23) in the CCW can be identified as a sensible compact symmetry compatible with the 5D spacetime symmetries. To answer this question, one should embed the continuum lagrangian (2.21) into a theory manifestly invariant under the 5D diffeomorphism and Lorentz symmetry. As we will see, there are two different embeddings of the continuum lagrangian (2.21) to a 5D diffeomorphism and Lorentz invariant theory, and for each embedding one can retain only different partial features of the DCW. Note that some issues in clockwork models, e.g. the periodicity (or field range) of axions and the natural size (or quantized nature) of the axion-instanton couplings κ_0, κ_π and the $\text{U}(1)$ gauge charges Q_0, Q_π , crucially depend on how the compact nature of the associated $\text{U}(1)$ symmetries are introduced in the underlying theory.

We remark that the above subtle feature of CCW partly originates from the fact that we replace the discrete field index i in the DCW model with the spatial coordinate y of extra spacial dimension in the continuum limit. While the index i is a frozen label in the DCW, the spatial coordinate y describes a dynamical extra dimension which imposes non-trivial restrictions on the clockwork lagrangian via the 5D spacetime symmetries. In the following, we will discuss two different prescriptions of CCW originally proposed in [5, 26], and subsequently introduce general continuum clockwork which can incorporate these two prescriptions within a common framework.

2.2.1 CCW-I: CW from localized zero mode profile

To embed the continuum lagrangian (2.21) in a 5D diffeomorphism and Lorentz invariant theory, we first notice following [26] that the lagrangian can be rewritten as follows by

$$\begin{aligned}
 \int dy & \left[-\frac{1}{2} \left((\partial_\mu \Phi)^2 + (\partial_y \Phi)^2 + (\mu^2 - \mu[\delta(y) - \delta(y - \pi R)]) \Phi^2 \right) \right. \\
 & - \frac{1}{4g_5^2} \left((F_{\mu\nu})^2 + (\partial_y A_\mu)^2 + (\tilde{\mu}^2 - \tilde{\mu}[\delta(y) - \delta(y - \pi R)]) A_\mu^2 \right) \\
 & + \frac{\Phi}{f_5^{3/2}} \left(\kappa_0 \delta(y) G_0 \tilde{G}_0 + \kappa_\pi \delta(y - \pi R) G_\pi \tilde{G}_\pi \right) \\
 & \left. + A_\mu \left(Q_0 \delta(y) \bar{\psi}_0 \bar{\sigma}^\mu \psi_0 + Q_\pi \delta(y - \pi R) \bar{\psi}_\pi \bar{\sigma}^\mu \psi_\pi \right) \right], \tag{2.24}
 \end{aligned}$$

where the continuum clockwork parameters $\mu, \tilde{\mu}$ appear as boundary and bulk masses. The above lagrangian can be easily made to be invariant under the 5D spacetime symmetries by introducing appropriate metric dependence:

$$\begin{aligned}
 \int d^5x \sqrt{-G} & \left[-\frac{1}{2} G^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{2} \left(\mu^2 - \mu \left[\frac{\delta(y)}{\sqrt{G_{55}}} - \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \right] \right) \Phi^2 \right. \\
 & - \frac{1}{4g_5^2} G^{MN} G^{PQ} F_{MP} F_{NQ} - \frac{\tilde{\mu}^2}{4g_5^2} G^{MN} A_M A_N - \frac{\tilde{\mu}}{4g_5^2} \left(\frac{\delta(y)}{\sqrt{G_{55}}} - \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \right) G^{\mu\nu} A_\mu A_\nu \\
 & + \frac{\Phi}{f_5^{3/2}} \left(\kappa_0 \frac{\delta(y)}{\sqrt{G_{55}}} G_0 \tilde{G}_0 + \kappa_\pi \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} G_\pi \tilde{G}_\pi \right) \\
 & \left. + A_\mu \left(Q_0 \frac{\delta(y)}{\sqrt{G_{55}}} \bar{\psi}_0 \bar{\sigma}^\mu \psi_0 + Q_\pi \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \bar{\psi}_\pi \bar{\sigma}^\mu \psi_\pi \right) \right], \tag{2.25}
 \end{aligned}$$

where the continuum lagrangian (2.24) can be obtained when the spacetime metric G_{MN} is replaced by the flat background:

$$ds^2 = \langle G_{MN} \rangle dx^M dx^N = \eta_{MN} dx^M dx^N. \tag{2.26}$$

However, once one includes the metric dependence, certain feature of the model is lost. For the bulk and boundary masses which are tuned to be equal as in (2.25), the CW symmetry (2.23) is respected when the metric G_{MN} is simply replaced by its background value η_{MN} , however *not* by the interactions of the metric fluctuation $h_{MN} = G_{MN} - \eta_{MN}$. For instance, upon ignoring the off-diagonal part of G_{MN} , the 5D action of Φ can be rewritten as

$$\int d^5x \sqrt{-G} \left[-\frac{1}{2} G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \left(\frac{1}{\sqrt{G_{55}}} \partial_y \Phi + \mu \Phi \right)^2 + \frac{1}{2} \frac{1}{\sqrt{-G}} \partial_y \left(\frac{\sqrt{-G}}{\sqrt{G_{55}}} \right) \mu \Phi^2 \right], \tag{2.27}$$

where the last term shows that the CW symmetry (2.23) is not respected by the y -dependent fluctuation of $G_{\mu\nu}$. This means that generically the continuum CW symmetry (2.23) cannot be identified as a good symmetry compatible with 5D spacetime symmetries. Rather, it should be regarded as an approximate accidental symmetry which holds

for a particular metric background, but is broken by higher dimensional interactions of the metric fluctuation. As a consequence, the particular relations among the bulk and boundary masses, which are crucial for the existence of massless modes protected by the CW symmetry (2.23), are potentially unstable against radiative corrections involving the couplings of the metric fluctuation.⁶ Nevertheless, if we accept it, while ignoring the fine tuning issue on the bulk and boundary masses, the 5D model (2.25) can reproduce many, although not all, features of the DCW model as we will show in detail in section 3.

As noticed before, one needs an appropriate periodicity condition for Φ to address the field range of zero mode axion and the natural size (or quantized nature) of the boundary axion-instanton couplings κ_0, κ_π . For this, one can simply impose the condition

$$\Phi \equiv \Phi + 2\pi f_5^{3/2}, \tag{2.28}$$

while making the following replacement in the 5D action (2.25):

$$\mu\Phi \rightarrow \mu f_5^{3/2} \sin\left(\frac{\Phi}{f_5^{3/2}}\right). \tag{2.29}$$

Then the infinitesimal CW symmetry (2.23) is modified as follows

$$[\text{U}(1)_{\text{CW}}]_{\text{global}} : \delta\Phi = f_5^{3/2} \alpha_0 e^{-\mu \int dy \cos(\Phi/f_5^{3/2})}, \tag{2.30}$$

and the zero mode of the fluctuation $\delta\Phi$ around $\Phi = 0$ is protected to be light by this symmetry as far as we are not concerned with the symmetry breaking by the gravitational couplings of metric fluctuation.

The 5D diffeomorphism invariant theory (2.25) with the replacement (2.29) reproduces the continuum lagrangian (2.24) for a small field fluctuation $\delta\Phi$ around $\Phi = 0$ when the 5D metric is fixed as the flat background (2.26). It is then expected that certain perturbative features of this 5D theory, e.g. the spectrum and couplings of small field fluctuations, are similar to those of the original DCW model. For instance, the bulk and boundary mass terms of $\delta\Phi(x, y)$ and $\delta A_\mu(x, y)$ enforce their zero modes to have y -dependent profile exponentially localized at one boundary, e.g. at $y = 0$, which results in an exponential hierarchy between the zero mode couplings at $y = 0$ and their counterpart couplings at $y = \pi R$, as well as a similar hierarchy between the zero mode coupling and the massive mode coupling to the same operator at $y = \pi R$. Although the 5D theory (2.25) reproduces certain perturbative features of the original DCW model, large field behavior of the theory such as the field range of the zero mode axion can be quite different from the DCW model. We will discuss these issues in section 3 in terms of the general CCW to be defined later. For a moment, we note the key weak point of the 5D diffeomorphism invariant CCW-I theory (2.25) which realizes certain features of the DCW model through localized zero mode profile: *the model has a problem with the CW symmetry (2.23) which is not respected by gravitational interactions.*

⁶It might be still possible that those relations are protected by other symmetry such as supersymmetry [32].

In fact, our notion of localized zero mode is not basis-independent, and the localized feature of the zero mode of $\delta\Phi$ disappears for instance if one makes the y -dependent field redefinition: $\Phi(x, y) \rightarrow e^{-\mu y}\Phi(x, y)$. To avoid an ambiguity arising from such field redefinition, one needs to specify the field basis used to address the localization of zero mode in the 5th dimension. Throughout this paper, we will consider the localization in a specific field basis defined in terms of the discrete shift symmetry which would ensure the periodicity of Φ and the compact 5D U(1) gauge symmetry for A_M , i.e. the localization of the zero-mode wavefunctions $\delta\Phi(x, y)/\delta\phi_0(x)$ and $\delta A_\mu(x, y)/\delta A_{0\mu}(x)$, where the 5D fields $\Phi(x, y)$ and $A_M(x, y)$ transform under those gauge symmetries as

$$\Phi \rightarrow \Phi + 2\pi f_5^{3/2}, \quad A_M \rightarrow A_M + \partial_M \Lambda, \quad (2.31)$$

where f_5 is an y -independent constant and the U(1) gauge transformation function $\Lambda(x, y)$ obeys the y -independent periodicity condition as $\Lambda \equiv \Lambda + 2\pi$. Note that in this standard field basis for 5D gauge field, the two boundary gauge charges Q_0 and Q_π are quantized in a *common* unit as $\Lambda(y=0)$ and $\Lambda(y=\pi R)$ have the common periodicity. The same is true for the axion-instanton couplings κ_0 and κ_π as $\Phi(y=0)$ and $\Phi(y=\pi R)$ has the common periodicity. One can then start with $Q_0 \sim Q_\pi = \mathcal{O}(1)$ and $\kappa_0 \sim \kappa_\pi = \mathcal{O}(1)$, which are perfectly natural in view of the underlying symmetries, and examine the exponential hierarchies between the effective couplings of zero modes at different boundaries.

2.2.2 CCW-II: CW from geometry

As we noticed in the previous subsection, in an approach to implement continuum CW via localized zero mode profile, the CW symmetry (2.23) is not respected by the couplings of metric fluctuation. This problem arises from non-vanishing bulk and boundary mass parameters which are identified as the origin of the CCW parameters $\mu, \tilde{\mu}$. In fact, one can easily avoid this problem by embedding (2.21) into a 5D theory without any bulk or boundary mass term [5]. The continuum lagrangian (2.21) can be written in the following form by redefining the fields and boundary couplings:

$$\begin{aligned} & \int dy \left[-\frac{1}{2} e^{-2\mu y} \left((\partial_\mu \hat{\Phi})^2 + (\partial_y \hat{\Phi})^2 \right) - \frac{1}{4g_5^2} e^{-2\tilde{\mu} y} \left((\hat{F}_{\mu\nu})^2 + (\partial_\mu \hat{A}_5 - \partial_y \hat{A}_\mu)^2 \right) \right. \\ & \quad + \frac{\hat{\Phi}}{f_5^{3/2}} \left(\hat{\kappa}_0 \delta(y) G_0 \tilde{G}_0 + \hat{\kappa}_\pi \delta(y - \pi R) G_\pi \tilde{G}_\pi \right) \\ & \quad \left. + \hat{A}_\mu \left(\hat{Q}_0 \delta(y) \bar{\psi}_0 \bar{\sigma}^\mu \psi_0 + \hat{Q}_\pi \delta(y - \pi R) \bar{\psi}_\pi \bar{\sigma}^\mu \psi_\pi \right) \right], \end{aligned} \quad (2.32)$$

where the redefined fields and couplings are related to the original fields and couplings as follows.

$$\hat{\Phi} = e^{\mu y} \Phi, \quad \hat{A}_\mu = e^{\tilde{\mu} y} A_\mu, \quad \hat{A}_5 = e^{\tilde{\mu} y} \Omega \quad (2.33)$$

$$\hat{\kappa}_0 = \kappa_0, \quad \hat{\kappa}_\pi = \kappa_\pi e^{-\mu\pi R}, \quad \hat{Q}_0 = Q_0, \quad \hat{Q}_\pi = Q_\pi e^{-\mu\pi R} \quad (2.34)$$

In this prescription, the CCW parameters $\mu, \tilde{\mu}$ appear in the wave function coefficients, while the bulk and boundary mass parameters are all vanishing. For the case with $\mu = \tilde{\mu}$,

the above form of lagrangian can be embedded into a 5D diffeomorphism and Lorentz invariant linear dilaton model as discussed in [5, 29]:

$$\int d^5x \sqrt{-G} \left[-\frac{1}{2} G^{MN} \partial_M \hat{\Phi} \partial_N \hat{\Phi} - \frac{1}{4g_5^2} e^{2S/3} G^{MN} G^{PQ} \hat{F}_{MP} \hat{F}_{NQ} \right. \\ \left. + \frac{\hat{\Phi}}{f_5^{3/2}} \left(\hat{\kappa}_0 \frac{\delta(y)}{\sqrt{G_{55}}} G_0 \tilde{G}_0 + \hat{\kappa}_\pi \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} G_\pi \tilde{G}_\pi \right) \right. \\ \left. + \hat{A}_\mu \left(\hat{Q}_0 \frac{\delta(y)}{\sqrt{G_{55}}} \bar{\psi}_0 \bar{\sigma}^\mu \psi_0 + \hat{Q}_\pi \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \bar{\psi}_\pi \bar{\sigma}^\mu \psi_\pi \right) \right], \quad (2.35)$$

where the continuum lagrangian (2.32) is reproduced when the metric and dilaton field are replaced by the following solution of the linear dilaton model:⁷

$$ds^2 = \langle G_{MN} \rangle dx^M dx^N = e^{-\frac{4}{3}\mu y} (dx^2 + dy^2), \quad \langle e^S \rangle = e^{-2\mu y}. \quad (2.36)$$

The CW symmetry (2.23) can be also expressed in terms of the redefined fields $\hat{\Phi}$ and \hat{A}_M as

$$[\text{U}(1)_{\text{CW}}]_{\text{global}}: \quad \delta \hat{\Phi} = f_5^{3/2} \alpha_0, \\ [\text{U}(1)_{\text{CW}}]_{\text{local}}: \quad \delta \hat{A}_\mu = \partial_\mu \beta_0(x), \quad \delta \hat{A}_5 = 0, \quad \delta \psi_0 = i \hat{Q}_0 \beta_0(x) \psi_0, \quad \delta \psi_\pi = i \hat{Q}_\pi \beta_0(x) \psi_\pi. \quad (2.37)$$

Now $[\text{U}(1)_{\text{CW}}]_{\text{global}}$ can be identified as a global shift symmetry for the redefined scalar field $\hat{\Phi}$, $[\text{U}(1)_{\text{CW}}]_{\text{local}}$ as the 4D subgroup of a 5D U(1) gauge symmetry for the redefined gauge field \hat{A}_M , and both CW symmetries are obviously compatible with the 5D spacetime symmetries, e.g. respected by the couplings of the spacetime metric and dilaton field. Moreover, under the assumption that the redefined boundary couplings $\hat{\kappa}_{0,\pi}$ and $\hat{Q}_{0,\pi}$ are integer-valued, one can impose the following periodicity condition without any difficulty:

$$\hat{\Phi} \equiv \hat{\Phi} + 2\pi f_5^{3/2}, \quad \hat{\Lambda}(x, y) \equiv \hat{\Lambda} + 2\pi, \quad (2.38)$$

where $\hat{\Lambda}$ is the transformation function of the underlying 5D U(1) gauge symmetry, under which

$$\hat{A}_M \rightarrow \hat{A}_M + \partial_M \hat{\Lambda}. \quad (2.39)$$

Thus it seems that this construction apparently solves all problems of the CCW-I.

However this construction also has its own limitation. Notice that the redefined couplings $\hat{\kappa}_0, \hat{\kappa}_\pi$ and \hat{Q}_0, \hat{Q}_π reveal an exponential hierarchy in (2.34) if the original couplings κ_0, κ_π and Q_0, Q_π were of order unity in order to reproduce the continuum version (2.21) of the original DCW model. On the other hand, in the 5D theory (2.35) of $\hat{\Phi}$ and \hat{A}_M obeying the periodicity condition (2.38), the redefined couplings $\hat{\kappa}_{0,\pi}$ and $\hat{Q}_{0,\pi}$, *not* the original couplings $\kappa_{0,\pi}$ and $Q_{0,\pi}$, are required to be integer-valued, so have natural size of order unity. Furthermore, the zero modes of $\hat{\Phi}$ and \hat{A}_M in (2.35) have flat profile over the

⁷We can have $\mu \neq \tilde{\mu}$ if the bulk dilaton couplings are different from the couplings of the original linear dilaton model. This can be also realized in the general CCW as we will discuss later.

5th dimension because of the absence of bulk and boundary masses.⁸ As a consequence, the *CCW-II cannot reproduce the exponential hierarchy among the zero mode couplings at different boundaries*, which is one of the primary features of the original DCW model. One may wonder whether the situation can be changed by introducing dilaton-dependent couplings ($\propto e^{-\gamma S}$) to the boundary operators in order to realize the desired exponential hierarchy among the zero mode couplings at different boundaries. However, again once one imposes the periodicity condition (2.38), such dilaton-dependent couplings are forbidden by the discrete shift symmetry of $\hat{\Phi}$ and the 5D gauge symmetry of \hat{A}_M . Nevertheless, as was stressed in [29], many of the other features of the DCW can be successfully reproduced in the CCW-II realized through the background geometry of the linear dilaton model.

The main reason for the above limitation of the CCW-II is attributed to the absence of localization of zero mode of 5D bosonic fields without bulk and boundary masses. There is still an example that localized zero mode profile occurs by warped geometry alone: the 5th component of a 5D gauge field C_M with odd orbifold-parity whose zero mode can be interpreted as a 4D axion field [33, 34]. In this case, there is a corresponding localized CW symmetry protecting the zero mode axion as we will discuss in section 3. However, as we will see, the 5D gauge symmetry of C_M forbids certain boundary couplings of C_5 , so still prohibits the full realization of the characteristic features of the DCW such as the exponentially enlarged axion field range and/or the exponential hierarchy among the zero mode axion couplings to instantons at different boundaries.

To summarize, both CCW-I and CCW-II have their own limitations and realize only certain partial features of the original DCW model. For appropriately chosen 5D metric backgrounds, i.e. the flat background (2.26) for CCW-I and the linear dilaton background (2.36) for CCW-II, two models are related to each other by the field and parameter redefinitions (2.33) and (2.34). This suggests that two schemes share certain features, for instance the KK spectrum which are independent of the boundary couplings in our approximation. Yet, CCW-I and CCW-II are totally different models. In CCW-I, the CW symmetry which is supposed to protect the zero mode is not respected by the gravitational interactions of the metric fluctuation, which would raise a question about the naturalness of the scheme. In CCW-II, zero modes have flat profile over the 5th dimension in the field basis for which the 5D U(1) and discrete gauge symmetries take the standard form as (2.31), so there is no hierarchy between the zero mode couplings at different boundaries. Related to this point, in CCW-I the *y-independent* periodicities of the 5D axion field and the 5D U(1) gauge symmetry transformation (2.31) are imposed on Φ and A_M , and as a consequence the corresponding boundary couplings $\kappa_{0,\pi}$ and $Q_{0,\pi}$ are quantized in the common unit, suggesting that they all have integer values of order unity. On the other hand, in CCW-II, the same applies for the redefined fields $\hat{\Phi}$ and \hat{A}_M and boundary couplings $\hat{\kappa}_{0,\pi}$ and $\hat{Q}_{0,\pi}$, *not* for the original fields and couplings. This is the reason why the two models predict different hierarchical patterns for the effective couplings of zero modes at different boundaries. In the following, we will introduce *general CCW* in which CCW-I

⁸Note that we are considering the localized zero mode profile in the field basis defined by the 5D gauge symmetries (2.31), and the corresponding field basis for CCW-II is provided by the redefined fields $\hat{\Phi}$ and \hat{A}_M , not by the original fields Φ and A_M .

and CCW-II are incorporated into common framework which can realize the continuum CW in a most general way.

2.2.3 General CCW

Inspired by the previous discussions, let us define the general continuum clockwork (general CCW) as an extra-dimensional setup yielding light mode by means of a localized symmetry. In this perspective, we can start with the following most general quadratic lagrangian which might be obtained after replacing the background geometry and dilaton with their vacuum expectation values which are supposed to have an exponential y -dependence:

$$-\frac{1}{2} \int_0^{\pi R} dy \left[e^{-2\mu_1 y} (\partial_\mu \Phi)^2 + e^{-2\mu_2 y} (\partial_y \Phi)^2 + e^{-2\mu_3 y} m_B^2 \Phi^2 + e^{-2\mu_4 y} (m_0 \delta(y) + m_\pi \delta(y - \pi R)) \Phi^2 \right]. \quad (2.40)$$

Requiring to be invariant under the following *localized infinitesimal* shift symmetry

$$\delta \Phi = c_0 e^{-\mu y}, \quad (2.41)$$

where c_0 is an infinitesimal constant, we find the model parameters should obey the following relations

$$\mu_2 = \mu_3 = \mu_4, \quad m_B^2 = \mu(\mu + 2\mu_2), \quad -m_0 = m_\pi = \mu, \quad (2.42)$$

for which the lagrangian can be written in a form which is manifestly invariant under (2.41):

$$-\frac{1}{2} \int dy \left[e^{-2\mu_1 y} (\partial_\mu \Phi)^2 + e^{-2\mu_2 y} (\partial_y \Phi + \mu \Phi)^2 \right]. \quad (2.43)$$

Similarly one can consider a general CCW theory for U(1) gauge bosons invariant under a localized infinitesimal U(1) gauge symmetry:

$$\delta A_\mu = e^{-\tilde{\mu} y} \partial_\mu \beta(x), \quad (2.44)$$

which result in

$$\int dy \left[-\frac{1}{4g_5^2} \left\{ e^{-\tilde{\mu}_1 y} (F_{\mu\nu})^2 + e^{-2\tilde{\mu}_2 y} (\partial_\mu \omega - \partial_y A_\mu - \tilde{\mu} A_\mu)^2 \right\} + \dots \right] \quad (2.45)$$

Yet we have a freedom to redefine the fields as

$$\Phi \rightarrow e^{\mu_0 y} \Phi, \quad (A_\mu, \omega) \rightarrow e^{\tilde{\mu}_0 y} (A_\mu, \omega), \quad (2.46)$$

while keeping the lagrangian and localized symmetry to take the same form, but with the redefined fields and parameters. This makes the physical interpretation (or the physical origin) of each of the CCW parameters, e.g. μ_1, μ_2, μ for the CCW axion, obscure as they are not invariant under the field redefinition (2.46).

On the other hand, we are interested in the possibility that the CCW lagrangians (2.43) and (2.45) originate from a 5D theory in which Φ can be identified as a periodic 5D scalar field associated with a discrete gauge shift symmetry of the form:

$$\Phi \rightarrow \Phi + 2\pi f_5^{3/2}, \quad (2.47)$$

and the 5D vector field A_M is introduced as the gauge field of compact (Z_2 even or odd) $U(1)$ gauge symmetry, under which

$$A_M \rightarrow A_M + \partial_M \Lambda \quad (\Lambda(x, y) \equiv \Lambda(x, y) + 2\pi). \quad (2.48)$$

Once Φ and A_M are identified as such field variables in the underlying UV theory, we do not have anymore a freedom to make the field redefinition (2.46), and can attempt to embed (2.43) and (2.45) into a 5D diffeomorphism and Lorentz invariant theory defined in the field basis fixed by the above form of 5D discrete shift and $U(1)$ gauge symmetries. As we will see, in such prescription, one can make an unambiguous physical distinction between μ_1, μ_2 and μ . Note that the 5D discrete shift and compact $U(1)$ gauge symmetries, which appear to be inevitable in any sensible UV completion of CCW, ensure the quantization of the $U(1)$ gauge charges and the axion-instanton couplings in the underlying UV theory, and therefore provide also a criterion for natural size of relevant UV parameters, which would be essential for a proper interpretation of the exponential hierarchies generated by the CW mechanism.

As will be explained in the next section in detail, the most natural way to implement the general CCW within the framework of 5D diffeomorphism and Lorentz invariant theory is the *generalized linear dilaton model*. For instance, the general CCW axion lagrangian (2.43) at quadratic order can be obtained from the following 5D action

$$- \int d^5x \sqrt{-G} \left[\frac{1}{2} G^{MN} \partial_M \Phi \partial_N \Phi + \frac{1}{2} e^{-2cS} f_5^3 V_B \left(\frac{\Phi}{f_5^{3/2}} \right) - e^{-cS} f_5^3 \left(\frac{\delta(y)}{\sqrt{G_{55}}} - \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \right) V_b \left(\frac{\Phi}{f_5^{3/2}} \right) \right], \quad (2.49)$$

where $V_B(\theta)$ and $V_b(\theta)$ are even functions of the periodic angle $\theta \equiv \theta + 2\pi$, which are manifestly invariant under the discrete gauge symmetry (2.47), while the background geometry and dilaton turn out to have the following expectation values:

$$ds^2 = \langle G_{MN} \rangle dx^M dx^N = e^{2k_1 y} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2k_2 y} dy^2, \quad \langle e^{cS} \rangle = e^{k_2 y}. \quad (2.50)$$

Then the CCW parameters μ_1, μ_2 are determined by the geometric parameters k_1 and k_2 describing the exponential warp factor and the exponentially large volume, respectively,

$$\mu_1 = - \left(k_1 + \frac{k_2}{2} \right), \quad \mu_2 = - \left(2k_1 - \frac{k_2}{2} \right), \quad (2.51)$$

while the CCW parameter μ for localized symmetry originates from the bulk and boundary masses for $V_B(\theta) \simeq V_{B0} + m_B^2 \theta^2$, $V_b(\theta) \simeq V_{b0} + m_b \theta^2$ in the vicinity of $\Phi = 0$, which are tuned to satisfy the following relations to have the localized symmetry (2.41):⁹

$$\mu = m_b, \quad m_B^2 = m_b (m_b + 4k_1 - k_2). \quad (2.52)$$

⁹For the arbitrary value of Φ , the condition becomes $V_B = \frac{1}{4}(V_b')^2 + (4k_1 - k_2)V_b + \text{Const.}$. The localized symmetry must be generalized as (2.30) for arbitrary value of Φ to have the discrete gauge symmetry (2.47).

Therefore, the generalized linear dilaton model implements the general CCW with three independent parameters k_1, k_2 and μ . We can also see that CCW-I corresponds to $\mu \neq 0$ and $k_1 = k_2 = 0$, while CCW-II corresponds to $\mu = 0$ and $k_1 = k_2 \neq 0$. Thus the general CCW incorporates the two schemes into a single framework and can exhibit continuous deformation among all possible implementations of the CCW.

As we will show in the next section, the three independent parameters are responsible for the following three characteristic features of the CW mechanism: i) the bulk and boundary mass parameter μ determines the hierarchy between the CW zero mode couplings at different boundaries, ii) a certain combination of μ and the warp factor parameter k_1 (for a given k_2) determines the hierarchy between the zero mode coupling and the massive KK mode couplings at the same boundary, and iii) the difference between the warp factor and the volume factor, i.e. $k_1 - k_2$, controls the mass gap between the massive gear modes (KK modes).

As noticed in the previous discussion, for non-zero $\mu, \tilde{\mu}$ the localized symmetries (2.41) and (2.44) to protect zero modes are not respected by the couplings of metric and dilaton fluctuations, while an exponential hierarchy between the zero mode couplings at different boundaries is possible only with non-zero $\mu, \tilde{\mu}$. On the other hand, if $\mu, \tilde{\mu}$ vanish, the zero modes are protected by an unbroken symmetry respected by the metric and dilaton fluctuations, while they lose the localized feature for generating an exponential hierarchy among the couplings at different boundaries.

Before closing this section, let us briefly summarize the main features of the continuum CW.

1. General CCW is defined as a generic 5D theory producing light 4D modes protected by a symmetry localized in extra dimension. It can be considered as a generalization of the naive continuum limit of the DCW models, which can accommodate both the CCW from geometry and the CCW associated with localized zero mode profile over the extra dimension, which is achieved by fine-tuned bulk and boundary masses. However, there are always certain limitations of CCW, so even general CCW can not produce all of the interesting features of DCW. For instance, general CCW could yield neither an exponentially enhanced trans-Planckian field range of the zero mode axion, nor an exponential hierarchy among the 4D gauge charges, while keeping the quantized nature of gauge charges of an unbroken 4D U(1) gauge symmetry.
2. Sensible UV theory for axions and Abelian gauge bosons should be based on a discrete gauge shift symmetry for the periodicity of axions and/or the compact U(1) gauge symmetries for gauge bosons, which are compatible with the 5D diffeomorphism and Lorentz symmetry. Specifically, one may start with a 5D theory invariant under

$$\begin{aligned}
 T_\Phi : \quad \Phi &\rightarrow \Phi + 2\pi f_5^{3/2}, \\
 U(1)_{5D} : \quad A_M &\rightarrow A_M + \partial_M \Lambda \quad \text{for} \quad \Lambda(x, y) \equiv \Lambda(x, y) + 2\pi.
 \end{aligned}
 \tag{2.53}$$

Then, in the above field basis defined in terms of T_Φ and $U(1)_{5D}$, the physical origin and meaning of the general CCW model parameters, including the couplings to

instantons and charged particles at the boundaries, are unambiguously identified. In this prescription, there are two model parameters μ_1, μ_2 describing the effects of background geometry and dilaton field profile, and one parameter μ (or $\tilde{\mu}$) describing the localization of zero mode axion (or gauge boson) by fine-tuned bulk and boundary masses.

3. Only when $\mu = \tilde{\mu} = 0$, the infinitesimal localized CW symmetries to protect zero modes can be embedded into a sensible 5D symmetry compatible with the 5D space-time symmetries.
4. For $\mu \neq 0$ (or $\tilde{\mu} \neq 0$), the localized infinitesimal CW symmetry for zero mode axion (or gauge boson) corresponds to an accidental approximate symmetry not respected by the metric and dilaton fluctuations over the background vacuum values.

3 5D models for continuum clockwork

In this section, we provide a concrete framework to realize general CCW discussed in the previous section. We first introduce the *generalized linear dilaton model* which is obtained by generalizing the dilaton couplings of the original linear dilaton model [28]. This model will provide the background geometry and dilaton profile necessary to implement general CCW. We then analyze the mass spectrum and couplings of the KK graviton modes in general CCW. It turns out that many of the results on KK gravitons apply also to the KK modes of CW axions and U(1) gauge bosons, reproducing many of the characteristic features of the DCW model. In the subsequent subsections, we examine more detailed features of the CCW axions originating from a 5D angular field and/or a 5D gauge field with odd orbifold-parity, and also the CCW photon originating from a 5D gauge field with even orbifold-parity.

3.1 Generalized linear dilaton model

We start from a five-dimensional (5D) model with the 5th dimension compactified on orbifold S_1/Z_2 with the fixed points at $y = \{0, \pi R\}$. All 5D fields in the model obey the boundary conditions

$$\Phi_{\pm}(y + 2\pi R) = \Phi_{\pm}(y), \quad \Phi_{\pm}(-y) = \pm\Phi_{\pm}(y), \tag{3.1}$$

where \pm denotes the orbifold parity. The most general form of 5D metric invariant under the 4D Poincare symmetry is given by¹⁰

$$ds^2 = \langle G_{MN} \rangle dx^M dx^N = \Omega_1(y) \eta_{\mu\nu} dx^\mu dx^\nu + \Omega_2(y) dy^2, \tag{3.2}$$

where Ω_1 corresponds to the warp factor of the 4D metric component, while $\Omega_2 = G_{55}$ defines the physical length of the 5th dimension: $L_5 = \int_0^{\pi R} dy \sqrt{\Omega_2}$. Throughout this

¹⁰The corresponding solution of the inflationary 4D spacetime for the linear dilaton model was studied in [8, 35]. This might be straightforwardly extended to our generalized model.

paper, we consider the case that underlying 5D gravity dynamics generates an exponential y -dependence in $\Omega_{1,2}$:

$$\Omega_1(y) = e^{2k_1 y}, \quad \Omega_2(y) = e^{2k_2 y}, \quad (3.3)$$

where k_1 and k_2 are generically independent mass parameters with the same sign. As we will see, the above form of 5D metric can be obtained as a solution of linear dilaton model which is appropriately generalized from the well known model of ref. [28].

Motivated by the UV origin of the linear dilaton model as a dual description of the little string theory [28] (see also [36, 37]), we consider the following 5D model with a universal dilaton coupling, which might be interpreted as the 5D action in the string frame:

$$S = \int d^4x \int_{-\pi R}^{\pi R} dy \sqrt{-G} e^{\xi S} \left(\frac{M_5^3}{2} R_5 + \frac{M_5^3}{2} \partial_M S \partial^M S - \Lambda_5 - \frac{1}{\sqrt{G_{55}}} [\Lambda_0 \delta(y) + \Lambda_\pi \delta(y - \pi R)] + \dots \right), \quad (3.4)$$

where M_5 is the 5D cut-off scale, $\Lambda_{5,0,\pi}$ are the bulk and boundary cosmological constants, and the normalization of the dimensionless dilaton S is fixed by the coefficient of the kinetic term. Notice that we introduce a general dilaton coupling ξ , which was taken to be 1 in the original model [28]. Here we assume that the underlying theory involves a small (string) coupling $g_{\text{st}}^2 \propto e^{-\xi S}$, which would assure that the above 5D action defined at the leading order in g_{st}^2 provides a sufficiently good approximation. As we go to the Einstein frame with

$$g_{MN} \rightarrow e^{-2\xi S/3} g_{MN}, \quad (3.5)$$

and subsequently rescale the dilaton field to arrive at the canonically normalized kinetic term, the action reads

$$S = \int d^5x \sqrt{-G} \left(\frac{M_5^3}{2} R_5 - \frac{M_5^3}{2} \partial_M S \partial^M S - e^{-2cS} \Lambda_5 - \frac{e^{-cS}}{\sqrt{G_{55}}} [\Lambda_0 \delta(y) + \Lambda_\pi \delta(y - \pi R)] + \dots \right), \quad (3.6)$$

where

$$c \equiv \frac{\xi}{\sqrt{12\xi^2 - 9}}. \quad (3.7)$$

We will see that the value of c different from $1/\sqrt{3}$ ($\xi = 1$) accounts for the background metric and dilaton profile necessary to realize general CCW.

Since the cosmological constants $\Lambda_{5,0,\pi}$ break the dilatonic shift symmetry

$$S \rightarrow S + \alpha, \quad (3.8)$$

one can take them to be soft symmetry breaking parameters whose scale is well below the 5D cut-off scale M_5 . Let us parameterize them as

$$\Lambda_5 = -2M_5^3 k^2, \quad \Lambda_0 = -4M_5^3 k_0, \quad \Lambda_\pi = 4M_5^3 k_\pi, \quad (3.9)$$

with $k, k_0, \pi < M_5$. Then the leading order action of the dilaton-graviton system is written as

$$S_{\text{gravity}} = \int d^5x \sqrt{-G} M_5^3 \left(\frac{R_5}{2} - \frac{1}{2} \partial_M S \partial^M S + 2e^{-2cS} k^2 + 4 \frac{e^{-cS}}{\sqrt{G_{55}}} [k_0 \delta(y) - k_\pi \delta(y - \pi R)] \right). \quad (3.10)$$

Now we can solve the equations of motion with the following ansatz¹¹

$$ds^2 = e^{2k_1|y|} (\eta_{\mu\nu} dx^\mu dx^\nu) + e^{2k_2|y|} dy^2, \quad S = k_3|y|. \quad (3.11)$$

We then find a solution with k_i ($i = 1, 2, 3$) given by

$$k_1 = \frac{2k}{\sqrt{12 - 9c^2}}, \quad k_2 = ck_3 = 3c^2 k_1, \quad (3.12)$$

while the two boundary parameters $k_{0,\pi}$ are fixed by the boundary conditions as

$$k_0 = k_\pi = \frac{3}{2} k_1. \quad (3.13)$$

Note that k is real (AdS₅ bulk space) for $c^2 < 4/3$, while it is imaginary (dS₅ bulk space) for $c^2 > 4/3$.

From the above solution, we can obtain the two well-known limits:

$$\begin{aligned} c = 0 : & \quad \text{RS geometry with} & \quad k_2 = 0, \\ c = \frac{1}{\sqrt{3}} : & \quad \text{Linear Dilaton geometry with} & \quad k_1 = k_2. \end{aligned} \quad (3.14)$$

The large extra dimension (LED) limit can be obtained also by taking $c \gg 1$, which yields $k_2 \pi R \gg 1 \gg k_1 \pi R$.

We are basically interested in the parameter space where the background geometry provides a solution to the weak scale hierarchy problem. This is addressed with an appropriate size of the extra dimension depending on the values of k_1, k_2 . The 4D Planck mass is evaluated as

$$M_{\text{Pl}}^2 = M_5^3 \int_{-\pi R}^{\pi R} dy e^{(2k_1+k_2)|y|} = \frac{M_5^3}{k_1 + k_2/2} \left(e^{2(k_1+k_2/2)\pi R} - 1 \right), \quad (3.15)$$

while the physical size of the extra dimension is given by

$$L_5 = \int_0^{\pi R} dy e^{k_2|y|} = \frac{1}{k_2} \left(e^{k_2 \pi R} - 1 \right). \quad (3.16)$$

¹¹By rescaling the soft parameters, $k \rightarrow e^{cS|_{y=0}} k$, we can set $S|_{y=0} = 0$. It is assumed that the additional boundary dynamics determines the radius, R , of the 5th dimension.

From (3.12), one can see that k_1 and k_2 must have the same sign for real c , and their sign is determined by the sign of k . If we take them to be positive, the boundary at $y = 0$ is identified as the IR boundary, while the boundary at $y = \pi R$ becomes the UV boundary. For negative $k_{1,2}$, the boundaries are flipped without changing physics. In the following, we will take the convention of $k_{1,2} \geq 0$. Then, combining (3.15) and (3.16), we can express the proper length L_5 of the 5-th dimension in terms of M_{Pl}, M_5, k_1 and k_2 :

$$L_5 = \frac{1}{k_2} \left[\left(\frac{(k_1 + \frac{k_2}{2})M_{\text{Pl}}^2}{M_5^3} + 1 \right)^{\frac{k_2}{2k_1+k_2}} - 1 \right] \tag{3.17}$$

$$\approx \frac{1}{k_2} \left[\left(\frac{M_{\text{Pl}}}{M_5} \right)^{\frac{2(k_2/k_1)}{2+(k_2/k_1)}} - 1 \right] \approx \begin{cases} \frac{1}{k_1} \ln \frac{M_{\text{Pl}}}{M_5} & \text{as } k_2/k_1 \rightarrow 0 \text{ (RS)} \\ \frac{1}{k_2} \frac{M_{\text{Pl}}^2}{M_5^2} & \text{as } k_2/k_1 \rightarrow \infty \text{ (LED)} \end{cases},$$

where $0 < k_2/k_1 < 4$ for real k . Imposing the experimental upper limit on $L_5 \lesssim 100 \mu\text{m}$ [38], we find

$$k_2 \lesssim 2k_1 \times \frac{1-b}{1+b}, \tag{3.18}$$

where

$$b \equiv \frac{\log_{10}(M_5/k_2) - 2 \log_{10}(M_5/\text{TeV})}{15 - \log_{10}(M_5/\text{TeV})}. \tag{3.19}$$

Therefore, if one wishes to address the hierarchy problem with the lowest possible $M_5 \sim \text{TeV}$, k_2 cannot be larger than about $2k_1$.

3.2 KK spectrum and couplings

Let us now discuss the spectrum and couplings of the bulk graviton KK modes in the generalized linear dilaton background. It turns out that much of the results apply also to the CW axions and U(1) gauge bosons.¹²

To quadratic order of the 4D metric fluctuation $h_{\mu\nu}$ ($\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + 2h_{\mu\nu}$) in the traceless transverse gauge $h^\mu_\mu = \partial^\mu h_{\mu\nu} = 0$, the action (3.10) gives rise to

$$S_{\text{gravity}} = -M_5^3 \int d^5x e^{(2k_1+k_2)|y|} \left[\frac{1}{2} (\partial_\rho h_{\mu\nu})^2 + \frac{1}{2} e^{2(k_1-k_2)|y|} (\partial_y h_{\mu\nu})^2 \right]. \tag{3.20}$$

Notice that the (bulk or boundary) graviton mass term is absent due to the 4D diffeomorphism invariance even in the presence of non-trivial quadratic order graviton couplings to the dilaton background in S_{gravity} . As we will show later, a generic quadratic action for 5D CW bosonic field Φ (= graviton, axion or U(1) gauge boson) takes the form:

$$S_\Phi = - \int d^5x e^{2k_\Phi|y|} \left[\frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2} e^{2(k_1-k_2)|y|} (\partial_y \Phi - \mu_\Phi \epsilon(y)\Phi)^2 \right], \tag{3.21}$$

¹²The spectrum and couplings for KK modes of the dilaton and also radion field depend on the additional boundary dynamics that stabilize both fields. Such a model dependence is studied in [39] for the linear dilaton model.

where k_Φ is some combination of k_1 and k_2 describing the effects of background metric and dilaton profile, and μ_Φ originates from the bulk and boundary masses. For graviton, $k_\Phi = k_1 + k_2/2$ and $\mu_\Phi = 0$ as can be seen from (3.20). Notice that this corresponds to the defining lagrangian (2.43) of general CCW whose parameters are identified as

$$\mu_1 = -k_\Phi, \quad \mu_2 = -(k_\Phi + k_1 - k_2), \quad \mu = -\mu_\Phi. \quad (3.22)$$

The field basis used in (3.21) corresponds to the one for which the 5D discrete shift or continuous U(1) gauge symmetries are defined as (2.53), and therefore both the natural size of the corresponding couplings and the origin of the mass parameters k_Φ , μ_Φ , $k_1 - k_2$ are unambiguously determined. To perform the KK analysis, however, it is convenient to use the other field basis with the field redefinition

$$\Phi_c = e^{k_\Phi|y|}\Phi, \quad (3.23)$$

for which the action (3.21) reads

$$S_\Phi = - \int d^5x \left[\frac{1}{2}(\partial_\mu \Phi_c)^2 + \frac{1}{2}e^{2\kappa|y|}(\partial_y \Phi_c - m_\Phi \epsilon(y)\Phi_c)^2 \right], \quad (3.24)$$

where

$$m_\Phi = k_\Phi + \mu_\Phi, \quad \kappa = k_1 - k_2. \quad (3.25)$$

The equation of motion for Φ_c is then given by

$$\left(\partial_y^2 + 2\kappa \epsilon(y)\partial_y + e^{-2\kappa|y|}\eta^{\mu\nu}\partial_\mu\partial_\nu - M_\Phi^2 \right) \Phi_c(x, y) = 2m_\Phi [\delta(y) - \delta(y - \pi R)] \Phi_c(x, y), \quad (3.26)$$

where

$$M_\Phi^2 \equiv m_\Phi^2 + 2\kappa m_\Phi. \quad (3.27)$$

Decomposing Φ_c as

$$\Phi_c(x, y) = \sum_{n=0}^{\infty} f_n(y) \phi_n(x), \quad (3.28)$$

the equation for the mode functions f_n are given by

$$\left(\partial_y^2 + 2\kappa \epsilon(y)\partial_y + e^{-2\kappa|y|}M_n^2 - M_\Phi^2 \right) f_n(y) = 2m_\Phi [\delta(y) - \delta(y - \pi R)] f_n(y), \quad (3.29)$$

where M_n denotes the 4D mass eigenvalues of the corresponding 4D field $\phi^{(n)}$:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi_n(x) = M_n^2\phi_n(x). \quad (3.30)$$

The solution is then found to be¹³

$$f_0(y) = \frac{1}{N_0} e^{m_\Phi |y|},$$

$$f_n(y) = \frac{1}{N_n} e^{-\kappa |y|} \left[J_{|\alpha|} \left(\frac{M_n}{|\kappa|} e^{-\kappa |y|} \right) + b_\alpha(M_n) Y_{|\alpha|} \left(\frac{M_n}{|\kappa|} e^{-\kappa |y|} \right) \right] \quad (n = 1, 2, \dots), \quad (3.32)$$

where N_n is the normalization factor, $J_{|\alpha|}(x)$ and $Y_{|\alpha|}(x)$ are the first and second kind Bessel functions, respectively, and

$$\alpha = 1 + \frac{m_\Phi}{\kappa}, \quad (3.33)$$

$$b_\alpha(M_n) = - \frac{J_{\text{sgn}(\alpha)(\alpha-1)} \left(\frac{M_n}{|\kappa|} \right)}{Y_{\text{sgn}(\alpha)(\alpha-1)} \left(\frac{M_n}{|\kappa|} \right)}. \quad (3.34)$$

The zero mode ϕ_0 is massless as expected, and the mass eigenvalues M_n of the massive modes ϕ_n ($n = 1, 2, 3, \dots$) can be determined by the following boundary condition:

$$b_\alpha(M_n e^{-\kappa \pi R}) = b_\alpha(M_n), \quad (3.35)$$

which results in

$$M_n \simeq \begin{cases} \left(n - \frac{1}{4} + \left| \frac{m_\Phi}{2\kappa} \right| \right) \pi \kappa, & \kappa \gtrsim |m_\Phi| \\ \sqrt{m_\Phi^2 + \frac{n^2}{R^2}}, & \kappa \simeq 0 \\ \left(n - \frac{1}{4} + \left| \frac{m_\Phi}{2\kappa} \right| \right) \pi |\kappa| e^{-|\kappa| \pi R}, & \kappa \lesssim -|m_\Phi| \end{cases} \quad (3.36)$$

The above result shows that the spectrum is similar to the case of RS for $\kappa \gtrsim |m_\Phi|$, while the overall mass scale is suppressed for $\kappa \lesssim -|m_\Phi|$ as in the LED case. In figure 1, we depict how the KK masses behave as a function of $(k_1 - k_2)/m_\Phi$ for fixed values of m_Φ and R .

In fact, the parameter point $\kappa \equiv k_1 - k_2 = 0$ is special in regard to the KK mass pattern. In figure 2, we plot the mass ratios M_n/M_1 as a function of $\kappa/|m_\Phi|$, again for fixed values of m_Φ and R . It shows that the mass gap among the KK states is minimal in the unit of the first KK mass M_1 for $\kappa = 0$. This feature is related to the fact that the CW gear masses in the DCW are nearly degenerate at the CW threshold scale. As we have shown in the previous section, the direct continuum limit of the DCW produces either CCW-I or CCW-II depending on which field basis is used to realize the 5D discrete shift or continuous U(1) gauge symmetry as in (2.53). Notice that both 5D models correspond to the special case with $\kappa \equiv k_1 - k_2 = 0$ in view of general CCW. Therefore, the relatively degenerate KK masses for a given KK threshold scale can be considered as a special feature of the CCW obtained as a direct continuum limit of the DCW, but not a generic feature of general CCW.

¹³For $\kappa = 0$, taking the limit $\kappa \rightarrow 0$ in (3.32) is rather tricky, and the corresponding explicit expression is given by

$$f_n(y)|_{\kappa=0} = \frac{1}{N_n} \left[\frac{n}{R} \cos \left(\frac{n}{R} |y| \right) + m_\Phi \sin \left(\frac{n}{R} |y| \right) \right]. \quad (3.31)$$

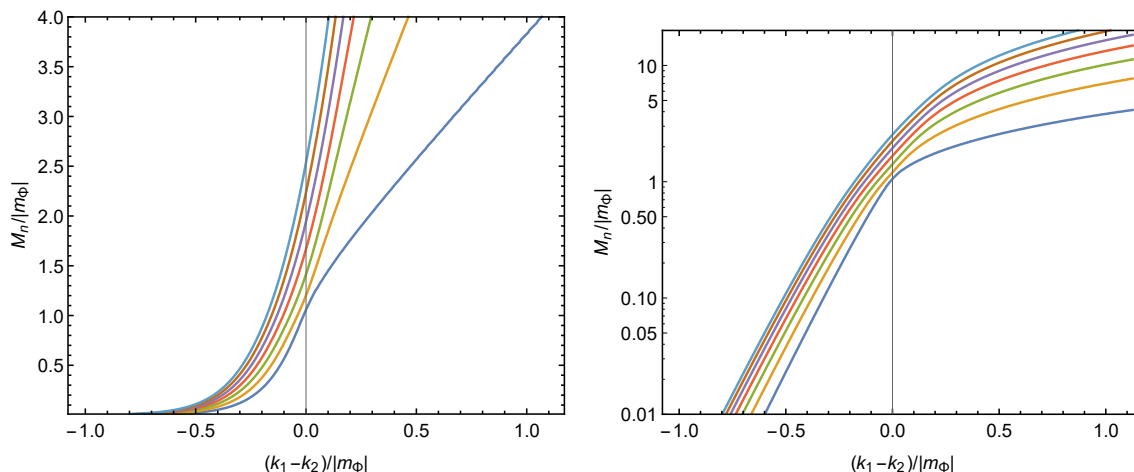


Figure 1. KK masses ($n = 1, 2, \dots, 7$) as a function of $\kappa/|m_\Phi| \equiv (k_1 - k_2)/|m_\Phi|$ for fixed values of m_Φ and R (left: linear, right: log). For graviton ($|m_\Phi| = k_1 + k_2/2$), the RS limit corresponds to $(k_1 - k_2)/|m_\Phi| = 1$, while the LED limit is $(k_1 - k_2)/|m_\Phi| = -2$.

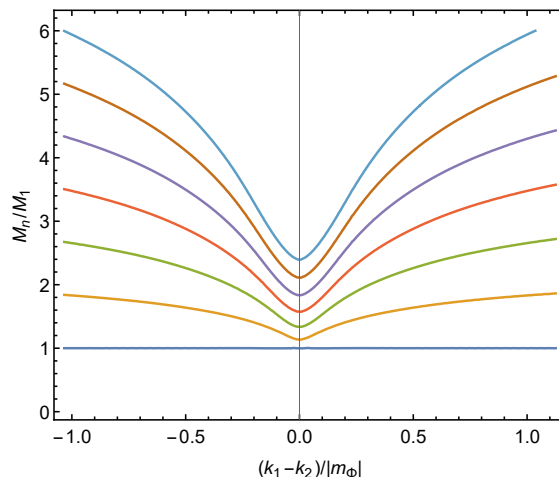


Figure 2. KK masses ($n = 1, 2, \dots, 7$) relative to the first KK mass M_1 as a function of $\kappa/|m_\Phi| \equiv (k_1 - k_2)/|m_\Phi|$ for fixed values of m_Φ and R .

Let us now turn to the couplings of the zero mode and KK modes to the boundary operators. For this, we should go back to the original field basis Φ rather than using the canonical basis Φ_c , since the natural size of the relevant boundary couplings are determined in the original field basis for which the 5D discrete shift and continuous U(1) gauge symmetries are defined as (2.53). Then the mode couplings to the boundary operators can be read off by the component of the corresponding mode in Φ . Explicitly, if we consider the couplings on the boundary at $y = y_b$, the mode coupling $C_n(y_b)$ is extracted from

$$\Phi(x, y = y_b) = \sum_n \sqrt{|m_\Phi|} C_n(y_b) \phi_n(x), \tag{3.37}$$

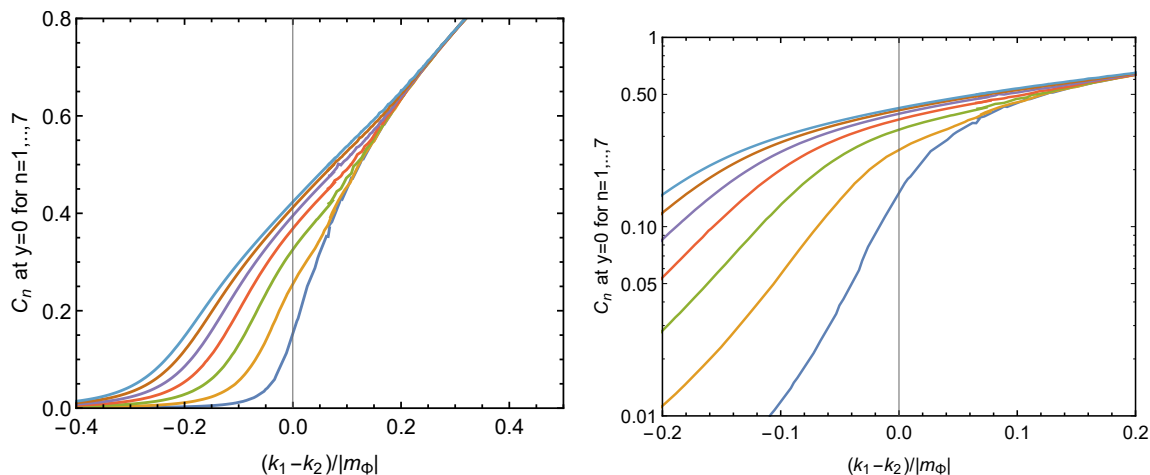


Figure 3. The couplings between the boundary localized fields (at $y = 0$) and the KK modes ($n = 1, 2, \dots, 7$) as a function of $\kappa/|m_\Phi| \equiv (k_1 - k_2)/|m_\Phi|$ for fixed value of m_Φ and R .

where $C_n(y_b)$ is determined to be

$$C_n(y_b) = \frac{f_n(y_b)}{\sqrt{|m_\Phi|}} e^{-k_\Phi |y|} \quad (3.38)$$

for $f_n(y)$ given by (3.32).

Let us first discuss the case $m_\Phi > 0$, which applies to the graviton case with $m_\Phi = k_1 + k_2/2 > 0$ as we take the convention $k_1, k_2 > 0$. The resulting couplings at $y = 0$ are calculated to be

$$C_0(0) = \frac{1}{\sqrt{|e^{2m_\Phi \pi R} - 1|}}, \quad (3.39)$$

$$C_n(0) \simeq \begin{cases} (-1)^{n-1} \sqrt{\frac{\kappa}{m_\Phi}}, & \kappa \gtrsim m_\Phi \\ \frac{1}{\sqrt{m_\Phi \pi R}} \frac{n}{\sqrt{m_\Phi^2 R^2 + n^2}}, & \kappa \simeq 0 \\ -\sqrt{\frac{|\kappa| \pi}{m_\Phi}} \frac{e^{-m_\Phi \pi R}}{\Gamma(m_\Phi/|\kappa|)} \left[\frac{\pi}{2} \left(n - \frac{1}{4} + \left| \frac{m_\Phi}{2\kappa} \right| \right) \right]^{\frac{m_\Phi}{|\kappa|} - \frac{1}{2}}, & \kappa \lesssim -m_\Phi \end{cases} \quad (3.40)$$

In figure 3, we depict the couplings of KK modes to boundary localized fields at $y = 0$ as a function of $\kappa/m_\Phi \equiv (k_1 - k_2)/m_\Phi$ for fixed value of m_Φ and R .¹⁴ Notice that there is an exponential hierarchy $\sim e^{-m_\Phi \pi R}$ between the zero mode couplings and the massive KK mode couplings at $y = 0$ for $\kappa \gtrsim 0$, which is one of the main features of the DCW. This hierarchy is controlled by $m_\Phi \equiv \mu_\Phi + k_\Phi$, i.e. the sum of the bulk and boundary mass parameter μ_Φ and the warp factor parameter k_Φ in the lagrangian (3.21). This means that both CCW-I ($\mu_\Phi \neq 0, k_\Phi = 0, \kappa = 0$) and CCW-II ($\mu_\Phi = 0, k_\Phi \neq 0, \kappa = 0$) possess this property. On the other hand, this exponential hierarchy disappears for $\kappa \lesssim -|m_\Phi|$ which

¹⁴In the figure, the absolute values $|C_n(0)|$ are depicted, since the sign of $C_n(0)$ is not physical so that it can be removed by redefining the mode field $\phi_n(x)$. On the other hand, note that the relative sign between the KK mode couplings at different boundaries has a physical meaning.

approaches to the LED limit. This can be easily understood by noting that deconstructing the 5D LED model gives a 4D lagrangian with the DCW parameter $q = 1$ [40, 41], i.e. there is no clockwork.

The zero mode and KK mode couplings at $y = \pi R$ for $m_\Phi > 0$ are determined by

$$C_0(\pi R) = C_0(0) e^{\mu_\Phi \pi R}, \tag{3.41}$$

$$C_n(\pi R) \simeq (-1)^n C_n(0) \begin{cases} \frac{\sqrt{\pi}}{\Gamma(m_\Phi/\kappa)} \left[\frac{\pi}{2} \left(n - \frac{1}{4} + \left| \frac{m_\Phi}{2\kappa} \right| \right) \right]^{\frac{m_\Phi}{\kappa} - \frac{1}{2}} e^{-(m_\Phi + k_\Phi)\pi R}, & \kappa \gtrsim m_\Phi \\ e^{-k_\Phi \pi R}, & \kappa \simeq 0 \\ \Gamma(m_\Phi/|\kappa|) \left[\frac{\pi}{2} \left(n - \frac{1}{4} + \left| \frac{m_\Phi}{2\kappa} \right| \right) \right]^{-\frac{m_\Phi}{|\kappa|} + \frac{1}{2}} e^{(m_\Phi - k_\Phi)\pi R}, & \kappa \lesssim -m_\Phi \end{cases} \tag{3.42}$$

At this boundary at $y = \pi R$, the exponential hierarchy between the zero mode couplings and the KK mode couplings vanishes for $\kappa \equiv k_1 - k_2 \simeq 0$ as expected from the DCW, while the KK mode couplings for $\kappa \gtrsim m_\Phi$ are exponentially smaller than the zero mode couplings as in the RS case.

For the case $m_\Phi < 0$, the couplings can be easily obtained using the relation

$$f_n(y)|_{\kappa, -m_\Phi} \simeq f_n(\pi R - y)|_{-\kappa, m_\Phi} \tag{3.43}$$

for $n \geq 1$. The resultant couplings of the KK modes ($n \geq 1$) are qualitatively similar to (3.40) and (3.42), but now m_Φ replaced by $|m_\Phi|$, while the expression for zero mode couplings are same as (3.39) and (3.41). In this case, the hierarchical structure of couplings turns out to be similar to flipping the boundaries to each other, while interchanging the RS limit and the LED limit in the case of $m_\Phi > 0$. Interestingly, this means that the KK couplings are similar to the LED limit while the KK spectrums are similar to the RS limit for $\kappa \gtrsim |m_\Phi|$, and vice versa for $\kappa \lesssim -|m_\Phi|$, when m_Φ is negative. Thus the sign of m_Φ for a given κ determines the pattern of the KK couplings, whereas the sign of κ determines the pattern of the KK spectrum.

Let us finally discuss the zero mode couplings at different boundaries. From (3.38), the zero mode coupling at $y = y_b$ is determined as

$$C_0(y_b) = \frac{e^{\mu_\Phi |y_b|}}{\sqrt{|e^{2m_\Phi \pi R} - 1|}}, \tag{3.44}$$

showing that a non-zero μ_Φ , which is obtained only by non-zero bulk and boundary masses, is necessary to obtain an exponential hierarchy between the zero mode couplings at different y_b . On the other hand, a non-zero $m_\Phi \equiv \mu_\Phi + k_\Phi$, which does not necessarily require non-zero μ_Φ , yields an exponential suppression of the zero mode coupling at $y_b = 0 (\pi R)$ for a positive (negative) m_Φ , which is responsible for the hierarchy between the zero mode coupling and the massive KK mode coupling at $y_b = 0 (\pi R)$.

In the following three subsections, we will examine how the lagrangian (3.21) can be obtained from a 5D diffeomorphism and Lorentz invariant theory for CCW axions and U(1) gauge bosons, and discuss further issues such as the field range of CCW axion zero mode.

3.3 CCW axion from 5D angular field

A natural candidate of 5D field that would yield a periodic 4D axion is a 5D angular field $\theta(x, y)$, with the periodicity:

$$\theta \equiv \theta + 2\pi. \quad (3.45)$$

Including a periodic potential of θ , as well as the couplings to gauge fields at the boundaries, the 5D action can be written as

$$S = - \int d^5x \sqrt{-G} \left[\frac{f_5^3}{2} G^{MN} \partial_M \theta \partial_N \theta + V(\theta) \right] \quad (3.46)$$

$$+ \frac{\theta}{32\pi^2} \left(\kappa_0 \frac{\delta(y)}{\sqrt{G_{55}}} G_0 \tilde{G}_0 + \kappa_\pi \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} G_\pi \tilde{G}_\pi \right), \quad (3.47)$$

where $G_{0,\pi}$ are the gauge field strengths at the boundaries $y = \{0, \pi R\}$, $\tilde{G}_{0,\pi}$ are their duals, $\kappa_{0,\pi}$ are restricted to be integers, and the dimensionful parameter f_5 is assumed to be well below the 5D Planck scale M_5 .

If $V(\theta) = 0$, there is an obvious massless 4D mode $\phi_0(x)$ protected by the shift symmetry $\theta \rightarrow \theta + \text{constant}$, and the corresponding mode function has a flat profile, i.e. $f_0(y) = \text{constant}$ for $\theta(x, y) = \sum_{n=0} f_n(y) \phi_n(x)$. In the presence of nonzero bulk and/or boundary potentials, generically there is no massless 4D mode. However, as we will show in the following, if the bulk and boundary potentials are appropriately related to each other, there can be a massless mode with exponentially localized profile over the 5th dimension, which reveals many features of the CW mechanism.

To achieve a 5D axion potential yielding such a zero mode, let us assume that both the dilatonic shift symmetry $S \rightarrow S + \text{constant}$ and the axionic shift symmetry $\theta \rightarrow \theta + \text{constant}$ are softly broken by a dimensionful mass parameter m through the combination

$$e^{-cS} e^{i\theta} m. \quad (3.48)$$

As the exponential dilaton dependence originates from the dilaton-dependent string coupling $g_{\text{st}}^2 \propto e^{-\xi S}$, at leading order in g_{st}^2 the bulk dilaton potential in the Einstein frame should scale as e^{-2cS} , while the boundary dilaton potential scales as e^{-cS} . (See eqs. (3.4) and (3.10).) In addition to the soft breaking by $e^{-cS+i\theta} m$ of both the dilatonic and axionic shift symmetries, the generalized linear dilaton model involves additional soft breaking by $k e^{-cS}$ of the dilatonic shift symmetry only, where k parametrizes the bulk cosmological constant as in (3.9). Then the leading order bulk and boundary potentials of the 5D angular field θ is generically given by

$$V(\theta) = -f_5^3 \left[e^{-2cS} (c_1 k m \cos \theta + c_2 m^2 \cos 2\theta) + \frac{e^{-cS}}{\sqrt{G_{55}}} m \cos \theta (c_0 \delta(y) - c_\pi \delta(y - \pi R)) \right], \quad (3.49)$$

where we assume the CP invariance under $\theta \rightarrow -\theta$, and $\{c_i\}$ are dimensionless coefficients of order unity. In order to have a localized massless mode under the metric and dilaton

background given by (3.11) and (3.12), one needs the following specific values of $\{c_i\}$:

$$c_1 = \frac{-4k_1 + k_2}{k} = -\sqrt{\frac{16}{3} - 4c^2}, \quad c_2 = \frac{1}{4}, \quad c_0 = c_\pi = 2. \quad (3.50)$$

Taking these particular parameter values, while replacing the spacetime metric and dilaton by their vacuum values, the 5D action (3.46) now takes the following form of general CCW:

$$S = -\frac{f_5^3}{2} \int d^5x e^{(2k_1+k_2)|y|} \left[(\partial_\mu \theta)^2 + e^{2(k_1-k_2)|y|} (\partial_y \theta - m \sin \theta)^2 \right] \quad (3.51)$$

which is manifestly invariant under the discrete shift $\theta \rightarrow \theta + 2\pi$.

To proceed, we can make a field redefinition

$$\varphi(x, y) \equiv \tan(\theta(x, y)/2), \quad (3.52)$$

and then the 5D action (3.51) is given by

$$-\frac{f_5^3}{2} \int d^5x \frac{4e^{(2k_1+k_2)|y|}}{(1+\varphi^2)^2} \left[(\partial_\mu \varphi)^2 + e^{2(k_1-k_2)|y|} (\partial_y \varphi - m\varphi)^2 \right]. \quad (3.53)$$

It is clear that the above action admits a localized massless 4D mode given by

$$\varphi(x, y) = e^{my} u(a(x)) \quad (3.54)$$

where $a(x)$ is the canonically normalized massless 4D axion, and $u(a)$ is a function of a introduced to make the 4D axion a have the standard canonical kinetic energy *over its entire field range*. For flat metric and trivial dilaton backgrounds, i.e. $k_1 = k_2 = 0$, which we will be focusing on, the kinetic term of a is given by

$$-\frac{f_5^3 (e^{2\pi m R} - 1) (\partial_\mu u)^2}{m(u^2 + 1)(e^{2\pi m R} u^2 + 1)} \equiv -\frac{1}{2} (\partial_\mu a)^2. \quad (3.55)$$

We then find

$$u(a) = ie^{-m\pi R} \operatorname{sn} \left(\frac{a}{2if} \middle| e^{-2m\pi R} \right), \quad (3.56)$$

where $f = \sqrt{f_5^3(1 - e^{-2m\pi R})/2m}$, and $\operatorname{sn}(x|z)$ is the Jacobi elliptic function which has two periods as $\operatorname{sn}(x|z) = (-1)^{n_1} \operatorname{sn}(x + 2n_1 K(z) + 2in_2 K(1-z)|z)$ for integer-valued n_1, n_2 and $K(z)$ being the complete elliptic integral of the first kind.

With the above model for CCW axion, we can address the following three issues:

- Wavefunction profile and the KK mass spectrum, $\{f_n(y), M_n\}$ for a given background value of $\langle \theta \rangle$.
- Field range of the canonically normalized 4D axion a .
- Effective potentials of the 4D axion a generated by the model parameters $\{c_i\}$ slightly deviated from the values of (3.50).

It is in principle straightforward to examine these issues for generic metric and dilaton background. However, for simplicity in the following we will focus on the simplest background with flat metric and constant dilaton, i.e. $k_1 = k_2 = 0$.

For the KK analysis, let us divide $\theta(x, y)$ into the background vacuum value $\langle \theta \rangle$ and a small fluctuation $\delta\theta$:

$$\theta(x, y) = \langle \theta \rangle + \delta\theta, \quad (3.57)$$

where the vacuum solution of θ is given by

$$\langle \theta \rangle = 2 \tan^{-1}[e^{my}\langle u(a) \rangle] = 2 \tan^{-1}[e^{m(y-y_0)}]. \quad (3.58)$$

Here the parameter y_0 is introduced to describe the vacuum value of the massless 4D axion a . As we will see later, y_0 describes also the position where the zero mode wavefunction is localized. Then the action (3.51) can be written as

$$-\frac{f_5^3}{2} \int d^5x \left[(\partial_\mu \delta\theta)^2 + (\partial_y \delta\theta + m \tanh[m(y-y_0)] \delta\theta)^2 + \mathcal{O}(\delta\theta^3) \right]. \quad (3.59)$$

At the quadratic order in $\delta\theta$, the above action reveals an infinitesimal shift symmetry

$$\delta\theta(x, y) \rightarrow \delta\theta(x, y) + c_0 \operatorname{sech}[m(y-y_0)] \quad (|c_0| \ll 1), \quad (3.60)$$

which insures that there exists a massless mode for any value of y_0 . Taking the mode expansion

$$\delta\theta(x, y) = \sum_{n=0}^{\infty} f_n(y) \delta\phi_n(x), \quad (3.61)$$

where $\delta\phi_n$ correspond to the canonically normalized 4D field fluctuations in mass eigenstates, we find the wavefunction profiles given by

$$\begin{aligned} f_0(y) &= \frac{1}{N_0} \operatorname{sech}[m(y-y_0)], \\ f_n(y) &= \frac{1}{N_n} \left[n \cos\left(\frac{ny}{R}\right) + mR \tanh[m(y-y_0)] \sin\left(\frac{ny}{R}\right) \right] \quad (n = 1, 2, \dots) \end{aligned} \quad (3.62)$$

and the mass eigenvalues:

$$M_0 = 0, \quad M_n = \sqrt{m^2 + \frac{n^2}{R^2}}. \quad (3.63)$$

In figure 4, we depict the wavefunction profiles of the zero and few KK modes for certain values of y_0 . Note that y_0 can have any value in the range $-\infty < y_0 < \infty$, while the orbifold coordinate y is defined only over the fundamental domain: $0 \leq y \leq \pi R$.

From the above wavefunction profile, we can obtain also the zero mode coupling to boundary operators. For this, let us introduce a boundary coupling of the form

$$\int dy \delta(y-y_b) \kappa_b \theta(x, y) \mathcal{O}(x), \quad (3.64)$$

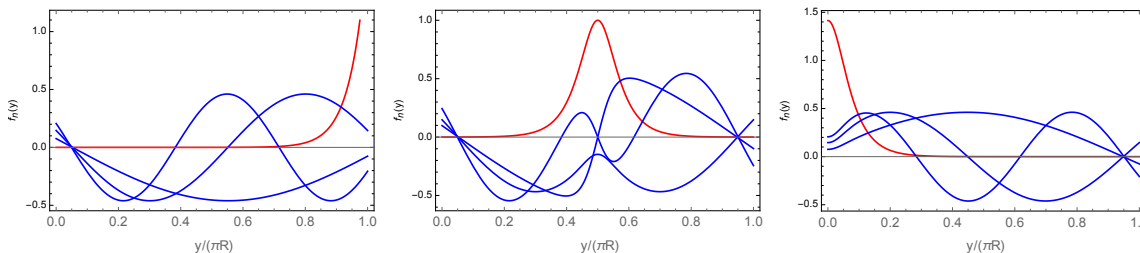


Figure 4. Wavefunction profile (3.62) for the zero mode fluctuation ($n = 0$: red) and massive mode fluctuations ($n=1, 2, 3$: blue) for $y_0 = 2\pi R$ (left), $y_0 = \pi R/2$ (middle) and $y_0 = 0$ (right). For illustration, we take $mR = 6$.

where $\mathcal{O}(x)$ is a local operator constructed by generic boundary fields at $y = y_b$. The resulting coupling of the zero mode fluctuation $\delta\phi_0 \equiv \delta a$, which corresponds to the small fluctuation of the canonically normalized 4D axion field a which was defined in eqs. (3.52), (3.54), and (3.56), is given by

$$\kappa_b \sqrt{\frac{m}{f_5^3}} \frac{\text{sech}[m(y_b - y_0)]}{\sqrt{\tanh[m(\pi R - y_0)] + \tanh(my_0)}} \delta a(x) \mathcal{O}(x). \quad (3.65)$$

This then yields

$$\frac{\kappa_b e^{-m(\pi R - y_b)}}{f} \delta a(x) \mathcal{O}(x) \quad (3.66)$$

in the limit $y_0 \rightarrow -\infty$ where the zero mode axion is exponentially localized at $y = \pi R$. With the above form of boundary couplings of the zero mode axion fluctuation δa , one can easily generate an exponential hierarchy between the zero mode couplings at different boundaries, i.e. $y_b = 0$ and $y_b = \pi R$. Although the analytic expression of the massive KK mode couplings to the boundary operator $\mathcal{O}(x)$ is rather complicated, one can generate also a similar hierarchy between the zero mode coupling and the KK mode coupling to the boundary fields at a given y_b with an appropriately chosen value of y_0 .

Let us now discuss the field range of the canonically normalized 4D axion a , which is embedded in the 5D angular field θ as

$$\theta = 2 \tan^{-1}[e^{my} u(a)] \quad (3.67)$$

where

$$u = ie^{-m\pi R} \text{sn} \left(\frac{a}{2if} \middle| e^{-2m\pi R} \right) \quad \text{for } f = \sqrt{\frac{f_5^3(1 - e^{-2m\pi R})}{2m}}. \quad (3.68)$$

From the periodicity of the Jacobi elliptic function:

$$\text{sn}(x|z) = (-1)^{n_1} \text{sn}(x + 2n_1 K(z) + 2in_2 K(1 - z)|z) \quad (n_1, n_2 = \text{integers}), \quad (3.69)$$

where $K(y)$ is the complete elliptic integral of the first kind, we find the field range of a , which is matched to the 2π range of θ , i.e. $\theta \equiv \theta + 2\pi$, is given by

$$a \equiv a + 2\pi f_{\text{eff}} \quad \text{for } 2\pi f_{\text{eff}} = 4fK(1 - e^{-2m\pi R}). \quad (3.70)$$

In the limit of small m ($m\pi R \ll 1$), this results in

$$f_{\text{eff}} \simeq \sqrt{f_5^3 \pi R}, \quad (3.71)$$

while for a large value of m ($m\pi R \gg 1$), the field range becomes

$$f_{\text{eff}} \simeq \sqrt{\frac{2mR}{\pi}} \sqrt{f_5^3 \pi R} = \sqrt{\frac{2mRf_5^3}{\pi M_5^3}} M_{\text{Pl}}. \quad (3.72)$$

Note that there is *no* exponential enhancement of the field range of a relative to the 5D axion scale f_5 . Yet, an exponentially localized profile of the zero mode fluctuation δa could be successfully realized, yielding an exponential hierarchy among the couplings of δa at different boundaries as dictated by (3.65) or (3.66), as well as a similar exponential hierarchy between the zero mode coupling and the KK mode couplings at the same boundary.

Exponential hierarchy among the couplings of δa *without* an exponentially enhanced range of a yields a rather unusual form of the effective potential of a , which might be generated by a deviation of model parameters from (3.50) or by an axion coupling to the confining gauge sector at the boundaries. To be specific, here we consider the case that the model parameter c_0 and c_π are slightly deviated from the values of (3.50), which would result in the following effective potential

$$\begin{aligned} V_{\text{eff}}(a) &= \sum_b -\Lambda_b^4 \cos \theta(x, y_b) + \dots \\ &= -2 \sum_b \Lambda_b^4 \left[\frac{e^{-2m(\pi R - y_b)} \text{sn}^2\left(\frac{a}{2if} | e^{-2m\pi R}\right)}{1 - e^{-2m(\pi R - y_b)} \text{sn}^2\left(\frac{a}{2if} | e^{-2m\pi R}\right)} \right] + \dots, \end{aligned} \quad (3.73)$$

where $y_b = \{0, \pi R\}$ and we ignored irrelevant constant parts. We stress that the above axion potential is valid over the full range of a given by (3.70). In figure 5, we depict this effective potential originating from $y_b = 0$ (red) and $y_b = \pi R$ (blue). The unusual shape of this axion potential can be understood from that the zero mode fluctuation $\delta a \equiv a - \langle a \rangle$ is exponentially localized at different position for different value of $\langle a \rangle$. For $\langle a \rangle = 0$, δa is localized at $y_b = \pi R$. However for $\langle a \rangle = \pi f_{\text{eff}}$, δa is localized at $y = 0$. For $0 < \langle a \rangle < \pi f_{\text{eff}}$, δa is localized at a certain position between 0 and πR . Since δa is localized at different position in the 5th dimension for different $\langle a \rangle$, its overlap with $\theta(x, y_b)$ also changes, which results in the change of the curvature of the scalar potential along different value of $\langle a \rangle$.

3.4 CCW axions from 5D gauge and Stückelberg fields

Another candidate 5D field that could give a periodic 4D axion is a 5D gauge field C_M with odd orbifold parity [10, 33, 34]. To realize continuum CW axions in more general context, in this section we consider the case that the 5D gauge field C_M has a Stückelberg mass involving an angular field θ which might originate from the phase degree of freedom of U(1)-charged complex scalar field. Under the U(1) gauge symmetry, C_M and θ transform as

$$U(1)_C : C_M \rightarrow C_M + \partial_M \Lambda, \quad \theta \rightarrow \theta + q\epsilon(y)\Lambda, \quad (3.74)$$

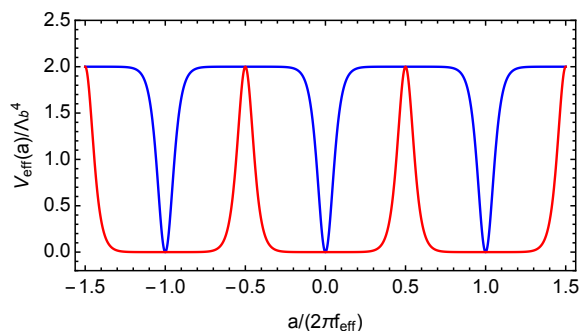


Figure 5. The axion potentials induced by small parameter deviation at $y_b = 0$ (red) and $y_b = \pi R$ (blue). For illustrative purposes, we take a moderated value of m giving $mR = 2$.

where θ and Λ are periodic as

$$\theta \equiv \theta + 2\pi, \quad \Lambda \equiv \Lambda + 2\pi, \quad (3.75)$$

q is an integer-valued charge, and the fields and transformation function obey the following orbifold boundary conditions:

$$\begin{aligned} C_\mu(x, y) = C_\mu(x, y + 2\pi R) = -C_\mu(x, -y), \quad C_5(x, y) = C_5(x, y + 2\pi R) = C_5(x, -y), \\ \theta(x, y) = \theta(x, y + 2\pi R) = \theta(x, -y), \quad \Lambda(x, y) = \Lambda(x, y + 2\pi R) = -\Lambda(x, -y). \end{aligned} \quad (3.76)$$

Note that the gauge transformation of θ involves a parity odd step function $\epsilon(y) = \epsilon(y + 2\pi R) = -\epsilon(-y)$ with $\epsilon(y) = 1$ for $0 < y < \pi R$, which is necessary to be compatible with the above orbifold boundary conditions.

As in section 3.1, one can start with a 5D action of C_M and θ at leading order in $g_{\text{st}}^2 \propto e^{-\xi S}$ in the context of generalized linear dilaton model in the string or Jordan frame, for which the 5D action including the boundary terms involves the overall dilaton factor $e^{\xi S}$. Moving to the Einstein frame, one finds the relevant part of the 5D action is given by

$$\begin{aligned} S_C = - \int d^5x \sqrt{-G} \left[\frac{e^{2cS}}{4g_C^2} C^{MN} C_{MN} + \frac{f_5^3}{2} (\partial^M \theta - q\epsilon(y)C^M) (\partial_M \theta - q\epsilon(y)C_M) \right. \\ \left. + \frac{\delta(y)}{\sqrt{G_{55}}} \left(\frac{\gamma_0}{M_5} \frac{C_{\mu 5}}{\sqrt{G_{55}}} \tilde{J}_0^\mu + \frac{\kappa_0}{32\pi^2} \theta G_0 \tilde{G}_0 + c_0 \partial_\mu \theta J_0^\mu \right) \right. \\ \left. + \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \left(\frac{\gamma_\pi}{M_5} \frac{C_{\mu 5}}{\sqrt{G_{55}}} \tilde{J}_\pi^\mu + \frac{\kappa_\pi}{32\pi^2} \theta G_\pi \tilde{G}_\pi + c_\pi \partial_\mu \theta J_\pi^\mu \right) \right], \end{aligned} \quad (3.77)$$

where $\tilde{J}_{0,\pi}^\mu, J_{0,\pi}^\mu$ are model-dependent boundary-localized current operators. As for the natural size of model parameters, the 5D gauge coupling g_C might be estimated as $1/g_C^2 \sim M_5$ for the 5D Planck mass M_5 , the bulk Stückelberg mass of C_M , i.e. $qg_C f_5^{3/2}$, is supposed to be well below M_5 , and the dimensionless boundary couplings $\gamma_{0,\pi}, \kappa_{0,\pi}$ and $c_{0,\pi}$ can be chosen to be of order unity. We note that the $U(1)_C$ invariance of the bulk kinetic term of θ enforces that the gauge transformation function Λ should be continuous at the boundaries, so

$$\Lambda(x, 0) = \Lambda(x, \pi R) = 0. \quad (3.78)$$

As a consequence, $\delta_{U(1)_C}\theta = 0$ at the boundaries, and therefore the $U(1)_C$ gauge symmetry admits an arbitrary boundary potential of θ at $y = 0, \pi R$. One can forbid such arbitrary boundary potential by assuming a global $U(1)$ symmetry at each boundary,

$$\begin{aligned} U(1)_0 &: \quad \delta(y)\theta(x, y) \rightarrow \delta(y)(\theta(x, y) + c_0), \\ U(1)_\pi &: \quad \delta(y - \pi R)\theta(x, y) \rightarrow \delta(y - \pi R)(\theta(x, y) + c_\pi), \end{aligned} \quad (3.79)$$

which is explicitly broken by the YM instantons at the boundaries. We already assumed such global $U(1)$ symmetries in the 5D action (3.77). Note that the axion-instanton couplings $\kappa_{0,\pi}$ should be integer-valued, while the derivative axion couplings $\gamma_{0,\pi}$ and $c_{0,\pi}$ can have any real value.

To proceed, we choose a specific gauge in the 5D action, introducing R_ξ gauge fixing term to remove the mixing between C_μ and $\{C_5, \theta\}$, and take the Feynman-'t Hooft gauge ($\xi = 1$). This gauge choice makes the 5D action to take CCW form, from which the KK mass spectrum and wavefunction profiles can be easily extracted. Specifically, we add

$$S_{g.f} = - \int d^5x \sqrt{-G} \left[\frac{e^{2cS}}{g_C^2} \left(g^{\mu\nu} \partial_\mu C_\nu + G^{55} \chi^{-1} \partial_y (\chi C_5) - e^{-2cS} g_C^2 q f_5^3 \theta \right)^2 \right] \quad (3.80)$$

to the action (3.77), where $\chi = e^{(2k_1 - k_2)|y|}$, and replace the 5D metric G_{MN} and the dilaton field S with their vacuum values (3.11). Although this removes the mixing between C_μ and $\{C_5, \theta\}$, the mixing between C_5 and θ is not removed yet. We then make the field redefinition:

$$\begin{aligned} \theta(x, y) &= e^{-(k_1 + \frac{k_2}{2})|y|} f_5^{-3/2} (\cos \beta \varphi_+(x, y) - \sin \beta \varphi_-(x, y)), \\ C_5(x, y) &= e^{-(k_1 + \frac{k_2}{2})|y|} g_C (\sin \beta \varphi_+(x, y) + \cos \beta \varphi_-(x, y)), \end{aligned} \quad (3.81)$$

which results in the following CCW form of 5D action:

$$\begin{aligned} S_C + S_{g.f} &= -\frac{1}{2} \int d^5x \left[(\partial_\mu \varphi_+)^2 + e^{2(k_1 - k_2)|y|} (\partial_y \varphi_+ - m_C \varphi_+)^2 \right. \\ &\quad \left. + (\partial_\mu \varphi_-)^2 + e^{2(k_1 - k_2)|y|} (\partial_y \varphi_- + m_C \varphi_-)^2 \right], \end{aligned} \quad (3.82)$$

where

$$m_C = \sqrt{q^2 g_C^2 f_5^3 + \left(k_1 + \frac{k_2}{2}\right)^2}, \quad \tan 2\beta = \frac{2q g_C f_5^{3/2}}{2k_1 - k_2}. \quad (3.83)$$

Obviously the above CCW lagrangian enjoys the localized shift symmetries

$$\varphi_\pm(x, y) \rightarrow \varphi_\pm(x, y) + c_\pm e^{\pm m_C y}, \quad (3.84)$$

which originate essentially from the global shift symmetry (3.79) at the boundaries, which insure the existence of two massless 4D axions.

One may do the standard KK analysis in this prescription with

$$\varphi_\pm = \sum_{n=0}^{\infty} f_{\pm n}(y) \delta \phi_{\pm n}(x), \quad (3.85)$$

and then the resulting mass eigenvalues and wavefunction profiles can be immediately read off from eq. (3.32) and (3.36). Here, the two zero modes $\delta\phi_{\pm 0}$ clearly correspond to massless 4D axions. The zero mode axion $\delta\phi_{+0}(x)$ has a wavefunction profile exponentially localized at $y = \pi R$, while $\delta\phi_{-0}(x)$ does at $y = 0$. With the localized shift symmetry (3.84) and the associated zero mode axions $\delta\phi_{\pm 0}$, one can achieve hierarchical axion couplings at different boundaries. We stress that the localized symmetry (3.84) for CCW axions is obtained as a simple consequence of the underlying symmetries of the model, i.e. $U(1)_C \times U(1)_0 \times U(1)_\pi$, *without* any fine tuning of model parameters.

As for the massive modes, one combination of $\delta\phi_{\pm n}$ ($n \geq 1$) corresponds to the longitudinal mode of massive 4D vector field, while the other is a massive 4D scalar. Since we are working at a particular gauge, at this level we can not identify which combination corresponds to the longitudinal mode of massive vector field. This can be done by solving the 5D equations of motion as in section 3.5, however we will not perform it here. Yet we can easily show that each massive vector and scalar modes have the same mass. For instance, for $k_1 = k_2$, the massive vector and scalar modes have a common mass $M_n = \sqrt{m_C^2 + n^2/R^2}$ for $n \geq 1$,

Although convenient for the KK analysis, the above prescription is not ideal for a discussion of the field range of the canonically normalized zero mode axions. A straightforward way to address this issue, as well as the low energy effective couplings of zero mode axions, is to integrate out massive KK modes in a gauge invariant manner and construct the effective action of zero mode axions. The equations of motion of C_μ and θ have a gauge-covariant solution obeying

$$C_\mu(x, y) = \partial_\mu \Gamma(x, y), \quad \partial_y \theta(x, y) = q\epsilon(y) C_5(x, y), \quad (3.86)$$

where $\Gamma(x, y)$ is a scalar field with the boundary condition $\Gamma(x, 0) = \Gamma(x, \pi R) = 0$, which transforms as $\Gamma \rightarrow \Gamma + \Lambda$ under $U(1)_C$. We then find that a gauge invariant solution given by

$$\theta(x, y) - q\epsilon(y)\Gamma(x, y) = e^{-(k_1 + \frac{k_2}{2})y} \left[\operatorname{csch}(m_C \pi R) \sinh[m_C(\pi R - y)] \theta_1(x) + \operatorname{csch}(m_C \pi R) \sinh(m_C y) \theta_2(x) \right], \quad (3.87)$$

which is described by the two gauge invariant 4D scalar degrees of freedom $\theta_i(x)$ ($i = 1, 2$). Note that the matching condition

$$\theta_1(x) = \theta(x, y = 0), \quad \theta_2(x) = e^{(k_1 + \frac{k_2}{2})\pi R} \theta(x, y = \pi R) \quad (3.88)$$

determines the periodicity of 4D fields $\theta_i(x)$ as

$$\theta_1 \equiv \theta_1 + 2\pi, \quad \theta_2 \equiv \theta_2 + 2\pi e^{(k_1 + \frac{k_2}{2})\pi R}. \quad (3.89)$$

The 4D effective Lagrangian of θ_i can be obtained by inserting the solution (3.87) into the original 5D action (3.77). The more conventional form can be obtained in terms of the canonically normalized axion fields $a_{1,2}$ (up to kinetic mixing):

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & -\frac{1}{2} \left(\partial_\mu a_1 \partial^\mu a_1 + \partial_\mu a_2 \partial^\mu a_2 + \hat{\epsilon} \partial_\mu a_1 \partial^\mu a_2 \right) + \frac{\kappa_0}{32\pi^2} \frac{a_1}{f_1} G_0 \tilde{G}_0 + \frac{\kappa_\pi}{32\pi^2} \frac{a_2}{f_2} G_\pi \tilde{G}_\pi \\ & + \frac{\partial_\mu a_1}{f_1} \left(c_{11} \tilde{J}_0^\mu + c_{12} \tilde{J}_\pi^\mu - c_0 J_0^\mu \right) - \frac{\partial_\mu a_2}{f_2} \left(c_{21} \tilde{J}_0^\mu + c_{22} \tilde{J}_\pi^\mu + c_\pi J_\pi^\mu \right), \end{aligned} \quad (3.90)$$

where the axion decay constants f_i define the axion periodicity as

$$a_i \equiv a_i + 2\pi f_i \quad (i = 1, 2). \quad (3.91)$$

Using the parameters defined as

$$\begin{aligned} Z_{11} &= m_C \coth(m_C \pi R) + (k_1 + k_2/2), \\ Z_{12} &= Z_{21} = -m_C \operatorname{csch}(m_C \pi R), \\ Z_{22} &= m_C \coth(m_C \pi R) - (k_1 + k_2/2), \end{aligned} \quad (3.92)$$

we obtain explicit expressions for the axion decay constants,

$$f_1 = \frac{\sqrt{Z_{11}}}{qg_C}, \quad f_2 = e^{(k_1 + \frac{k_2}{2})\pi R} \frac{\sqrt{Z_{22}}}{qg_C}, \quad (3.93)$$

and the nontrivial kinetic mixing coefficient $\hat{\epsilon}$,

$$\hat{\epsilon} = \frac{2Z_{12}}{\sqrt{Z_{11}Z_{22}}}, \quad (3.94)$$

and the dimensionless couplings c_{ij} between the axions and the currents,

$$\begin{aligned} c_{11} &= \frac{\gamma_0 Z_{11}}{M_5}, & c_{12} &= e^{-(k_1 + \frac{k_2}{2})\pi R} \frac{\gamma_\pi Z_{12}}{M_5}, \\ c_{22} &= e^{-k_2 \pi R} \frac{\gamma_\pi Z_{22}}{M_5}, & c_{21} &= e^{(k_1 + \frac{k_2}{2})\pi R} \frac{\gamma_0 Z_{21}}{M_5}. \end{aligned} \quad (3.95)$$

To examine the possible hierarchical structure of axion decay constants and the couplings, let us take the limit $m_C \pi R \gg 1$, and assume the following natural size of the underlying model parameters:

$$k_i \lesssim m_C < M_5, \quad \gamma_{0,\pi} \sim c_{0,\pi} \sim \kappa_{0,\pi} = \mathcal{O}(1). \quad (3.96)$$

The resulting low energy axion decay constants are estimated as

$$f_1 \sim \sqrt{m_C M_5}, \quad \frac{f_2}{f_1} \sim e^{(k_1 + \frac{k_2}{2})\pi R}, \quad \frac{f_2}{M_{\text{Pl}}} \sim \frac{\sqrt{m_C k_i}}{M_5} \quad \text{or} \quad \sqrt{\frac{m_C}{M_5^2 \pi R}}. \quad (3.97)$$

The kinetic mixing is exponentially suppressed, $\hat{\epsilon} \sim e^{-m_C \pi R}$, and the axion couplings have the following hierarchical structure:

$$\begin{aligned} c_{11} &\sim \frac{m_C}{M_5}, & c_{12} &\sim e^{-(m_C + k_1 + \frac{3}{2}k_2)\pi R} \frac{m_C}{M_5}, \\ c_{22} &\sim e^{-(m_C + k_2)\pi R} \frac{m_C}{M_5}, & c_{21} &\sim e^{-(m_C - k_1 - \frac{k_2}{2})\pi R} \frac{m_C}{M_5}. \end{aligned} \quad (3.98)$$

The above results show that the 5D model (3.77) for CCW axions can generate a variety of exponential hierarchies among the axion derivative couplings, as well as an exponentially small kinetic mixing. The model can also generate an exponential hierarchy between the two axion scales f_i ($i = 1, 2$) which define the field range of the canonically normalized

4D axions a_i . However, as long as the underlying 5D mass parameters are sub-Planckian, e.g. $m_C, k_i < M_5$, the model can *not* generate a super-Planckian axion field range in the effective 4D theory, i.e. $f_i < M_{\text{Pl}}$. Note that a super-Planckian effective axion scale could be easily achieved in the original DCW axion. In regard to this point, the key difference between DCW and CCW is that the extra spacial dimension for CCW should be a part of gravitational dynamics, while the linear quiver for DCW is completely decoupled from gravity. It is thus likely that generically CCW can not provide a super-Planckian axion field range. Our specific 5D model (3.77) does not generate also an exponential hierarchy among the axion-instanton couplings.

Finally, we provide the relation between the canonically normalized zero modes $\delta\phi_{\pm 0}(x)$ defined in (3.85) and the axion fluctuation $\delta a_1 \equiv f_1 \delta\theta_1$ and $\delta a_2 \equiv f_2 e^{-(k_1 + \frac{k_2}{2})\pi R} \delta\theta_2$ for θ_i defined as the massless fluctuation of the gauge-invariant combination (3.87):

$$\begin{aligned} \delta\phi_{+0} &= \sqrt{\frac{f_5}{(1 - e^{-2m_C\pi R})(2m_C + 2k_1 + k_2)}} \left(\frac{f_5 e^{(k_1 + \frac{k_2}{2})\pi R}}{f_2} \delta a_2 - \frac{f_5 e^{-m_C\pi R}}{f_1} \delta a_1 \right), \\ \delta\phi_{-0} &= -\sqrt{\frac{f_5}{(1 - e^{-2m_C\pi R})(2m_C - 2k_1 - k_2)}} \left(\frac{f_5}{f_1} \delta a_1 - \frac{f_5 e^{-(m_C - k_1 - \frac{k_2}{2})\pi R}}{f_2} \delta a_2 \right). \end{aligned} \quad (3.99)$$

3.5 CCW photon from 5D gauge field with even orbifold-parity

In this section, we study a 5D model for CCW U(1) gauge boson. The model involves a U(1) gauge symmetry and the associated gauge field $A_M = \{A_\mu, A_5\}$ which transforms as

$$U(1)_A : A_M \rightarrow A_M + \partial_M \Lambda. \quad (3.100)$$

To have a light 4D vector boson with localized profile over the extra dimension, we need to introduce the bulk and boundary masses of A_M , which may take the Stückelberg form for simplicity. Then, again at leading order in $g_{\text{st}}^2 \propto e^{-\xi S}$ in the generalized linear dilaton model, the 5D action is given by

$$\begin{aligned} S_A = - \int d^5x \sqrt{-G} & \left[\frac{e^{2cS}}{4g_A^2} F^{MN} F_{MN} + \frac{m_B^2}{2g_A^2} (\partial^M \theta - A^M)(\partial_M \theta - A_M) \right. \\ & \left. + \frac{e^{cS} m_b}{g_A^2} (\partial^\mu \theta - A^\mu)(\partial_\mu \theta - A_\mu) \left(\frac{\delta(y)}{\sqrt{G_{55}}} - \frac{\delta(y - \pi R)}{\sqrt{G_{55}}} \right) \right], \end{aligned} \quad (3.101)$$

where θ is a periodic Goldstone field for gauge invariant Stückelberg mass, which transforms under $U(1)_A$ as

$$\theta \rightarrow \theta + \Lambda, \quad (3.102)$$

and m_B and m_b are the Stückelberg bulk and boundary mass terms, respectively. The field variables and the gauge transformation function obey the following orbifold boundary

condition and the periodicity condition:

$$\begin{aligned}
A_\mu(x, y) &= A_\mu(x, y + 2\pi R) = A_\mu(x, -y), \\
A_5(x, y) &= A_5(x, y + 2\pi R) = -A_5(x, -y), \\
\theta(x, y) &= \theta(x, y + 2\pi R) = \theta(x, -y) = \theta(x, y) + 2\pi, \\
\Lambda(x, y) &= \Lambda(x, y + 2\pi R) = \Lambda(x, -y) = \Lambda(x, y) + 2\pi.
\end{aligned} \tag{3.103}$$

For a gauge covariant KK analysis, we decompose the relevant 5D fields as

$$\begin{aligned}
A_\mu(x, y) &= \sum_{n=0}^{\infty} f_n(y) A_{n\mu}(x), \\
A_5(x, y) &= \sum_{n=0}^{\infty} f'_n(y) \pi_n(x) + \sum_{n=1}^{\infty} (h_n(y) + g'_n(y)) \phi_n(x), \\
\theta(x, y) &= \sum_{n=0}^{\infty} f_n(y) \pi_n(x) + \sum_{n=1}^{\infty} g_n(y) \phi_n(x),
\end{aligned} \tag{3.104}$$

and solve the equations of motion. One can similarly decompose the gauge transformation function as

$$\Lambda(x, y) = \sum_{n=0}^{\infty} f_n(y) \alpha_n(x), \tag{3.105}$$

and then the 4D fields $A_{n\mu}, \pi_n, \phi_n$ transform under $U(1)_A$ as

$$A_{n\mu} \rightarrow A_{n\mu} + \partial_\mu \alpha_n, \quad \pi_n \rightarrow \pi_n + \alpha_n, \quad \phi_n \rightarrow \phi_n. \tag{3.106}$$

This shows that π_n can be identified as the longitudinal component of the massive 4D vector fields $A_{n\mu}$, while the gauge invariant ϕ_n denote physical massive 4D scalar fields. After integrating over y , the lagrangian of all 4D fields take the form

$$\mathcal{L} = - \sum_{n=0}^{\infty} \left[\frac{1}{4} (F_{n\mu\nu})^2 + \frac{M_n^2}{2} (\partial_\mu \pi_n - A_{n\mu})^2 \right] - \sum_{n=1}^{\infty} \left[\frac{1}{2} (\partial_\mu \phi_n)^2 + \frac{M_n^2}{2} \phi_n^2 \right], \tag{3.107}$$

where the mass eigenvalues M_n will be determined later. Here we assume that M_n (including $n = 0$) is nonzero in order to take into account all physical degrees of freedom correctly. We then find from the 5D equations of motion that the mode functions f_n, h_n and g_n satisfy

$$\begin{aligned}
&\left[\partial_y^2 - m_B^2 - \left(k_1 + \frac{k_2}{2} \right)^2 + e^{-2(k_1 - k_2)y} M_n^2 \right] \left(e^{(k_1 + \frac{k_2}{2})y} f_n(y) \right) = 0, \\
&\left[\partial_y^2 - m_B^2 - \left(k_1 + \frac{k_2}{2} \right)^2 + e^{-2(k_1 - k_2)y} M_n^2 \right] \left(e^{3(k_1 - \frac{k_2}{2})y} h_n(y) \right) = 0, \\
&\left[\partial_y + k_1 + \frac{k_2}{2} \right] \left(e^{3(k_1 - \frac{k_2}{2})y} h_n(y) \right) - M_n^2 \left(e^{(k_1 + \frac{k_2}{2})y} g_n(y) \right) = 0,
\end{aligned} \tag{3.108}$$

together with the boundary conditions

$$\begin{aligned} \left[\partial_y - m_V - k_1 - \frac{k_2}{2} \right] (e^{(k_1 + \frac{k_2}{2})y} f_n(y))|_{y=0, \pi R} &= 0, \\ \left[\partial_y - \frac{m_B^2}{m_b} + k_1 + \frac{k_2}{2} \right] (e^{3(k_1 - \frac{k_2}{2})y} h_n(y))|_{y=0, \pi R} &= 0. \end{aligned} \quad (3.109)$$

Although the boundary conditions for $f_n(y)$ and $h_n(y)$ are different, it could be verified that they share the same mass spectrum except the mass M_0 of $A_{0\mu}$. For generic values of $\{k_1, k_2, m_B, m_b\}$, M_0 can be even larger than M_n ($n \geq 1$). However there exists a wide range of model parameters yielding $M_0 \ll M_n$ ($n \geq 1$). In particular, for $m_B^2 = m_b(m_b + 2k_1 + k_2)$, we have a massless 4D vector ($M_0 = 0$) with a localized wave function $f_0(y) \propto e^{m_b y}$.

For simplicity, let us consider the case $k_1 = k_2$ for which it is straightforward to obtain the mode functions and mass eigenvalues. We then find

$$\begin{aligned} f_0(y) &= e^{m_b y} \sqrt{\frac{m_b + \frac{3}{2}k_1}{e^{(2m_b + 3k_1)\pi R} - 1}} \\ f_n(y) &= \frac{e^{-\frac{3k_1}{2}y}}{\sqrt{\pi R((m_b + \frac{3}{2}k_1)^2 R^2 + n^2)}} \left[n \cos\left(\frac{ny}{R}\right) + \left(m_b + \frac{3}{2}k_1\right) R \sin\left(\frac{ny}{R}\right) \right], \\ h_n(y) &= \frac{e^{-\frac{3}{2}k_1 y}}{\sqrt{\pi R\left(\left(\frac{m_B^2}{m_b} - \frac{3}{2}k_1\right)^2 R^2 + n^2\right)}} \left[n \cos\left(\frac{ny}{R}\right) + \left(\frac{m_B^2}{m_b} - \frac{3}{2}k_1\right) R \sin\left(\frac{ny}{R}\right) \right], \end{aligned} \quad (3.110)$$

for $n \geq 1$, and the corresponding mass eigenvalues are given by

$$\begin{aligned} M_0^2 &= m_B^2 - m_b(m_b + 3k_1), \\ M_n^2 &= m_B^2 + \left(\frac{3}{2}k_1\right)^2 + \frac{n^2}{R^2} \quad (n \geq 1). \end{aligned} \quad (3.111)$$

Focusing on the exponential y -dependence in the limit $m\pi R \gg 1$, we have

$$f_0(y) \sim e^{-\frac{3}{2}k_1 \pi R} e^{m_b(y - \pi R)}, \quad f_n(y) \sim h_n(y) \sim e^{-\frac{3}{2}k_1 y}. \quad (3.112)$$

This behavior of mode functions shows clearly the role of each parameter. $A_{0\mu}$ has an exponentially localized profile only when the boundary Stückelberg mass m_b is non-vanishing. On the other hand, the mode function ratio $f_0(y)/f_n(y) \sim e^{(m_b + \frac{3}{2}k_1)(y - \pi R)}$ ($n \geq 1$) indicates that a possible exponential hierarchy between the zero mode ($n = 0$) coupling and the non-zero mode ($n \geq 1$) coupling at a boundary with $y = 0$ depends on both the boundary mass m_b and the background geometry described by k_1 . Actually, all KK modes are exponentially localized at $y = 0$ in our field basis due to the background geometry. We also note that the hierarchical structure of the boundary couplings of mass eigenstate modes is independent of the existence of massless or particularly light mode, i.e. it is valid even when $M_0 \sim M_1$.

Related to the CCW, the most interesting parameter region of the model is the one yielding¹⁵

$$M_0 = \sqrt{m_B^2 - m_b(m_b + 3k_1)} \ll 1/R \ll m_B. \quad (3.113)$$

In such case, the 5D model (3.101) successfully provides a light $A_{0\mu}$ with localized profile whose boundary couplings at different boundaries reveal an exponential hierarchy¹⁶ as

$$\frac{g_0(y=0)}{g_0(y=\pi R)} \propto e^{-m_b\pi R}, \quad (3.114)$$

together with approximately degenerate massive KK modes whose boundary couplings at $y=0$ reveal another exponential hierarchy

$$\frac{g_0(y=0)}{g_n(y=0)} \propto e^{-(m_b + \frac{3}{2}k_1)\pi R} \quad (n \geq 1), \quad (3.115)$$

thereby realizing the key features of the CW U(1) gauge bosons. Note that the mass eigenvalues of non-zero modes, M_n for $n \geq 1$, are independent of the boundary Stückelberg mass m_b , and the shape of mode functions is rather insensitive to the bulk Stückelberg mass m_B .

4 Conclusions

In this paper, we have studied the continuum limit of the discrete clockwork (DCW) mechanism, dubbed the continuum clockwork (CCW). To be specific, we focused on the possible continuum realizations of CW axions and U(1) gauge bosons. There are two different prescriptions for the corresponding CCW lagrangian as previously discussed in [5, 26]: CCW-I associated with localized profile of zero modes in a field basis for which the 5D U(1) and discrete gauge symmetries take the standard form as (2.31), which is induced by appropriately tuned bulk and boundary masses, and CCW-II associated with the background geometry and dilaton profile of the linear dilaton model [28]. We discussed in detail that each prescription has its own limitation. For instance, assuming a discrete shift symmetry which assures that the involved axion is a periodic scalar, and also taking into account the U(1) gauge symmetry associated with the involved 5D vector field, in CCW either the localized CW symmetries to protect zero modes are not respected by gravitational interactions (CCW-I), or an exponential hierarchy between the zero mode couplings at different boundaries can not be generated (CCW-II). As other limitations relative to the DCW, CCW could yield neither an exponentially enhanced trans-Planckian field range of the zero

¹⁵One may even take the limit $M_0 \rightarrow 0$, while keeping M_n ($n \geq 1$) well above the characteristic energy scale. It is in fact not clear if the Goldstone zero mode π_0 has a sensible behavior when $M_0 \rightarrow 0$ as its kinetic term vanishes, while still there could be non-vanishing interactions. Such potentially singular behavior might be a consequence of another massless degree of freedom which is already integrated out at the starting point. In this paper, we do not pursue this issue further as our major concern is the CCW features of $A_{0\mu}$.

¹⁶Such exponential hierarchy among the couplings of $A_{0\mu}$ suggests that there is no globally well defined compact 4D U(1) gauge symmetry associated with $A_{0\mu}$.

mode axion, nor an exponential hierarchy among the 4D gauge charges, while keeping the quantized nature of gauge charges of an unbroken 4D U(1) gauge symmetry.

Motivated by those two prescriptions, we examined a more general CCW which can incorporate both CCW-I and CCW-II within a common framework. For this, we proposed a generalized 5D linear dilaton model providing a concrete realisation of the setup, in which the three independent parameters describing the general CCW are identified as geometric warp factor, large volume, and bulk/boundary mass responsible for localized zero mode profile. We then analyzed the KK spectra and the couplings of zero and massive KK modes to boundary-localized operators. It is found that many of the characteristic features of the DCW can be reproduced in a certain parameter region which might be identified as the CCW limit, and our setup allows a continuous deformation from the CCW limit to other limits such as the well known Randall-Sundrum and large extra dimension limits.

We subsequently discussed concrete 5D models for CCW axions and U(1) gauge bosons in generalized linear dilaton background. The 4D zero mode axion originating from a 5D angular field has quite different properties from the DCW axion since the corresponding CW symmetry to protect zero mode has a different form due to the periodicity of the 5D angular field. As a result, the couplings of the zero mode axion fluctuation to different boundary operators can exhibit an exponential hierarchy, while there is no exponentially enlarged field range of the corresponding zero mode axion. We also examined the CCW axions originating from a 5D gauge field having nonzero Stückelberg mass and odd orbifold parity. It turns out that the resulting zero mode axions can have an exponential hierarchy among the *derivative* couplings at different boundaries, while there is no such hierarchy for the axion-instanton couplings due to the restriction imposed by the underlying 5D gauge symmetry. One can obtain also a light 4D vector field and its KK excitations, which originate from a 5D gauge field with even orbifold parity and nonzero bulk and boundary Stückelberg masses, and reproduce many of the characteristic features of the DCW U(1) gauge bosons.

Acknowledgments

The work of [KC] and [CSS] was supported by IBS under the project code, IBS-R018-D1. [CSS] acknowledges the support from the Korea Ministry of Education, Science and Technology, Gyeongsangbuk-Do and Pohang City for Independent Junior Research Groups at the Asia Pacific Center for Theoretical Physics. [CSS] was also supported by the Basic Science Research Program through the NRF Grant (No. 2017R1D1A1B04032316). [SHI] was supported by the German Science Foundation (DFG) within the SFB-Transregio TR33 “The Dark Universe”.

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