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Classification of non-Riemannian doubled-yet-gauged spacetime

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Abstract Assuming $\mathbf{O}(D, D)$ covariant fields as the 'fundamental' variables, double field theory can accommodate novel geometries where a Riemannian metric cannot be defined, even locally. Here we present a complete classification of such non-Riemannian spacetimes in terms of two non-negative integers, $(n, \bar{n})$, $0 \leq n + \bar{n} \leq D$. Upon these backgrounds, strings become chiral and anti-chiral over n and $\bar{n}$ directions, respectively, while particles and strings are frozen over the $n + \bar{n}$ directions. In particular, we identify $(0, 0)$ as Riemannian manifolds, $(1, 0)$ as non-relativistic spacetime, $(1, 1)$ as Gomis–Ooguri non-relativistic string, $(D-1, 0)$ as ultra-relativistic Carroll geometry, and $(D, 0)$ as Siegel's chiral string. Combined with a covariant Kaluza–Klein ansatz which we further spell, $(0, 1)$ leads to Newton–Cartan gravity. Alternative to the conventional string compactifications on small manifolds, non-Riemannian spacetime such as $D = 10$, $(3, 3)$ may open a new scheme for the dimensional reduction from ten to four.

Contents

1 Introduction

Ever since Einstein formulated his theory of gravity, i.e. general relativity (GR), by employing differential geometry *à la* Riemann, the Riemannian metric, $g_{\mu\nu}$, has been privileged to be the only geometric and thus gravitational field. All other fields are meant to be 'extra matters'. On the other hand, string theory suggests us to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric. Forming the massless sector of closed strings, this triplet of objects is ubiquitous in all string theories. Further, a genuine stringy symmetry, T-duality, can mix the three of them [1,2], thus hinting at the existence of *stringy gravity* which should take the entire closed string massless sector as geometric and gravitational. After a series of pioneering works on 'doubled sigma models' [3–8] and 'double field theory' (DFT) [9–13] (cf. [14–16] for reviews), such an idea of stringy gravity has materialized.[1]

The word 'double' above refers to the fact that doubled $(D+D)$-dimensional coordinates are used for the descrip-

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[1] Strictly speaking, string theory does not predict general relativity but its own gravity, i.e. stringy gravity.

tion of D-dimensional physical spacetime. While such a usage was historically first made in the case of a torus background—by introducing a dual coordinate conjugate to the string winding momentum—the doubled coordinates are far more general and can be applied to any compact or non-compact spacetime, and not only to string but also to particle theories.

Stringy gravity of our interest adopts the *doubled-yet-gauged coordinate system* [17] which meets two properties. Firstly, an $\mathbf{O}(D,D)$ group is a priori postulated, having the invariant constant "metric",

$$\mathcal{J}_{AB}=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{1.1}$$

Along with its inverse, $\mathcal{J}^{AB}$, the invariant metric can be used to freely raise and lower the $\mathbf{O}(D,D)$ vector indices (capital letters, $A,B,\ldots$). Secondly, the doubled coordinates are gauged by an equivalence relation,

$$x^A\ \sim\ x^A+\Delta^A(x), \tag{1.2}$$

where Δ^A is an arbitrary 'derivative-index-valued' vector. This means that its superscript index must be identifiable as that of a derivative, $\partial^A=\mathcal{J}^{AB}\partial_B$. For example, with arbitrary functions, Φ_1, Φ_2 belonging to the theory, $\Delta^A=\Phi_1\partial^A\Phi_2$. The equivalence relation can be realized by requiring that all the fields or functions in stringy gravity—such as Φ_1,Φ_2, physical fields, local symmetry parameters, and their arbitrary derivatives—should be invariant under the coordinate gauge symmetry shift,

$$\Phi(x+\Delta)=\Phi(x)\quad\Longleftrightarrow\quad\Delta^A\partial_A=0. \tag{1.3}$$

In this way, a single physical point is not represented by a point, as in ordinary Riemannian geometry, but as a gauge orbit in the doubled coordinate system.

The above coordinate gauge symmetry invariance is equivalent to the so-called 'section condition' in DFT,

$$\partial_A\partial^A=0. \tag{1.4}$$

With respect to the off-block-diagonal form of the $\mathbf{O}(D,D)$ metric (1.1), the doubled coordinates split into two parts: $x^A=(\tilde{x}_\mu,x^\nu)$ and $\partial_A=(\tilde{\partial}^\mu,\partial_\nu)$, such that $\partial_A\partial^A=2\partial_\mu\tilde{\partial}^\mu$. The general solution to the section condition is then given by $\tilde{\partial}^\mu\equiv 0$, up to $\mathbf{O}(D,D)$ rotations [10,11].

Diffeomorphism covariance in doubled-yet-gauged spacetime reads

$$\delta x^A=\xi^A,\quad \delta\partial_A=-\partial_A\xi^B\partial_B=(\partial^B\xi_A-\partial_A\xi^B)\partial_B, \tag{1.5}$$

and for a covariant tensor (or tensor density with weight ω),

$$\delta T_{A_1\cdots A_n}=-\omega\partial_B\xi^B T_{A_1\cdots A_n}+\sum_{i=1}^{n}(\partial_B\xi_{A_i}-\partial_{A_i}\xi_B)T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}. \tag{1.6}$$

The latter corresponds to the passive counterpart of the "generalized Lie derivative", $\hat{\mathcal{L}}_\xi$, *à la* Siegel [10].

The whole massless sector of closed strings, or *stringy gravitons*, can be represented by a unit-weighted scalar density, e^{-2d}, and a symmetric projector,

$$P_{AB}=P_{BA},\quad P_A{}^BP_B{}^C=P_A{}^C. \tag{1.7}$$

The complementary, orthogonal projector, $\bar{P}_{AB}=\mathcal{J}_{AB}-P_{AB}$, satisfies, from (1.7), $P\bar{P}=0$, $\bar{P}^2=\bar{P}$. Covariant derivatives, $\nabla_A=\partial_A+\Gamma_A$, scalar curvature $S_{(0)}$ and "Ricci-like" curvature $P_A{}^C\bar{P}_B{}^DS_{CD}$ are then expressed in terms of $\{P_{AB},\bar{P}_{AB},d\}$ and their derivatives or equivalently in terms of the stringy analog of the Christoffel symbol, Γ_{ABC} (2.17) [18].[2]

The difference of the two projectors sets a symmetric $\mathbf{O}(D,D)$ element, known as the DFT-metric (or "generalized metric"),

$$\mathcal{H}_{AB}=\mathcal{H}_{BA}=P_{AB}-\bar{P}_{AB}\quad\text{satisfying}\quad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD}=\mathcal{J}_{AB}. \tag{1.8}$$

These $\mathbf{O}(D,D)$ covariant defining properties of the stringy gravitational fields can be conveniently solved by the conventional variables,

$$\mathcal{H}_{MN}=\begin{pmatrix}g^{\mu\nu}&-g^{\mu\sigma}B_{\sigma\lambda}\\B_{\kappa\rho}g^{\rho\nu}&g_{\kappa\lambda}-B_{\kappa\rho}g^{\rho\sigma}B_{\sigma\lambda}\end{pmatrix}. \tag{1.9}$$

However, this is not the most general solution: counter examples have been reported where the upper left $D\times D$ block of $\mathcal{H}_{AB}$ is degenerate [20–22], and have been shown to provide a natural geometry for the non-relativistic string theory *à la* Gomis and Ooguri [23].[3] Namely, by assuming the $\mathbf{O}(D,D)$ covariant variables as *the* fundamental fields, DFT or stringy gravity becomes more general than GR: it encompasses 'non-Riemannian' spacetime where the Riemannian metric, $g_{\mu\nu}$, cannot be defined, even locally. This includes various 'singular' limits of the Riemannian metric of which the inverse, $g^{\mu\nu}$, becomes degenerate; cf. T-fold, 'non-geometries' or 'waves' in the global sense [6–8,26–32]).

[2] However, a fully covariant four-indexed "Riemann-like" curvature has been argued not to exist [18,19]. This absence is, in a way, consistent with the fact that there exists no locally inertial frame for an extended object, i.e. a string, where the stringy Christoffel connection (2.17) might vanish completely: the equivalence principle holds for particles, not strings.

[3] For U-duality analogs, cf. [24,25].

Scope of the paper
It is the purpose of the present paper to classify completely DFT backgrounds, by deriving the most general solution to the defining property of the stringy gravitational field, or (1.8). Our classification is given in terms of two non-negative integers, $(n, \bar{n})$, $0 \le n + \bar{n} \le D$. Except for the (0, 0) case, these are generically non-Riemannian.

Since various DFTs and the relevant doubled sigma models have been constructed, strictly in terms of the $\mathbf{O}(D, D)$ covariant fields without referring to the Riemannian ones $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$,[4] our result can be *readily and unambiguously* applicable to these models which include, e.g. coupling to the standard model [33], higher spin [34], fluctuation or Noether analyses [21,35–37], the doubled-yet-gauged Green–Schwarz superstring action [22] and the maximally supersymmetric $D = 10$ DFT [38]. In particular, this last example, with the Killing spinor equations therein, may lead to a new scheme for dimensional reduction from ten to four, by assuming the six-dimensional internal space to be non-Riemannian, alternative to the traditional string compactification on 'small' Riemannian manifolds [39]. Further applications can be found in the holographic correspondences between Newton–Cartan gravity and condensed matter physics; see e.g. [40,41].

Organization of the paper
The rest of the paper is structured as follows. In Sect. 2, we classify the DFT-metric and spell the corresponding DFT-vielbeins. We discuss the dynamics of point particle and string upon these backgrounds. We also spell a path integral definition of the proper length in doubled-yet-gauged spacetime as well as a covariant Kaluza–Klein ansatz for DFT. In Sect. 3, we discuss various applications, such as Gomis–Ooguri non-relativistic string, non-relativistic and ultra-relativistic geometries, Siegel's chiral string and Newton–Cartan gravity. The appendix contains the technical derivation of the main result.

2 General results

2.1 Classification of the DFT-metric

As recalled in the introduction, the DFT-metric is by definition a symmetric $\mathbf{O}(D, D)$ element, satisfying the following relation:

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}, \quad \mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A{}^C. \tag{2.1}$$

Our main result consists in providing a full classification for DFT-metrics by solving the above defining properties: the most general solution is characterized by two non-negative integers, $(n, \bar{n})$, $0 \le n + \bar{n} \le D$, and assumes the following form:

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma} B_{\sigma\lambda} + Y_i^\mu X_\lambda^i - \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho} H^{\rho\nu} + X_\kappa^i Y_i^\nu - \bar{X}_\kappa^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda} - B_{\kappa\rho} H^{\rho\sigma} B_{\sigma\lambda} + 2X_{(\kappa}^i B_{\lambda)\rho} Y_i^\rho - 2\bar{X}_{(\kappa}^{\bar{\imath}} B_{\lambda)\rho} \bar{Y}_{\bar{\imath}}^\rho \end{pmatrix} \tag{2.2}$$

where $i, j = 1, 2, \ldots, n$ and $\bar{\imath}, \bar{\jmath} = 1, 2, \ldots, \bar{n}$. The variables, $\left\{H^{\mu\nu}, K_{\mu\nu}, B_{\mu\nu}, X_\mu^i, Y_j^\nu, \bar{X}_\mu^{\bar{\imath}}, \bar{Y}_{\bar{\jmath}}^\nu\right\}$, must meet the following properties:

– $H^{\mu\nu}$ and $K_{\mu\nu}$ are symmetric tensors

$$H^{\mu\nu} = H^{\nu\mu}, \quad K_{\mu\nu} = K_{\nu\mu}, \tag{2.3}$$

whose kernels are spanned by $\{X_\mu^i, \bar{X}_\nu^{\bar{\imath}}\}$ and $\{Y_j^\mu, \bar{Y}_{\bar{\jmath}}^\nu\}$, respectively,

$$\begin{aligned} &H^{\mu\nu} X_\nu^i = 0, \quad H^{\mu\nu} \bar{X}_\nu^{\bar{\imath}} = 0; \\ &K_{\mu\nu} Y_j^\nu = 0, \quad K_{\mu\nu} \bar{Y}_{\bar{\jmath}}^\nu = 0; \end{aligned} \tag{2.4}$$

– a completeness relation,

$$H^{\mu\rho} K_{\rho\nu} + Y_i^\mu X_\nu^i + \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\nu^{\bar{\imath}} = \delta^\mu{}_\nu; \tag{2.5}$$

– the skew-symmetry of the B-field,

$$B_{\mu\nu} = -B_{\nu\mu}. \tag{2.6}$$

While the derivation is carried out in the appendix, some comments are in order. From (2.4), (2.5) and the linear independency of $\{X_\mu^i, \bar{X}_\nu^{\bar{\imath}}\}$, orthonormal as well as algebraic relations follow,

$$\begin{aligned} &Y_i^\mu X_\mu^j = \delta_i{}^j, \quad \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\mu^{\bar{\jmath}} = \delta_{\bar{\imath}}{}^{\bar{\jmath}}, \quad Y_i^\mu \bar{X}_\mu^{\bar{\jmath}} = \bar{Y}_{\bar{\imath}}^\mu X_\mu^j = 0, \\ &H^{\rho\mu} K_{\mu\nu} H^{\nu\sigma} = H^{\rho\sigma}, \quad K_{\rho\mu} H^{\mu\nu} K_{\nu\sigma} = K_{\rho\sigma}. \end{aligned} \tag{2.7}$$

With the choice of the section, $\tilde{\partial}^\mu \equiv 0$, the doubled-yet-gauged diffeomorphisms (1.5), (1.6), or the generalized Lie derivative of the DFT-metric, cf. (A.4), imply that the variables transform covariantly as

$$\begin{aligned} \delta X_\mu^i &= \mathcal{L}_\xi X_\mu^i, \quad \delta \bar{X}_\mu^{\bar{\imath}} = \mathcal{L}_\xi \bar{X}_\mu^{\bar{\imath}}, \\ \delta Y_j^\nu &= \mathcal{L}_\xi Y_j^\nu, \quad \delta \bar{Y}_{\bar{\jmath}}^\nu = \mathcal{L}_\xi \bar{Y}_{\bar{\jmath}}^\nu, \\ \delta H^{\mu\nu} &= \mathcal{L}_\xi H^{\mu\nu}, \quad \delta K_{\mu\nu} = \mathcal{L}_\xi K_{\mu\nu}, \\ \delta B_{\mu\nu} &= \mathcal{L}_\xi B_{\mu\nu} + \partial_\mu \tilde{\xi}_\nu - \partial_\nu \tilde{\xi}_\mu, \end{aligned} \tag{2.8}$$

where $\mathcal{L}_\xi$ denotes the ordinary, i.e. undoubled, Lie derivative with the local parameter, ξ^ν, being part of the doubled

[4] Yet, there are quite a few works in the literature which do not meet this criterion, relying explicitly on the Riemannian variables. Our results are thus not applicable therein.

vector field, $\xi^A = (\tilde{\xi}_\mu, \xi^\nu)$. Our $(n,\bar{n})$-classification of the DFT-metric having the explicit parametrization (2.2) is particularly useful for the choice of the section, $\tilde{\partial}^\mu \equiv 0$. For example, the action for a massless scalar field reads (cf. [43])

$$\int_{\text{section}} e^{-2d}\,\mathcal{H}^{AB}\partial_A\Phi\partial_B\Phi \;\equiv\; \int \mathrm{d}^D x\; e^{-2d}\; H^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi. \tag{2.9}$$

For couplings to generic tensors or Yang–Mills fields, we refer to [18,21,36,44,45]. However, if the $(n,\bar{n})$ DFT-metric (2.2) admits an isometry direction, there appears arbitrariness in choosing the section. In this case, our parametrization (2.2) may be modified; see e.g. [32,46].

Clearly, constant $(n,\bar{n})$ DFT-metric (2.2) and DFT-dilaton, d, solve the equations of motion of DFT. Thus, our $(n,\bar{n})$ classification also accounts for non-Riemannian 'flat' backgrounds. It is worthwhile to note that the characteristic value, $(n,\bar{n})$, may change point-wise in a given doubled-yet-gauged curved spacetime, typically at a "Riemannian singular point". Further, $\mathbf{O}(D,D)$ rotations (along isometry directions) can also change the value of $(n,\bar{n})$, for example, the (0, 0) fundamental string background *à la* Dabholkar et al. [47] can be mapped to (1, 1) by certain $\mathbf{O}(D,D)$ rotations [20] (cf. [24]). However, the trace of a DFT-metric,

$$\mathcal{H}_A{}^A = 2(n-\bar{n}), \tag{2.10}$$

remains invariant under $\mathbf{O}(D,D)$ rotations and further point-wise if we fix the underlying spin group (2.50).

It is instructive to note that the B-field contributes to the DFT-metric by an $\mathbf{O}(D,D)$ conjugation,

$$\mathcal{H}_{AB} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} H & Y_i(X^i)^T - \bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T \\ X^i(Y_i)^T - \bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T & K \end{pmatrix} \times \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \tag{2.11}$$

such that the contribution is 'Abelian', in the following sense:

$$\begin{pmatrix} 1 & 0 \\ B_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ B_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B_1+B_2 & 1 \end{pmatrix}. \tag{2.12}$$

Further, the precise expression of the $(n,\bar{n})$ DFT-metric (2.2) as well as the fundamental algebraic relations (2.4), (2.5), (2.6), are invariant under several transformations.

Firstly under obvious $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ rotations,

$$\left(X^i_\mu,\, Y_i^\mu,\, \bar{X}^{\bar{\imath}}_\mu,\, \bar{Y}_{\bar{\imath}}^\nu\right) \longmapsto \left(X^j_\mu\, R_j{}^i,\; R^{-1}{}_i{}^j\, Y_j^\nu,\; \bar{X}^{\bar{\jmath}}_\mu\, \bar{R}_{\bar{\jmath}}{}^{\bar{\imath}},\; \bar{R}^{-1}{}_{\bar{\imath}}{}^{\bar{\jmath}}\, \bar{Y}_{\bar{\jmath}}^\nu\right)\;; \tag{2.13}$$

secondly under the transformation of only the B-field having two arbitrary skew-symmetric local parameters, $m_{ij} = -m_{ji}, \bar{m}_{\bar{\imath}\bar{\jmath}} = -\bar{m}_{\bar{\jmath}\bar{\imath}}$,

$$B_{\mu\nu} \quad\longmapsto\quad B_{\mu\nu} + X^i_\mu X^j_\nu m_{ij} + \bar{X}^{\bar{\imath}}_\mu \bar{X}^{\bar{\jmath}}_\nu \bar{m}_{\bar{\imath}\bar{\jmath}}; \tag{2.14}$$

and lastly under the following somewhat less trivial transformations of $\{Y_i^\mu,\ \bar{Y}_{\bar{\imath}}^\mu,\ K_{\mu\nu},\ B_{\mu\nu}\}$:

$$\begin{aligned}
Y_i^\mu &\longmapsto Y_i^\mu + H^{\mu\nu}V_{\nu i},\\
\bar{Y}_{\bar{\imath}}^\mu &\longmapsto \bar{Y}_{\bar{\imath}}^\mu + H^{\mu\nu}\bar{V}_{\nu\bar{\imath}},\\
K_{\mu\nu} &\longmapsto K_{\mu\nu} - 2X^i_{(\mu}K_{\nu)\rho}H^{\rho\sigma}V_{\sigma i} - 2\bar{X}^{\bar{\imath}}_{(\mu}K_{\nu)\rho}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\\
&\qquad + \left(X^i_\mu V_{\rho i} + \bar{X}^{\bar{\imath}}_\mu \bar{V}_{\rho\bar{\imath}}\right) H^{\rho\sigma}\left(X^j_\nu V_{\sigma j} + \bar{X}^{\bar{\jmath}}_\nu \bar{V}_{\sigma\bar{\jmath}}\right),\\
B_{\mu\nu} &\longmapsto B_{\mu\nu} - 2X^i_{[\mu}K_{\nu]\rho}H^{\rho\sigma}V_{\sigma i} + 2\bar{X}^{\bar{\imath}}_{[\mu}K_{\nu]\rho}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\\
&\qquad + 2X^i_{[\mu}\bar{X}^{\bar{\imath}}_{\nu]}V_{\rho i}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}},
\end{aligned} \tag{2.15}$$

where $V_{\mu i}$ and $\bar{V}_{\mu\bar{\imath}}$ are arbitrary local parameters. In fact, the latter two transformations, (2.14) and (2.15), can be unified into

$$\begin{aligned}
Y_i^\mu &\longmapsto Y_i^\mu + H^{\mu\nu}V_{\nu i},\\
\bar{Y}_{\bar{\imath}}^\mu &\longmapsto \bar{Y}_{\bar{\imath}}^\mu + H^{\mu\nu}\bar{V}_{\nu\bar{\imath}},\\
K_{\mu\nu} &\longmapsto K_{\mu\nu} - 2X^i_{(\mu}K_{\nu)\rho}H^{\rho\sigma}V_{\sigma i} - 2\bar{X}^{\bar{\imath}}_{(\mu}K_{\nu)\rho}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\\
&\qquad + \left(X^i_\mu V_{\rho i} + \bar{X}^{\bar{\imath}}_\mu \bar{V}_{\rho\bar{\imath}}\right) H^{\rho\sigma}\left(X^j_\nu V_{\sigma j} + \bar{X}^{\bar{\jmath}}_\nu \bar{V}_{\sigma\bar{\jmath}}\right),\\
B_{\mu\nu} &\longmapsto B_{\mu\nu} - 2X^i_{[\mu}V_{\nu]i} + 2\bar{X}^{\bar{\imath}}_{[\mu}\bar{V}_{\nu]\bar{\imath}}\\
&\qquad + 2X^i_{[\mu}\bar{X}^{\bar{\imath}}_{\nu]}\left(Y_i^\rho\bar{V}_{\rho\bar{\imath}} + \bar{Y}_{\bar{\imath}}^\rho V_{\rho i} + V_{\rho i}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\right).
\end{aligned} \tag{2.16}$$

Note that in (2.15) the local parameters appear only through the contractions with $H^{\mu\nu}$, i.e $H^{\mu\nu}V_{\nu i}$ and $H^{\mu\nu}\bar{V}_{\nu i}$. On the other hand in (2.16), the B-field transformation contains orthogonal contributions. Substituting $V_{\mu i} = -\frac{1}{2}m_{ij}X^j_\mu$ and $\bar{V}_{\mu\bar{\imath}} = \frac{1}{2}\bar{m}_{\bar{\imath}\bar{\jmath}}\bar{X}^{\bar{\jmath}}_\mu$ into (2.16) reproduces (2.14). Alternatively, if we replace $V_{\mu i}$ and $\bar{V}_{\mu\bar{\imath}}$ in (2.16) by $K_{\mu\nu}H^{\nu\rho}V_{\rho i}$ and $K_{\mu\nu}H^{\nu\rho}\bar{V}_{\rho\bar{\imath}}$, we recover (2.15).

The dynamics of the DFT-metric and the DFT-dilaton is dictated by the Euler–Lagrange equations of DFT. The expression of the $(n,\bar{n})$ DFT-metric (2.2) may then be inserted into the known stringy extension of the Christoffel symbol to lead to covariant derivatives and curvatures [18]. Yet, the trace (2.10) of the $(n,\bar{n})$ DFT-metric can be nontrivial, and this calls for some revision of the previous result:

$$\begin{aligned}
\Gamma_{CAB} &= 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\\
&\quad\times \partial_D P_{EC} - 4\left(\tfrac{1}{P_M{}^M - 1}P_{C[A}P_{B]}{}^D\right.\\
&\quad\left. + \tfrac{1}{\bar{P}_M{}^M - 1}\bar{P}_{C[A}\bar{P}_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right),
\end{aligned} \tag{2.17}$$

which now allows for generic values for the traces of the projectors,

$$P_M{}^M = D + n - \bar{n}, \quad \bar{P}_M{}^M = D - n + \bar{n}. \tag{2.18}$$

2.2 Particle and string on $(n, \bar{n})$ doubled-yet-gauged spacetime

While the notion of doubled-yet-gauged spacetime might sound somewhat mysterious, it is possible to define *proper length* and hence to show that it is a 'metric space'. To do so, we first note that the usual infinitesimal one-form, $\mathrm{d}x^A$, is neither diffeomorphism covariant (1.5), (1.6),

$$\delta(\mathrm{d}x^A) = \mathrm{d}x^B\partial_B V^A \neq \mathrm{d}x^B(\partial_B V^A - \partial^A V_B), \tag{2.19}$$

nor coordinate gauge symmetry invariant (1.3), since

$$\mathrm{d}\Delta^A = \mathrm{d}x^B\partial_B\Delta^A \neq 0. \tag{2.20}$$

Thus, the naive contraction with the DFT-metric, $\mathrm{d}x^A\mathrm{d}x^B \mathcal{H}_{AB}$, cannot give any sensible definition of proper length in doubled-yet-gauged spacetime. To cure the problem, we need to gauge $\mathrm{d}x^A$ explicitly, introducing a connection, $\mathcal{A}^A$, which should satisfy the same property as the coordinate gauge symmetry generator, Δ^A (1.3),

$$\mathrm{D}x^A := \mathrm{d}x^A - \mathcal{A}^A, \quad \mathcal{A}^A\partial_A = 0, \quad \mathcal{A}^A\mathcal{A}_A = 0. \tag{2.21}$$

Provided the connection transforms appropriately, $\mathrm{D}x^A$ becomes a well-behaved i.e. covariant vector [20],

$$\begin{aligned}
&\delta x^A = \xi^A,\\
&\delta\mathcal{A}^A = \partial^A\xi_B(\mathrm{d}x^B - \mathcal{A}^B)\\
&\qquad \Longrightarrow \quad \delta(\mathrm{D}x^A) = (\partial_B\xi^A - \partial^A\xi_B)\mathrm{D}x^B;\\
&\delta x^A = \Delta^A,\\
&\delta\mathcal{A}^A = \mathrm{d}\Delta^A \quad \Longrightarrow \quad \delta(\mathrm{D}x^A) = 0.
\end{aligned} \tag{2.22}$$

We propose then to define the proper distance in doubled-yet-gauged spacetime by the *path integral* [48],

$$\|x_1, x_2\| := -\ln\left[\int \mathcal{D}\mathcal{A}\,\exp\left(-\int_1^2 \sqrt{\mathrm{D}x^A\mathrm{D}x^B\mathcal{H}_{AB}}\right)\right]. \tag{2.23}$$

By letting $\tilde{\partial}^\mu \equiv 0$ and therefore $\mathcal{A}^A \equiv (A_\mu, 0)$, we may solve the constraints and write

$$\mathrm{D}x^A \equiv \left(\mathrm{d}\tilde{x}_\mu - A_\mu,\ \mathrm{d}x^\nu\right). \tag{2.24}$$

That is to say, only half of the doubled coordinates, i.e. the $\tilde{x}_\mu$ directions, are gauged. Furthermore, with the Riemannian DFT-metric (1.9), we get [20]

$$\begin{aligned}
\mathrm{D}x^A\mathrm{D}x^B\mathcal{H}_{AB} \equiv\ & \mathrm{d}x^\mu\mathrm{d}x^\nu g_{\mu\nu} + \left(\mathrm{d}\tilde{x}_\mu - A_\mu + \mathrm{d}x^\rho B_{\rho\mu}\right)\\
&\times\left(\mathrm{d}\tilde{x}_\nu - A_\nu + \mathrm{d}x^\sigma B_{\sigma\nu}\right) g^{\mu\nu}.
\end{aligned} \tag{2.25}$$

Thus, after integrating out the auxiliary connection, our proposal (2.23) reduces—at least classically—to the conventional, i.e. Riemannian proper distance, $\left\|x_1^\mu, x_2^\mu\right\| = \int_1^2\sqrt{\mathrm{d}x^\mu\mathrm{d}x^\nu g_{\mu\nu}}$. Being independent of the gauged $\tilde{x}_\mu$ coordinates, i.e. $\left\|x_1^A, x_2^A\right\| \equiv \left\|x_1^\mu, x_2^\mu\right\|$, indeed Eq. (2.23) measures the distance between two 'gauge orbits'.

The exponent in (2.23) immediately sets the action for a point particle in doubled-yet-gauged spacetime, or its square-root free einbein formulation [42],

$$S_{\text{particle}} = \int \mathrm{d}\tau\ e^{-1}\mathrm{D}_\tau x^A\mathrm{D}_\tau x^B\mathcal{H}_{AB}(x) - \tfrac{1}{4}m^2 e. \tag{2.26}$$

It also easily extends to (Nambu–Goto type) area and volume, which in turn provides the doubled-yet-gauged string action [20] (cf. [8] and also [22] for an extension to the Green–Schwarz superstring),

$$\begin{aligned}
S_{\text{string}} &= \tfrac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\text{string}},\\
\mathcal{L}_{\text{string}} &= -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\mathrm{D}_\alpha x^A\mathrm{D}_\beta x^B\mathcal{H}_{AB}(x) - \epsilon^{\alpha\beta}\mathrm{D}_\alpha x^A\mathcal{A}_{\beta A}.
\end{aligned} \tag{2.27}$$

These two actions are fully covariant under $\mathbf{O}(D, D)$ rotations, coordinate gauge symmetry (1.3), target-spacetime diffeomorphisms (1.6), world-volume diffeomorphisms and Weyl symmetry in the string case.

Besides the constraint imposed by the auxiliary potential, $\mathcal{A}^A$, the equation of motion of the former particle action can be spelled in terms of the stringy Christoffel connection (2.17),

$$e\frac{\mathrm{d}}{\mathrm{d}\tau}(e^{-1}\mathcal{H}_{AB}\mathrm{D}_\tau x^B) + 2\Gamma_{ABC}(\bar{P}\mathrm{D}_\tau x)^B(P\mathrm{D}_\tau x)^C = 0. \tag{2.28}$$

On the other hand, for a string propagating on the (0, 0) Riemannian background, the auxiliary potential, $\mathcal{A}^A$, implies the self-duality (i.e. chirality) over the entire doubled spacetime [20],

$$\mathrm{D}_\alpha x_A + \tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^{j\beta}\mathcal{H}_A{}^B\mathrm{D}_\beta x_B = 0, \tag{2.29}$$

and the Euler–Lagrangian equation of x^A gets simplified to give the *stringy geodesic equation*,

$$\tfrac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}\mathcal{H}_{AB}\mathrm{D}^\alpha x^B) + \Gamma_{ABC}(\bar{P}\mathrm{D}_\alpha x)^B(P\mathrm{D}^\alpha x)^B = 0, \tag{2.30}$$

which extends (2.28), yet with a different numerical factor in front of the connection, 2 *versus* 1.

For a generic non-Riemannian background, the analysis is more subtle, which we investigate hereafter. We substitute the generic $(n, \bar{n})$ DFT-metric (2.2) into the covariant actions, and move from doubled to undoubled formalism. One useful identity which generalizes (2.25) from Riemannian (0, 0) to a generic $(n,\bar{n})$ case is, with $\mathrm{D}_\alpha x^A=(\partial_\alpha\tilde{x}_\mu-A_{\alpha\mu},\partial_\alpha x^\nu)$,

$$\begin{aligned}\mathrm{D}_\alpha x^M\mathrm{D}_\beta x^N\mathcal{H}_{MN}=&\,\partial_\alpha x^\mu\partial_\beta x^\nu K_{\mu\nu}+\left(\mathrm{D}_\alpha\tilde{x}_\mu-B_{\mu\kappa}\partial_\alpha x^\kappa\right)\\&\times\left(\mathrm{D}_\beta\tilde{x}_\nu-B_{\nu\lambda}\partial_\beta x^\lambda\right)H^{\mu\nu}\\&+2X^i_\mu\partial_{(\alpha}x^\mu\left[\mathrm{D}_{\beta)}\tilde{x}_\nu-B_{\nu\rho}\partial_{\beta)}x^\rho\right]Y_i^\nu\\&-2\bar{X}^{\bar{\imath}}_\mu\partial_{(\alpha}x^\mu\left[\mathrm{D}_{\beta)}\tilde{x}_\nu-B_{\nu\rho}\partial_{\beta)}x^\rho\right]\bar{Y}_{\bar{\imath}}^\nu,\end{aligned}\tag{2.31}$$

which reads more explicitly for particles,

$$\begin{aligned}\mathrm{D}_\tau x^M\mathrm{D}_\tau x^N\mathcal{H}_{MN}=&\,\dot{x}^\mu\dot{x}^\nu K_{\mu\nu}+\left(\dot{\tilde{x}}_\mu-A_{\tau\mu}-B_{\mu\kappa}\dot{x}^\kappa\right)\\&\times\left(\dot{\tilde{x}}_\nu-A_{\tau\nu}-B_{\nu\lambda}\dot{x}^\lambda\right)H^{\mu\nu}\\&+2X^i_\mu\dot{x}^\mu\left(\dot{\tilde{x}}_\nu-A_{\tau\nu}-B_{\nu\rho}\dot{x}^\rho\right)Y_i^\nu\\&-2\bar{X}^{\bar{\imath}}_\mu\dot{x}^\mu\left(\dot{\tilde{x}}_\nu-A_{\tau\nu}-B_{\nu\rho}\dot{x}^\rho\right)\bar{Y}_{\bar{\imath}}^\nu.\end{aligned}\tag{2.32}$$

Note that, in accordance with the completeness relation (2.5), the auxiliary vector potential decomposes as

$$A_{\alpha\mu}=K_{\mu\nu}\left(H^{\nu\rho}A_{\alpha\rho}\right)+X^i_\mu\left(Y_i^\rho A_{\alpha\rho}\right)+\bar{X}^{\bar{\imath}}_\mu\left(\bar{Y}_{\bar{\imath}}^\rho A_{\alpha\rho}\right).\tag{2.33}$$

- *Particle dynamics.*
  Integrating out $H^{\mu\nu}A_{\tau\nu}$ gives the on-shell relation,

  $$H^{\mu\nu}A_{\tau\nu}\equiv H^{\mu\nu}\left(\dot{\tilde{x}}_\nu-B_{\nu\lambda}\dot{x}^\lambda\right)\quad\text{or equivalently}\quad H^{\mu\nu}\left(\mathrm{D}_\tau\tilde{x}_\nu-B_{\nu\lambda}\dot{x}^\lambda\right)\equiv0,\tag{2.34}$$

  which implies that the 'dual' conjugate momenta are trivial along $D-n-\bar{n}$ of the $\tilde{x}_\mu$ directions.
  On the other hand, integrating out the remaining components, $Y_i^\rho A_{\tau\rho}$ and $\bar{Y}_{\bar{\imath}}^\rho A_{\tau\rho}$, we acquire constraints on the x^μ coordinates,

  $$X^i_\mu\dot{x}^\mu\equiv0,\quad\bar{X}^{\bar{\imath}}_\mu\dot{x}^\mu\equiv0.\tag{2.35}$$

  Namely, the particle freezes over $n+\bar{n}$ directions on the physical section formed by x^μ coordinates.
- *String dynamics.*
  In the string case, combining the useful identity (2.31) with the topological term in the action (2.27), we can reduce the world-sheet Lagrangian,

$$\begin{aligned}\tfrac{1}{4\pi\alpha'}\mathcal{L}_{\text{string}}=&\,\tfrac{1}{2\pi\alpha'}\mathcal{L}'_{\text{string}},\\\mathcal{L}'_{\text{string}}=&-\tfrac12\sqrt{-h}h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu K_{\mu\nu}\\&+\tfrac12\epsilon^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu B_{\mu\nu}+\tfrac12\epsilon^{\alpha\beta}\partial_\alpha\tilde{x}_\mu\partial_\beta x^\mu\\&-\tfrac12\sqrt{-h}h^{\alpha\gamma}\left[X^i_\mu\left(\partial_\alpha x^\mu+\tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right)\right]\\&\times\left(\partial_\gamma\tilde{x}_\nu-A_{\gamma\nu}-B_{\nu\rho}\partial_\gamma x^\rho\right)Y_i^\nu\\&+\tfrac12\sqrt{-h}h^{\alpha\gamma}\left[\bar{X}^{\bar{\imath}}_\mu\left(\partial_\alpha x^\mu-\tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right)\right]\\&\times\left(\partial_\gamma\tilde{x}_\nu-A_{\gamma\nu}-B_{\nu\rho}\partial_\gamma x^\rho\right)\bar{Y}_{\bar{\imath}}^\nu\\&-\tfrac14\sqrt{-h}h^{\alpha\beta}\left(\mathcal{C}_{\alpha\mu}-A_{\alpha\mu}\right)\left(\mathcal{C}_{\beta\nu}-A_{\beta\nu}\right)H^{\mu\nu},\end{aligned}\tag{2.36}$$

where for short notation we set

$$\mathcal{C}_{\alpha\mu}:=\partial_\alpha\tilde{x}_\mu-B_{\mu\nu}\partial_\alpha x^\nu+\tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta K_{\mu\nu}\partial_\beta x^\nu.\tag{2.37}$$

Now, integrating out $H^{\mu\nu}A_{\alpha\nu}$ we obtain the on-shell relation,

$$\begin{aligned}&H^{\mu\nu}A_{\alpha\nu}\equiv H^{\mu\nu}\mathcal{C}_{\alpha\nu}\quad\text{or equivalently}\\&H^{\mu\nu}\left(\mathrm{D}_\alpha\tilde{x}_\mu-B_{\mu\nu}\partial_\alpha x^\nu+\tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta K_{\mu\nu}\partial_\beta x^\nu\right)\equiv0,\end{aligned}\tag{2.38}$$

and integrating out $Y_i^\nu A_{\alpha\nu}$, $\bar{Y}_{\bar{\imath}}^\nu A_{\alpha\nu}$, we obtain chiral constraints,

$$\begin{aligned}&X^i_\mu\left(\partial_\alpha x^\mu+\tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right)\equiv0,\\&\bar{X}^{\bar{\imath}}_\mu\left(\partial_\alpha x^\mu-\tfrac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right)\equiv0.\end{aligned}\tag{2.39}$$

Namely, the string becomes chiral over n directions and anti-chiral over $\bar{n}$ directions on the section coordinatized by x^μ. The chirality further implies that strings which meet boundary conditions (periodic, Neumann or Dirichlet) are also frozen, or localized, over the $n+\bar{n}$ directions, similarly to the particle case (2.35).

2.3 DFT-vielbeins for $(n,\bar{n})$ doubled-yet-gauged spacetime

In order to couple to fermions [33,49] or to the RR-sector [50], as well as for supersymmetrizations [38,51,52], it is necessary to introduce *a pair of* DFT-vielbeins, V_{Ap} and $\bar{V}_{A\bar{p}}$, from which one can construct the projectors,

$$\begin{aligned}P_{AB}&=\tfrac12(\mathcal{J}_{AB}+\mathcal{H}_{AB})=V_{Ap}V_{Bq}\eta^{pq},\\\bar{P}_{AB}&=\tfrac12(\mathcal{J}_{AB}-\mathcal{H}_{AB})=\bar{V}_{A\bar{p}}\bar{V}_{B\bar{q}}\bar{\eta}^{\bar{p}\bar{q}},\end{aligned}\tag{2.40}$$

where η_{pq} and $\bar{\eta}_{\bar{p}\bar{q}}$ are the two constant metrics of twofold local Lorentz symmetries for two distinct locally inertial frames, one for the left and the other for the right closed string modes [53].[5] To ensure the symmetric, orthogonal and completeness properties of the projectors (1.7), the DFT-vielbeins must satisfy their own defining properties:

$$V_{Ap}V^{A}{}_{q}=\eta_{pq},\quad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}},\quad V_{Ap}\bar{V}^{A\bar{p}}=0,$$
$$V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}. \tag{2.41}$$

Essentially, with $\mathcal{H}_{AB}=V_{Ap}V_{B}{}^{p}-\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}$, the DFT-vielbeins diagonalize $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$ simultaneously, with the eigenvalues $(\eta,+\bar{\eta})$ and $(\eta,-\bar{\eta})$.

The main result of this subsection is the construction of the DFT-vielbeins, V_{Ap} and $\bar{V}_{A\bar{p}}$, for the general $(n,\bar{n})$ DFT-metric (2.2). They are given by $2D\times(D+n-\bar{n})$ and $2D\times(D-n+\bar{n})$ matrices, respectively,

$$V_{Ap}=\frac{1}{\sqrt{2}}\begin{pmatrix}\mathcal{Y}_{p}{}^{\mu}\\ \mathcal{X}_{\nu}{}^{q}\eta_{qp}+B_{\nu\sigma}\mathcal{Y}_{p}{}^{\sigma}\end{pmatrix},$$
$$\bar{V}_{A\bar{p}}=\frac{1}{\sqrt{2}}\begin{pmatrix}\bar{\mathcal{Y}}_{\bar{p}}{}^{\mu}\\ \bar{\mathcal{X}}_{\nu}{}^{\bar{q}}\bar{\eta}_{\bar{q}\bar{p}}+B_{\nu\sigma}\bar{\mathcal{Y}}_{\bar{p}}{}^{\sigma}\end{pmatrix}. \tag{2.42}$$

Here $\mathcal{X}_{\mu}{}^{p},\mathcal{Y}_{p}{}^{\mu},\bar{\mathcal{X}}_{\nu}{}^{\bar{q}}$ and $\bar{\mathcal{Y}}_{\bar{q}}{}^{\nu}$ are, respectively, $D\times(D+n-\bar{n})$, $(D+n-\bar{n})\times D$, $D\times(D-n+\bar{n})$ and $(D-n+\bar{n})\times D$ matrices, such that $1\leq p\leq D+n-\bar{n}$ and $1\leq\bar{p}\leq D-n+\bar{n}$. Explicitly, with the smaller range of indices, $1\leq a,\bar{a}\leq D-n-\bar{n}$ and $1\leq i\leq n,\ 1\leq\bar{\imath}\leq\bar{n}$ as before, the matrices read

$$\mathcal{X}_{\mu}{}^{p}:=\begin{pmatrix}k_{\mu}{}^{a} & X^{i}_{\mu} & X^{j}_{\mu}\end{pmatrix},$$
$$\bar{\mathcal{X}}_{\mu}{}^{\bar{p}}:=\begin{pmatrix}\bar{k}_{\mu}{}^{\bar{a}} & \bar{X}^{\bar{\imath}}_{\mu} & \bar{X}^{\bar{\jmath}}_{\mu}\end{pmatrix},$$
$$\mathcal{Y}_{p}{}^{\mu}:=\begin{pmatrix}h_{a}{}^{\mu}\\ Y^{\mu}_{i}\\ Y^{\mu}_{j}\end{pmatrix}, \tag{2.43}$$
$$\bar{\mathcal{Y}}_{\bar{p}}{}^{\mu}:=\begin{pmatrix}\bar{h}_{\bar{a}}{}^{\mu}\\ \bar{Y}^{\mu}_{\bar{\imath}}\\ \bar{Y}^{\mu}_{\bar{\jmath}}\end{pmatrix},$$

where $\{h_{a}{}^{\mu},k_{\nu}{}^{b}\}$ and $\{\bar{h}_{\bar{a}}{}^{\mu},\bar{k}_{\nu}{}^{\bar{b}}\}$ are two sets of the "square-roots" of $H^{\mu\nu}$ and $K_{\mu\nu}$,

$$H^{\mu\nu}=\eta^{ab}h_{a}{}^{\mu}h_{b}{}^{\nu}=-\bar{\eta}^{\bar{a}\bar{b}}\bar{h}_{\bar{a}}{}^{\mu}\bar{h}_{\bar{b}}{}^{\nu},$$
$$K_{\mu\nu}=k_{\mu}{}^{a}k_{\nu}{}^{b}\eta_{ab}=-\bar{k}_{\mu}{}^{\bar{a}}\bar{k}_{\nu}{}^{\bar{b}}\bar{\eta}_{\bar{a}\bar{b}}. \tag{2.44}$$

The 'total' twofold local Lorentz symmetry group is clearly $\mathbf{Spin}(t+n,s+n)\times\mathbf{Spin}(s+\bar{n},t+\bar{n})$, with $t+s+n+\bar{n}=D$, where (t,s) is the signature of $H^{\mu\nu}$ and $K_{\mu\nu}$. The corresponding constant metrics are η_{pq} and $\bar{\eta}_{\bar{p}\bar{q}}$, respectively, while η_{ab} and $\bar{\eta}_{\bar{a}\bar{b}}$ are $(t+s)\times(t+s)$ sub-blocks of them, of which the signatures are numerically opposite to each other [18],

$$\eta_{pq}=\begin{pmatrix}\eta_{ab} & 0 & 0\\ 0 & -\delta_{ij} & 0\\ 0 & 0 & +\delta_{ij}\end{pmatrix},$$
$$\eta_{ab}=\mathrm{diag}(\underbrace{--\ldots--}_{t}\underbrace{++\ldots++}_{s}),$$
$$\bar{\eta}_{\bar{p}\bar{q}}=\begin{pmatrix}\bar{\eta}_{\bar{a}\bar{b}} & 0 & 0\\ 0 & +\delta_{\bar{\imath}\bar{\jmath}} & 0\\ 0 & 0 & -\delta_{\bar{\imath}\bar{\jmath}}\end{pmatrix},$$
$$\bar{\eta}_{\bar{a}\bar{b}}=\mathrm{diag}(\underbrace{++\ldots++}_{t}\underbrace{--\ldots--}_{s}). \tag{2.45}$$

There are defining properties of $\{h_{a}{}^{\mu},k_{\nu}{}^{b}\}$ and $\{\bar{h}_{\bar{a}}{}^{\mu},\bar{k}_{\nu}{}^{\bar{b}}\}$, in accordance with (2.4) and (2.5):

$$h_{a}{}^{\mu}X^{i}_{\mu}=0,\quad h_{a}{}^{\mu}\bar{X}^{\bar{\imath}}_{\mu}=0,\quad Y^{\mu}_{i}k_{\mu}{}^{a}=0,\quad \bar{Y}^{\mu}_{\bar{\imath}}k_{\mu}{}^{a}=0,$$
$$\bar{h}_{\bar{a}}{}^{\mu}X^{i}_{\mu}=0,\quad \bar{h}_{\bar{a}}{}^{\mu}\bar{X}^{\bar{\imath}}_{\mu}=0,\quad Y^{\mu}_{i}\bar{k}_{\mu}{}^{\bar{a}}=0,\quad \bar{Y}^{\mu}_{\bar{\imath}}\bar{k}_{\mu}{}^{\bar{a}}=0,$$
$$k_{\mu}{}^{a}h_{a}{}^{\nu}+X^{i}_{\mu}Y^{\nu}_{i}+\bar{X}^{\bar{\imath}}_{\mu}\bar{Y}^{\nu}_{\bar{\imath}}=\delta_{\mu}{}^{\nu},\quad h_{a}{}^{\mu}k_{\mu}{}^{b}=\delta_{a}{}^{b},$$
$$\bar{k}_{\mu}{}^{\bar{a}}\bar{h}_{\bar{a}}{}^{\nu}+X^{i}_{\mu}Y^{\nu}_{i}+\bar{X}^{\bar{\imath}}_{\mu}\bar{Y}^{\nu}_{\bar{\imath}}=\delta_{\mu}{}^{\nu},\quad \bar{h}_{\bar{a}}{}^{\mu}\bar{k}_{\mu}{}^{\bar{b}}=\delta_{\bar{a}}{}^{\bar{b}}. \tag{2.46}$$

It follows that

$$\mathcal{X}_{\mu}{}^{p}\mathcal{Y}_{p}{}^{\nu}=\delta_{\mu}{}^{\nu}+X^{i}_{\mu}Y^{\nu}_{i}-\bar{X}^{\bar{\imath}}_{\mu}\bar{Y}^{\nu}_{\bar{\imath}},$$
$$\bar{\mathcal{X}}_{\mu}{}^{\bar{p}}\bar{\mathcal{Y}}_{\bar{p}}{}^{\nu}=\delta_{\mu}{}^{\nu}-X^{i}_{\mu}Y^{\nu}_{i}+\bar{X}^{\bar{\imath}}_{\mu}\bar{Y}^{\nu}_{\bar{\imath}},$$
$$\mathcal{Y}_{p}{}^{\lambda}\mathcal{X}_{\lambda}{}^{q}=\begin{pmatrix}\delta_{a}{}^{b} & 0 & 0\\ 0 & \delta_{i}{}^{k} & \delta_{i}{}^{l}\\ 0 & \delta_{j}{}^{k} & \delta_{j}{}^{l}\end{pmatrix},$$
$$\bar{\mathcal{Y}}_{\bar{p}}{}^{\lambda}\bar{\mathcal{X}}_{\lambda}{}^{\bar{q}}=\begin{pmatrix}\delta_{\bar{a}}{}^{\bar{b}} & 0 & 0\\ 0 & \delta_{\bar{\imath}}{}^{\bar{k}} & \delta_{\bar{\imath}}{}^{\bar{l}}\\ 0 & \delta_{\bar{\jmath}}{}^{\bar{k}} & \delta_{\bar{\jmath}}{}^{\bar{l}}\end{pmatrix}, \tag{2.47}$$

and

[5] Ensuring the twofold spin groups in the maximally supersymmetric DFT [38] and the doubled-yet-gauged Green–Schwarz superstring action [22], the conventional IIA and IIB theories are unified into a single theory that is chiral with respect to both spin groups. The distinction of IIA and IIB then refers to two different types of (Riemannian) 'solutions' rather than 'theories'.

$$
\begin{aligned}
P_{AB} &= \begin{pmatrix} \frac{1}{2}H & Y_i(X^i)^T + \frac{1}{2}H(K-B) \\ X^i(Y_i)^T + \frac{1}{2}(K+B)H & \frac{1}{2}(K+B)H(K-B) + BY_i(X^i)^T - X^i(Y_i)^T B \end{pmatrix}, \\
\bar{P}_{AB} &= \begin{pmatrix} -\frac{1}{2}H & \bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T + \frac{1}{2}H(K+B) \\ \bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T + \frac{1}{2}(K-B)H & -\frac{1}{2}(K-B)H(K+B) + B\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T - \bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T B \end{pmatrix},
\end{aligned} \tag{2.48}
$$

where the superscript T converts column vectors to row ones. As expected, P_{AB} and $\bar{P}_{AB}$ are, respectively, free of the barred and unbarred variables, $\{\bar{X}^{\bar{\imath}}, \bar{Y}_{\bar{\jmath}}\}$ and $\{X^i, Y_j\}$.

In a parallel manner to (2.11), the B-field contributes to the DFT-vielbeins through $\mathbf{O}(D, D)$ multiplications,

$$
\begin{aligned}
V_{Mp} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} \mathcal{Y}^T \\ \mathcal{X}\eta \end{pmatrix}, \\
\bar{V}_{M\bar{p}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} \bar{\mathcal{Y}}^T \\ \bar{\mathcal{X}}\bar{\eta} \end{pmatrix}.
\end{aligned} \tag{2.49}
$$

For consistency, the trace of the DFT-metric reads

$$
\begin{aligned}
\mathcal{H}_A{}^A &= \eta_p{}^p - \bar{\eta}_{\bar{p}}{}^{\bar{p}} = (t+s+2n) - (t+s+2\bar{n}) \\
&= (D+n-\bar{n}) - (D-n+\bar{n}) = 2(n-\bar{n}).
\end{aligned} \tag{2.50}
$$

The symmetry of the DFT-metric (2.16) extends to DFT-vielbeins:

$$
\begin{aligned}
Y_i^\mu &\longmapsto Y_i^\mu + H^{\mu\nu}V_{\nu i}, \\
\bar{Y}_{\bar{\imath}}^\mu &\longmapsto \bar{Y}_{\bar{\imath}}^\mu + H^{\mu\nu}\bar{V}_{\nu\bar{\imath}}, \\
k_\mu{}^a &\longmapsto k_\mu{}^a - X_\mu^i \eta^{ab} h_b{}^\nu V_{\nu i} - \bar{X}_\mu^{\bar{\imath}} \eta^{ab} h_b{}^\nu \bar{V}_{\nu\bar{\imath}}, \\
\bar{k}_\mu{}^{\bar{a}} &\longmapsto \bar{k}_\mu{}^{\bar{a}} - X_\mu^i \bar{\eta}^{\bar{a}\bar{b}} \bar{h}_{\bar{b}}{}^\nu V_{\nu i} - \bar{X}_\mu^{\bar{\imath}} \bar{\eta}^{\bar{a}\bar{b}} \bar{h}_{\bar{b}}{}^\nu \bar{V}_{\nu\bar{\imath}}, \\
B_{\mu\nu} &\longmapsto B_{\mu\nu} - 2X^i_{[\mu}V_{\nu]i} + 2\bar{X}^{\bar{\imath}}_{[\mu}\bar{V}_{\nu]\bar{\imath}} \\
&\quad + 2X^i_{[\mu}\bar{X}^{\bar{\imath}}_{\nu]}\left(Y_i^\rho \bar{V}_{\rho\bar{\imath}} + \bar{Y}_{\bar{\imath}}^\rho V_{\rho i} + V_{\rho i}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\right).
\end{aligned} \tag{2.51}
$$

As seen from the doubled-yet-gauged actions for particle and string (2.32), (2.36), as well as the coupling to a scalar field (2.9), it is not the full signatures of the spin group,

$$
\mathbf{Spin}(t+n, s+n) \times \mathbf{Spin}(s+\bar{n}, t+\bar{n}), \tag{2.52}
$$

but the signature of $K_{\mu\nu}$ and $H^{\mu\nu}$, i.e. (t, s), that matters for *unitarity*. The choice of $t = 1$ then amounts to the usual Minkowskian spacetime.

$$
\hat{g} = \begin{pmatrix} g' + aga^T & ag \\ ga^T & g \end{pmatrix} = \exp\left[\hat{a}\right] \begin{pmatrix} g' & 0 \\ 0 & g \end{pmatrix} \exp\left[\hat{a}^T\right]
$$

$$
\text{where} \quad \hat{a}_{\hat{\mu}}{}^{\hat{\nu}} = \begin{pmatrix} 0 & a_{\mu'}{}^\nu \\ 0 & 0 \end{pmatrix}. \tag{2.53}
$$

In a similar fashion, we propose the Kaluza–Klein ansatz for the DFT-metric, $\hat{\mathcal{H}}_{\hat{M}\hat{N}}$,

$$
\hat{\mathcal{H}} = \exp\left[\hat{W}\right] \begin{pmatrix} \mathcal{H}' & 0 \\ 0 & \mathcal{H} \end{pmatrix} \exp\left[\hat{W}^T\right], \tag{2.54}
$$

for which we decompose $\hat{D} = D' + D$, such that

$$
\mathbf{O}(\hat{D}, \hat{D}) \;\rightarrow\; \mathbf{O}(D', D') \times \mathbf{O}(D, D), \quad \hat{\mathcal{J}} = \begin{pmatrix} \mathcal{J}' & 0 \\ 0 & \mathcal{J} \end{pmatrix}, \tag{2.55}
$$

and we set an off-block-diagonal $\mathfrak{so}(D, D)$ element,

$$
\begin{aligned}
\hat{W} &= \begin{pmatrix} 0 & -W \\ \bar{W} & 0 \end{pmatrix} \in \mathfrak{so}(D, D), \\
\bar{W}_M{}^{M'} &:= W^{M'}{}_M = \mathcal{J}_{MN} W_{N'}{}^N \mathcal{J}'^{N'M'}.
\end{aligned} \tag{2.56}
$$

Further, we impose a constraint on the $2D' \times 2D$ matrix, $W_{M'}{}^N$,

$$
\bar{W}W = 0 \quad \text{or explicitly} \quad W_{L'M}W^{L'N} = 0, \tag{2.57}
$$

which sets half of its components trivial. At least for the Riemannian, i.e. $(0, 0)$ case, this constraint makes the counting of the degrees of freedom consistent: $g_{\mu\nu}$ and $B_{\mu\nu}$ have D^2 degrees of freedom, while $W_{M'}{}^N$ has $2D'D$ degrees, such that

$$
\hat{D}^2 = (D'+D)^2 = D'^2 + D^2 + 2D'D, \tag{2.58}
$$

matching the degrees of freedom between $\hat{\mathcal{H}}$ and $\{\mathcal{H}', \mathcal{H}, W\}$. Essentially, $\hat{g}_{\mu'\nu}$ and $\hat{B}_{\mu'\nu}$ constitute $W_{M'}{}^N$.

Explicitly, we have $\hat{W}^3 = 0$ and

$$
\hat{\mathcal{H}} = \begin{pmatrix} (1-\frac{1}{2}W\bar{W})\mathcal{H}'(1-\frac{1}{2}W\bar{W})^T + W\mathcal{H}W^T & -W\mathcal{H} + (1-\frac{1}{2}W\bar{W})\mathcal{H}'\bar{W}^T \\ -\mathcal{H}W^T + \bar{W}\mathcal{H}'(1-\frac{1}{2}W\bar{W})^T & \mathcal{H} + \bar{W}\mathcal{H}'\bar{W}^T \end{pmatrix}, \tag{2.59}
$$

which is classified by four non-negative integers: $(n, \bar{n})$ for $\mathcal{H}_{AB}$ and $(n', \bar{n}')$ for $\mathcal{H}'_{A'B'}$, with the total trace, $\hat{\mathcal{H}}_{\hat{A}}{}^{\hat{A}} = 2(n+n'-\bar{n}-\bar{n}')$.

2.4 Kaluza–Klein ansatz for DFT

The ordinary Kaluza–Klein ansatz for a Riemannian metric can be 'block-diagonalized', Especially, in the maximally non-Riemannian case of $\mathcal{H}' = \mathcal{J}'$, i.e. $(n', \bar{n}') = (D', 0)$, the above expression dramatically simplifies

$$\hat{\mathcal{H}} = \begin{pmatrix} \mathcal{J}' - 2W\bar{P}W^T & 2W\bar{P} \\ 2\bar{P}W^T & \mathcal{H} \end{pmatrix}. \tag{2.60}$$

Intriguingly, the resulting field content, $\mathcal{H}_{AB}$, $\bar{P}_{AB}W_{A'}{}^{B}$, coincides with the ansatz for heterotic DFT proposed by Hohm et al. [54]. We leave it as a future work to explore the tantalizing connection between heterotic string and non-Riemannian doubled-yet-gauged spacetime, possibly using the Scherk–Schwartz reduction scheme in DFT [52,55–63].

3 Applications

The case of $(0, 0)$ admits a well-defined Riemannian metric and hence corresponds to Riemannian geometry, or to "generalized geometry" [64–69] when equipped with the pair of DFT-vielbeins. In this section, we discuss various applications of other $(n, \bar{n})$ backgrounds and identify the corresponding geometries.

3.1 Maximally non-Riemannian $(D, 0)$: Siegel's chiral string

In the maximally non-Riemannian case of $(D, 0)$, with $i = 1, 2, \ldots, D$, we can view X^i_μ as a non-degenerate $D \times D$ square matrix. Then, from (2.7) and

$$\left(X^j_\lambda Y^\mu_j\right) X^i_\mu = X^i_\lambda, \tag{3.1}$$

we conclude that $X^j_\lambda Y^\mu_j$ is actually an identity,

$$X^j_\lambda Y^\mu_j = \delta_\lambda{}^\mu. \tag{3.2}$$

Thus, in the case of $(D, 0)$, we have

$$\mathcal{J}_{AB} = \mathcal{H}_{AB} = P_{AB}, \quad \bar{P}_{AB} = 0. \tag{3.3}$$

The corresponding DFT-vielbein, V_{Ap} (2.42) and the $\mathbf{Spin}(D, D)$ metric are also $2D \times 2D$ square matrices,

$$V_{Ap} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \eta_{pq} = \begin{pmatrix} -\delta_{ij} & 0 \\ 0 & +\delta_{ij} \end{pmatrix}. \tag{3.4}$$

On the other hand, $\bar{V}_{A\bar{p}}$ is trivial.

The resulting string action is completely chiral on the D-dimensional section (2.36) [20],

$$S_{\text{string}} = \tfrac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \;\; \epsilon^{\alpha\beta} \partial_\alpha \tilde{x}_\mu \partial_\beta x^\mu,$$
$$\partial_\alpha x^\mu + \tfrac{1}{\sqrt{-h}} \epsilon_\alpha{}^\beta \partial_\beta x^\mu = 0. \tag{3.5}$$

From the conventional $(0, 0)$ set-up, noting the sign difference,

$$\mathcal{J}_{AB} = V_A{}^p V_B{}^q \eta_{pq} + \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}},$$
$$\mathcal{H}_{AB} = V_A{}^p V_B{}^q \eta_{pq} - \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}}, \tag{3.6}$$

we may regard the substitution of the $\mathbf{O}(D, D)$ invariant metric, $\mathcal{J}_{AB}$, into the DFT-metric, $\mathcal{H}_{AB}$, inside the doubled-yet-gauged string action (2.27) as the flipping of the spin group signature,

$$\bar{\eta}_{\bar{p}\bar{q}} \longrightarrow -\bar{\eta}_{\bar{p}\bar{q}}, \tag{3.7}$$

such that η_{pq} and $-\bar{\eta}_{\bar{p}\bar{q}}$ assume not opposite (as in (2.45)) but rather identical signatures. That is to say, there are no right modes: only left modes exist. This is consistent with (3.5), and it realizes the chiral string theory *à la* Siegel [70][6] in a rather geometric set-up.

3.2 $D = 10$, $(3, 3)$: non-Riemannian dimensional reduction from ten to four

If we set $n = \bar{n}$, then the DFT-metric is traceless and the two spin groups become commonly D-dimensional,

$$\mathbf{Spin}(t + n, s + n) \times \mathbf{Spin}(s + n, t + n) \quad \text{where}$$
$$t + s + 2n = D. \tag{3.8}$$

Thus, the maximally supersymmetric $D = 10$ DFT [38] and the doubled-yet-gauged Green–Schwarz superstring [22], both of which assume the Minkowskian spin group, $\mathbf{Spin}(1, 9) \times \mathbf{Spin}(9, 1)$, can accommodate $(0, 0)$ and $(1, 1)$. However, the theories constructed in [22,38] can be readily generalized to an arbitrary signature, $\mathbf{Spin}(\hat{t}, \hat{s}) \times \mathbf{Spin}(\hat{s}, \hat{t})$, with $\hat{t} + \hat{s} = 10$, by relaxing the Majorana condition on the spinors and employing their charge conjugations only, without involving the complex Dirac conjugations. In this case, the theory can describe (n, n) non-Riemannian doubled-yet-gauged spacetime with $n = 0, 1, 2, \ldots, \min\left(\hat{t}, \hat{s}\right)$.

An interesting choice then appears to be $\mathbf{Spin}(4, 6) \times \mathbf{Spin}(6, 4)$. Such a choice can encompass six-dimensional $(3, 3)$ non-Riemannian 'internal' spacetime, while maintaining the ordinary four-dimensional Minkowskian 'external' spacetime. As analyzed in Sect. 2.2, point particles and strings freeze on the $(3, 3)$ internal spacetime and this may imply a natural dimensional reduction of string theory from ten to four, alternative to the conventional compactification on 'small' Riemannian manifolds; e.g. CY_3. The latter will be of interest in order to analyse the Killing spinor equations [38] for the $D = 10$ $(3, 3)$ DFT-vielbeins (2.42). Certainly, constant 'flat' backgrounds are maximally supersymmetric.

[6] See also [71,72].

3.3 (1, 1): non-relativistic limit *à la* Gomis–Ooguri

In this subsection, we identify (1, 1) as the non-relativistic limit *à la* Gomis–Ooguri [23]. We start by considering a *generic* Riemannian metric which depends explicitly on the speed of light, c,

$$g_{\mu\nu} = -c^2 T_\mu T_\nu (1 - S^\rho S^\sigma \Phi_{\rho\sigma}) + 2cT_{(\mu}\Phi_{\nu)\rho}S^\rho + \Phi_{\mu\nu}, \tag{3.9}$$

where T_μ and S^ν are orthogonal time-like and space-like vectors,

$$T_\mu S^\mu = 0. \tag{3.10}$$

Essentially, (3.9) is the 'covariantized' form of the ordinary Kaluza–Klein ansatz for the Riemannian metric (2.53) as

$$\begin{aligned} &g_{\mu\nu} = (\delta_\mu{}^\rho + cT_\mu S^\rho)(\delta_\nu{}^\sigma + cT_\nu S^\sigma)(-c^2 T_\rho T_\sigma + \Phi_{\rho\sigma}),\\ &\quad \left[\exp(cT{\cdot}S)\right]_\mu{}^\nu = \delta_\mu{}^\nu + cT_\mu S^\nu. \end{aligned} \tag{3.11}$$

The inverse of the metric is then given by

$$\begin{aligned} g^{\mu\nu} &= \Upsilon^{\mu\nu} - S^\mu S^\nu + \tfrac{2}{c} N^{(\mu}S^{\nu)} - \tfrac{1}{c^2} N^\mu N^\nu \\ &= (\delta^\mu{}_\rho - cS^\mu T_\rho)(\delta^\nu{}_\sigma - cS^\nu T_\sigma)(-\tfrac{1}{c^2} N^\rho N^\sigma + \Upsilon^{\rho\sigma}), \end{aligned} \tag{3.12}$$

where the variables, N^ν and $\Upsilon^{\mu\nu}$, meet by construction,[7]

$$\begin{aligned} &T_\mu N^\mu = 1, \quad T_\mu \Upsilon^{\mu\nu} = 0, \quad N^\mu \Phi_{\mu\nu} = 0,\\ &\quad T_\mu N^\nu + \Phi_{\mu\rho}\Upsilon^{\rho\nu} = \delta_\mu{}^\nu. \end{aligned} \tag{3.13}$$

Now, we introduce an ansatz for the B-field in a similar manner,

$$B_{\mu\nu} = 2cT_{[\mu}B_{\nu]} + B^0_{\mu\nu}, \tag{3.14}$$

and we require that the Riemannian DFT-metric (1.9) should be non-singular in the non-relativistic, large c limit. In (3.14), without loss of generality we may set B_μ to be orthogonal to N^ν, i.e. $N^\mu B_\mu = 0$. Further, $B^0_{\mu\nu}$ denotes the zeroth order in c, which is arbitrary and should survive once the limit is taken, as expected from the 'Abelian' nature of the B-field from (2.11) and (2.12).

Clearly in the limit of $c \to \infty$, the inverse, $g^{\mu\nu}$, is regular. We only need to ensure both $g^{-1}B$ and $g - Bg^{-1}B$ to be non-singular. The former implies

$$\begin{aligned} &\left(\Upsilon^{\mu\nu} - S^\mu S^\nu\right) B_\mu = 0, \lim_{c\to\infty} g^{\mu\rho}B_{\rho\nu} = (\Upsilon^{\mu\rho} - S^\mu S^\rho)B^0_{\rho\nu}\\ &\quad + S^\mu B_\nu - (S^\rho B_\rho)N^\mu T_\nu. \end{aligned} \tag{3.15}$$

In turn, $Bg^{-1}B$ cannot be quadratically singular, and hence, for the regularity of $g - Bg^{-1}B$, the leading power of g must be first order in c, i.e. the apparent second order term in (3.9) must be trivial,

$$S^\rho S^\sigma \Phi_{\rho\sigma} = 1. \tag{3.16}$$

Therefore, the nontrivial cancellation of diverging terms inside $g - Bg^{-1}B$ takes place at the first order, reading

$$\begin{aligned} &c \times \Big[\left(\Phi_{\mu\rho}S^\rho - B_\rho S^\rho B_\mu\right) T_\nu \\ &\quad + \left(\Phi_{\nu\rho}S^\rho - B_\rho S^\rho B_\nu\right) T_\mu \Big] = 0. \end{aligned} \tag{3.17}$$

Contraction of the quantity inside the square bracket with N^ν gives

$$B_\rho S^\rho B_\mu = \Phi_{\mu\rho}S^\rho. \tag{3.18}$$

Hence from (3.16) and (3.18), we obtain

$$B_\rho S^\rho = \pm 1. \tag{3.19}$$

It follows that $g - Bg^{-1}B$ is non-singular as

$$\begin{aligned} &\lim_{c\to\infty} \left(g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu}\right)\\ &\quad = \Phi_{\mu\nu} - B_\mu B_\nu - B^0_{\mu\rho}(\Upsilon^{\rho\sigma} - S^\rho S^\sigma)B^0_{\sigma\nu}\\ &\qquad \mp (T_\mu N^\sigma - \Phi_{\mu\rho}S^\rho S^\sigma)B^0_{\sigma\nu}\\ &\qquad \pm B^0_{\mu\sigma}(N^\sigma T_\nu - S^\sigma S^\rho \Phi_{\rho\nu}). \end{aligned} \tag{3.20}$$

After all, the DFT-metric becomes completely regular,

$$\begin{aligned} \mathcal{H}_{AB} = &\begin{pmatrix} 1 & 0 \\ B^0 & 1 \end{pmatrix}\\ &\times \begin{pmatrix} \Upsilon - SS^T & \pm\left(SS^T\Phi - NT^T\right) \\ \pm\left(\Phi SS^T - TN^T\right) & \Phi - \Phi SS^T \Phi \end{pmatrix}\\ &\times \begin{pmatrix} 1 & -B^0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{3.21}$$

which can easily and precisely be identified as the (1, 1) type of the classification (2.2) as

$$\begin{aligned} &\Upsilon^{\mu\nu} - S^\mu S^\nu \equiv H^{\mu\nu}, \quad \Phi_{\mu\nu} - \Phi_{\mu\rho}S^\rho \Phi_{\nu\sigma}S^\sigma \equiv K_{\mu\nu},\\ &\quad \left\{T_\mu,\ \Phi_{\nu\rho}S^\rho\right\} \equiv \left\{X_\mu,\ \bar{X}_\nu\right\}. \end{aligned} \tag{3.22}$$

As demonstrated in [20], a constant flat background belonging to this type generates the Gomis–Ooguri non-relativistic string [23] (see also [22] for its Green–Schwarz superstring extension). Thus, a generic (1, 1) DFT-metric provides a curved spacetime generalization of the non-relativistic string.

3.4 $(D-1, 0)$: ultra-relativistic or Carroll

The Riemannian metric (3.9) in the previous section defines the *proper length*. Rescaling the metric by an overall factor of c^{-2}, it becomes the Riemannian metric for the *proper time*:

[7] In Sects. 3.4 and 3.5, $\{T_\mu, N^\nu, \Upsilon^{\mu\nu}, \Phi_{\mu\nu}\}$ will be identified as either Carroll or Newton–Cartan variables.

$$g_{\mu\nu} = -T_\mu T_\nu(1 - S^\rho S^\sigma \Phi_{\rho\sigma}) + \frac{2}{c} T_{(\mu}\Phi_{\nu)\rho} S^\rho + \frac{1}{c^2}\Phi_{\mu\nu},$$
$$g^{\mu\nu} = c^2\left(\Upsilon^{\mu\nu} - S^\mu S^\nu\right) + 2cN^{(\mu}S^{\nu)} - N^\mu N^\nu, \qquad (3.23)$$

where the variables should satisfy (3.10) and (3.13), which we recall here:

$$T_\mu S^\mu = 0, \quad T_\mu N^\mu = 1, \quad T_\mu \Upsilon^{\mu\nu} = 0,$$
$$N^\mu \Phi_{\mu\nu} = 0, \quad T_\mu N^\nu + \Phi_{\mu\rho}\Upsilon^{\rho\nu} = \delta_\mu{}^\nu. \qquad (3.24)$$

Clearly, the expression of $g^{\mu\nu}$ in (3.23) indicates the possibility of taking a small c i.e. ultra-relativistic limit, as the inverse remains non-singular, yet degenerate having rank one,

$$\lim_{c\to 0} g^{\mu\nu} = -N^\mu N^\nu. \qquad (3.25)$$

In this subsection, we propose a $(D-1,0)$ DFT-metric as the ultra-relativistic 'completion' of the above degenerate inverse (3.25),

$$\mathcal{H}_{AB} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} -NN^T & \Upsilon\Phi \\ \Phi\Upsilon & -TT^T \end{pmatrix}\begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \qquad (3.26)$$

where all the variables are from (3.24). It is easy to check that this ansatz satisfies the defining properties of the DFT-metric (2.1) and $\mathcal{H}_A{}^A = 2\Upsilon^{\mu\nu}\Phi_{\mu\nu} = 2(D-1)$. Note the identification

$$H^{\mu\nu} \equiv -N^\mu N^\nu, \quad K_{\mu\nu} \equiv -T_\mu T_\nu,$$
$$\sum_{i=1}^{D-1} X^i_\mu Y^\nu_i \equiv \Phi_{\mu\rho}\Upsilon^{\rho\nu}. \qquad (3.27)$$

From (2.35), particles freeze over almost all the directions except one,

$$\Phi_{\mu\nu}\dot{x}^\nu \equiv 0. \qquad (3.28)$$

This is in agreement with the ultra-relativistic limit of Riemannian geodesics *à la* Bergshoeff et al. [73]. Namely, particles cannot move faster than light, and thus must freeze in the ultra-relativistic limit, $c \to 0$.

In fact, $(D-1,0)$ forms a Carroll structure [74,75]: $\Phi_{\mu\nu}$ is known as a *Carrollian metric*, i.e. a rank $(D-1)$ covariant metric whose kernel is spanned by the *Carroll vector*, N^ν, and T_μ is a *principal connection*. The *Carrollian boost* symmetry [75] is given, with an arbitrary local parameter, V^μ, by

$$T_\mu \longmapsto T_\mu + \Phi_{\mu\nu}V^\nu,$$
$$\Upsilon^{\mu\nu} \longmapsto \Upsilon^{\mu\nu} - 2N^{(\mu}\Upsilon^{\nu)\rho}\Phi_{\rho\sigma}V^\sigma + N^\mu N^\nu \Phi_{\rho\sigma}V^\rho V^\sigma,$$
$$B_{\mu\nu} \longmapsto B_{\mu\nu} + 2T_{[\mu}\Phi_{\nu]\rho}V^\rho, \qquad (3.29)$$

which leaves our $(D-1,0)$ DFT-metric (3.26) invariant, and can be identified with the symmetry of the DFT-vielbein (2.51) for the case of $(D-1,0)$.

3.5 Least non-Riemannian (1, 0) or (0, 1): Non-relativistic or Newton–Cartan

The ordinary Kaluza–Klein ansatz (2.53) treats the two block-diagonal Riemannian metrics, g and g', asymmetrically. Exchanging the two will lead to an alternative Kaluza–Klein ansatz. In this subsection, we consider such an alternative ansatz for the Riemannian metric (3.9), which reads

$$g_{\mu\nu} = (\delta_\mu{}^\rho - c^{-1}\Phi_{\mu\kappa}U^\kappa N^\rho)(\delta_\nu{}^\sigma - c^{-1}\Phi_{\nu\lambda}U^\lambda N^\sigma)$$
$$\times\ (-c^2 T_\rho T_\sigma + \Phi_{\rho\sigma})$$
$$= -c^2 T_\mu T_\nu + 2cT_{(\mu}\Phi_{\nu)\rho}U^\rho + \Phi_{\mu\nu} - \Phi_{\mu\rho}U^\rho\Phi_{\nu\sigma}U^\sigma, \qquad (3.30)$$

with the inverse,

$$g^{\mu\nu} = (\delta^\mu{}_\rho + c^{-1}N^\mu U^\kappa\Phi_{\kappa\rho})(\delta^\nu{}_\sigma + c^{-1}N^\nu U^\lambda\Phi_{\lambda\sigma})$$
$$\times\ (-c^{-2}N^\rho N^\sigma + \Upsilon^{\rho\sigma})$$
$$= \Upsilon^{\mu\nu} + 2c^{-1}N^{(\mu}U^{\nu)} - c^{-2}N^\mu N^\nu$$
$$\times\ \left(1 - U^\rho\Phi_{\rho\sigma}U^\sigma + 2cT_\rho U^\rho\right). \qquad (3.31)$$

Clearly the inverse of the Riemannian metric allows a non-singular large c limit,

$$\lim_{c\to\infty} g^{\mu\nu} = \Upsilon^{\mu\nu}, \qquad (3.32)$$

of which the rank is $D-1$.

The DFT-metric which completes this degenerate inverse is then

$$\mathcal{H}_{AB} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} \Upsilon & \pm NT^T \\ \pm TN^T & \Phi \end{pmatrix}\begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}, \qquad (3.33)$$

with $\mathcal{H}_A{}^A = \pm 2$. Here the upper and lower signs correspond to $(1,0)$ and $(0,1)$, respectively.

Satisfying (3.13), which we recall,

$$T_\mu N^\mu = 1, \quad T_\mu\Upsilon^{\mu\nu} = 0, \quad N^\mu\Phi_{\mu\nu} = 0,$$
$$T_\mu N^\nu + \Phi_{\mu\rho}\Upsilon^{\rho\nu} = \delta_\mu{}^\nu, \qquad (3.34)$$

$\{T_\lambda, \Upsilon^{\mu\nu}\}$ forms a Leibnizian structure (cf. e.g. [76,77]): T_λ is the *absolute clock* and $\Upsilon^{\mu\nu}$ is a collection of *absolute rulers* with non-negative signature, i.e. $\eta_{ab} = \delta_{ab}$ from (2.44). Further, the vector N^μ corresponds to a field of *observers*, and the covariant rank $D-1$ metric, $\Phi_{\mu\nu}$, provides the associated *transverse metric*. The transformation (2.15) reduces to

$$\begin{aligned}
&N^{\mu} \longmapsto N^{\mu} + U^{\mu},\\
&\Phi_{\mu\nu} \longmapsto \Phi_{\mu\nu} - 2T_{(\mu}\Phi_{\nu)\rho}U^{\rho} + U^{\rho}\Phi_{\rho\sigma}U^{\sigma}T_{\mu}T_{\nu},\\
&B_{\mu\nu} \longmapsto B_{\mu\nu} \mp 2T_{[\mu}\Phi_{\nu]\rho}U^{\rho},
\end{aligned} \tag{3.35}$$

with $U^{\mu} = \Upsilon^{\mu\nu}V_{\nu} \in \mathrm{Ker}(T)$; this is sometimes referred to as a Milne transformation or a Galilean boost in the literature [78].

From (2.35), particles freeze over the time direction only,

$$T_{\mu}\dot{x}^{\mu} = 0. \tag{3.36}$$

Thus, the observer $\dot{x}^{\mu}$ is said to be *space-like*. This is naturally dual to the ultra-relativistic Carroll dynamics (3.28) where time flows but all spatial directions freeze.

In order to account for the dynamics of *time-like* observers (for which time flows), one needs to introduce external forces, as done in the following subsection within the ambient framework of a null Kaluza–Klein reduction.

3.6 Embedding (0, 1) into ambient (0, 0) Kaluza–Klein ansatz: Carroll or Newton–Cartan

We start by considering the $\hat{D} = 1 + D$ Kaluza–Klein ansatz (2.60) for a Riemannian ambient DFT-metric, i.e. $(\hat{n}, \hat{\bar{n}}) = (0, 0)$. As for the 'internal' space, we assume $D' = 1$, $(n', \bar{n}') = (1, 0)$ with $\mathcal{H}'_{A'B'} \equiv \mathcal{J}'_{A'B'}$. Then the 'external' DFT-metric, $\mathcal{H}_{AB}$, must be of the $(n, \bar{n}) = (0, 1)$ type[8], i.e. the lower sign in (3.33), which ensures $\hat{\mathcal{H}}_{\hat{A}}{}^{\hat{A}} = 2(\hat{n} - \hat{\bar{n}}) = 2(n + n' - \bar{n} - \bar{n}') = 0$. We let $(\tilde{y}, y)$ denote the primed coordinates, $(\tilde{x}'_1, x'^1)$, and write for the ambient doubled coordinates

$$\begin{aligned}
&\mathrm{D}_{\tau}x^{\hat{A}} = (\mathrm{D}_{\tau}\tilde{y},\ \dot{y},\\
&\mathrm{D}_{\tau}x^{A}) = (\dot{\tilde{y}} - A_{\tau\tilde{y}},\ \dot{y},\ \dot{\tilde{x}}_{\mu} - A_{\tau\mu},\ \dot{x}^{\nu}).
\end{aligned} \tag{3.37}$$

We solve the constraint on $W_{M'}{}^{N}$ (2.57) by putting $W_{\mu'}{}^{N} \equiv 0$, such that, for the present case of $D' = 1$, we simply have

$$W_{M'}{}^{N} \equiv \left(W^{N},\ 0\right), \tag{3.38}$$

where the $\mathbf{O}(D, D)$ vector, W^{N}, carries no hidden index. By choosing this—instead of letting e.g. $W^{\mu' N}$ vanish—we ensure a *null Killing vector*, $\xi^{\hat{A}} = (\tilde{\xi}_{\hat{\mu}},\ \xi^{\hat{\nu}})$ (2.8), (A.4) with $\xi^{\hat{\mu}}\partial_{\hat{\mu}} = \partial_{y}$, satisfying from (2.23),[9]

$$\ln\left[\int \mathcal{D}\mathcal{A}\ \exp\left(-\sqrt{(\xi^{\hat{A}} - \mathcal{A}^{\hat{A}})(\xi^{\hat{B}} - \mathcal{A}^{\hat{B}})\hat{\mathcal{H}}_{\hat{A}\hat{B}}}\right)\right] = 0. \tag{3.39}$$

The ambient DFT-metric (2.60) then takes the following form:

$$\hat{\mathcal{H}}_{\hat{A}\hat{B}} = \begin{pmatrix} -2W_{\bar{p}}W^{\bar{p}} & 1 & 2\bar{V}_{B\bar{p}}W^{\bar{p}} \\ 1 & 0 & 0 \\ 2\bar{V}_{A\bar{p}}W^{\bar{p}} & 0 & \mathcal{H}_{AB} \end{pmatrix}, \tag{3.40}$$

where, using the notations of Sect. 2.3, we set a $(D+1)$-dimensional $\mathbf{Spin}(s+1, t+1)$ vector,[10]

$$\begin{aligned}
W^{\bar{p}} &= W^{A}\bar{V}_{A}{}^{\bar{p}}\\
&\equiv \left(W^{\bar{a}},\ \tfrac{1}{\sqrt{2}}(W_{+} + W_{-}),\ \tfrac{1}{\sqrt{2}}(W_{+} - W_{-})\right),
\end{aligned} \tag{3.41}$$

such that, from (2.42),

$$\begin{aligned}
\bar{P}^{A}{}_{B}W^{B} &= \bar{V}^{A}{}_{\bar{p}}W^{\bar{p}} = \Big(\tfrac{1}{\sqrt{2}}\bar{k}_{\mu\bar{a}}W^{\bar{a}} + T_{\mu}W_{-}\\
&\quad + B_{\mu\rho}\left(\tfrac{1}{\sqrt{2}}W^{\bar{b}}\bar{h}_{\bar{b}}{}^{\rho} + W_{+}N^{\rho}\right),\\
&\quad \tfrac{1}{\sqrt{2}}W^{\bar{b}}\bar{h}_{\bar{b}}{}^{\nu} + W_{+}N^{\nu}\Big),\\
W_{\bar{p}}W^{\bar{p}} &= W_{A}W_{B}\bar{P}^{AB} = W_{\bar{a}}W^{\bar{a}} + 2W_{+}W_{-}.
\end{aligned} \tag{3.42}$$

It is also convenient to define based on (2.43) and (2.45)

$$W_{\mu} := \sqrt{2}\bar{\mathcal{X}}_{\mu}{}^{\bar{p}}W_{\bar{p}} = \sqrt{2}\bar{k}_{\mu\bar{a}}W^{\bar{a}} + 2W_{-}T_{\mu}. \tag{3.43}$$

Note the identification of the conventions

$$\begin{aligned}
\Phi_{\mu\nu} &\equiv K_{\mu\nu} = -\bar{k}_{\mu}{}^{\bar{a}}\bar{k}_{\nu}{}^{\bar{b}}\bar{\eta}_{\bar{a}\bar{b}},\\
\Upsilon^{\mu\nu} &\equiv H^{\mu\nu} = -\bar{\eta}^{\bar{a}\bar{b}}\bar{h}_{\bar{a}}{}^{\mu}\bar{h}_{\bar{b}}{}^{\nu}.
\end{aligned} \tag{3.44}$$

Now, with the lower sign choice of (3.33), plugging (3.40) into the master doubled-yet-gauged action for a point particle (2.26), we obtain in a similar fashion to (2.32),

$$\begin{aligned}
S &= \int \mathrm{d}\tau\ e^{-1}\mathrm{D}_{\tau}x^{\hat{A}}\mathrm{D}_{\tau}x^{\hat{B}}\hat{\mathcal{H}}_{\hat{A}\hat{B}} - \tfrac{1}{4}m^{2}e\\
&= \int \mathrm{d}\tau\ e^{-1}\Big[2\mathrm{D}_{\tau}\tilde{y}\left(\dot{y} + 2\mathrm{D}_{\tau}x^{A}\bar{V}_{A\bar{p}}W^{\bar{p}} - \mathrm{D}_{\tau}\tilde{y}W_{\bar{p}}W^{\bar{p}}\right) + \mathrm{D}_{\tau}x^{A}\mathrm{D}_{\tau}x^{B}\mathcal{H}_{AB}\Big] - \tfrac{1}{4}m^{2}e\\
&= \int \mathrm{d}\tau \left[\begin{array}{l} e^{-1}\Big[\dot{x}^{\mu}\dot{x}^{\nu}\Phi_{\mu\nu} + 2\mathrm{D}_{\tau}\tilde{y}W_{\mu}\dot{x}^{\mu} - 4(\mathrm{D}_{\tau}\tilde{y})^{2}W_{+}W_{-} + 2\dot{y}\mathrm{D}_{\tau}\tilde{y}\Big] - \tfrac{1}{4}m^{2}e \\ -2e^{-1}\left(T_{\mu}\dot{x}^{\mu} - 2\mathrm{D}_{\tau}\tilde{y}W_{+}\right)\Lambda - e^{-1}\bar{h}_{\bar{a}}{}^{\mu}\Lambda_{\mu}\bar{h}^{\bar{a}\nu}\Lambda_{\nu} \end{array}\right],
\end{aligned} \tag{3.45}$$

[8] The alternative choice of $(n', \bar{n}') = (0, 1)$ obtained by setting $\mathcal{H}'_{A'B'} \equiv -\mathcal{J}'_{A'B'}$ will involve replacing $\bar{P}$ by $-P$ in (2.60), and accordingly the external DFT-metric, $\mathcal{H}$, will need to be of (1, 0) type.

[9] In terms of the ordinary Lie derivative, $\mathcal{L}_{\xi}\hat{g}_{\hat{\mu}\hat{\nu}} = 0$, $\mathcal{L}_{\xi}\hat{B}_{\hat{\mu}\hat{\nu}} = -2\partial_{[\hat{\mu}}\tilde{\xi}_{\hat{\nu}]}$, and (3.39) means $\xi^{\hat{\mu}}\xi^{\hat{\nu}}\hat{g}_{\hat{\mu}\hat{\nu}} = 0$.

[10] If we had chosen $(n, \bar{n}) = (1, 0)$, from (2.42), the expression (3.41) would have reduced to '$W^{\bar{p}} = W^{\bar{a}}$' without $W_{\pm}$.

where we set for short notation as well as for a convenient field redefinition to replace $A_{\tau\mu}$,

$$\Lambda_\mu := \dot{\tilde{x}}_\mu - A_{\tau\mu} - B_{\mu\kappa}\dot{x}^\kappa - \mathrm{D}_\tau\tilde{y}W_\mu,$$
$$\Lambda := \Lambda_\mu N^\mu + 2\mathrm{D}_\tau\tilde{y}W_-. \tag{3.46}$$

Note that the very last term in (3.45) is a perfect square which vanishes after $\bar{h}_{\bar{a}}{}^\mu\Lambda_\mu$ having been integrated out as

$$\bar{h}_{\bar{a}}{}^\mu\Lambda_\mu \equiv 0. \tag{3.47}$$

Since y is the coordinate for the isometry direction, it serves as a Lagrange multiplier: it forces the conjugate momentum of y, or p, to be constant,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\mathrm{D}_\tau\tilde{y}\right) \equiv 0 \quad\Longrightarrow\quad 2\mathrm{D}_\tau\tilde{y} = ep \quad \text{with constant } p. \tag{3.48}$$

Integrating out Λ gives a constraint,

$$\mathcal{E}_\Lambda := T_\mu\dot{x}^\mu - epW_+ \equiv 0, \tag{3.49}$$

such that the time is generically not frozen; cf. (3.36). Further, integrating out the auxiliary field, $A_{\tau\tilde{y}}$, inside $\mathrm{D}_\tau\tilde{y}$ determines the velocity, with (3.47), (3.48), and (3.49),

$$\dot{y} = epW_{\bar{p}}W^{\bar{p}} - 2\mathrm{D}_\tau x^A\bar{V}_{A\bar{p}}W^{\bar{p}}$$
$$= -2W_+\Lambda - W_\mu\dot{x}^\mu + 2epW_+W_-. \tag{3.50}$$

The einbein imposes the Hamiltonian constraint,

$$\mathcal{E}_e := \Phi_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + e^2p^2W_+W_- - 2epW_+\Lambda + \tfrac{1}{4}e^2m^2 \equiv 0. \tag{3.51}$$

From (3.50) and (3.51), it follows that

$$-p\dot{y} = e^{-1}\Phi_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + pW_\mu\dot{x}^\mu - ep^2W_+W_- + \tfrac{1}{4}m^2e. \tag{3.52}$$

That is to say, whenever $p \neq 0$, $\dot{y}$ is completely fixed by the dynamics of the x^μ coordinates. The auxiliary variable, Λ, is fixed in the same manner.

Making use of the world-line diffeomorphisms, we hereafter normalize the einbein:

$$e \equiv 1, \tag{3.53}$$

such that τ coincides with the proper length.

The equation of motion for x^μ reads now

$$\mathcal{E}_\mu := \Phi_{\mu\nu}\ddot{x}^\nu + \left(\partial_\rho\Phi_{\sigma\mu} - \tfrac{1}{2}\partial_\mu\Phi_{\rho\sigma}\right)\dot{x}^\rho\dot{x}^\sigma$$
$$+ \left(T_{\mu\nu}\Lambda - \tfrac{1}{2}pW_{\mu\nu}\right)\dot{x}^\nu + \tfrac{1}{2}p^2\partial_\mu(W_+W_-)$$
$$- p\Lambda\partial_\mu W_+ - T_\mu\dot{\Lambda}, \tag{3.54}$$

where we defined for simplicity the field strengths

$$T_{\mu\nu} := \partial_\mu T_\nu - \partial_\nu T_\mu, \quad W_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu. \tag{3.55}$$

Computing the contractions, $N^\mu\mathcal{E}_\mu$, $\dot{x}^\mu\mathcal{E}_\mu$, respectively, we obtain the time derivative of the auxiliary variable,

$$\dot{\Lambda} = N^\mu\Big[\left(\partial_\rho\Phi_{\sigma\mu} - \tfrac{1}{2}\partial_\mu\Phi_{\rho\sigma}\right)\dot{x}^\rho\dot{x}^\sigma + \left(T_{\mu\nu}\Lambda - \tfrac{1}{2}pW_{\mu\nu}\right)\dot{x}^\nu$$
$$+ \tfrac{1}{2}p^2\partial_\mu(W_+W_-) - p\Lambda\partial_\mu W_+\Big], \tag{3.56}$$

and a consistency relation among the constraints (3.49) and (3.51),

$$\dot{x}^\mu\mathcal{E}_\mu + \dot{\Lambda}\mathcal{E}_\Lambda - \tfrac{1}{2}\dot{\mathcal{E}}_e = 0. \tag{3.57}$$

While (3.54) determines partially the acceleration, $\ddot{x}^\mu$, the time derivative of the constraint (3.49) can provide the missing component,

$$\dot{\mathcal{E}}_\Lambda = T_\mu\ddot{x}^\mu + \partial_{(\mu}T_{\nu)}\dot{x}^\mu\dot{x}^\nu - p\dot{x}^\mu\partial_\mu W_+ = 0. \tag{3.58}$$

All together, the combination $\Upsilon^{\lambda\mu}\mathcal{E}_\mu + N^\lambda\dot{\mathcal{E}}_\Lambda$ fully determines the acceleration,

$$\ddot{x}^\lambda + \gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + \left[\Upsilon^{\lambda\mu}T_{\mu\nu}\Lambda - p\left(N^\lambda\partial_\nu W_+\right.\right.$$
$$\left.\left.+ \tfrac{1}{2}\Upsilon^{\lambda\mu}W_{\mu\nu}\right)\right]\dot{x}^\nu + \tfrac{1}{2}p^2\Upsilon^{\lambda\mu}\partial_\mu(W_+W_-)$$
$$- p\Lambda\Upsilon^{\lambda\mu}\partial_\mu W_+ = 0, \tag{3.59}$$

where $\gamma^\lambda_{\mu\nu}$ denotes the following coefficients:

$$\gamma^\lambda_{\mu\nu} := N^\lambda\partial_{(\mu}T_{\nu)} + \tfrac{1}{2}\Upsilon^{\lambda\rho}\left(\partial_\mu\Phi_{\nu\rho} + \partial_\nu\Phi_{\rho\mu} - \partial_\rho\Phi_{\mu\nu}\right). \tag{3.60}$$

We emphasize that the dynamics of the D-dimensional coordinates x^μ as prescribed by (3.59) is independent of the Kaluza–Klein direction, y. Geometrically, this means that one can interpret x^μ as coordinates on the *quotient manifold* of the ambient spacetime by the light-like direction along the vector field, $\xi^{\hat{\mu}}\partial_{\hat{\mu}} = \partial_y$.

In the special case where $T_{\mu\nu}$ vanishes (i.e. the one-form, T_μ, is closed) and W_+ is a (non-vanishing) constant, Eq. (3.59) simplifies to

$$\ddot{x}^\lambda + \gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \tfrac{1}{2}p\Upsilon^{\lambda\mu}\left[W_{\mu\nu}\dot{x}^\nu - p\partial_\mu(W_+W_-)\right], \tag{3.61}$$

of which the right-hand side can be interpreted as the Lorentz plus Coulomb forces. In this particular case, the coefficients (3.60) are the ones associated to the so-called 'special Galilean connection' for the field of observers, N^μ, (cf. e.g. [77]). In accordance with the usual Riemannian ambient approach of [80–82] (cf. also [75,83,85,86]), the resulting dynamical trajectories (3.61) can be interpreted as Newton–Cartan geodesics. These are of two different types, depending on the value of p:

- $p = 0$ (space-like observer).
  In this case, the constraint,[11] $T_\mu \dot{x}^\mu = 0$, holds as a consequence of (3.49), so that we recover the case investigated in Sect. 3.5 for which time freezes. Geometrically, the observer trajectory is restricted to a $(D-1)$-dimensional hypersurface (absolute space). The absolute spaces are Riemannian spaces (of Euclidean signature), since the degenerate metric $\Phi_{\mu\nu}$ becomes invertible on Ker T (cf. e.g. [77]). Equation (3.61) thus describes geodesics associated to the spatial Riemannian metric and the Hamiltonian constraint (3.51) can be solved as $e = \frac{2}{|m|}\sqrt{\Phi_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \equiv 1$.
- $p \neq 0$ (time-like observer).
  In this case, τ is parametrized to ensure $e = \frac{1}{pW_+}T_\mu\dot{x}^\mu \equiv 1$ such that the observer $\dot{x}^\mu$ is time-like. Equation (3.61) can thus be reformulated as

$$\ddot{x}^\lambda + \hat{\gamma}^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{3.62}$$

  where the coefficients $\hat{\gamma}^\lambda_{\mu\nu}$ are defined as

$$\hat{\gamma}^\lambda_{\mu\nu} := \gamma^\lambda_{\mu\nu} + \Upsilon^{\lambda\rho}\, T_{(\mu}F_{\nu)\rho}, \tag{3.63}$$

  with $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu := \frac{1}{2\,W_+}\big(W_\mu - W_-\,T_\mu\big)$.
  The connection associated to the coefficients (3.63) is naturally interpreted as a Newtonian connection [79], i.e. a torsion-free connection compatible with the Leibnizian structure $(\Upsilon^{\mu\nu}, T_\mu)$ such that the associated field strength, $F_{\mu\nu}$, is closed.

In summary, assuming the triviality of $T_{\mu\nu}$ and W_+, the doubled-yet-gauged particle action (2.26) with the ambient $(D+1)$-dimensional Kaluza–Klein ansatz (2.60) reproduces the full content of Newtonian dynamics (unifying the space-like and time-like cases) on the D-dimensional manifold quotient along the light-like direction, y.

In principle, the assumption regarding the triviality of the variables, $T_{\mu\nu}$ and W_+, should be examined by considering the on-shell dynamics of the DFT-metric, i.e. the Euler–Lagrangian equations of DFT. In the present work, we have focused on the kinematical side of the DFT-metric and the subsequent particle and string dynamics on the background. We leave the investigation of the dynamical aspect of the $(n,\bar{n})$ DFT-metric for future work. From our perspective, the DFT action and its full equations of motion determine universally and unambiguously all the dynamics of the $(n,\bar{n})$ backgrounds, including $(0,0)$ Riemannian general relativity and $(0,1)$ Newton–Cartan gravity, in a unifying manner.

[11] From the ambient perspective, the constraint, $T_\mu\dot{x}^\mu = 0$, implies that the dynamics becomes restricted to a D-dimensional hypersurface of the ambient manifold, transverse to the null isometry vector field, $\xi^{\hat{\mu}}\partial_{\hat{\mu}} = \partial_y$. Such a light-like hypersurface is naturally endowed with a Carrollian metric structure [74], and the equations of motion (3.61) together with (3.50) and (3.56) can be naturally interpreted as geodesics associated to a suitable Carrollian connection induced by the ambient metric structure (cf. [74,84] for details). The role of Carrollian time is then played by y and the 'space-like' directions are generically unfrozen, thus generalizing the Carrollian dynamics discussed in Sect. 3.4.

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Appendix: Derivation of the most general form of the DFT-metric, Eq. (2.2)

By definition, see (2.1), the DFT-metric is a symmetric $\mathbf{O}(D,D)$ element, such that it satisfies

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}, \quad \mathcal{H}_L{}^M\mathcal{H}_M{}^N = \delta_L{}^N. \tag{A.1}$$

With respect to the $\mathbf{O}(D,D)$ metric (1.1) and the choice of the section, $\tilde{\partial}^\mu \equiv 0$, we decompose the DFT-metric,

$$\mathcal{H}_{MN} = \begin{pmatrix} \mathcal{H}^{\mu\nu} & \mathcal{H}^\mu{}_\lambda \\ \mathcal{H}_\kappa{}^\nu & \mathcal{H}_{\kappa\lambda} \end{pmatrix}. \tag{A.2}$$

The defining condition (A.1) reads explicitly

$$\begin{aligned}
&\mathcal{H}^{\mu\nu} = \mathcal{H}^{\nu\mu}, \quad \mathcal{H}_{\mu\nu} = \mathcal{H}_{\nu\mu}, \quad \mathcal{H}_\mu{}^\nu = \mathcal{H}^\nu{}_\mu, \\
&\mathcal{H}^{(\mu}{}_\rho\mathcal{H}^{\nu)\rho} = 0, \quad \mathcal{H}_{\rho(\mu}\mathcal{H}^\rho{}_{\nu)} = 0, \\
&\mathcal{H}^\mu{}_\rho\mathcal{H}^\rho{}_\nu + \mathcal{H}^{\mu\rho}\mathcal{H}_{\rho\nu} = \delta^\mu{}_\nu.
\end{aligned} \tag{A.3}$$

The generalized Lie derivative of the DFT-metric, cf. (1.6),

$$\begin{aligned}
\hat{\mathcal{L}}_\xi\mathcal{H}_{AB} &= \xi^C\partial_C\mathcal{H}_{AB} + (\partial_A\xi_C - \partial_C\xi_A)\mathcal{H}^C{}_B \\
&\quad + (\partial_B\xi_C - \partial_C\xi_B)\mathcal{H}_A{}^C,
\end{aligned} \tag{A.4}$$

leads to

$$\begin{aligned}
\delta\mathcal{H}^{\mu\nu} &= \mathcal{L}_\xi\mathcal{H}^{\mu\nu}, \\
\delta\mathcal{H}_{\kappa\lambda} &= \mathcal{L}_\xi\mathcal{H}_{\kappa\lambda} + (\partial_\kappa\tilde{\xi}_\rho - \partial_\rho\tilde{\xi}_\kappa)\mathcal{H}^\rho{}_\lambda \\
&\quad - \mathcal{H}_\kappa{}^\rho(\partial_\rho\tilde{\xi}_\lambda - \partial_\lambda\tilde{\xi}_\rho), \\
\delta\mathcal{H}^\mu{}_\lambda &= \mathcal{L}_\xi\mathcal{H}^\mu{}_\lambda - \mathcal{H}^{\mu\rho}(\partial_\rho\tilde{\xi}_\lambda - \partial_\lambda\tilde{\xi}_\rho), \\
\delta\mathcal{H}_\kappa{}^\nu &= \mathcal{L}_\xi\mathcal{H}_\kappa{}^\nu + (\partial_\kappa\tilde{\xi}_\rho - \partial_\rho\tilde{\xi}_\kappa)\mathcal{H}^{\rho\nu}.
\end{aligned} \tag{A.5}$$

Viewed as a $D \times D$ matrix, if $\mathcal{H}^{\mu\nu}$ is non-degenerate, we may identify it as the inverse of a Riemannian metric. It is easy to see then that the remaining constraints are all solved by a skew-symmetric B-field, such that the most general DFT-metric in this case takes the well-known form (1.9). Henceforth, we look for the most general form of the DFT-metric, where $\mathcal{H}^{\mu\nu}$ is degenerate. Firstly, we focus on the case where the rank of $\mathcal{H}^{\mu\nu}$ is $D-1$, admitting only one zero-eigenvector, X_μ,

$$\mathcal{H}^{\mu\nu} \equiv H^{\mu\nu}, \quad H^{\mu\nu}X_\nu = 0. \tag{A.6}$$

From (A.3), $\mathcal{H}^\mu{}_\rho H^{\rho\nu}$ is skew-symmetric, and hence

$$X_\mu \mathcal{H}^\mu{}_\rho H^{\rho\nu} = -\mathcal{H}^\nu{}_\rho H^{\rho\mu}X_\mu = 0. \tag{A.7}$$

Without loss of generality then, introducing a skew-symmetric B-field,[12] we may put

$$\mathcal{H}^\mu{}_\rho H^{\rho\nu} \equiv -H^{\mu\rho}B_{\rho\sigma}H^{\sigma\nu}, \quad B_{\mu\nu} = -B_{\nu\mu}. \tag{A.8}$$

It follows that, with some vector field, Y^μ, $\mathcal{H}^\mu{}_\nu$ takes the form

$$\mathcal{H}^\mu{}_\nu = -H^{\mu\rho}B_{\rho\nu} + Y^\mu X_\nu. \tag{A.9}$$

We proceed with a new symmetric variable, $K_{\mu\nu} = K_{\nu\mu}$,

$$\mathcal{H}_{\mu\nu} \equiv K_{\mu\nu} - B_{\mu\rho}H^{\rho\sigma}B_{\sigma\nu} + 2X_{(\mu}B_{\nu)\rho}Y^\rho. \tag{A.10}$$

The last relation in (A.3) gives

$$H^{\mu\rho}K_{\rho\nu} + (Y^\rho X_\rho)Y^\mu X_\nu = \delta^\mu{}_\nu. \tag{A.11}$$

Contracting this with X_μ shows

$$Y^\mu X_\mu = \pm 1. \tag{A.12}$$

Lastly we impose the skew-symmetric condition of $\mathcal{H}_{\mu\rho}\mathcal{H}^\rho{}_\nu$, which gives with (A.11)

$$K_{\mu\rho}Y^\rho X_\nu + K_{\nu\rho}Y^\rho X_\mu = 0, \tag{A.13}$$

and hence in particular, contracting with Y^μ, we have

$$K_{\nu\rho}Y^\rho = \mp(Y^\mu K_{\mu\rho}Y^\rho)X_\nu. \tag{A.14}$$

Substituting this back into (A.13), we conclude that $Y^\mu K_{\mu\rho}Y^\rho$ must be trivial, and hence in fact from (A.14),

$$K_{\nu\rho}Y^\rho = 0. \tag{A.15}$$

We may perform a field redefinition, $Y^\mu \to \pm Y^\mu$, in order to remove the sign factor in the normalization of (A.12). After all, the most general form of the DFT-metric in the 'least' degenerate case takes the form

$$\mathcal{H}_{MN} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} \pm Y^\mu X_\lambda \\ B_{\kappa\rho}H^{\rho\nu} \pm X_\kappa Y^\nu & K_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} \pm 2X_{(\kappa}B_{\lambda)\rho}Y^\rho \end{pmatrix}, \tag{A.16}$$

of which the variables must meet

$$\begin{aligned} &H^{\mu\nu}X_\nu = 0, \quad K_{\mu\nu}Y^\nu = 0, \quad Y^\mu X_\mu = 1, \\ &H^{\lambda\mu}K_{\mu\nu} + Y^\lambda X_\nu = \delta^\lambda{}_\nu, \quad B_{\mu\nu} = -B_{\nu\mu}. \end{aligned} \tag{A.17}$$

The above analysis can be straightforwardly extended to the most general degenerate cases, where there are N linearly independent zero-eigenvectors, X^i_μ, $i = 1, 2, \ldots, N$, such that the rank of $\mathcal{H}^{\mu\nu}$ is $D - N$,

$$\mathcal{H}^{\mu\nu} \equiv H^{\mu\nu}, \qquad H^{\mu\nu}X^i_\nu = 0. \tag{A.18}$$

From

$$\begin{aligned} &\mathcal{H}^{(\mu}{}_\rho H^{\nu)\rho} = 0, \quad \mathcal{H}^\mu{}_\rho H^{\rho\nu}X^i_\nu = 0, \\ &X^i_\mu \mathcal{H}^\mu{}_\rho H^{\rho\nu} = 0, \end{aligned} \tag{A.19}$$

Eqs. (A.9) and (A.10) generalize, defining Y^μ_i and $M_{\mu\nu}$, to

$$\begin{aligned} &\mathcal{H}^\mu{}_\nu \equiv -H^{\mu\rho}B_{\rho\sigma} + Y^\mu_i X^i_\nu, \\ &\mathcal{H}_{\mu\nu} \equiv M_{\mu\nu} - B_{\mu\rho}H^{\rho\sigma}B_{\sigma\nu} + 2X^i_{(\mu}B_{\nu)\rho}Y^\rho_i, \end{aligned} \tag{A.20}$$

such that the DFT-metric assumes the following intermediate form:

$$\mathcal{H}_{MN} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} + Y^\mu_i X^i_\lambda \\ B_{\kappa\rho}H^{\rho\nu} + X^i_\kappa Y^\nu_i & M_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} + 2X^i_{(\kappa}B_{\lambda)\rho}Y^\rho_i \end{pmatrix}. \tag{A.21}$$

In the above, the repeated index, i, is surely summed from 1 to N. The remaining constraints in (A.3) give

$$H^{\mu\rho}M_{\rho\nu} + (Y^\rho_i X^j_\rho)Y^\mu_j X^i_\nu = \delta^\mu{}_\nu, \tag{A.22}$$

$$M_{\mu\rho}Y^\rho_i X^i_\nu + M_{\nu\rho}Y^\rho_i X^i_\mu = 0. \tag{A.23}$$

Contraction of (A.22) with X^k_μ leads to

$$X^i_\nu(Y_i{\cdot}X^j\, Y_j{\cdot}X^k) = X^k_\nu, \tag{A.24}$$

where we set $Y_i{\cdot}X^j \equiv Y^\mu_i X^j_\mu$. Since $k = 1, 2, \ldots, N$ is arbitrary and the X^k_ν are independent, the above result actually implies that $Y_i{\cdot}X^j$ is an involutory $N \times N$ matrix,

$$Y_i{\cdot}X^j\, Y_j{\cdot}X^k = \delta_i{}^k. \tag{A.25}$$

On the other hand, contraction of (A.23) with $(Y_j{\cdot}X^k)Y^\mu_k$ leads to

$$M_{\nu\rho}Y^\rho_j = -(Y_j{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_i)X^i_\nu, \tag{A.26}$$

[12] The ambiguity in introducing the B-field through (A.8) amounts to the symmetry of the final result (2.16).

where we set $Y_k{\cdot}M{\cdot}Y_i \equiv Y_k^\rho M_{\rho\sigma} Y_i^\sigma$ for short notation. Substituting into (A.23), we get

$$(Y_j{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_i)X^i_\mu X^j_\nu + (Y_j{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_i)X^i_\nu X^j_\mu = 0, \tag{A.27}$$

which, after contraction with $Y_l^\mu Y_m^\nu$ and from (A.25), can be seen to be equivalent to

$$(Y_l{\cdot}X^i)(Y_i{\cdot}M{\cdot}Y_m) = -(Y_m{\cdot}X^i)(Y_i{\cdot}M{\cdot}Y_l). \tag{A.28}$$

It follows from (A.22), (A.25), and (A.26) that $(Y_i{\cdot}X^j)Y_j^\mu X^i_\nu$ and $H^{\mu\rho}M_{\rho\nu}$ are mutually orthogonal and complementary (A.22) projection matrices,

$$\begin{aligned}
&(Y_i{\cdot}X^j)Y_j^\lambda X^i_\mu\, (Y_k{\cdot}X^l)Y_l^\mu X^k_\nu = (Y_i{\cdot}X^j)Y_j^\lambda X^i_\nu,\\
&H^{\lambda\rho}M_{\rho\mu}\, H^{\mu\sigma}M_{\sigma\nu} = H^{\lambda\rho}M_{\rho\nu},\\
&(Y_i{\cdot}X^j)Y_j^\lambda X^i_\mu\, H^{\mu\sigma}M_{\sigma\nu} = 0,\\
&H^{\lambda\rho}M_{\rho\mu}\, (Y_k{\cdot}X^l)Y_l^\mu X^k_\nu = 0.
\end{aligned} \tag{A.29}$$

Now, we may recast (A.26) into

$$\left[M_{\mu\nu} + X^i_\mu\left\{X^l_\nu(Y_l{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_j)X^j_\rho\right\}Y_i^\rho\right]Y_m^\nu = 0. \tag{A.30}$$

It is crucial to note, from the symmetric property, $Y_k{\cdot}M{\cdot}Y_j = Y_j{\cdot}M{\cdot}Y_k$, that the free indices, μ and ν, are symmetric in

$$\begin{aligned}
&X^i_\mu\left\{X^l_\nu(Y_l{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_j)X^j_\rho\right\}Y_i^\rho\\
&\quad = X^i_\nu\left\{X^l_\mu(Y_l{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_j)X^j_\rho\right\}Y_i^\rho,
\end{aligned} \tag{A.31}$$

and furthermore, from the skew-symmetric property (A.28), that the free indices, ν and ρ, are skew-symmetric in

$$X^l_\nu(Y_l{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_j)X^j_\rho = -X^l_\rho(Y_l{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_j)X^j_\nu. \tag{A.32}$$

Therefore, if we perform a field redefinition,

$$B_{\mu\nu} \quad\longrightarrow\quad B_{\mu\nu} + \tfrac{1}{2}X^i_\mu(Y_i{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_j)X^j_\nu, \tag{A.33}$$

among the components of the DFT-metric spelled in (A.21), $H^{\mu\sigma}B_{\sigma\lambda}$, $B_{\kappa\rho}H^{\rho\sigma}$, and $B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}$ remain invariant, but $M_{\kappa\lambda} + 2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho$ transforms as follows:

$$\begin{aligned}
&M_{\kappa\lambda} + 2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho \longrightarrow M_{\kappa\lambda}\\
&\quad +\ X^i_{(\kappa}\left\{X^j_{\lambda)}(Y_j{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_l)X^l_\rho\right\}Y_i^\rho + 2X^i_{(\kappa}B_{\lambda)\rho}Y_i^\rho.
\end{aligned} \tag{A.34}$$

We then let

$$K_{\kappa\lambda} := M_{\kappa\lambda} + X^i_{(\kappa}\left\{X^j_{\lambda)}(Y_j{\cdot}X^k)(Y_k{\cdot}M{\cdot}Y_l)X^l_\rho\right\}Y_i^\rho, \tag{A.35}$$

which nicely satisfies

$$K_{\kappa\lambda}Y_i^\lambda = 0, \quad H^{\mu\rho}K_{\rho\nu} + (Y_i{\cdot}X^j)Y_j^\mu X^i_\nu = \delta^\mu{}_\nu. \tag{A.36}$$

Finally, we perform a similarity transformation, $(X^i_\mu, Y_j^\nu) \to (S^i{}_k X^k_\mu, Y_k^\nu S^{-1k}{}_j)$, which leaves $Y_i^\mu X^i_\nu$ invariant but diagonalizes $Y_i^\rho X^j_\rho$ with the eigenvalues of either $+1$ or -1. We then let $N = n + \bar{n}$ in order to denote the numbers of the $+1$ and -1 eigenvalues of $Y_i^\rho X^j_\rho$. If the corresponding eigenvalue is -1, we furthermore perform a field redefinition, $(\bar{X}^{\bar{i}}_\mu, \bar{Y}_{\bar{i}}^\nu) := (X^i_\mu, -Y_i^\nu)$, which involves the change of the index from i to $\bar{\imath}$. In this way, we arrive at the most general form of the DFT-metric, (2.2), classified by two non-negative integers, $n, \bar{n}$.

It is also worthwhile to decompose the B-field utilizing the completeness relation (2.5),

$$\begin{aligned}
B_{\mu\nu} = \beta_{\mu\nu} &+ B_{\mu j}X^j_\nu - B_{\nu j}X^j_\mu + \bar{B}_{\mu\bar{\jmath}}\bar{X}^{\bar{\jmath}}_\nu\\
&- \bar{B}_{\nu\bar{\jmath}}\bar{X}^{\bar{\jmath}}_\mu + X^i_\mu X^j_\nu b_{ij} + \bar{X}^{\bar{\imath}}_\mu \bar{X}^{\bar{\jmath}}_\nu b_{\bar{\imath}\bar{\jmath}}\\
&+ 2X^i_{[\mu}\bar{X}^{\bar{\jmath}}_{\nu]}b_{i\bar{\jmath}},
\end{aligned} \tag{A.37}$$

for which we set

$$\begin{aligned}
&\beta_{\mu\nu} := (KH)_\mu{}^\rho(KH)_\nu{}^\sigma B_{\rho\sigma}, \quad b_{ij} := Y_i^\mu Y_j^\nu B_{\mu\nu},\\
&B_{\mu i} := B_{\mu\nu}Y_i^\nu - X^j_\mu b_{ji} + \bar{X}^{\bar{\jmath}}_\mu b_{i\bar{\jmath}},\\
&b_{i\bar{\jmath}} := Y_i^\mu \bar{Y}_{\bar{\jmath}}^\nu B_{\mu\nu}, \quad b_{\bar{\imath}\bar{\jmath}} := \bar{Y}_{\bar{\imath}}^\mu \bar{Y}_{\bar{\jmath}}^\nu B_{\mu\nu},\\
&\bar{B}_{\mu\bar{\imath}} := B_{\mu\nu}\bar{Y}_{\bar{\imath}}^\nu - \bar{X}^{\bar{\jmath}}_\mu b_{\bar{\jmath}\bar{\imath}} - X^j_\mu b_{j\bar{\imath}}.
\end{aligned} \tag{A.38}$$

The variables, $B_{\mu i}$, $\bar{B}_{\mu\bar{\imath}}$ and $\beta_{\mu\nu}$ are completely orthogonal to the vectors, Y_j^μ and $\bar{Y}_{\bar{\jmath}}^\mu$,

$$\begin{aligned}
&B_{\mu i}Y_j^\mu = 0, \quad B_{\mu i}\bar{Y}_{\bar{\jmath}}^\mu = 0, \quad \bar{B}_{\mu\hat{\imath}}Y_j^\mu = 0,\\
&\bar{B}_{\mu\hat{\imath}}\bar{Y}_{\bar{\jmath}}^\mu = 0, \quad \beta_{\mu\nu}Y_j^\mu = 0, \quad \beta_{\mu\nu}\bar{Y}_{\bar{\jmath}}^\mu = 0.
\end{aligned} \tag{A.39}$$

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