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Antisymplectic involution and Floer cohomology

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The main purpose of the present paper is a study of orientations of the moduli spaces of pseudoholomorphic discs with boundary lying on a *real* Lagrangian submanifold, ie the fixed point set of an antisymplectic involution τ on a symplectic manifold. We introduce the notion of τ–relative spin structure for an antisymplectic involution τ and study how the orientations on the moduli space behave under the involution τ. We also apply this to the study of Lagrangian Floer theory of real Lagrangian submanifolds. In particular, we study unobstructedness of the τ–fixed point set of symplectic manifolds and, in particular, prove its unobstructedness in the case of Calabi–Yau manifolds. We also do explicit calculation of Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$, which provides an example whose Floer cohomology is not isomorphic to its classical cohomology. We study Floer cohomology of the diagonal of the square of a symplectic manifold, which leads to a rigorous construction of the quantum Massey product of a symplectic manifold in complete generality.

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1 Introduction and statement of results

An *antisymplectic involution* τ on a symplectic manifold (M,ω) is an involution on M which satisfies $\tau^*\omega = -\omega$. Two prototypes of antisymplectic involutions are the complex conjugation of a complex projective space with respect to the Fubini–Study metric and the canonical reflection along the zero section on the cotangent bundle. (See also Castaño-Bernard, Matessi and Solomon [2] for a construction of an interesting class of antisymplectic involutions on Lagrangian torus fibrations.) The fixed point set of τ, if it is nonempty, gives an example of Lagrangian submanifolds. For instance, the set of real points of a complex projective manifold defined over $\mathbb{R}$ belongs to this class. In this paper, we study Lagrangian intersection Floer theory for the fixed point set of an antisymplectic involution.

Let (M,ω) be a compact, or more generally, tame, $2n$–dimensional symplectic manifold and L an oriented closed Lagrangian submanifold of M. It is well known by now that the Floer cohomology of a Lagrangian submanifold L cannot be defined in general. The phenomenon of bubbling-off holomorphic discs is the main source of troubles in defining Floer cohomology of Lagrangian submanifolds. In our books [9] and [10], we developed the general theory of obstructions and deformations of Lagrangian intersection Floer cohomology based on the theory of filtered A_∞ algebras which we associate to each Lagrangian submanifold. However, it is generally very hard to formulate the criterion for unobstructedness to defining Floer cohomology, let alone to calculate Floer cohomology for a given Lagrangian submanifold. In this regard, Lagrangian torus fibers in toric manifolds provide good test cases for these problems, which we studied in [11] and [12] in detail. For this class of Lagrangian submanifolds, we can do many explicit calculations of various notions and invariants that are introduced in the books [9] and [10].

Another important class of Lagrangian submanifolds is that of the fixed point set of an antisymplectic involution. Actually, the set of real points in Calabi–Yau manifolds plays an important role in the homological mirror symmetry conjecture. (See Walcher [23], Pandharipande, Solomon and Walcher [20], and Fukaya [5]. See also Welschinger [24] for related topics of real points.) The purpose of the present paper is to study Floer cohomology of this class of Lagrangian submanifolds. For example, we prove unobstructedness for such Lagrangian submanifolds in Calabi–Yau manifolds, and also provide some other examples of explicit calculations of Floer cohomology. The main ingredient of this paper is a careful study of orientations of the moduli spaces of pseudoholomorphic discs.

Take an ω–compatible almost complex structure J on (M,ω). We consider the moduli space $\mathcal{M}(J;\beta)$ of J–holomorphic stable maps from a bordered Riemann surface

$(\Sigma, \partial\Sigma)$ of genus 0 to (M, L) which represents a class $\beta \in \Pi(L) = \pi_2(M, L)/\sim$, where $\beta \sim \beta' \in \pi_2(M, L)$ if and only if $\omega(\beta) = \omega(\beta')$ and $\mu_L(\beta) = \mu_L(\beta')$. Here $\mu_L\colon \pi_2(M, L) \to \mathbb{Z}$ is the Maslov index homomorphism. The values of μ_L are even integers if L is oriented. When the domain Σ is a 2–disc D^2, we denote by $\mathcal{M}^{\text{reg}}(J;\beta)$ the subset of $\mathcal{M}(J;\beta)$ consisting of *smooth* maps, that is, pseudoholomorphic maps from the disc without disc- or sphere-bubbles. The moduli space $\mathcal{M}(J;\beta)$ has a Kuranishi structure; see Proposition 7.1.1 in [10]. However, it is not orientable in the sense of a Kuranishi structure in general. In Chapter 8 of [10], we introduce the notion of relative spin structure on $L \subset M$ and its stable conjugacy class, and prove that if L carries a relative spin structure (V, σ), its stable conjugacy class $[(V, \sigma)]$ determines an orientation on the moduli space $\mathcal{M}(J;\beta)$ (see Sections 2 and 3 for the precise definitions and notations). We denote it by $\mathcal{M}(J;\beta)^{[(V,\sigma)]}$ when we want to specify the stable conjugacy class of the relative spin structure. If we have a diffeomorphism $f\colon M \to M$ satisfying $f(L) = L$, we can define the pull-back $f^*[(V, \sigma)]$ of the relative spin structure. (See also Section 3.1.)

Now we consider the case that $\tau\colon M \to M$ is an antisymplectic involution and

$$L = \operatorname{Fix}\tau.$$

We assume L is nonempty, oriented and relatively spin. Take an ω–compatible almost complex structure J satisfying $\tau^* J = -J$. Such J we call *τ–anti-invariant*. Then we find that τ induces a map

$$\tau_*\colon \mathcal{M}^{\text{reg}}(J;\beta) \to \mathcal{M}^{\text{reg}}(J;\beta)$$

which satisfies $\tau_* \circ \tau_* = \text{Id}$. (See Definition 4.2 and Lemma 4.4.) Here we note that $\tau_*(\beta) = \beta$ in $\Pi(L)$ (see Remark 4.3). We pick a conjugacy class of relative spin structure $[(V, \sigma)]$ and consider the pull-back $\tau^*[(V, \sigma)]$. Then we have an induced map

$$\tau_*\colon \mathcal{M}^{\text{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\text{reg}}(J;\beta)^{[(V,\sigma)]}.$$

We will prove in Proposition 4.5 that τ_* is induced by an automorphism of $\mathcal{M}^{\text{reg}}(J;\beta)$ as a space with Kuranishi structure; see Definition A.3. The definition for an automorphism to be orientation-preserving in the sense of Kuranishi structure is given in Definition A.6. The first problem we study is the question whether τ_* respects the orientation or not. The following theorem plays a fundamental role in this paper.

Theorem 1.1 (Theorem 4.6) *Let L be a fixed point set of an antisymplectic involution τ on (M, ω) and J a τ–anti-invariant almost complex structure compatible with ω. Suppose that L is oriented and carries a relative spin structure (V, σ). Then the map $\tau_*\colon \mathcal{M}^{\text{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\text{reg}}(J;\beta)^{[(V,\sigma)]}$ is orientation-preserving if $\mu_L(\beta) \equiv 0 \bmod 4$, and is orientation-reversing if $\mu_L(\beta) \equiv 2 \bmod 4$.*

Remark 1.2 If L has a *τ–relative spin structure* (see Definition 3.11), then

$$\mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} = \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$$

as spaces with oriented Kuranishi structures. Corollary 4.7 is nothing but this case. If L is spin, then it is automatically τ–relatively spin (see Example 3.12). Later in Proposition 3.14 we show that there is an example of Lagrangian submanifold L which is relatively spin but not τ–relatively spin.

Including marked points, we consider the moduli space $\mathcal{M}_{k+1,m}(J;\beta)$ of J–holomorphic stable maps to (M,L) from a bordered Riemann surface $(\Sigma,\partial\Sigma)$ in class $\beta\in\Pi(L)$ of genus 0 with $(k+1)$ boundary marked points and m interior marked points. The antisymplectic involution τ also induces a map τ_* on the moduli space of J–holomorphic maps with marked points. See Theorem 4.10. Then we have:

Theorem 1.3 (Theorem 4.10) *The induced map*

$$\tau_*\colon \mathcal{M}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$$

is orientation-preserving if and only if $\frac{1}{2}\mu_L(\beta)+k+1+m$ is even.

When we construct the filtered A_∞ algebra $(C(L,\Lambda_{0,\mathrm{nov}}),\mathfrak{m})$ associated to a relatively spin Lagrangian submanifold L, we use the component of $\mathcal{M}_{k+1}(J;\beta)$ consisting of the elements whose boundary marked points lie in counterclockwise cyclic order on $\partial\Sigma$. We also involve interior marked points. For the case of $(k+1)$ boundary marked points on $\partial\Sigma$ and m interior marked points in $\mathrm{Int}\,\Sigma$, we denote the corresponding component, which we call the *main component*, by $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$. Moreover, we consider the moduli space $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;P_1,\dots,P_k)$, which is defined by taking a fiber product of $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$ with smooth singular simplices $P_1,\dots,P_k$ of L. (This is nothing but the main component of (2-4) with $m=0$.) A stable conjugacy class of a relative spin structure determines orientations on these spaces as well. See Sections 2 and 3 for the definitions and a precise description of their orientations. Here we should note that τ_* above does *not* preserve the cyclic ordering of boundary marked points, and so it does not preserve the main component. However, we can define the maps denoted by

$$\tau_*^{\mathrm{main}}\colon \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$$

and

$$\tau_*^{\mathrm{main}}\colon \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]}. \tag{1-1}$$

See (4-6), (4-7) and (4-10) for the definitions. We put $\deg' P=\deg P-1$, which is the shifted degree of P as a singular cochain of L (ie $\deg P=\dim L-\dim P$.) Then we show the following:

Theorem 1.4 (Theorem 4.12) *Let*

$$\epsilon = \tfrac{1}{2}\mu_L(\beta) + k + 1 + m + \sum_{1\le i<j\le k} \deg' P_i \deg' P_j .$$

Then the map induced by the involution τ,

$$\tau_*^{\mathrm{main}} \colon \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]},$$

is orientation-preserving if ϵ *is even, and orientation-reversing if* ϵ *is odd.*

See Theorem 5.2 for a more general statement involving the fiber product with singular simplices Q_j $(j=1,\dots,m)$ of M.

These results give rise to some nontrivial applications to Lagrangian intersection Floer theory for the case $L = \operatorname{Fix}\tau$. We briefly describe some consequences in the rest of this section.

In the books [9] and [10], using the moduli spaces $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;P_1,\dots,P_k)$, we construct a filtered A_∞ algebra $(C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}),\mathfrak{m})$ with $\mathfrak{m}=\{\mathfrak{m}_k\}_{k=0,1,2,\dots}$ for any relatively spin closed Lagrangian submanifold L of (M,ω) (see Theorem 6.2) and develop the obstruction and deformation theory of Lagrangian intersection Floer cohomology. Here $\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$ is the universal Novikov ring over $\mathbb{Q}$; see (6-1). In particular, we formulate the unobstructedness to defining Floer cohomology of L as the existence of solutions of the Maurer–Cartan equation for the filtered A_∞ algebra; see Definition 6.3. We denote by $\mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ the set of such solutions. By definition, when $\mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}) \neq \varnothing$, we can use any element $b \in \mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ to deform the Floer's "boundary" map and define a deformed Floer cohomology $HF((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$. See Section 6.1 for a short review of this process. Now, for the case $L=\operatorname{Fix}\tau$, Theorem 1.4 yields the following particular property of the filtered A_∞ algebra:

Theorem 1.5 *Let* M *be a compact, or tame, symplectic manifold and* τ *an antisymplectic involution. If* $L=\operatorname{Fix}\tau$ *is nonempty, compact, oriented and* τ*–relatively spin, then the filtered* A_∞ *algebra* $(C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}),\mathfrak{m})$ *can be chosen so that*

$$\mathfrak{m}_{k,\beta}(P_1,\dots,P_k) = (-1)^{\epsilon_1}\mathfrak{m}_{k,\tau_*\beta}(P_k,\dots,P_1), \tag{1-2}$$

where

$$\epsilon_1 = \tfrac{1}{2}\mu_L(\beta) + k + 1 + \sum_{1\le i<j\le k} \deg' P_i \deg' P_j .$$

Using the results from [9] and [10], we derive that Theorem 1.5 implies unobstructedness of $L=\operatorname{Fix}\tau$ in the following cases:

Corollary 1.6 *Let τ and $L = \mathrm{Fix}\,\tau$ be as in Theorem 1.5. In addition, we assume that either*

(1) $c_1(TM)|_{\pi_2(M)} \equiv 0 \bmod 4$, *or*

(2) $c_1(TM)|_{\pi_2(M)} \equiv 0 \bmod 2$ *and* $i_*\colon \pi_1(L) \to \pi_1(M)$ *is injective. (Here i_* is the natural map induced by the inclusion $i\colon L \to M$.)*

Then L is unobstructed over $\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$ (ie $\mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}) \neq \varnothing$), and so

$$\mathrm{HF}((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$$

is defined for any $b \in \mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$. Moreover, we may choose $b \in \mathcal{M}(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ so that the map

$$(-1)^{k(\ell+1)}(\mathfrak{m}_2)_*\colon \mathrm{HF}^k\big((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}\big) \otimes \mathrm{HF}^\ell\big((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}\big) \\ \to \mathrm{HF}^{k+\ell}((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$$

induces a graded commutative product.

Remark 1.7 By symmetrizing the filtered A_∞ structure $\mathfrak{m}_k$ of $(C(L,\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}),\mathfrak{m})$, we obtain a filtered L_∞ algebra $(C(L,\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}),\mathfrak{l}) = (C(L,\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}),\{\mathfrak{l}_k\}_{k=0,1,2,\dots})$. See Section A3 of [10] for the definitions of the symmetrization and of the filtered L_∞ structure. In the situation of Corollary 1.6, the same proof shows that we have $\mathfrak{l}_k = \bar{\mathfrak{l}}_k \otimes \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$ if k is even. Here $\bar{\mathfrak{l}}_k$ is the (unfiltered) L_∞ structure obtained by the reduction of the coefficient of $(C(L,\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}),\mathfrak{l})$ to $\mathbb{Q}$. Note that *over* $\mathbb{R}$ we may choose $\bar{\mathfrak{l}}_k = 0$ for $k \geq 3$ by Theorem X in Chapter 1 of [9]. On the other hand, Theorem A3.19 in [10] shows that $\bar{\mathfrak{l}}_k = 0$ for $H(L;\mathbb{Q})$.

We note that we do not assert that the Floer cohomology $\mathrm{HF}((L,b),(L,b);\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ is isomorphic to $H^*(L;\mathbb{Q}) \otimes \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$, in general. (Namely, we do not assert that $\mathfrak{m}_1 = \overline{\mathfrak{m}}_1 \otimes \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$.) Indeed, we show in Section 6.4 that for the case $L = \mathbb{R}P^{2n+1}$ in $\mathbb{C}P^{2n+1}$, the Floer cohomology group is *not* isomorphic to the classical cohomology group. (See Theorem 6.71.)

Moreover, if we also assume that $c_1(TM)|_{\pi_2(M)} = 0$, we can show the following nonvanishing theorem of Floer cohomology:

Corollary 1.8 *Let τ and $L = \mathrm{Fix}\,\tau$ be as in Theorem 1.5. Assume $c_1(TM)|_{\pi_2(M)} = 0$. Then L is unobstructed over $\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$, and*

$$\mathrm{HF}\big((L,b),(L,b);\Lambda^{\mathbb{Q}}_{\mathrm{nov}}\big) \neq 0$$

for any $b \in \mathcal{M}(L;\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$. In particular, we have

$$\psi(L)\cap L \neq \varnothing$$

for any Hamiltonian diffeomorphism $\psi\colon M\to M$.

Theorem 1.5 and Corollaries 1.6 and 1.8 can be applied to the real point set L of any Calabi–Yau manifold (defined over $\mathbb{R}$) if it is oriented and τ–relative spin.

Another application of Theorem 1.5 and Corollary 1.6 is a ring isomorphism between quantum cohomology and Lagrangian Floer cohomology for the case of the diagonal of a square of a symplectic manifold. Let (N,ω_N) be a closed symplectic manifold. We consider the product

$$(M,\omega_M)=(N\times N,-\mathrm{pr}_1^*\omega_N+\mathrm{pr}_2^*\omega_N),$$

where pr_i is the projection to the i^{th} factor. The involution $\tau\colon M\to M$ defined by $\tau(x,y)=(y,x)$ is antisymplectic, and its fixed point set L is the diagonal

$$\Delta_N=\{(x,x)\mid x\in N\}\cong N.$$

As we will see in the proof of Theorem 1.9, the diagonal set is always unobstructed. Moreover, we note that the natural map $i_*\colon H_*(\Delta_N,\mathbb{Q})\to H_*(N\times N;\mathbb{Q})$ is injective, and so the spectral sequence constructed in Chapter 6 of [9] collapses at the E_2–term by Theorem D (D.3) [9], which in turn induces the natural isomorphism $H(N;\mathbb{Q})\otimes\Lambda_{0,\mathrm{nov}}\cong \mathrm{HF}((L,b),(L,b);\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ for any $b\in\mathcal{M}(L;\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$. In the proof of Theorem 1.9 in Section 6.2 we prove that $\mathfrak{m}_2$ also derives a graded commutative product

$$\cup_Q\colon \mathrm{HF}\big((L,b),(L,b);\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big)\otimes \mathrm{HF}\big((L,b),(L,b);\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big)\to \mathrm{HF}\big((L,b),(L,b);\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big).$$

In fact, we can prove that the following stronger statement.

Theorem 1.9 *Let (N,ω_N) be a closed symplectic manifold.*

(1) *The diagonal set of $(N\times N,-\mathrm{pr}_1^*\omega_N+\mathrm{pr}_2^*\omega_N)$ is unobstructed over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$.*

(2) *There exists a bounding cochain b such that product $\cup_Q$ coincides with the quantum cup product on (N,ω_N) under the natural isomorphism*

$$\mathrm{HF}\big((L,b),(L,b);\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big)\cong H(N;\mathbb{Q})\otimes\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}.$$

If we use Corollary 3.8.43 in [9], we can easily find that the diagonal set is *weakly unobstructed* in the sense of Definition 3.6.29 in [9]. See also Remark 6.72. We also note that for the case of diagonals, the $\mathfrak{m}_k$ $(k \geq 3)$ define a quantum (higher) Massey product. It was discussed formally in Fukaya [4]. We have made it rigorous here:

Corollary 1.10 *For any closed symplectic manifold* (N, ω_N)*, there exists a filtered* A_∞ *structure* $\mathfrak{m}_k$ *on* $H(N; \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}) = H(N; \mathbb{Q}) \otimes \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$ *such that*

(1) $\mathfrak{m}_0 = \mathfrak{m}_1 = 0$;

(2) $\cup_Q$ *defined by (6-13) using* $\mathfrak{m}_2$ *coincides with the quantum cup product;*

(3) *the* $\mathbb{R}$*–reduction* $(H(N; \mathbb{Q}), \overline{\mathfrak{m}}) \otimes_{\mathbb{Q}} \mathbb{R}$ *of the filtered* A_∞ *algebra is homotopy equivalent to the de Rham complex of* N *as an* A_∞ *algebra, where* $(H(N; \mathbb{Q}), \overline{\mathfrak{m}})$ *is the reduction of the coefficient of* $(H(N; \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}), \mathfrak{m})$ *to* $\mathbb{Q}$.

The paper is organized as follows: In Section 2, we briefly recall some basic material on the moduli space of stable maps from a bordered Riemann surface of genus 0. In Section 3, we also recall from [10] the notion of relative spin structure, its stable conjugacy class and the orientation of the moduli space of pseudoholomorphic discs. We describe how the stable conjugacy class of relative spin structure determines an orientation on the moduli space. We introduce here the notion of τ–relative spin structure for an antisymplectic involution $\tau\colon M \to M$, and also give some examples which are relatively spin but not τ–relatively spin Lagrangian submanifolds. In Section 4, we define the map τ_* on the moduli space induced by τ and study how the induced map τ_* on various moduli spaces changes or preserves the orientations. Assuming Theorem 1.1 holds, we prove Theorem 1.3 in this section. The fundamental theorems Theorem 1.1 and Theorem 1.4 are proved in Section 5. Section 6 is devoted to various applications of the results obtained above to Lagrangian Floer cohomology. After a short review of the general story of Lagrangian intersection Floer theory laid out in [9] and [10], we prove Theorem 1.5, Corollary 1.6 and Corollary 1.8 in Section 6.2. Section 6.3 is devoted to the proofs of Theorem 1.9 and Corollary 1.10. In particular, we introduce stable maps with admissible system of circles and study their moduli spaces in Section 6.3.4. In Section 6.4, we calculate Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ coefficients by studying orientations in detail. The calculation shows that the Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ is not isomorphic to the usual cohomology. This result contrasts with Oh's earlier calculation [17] of the Floer cohomology of real projective spaces over $\mathbb{Z}_2$ coefficients, where the Floer cohomology is isomorphic to the usual cohomology over $\mathbb{Z}_2$. In the first two subsections of the Appendix, we briefly recall form [10] the definition of orientation on the space with Kuranishi structure and the notion of group action on a space with Kuranishi structure. In the third subsection of the Appendix, we present how to promote filtered $A_{n,K}$–structures keeping the invariance under the involution.

Originally, the content of this paper appeared as a part of Chapter 8 in the preprint version [7] of the books [9] and [10], and was intended to be published in a part of the book. However, due to the publisher's page restriction on the AMS/IP Advanced Math Series, we took out two chapters, Chapter 8 and Chapter 10 from the preprint [7] and published the book without those two chapters. The content of this paper is based on the parts extracted from Chapter 8 (Floer theory of Lagrangian submanifolds over $\mathbb{Z}$) and Chapter 9 (Orientation) in the preprint [7]. We also note that this is a part of the paper cited as [FOOO09I] in the books [9] and [10]. The half of the remaining part of Chapter 8 of [7] is published as [14].

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2 Preliminaries

In this section, we prepare some basic notation we use in this paper. We refer to Section 2.1 of [9] and Section A1 of [10] for more detailed explanations of moduli spaces and the notion of Kuranishi structure, respectively. Let L be an oriented compact Lagrangian submanifold of (M,ω). Take an ω–compatible almost complex structure J on M. We recall Definition 2.4.17 in [9] where we introduce the relation $\sim$ in $\pi_2(M,L)$: We define $\beta \sim \beta'$ in $\pi_2(M,L)$ if and only if $\omega(\beta) = \omega(\beta')$ and $\mu_L(\beta) = \mu_L(\beta')$. We denote the quotient group by

$$\Pi(L) = \pi_2(M,L)/\sim . \tag{2-1}$$

This is an abelian group. Let $\beta \in \Pi(L)$. *A stable map from a bordered Riemann surface of genus zero with $(k+1)$ boundary marked points and m interior marked points* is a pair $((\Sigma,\vec{z},\vec{z}^{\,+}),w) = ((\Sigma,z_0,\ldots,z_k,z_1^+,\ldots,z_m^+),w)$ such that $(\Sigma,\vec{z},\vec{z}^{\,+})$ is a bordered semistable curve of genus zero with $(k+1)$ boundary marked points and m interior marked points, and $w\colon (\Sigma, \partial\Sigma) \to (M, L)$ is a J–holomorphic map such that its automorphism group, ie the set of biholomorphic maps $\psi\colon \Sigma \to \Sigma$ satisfying $\psi(z_i) = z_i$, $\psi(\vec{z}_i^{\,+}) = \vec{z}_i^{\,+}$ and $w \circ \psi = w$, is finite. We say that $((\Sigma, \vec{z}, \vec{z}^{\,+}), w)$ is isomorphic to $((\Sigma', \vec{z}', \vec{z}^{\,+\prime}), w')$ if there exists a biholomorphic map $\psi\colon \Sigma \to \Sigma'$ satisfying $\psi(z_i) = z_i'$, $\psi(\vec{z}_i^{\,+}) = \vec{z}_i^{\,+\prime}$ and $w' \circ \psi = w$. We denote by $\mathcal{M}_{k+1,m}(J;\beta)$ the set of the isomorphism classes of stable maps in class $\beta \in \Pi(L)$ from a bordered Riemann surface of genus zero with $(k+1)$ boundary marked points and m interior marked points. When the domain curve Σ is a smooth 2–disc D^2, we denote the corresponding subset by $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$. We note that $\mathcal{M}_{k+1,m}(J;\beta)$ is a compactification of $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$. The virtual real dimension is

$$\dim_{\mathbb{R}} \mathcal{M}_{k+1,m}(J;\beta) = n + \mu_L(\beta) + k + 1 + 2m - 3,$$

where $n = \dim L$ and $\mu_L(\beta)$ is the Maslov index (which is an even integer for an oriented Lagrangian submanifold L). When we do not consider interior marked points, we denote them by $\mathcal{M}_{k+1}(J;\beta)$ and $\mathcal{M}^{\mathrm{reg}}_{k+1}(J;\beta)$, and when we do not consider any marked points, we simply denote them by $\mathcal{M}(J;\beta)$ and $\mathcal{M}^{\mathrm{reg}}(J;\beta)$. Furthermore, we define a component $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$ of $\mathcal{M}_{k+1,m}(J;\beta)$ by

$$\begin{aligned}\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta) = \big\{((\Sigma, \vec{z}, \vec{z}^{\,+}), w) \in \mathcal{M}_{k+1,m}(J;\beta) \mid \\ (z_0, z_1, \dots, z_k) \text{ is in counterclockwise cyclic order on } \partial\Sigma\big\},\end{aligned}$$

which we call the *main component*. We define $\mathcal{M}^{\mathrm{main,reg}}_{k+1,m}(J;\beta)$, $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta)$ and $\mathcal{M}^{\mathrm{main,reg}}_{k+1}(J;\beta)$ in a similar manner.

We have a Kuranishi structure on $\mathcal{M}_{k+1,m}(J;\beta)$ so that the evaluation maps

$$\begin{aligned} \mathrm{ev}_i\colon \mathcal{M}_{k+1,m}(J;\beta) &\to +L, \quad i = 0, 1, \dots, k, \\ \mathrm{ev}_j^+\colon \mathcal{M}_{k+1,m}(J;\beta) &\to M, \qquad j = 1, \dots, m \end{aligned} \tag{2-2}$$

defined by $\mathrm{ev}_i((\Sigma, \vec{z}, \vec{z}^{\,+}), w) = w(z_i)$ and $\mathrm{ev}_j^+((\Sigma, \vec{z}, \vec{z}^{\,+}), w) = w(z_j^+)$ are weakly submersive. (See Section 5 of [16] and Section A1.1 of [10] for the definitions of Kuranishi structure and weakly submersive maps.) Then for given smooth singular simplices $(f_i\colon P_i \to L)$ of L and $(g_j\colon Q_j \to M)$ of M, we can define the fiber product in the sense of Kuranishi structure:

$$\begin{aligned} &\mathcal{M}_{k+1,m}(J;\beta;\vec{Q},\vec{P}) \\ &:= \mathcal{M}_{k+1,m}(J;\beta)_{(\mathrm{ev}_1^+,\dots,\mathrm{ev}_m^+,\mathrm{ev}_1,\dots,\mathrm{ev}_k)} \times_{g_1\times\cdots\times f_k} \Big(\prod_{j=1}^m Q_j \times \prod_{i=1}^k P_i\Big). \end{aligned} \tag{2-3}$$

See Section A1.2 of [10] for the definition of fiber product of Kuranishi structures. We define $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;\vec{Q},\vec{P})$ in a similar way. When we do not consider the interior marked points, we denote the corresponding moduli spaces by $\mathcal{M}_{k+1}(J;\beta;\vec{P})$ and $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;\vec{P})$, respectively. Namely,

$$\mathcal{M}_{k+1}(J;\beta;\vec{P}) := \mathcal{M}_{k+1}(J;\beta)_{(\mathrm{ev}_1,\dots,\mathrm{ev}_k)} \times_{f_1\times\cdots\times f_k} \left(\prod_{i=1}^{k} P_i\right). \tag{2-4}$$

In Section 3.3, we describe the orientations on these spaces precisely.

3 τ–relative spin structure and orientation

3.1 Review of relative spin structure and orientation

It is known that the moduli space of pseudoholomorphic discs with Lagrangian boundary condition is not always orientable. To discuss orientability and orientation of the moduli space, we first recall the notion of relative spin structure and its stable conjugacy class introduced in [10]; we also briefly review how the stable conjugacy class of relative spin structure determines an orientation of the moduli space of pseudoholomorphic discs with Lagrangian boundary condition. See Section 8.1 of [10] for more details. See also V de Silva's work [21].

Definition 3.1 An oriented Lagrangian submanifold L of M is called *relatively spin* if there exists a class $st \in H^2(M;\mathbb{Z}_2)$ such that $st|_L = w_2(TL)$.

A pair of oriented Lagrangian submanifolds $(L^{(1)}, L^{(0)})$ is called *relatively spin* if there exists a class $st \in H^2(M;\mathbb{Z}_2)$ satisfying $st|_{L^{(i)}} = w_2(TL^{(i)})$ $(i = 0, 1)$ simultaneously.

Remark 3.2 Using the relative pin structure, J Solomon [22] generalized our results about the orientation problem studied in [10] to the case of nonorientable Lagrangian submanifolds.

Let L be a relatively spin Lagrangian submanifold of M. We fix a triangulation of M such that L is a subcomplex. A standard obstruction theory yields that we can take an oriented real vector bundle V over the 3–skeleton $M_{[3]}$ of M which satisfies $w_2(V) = st$. Then $w_2(TL|_{L_{[2]}} \oplus V|_{L_{[2]}}) = 0$, and so $TL \oplus V$ carries a spin structure on the 2–skeleton $L_{[2]}$ of L.

Definition 3.3 The choice of an orientation on L, a cohomology class $st \in H^2(M;\mathbb{Z}_2)$, an oriented real vector bundle V over the 3–skeleton $M_{[3]}$ satisfying $w_2(V) = st$, and a spin structure σ on $(TL \oplus V)|_{L_{[2]}}$ is called a *relative spin structure* on $L \subset M$.

A *relative spin structure* on the pair $(L^{(1)}, L^{(0)})$ is the choice of orientations on $L^{(i)}$, a cohomology class $st \in H^2(M;\mathbb{Z}_2)$, an oriented real vector bundle V over the 3–skeleton $M_{[3]}$ satisfying $w_2(V) = st$, and spin structures on $(TL^{(i)} \oplus V)|_{L^{(i)}_{[2]}}$ $(i = 0, 1)$.

In this paper we fix an orientation on L. If L is spin, we have an associated relative spin structure for each spin structure on L as follows: Take $st = 0$ and V to be trivial. Then the spin structure on L naturally induces the spin structure on $TL \oplus V$.

Definition 3.3 depends on the choices of V and the triangulation of M. We introduce an equivalence relation called *stable conjugacy* on the set of relative spin structures so that the stable conjugacy class is independent of such choices.

Definition 3.4 We say that two relative spin structures (st_i, V_i, σ_i) $(i = 1, 2)$ on L are *stably conjugate* if there exist integers k_i and an orientation-preserving bundle isomorphism $\varphi\colon V_1 \oplus \mathbb{R}^{k_1} \to V_2 \oplus \mathbb{R}^{k_2}$ such that by $1 \oplus \varphi|_{L_{[2]}}\colon (TL \oplus V_1)_{L_{[2]}} \oplus \mathbb{R}^{k_1} \to (TL \oplus V_2)_{L_{[2]}} \oplus \mathbb{R}^{k_2}$, the spin structure $\sigma_1 \oplus 1$ induces the spin structure $\sigma_2 \oplus 1$.

Here $\mathbb{R}^{k_i}$ denotes a trivial vector bundle of rank k_i $(i = 1, 2)$. We note that in Definition 3.4, we still fix a triangulation $\mathfrak{T}$ of M such that L is a subcomplex. However, by Proposition 8.1.6 in [10], we find that the stable conjugacy class of relative spin structure is actually independent of the choice of a triangulation of M as follows: We denote by $\mathrm{Spin}(M, L; \mathfrak{T})$ the set of all the stable conjugacy classes of relative spin structures on $L \subset M$.

Proposition 3.5 [10, Proposition 8.1.6] (1) *There is a simply transitive action of* $H^2(M, L; \mathbb{Z}_2)$ *on* $\mathrm{Spin}(M, L; \mathfrak{T})$.

(2) *For two triangulations $\mathfrak{T}$ and $\mathfrak{T}'$ of M such that L is a subcomplex, there exists a canonical isomorphism* $\mathrm{Spin}(M, L; \mathfrak{T}) \cong \mathrm{Spin}(M, L; \mathfrak{T}')$ *compatible with the above action.*

In particular, if a spin structure of L is given, there is a canonical isomorphism $\mathrm{Spin}(M, L; \mathfrak{T}) \cong H^2(M, L; \mathbb{Z}_2)$. Thus, hereafter, we denote by $\mathrm{Spin}(M, L)$ the set of the stable conjugacy classes of relative spin structures on L without specifying any triangulation of M.

Since the class st is determined by V, we simply write the stable conjugacy class of relative spin structure as $[(V, \sigma)]$, where σ is a spin structure on $(TL \oplus V)|_{L_{[2]}}$.

The following theorem is proved in Section 8.1 of [10]. We denote by $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta)$ the set of all J–holomorphic maps from $(D^2, \partial D^2)$ to (M, L) representing a class β. We note that $\mathcal{M}^{\mathrm{reg}}(J; \beta) = \widetilde{\mathcal{M}}^{\mathrm{reg}}(J; \beta) / \mathrm{PSL}(2, \mathbb{R})$.

Theorem 3.6 *If L is a relatively spin Lagrangian submanifold, $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ is orientable. Furthermore, the choice of stable conjugacy class of relative spin structure on L determines an orientation on $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ canonically for all $\beta \in \pi_2(M,L)$.*

Remark 3.7 (1) Following Convention 8.2.1 in [10], we have an induced orientation on the quotient space. Thus Theorem 3.6 holds for the quotient space $\mathcal{M}^{\mathrm{reg}}(J;\beta) = \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)/\operatorname{PSL}(2,\mathbb{R})$ as well. Here we use the orientation of $\operatorname{PSL}(2,\mathbb{R})$ as in Convention 8.3.1 in [10].

(2) Since $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ is the top-dimensional stratum of $\mathcal{M}(J;\beta)$, the orientation on $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ determines one on $\mathcal{M}(J;\beta)$. In this sense, it is enough to consider $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ when we discuss orientation on $\mathcal{M}(J;\beta)$. The same remark applies to other moduli spaces including marked points and fiber products with singular simplices.

(3) The moduli space $\mathcal{M}(J,\beta)$ may not contain a smooth holomorphic disc, ie $\mathcal{M}^{\mathrm{reg}}(J,\beta)=\varnothing$. However, the orientation issue can be discussed as if $\mathcal{M}^{\mathrm{reg}}(J,\beta)\neq\varnothing$. This is because we consider the orientation of Kuranishi structure, ie the orientation of $\det E^* \otimes \det TV$, where V is a Kuranishi neighborhood around a point $p = [w\colon (\Sigma,\partial\Sigma)\to(M,L)]$ in $\mathcal{M}(J,\beta)$ and $E\to V$ is the obstruction bundle. Even though p is not represented by a bordered stable map with an irreducible domain, ie a disc, V contains a solution of $\overline{\partial}u \equiv 0 \bmod E$ for $u\colon (D^2,\partial D^2)\to(M,L)$. The determinant bundle of the linearized $\overline{\partial}$–operators parametrized by V is trivialized around p, hence the orientation of the determinant line at $[u]$ determines the one at p.

We recall from Section 8.1 of [10] how each stable conjugacy class of relative spin structures determines an orientation on the moduli space of holomorphic discs. Once we know the orientability of $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$, it suffices to give an orientation on the determinant of the tangent space at a point $w\in\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ for each stable conjugacy class of relative spin structures. We consider the linearized operator of the pseudoholomorphic curve equation

$$D_w\overline{\partial}\colon W^{1,p}(D^2,\partial D^2; w^*TM,\ell^*TL)\to L^p(D^2;w^*TM\otimes\Lambda^{0,1}_{D^2}). \tag{3-1}$$

Here $\ell = w|_{\partial D^2}$ and $p>2$. Since it has the same symbol as the Dolbeault operator

$$\overline{\partial}_{(w^*TM,\ell^*TL)}\colon W^{1,p}(D^2,\partial D^2; w^*TM,\ell^*TL)\to L^p(D^2;w^*TM\otimes\Lambda^{0,1}_{D^2}),$$

we may consider the determinant of the index of this Dolbeault operator $\overline{\partial}_{(w^*TM,\ell^*TL)}$ instead. We can deform $w\colon (D^2,\partial D^2)\to(M,L)$ to $w_0\colon (D^2,\partial D^2)\to(M_{[2]},L_{[1]})$ by the simplicial approximation theorem. We put $\ell_0 = w_0|_{\partial D^2}$.

Now pick $[(V,\sigma)]\in\operatorname{Spin}(M,L)$. It determines the stable homotopy class of the trivialization of $\ell_0^*(TL\oplus V)$. The existence of the oriented bundle V on $M_{[3]}$ induces a unique homotopy class of the trivialization of $\ell_0^* V$. Thus, we have a unique homotopy class of trivialization of $\ell_0^* TL$. Using this trivialization and Proposition 3.8 below (applied to the pair of (E,λ) with $E = w_0^* TM$, $\lambda = \ell_0^* TL$), we can assign an orientation on the determinant of the index,

$$\det \mathrm{Index}\, \overline{\partial}_{(w_0^* TM, \ell_0^* TL)} := \det(\mathrm{coker}\, \overline{\partial}_{(w_0^* TM, \ell_0^* TL)})^* \otimes \det \ker \overline{\partial}_{(w_0^* TM, \ell_0^* TL)}.$$

This process is invariant under the stably conjugate relation of relative spin structures. Therefore, we obtain an orientation on $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$, and so on $\mathcal{M}^{\mathrm{reg}}(J;\beta)$, for each stable conjugacy class of relative spin structure $[(V,\sigma)]$.

Proposition 3.8 [10, Proposition 8.1.4] *Let E be a complex vector bundle over D^2 and λ a maximally totally real bundle over ∂D^2 with an isomorphism*

$$E|_{\partial D^2} \cong \lambda \otimes \mathbb{C}.$$

Suppose that λ is trivial. Then each trivialization on λ canonically induces an orientation on $\mathrm{Index}\, \overline{\partial}_{(E,\lambda)}$. *Here $\overline{\partial}_{(E,\lambda)}$ is the Dolbeault operator on $(D^2, \partial D^2)$ with coefficient (E,λ):*

$$\overline{\partial}_{(E,\lambda)} \colon W^{1,p}(D^2, \partial D^2; E, \lambda) \to L^p(D^2; E \otimes \Lambda^{0,1}_{D^2}).$$

Remark 3.9 To explain some part of the proof of Theorem 1.1 given in Section 5 in a self-contained way, we briefly recall the outline of the proof of Proposition 3.8. See Section 8.1.1 of [10] for more detail. For $0 < \epsilon < 1$, we put

$$A(\epsilon) = \{z \in D^2 \mid 1-\epsilon \le |z| \le 1\} \quad \text{and} \quad C_{1-\epsilon} = \{z \in D^2 \mid |z| = 1-\epsilon\}.$$

By pinching the circle C to a point, we have a union of a 2–disc D^2 and a 2–sphere $\mathbb{C}P^1$ with the center $O \in D^2$ identified with a point $p \in \mathbb{C}P^1$. The resulting space $\Sigma = D^2 \cup \mathbb{C}P^1$ naturally has a structure of a nodal curve where $O = p$ is the nodal point. Under the situation of Proposition 3.8, the trivial bundle $\lambda \to \partial D^2$ trivially extends to $A(\epsilon)$, and the complexification of each trivialization of $\lambda \to \partial D^2$ gives a trivialization on $E|_{A(\epsilon)} \to A(\epsilon)$. Thus the bundle $E \to D^2$ descends to a bundle over the nodal curve Σ together with a maximally totally real bundle over $\partial\Sigma = \partial D^2$. We denote them by $E' \to \Sigma$ and $\lambda' \to \partial\Sigma$, respectively. We also denote by $W^{1,p}(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$ the space of $W^{1,p}$–sections of $E'|_{\mathbb{C}P^1} \to \mathbb{C}P^1$, and by $W^{1,p}(D^2; E'|_{D^2}, \lambda')$ the space of $W^{1,p}$–sections ξ_{D^2} of $E'|_{D^2} \to D^2$ satisfying $\xi_{D^2}(z) \in \lambda'_z$ and $z \in \partial D^2 = \partial\Sigma$. We consider a map

$$\mathrm{diff} \colon W^{1,p}(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) \oplus W^{1,p}(D^2, \partial D^2; E'|_{D^2}, \lambda') \to \mathbb{C}^n,$$
$$(\xi_{\mathbb{C}P^1}, \xi_{D^2}) \mapsto \xi_{\mathbb{C}P^1}(p) - \xi_{D^2}(O).$$

We put $W^{1,p}(E',\lambda') := \mathrm{diff}^{-1}(0)$ and consider the index of operator

$$\overline{\partial}_{(E',\lambda')}\colon W^{1,p}(E',\lambda') \to L^p(\mathbb{C}P^1; E'|_{\mathbb{C}P^1} \otimes \Lambda^{0,1}_{\mathbb{C}P^1}) \oplus L^p(D^2, \partial D^2; E'|_{D^2} \otimes \Lambda^{0,1}_{D^2}).$$

Then the orientation problem for $\mathrm{Index}\,\overline{\partial}_{(E,\lambda)}$ on $(D^2, \partial D^2)$ is translated into the problem for $\mathrm{Index}\,\overline{\partial}_{(E',\lambda')}$ on $(\Sigma, \partial\Sigma)$. Firstly, we note that the operator

$$\overline{\partial}_{(E'|_{D^2},\lambda'|_{\partial D^2})}\colon W^{1,p}(D^2, \partial D^2; E'|_{D^2}, \lambda') \to L^p(D^2; E'|_{D^2} \otimes \Lambda^{0,1}_{D^2})$$

is surjective. Each trivialization of $\lambda \to \partial D^2$ gives an identification

$$\ker \overline{\partial}_{(E'|_{D^2},\lambda'|_{\partial D^2})} \cong \ker \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)} \cong \mathbb{R}^n, \tag{3-2}$$

where $\ker \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)}$ is the space of solutions $\xi\colon D^2 \to \mathbb{C}^n$ of the Cauchy–Riemann equation with boundary condition

$$\overline{\partial}\xi = 0, \quad \xi(z) \in \lambda'_z \equiv \mathbb{R}^n, \quad z \in \partial D^2.$$

Thus the solution must be a real constant vector. This implies that we have a canonical isomorphism in (3-2). Then the argument in Section 8.1.1 of [10] shows that the orientation problem can be reduced to the orientation on $\ker \overline{\partial}_{(E'|_{D^2},\lambda'|_{\partial D^2})}$ and $\mathrm{Index}\,\overline{\partial}_{E'|_{\mathbb{C}P^1}}$. The latter one has a complex orientation. By taking a finite-dimensional complex vector space $W \subset L^p(\mathbb{C}P^1; E'|_{\mathbb{C}P^1} \otimes \Lambda^{0,1}_{\mathbb{C}P^1})$ such that

$$L^p(\mathbb{C}P^1; E'|_{\mathbb{C}P^1} \otimes \Lambda^{0,1}_{\mathbb{C}P^1}) = \mathrm{Im}\,\overline{\partial}_{E'|_{\mathbb{C}P^1}} + W,$$

a standard argument (see the paragraphs after Remark 8.1.3 in [10], for example) shows that the orientation problem on $\mathrm{Index}\,\overline{\partial}_{E'|_{\mathbb{C}P^1}}$ is further reduced to one on $\ker \overline{\partial}_{E'|_{\mathbb{C}P^1}}$ which is the space of holomorphic sections of $E'|_{\mathbb{C}P^1} \to \mathbb{C}P^1$, denoted by $\mathrm{Hol}(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$.

We next describe how the orientation behaves under the change of stable conjugacy classes of relative spin structures. Proposition 3.5 shows that the difference of relative spin structures is measured by an element $\mathfrak{x}$ in $H^2(M, L; \mathbb{Z}_2)$. We denote the simply transitive action of $H^2(M, L; \mathbb{Z}_2)$ on $\mathrm{Spin}(M, L)$ by

$$(\mathfrak{x}, [(V, \sigma)]) \mapsto \mathfrak{x} \cdot [(V, \sigma)].$$

When we change the relative spin structure by $\mathfrak{r} \in H^2(M, L; \mathbb{Z}_2)$, then we find that the orientation on the index of the operator $D_w\overline{\partial}$ in (3-1) changes by $(-1)^{\mathfrak{x}[w]}$. The following result is proved in Proposition 8.1.16 in [10] and is also obtained by Cho [3] and Solomon [22]. This proposition is used in Sections 6.3 and 6.4.

Proposition 3.10 *The identity map*

$$\mathcal{M}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}(J;\beta)^{\mathfrak{x}\cdot[(V,\sigma)]}$$

is orientation-preserving if and only if $\mathfrak{x}[\beta]=0$.

For a diffeomorphism $\psi\colon (M,L)\to(M',L')$ satisfying $\psi(L)=L'$, we define the pull-back map

$$\psi^*\colon \mathrm{Spin}(M',L')\to \mathrm{Spin}(M,L) \tag{3-3}$$

by $\psi^*[(V',\sigma')]=[(\psi^*V',\psi^*\sigma')]$. That is, we take a triangulation on M' such that L' is its subcomplex and $\psi\colon (M,L)\to(M',L')$ is a simplicial map. Then ψ^*V' is a real vector bundle over $M_{[3]}$, and σ' induces a spin structure on $(TL\oplus\psi^*V')|_{L_{[2]}}$. Then it is easy to see that

$$\psi_*\colon \mathcal{M}(\beta;M,L;J)^{\psi^*[(V',\sigma')]} \to \mathcal{M}(\psi_*\beta;M',L';\psi_*J)^{[(V',\sigma')]}$$

is orientation-preserving.

3.2 τ–relative spin structure and an example

After these general results are prepared in the previous subsection, we focus ourselves on the case $L=\mathrm{Fix}\,\tau$, the fixed point set of an antisymplectic involution τ of M. We define the notion of τ–relative spin structure and discuss its relationship with the orientation of the moduli space. Note that for τ such that $L=\mathrm{Fix}\,\tau$ is relative spin, τ induces an involution τ^* on the set of relative spin structures (V,σ) by pull-back (3-3).

Definition 3.11 A *τ–relative spin structure* on L is a relative spin structure (V,σ) on L such that $\tau^*(V,\sigma)$ is stably conjugate to (V,σ), ie $\tau^*[(V,\sigma)]=[(V,\sigma)]$ in $\mathrm{Spin}(M,L)$. We say that L is *τ–relatively spin* if it carries a τ–relative spin structure, ie if the involution $\tau^*\colon \mathrm{Spin}(M,L)\to\mathrm{Spin}(M,L)$ has a fixed point.

Example 3.12 If L is spin, then it is τ–relatively spin: Obviously, τ preserves the spin structure of L since it is the identity on L. And we may take V as needed in the definition of relative spin structure to be the trivial vector bundle.

Remark 3.13 We would like to emphasize that a relative spin structure (V,σ,st) satisfying $\tau^*st=st$ is *not necessarily* a τ–relative spin structure in the sense of Definition 3.11. See Proposition 3.14 below.

Now we give an example of $L = \mathrm{Fix}\ \tau$ that is relatively spin but *not* τ–relatively spin.

Consider $M = \mathbb{C}P^n$ with the standard symplectic and complex structures and $L = \mathbb{R}P^n \subset \mathbb{C}P^n$, the real point set. The real projective space $\mathbb{R}P^n$ is oriented if and only if n is odd. We take the tautological real line bundle ξ on $\mathbb{R}P^n$ such that the 1st Stiefel–Whitney class $w_1(\xi) := x$ is a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$. Then we have

$$T\mathbb{R}P^n \oplus \mathbb{R} \cong \xi^{\oplus(n+1)},$$

and so the total Stiefel–Whitney class is given by

$$w(T\mathbb{R}P^n) = (1+x)^{n+1}.$$

Therefore, we have

$$w_2(\mathbb{R}P^{2n+1}) = \begin{cases} x^2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3-4}$$

From this, it follows that $\mathbb{R}P^{4n+3}$ $(n \geq 0)$ and $\mathbb{R}P^1$ are spin and hence are τ–relatively spin by Example 3.12. On the other hand, we prove:

Proposition 3.14 *The real projective space $\mathbb{R}P^{4n+1} \subset \mathbb{C}P^{4n+1}$ $(n \geq 1)$ is relatively spin but not τ–relatively spin.*

Proof The homomorphism

$$H^2(\mathbb{C}P^{4n+1}; \mathbb{Z}_2) \to H^2(\mathbb{R}P^{4n+1}; \mathbb{Z}_2)$$

is an isomorphism. We can construct an isomorphism explicitly as follows: Let η be the tautological complex line bundle on $\mathbb{C}P^{4n+1}$ such that

$$c_1(\eta) = y \in H^2(\mathbb{C}P^{4n+1}; \mathbb{Z})$$

is a generator. We can easily see that

$$\eta|_{\mathbb{R}P^{4n+1}} = \xi \oplus \xi,$$

where ξ is the real line bundle over $\mathbb{R}P^{4n+1}$ chosen as above. Since $c_1(\eta)$ is the Euler class which reduces to the second Stiefel–Whitney class under $\mathbb{Z}_2$–reduction, $y \mapsto x^2$ under the above isomorphism. But (3-4) shows that $x^2 = w_2(\mathbb{R}P^{4n+1})$. This proves that $\mathbb{R}P^{4n+1}$ is relatively spin: for st, we take $st = y$.

Now we examine the relative spin structures of $\mathbb{R}P^{4n+1}$. It is easy to check that $H^2(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and so there are two inequivalent relative spin structures by Proposition 3.5. Let $st = y$ and take

$$V = \eta^{\oplus 2n+1} \oplus \mathbb{R}$$

for the vector bundle V, noting $w_2(V) \equiv c_1(V) = (2n+1)y = y = st \mod 2$.

Next we have the isomorphism

$$\widetilde{\sigma}\colon T\mathbb{R}P^{4n+1} \oplus \mathbb{R}^2 \cong \xi^{\oplus(4n+2)} \oplus \mathbb{R} \cong \eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R},$$

and so it induces a trivialization of

$$\begin{aligned}(T\mathbb{R}P^{4n+1} \oplus V) \oplus \mathbb{R}^2 &\cong T\mathbb{R}P^{4n+1} \oplus \mathbb{R}^2 \oplus V \\ &\cong (\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}) \oplus (\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}).\end{aligned}$$

We note that on a 2–dimensional CW complex, any stable isomorphism between two oriented real vector bundles V_1 and V_2 induces a stable trivialization of $V_1 \oplus V_2$. In particular, $(\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R}) \oplus (\eta^{\oplus(2n+1)}|_{\mathbb{R}P^{4n+1}} \oplus \mathbb{R})$ has a canonical stable trivialization on the 2–skeleton of $\mathbb{R}P^{4n+1}$, which in turn provides a spin structure on $T\mathbb{R}P^{4n+1} \oplus V$ denoted by σ. This provides a relative spin structure on $\mathbb{R}P^{4n+1} \subset \mathbb{C}P^{4n+1}$.

Next we study the question on the τ–relatively spin property. By the definition of the tautological line bundle η on $\mathbb{C}P^{4n+1}$, the involution τ lifts to an anticomplex linear isomorphism of η, which we denote

$$c\colon \eta \to \eta.$$

Then

$$c^{\oplus(2n+1)} \oplus (-1)\colon \eta^{\oplus(2n+1)} \oplus \mathbb{R} \to \eta^{\oplus(2n+1)} \oplus \mathbb{R}$$

is an isomorphism which covers τ. Therefore, we may identify

$$\tau^* V = V = \eta^{\oplus(2n+1)} \oplus \mathbb{R}$$

on $\mathbb{R}P^{4n+1}$, and also

$$\tau^* = c^{\oplus(2n+1)} \oplus (-1).$$

Then we have

$$\tau^*(V, \sigma, st) = (V, \sigma', st),$$

where the spin structure σ' corresponds to the isomorphism

$$\big(c^{\oplus(2n+1)} \oplus (-1)\big) \circ \widetilde{\sigma}.$$

Therefore, to complete the proof of Proposition 3.14 it suffices to show that the restriction of $c^{\oplus(2n+1)} \oplus (-1)$ to $(\mathbb{R}P^{4n+1})_{[2]}$ is not stably homotopic to the identity map as a bundle isomorphism.

Note that the 2–skeleton $(\mathbb{R}P^{4n+1})_{[2]}$ is $\mathbb{R}P^2$. We have $\pi_1(SO(m)) \cong \mathbb{Z}_2$ and $\pi_2(SO(m)) = 1$ for $m > 2$. Hence an oriented isomorphism of real vector bundles on $(\mathbb{R}P^{4n+1})_{[2]}$ is stably homotopic to the identity if it is so on the 1–skeleton $S^1 = (\mathbb{R}P^{4n+1})_{[1]}$.

It is easy to see that $c \oplus c$ is homotopic to the identity. So it remains to consider $c \oplus -1\colon \eta \oplus \mathbb{R} \to \eta \oplus \mathbb{R}$ on S^1. Note that $\eta|_{S^1} = \xi \oplus \xi$ and this bundle is trivial. The splitting corresponds to the basis $(\cos t/2, \sin t/2)$, $(-\sin t/2, \cos t/2)$. (Here $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$.) The map c is given by $c = (1,-1)\colon \xi \oplus \xi \to \xi \oplus \xi$. So when we identify $\eta \oplus \mathbb{R} \cong \mathbb{R}^3$ on S^1, the isomorphism $c \oplus -1$ is represented by the matrix

$$\begin{pmatrix} \cos t/2 & \sin t/2 & 0 \\ -\sin t/2 & \cos t/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos t/2 & -\sin t/2 & 0 \\ \sin t/2 & \cos t/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos t & -\sin t & 0 \\ -\sin t & -\cos t & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This loop represents the nontrivial homotopy class in $\pi_1(SO(3)) \cong \mathbb{Z}_2$. This proves that the involution $\tau^*\colon \mathrm{Spin}(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}) \to \mathrm{Spin}(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1})$ is nontrivial. Since $\mathrm{Spin}(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}) \cong \mathbb{Z}_2$, the proof of Proposition 3.14 is complete. □

Using the results in this section, we calculate Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ in Section 6.4 (see (6-2) for the definition of $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$), which provides an example of Floer cohomology that is *not* isomorphic to the ordinary cohomology.

3.3 Orientations on $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta;\vec{Q},\vec{P})$

In this subsection, we recall the definitions of the orientations of $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta;\vec{Q},\vec{P})$ from Sections 8.4 and 8.10.2 of [10]. Here L is not necessarily the fixed point set of an antisymplectic involution τ.

When we discuss the orientation problem, it suffices to consider the regular parts of the moduli spaces. See Remark 3.7(2). By Theorem 3.6, we have an orientation on $\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ for each stable conjugacy class of relative spin structure. Including marked points, we define an orientation on $\mathcal{M}_{k+1,m}^{\mathrm{reg}}(J;\beta)$ by

$$\begin{aligned}\mathcal{M}_{k+1,m}^{\mathrm{reg}}&(J;\beta)\\ &= \big(\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta) \times \partial D_0^2 \times D_1^2 \times \cdots \times D_m^2 \times \partial D_{m+1}^2 \times \cdots \times \partial D_{m+k}^2\big)/\,\mathrm{PSL}(2,\mathbb{R}).\end{aligned}$$

Here the subindices in ∂D^2_0 and ∂D^2_{m+i} (resp. D^2_j) stand for the positions of the marked points z_0 and z_i (resp. z_j^+). (In Section 8.10.2 of [10] we write the above space as $\mathcal{M}_{(1,k),m}(\beta)$.) Strictly speaking, since the marked points are required to be distinct, the left-hand side above is not exactly equal to the right-hand side but is an open subset. However, when we discuss the orientation problem, we sometimes do not distinguish them when no confusion can occur.

In (2-3) and (2-4), we define $\mathcal{M}_{k+1}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}(J;\beta;\vec{Q},\vec{P})$ by fiber products. Now we equip the right-hand sides in (2-3) and (2-4) with the fiber product orientations using Convention 8.2.1(3) in [10]. However, we do not use the fiber product orientations themselves as the orientations on $\mathcal{M}_{k+1}(J;\beta;\vec{P})$ and $\mathcal{M}_{k+1,m}(J;\beta;\vec{Q},\vec{P})$, but we use the following orientations twisted from the fiber product orientation: We put $\deg P_i = n - \dim P_i$ and $\deg Q_j = 2n - \dim Q_j$ for smooth singular simplices $f_i\colon P_i \to L$ and $g_j\colon Q_j \to M$.

Definition 3.15 [10, Definition 8.4.1] For given smooth singular simplices $f_i\colon P_i \to L$, we define an orientation on $\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_k)$ by

$$\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_k) := (-1)^{\epsilon(P)}\mathcal{M}_{k+1}(J;\beta)_{(\mathrm{ev}_1,\dots,\mathrm{ev}_k)} \times_{f_1\times\cdots\times f_k} \Big(\prod_{i=1}^k P_i\Big),$$

where

$$\epsilon(P) = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j} \deg P_i.$$

Definition 3.16 [10, Definition 8.10.2] Given smooth singular simplices $f_i\colon P_i \to L$ in L and $g_j\colon Q_j \to M$ in M, we define

$$\begin{aligned}&\mathcal{M}_{k+1,m}(J;\beta;Q_1,\dots,Q_m;P_1,\dots,P_k)\\ &\quad := (-1)^{\epsilon(P,Q)}\mathcal{M}_{k+1,m}(J;\beta)_{(\mathrm{ev}_1^+,\dots,\mathrm{ev}_m^+,\mathrm{ev}_1,\dots,\mathrm{ev}_k)} \times_{g_1\times\cdots\times f_k} \Big(\prod_{j=1}^m Q_j \times \prod_{i=1}^k P_i\Big),\end{aligned}$$

where

$$\epsilon(P,Q) = (n+1)\sum_{j=1}^{k-1}\sum_{i=1}^{j}\deg P_i + \big((k+1)(n+1)+1\big)\sum_{j=1}^{m}\deg Q_j. \tag{3-5}$$

Replacing $\mathcal{M}_{k+1}(J;\beta)$ and $\mathcal{M}_{k+1,m}(J;\beta)$ on the right-hand sides of above definitions by $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta)$ and $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$ respectively, we similarly define orientations on the main components

$$\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;P_1,\dots,P_k) \quad \text{and} \quad \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta;Q_1,\dots,Q_m;P_1,\dots,P_k).$$

When we do not consider the fiber product with $g_j\colon Q_j \to M$, we drop the second term in (3-5). Thus when $m = 0$, the moduli space in Definition 3.16 is nothing but $\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_k)$ equipped with the orientation given by Definition 3.15.

When we study the map τ_*^{main} in (1-1), we have to change the ordering of boundary marked points. Later we use the following lemma which describes the behavior of orientations under the change of ordering of boundary marked points:

Lemma 3.17 [10, Lemma 8.4.3] *Let σ be the transposition element $(i, i+1)$ in the k^{th} symmetric group $\mathfrak{S}_k$ for $i = 1,\dots,k-1$. Then the action of σ on the moduli space $\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_i,P_{i+1},\dots,P_k)$ changing the order of marked points induces an orientation-preserving isomorphism*

$$\begin{aligned}\sigma\colon \mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_i,P_{i+1},\dots,P_k)&\\ \to (-1)^{(\deg P_i+1)(\deg P_{i+1}+1)}&\mathcal{M}_{k+1}(J;\beta;P_1,\dots,P_{i+1},P_i,\dots,P_k).\end{aligned}$$

4 The induced maps τ_* and τ_*^{main}

Let (M,ω) be a compact, or tame, symplectic manifold, and let $\tau\colon M \to M$ be an antisymplectic involution, ie a map satisfying $\tau^2 = \mathrm{id}$ and $\tau^*(\omega) = -\omega$. We also assume that the fixed point set $L = \mathrm{Fix}\,\tau$ is nonempty, oriented and compact.

Let $\mathcal{J}_\omega$ be the set of all ω–compatible almost complex structures and $\mathcal{J}_\omega^\tau$ its subset consisting of τ–anti-invariant almost complex structures J satisfying $\tau_*J = -J$.

Lemma 4.1 ([6, Lemma 11.3]; see also [24, Proposition 1.1]) *The space $\mathcal{J}_\omega^\tau$ is nonempty and contractible. It becomes an infinite-dimensional (Fréchet) manifold.*

Proof For a given $J \in \mathcal{J}_\omega^\tau$, its tangent space $T_J\mathcal{J}_\omega^\tau$ consists of sections Y of the bundle $\mathrm{End}(TM)$ whose fiber at $p \in M$ is the space of linear maps $Y\colon T_pM \to T_pM$ such that

$$YJ + JY = 0, \quad \omega(Yv,w) + \omega(v,Yw) = 0, \quad \tau^*Y = -Y.$$

Note that the second condition means that JY is a symmetric endomorphism with respect to the metric $g_J = \omega(\cdot, J\cdot)$. It immediately follows that $\mathcal{J}_\omega^\tau$ becomes a manifold. The fact that $\mathcal{J}_\omega^\tau$ is nonempty (and contractible) follows from the polar decomposition theorem by choosing a τ–invariant Riemannian metric on M. □

4.1 The map τ_* and orientation

We recall the definition of $\Pi(L) = \pi_2(M, L)/\sim$ where the equivalence relation is defined by $\beta \sim \beta' \in \pi_2(M, L)$ if and only if $\omega(\beta) = \omega(\beta')$ and $\mu_L(\beta) = \mu_L(\beta')$; see (2-1). We notice that for each $\beta \in \Pi(L)$, we defined the moduli space $\mathcal{M}(J;\beta)$ as the union

$$\mathcal{M}(J;\beta) = \bigcup_{\substack{B\in\pi_2(M,L)\\ [B]=\beta\in\Pi(L)}} \mathcal{M}(J;B). \tag{4-1}$$

We put $D^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$, and $\overline{z}$ denotes the complex conjugate.

Definition 4.2 Let $J \in \mathcal{J}_\omega^\tau$. For J–holomorphic curves $w\colon (D^2, \partial D^2) \to (M, L)$ and $u\colon S^2 \to M$, we define $\widetilde{w}$ and $\widetilde{u}$ by

$$\widetilde{w}(z) = (\tau\circ w)(\overline{z}), \quad \widetilde{u}(z) = (\tau\circ u)(\overline{z}). \tag{4-2}$$

For $(D^2, w) \in \mathcal{M}^{\mathrm{reg}}(J;\beta)$ and $((D^2, \vec{z}, \vec{z}^{+}), w) \in \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$, we define

$$\tau_*((D^2, w)) = (D^2, \widetilde{w}), \quad \tau_*\big(((D^2, \vec{z}, \vec{z}^{+}), w)\big) = ((D^2, \vec{\overline{z}}, \vec{\overline{z}}^{+}), \widetilde{w}), \tag{4-3}$$

where

$$\vec{\overline{z}} = (\overline{z}_0, \dots, \overline{z}_k), \quad \vec{\overline{z}}^{+} = (\overline{z}_0^{+}, \dots, \overline{z}_m^{+}).$$

Remark 4.3 For $\beta = [w]$, we put $\tau_*\beta = [\widetilde{w}]$. Note if $\tau_\sharp\colon \pi_2(M, L) \to \pi_2(M, L)$ is the natural homomorphism induced by τ, then

$$\tau_*\beta = -\tau_\sharp\beta.$$

This is because $z \mapsto \overline{z}$ is of degree -1. In fact, we have

$$\tau_*(\beta) = \beta$$

in $\Pi(L)$ since τ_* preserves both the symplectic area and the Maslov index.

Lemma 4.4 *The definition (4-3) induces the maps*

$$\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta) \to \mathcal{M}^{\mathrm{reg}}(J;\beta), \quad \tau_*\colon \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta) \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta),$$

which satisfy $\tau_*\circ\tau_* = \mathrm{id}$.

Proof If $(w, (z_0, \dots, z_k), (z_1^{+}, \dots, z_m^{+})) \sim (w', (z'_0, \dots, z'_k), (z'^{+}_1, \dots, z'^{+}_m))$, we have $\varphi \in \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Aut}(D^2)$ such that $w' = w\circ\varphi^{-1}$, $z'_i = \varphi(z_i)$ and $z'^{+}_i = \varphi(z_i^{+})$ by definition. We define $\overline{\varphi}\colon D^2 \to D^2$ by

$$\overline{\varphi}(z) = \overline{(\varphi(\overline{z}))}. \tag{4-4}$$

Then $\overline{\varphi} \in \mathrm{PSL}(2,\mathbb{R})$, $\widetilde{w}' = \widetilde{w} \circ \overline{\varphi}^{-1}$, $\overline{z}'_i = \overline{\varphi}(\overline{z}_i)$ and $\overline{z}'^{+}_i = \overline{\varphi}(\overline{z}^{+}_i)$. The property $\tau_* \circ \tau_* = \mathrm{id}$ is straightforward. □

We note that the mapping $\varphi \mapsto \overline{\varphi}$, $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is orientation-preserving.

Proposition 4.5 *The involution τ_* is induced by an automorphism of $\mathcal{M}^{\mathrm{reg}}(J;\beta)$ as a space with Kuranishi structure.*

Proposition 4.5 is a special case of Theorem 4.10(1). The proof of Theorem 4.10(1) is similar to the proof of Theorem 4.11 which will be proved in Section 5. See Definition A.3 for the definition of an automorphism of a space with Kuranishi structure and Definition A.6 for the definition for an automorphism that is *orientation-preserving* in the sense of Kuranishi structure. In this paper, we use the terminology *orientation-preserving* only in the sense of Kuranishi structure. We refer to Section A1.3 of [10] for a more detailed explanation of the group action on a space with Kuranishi structure.

In Section 3, we explained that a choice of stable conjugacy class $[(V,\sigma)] \in \mathrm{Spin}(M,L)$ of relative spin structure on L induces an orientation on $\mathcal{M}_{k+1,m}(J;\beta)$ for any given $\beta \in \Pi(L)$. Hereafter, we equip $\mathcal{M}_{k+1,m}(J;\beta)$ with this orientation when we regard it as a space with oriented Kuranishi structure. We write it as $\mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ when we specify the stable conjugacy class of relative spin structure.

For an antisymplectic involution τ of (M,ω), we have the pull-back $\tau^*[(V,\sigma)]$ of the stable conjugacy class of relative spin structure $[(V,\sigma)]$; see (3-3). Then from the definition of the map τ_* in Lemma 4.4, we obtain the maps

$$\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]},$$
$$\tau_*\colon \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}.$$

Here we note that $\tau_* J = -J$, and we use the same τ–antisymmetric almost complex structure J in both the source and the target spaces of the map τ_*. If $[(V,\sigma)]$ is τ–relatively spin (ie $\tau^*[(V,\sigma)] = [(V,\sigma)]$), τ_* defines involutions of $\mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$ and $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ with Kuranishi structures.

Theorem 4.6 *Let L be a fixed point set of an antisymplectic involution τ and $J \in \mathcal{J}^{\tau}_{\omega}$. Suppose that L is oriented and carries a relative spin structure (V,σ). Then the map $\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$ is orientation-preserving if $\mu_L(\beta) \equiv 0 \bmod 4$ and is orientation-reversing if $\mu_L(\beta) \equiv 2 \bmod 4$.*

Corollary 4.7 *Let L be as in Theorem 4.6. If, in addition, L carries a τ–relative spin structure $[(V,\sigma)]$, then the map $\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$ is orientation-preserving if $\mu_L(\beta) \equiv 0 \bmod 4$ and is orientation-reversing if $\mu_L(\beta) \equiv 2 \bmod 4$.*

We prove Theorem 4.6 in Section 5. Here we give a couple of examples.

Example 4.8 (1) Consider the case of $M = \mathbb{C}P^n$, $L = \mathbb{R}P^n$. In this case, each Maslov index $\mu_L(\beta)$ has the form

$$\mu_L(\beta) = \ell_\beta(n+1),$$

where $\beta = \ell_\beta$ times the generator. We know that when n is even, L is not orientable, and so we consider only the case where n is odd. On the other hand, when n is odd, L is relatively spin. Moreover, we have proved in Proposition 3.14 that $\mathbb{R}P^{4n+3}$ $(n \geq 0)$ is τ–relatively spin (indeed, $\mathbb{R}P^{4n+3}$ is spin), but $\mathbb{R}P^{4n+1}$ $(n \geq 1)$ is *not* τ–relatively spin. Then using the above formula for the Maslov index, we can conclude from Theorem 4.6 that the map $\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$ is always an orientation-preserving involution for any τ–relative spin structure $[(V,\sigma)]$ of $\mathbb{R}P^{4n+3}$.

Of course, $\mathbb{R}P^1$ is spin and so τ–relatively spin. The map τ_* is an orientation-preserving involution if ℓ_β is even, and an orientation-reversing involution if ℓ_β is odd.

(2) Let M be a Calabi–Yau 3–fold, and let $L \subset M$ be the set of real points (ie the fixed point set of an antiholomorphic involutive isometry). In this case, L is orientable (because it is a special Lagrangian) and spin (because any orientable 3–manifold is spin). Furthermore, $\mu_L(\beta) = 0$ for any $\beta \in \pi_2(M, L)$. Therefore, Theorem 4.6 implies that the map $\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$ is orientation-preserving for any τ–relative spin structure $[(V,\sigma)]$.

We next include marked points. We consider the moduli space $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$.

Proposition 4.9 *The map $\tau_*\colon \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ is orientation-preserving if and only if $\mu_L(\beta)/2 + k + 1 + m$ is even.*

Proof Assuming Theorem 4.6, we prove Proposition 4.9. Let us consider the diagram:

$$\begin{tikzcd}
(S^1)^{k+1} \times (D^2)^m \arrow[r, "c"] & (S^1)^{k+1} \times (D^2)^m \\
((S^1)^{k+1} \times (D^2)^m)_0 \arrow[r, "c"] \arrow[u, "\text{inclusion}"] \arrow[d] & ((S^1)^{k+1} \times (D^2)^m)_0 \arrow[u, "\text{inclusion}"'] \arrow[d] \\
\widetilde{\mathcal{M}}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \arrow[r, "\text{Proposition 4.9}"] \arrow[d, "\mathfrak{forget}"'] & \widetilde{\mathcal{M}}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]} \arrow[d, "\mathfrak{forget}"'] \\
\widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \arrow[r, "\text{Theorem 4.6}"] & \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}
\end{tikzcd}$$

Here c is defined by

$$c(z_0, z_1, \dots, z_k, z_1^+, \dots, z_m^+) = (\overline{z}_0, \overline{z}_1, \dots, \overline{z}_k, \overline{z}_1^+, \dots, \overline{z}_m^+),$$

the $\mathfrak{forget}$ are forgetful maps of marked points, and we let $((S^1)^{k+1} \times (D^2)^m)_0$ denote the set of all $c(z_0, z_1, \dots, z_k, z_1^+, \dots, z_m^+)$ such that $z_i \neq z_j$ and $z_i^+ \neq z_j^+$ for $i \neq j$.

Proposition 4.9 then follows from Theorem 4.6 and the fact that the $\mathbb{Z}_2$–action $\varphi \mapsto \overline{\varphi}$ on $\mathrm{PSL}(2, \mathbb{R})$ given by (4-4) is orientation-preserving. $\square$

We next extend τ_* to the compactification $\mathcal{M}_{k+1,m}(J;\beta)$ of $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$ and define the continuous map (4-5).

Theorem 4.10 (1) *The map* $\tau_*\colon \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ *extends to an automorphism* τ_* *between spaces with Kuranishi structures, denoted by the same symbol:*

$$\tau_*\colon \mathcal{M}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}. \tag{4-5}$$

(2) *It preserves orientation if and only if* $\frac{1}{2}\mu_L(\beta) + k + 1 + m$ *is even. In particular, if* $[(V,\sigma)]$ *is a* τ*–relative spin structure, it can be regarded as an involution on the space* $\mathcal{M}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ *with Kuranishi structure.*

Proof (1) The proof of (1) is given right after the proof of Theorem 4.11 in Section 5.

(2) The statement follows from the corresponding statement on $\mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$ in Proposition 4.9. For $((\Sigma, \vec{z}, \vec{z}^{\,+}), w) \in \mathcal{M}_{k+1,m}(J,\beta)$, we denote by

$$((\Sigma_i, \vec{z}^{\,(i)}, \vec{z}^{\,+(i)}), w_{(i)}) \in \mathcal{M}^{\mathrm{reg}}_{k_i+1,m_i}(J, \beta_{(i)}) \quad \text{and} \quad ((\Sigma_j^S, \vec{z}^{\,+(j)S}), u_{(j)}) \in \mathcal{M}^{\mathrm{sph}}_{\ell_j}(\alpha)$$

the irreducible disc components and the irreducible sphere components, respectively.

- By Proposition 4.9, we find that τ_* respects the orientation of $\mathcal{M}^{\mathrm{reg}}_{k_i+1,m_i}(J,\beta_{(i)})$ if and only if $\frac{1}{2}\mu(\beta_i) + k_i + 1 + m_i$ is even.
- In the same way, we find that τ_* respects the orientation of $\mathcal{M}^{\mathrm{sph}}_{\ell_j}(\alpha)$ if and only if $n + c_1(M)[\alpha] + \ell_j - 3$ is even.
- $m \equiv \sum_i m_i + \sum_j \ell_j \bmod 2$, and $k + 1 \equiv \sum_i (k_i + 1) \bmod 2$.
- The number of interior nodes is equal to the number of sphere components since Σ is a bordered stable curve of genus 0 such that $\partial\Sigma$ is connected.
- The involution τ_* acts on the space of parameters for smoothing interior nodes, and is orientation-preserving if and only if the number of interior nodes is even.
- The fiber product is taken over either L or M. The involution τ respects the orientation of M if and only if n is even.

Combining these with Lemma 8.2.3(4) in [10], we obtain that τ_* respects the orientation on $\mathcal{M}_{k+1,m}(J,\beta)$ if and only if $\frac{1}{2}\mu(\beta)+k+1+m$ is even. Hence we obtain the second statement of the theorem. □

4.2 The map τ_*^{main} and orientation

We next restrict our maps to the main component of $\mathcal{M}_{k+1,m}(J;\beta)$. As we mentioned before, we observe that the induced map $\tau_*\colon \mathcal{M}_{k+1,m}(J;\beta)\to\mathcal{M}_{k+1,m}(J;\beta)$ does *not* preserve the main component for $k>1$. On the other hand, the assignment given by

$$(w,\vec{z},\vec{z}^{+}) = \big(w,(z_0,z_1,z_2,\dots,z_{k-1},z_k),(z_1^+,\dots,z_m^+)\big) \mapsto (\widetilde{w},\vec{\overline{z}}^{\mathrm{rev}},\vec{\overline{z}}^{+}) = \big(\widetilde{w},(\overline{z}_0,\overline{z}_k,\overline{z}_{k-1},\dots,\overline{z}_2,\overline{z}_1),(\overline{z}_1^+,\dots,\overline{z}_m^+)\big) \tag{4-6}$$

respects the counterclockwise cyclic order of $S^1=\partial D^2$ and so preserves the main component, where $\widetilde{w}$ is as in (4-2). Therefore, we consider this map instead, which we denote by

$$\tau_*^{\mathrm{main}}\colon \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]}\to\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}. \tag{4-7}$$

We note that for $k=0,1$, we have

$$\tau_*^{\mathrm{main}}=\tau_*. \tag{4-8}$$

Theorem 4.11 *The map τ_*^{main} is induced by an automorphism between the spaces with Kuranishi structures and satisfies $\tau_*^{\mathrm{main}}\circ\tau_*^{\mathrm{main}}=\mathrm{id}$. In particular, if $[(V,\sigma)]$ is τ–relatively spin, it defines an involution of the space $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ with Kuranishi structure.*

The proof will be given in Section 5.

We now have the following commutative diagram:

$$\begin{tikzcd}
(S^1)^{k+1}\times(D^2)^m \arrow[r, "c'"] & (S^1)^{k+1}\times(D^2)^m \\
((S^1)^{k+1}\times(D^2)^m)_{00} \arrow[r, "c'"] \arrow[u, "\text{inclusion}"] \arrow[d] & ((S^1)^{k+1}\times(D^2)^m)_{00} \arrow[u, "\text{inclusion}"'] \arrow[d] \\
\widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \arrow[r, "\tau_*^{\mathrm{main}}"] \arrow[d, "\mathfrak{forget}"'] & \widetilde{\mathcal{M}}^{\mathrm{main}}_{k+1,m}(J;\beta)^{[(V,\sigma)]} \arrow[d, "\mathfrak{forget}"'] \\
\widetilde{\mathcal{M}}(J;\beta)^{\tau^*[(V,\sigma)]} \arrow[r, "\tau_*"] & \widetilde{\mathcal{M}}(J;\beta)^{[(V,\sigma)]}
\end{tikzcd}$$

Here c' is defined by

$$c'(z_0, z_1, \dots, z_k, z_1^+, \dots, z_m^+) = (\overline{z}_0, \overline{z}_k, \dots, \overline{z}_1, \overline{z}_1^+, \dots, \overline{z}_m^+),$$

and the $\mathfrak{forget}$ are the forgetful maps of marked points. Also, $((S^1)^{k+1} \times (D^2)^m)_{00}$ is the open subset of $(S^1)^{k+1} \times (D^2)^m$ consisting of the points such that all the z_i and z_j^+ are respectively distinct.

Let $\mathrm{Rev}_k\colon L^{k+1} \to L^{k+1}$ be the map defined by

$$\mathrm{Rev}_k(x_0, x_1, \dots, x_k) = (x_0, x_k, \dots, x_1).$$

It is easy to see that

$$\mathrm{ev} \circ \tau_*^{\mathrm{main}} = \mathrm{Rev}_k \circ \mathrm{ev}\,. \tag{4-9}$$

We note again that $\mathrm{Rev}_k = \mathrm{id}$ and $\tau_*^{\mathrm{main}} = \tau_*$ for $k = 0, 1$.

Let $P_1, \dots, P_k$ be smooth singular simplices on L. By taking the fiber product and using (4-6), we obtain a map

$$\tau_*^{\mathrm{main}}\colon \mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta;P_k,\dots,P_1)^{[(V,\sigma)]} \tag{4-10}$$

which satisfies $\tau_*^{\mathrm{main}} \circ \tau_*^{\mathrm{main}} = \mathrm{id}$. We put

$$\epsilon = \tfrac{1}{2}\mu_L(\beta) + k + 1 + m + \sum_{1 \le i < j \le k} \deg' P_i \deg' P_j\,. \tag{4-11}$$

Theorem 4.12 *The map (4-10) preserves orientation if ϵ is even and reverses orientation if ϵ is odd.*

The proof of Theorem 4.12 is given in Section 5.

5 Proofs of Theorems 4.6, 4.10(1), 4.11 and 4.12

In this section, we prove Theorems 4.6 (1.1), 4.10(1), 4.11 and 4.12 (1.4) stated in the previous sections.

Proof of Theorem 4.6 Pick $J \in \mathcal{J}_\omega^\tau$, a τ–anti-invariant almost complex structure compatible with ω. For a J–holomorphic curve $w\colon (D^2, \partial D^2) \to (M, L)$, we recall that we define $\widetilde{w}$ by

$$\widetilde{w}(z) = (\tau \circ w)(\overline{z}).$$

Moreover, for $(D^2, w) \in \mathcal{M}^{\mathrm{reg}}(J;\beta)$ and $((D^2, \vec{z}, \vec{z}^+), w) \in \mathcal{M}^{\mathrm{reg}}_{k+1,m}(J;\beta)$, we define

$$\tau_*((D^2, w)) = (D^2, \widetilde{w}), \quad \tau_*\big(((D^2, \vec{z}, \vec{z}^+), w)\big) = ((D^2, \vec{\overline{z}}, \vec{\overline{z}}^+), \widetilde{w}),$$

where

$$\vec{\overline{z}} = (\overline{z}_0, \dots, \overline{z}_k), \quad \vec{\overline{z}}^+ = (\overline{z}_0^+, \dots, \overline{z}_m^+).$$

Let $[D^2, w] \in \mathcal{M}^{\mathrm{reg}}(J;\beta)$. We consider the deformation complexes

$$D_w\overline{\partial}\colon \Gamma(D^2, \partial D^2 : w^*TM, w|_{\partial D^2}^* TL) \to \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM), \tag{5-1}$$

$$D_{\widetilde{w}}\overline{\partial}\colon \Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL) \to \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM), \tag{5-2}$$

where $D_w\overline{\partial}$ is the linearized operator of the pseudoholomorphic curve equation as in (3-1). (Here and hereafter, $\Lambda^1 = \Lambda^{1,0} \oplus \Lambda^{0,1}$ is the decomposition of the complexified cotangent bundle of the *domain* of pseudoholomorphic curves.)

We have the commutative diagram

$$\begin{tikzcd} (w^*TM, w|_{\partial D^2}^* TL) \arrow[r, "T\tau"] \arrow[d] & (\widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL) \arrow[d] \\ (D^2, \partial D^2) \arrow[r, "c"] & (D^2, \partial D^2) \end{tikzcd} \tag{5-3}$$

where $c(z) = \overline{z}$, and we denote by $T\tau$ the differential of τ. It induces a bundle map

$$\mathrm{Hom}_{\mathbb{R}}(TD^2, w^*TM) \to \mathrm{Hom}_{\mathbb{R}}(TD^2, \widetilde{w}^*TM),$$

which covers $z \mapsto \overline{z}$. This bundle map is fiberwise anticomplex linear, ie

$$\mathrm{Hom}_{\mathbb{R}}(T_z D^2, T_{w(z)} M) \to \mathrm{Hom}_{\mathbb{R}}(T_{\overline{z}} D^2, T_{\tau(w(z))} M)$$

is anticomplex linear at each $z \in D^2$ with respect to both of the complex structures $a \mapsto J \circ a$ and $a \mapsto a \circ j$. Therefore, it preserves the decomposition

$$\mathrm{Hom}_{\mathbb{R}}(TD^2, w^*TM) \otimes \mathbb{C} = (\Lambda^{1,0} \otimes w^*TM) \oplus (\Lambda^{0,1} \otimes w^*TM) \tag{5-4}$$

since (5-4) is the decomposition to the complex and anticomplex linear parts. Hence we obtain a map

$$(T_{w,1}\tau)_*\colon \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \to \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM), \tag{5-5}$$

which is anticomplex linear. In a similar way, we obtain an anticomplex linear map

$$(T_{w,0}\tau)_*\colon \Gamma(D^2, \partial D^2 : w^*TM, w|_{\partial D^2}^* TL) \to \Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|_{\partial D^2}^* TL).$$

Since τ is an isometry, it commutes with the covariant derivative. This gives rise to the following commutative diagram:

$$\begin{array}{ccc}
\Gamma(D^2, \partial D^2 : w^*TM, w|^*_{\partial D^2}TL) & \xrightarrow{D_w\overline{\partial}} & \Gamma(D^2; \Lambda^{0,1} \otimes w^*TM) \\
{\scriptstyle (T_{w,0}\tau)_*}\big\downarrow & & \big\downarrow{\scriptstyle (T_{w,1}\tau)_*} \\
\Gamma(D^2, \partial D^2 : \widetilde{w}^*TM, \widetilde{w}|^*_{\partial D^2}TL) & \xrightarrow{D_{\widetilde{w}}\overline{\partial}} & \Gamma(D^2; \Lambda^{0,1} \otimes \widetilde{w}^*TM)
\end{array}$$

We study the orientation. Let $w \in \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$ and consider $\widetilde{w} \in \widetilde{\mathcal{M}}^{\mathrm{reg}}(J;\beta)$. We consider the commutative diagram (5-3). A trivialization

$$\Phi\colon (w^*TM, w|^*_{\partial D^2}TL) \to (D^2, \partial D^2; \mathbb{C}^n, \Lambda)$$

naturally induces a trivialization

$$\widetilde{\Phi}\colon (\widetilde{w}^*TM, \widetilde{w}|^*_{\partial D^2}TL) \to (D^2, \partial D^2; \mathbb{C}^n, \widetilde{\Lambda}),$$

where $\Lambda\colon S^1 \simeq \partial D^2 \to \Lambda(\mathbb{C}^n)$ is a loop of Lagrangian subspaces given by $\Lambda(z) := T_{w(z)}L$ in the trivialization, and $\widetilde{\Lambda}$ is defined by

$$\widetilde{\Lambda}(z) = \overline{\Lambda(\overline{z})}. \tag{5-6}$$

With respect to these trivializations, we have the commutative diagram

$$\begin{array}{ccc}
(D^2, \partial D^2; \mathbb{C}^n, \Lambda) & \xrightarrow{\widetilde{\Phi}\circ T\tau\circ\Phi^{-1}} & (D^2, \partial D^2; \mathbb{C}^n, \widetilde{\Lambda}) \\
\big\downarrow & & \big\downarrow \\
(D^2, \partial D^2) & \xrightarrow{\quad c \quad} & (D^2, \partial D^2)
\end{array}$$

and the elliptic complexes (5-1) and (5-2) are identified with $\overline{\partial}_{(D^2,\partial D^2;\mathbb{C}^n,\Lambda)}$ and $\overline{\partial}_{(D^2,\partial D^2;\mathbb{C}^n,\widetilde{\Lambda})}$, respectively. The relative spin structure $\tau^*[(V,\sigma)]$ (resp. $[(V,\sigma)]$) determines a trivialization $\Lambda \cong \partial D^2 \times \mathbb{R}^n$ (resp. $\widetilde{\Lambda} \cong \partial D^2 \times \mathbb{R}^n$) unique up to homotopy. These trivializations are compatible with $\widetilde{\Phi}\circ T\tau\circ\Phi^{-1}$ in the above diagram.

We recall the argument explained in Remark 3.9. We have the complex vector bundle E' over the nodal curve $\Sigma = D^2 \cup \mathbb{C}P^1$ with a nodal point $D^2 \ni O = p \in \mathbb{C}P^1$. The topology of the bundle $E'|_{\mathbb{C}P^1} \to \mathbb{C}P^1$ is determined by the loop Λ of Lagrangian subspaces defined by $\Lambda(z) = T_{w(z)}L$ in the trivialization. The Cauchy–Riemann operator $\overline{\partial}_{(D^2\times\mathbb{C}^n,\partial D^2\times\mathbb{R}^n)}$ is surjective, and the Cauchy–Riemann operator

$$\overline{\partial}_{E'|_{\mathbb{C}P^1}}\colon \Gamma(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) \to \Gamma(\mathbb{C}P^1; \Lambda^{0,1}\otimes E'|_{\mathbb{C}P^1})$$

is approximated by a finite-dimensional model, namely 0–map: $H^0(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) \to H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$, where $H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$ is regarded as the obstruction bundle.

For a later purpose, we take a *stabilization* of this finite-dimensional model so that the evaluation at p is surjective to E'_p. Namely, we take a finite-dimensional complex linear subspace $\mathbb{V}^+ \subset \Gamma(\mathbb{C}P^1; \Lambda^{0,1} \otimes E'|_{\mathbb{C}P^1})$ such that the Cauchy–Riemann operator $\overline{\partial}_{E'|_{\mathbb{C}P^1}}$ is surjective modulo $\mathbb{V}^+$, and the evaluation $\mathrm{ev}_p\colon (\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V}^+) \to E'|_p$ at p is surjective. Set

$$\mathbb{V} = \mathbb{V}^+ \cap \Im\overline{\partial}_{E'|_{\mathbb{C}P^1}} \subset \Gamma(\mathbb{C}P^1; \Lambda^{0,1} \otimes E'|_{\mathbb{C}P^1}).$$

Then we have an isomorphism $\mathbb{V}^+ \cong \mathbb{V} \oplus H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})$, and $(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V}^+) = (\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})$. The Cauchy–Riemann operator $\overline{\partial}_{E'|_{\mathbb{C}P^1}}$ has a finite-dimensional approximation by

$$s\colon (\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V}) \to \mathbb{V} \oplus H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}), \quad s(\xi) = (\overline{\partial}_{E'|_{\mathbb{C}P^1}}(\xi), 0).$$

We use the notation in Convention 8.2.1(3)(4) in [10] to describe the kernel of the operator $\overline{\partial}_{(E',\lambda')}$ as the zero set of s in the fiber product of the kernel of $\overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)}$ and $(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})$. We decompose it as follows:

$$(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V}) = E'_p \times {}^{\circ}(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V}).$$

Here ${}^{\circ}(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})$ is the space of sections in $(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})$, which vanish at p. (See (8.2.1.6) in [10] for the notation used here.)

Since $\ker \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)} \cong \mathbb{R}^n$ by (3-2), the complex conjugate induces the trivial action on $\ker \overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)}$. Therefore, $(T\tau)_*\colon \det D_w\overline{\partial} \to \det D_{\widetilde{w}}\overline{\partial}$ is orientation-preserving or orientation-reversing if and only if the same is true of the complex conjugation action on

$$\det(\mathbb{V} \oplus H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}))^* \otimes \det {}^{\circ}\big((\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})\big),$$

which is isomorphic to

$$\det(\mathbb{V} \oplus H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}))^* \otimes \det(E'|_p^*) \otimes \det\big((\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})\big).$$

On the other hand, we observe that

$$\det(\mathbb{V} \oplus H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}))^* \otimes \det\big((\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})\big)$$

is isomorphic to

$$\det(H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}))^* \otimes \det(H^0(\mathbb{C}P^1; E'|_{\mathbb{C}P^1})),$$

on which the complex conjugation acts by the multiplication by $(-1)^{\mu(\Lambda)/2+n}$, since

$$\dim_{\mathbb{C}} H^0(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) - \dim_{\mathbb{C}} H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}) = \tfrac{1}{2}\mu(\Lambda) + n.$$

Here n is the rank of E' as a complex vector bundle. Note also that complex conjugation acts on $E'|_p$ via multiplication by $(-1)^n$. Combining these, we find that complex conjugation acts on

$$\det(\mathbb{V} \oplus H^1(\mathbb{C}P^1; E'|_{\mathbb{C}P^1}))^* \otimes \det{}^\circ\big((\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V})\big)$$

via multiplication by $(-1)^{\mu(\Lambda)/2}$. Note that the action of $T_w\tau$ on the determinant bundle of $D_w\overline{\partial}$ is isomorphic to the conjugation action explained above. Therefore, this map is orientation-preserving if and only if $\frac{1}{2}\mu(\Lambda) \equiv 0 \bmod 2$, ie $\mu(\Lambda) \equiv 0 \bmod 4$. We note that $\mu_L(\beta) = \mu(\Lambda)$ by definition. This finishes the proof of Theorem 4.6. □

The argument above is an adaptation of Lemma 8.3.2(4) in [10] with

$$X_1 = \mathrm{Index}\,\overline{\partial}_{(D^2\times\mathbb{C}^n, \partial D^2\times\mathbb{R}^n)}, \quad X_2 = \mathrm{Index}(\overline{\partial}_{E'|_{\mathbb{C}P^1}})^{-1}(\mathbb{V}) \quad \text{and} \quad Y = E'|_p.$$

The complex conjugation action on X_1 (resp. X_2 or Y) is a $+1$–oriented (resp. $(-1)^{\mu(\Lambda)/2+n}$– or $(-1)^n$–oriented) isomorphism. Hence the action on $X_1 \times_Y X_2$ is a $(-1)^{\mu(\Lambda)/2}$–oriented isomorphism.

Proof of Theorem 4.11 We will extend the map τ_*^{main} (see (4-7)) to an automorphism of Kuranishi structure by a triple induction over $\omega(\beta) = \int_\beta \omega$, k and m. Namely, we define an order on the set of triples (β, k, m) by the relation

(5-7a) $$\omega(\beta') < \omega(\beta);$$

(5-7b) $$\omega(\beta') = \omega(\beta), \quad k' < k;$$

(5-7c) $$\omega(\beta') = \omega(\beta), \quad k' = k, \quad m' < m.$$

We will define the extension of (4-7) for (β, k, m) to an automorphism of Kuranishi structure under the assumption that such an extension is already defined for all (β', k', m') smaller than (β, k, m) with respect to this order.

First, we consider the case that the domain is irreducible, ie $w\colon (D^2, \partial D^2) \to (M, L)$ is a pseudoholomorphic map. Let $((D^2, \vec{z}, \vec{z}^+), w)$ be an element of $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$.

We consider $((D^2, \vec{z}', \vec{z}^{+\prime}), w')$, where $(D^2, \vec{z}', \vec{z}^{+\prime})$ is close to $(D^2, \vec{z}, \vec{z}^+)$ in the moduli space of discs with $k+1$ boundary and m interior marked points. We use a local trivialization of this moduli space to take a diffeomorphism $(D^2, \vec{z}) \cong (D^2, \vec{z}')$. (In case $2m + k < 2$, we take additional interior marked points to stabilize the domain. See the Appendix of [16] and Definition 18.9 in [13].) We assume $w'\colon (D^2, \partial D^2) \to (M, L)$ is C^1–close to w using this identification. To define a Kuranishi chart in a neighborhood of $[((D^2,\vec z,\vec z^+),w)]$, we take a family of finite-dimensional subspaces $E_{[((D^2,\vec z,\vec z^+),w)]}((D^2,\vec z\,',\vec z^{+\prime}),w')$ of $\Gamma(D^2;\Lambda^{0,1}\otimes(w')^*TM)$ such that

$$\operatorname{Im} D_{w'}\overline{\partial}+E_{[((D^2,\vec z,\vec z^+),w)]}((D^2,\vec z\,',\vec z^{+\prime}),w')=\Gamma(D^2;\Lambda^{0,1}\otimes(w')^*TM).$$

The Kuranishi neighborhood $V_{[((D^2,\vec z,\vec z^+),w)]}$ is constructed in Section 7.1 of [10], and is given by the set of solutions of the equation

$$\overline{\partial}w'\equiv 0 \bmod E_{[((D^2,\vec z,\vec z^+),w)]}((D^2,\vec z\,',\vec z^{+\prime}),w'). \tag{5-8}$$

Moreover, we will take it so that the evaluation map

$$V_{[((D^2,\vec z,\vec z^+),w)]}\to L^{k+1}$$

defined by $((D^2,\vec z\,',\vec z^{+\prime}),w')\mapsto(w'(z'_1),\dots,w'(z'_k),w'(z'_0))$ is a submersion.

We choose $E_{[((D^2,\vec z,\vec z^+),w)]}((D^2,\vec z\,',\vec z^{+\prime}),w')$ so that it is invariant under τ^{main} in the following sense. We define $\widetilde w$, $\widetilde w'$, $\vec{\overline{z}}^{+}$ and $\vec{\overline{z}}^{+\prime}$ as in (4-3). We also define $\vec{\overline{z}}^{\mathrm{rev}}$ and $\vec{\overline{z}}^{\prime\mathrm{rev}}$ as in (4-6). (So $\tau_*^{\mathrm{main}}([((D^2,\vec z,\vec z^+),w)])=[((D^2,\vec{\overline{z}}^{\mathrm{rev}},\vec{\overline{z}}^{+}),\widetilde w)]$.) Then we require:

$$\begin{aligned}E_{[((D^2,\vec{\overline{z}}^{\mathrm{rev}},\vec{\overline{z}}^{+}),\widetilde w)]}&([((D^2,\vec{\overline{z}}^{\prime\mathrm{rev}},\vec{\overline{z}}^{+\prime}),\widetilde w')])\\&=(T_{w',1}\tau)_*(E_{[((D^2,\vec z,\vec z^+),w)]}((D^2,\vec z\,',\vec z^{+\prime}),w')).\end{aligned}\tag{5-9}$$

Here $(T_{w',1}\tau)_*$ is as in (5-5). If (5-8) is satisfied, then it is easy to see the following:

($*$) If w' satisfies (5-8) then $\overline{\partial}\widetilde w'\equiv 0 \bmod E_{[((D^2,\vec{\overline{z}}^{\mathrm{rev}},\vec{\overline{z}}^{+}),\widetilde w)]}$.

This implies that the map $[((D^2,\vec z\,',\vec z^{+\prime}),w')]\mapsto[((D^2,\vec{\overline{z}}^{\prime\mathrm{rev}},\vec{\overline{z}}^{+\prime}),\widetilde w')]$ defines a diffeomorphism from a Kuranishi neighborhood of $[((D^2,\vec z,\vec z^+),w)]$ to that of $[((D^2,\vec{\overline{z}}^{\mathrm{rev}},\vec{\overline{z}}^{+}),\widetilde w)]$. Moreover, the Kuranishi map

$$[((D^2,\vec z\,',\vec z^{+\prime}),w')]\mapsto s([((D^2,\vec z\,',\vec z^{+\prime}),w')])=\overline{\partial}w'$$

commutes with τ_*^{main}. Therefore, τ_*^{main} induces an isomorphism of our Kuranishi structure on $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$.

In order to find $E_{[((D^2,\vec z,\vec z^+),w)]}((D^2,\vec z\,',\vec z^{+\prime}),w')$ satisfying (5-9) in addition, we proceed as follows. First we briefly recall the way we defined it in the Appendix of [16], pages 423–424 of [10], and Section 18 of [13]. We take a sufficiently dense subset $\{((D^2_{\mathfrak a},\vec z_{\mathfrak a},\vec z^{+}_{\mathfrak a}),w_{\mathfrak a})\mid \mathfrak a\in\mathfrak A\}\subset\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$. We choose a finite-dimensional vector space

$$E_{\mathfrak a}\subset\Gamma(D^2;\Lambda^{0,1}\otimes(w_{\mathfrak a})^*TM)$$

of smooth sections whose supports are away from node or marked points, and satisfies

$$\operatorname{Im} D_{w_{\mathfrak{a}}}\overline{\partial} + E_{\mathfrak{a}} = \Gamma(D^2; \Lambda^{0,1} \otimes (w_{\mathfrak{a}})^* TM).$$

Moreover, we assume

$$\bigoplus \mathrm{dev}_{z_i}\colon (\operatorname{Im} D_{w_{\mathfrak{a}}}\overline{\partial})^{-1}(E_{\mathfrak{a}}) \to \bigoplus_{i=0,\dots,k} T_{w_{\mathfrak{a}}(z_i)}L$$

is surjective. See Lemma 7.1.18 in [10].

We take a sufficiently small neighborhood $W_{\mathfrak{a}}$ of each $w_{\mathfrak{a}}$ such that

$$\bigcup_{\mathfrak{a}\in\mathfrak{A}} W_{\mathfrak{a}} = \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta).$$

Let $[((D^2,\vec{z},\vec{z}^{\,+}),w)] \in W_{\mathfrak{a}}$, $((D^2,\vec{z}^{\,\prime},\vec{z}^{\,+\prime}),w')$ be as above. We take an isomorphism

(5-10) $I_{w',\mathfrak{a}}\colon \Gamma(D^2,\partial D^2 \colon w_{\mathfrak{a}}^* TM, w_{\mathfrak{a}}|^*_{\partial D^2} TL) \cong \Gamma(D^2,\partial D^2 \colon (w')^* TM, w'|^*_{\partial D^2} TL).$

Then we define

$$E_{[((D^2,\vec{z},\vec{z}^{\,+}),w)]}((D^2,\vec{z}^{\,\prime},\vec{z}^{\,+\prime}),w') = \bigoplus_{\mathfrak{a}\in\mathfrak{A},\, w'\in W_{\mathfrak{a}}} I_{w',\mathfrak{a}}(E_{\mathfrak{a}}). \tag{5-11}$$

(It is easy to see that we can perturb $E_{\mathfrak{a}}$ ($\mathfrak{a}\in\mathfrak{A}$) a bit so that the right-hand side is a direct sum. See Section 27 of [13], for example.)

We now explain the way we take $E_{[((D^2,\vec{z},\vec{z}^{\,+}),w)]}((D^2,\vec{z}^{\,\prime},\vec{z}^{\,+\prime}),w')$ so that (5-9) is satisfied.

We first require that:

- (i) The set $\{((D^2_{\mathfrak{a}},\vec{z}_{\mathfrak{a}},\vec{z}^{\,+}_{\mathfrak{a}}),w_{\mathfrak{a}}) \mid \mathfrak{a}\in\mathfrak{A}\}$ is invariant under the τ^{main}_* action.
- (ii) If $((D^2_{\mathfrak{b}},\vec{z}_{\mathfrak{b}},\vec{z}^{\,+}_{\mathfrak{b}}),w_{\mathfrak{b}}) = \tau^{\mathrm{main}}_*((D^2_{\mathfrak{a}},\vec{z}_{\mathfrak{a}},\vec{z}^{\,+}_{\mathfrak{a}}),w_{\mathfrak{a}})$ for $\mathfrak{a}\neq\mathfrak{b}$, then we have $E_{\mathfrak{b}} = (T_{w_{\mathfrak{a}},1}\tau)_*(E_{\mathfrak{a}})$. Moreover, $\tau^{\mathrm{main}}_*(W_{\mathfrak{a}}) = W_{\mathfrak{b}}$.
- (iii) If $((D^2_{\mathfrak{a}},\vec{z}_{\mathfrak{a}},\vec{z}^{\,+}_{\mathfrak{a}}),w_{\mathfrak{a}}) = \tau^{\mathrm{main}}_*((D^2_{\mathfrak{a}},\vec{z}_{\mathfrak{a}},\vec{z}^{\,+}_{\mathfrak{a}}),w_{\mathfrak{a}})$, then $E_{\mathfrak{a}} = (T_{w_{\mathfrak{a}},1}\tau)_*(E_{\mathfrak{a}})$. Moreover, $\tau^{\mathrm{main}}_*(W_{\mathfrak{a}}) = W_{\mathfrak{a}}$.

It is easy to see that such a choice exists.

Next we choose $I_{w',\mathfrak{a}}$ such that the following hold:

- (I) If $((D^2_{\mathfrak{b}},\vec{z}_{\mathfrak{b}},\vec{z}^{\,+}_{\mathfrak{b}}),w_{\mathfrak{b}}) = \tau^{\mathrm{main}}_*((D^2_{\mathfrak{a}},\vec{z}_{\mathfrak{a}},\vec{z}^{\,+}_{\mathfrak{a}}),w_{\mathfrak{a}})$ for $\mathfrak{a}\neq\mathfrak{b}$, then

$$(T_{w',1}\tau)_* \circ I_{w',\mathfrak{a}} = I_{\widetilde{w}',\mathfrak{b}} \circ (T_{w_{\mathfrak{a}},1}\tau)_*. \tag{5-12}$$

(II) If $((D^2_{\mathfrak a}, \vec z_{\mathfrak a}, \vec z^{\,+}_{\mathfrak a}), w_{\mathfrak a}) = \tau_*^{\mathrm{main}}((D^2_{\mathfrak a}, \vec z_{\mathfrak a}, \vec z^{\,+}_{\mathfrak a}), w_{\mathfrak a})$, then

$$(T_{w',1}\tau)_* \circ I_{w',\mathfrak a} = I_{\widetilde{w}',\mathfrak a} \circ (T_{w_{\mathfrak a},1}\tau)_*. \tag{5-13}$$

We can find such $I_{w',\mathfrak a}$ by taking various data to define this isomorphism (see Definition 17.7 in [13]) to be invariant under the τ action.

It is easy to see that (i), (ii), (iii), (I) and (II) imply (5-9).

We have thus constructed the Kuranishi neighborhood of the point $\mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$ corresponding to a pseudoholomorphic map from a disc (without disc or sphere bubbles).

Let us consider an element $((\Sigma, \vec z, \vec z^{\,+}), w) \in \mathcal{M}^{\mathrm{main}}_{k+1,m}(J;\beta)$ such that Σ is not irreducible. (Namely, Σ is not D^2.)

Lemma-Definition 5.1 In order to obtain $((\Sigma, \vec z, \vec z^{\,+}), w)$, we glue elements of $\mathcal{M}^{\mathrm{main}}_{k'+1,m'}(J;\beta')$ with $(\beta', k', m') < (\beta, k, m)$ and sphere bubbles. Also, we define $\tau_*^{\mathrm{main}}((\Sigma, \vec z, \vec z^{\,+}), w)$ using τ_*^{main} on $\mathcal{M}^{\mathrm{main}}_{k'+1,m'}(J;\beta')$ with $(\beta', k', m') < (\beta, k, m)$.

Proof First we suppose that Σ has a sphere bubble $S^2 \subset \Sigma$. We remove it from Σ to obtain Σ_0. We add one more marked point to Σ_0 at the location where the sphere bubble used to be attached. Then we obtain an element

$$((\Sigma_0, \vec z, \vec z^{\,(0)}), w_0) \in \mathcal{M}^{\mathrm{main}}_{k+1,m+1-\ell}(J;\beta').$$

Here ℓ is the number of marked points on S^2. By the induction hypothesis, τ_*^{main} is already defined on $\mathcal{M}^{\mathrm{main}}_{k+1,m+1-\ell}(J;\beta')$ since $\omega(\beta) \ge \omega(\beta')$, and if $\omega(\beta) = \omega(\beta')$, $\ell \ge 2$. We define

$$((\Sigma_0', \vec z\,', \vec z^{\,(0)\prime}), w_0') := \tau_*\big(((\Sigma_0, \vec z, \vec z^{\,(0)}), w_0)\big).$$

We define $v\colon S^2 \to M$ by

$$v(z) = \tau \circ w|_{S^2}(\overline{z}).$$

We assume that the nodal point in $\Sigma_0 \cap S^2$ corresponds to $0 \in \mathbb{C} \cup \{\infty\} \cong S^2$. We also map ℓ marked points on S^2 by $z \mapsto \overline{z}$ whose images we denote by $\vec z^{\,(1)} \in S^2$. We then glue $((S^2, \vec z^{\,(1)}), v)$ to $((\Sigma_0', \vec z\,', \vec z^{\,(0)\prime}), w_0')$ at the point $0 \in S^2$ and at the last marked point of $(\Sigma_0', \vec z\,', \vec z^{\,(0)\prime})$, and we obtain a curve which is to be the definition of $\tau_*^{\mathrm{main}}(((\Sigma, \vec z, \vec z^{\,+}), w))$.

Next suppose that there is no sphere bubble on Σ. Let Σ_0 be the component containing the 0^{th} marked point. If there is only one irreducible component of Σ, then τ_*^{main} is already defined there. So we assume that there is at least one other disc component. Then Σ is a union of Σ_0 and Σ_i for $i = 1, \dots, m$ $(m \ge 1)$. We regard the unique point in $\Sigma_0 \cap \Sigma_i$ as a marked point of Σ_0 for $i = 1, \dots, m$. Here each Σ_i itself is a union of disc components and is connected. We also regard the point in $\Sigma_0 \cap \Sigma_i$ as $1 \in D^2 \cong \mathbb{H} \cup \{\infty\}$ where D^2 is the irreducible component of Σ_i joined to Σ_0, and also as one of the marked points of Σ_i. This defines an element $((\Sigma_i, \vec{z}^{(i)}, \vec{z}^{(i)+}), w_{(i)})$ for each $i = 0, \dots, m$. By easy combinatorics and the induction hypothesis, we can show that τ_*^{main} is already constructed on them. Now we define $\tau_*^{\mathrm{main}}(((\Sigma, \vec{z}, \vec{z}^+), w))$ by gluing $\tau_*^{\mathrm{main}}(((\Sigma_i, \vec{z}^{(i)}, \vec{z}^{(i)+}), w_{(i)}))$ to $\tau_*^{\mathrm{main}}((\Sigma_0, \vec{z}^{(0)}, \vec{z}^{(0)+}), w_{(0)})$. □

Thus we proved that, if Σ is not irreducible, then $((\Sigma, \vec{z}, \vec{z}^+), w)$ is obtained by gluing some elements corresponding to $(\beta', k', m') < (\beta, k, m)$ and sphere bubbles.

We define the map $u \mapsto \tilde{u}$ by the same formula as (4-2) on the moduli space of *spheres*. Then we can regard this map as an involution on the space with Kuranishi structure in the same way as in the case of discs. (In other words, we construct τ–invariant Kuranishi structures on the moduli space of spheres before starting the construction of the Kuranishi structures on the moduli space of discs.)

Let us consider an element $((\Sigma, \vec{z}, \vec{z}^+), w) \in \mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta)$ such that Σ is not irreducible. By Lemma-Definition 5.1 and the above remark, the involution of its Kuranishi neighborhood is constructed by the induction hypothesis on each irreducible component (which is either a disc or a sphere). A Kuranishi neighborhood of $((\Sigma, \vec{z}, \vec{z}^+), w)$ is a fiber product of the Kuranishi neighborhoods of the gluing pieces and the space of the smoothing parameters of the singular points. By definition, our involution obviously commutes with the process to take the fiber product. For the parameter space of smoothing the interior singularities, the action of the involution is complex conjugation. For the parameter space of smoothing the boundary singularities, the action of involution is trivial. The fiber product corresponding to a boundary node is taken over the Lagrangian submanifold L, and the fiber product corresponding to an interior node is taken over the ambient symplectic manifold M. Hence the fiber product construction can be carried out in a τ–invariant way. It is easy to see that the analysis we worked out in Section 7.1 of [10] (see also Parts 2 and 3 of [13] for more detail) of the gluing is compatible with the involution. Thus τ_*^{main} defines an involution on $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta)$ with Kuranishi structure. The proof of Theorem 4.11 is complete. □

Proof of Theorem 4.10(1). The proof is the same as the proof of Theorem 4.11 except for the following point. Instead of $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta)$, we will construct a Kuranishi structure on $\mathcal{M}_{k+1,m}(J;\beta)$. We want it invariant under τ_* instead of τ_*^{main}. (Note $\mathcal{M}_{k+1,m}^{\mathrm{main}}(J;\beta)$ is not invariant under τ_*.) Taking this point into account, the proof of Theorem 4.10(1) goes in the same way as the proof of Theorem 4.11. □

Note we actually do not use Theorem 4.10(1) to prove our main results. (It is Theorem 4.11 that we actually use.) So we do not discuss its proof in more detail.

Proof of Theorem 4.12 To prove the assertion on orientation, it is enough to consider the orientation on the regular part $\mathcal{M}^{\mathrm{main,reg}}_{k+1,m}(J;\beta;P_1,\ldots,P_k)$. See Remark 3.7(2). By Theorem 4.6,

$$\tau_*\colon \mathcal{M}^{\mathrm{reg}}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{reg}}(J;\beta)^{[(V,\sigma)]}$$

is orientation-preserving if and only if $\frac{1}{2}\mu_L(\beta)$ is even. Recall that when we consider the main component $\mathcal{M}^{\mathrm{main,reg}}_{k+1,m}(J;\beta)$, the boundary marked points are in counterclockwise cyclic ordering. However, by the involution τ_* in Theorem 4.12, each boundary marked point z_i is mapped to $\overline{z}_i$, and each interior marked point z_j^+ is mapped to $\overline{z_j^+}$. Thus the order of the boundary marked points changes to clockwise ordering. Denote by $\mathcal{M}^{\mathrm{clock,reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ the moduli space with the boundary marked points $(z_0,z_1,\ldots,z_k)$ with the *clockwise* orientation and interior marked points $z_1^+,\ldots,z_m^+$. Since $z\mapsto\overline{z}$ reverses the orientation on the boundary and $z^+\mapsto\overline{z^+}$ reverses the orientation on the interior, the argument in the proof of Proposition 4.9 shows that $\tau_*\colon \mathcal{M}^{\mathrm{main,reg}}_{k+1,m}(J;\beta)^{\tau^*[(V,\sigma)]} \to \mathcal{M}^{\mathrm{clock,reg}}_{k+1,m}(J;\beta)^{[(V,\sigma)]}$ respects the orientation if and only if $\frac{1}{2}\mu_L(\beta)+k+1+m$ is even. Thus we have

$$\begin{aligned}\mathcal{M}^{\mathrm{main,reg}}_{k+1,m}&(J;\beta;P_1,\ldots,P_k)^{\tau^*[(V,\sigma)]}\\ &= (-1)^{\mu_L(\beta)/2+k+1+m}\mathcal{M}^{\mathrm{clock,reg}}_{k+1,m}(J;\beta;P_1,\ldots,P_k)^{[(V,\sigma)]}.\end{aligned}$$

Recall that Lemma 3.17 describes how the orientation of $\mathcal{M}_{k+1,m}(J;\beta;P_1,\ldots,P_k)$ changes by changing the ordering of boundary marked points. Thus, using Lemma 3.17, we obtain Theorem 4.12 immediately. □

Since the map τ_*^{main} preserves the ordering of interior marked points, we also obtain the following. When we study bulk deformations [7; 9] of the A_∞ algebra for $L=\mathrm{Fix}\,\tau$, which we do not discuss in this article, we need the next theorem.

Theorem 5.2 *Let $Q_1,\ldots,Q_m$ be smooth singular simplices of M. Then the map*

$$\begin{aligned}\tau_*^{\mathrm{main}}\colon \mathcal{M}^{\mathrm{main}}_{k+1,m}&\big(J;\beta;Q_1,\ldots,Q_m;P_1,\ldots,P_k\big)^{\tau^*[(V,\sigma)]}\\ &\to \mathcal{M}^{\mathrm{main}}_{k+1,m}\big(J;\beta;\tau(Q_1),\ldots,\tau(Q_m);P_k,\ldots,P_1\big)^{[(V,\sigma)]}\end{aligned}$$

preserves orientation if and only if

$$\epsilon = \tfrac{1}{2}\mu_L(\beta)+k+1+m+nm+\sum_{1\le i<j\le k}\deg' P_i\deg' P_j$$

is even.

Proof Note that τ preserves the orientation on M if and only if $\frac{1}{2}\dim_{\mathbb{R}} M = n$ is even. Taking this into account, Theorem 5.2 follows from Theorem 4.12. See also Lemma 6.76. $\square$

6 Applications

Using the results obtained in the previous sections, we prove Theorem 1.5, Corollary 1.6 and Corollary 1.8 in Section 6.2, we prove Theorem 1.9 and Corollary 1.10 in Section 6.3 and we calculate Floer cohomology of $\mathbb{R}P^{2n+1}$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ in Section 6.4.

6.1 Filtered A_∞ algebra and Lagrangian Floer cohomology

In order to explain how our study of orientations can be used for the applications to Lagrangian Floer theory, we briefly recall the construction of the filtered A_∞ algebra for the relatively spin Lagrangian submanifold and its obstruction/deformation theory developed in our books [9; 10].

See the survey paper [19] for a more detailed review.

Let R be a commutative ring with unity. Let e and T be formal variables of degree 2 and 0, respectively. We use the *universal Novikov ring* over R as our coefficient ring:

$$\Lambda_{\mathrm{nov}}^{R} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \;\middle|\; a_i \in R,\ \mu_i \in \mathbb{Z},\ \lambda_i \in \mathbb{R},\ \lim_{i\to\infty} \lambda_i = +\infty \right\}, \tag{6-1}$$

$$\Lambda_{0,\mathrm{nov}}^{R} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{\mathrm{nov}} \;\middle|\; \lambda_i \geq 0 \right\}. \tag{6-2}$$

We define a filtration $F^\lambda \Lambda_{0,\mathrm{nov}}^R = T^\lambda \Lambda_{0,\mathrm{nov}}^R$ ($\lambda \in \mathbb{R}_{\geq 0}$) on $\Lambda_{0,\mathrm{nov}}^R$ which induces a filtration $F^\lambda \Lambda_{\mathrm{nov}}^R$ ($\lambda \in \mathbb{R}$) on Λ_{nov}^R. We call this filtration the *energy filtration*. Given these filtrations, both $\Lambda_{0,\mathrm{nov}}^R$ and Λ_{nov}^R become graded filtered commutative rings. In the rest of this subsection and the next, we take $R = \mathbb{Q}$. We use the case $R = \mathbb{Z}$ in Section 6.4.

In Section 2, we define $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\ldots,P_k)$ for smooth singular simplices (P_i, f_i) of L. By the result of Section 7.1 of [10], it has a Kuranishi structure. Here we use the same notation for the Kuranishi structure as we use in the Appendix of the present paper. The space $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\ldots,P_k)$ is locally described by $s_p^{-1}(0)/\Gamma_p$. If the Kuranishi map s_p is transverse to the zero section, it is locally an orbifold. However, if Γ_p is nontrivial, we can not perturb s_p to a Γ_p–equivariant section transverse to the zero section in general. Instead of single valued sections, we take a

Γ_p–equivariant *multivalued* section (multisection) $\mathfrak{s}_p$ of $E_p \to V_p$ so that each branch of the multisection is transverse to the zero section and $\mathfrak{s}_p^{-1}(0)/\Gamma_p$, and sufficiently close to $s_p^{-1}(0)/\Gamma_p$. (See Sections 7.1 and 7.2 of [10] for the precise statement.) We denote the perturbed zero locus (divided by Γ_p) by $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}$. We have the evaluation map at the 0^{th} marked point for the perturbed moduli space:

$$\mathrm{ev}_0\colon \mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}} \to L.$$

Then such a system $\mathfrak{s} = \{\mathfrak{s}_p\}$ of multivalued sections gives rise to the virtual fundamental chain over $\mathbb{Q}$ as follows: By Lemma 6.9 in [16] and Lemma A1.26 in [10], $\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}$ carries a smooth triangulation. We take a smooth triangulation on it. Each simplex Δ_a^d of dimension $d = \dim \mathfrak{s}_p^{-1}(0)$ in the triangulation comes with multiplicity $\mathrm{mul}_{\Delta_a^d} \in \mathbb{Q}$. (See Definition A1.27 in [10] for the definition of multiplicity.) Restricting ev_0 to Δ_a^d, we have a singular simplex of L denoted by $(\Delta_a^d, \mathrm{ev}_0)$. Then the virtual fundamental chain over $\mathbb{Q}$, which we denote by $(\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}, \mathrm{ev}_0)$, is defined by

$$(\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}, \mathrm{ev}_0) = \sum_a \mathrm{mul}_{\Delta_a^d} \cdot (\Delta_a^d, \mathrm{ev}_0).$$

When the virtual dimension is zero, ie when $d = 0$, we let

$$\#\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}} = \sum_a \mathrm{mul}_{\Delta_a^0} \in \mathbb{Q}.$$

Now put $\Pi(L)_0 = \{(\omega(\beta), \mu_L(\beta)) \mid \beta \in \Pi(L),\ \mathcal{M}(J;\beta) \neq \varnothing\}$; see (2-1) for $\Pi(L)$. Let $G(L)$ be a submonoid of $\mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ generated by $\Pi(L)_0$. Put $\beta_0 = (0,0) \in G(L)$.

Definition 6.1 For smooth singular simplices P_i of L and $\beta \in G(L)$, we define a series of maps $\mathfrak{m}_{k,\beta}$ by

$$\mathfrak{m}_{0,\beta}(1) = \begin{cases} (\mathcal{M}_1(J;\beta)^{\mathfrak{s}}, \mathrm{ev}_0) & \text{for } \beta \neq \beta_0, \\ 0 & \text{for } \beta = \beta_0, \end{cases}$$

$$\mathfrak{m}_{1,\beta}(P) = \begin{cases} (\mathcal{M}_2^{\mathrm{main}}(J;\beta;P)^{\mathfrak{s}}, \mathrm{ev}_0) & \text{for } \beta \neq \beta_0, \\ (-1)^n \partial P & \text{for } \beta = \beta_0, \end{cases}$$

and

$$\mathfrak{m}_{k,\beta}(P_1,\dots,P_k) = (\mathcal{M}_{k+1}^{\mathrm{main}}(J;\beta;P_1,\dots,P_k)^{\mathfrak{s}}, \mathrm{ev}_0), \quad k \geq 2.$$

Here ∂ is the usual boundary operator, and $n = \dim L$. Then one of main results proved in [9] and [10] is as follows: For a smooth singular chain P on L, we put the cohomological grading $\deg P = n - \dim P$ and regard a smooth singular chain complex $S_*(L;\mathbb{Q})$ as a smooth singular cochain complex $S^{n-*}(L;\mathbb{Q})$. For a subcomplex

$C(L;\mathbb{Q})$ of $S(L;\mathbb{Q})$, we denote by $C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ a completion of $C(L;\mathbb{Q})\otimes\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$ with respect to the filtration induced from the one on $\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$ introduced above. We shift the degree by 1; ie define

$$C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})[1]^{\bullet}=C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})^{\bullet+1},$$

where we define $\deg(PT^{\lambda}e^{\mu})=\deg P+2\mu$ for $PT^{\lambda}e^{\mu}\in C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$. We put

$$\mathfrak{m}_k=\sum_{\beta\in G(L)}\mathfrak{m}_{k,\beta}\otimes T^{\omega(\beta)}e^{\mu_L(\beta)/2},\quad k=0,1,\dots. \tag{6-3}$$

To simplify notation, we write $C=C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$. Put

$$B_k(C[1])=\underbrace{C[1]\otimes\cdots\otimes C[1]}_{k}, \tag{6-4}$$

and take its completion with respect to the energy filtration. By an abuse of notation, we denote the completion by the same symbol. We define the *bar complex* $B(C[1])=\bigoplus_{k=0}^{\infty}B_k(C[1])$, and extend $\mathfrak{m}_k$ to the graded coderivation $\widehat{\mathfrak{m}}_k$ on $B(C[1])$ by

$$\widehat{\mathfrak{m}}_k(x_1\otimes\cdots\otimes x_n)=\sum_{i=1}^{n-k+1}(-1)^{*}x_1\otimes\cdots\otimes\mathfrak{m}_k(x_i,\dots,x_{i+k-1})\otimes\cdots\otimes x_n, \tag{6-5}$$

where $*=\deg x_1+\cdots+\deg x_{i-1}+i-1$. We put

$$\widehat{d}=\sum_{k=0}^{\infty}\widehat{\mathfrak{m}}_k\colon B(C[1])\to B(C[1]). \tag{6-6}$$

Theorem 6.2 [9, Theorem 3.5.11] *For any closed relatively spin Lagrangian submanifold L of M, there exist a countably generated subcomplex $C(L;\mathbb{Q})$ of a smooth singular cochain complex of L whose cohomology is isomorphic to $H(L;\mathbb{Q})$, and a system of multisections $\mathfrak{s}$ of $\mathcal{M}^{\mathrm{main}}_{k+1}(J;\beta;P_1,\dots,P_k)$ ($\mathfrak{s}$ is chosen depending on $P_i\in C(L;\mathbb{Q})$) such that*

$$\mathfrak{m}_k\colon\underbrace{C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})[1]\otimes\cdots\otimes C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})[1]}_{k}\to C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})[1],\quad k=0,1,\dots$$

are defined and satisfy $\widehat{d}\circ\widehat{d}=0$.

The equation $\widehat{d}\circ\widehat{d}=0$ is equivalent to

$$\sum_{k_1+k_2=k+1}\sum_{i}(-1)^{\deg' x_1+\cdots+\deg' x_{i-1}}\mathfrak{m}_{k_1}(x_1,\dots,\mathfrak{m}_{k_2}(x_i,\dots,x_{i+k_2-1}),\dots,x_k)=0,$$

which we call the A_∞ *formulas* or *relations*. Here $\deg' x_i = \deg x_i - 1$, the shifted degree. In particular, the A_∞ formulas imply an equality

$$\mathfrak{m}_2(\mathfrak{m}_0(1), x) + (-1)^{\deg' x}\mathfrak{m}_2(x, \mathfrak{m}_0(1)) + \mathfrak{m}_1\mathfrak{m}_1(x) = 0,$$

which shows $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$ may not hold unless $\mathfrak{m}_0(1) = 0$. So $\mathfrak{m}_0$ gives an obstruction to defining $\mathfrak{m}_1$–cohomology.

Definition 6.3 An element $b \in C(L; \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})[1]^0$ with $b \equiv 0 \bmod \Lambda^+_{0,\mathrm{nov}}$ is called a *solution of the Maurer–Cartan equation* or a *bounding cochain*, if it satisfies the Maurer–Cartan equation

$$\mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b, b) + \mathfrak{m}_3(b, b, b) + \cdots = 0.$$

Here $\Lambda^+_{0,\mathrm{nov}} = \{\sum a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{0,\mathrm{nov}} \mid \lambda_i > 0\}$. We denote by $\mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ the set of bounding cochains. We say L is *unobstructed* if $\mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}) \neq \varnothing$.

Remark 6.4 We do not introduce the notion of gauge equivalence of bounding cochains (Definition 4.3.1 in [9]) because we do not use it in this paper.

If $\mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}) \neq \varnothing$, then by using any $b \in \mathcal{M}(L; \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$, we can deform the A_∞ structure $\mathfrak{m}$ to $\mathfrak{m}^b$ by

$$\mathfrak{m}^b_k(x_1, \ldots, x_k) = \sum_{\ell_0,\ldots,\ell_k} \mathfrak{m}_{k+\sum \ell_i}(\underbrace{b, \ldots, b}_{\ell_0}, x_1, \underbrace{b, \ldots, b}_{\ell_1}, \ldots, \underbrace{b, \ldots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \ldots, b}_{\ell_k})$$

so that $\mathfrak{m}^b_1 \circ \mathfrak{m}^b_1 = 0$ (Proposition 3.6.10 in [9]). Then we can define

$$\mathrm{HF}((L, b); \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}) := H(C(L; \Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}), \mathfrak{m}^b_1),$$

which we call *Floer cohomology of L* (deformed by b).

In the actual proof of Theorem 6.2 given in Section 7.2 of [10], we do not construct the filtered A_∞ structure at once, but we first construct a *filtered $A_{n,K}$ structure* for any nonnegative integers n and K, and promote it to a filtered A_∞ structure by developing certain obstruction theory (Sections 7.2.6–7.2.10 of [10]). Here we recall the notion of a filtered $A_{n,K}$ structure from Section 7.2.6 of [10], which is mentioned later in the proof of Theorem 1.5. See also Sections 2.6 and 4 of [8] for a quick review. We briefly summarize the obstruction theory in the Appendix. Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a monoid such that $\mathrm{pr}_1^{-1}([0, c])$ is finite for any $c \geq 0$ and $\mathrm{pr}_1^{-1}(0) = \{\beta_0 = (0, 0)\}$, where pr_i denotes the projection to the i^{th} factor. We note that in the geometric situation, we take $G = G(L)$ introduced above. For $\beta \in G$, we define

$$\|\beta\| = \begin{cases} \sup\{n \mid \exists \beta_i \in G \setminus \{\beta_0\},\ \sum_{i=1}^n \beta_i = \beta\} + [\mathrm{pr}_1(\beta)] - 1 & \text{if } \beta \neq \beta_0, \\ -1 & \text{if } \beta = \beta_0, \end{cases} \tag{6-7}$$

where $[\mathrm{pr}_1(\beta)]$ stands for the largest integer less than or equal to $\mathrm{pr}_1(\beta)$. Using this, we introduce a partial order on $(G \times \mathbb{Z}_{\geq 0}) \setminus \{(\beta_0, 0)\}$ by $(\beta_1, k_1) < (\beta_2, k_2)$ if and only if either

$$\|\beta_1\| + k_1 < \|\beta_2\| + k_2$$

or

$$\|\beta_1\| + k_1 = \|\beta_2\| + k_2 \quad \text{and} \quad \|\beta_1\| < \|\beta_2\|.$$

We write $(\beta_1, k_1) \sim (\beta_2, k_2)$, when

$$\|\beta_1\| + k_1 = \|\beta_2\| + k_2 \quad \text{and} \quad \|\beta_1\| = \|\beta_2\|.$$

We write $(\beta_1, k_1) \lesssim (\beta_2, k_2)$ if either $(\beta_1, k_1) < (\beta_2, k_2)$ or $(\beta_1, k_1) \sim (\beta_2, k_2)$. For nonnegative integers n, n', k and k', we also use the notation $(\beta, k) < (n, k)$, $(n, k) < (n', k')$, $(\beta, k) \lesssim (n, k)$, $(n, k) \lesssim (n', k')$, etc in a similar way.

Let $\overline{C}$ be a cochain complex over R and $C = \overline{C} \otimes \Lambda^R_{0,\mathrm{nov}}$. Suppose that there is a sequence of R–linear maps

$$\mathfrak{m}_{k,\beta} \colon B_k(\overline{C}[1]) \to \overline{C}[1]$$

for $(\beta, k) \in (G \times \mathbb{Z}) \setminus \{(\beta_0, 0)\}$ with $(\beta, k) < (n, K)$. We also suppose that $\mathfrak{m}_{1,\beta_0}$ is the boundary operator of the cochain complex $\overline{C}$.

Definition 6.5 We call $(C, \{\mathfrak{m}_{k,\beta}\})$ a (G–gapped) *filtered $A_{n,K}$ algebra* if the identity

$$\sum_{\substack{\beta_1+\beta_2=\beta \\ k_1+k_2=k+1}} \sum_i (-1)^{\deg' \boldsymbol{x}_i^{(1)}} \mathfrak{m}_{k_2,\beta_2}\big(\boldsymbol{x}_i^{(1)}, \mathfrak{m}_{k_1,\beta_1}(\boldsymbol{x}_i^{(2)}), \boldsymbol{x}_i^{(3)}\big) = 0 \tag{6-8}$$

holds for all $(\beta, k) < (n, K)$, where

$$\Delta^2(\boldsymbol{x}) = \sum_i \boldsymbol{x}_i^{(1)} \otimes \boldsymbol{x}_i^{(2)} \otimes \boldsymbol{x}_i^{(3)}.$$

Here Δ is the coproduct of the tensor coalgebra.

Let C and C' be filtered $A_{n,K}$ algebras. We consider a sequence of R–linear maps of degree zero,

$$\mathfrak{f}_{k,\beta} \colon B_k(\overline{C}[1]) \to \overline{C}'[1],$$

satisfying $\mathfrak{f}_{0,\beta_0} = 0$.

Definition 6.6 We call $\{\mathfrak{f}_{k,\beta}\}$ a *filtered $A_{n,K}$ homomorphism* if the identity

$$\sum_{m,i} \sum_{\beta'+\beta_1+\cdots+\beta_m=\beta} \sum_{k_1+\cdots+k_m=k} \mathfrak{m}_{m,\beta'}\big(\mathfrak{f}_{k_1,\beta_1}(\boldsymbol{x}_i^{(1)}),\dots,\mathfrak{f}_{k_m,\beta_m}(\boldsymbol{x}_i^{(m)})\big) = \sum_{\substack{\beta_1+\beta_2=\beta\\ k_1+k_2=k+1}} \sum_i (-1)^{\deg' \boldsymbol{x}_i^{(1)}} \mathfrak{f}_{k_2,\beta_2}\big(\boldsymbol{x}_i^{(1)},\mathfrak{m}_{k_1,\beta_1}(\boldsymbol{x}_i^{(2)}),\boldsymbol{x}_i^{(3)}\big)$$

holds for $(\beta,k) \lesssim (n,K)$.

We also have the notion of filtered $A_{n,K}$ homotopy equivalences in a natural way. (See Section 7.2.6 of [10].)

Theorem 6.7 [10, Theorem 7.2.72] *Let C_1 be a filtered $A_{n,K}$ algebra and C_2 a filtered $A_{n',K'}$ algebra such that $(n,K) < (n',K')$. Let $\mathfrak{h}\colon C_1 \to C_2$ be a filtered $A_{n,K}$ homomorphism. Suppose that $\mathfrak{h}$ is a filtered $A_{n,K}$ homotopy equivalence. Then there exist a filtered $A_{n',K'}$ algebra structure on C_1 extending the given filtered $A_{n,K}$ algebra structure, and a filtered $A_{n',K'}$ homotopy equivalence $C_1 \to C_2$ extending the given filtered $A_{n,K}$ homotopy equivalence $\mathfrak{h}$.*

Next, let $(L^{(1)},L^{(0)})$ be a relatively spin pair of closed Lagrangian submanifolds. We first assume that $L^{(0)}$ is transverse to $L^{(1)}$. Let $C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ be the free $\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$ module generated by the intersection points $L^{(1)} \cap L^{(0)}$. Then we can construct a filtered A_∞ bimodule structure $\{\mathfrak{n}_{k_1,k_0}\}_{k_1,k_0=0,1,\dots}$ on $C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ over the pair $(C(L^{(1)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}),C(L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}))$ of A_∞ algebras as follows. Here we briefly describe the map

$$\mathfrak{n}_{k_1,k_0}\colon B_{k_1}(C(L^{(1)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1]) \otimes C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}) \otimes B_{k_0}(C(L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1]) \to C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}).$$

A typical element of the tensor product above is written as

$$\big(P_1^{(1)}T^{\lambda_1^{(1)}}e^{\mu_1^{(1)}} \otimes\cdots\otimes P_{k_1}^{(1)}T^{\lambda_{k_1}^{(1)}}e^{\mu_{k_1}^{(1)}}\big) \otimes T^{\lambda}e^{\mu}\langle p\rangle \otimes \big(P_1^{(0)}T^{\lambda_1^{(0)}}e^{\mu_1^{(0)}} \otimes\cdots\otimes P_{k_0}^{(0)}T^{\lambda_{k_0}^{(0)}}e^{\mu_{k_0}^{(0)}}\big)$$

for $p \in L^{(1)} \cap L^{(0)}$. Then $\mathfrak{n}_{k_1,k_0}$ maps it to

$$\sum_{q,B} \#\big(\mathcal{M}(p,q;B;P_1^{(1)},\dots,P_{k_1}^{(1)};P_1^{(0)},\dots,P_{k_0}^{(0)})\big)T^{\lambda'}e^{\mu'}\langle q\rangle$$

with $\lambda' = \omega(B) + \sum \lambda_i^{(1)} + \lambda + \sum \lambda_i^{(0)}$ and $\mu' = \mu_L(B) + \sum \mu_i^{(1)} + \mu + \sum \mu_i^{(0)}$. Here B is the homotopy class of Floer trajectories connecting p and q, and the sum is taken over all (q,B) such that the virtual dimension of the moduli space $\mathcal{M}(p,q;B;P_1^{(1)},\dots,P_{k_1}^{(1)};P_1^{(0)},\dots,P_{k_0}^{(0)})$ of Floer trajectories is zero. See Section 3.7.4 of [9] for the precise definition of

$$\mathcal{M}(p,q;B;P_1^{(1)},\dots,P_{k_1}^{(1)};P_1^{(0)},\dots,P_{k_0}^{(0)}).$$

Strictly speaking, we also need to take a suitable system of multisections on this moduli space to obtain the virtual fundamental chain that enters in the construction of the operators $\mathfrak{n}_{k_1,k_0}$ defining the desired A_∞ bimodule structure. Because of the usage of multisections, the counting number with sign

$$\#\big(\mathcal{M}(p,q;B;P_1^{(1)},\dots,P_{k_1}^{(1)};P_1^{(0)},\dots,P_{k_0}^{(0)})\big)$$

is a rational number, in general.

Now let $B(C(L^{(1)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1])\otimes C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})\otimes B(C(L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1])$ be the completion of

$$\bigoplus_{k_0\ge 0,k_1\ge 0} B_{k_1}(C(L^{(1)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1])\otimes C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})\otimes B_{k_0}(C(L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1])$$

with respect to the induced energy filtration. We extend $\mathfrak{n}_{k_1,k_0}$ to a bicoderivation on $B(C(L^{(1)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1])\otimes C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})\otimes B(C(L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})[1])$ which is given by the formula

$$\begin{aligned}
&\widehat{d}(x_1^{(1)}\otimes\cdots\otimes x_{k_1}^{(1)}\otimes y\otimes x_1^{(0)}\otimes\cdots\otimes x_{k_0}^{(0)})\\
&\quad=\sum_{\substack{k_1'\le k_1\\ k_0'\le k_0}}\Big((-1)^{\deg' x_1^{(1)}+\cdots+\deg' x_{k_1-k_1'}^{(1)}}\, x_1^{(1)}\otimes\cdots\otimes x_{k_1-k_1'}^{(1)}\\
&\qquad\qquad\otimes\mathfrak{n}_{k_1',k_0'}(x_{k_1-k_1'+1}^{(1)},\dots,y,\dots,x_{k_0'}^{(0)})\otimes x_{k_0'+1}^{(0)}\otimes\cdots\otimes x_{k_0}^{(0)}\Big)\\
&\qquad+\widehat{d}^{(1)}(x_1^{(1)}\otimes\cdots\otimes x_{k_1}^{(1)})\otimes y\otimes x_1^{(0)}\otimes\cdots\otimes x_{k_0}^{(0)}\\
&\qquad+(-1)^{\Sigma\deg' x_i^{(1)}+\deg' y}\,x_1^{(1)}\otimes\cdots\otimes x_{k_1}^{(1)}\otimes y\otimes\widehat{d}^{(0)}(x_1^{(0)}\otimes\cdots\otimes x_{k_0}^{(0)}).
\end{aligned}$$

Here $\widehat{d}^{(i)}$ is defined by (6-5) and (6-6) using the filtered A_∞ structure $\mathfrak{m}^{(i)}$ of $(C(L^{(i)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}),\mathfrak{m}^{(i)})$ $(i=0,1)$.

Theorem 6.8 [9, Theorem 3.7.21] *For any relatively spin pair $(L^{(1)},L^{(0)})$ of closed Lagrangian submanifolds, the family of maps $\{\mathfrak{n}_{k_1,k_0}\}_{k_1,k_0}$ defines a filtered A_∞ bimodule structure on $C(L^{(1)},L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ over $(C(L^{(1)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}),C(L^{(0)};\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}))$. Namely, $\widehat{d}\circ\widehat{d}=0$, where $\widehat{d}$ is the bicoderivation defined just above.*

Since the equation $\widehat{d} \circ \widehat{d} = 0$ implies, in particular,

$$\mathfrak{n}_{0,0} \circ \mathfrak{n}_{0,0}(y) + \mathfrak{n}_{1,0}(\mathfrak{m}_0^{(1)}(1), y) + (-1)^{\deg' y} \mathfrak{n}_{0,1}(y, \mathfrak{m}_0^{(0)}(1)) = 0,$$

we have $\mathfrak{n}_{0,0} \circ \mathfrak{n}_{0,0} \neq 0$, in general. However, if both of $L^{(0)}$ and $L^{(1)}$ are unobstructed in the sense of Definition 6.3, we can deform the filtered A_∞ bimodule structure $\mathfrak{n}$ by $b_i \in \mathcal{M}(L^{(i)}; \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ so that

$$^{b_1}\mathfrak{n}_{0,0}^{b_0}(y) := \sum_{k_1,k_0} \mathfrak{n}_{k_1,k_0}(\underbrace{b_1, \dots, b_1}_{k_1}, y, \underbrace{b_0, \dots, b_0}_{k_0})$$

satisfies $^{b_1}\mathfrak{n}_{0,0}^{b_0} \circ {}^{b_1}\mathfrak{n}_{0,0}^{b_0} = 0$ (Lemma 3.7.14 in [9]). Then we can define

$$\mathrm{HF}\big((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big) := H\big(C(L^{(1)}, L^{(0)}; \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}), {}^{b_1}\mathfrak{n}_{0,0}^{b_0}\big), \tag{6-9}$$

which we call *Floer cohomology of a pair* $(L^{(1)}, L^{(0)})$ (deformed by b_1, b_0).

So far, we assume that $L^{(0)}$ is transverse to $L^{(1)}$. But we can generalize the story to the Bott–Morse case, that is, where each component of $L^{(0)} \cap L^{(1)}$ is a smooth manifold. Especially for the case $L^{(1)} = L^{(0)}$, we have $\mathfrak{n}_{k_1,k_0} = \mathfrak{m}_{k_1+k_0+1}$ (see Example 3.7.6 in [9]) and an isomorphism

$$\mathrm{HF}\big((L, b), (L, b); \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big) \cong \mathrm{HF}\big((L, b); \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}\big) \tag{6-10}$$

for any $b \in \mathcal{M}(L; \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ by Theorem G (G.1) in [9]. Moreover, if we extend the coefficient ring $\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$ to $\Lambda_{\mathrm{nov}}^{\mathbb{Q}}$, we can find that Hamiltonian isotopies $\Psi_i^s \colon M \to M$ ($i = 0, 1$, $s \in [0, 1]$) with $\Psi_i^0 = \mathrm{id}$ and $\Psi_i^1 = \Psi_i$ induce an isomorphism

$$\begin{aligned} &\mathrm{HF}\big((L^{(1)}, b_1), (L^{(0)}, b_0); \Lambda_{\mathrm{nov}}^{\mathbb{Q}}\big) \\ &\qquad \cong \mathrm{HF}\big((\Psi_1(L^{(1)}), \Psi_{1*}b_1), (\Psi_0(L^{(0)}), \Psi_{0*}b_0); \Lambda_{\mathrm{nov}}^{\mathbb{Q}}\big) \end{aligned} \tag{6-11}$$

by Theorem G (G.4) in [9]. This shows invariance of Floer cohomology of a pair $(L^{(1)}, L^{(0)})$ over $\Lambda_{\mathrm{nov}}^{\mathbb{Q}}$ under Hamiltonian isotopies.

6.2 Proofs of Theorem 1.5, Corollary 1.6 and Corollary 1.8

Proof of Theorem 1.5 We consider the map (4-10) for the case $m = 0$. It is an automorphism of order 2. We first take its quotient by Lemma A.8 (Lemma A1.49 in [10]) in the sense of Kuranishi structure, and take a perturbed multisection of the quotient space, which is transverse to the zero section. After that, we lift the perturbed multisection. Then we can obtain a system of multisections $\mathfrak{s}$ on the moduli space $\mathcal{M}_{k+1}^{\mathrm{main}}(J; \beta; P_1, \dots, P_k)$ which is preserved by (4-10). Then Definition 6.1 yields the operators $\{\mathfrak{m}_{k,\beta}\}_{k,\beta}$ which satisfy the filtered $A_{n,K}$ relations (6-8) together with (1-2). The sign in (1-2) follows from Theorem 4.12. To complete the proof, we need to promote the filtered $A_{n,K}$ structure to a filtered A_∞ structure keeping the symmetry (1-2). This follows from an involution invariant version (see Theorem A.12) of Theorem 6.7 (Theorem 7.2.72 in [10]). Although the proof of the invariant version is a straightforward modification of that of Theorem 7.2.72 in [10], we give the outline of the argument in the Appendix for readers' convenience. □

Remark 6.9 In general, it is not possible to perturb a section of the obstruction bundle transversal to the zero section by a single-valued perturbation. Using a multivalued perturbation, we can take the perturbation to be transversal to the zero section and invariant under the action of the stabilizer as well as another finite group action coming from the symmetry of the problem. In our case, there may be a fixed point of (4-10). But this does not cause any problem as long as we work with *multi*sections and study virtual fundamental chains over $\mathbb{Q}$ as in the proof of Theorem 1.5.

Proof of Corollary 1.6 For $w\colon (D^2, \partial D^2) \to (M, L) = (M, \operatorname{Fix}\tau)$, we define its double $v\colon S^2 \to M$ by

$$v(z) = \begin{cases} w(z) & \text{for } z \in \mathbb{H}, \\ \tau \circ w(\overline{z}) & \text{for } z \in \mathbb{C} \setminus \mathbb{H}, \end{cases}$$

where $(D^2, (-1, 1, \sqrt{-1}))$ is identified with the upper half plane $(\mathbb{H}, (0, 1, \infty))$, and $S^2 = \mathbb{C} \cup \{\infty\}$. Then it is easy to see that $c_1(TM)[v] = \mu_L([w])$; see Example 4.8(1). Then the assumption (1) implies that $\mu_L \equiv 0 \bmod 4$. Next, we note the following general lemma.

Lemma 6.10 *Let L be an oriented Lagrangian submanifold of M. The composition*

$$\pi_2(M) \to \pi_2(M, L) \xrightarrow{\mu_L} \mathbb{Z}$$

is then equal to $2c_1(TM)[\alpha]$ for $[\alpha] \in \pi_2(M)$.

The proof is easy, and so it is omitted. Then by this lemma, the assumption (2) also implies that the Maslov index of L modulo 4 is trivial. Therefore, in either case (1) or (2), Theorem 1.5 implies $\mathfrak{m}_{0,\tau_*\beta}(1) = -\mathfrak{m}_{0,\beta}(1)$. On the other hand, we have

$$\mathfrak{m}_0(1) = \sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{0,\beta}(1) T^{\omega(\beta)} e^{\mu(\beta)/2}$$

by Definition 6.1 and (6-3), which is also the same as

$$\sum_{\beta \in \pi_2(M,L)} \mathfrak{m}_{0,\tau_*\beta}(1) T^{\omega(\tau_*\beta)} e^{\mu(\tau_*\beta)/2}$$

because $\tau_*^2 = \mathrm{id}$ and $\tau_*\colon \pi_2(M,L) \to \pi_2(M,L)$ is a one-to-one correspondence. Therefore, since $\omega(\beta) = \omega(\tau_*\beta)$ and $\mu(\beta) = \mu(\tau_*\beta)$, we can rewrite $\mathfrak{m}_0(1)$ as

$$\mathfrak{m}_0(1) = \frac{1}{2}\sum_\beta \bigl(\mathfrak{m}_{0,\beta}(1) + \mathfrak{m}_{0,\tau_*\beta}(1)\bigr) T^{\omega(\beta)} e^{\mu(\beta)/2},$$

which becomes 0 by the above parity consideration. Hence L is unobstructed. Actually, we find that 0 is a bounding cochain; $0 \in \mathcal{M}(L;\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$. Furthermore, (1-2) implies

$$\mathfrak{m}_{2,\beta}(P_1,P_2) = (-1)^{1+\deg' P_1 \deg' P_2}\mathfrak{m}_{2,\tau_*\beta}(P_2,P_1). \tag{6-12}$$

We let

$$P_1 \cup_Q P_2 := (-1)^{\deg P_1(\deg P_2+1)} \sum_\beta \mathfrak{m}_{2,\beta}(P_1,P_2) T^{\omega(\beta)} e^{\mu(\beta)/2}. \tag{6-13}$$

Then a simple calculation shows that (6-12) gives rise to

$$P_1 \cup_Q P_2 = (-1)^{\deg P_1 \deg P_2} P_2 \cup_Q P_1.$$

Hence $\cup_Q$ is graded commutative. □

Proof of Corollary 1.8 Let L be as in Corollary 1.8. By Corollary 1.6, L is unobstructed. Since $L = \mathrm{Fix}\,\tau$, we find that $c_1(TM)|\pi_2(M) = 0$ implies $\mu_L = 0$. Then Theorem E and Theorem 6.1.9 in [9] show that the Floer cohomology of L over $\Lambda_{\mathrm{nov}}^{\mathbb{Q}}$ does not vanish for any $b \in \mathcal{M}(L;\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$:

$$\mathrm{HF}((L,b);\Lambda_{\mathrm{nov}}^{\mathbb{Q}}) \neq 0.$$

(Note that Theorem E holds not only over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$ but also over $\Lambda_{\mathrm{nov}}^{\mathbb{Q}}$. See Theorem 6.1.9.) By extending the isomorphism (6-10) to $\Lambda_{\mathrm{nov}}^{\mathbb{Q}}$ coefficients (by taking the tensor product with $\Lambda_{\mathrm{nov}}^{\mathbb{Q}}$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}}$), we also have $\mathrm{HF}((L,b),(L,b);\Lambda_{\mathrm{nov}}^{\mathbb{Q}}) \neq 0$. Therefore by (6-11) we obtain

$$\mathrm{HF}((\psi(L),\psi_*b),(L,b);\Lambda_{\mathrm{nov}}^{\mathbb{Q}}) \neq 0,$$

which implies $\psi(L) \cap L \neq \varnothing$. □

6.3 Proofs of Theorem 1.9 and Corollary 1.10

6.3.1 Proof of Theorem 1.9(1) Let (N,ω) be a symplectic manifold, $M = N \times N$ and $\omega_M = -\mathrm{pr}_1^*\omega_N + \mathrm{pr}_2^*\omega_N$. We consider an antisymplectic involution $\tau\colon M \to M$ defined by $\tau(x,y) = (y,x)$. Then $L = \mathrm{Fix}\,\tau \cong N$. Let J_N be a compatible almost structure on N, and $J_M = -J_N \otimes 1 + 1 \otimes J_N$. The almost complex structure J_M is compatible with ω_M. Note that $w_2(T(N \times N)) = \mathrm{pr}_1^* w_2(TN) + \mathrm{pr}_2^* w_2(TN)$.

If N is spin, then $L = \mathrm{Fix}\,\tau \cong N$ is τ–relatively spin by Example 3.12 and we have $c_1(T(N \times N)) \equiv w_2(T(N \times N)) \equiv 0 \bmod 2$. Since $\pi_1(L) \to \pi_1(N \times N)$ is injective, Corollary 1.6 shows that L is unobstructed and $\mathfrak{m}_2$ defines a graded commutative product structure $\cup_Q$ by (6-13).

Suppose that N is not spin. Take a relative spin structure (V, σ) on $L = \mathrm{Fix}\,\tau \cong N$ such that $V = \mathrm{pr}_1^*(TN)$ and σ is the following spin structure on $(TL \oplus V)|_L \cong (TL \oplus TL)|_L$. Since the composition of the diagonal embedding $SO(n) \to SO(n) \times SO(n)$ and the inclusion $SO(n) \times SO(n) \to SO(2n)$ admits a unique lifting $SO(n) \to Spin(2n)$, we can equip the bundle $TL \oplus TL$ with a canonical spin structure. It determines the spin structure σ on $(TL \oplus V)|_L$. (In this case, $st = \mathrm{pr}_1^* w_2(TN)$.) Then clearly we find that $\tau^* V = \mathrm{pr}_2^* TN$. Note that $\mathrm{pr}_1^* TN$ and $\mathrm{pr}_2^* TN$ are canonically isomorphic to TL by the differentials of the projections pr_1 and pr_2, respectively.

On the other hand, since $(TL \oplus \tau^* V)|_L \cong (TL \oplus TL)|_L \cong (TL \oplus V)|_L$, the spin structure σ is preserved by τ. Therefore, the difference of the conjugacy classes of two relative spin structures $[(V, \sigma)]$ and $\tau^*[(V, \sigma)]$ is measured by $w_2(V \oplus \tau^* V) = w_2(\mathrm{pr}_1^* TN \oplus \mathrm{pr}_2^* TN)$. Using the canonical spin structure on $TL \oplus TL$ mentioned above, we can give a trivialization of $V \oplus \tau^* V$ over the 2–skeleton of L. Hence $w_2(V \oplus \tau^* V)$ is regarded as a class in $H^2(N \times N, L; \mathbb{Z}_2)$. Since

$$w_2(\mathrm{pr}_1^* TN \oplus \mathrm{pr}_2^* TN) = w_2(T(N \times N)) \equiv c_1(T(N \times N)) \bmod 2$$

and $\pi_2(N \times N) \to \pi_2(N \times N, L)$ is surjective, Lemma 6.10 shows that the class is equal to $\mu_L/2$. Hence by Proposition 3.10 we obtain the following:

Lemma 6.11 *In the above situation, the identity map*

$$\mathcal{M}(J; \beta)^{[(V,\sigma)]} \to \mathcal{M}(J; \beta)^{\tau^*[(V,\sigma)]}$$

is orientation-preserving if and only if $\mu_L(\beta)/2$ is even.

Combining Theorem 1.4, we find that the composition

$$\begin{aligned}\mathcal{M}^{\mathrm{main}}_{k+1}(J; \beta; P_1, \ldots, P_k)^{[(V,\sigma)]} &\to \mathcal{M}^{\mathrm{main}}_{k+1}(J; \beta; P_1, \ldots, P_k)^{\tau^*[(V,\sigma)]} \\ &\to \mathcal{M}^{\mathrm{main}}_{k+1}(J; \beta; P_k, \ldots, P_1)^{[(V,\sigma)]}\end{aligned}$$

is orientation-preserving if and only if

$$k + 1 + \sum_{1 \le i < j \le k} \deg' P_i \deg' P_j$$

is even. It follows that $\mathfrak{m}_{0,\tau_*\beta}(1) = -\mathfrak{m}_{0,\beta}(1)$ and hence we find that $L = \mathrm{Fix}\,\tau \cong N$ is unobstructed. This finishes the proof of the assertion (1). Moreover, we also find that $\mathfrak{m}_2$ satisfies Theorem 1.9 (6-12), which induces the graded commutative product $\cup_Q$ as well for the nonspin case. □

6.3.2 Proof of Theorem 1.9(2), I: Preliminaries Before starting the proof, we clarify the choice of the bounding cochain b for which this statement holds. Note we constructed a filtered A_∞ structure on $C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ using τ–invariant Kuranishi structure and τ–invariant perturbation. As we proved in Section 6.3.1, $b = 0$ is the bounding cochain of this filtered A_∞ structure. In fact, $\mathfrak{m}_0$ becomes 0 in the chain level by cancellation. This choice $b = 0$ is one for which the conclusion of Theorem 1.9(2) holds.

Remark 6.12 This particular choice $b = 0$ does not make sense unless we specify the particular way to construct our filtered A_∞ structure. Suppose we define a filtered A_∞ structure on $C(L;\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}})$ using a different perturbation. The set of the gauge equivalence classes of the bounding cochains are independent of the choice up to isomorphism, so there exists a certain bounding cochain which corresponds to the 0 of the filtered A_∞ structure defined by the τ–invariant perturbation. The conclusion of Theorem 1.9(2) holds for that b and that filtered A_∞ structure. However $b = 0$ may not hold in this different filtered A_∞ structure.

We now start the proof of Theorem 1.9(2).

Firstly we explain the proof of Theorem 1.9(2) under the hypothesis that there do not appear holomorphic disc bubbles.

Let $v\colon S^2 \to N$ be a J_N–holomorphic map. We fix three marked points $0, 1, \infty \in S^2 = \mathbb{C} \cup \{\infty\}$. Then we consider the upper half plane $\mathbb{H} \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\}$ and define a map $I(v)\colon \mathbb{H} \to M$ by

$$I(v)(z) = (v(\bar{z}), v(z)).$$

Identifying $(\mathbb{H}, (0,1,\infty))$ with $(D^2, (-1, 1, \sqrt{-1}))$ where $(-1, 1, \sqrt{-1}) \in \partial D^2$, we obtain a map from $(D^2, \partial D^2)$ to (M, L) which we also denote by $I(v)$. One can easily check the converse: For any given J_M–holomorphic map $w\colon (D^2, \partial D^2) \cong (\mathbb{H}, \mathbb{R} \cup \{\infty\}) \to (M, L) = (N \times N, \Delta_N)$, the assignment

$$v(z) = \begin{cases} \mathrm{pr}_2 \circ w(z) & \text{for } z \in \mathbb{H}, \\ \mathrm{pr}_1 \circ w(\bar{z}) & \text{for } z \in \mathbb{C} \setminus \mathbb{H} \end{cases}$$

defines a J_N–holomorphic sphere on N. Therefore the map $v \mapsto I(v)$ gives an isomorphism between the moduli spaces of J_N–holomorphic spheres and J_M–holomorphic discs with boundary in N. We can easily check that this map is induced by the isomorphism of Kuranishi structures.

We remark however that this construction works only at the interior of the moduli spaces of pseudoholomorphic discs and of pseudoholomorphic spheres, that is, the moduli spaces of those without bubble. To study the relationship between their compactifications, we need an extra argument, which will be explained later in Section 6.3.

Next we compare the orientations on these moduli spaces. The moduli spaces of holomorphic spheres have a canonical orientation; see, eg Section 16 of [16]. In Chapter 8 of [10], we proved that a relative spin structure determines a system of orientations on the moduli spaces of bordered stable maps of genus 0. We briefly review a crucial step for comparing orientations in our setting. See page 677 of [10].

Let $w\colon (D^2, \partial D^2) \to (M, L)$ be a J_M–holomorphic map. Denote by ℓ the restriction of w to ∂D^2. Consider the Dolbeault operator

$$\overline{\partial}_{(w^*TM, \ell^*TL)}\colon W^{1,p}(D^2, \partial D^2; w^*TM, \ell^*TL) \to L^p(D^2; w^*TM \otimes \Lambda^{0,1}_{D^2})$$

with $p > 2$. We deform this operator to an operator on the union Σ of D^2 and $\mathbb{C}P^1$ with the origin O of D^2 and the "south pole" S of $\mathbb{C}P^1$ identified. The spin structure σ on $TL \oplus V|_L$ gives a trivialization of $\ell^*(TL \oplus V|_L)$. Since w^*V is a vector bundle on the disc, it has a unique trivialization up to homotopy. Hence ℓ^*V inherits a trivialization which is again unique up to homotopy. Using this trivialization, the vector bundle $E = w^*TM$ descends to E' on Σ.

The index problem is reduced to the one for the Dolbeault operator on Σ. Namely, the restriction of the direct sum of the following two operators to the fiber product of the domains with respect to the evaluation maps at O and S. On D^2, we have the Dolbeault operator for the trivial vector bundle pair $(\underline{\mathbb{C}^n}, \underline{\mathbb{R}^n})$. On $\mathbb{C}P^1$, we have the Dolbeault operator for the vector bundle $E'|_{\mathbb{C}P^1}$. The former operator is surjective, and its kernel is the space of constant sections in $\underline{\mathbb{R}^n}$. The latter has a natural orientation, since it is a Dolbeault operator twisted by $E'|_{\mathbb{C}P^1}$ on a closed Riemann surface. Since the fiber product of kernels is taken on a complex vector space, the orientation of the index is determined by the orientations of the two operators.

Now we go back to our situation. Pick a 1–parameter family $\{\phi_t\}$ of dilations on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ such that $\lim_{t \to +\infty} \phi_t(z) = -\sqrt{-1}$ for $z \in \mathbb{C} \cup \{\infty\} \setminus \{\sqrt{-1}\}$. Here $\sqrt{-1}$ in the upper half plane and $-\sqrt{-1}$ in the lower half plane correspond to the north pole and the south pole of $\mathbb{C}P^1$, respectively. As $t \to +\infty$, the boundary of the second factor of the disc $I(v \circ \phi_t)$ contracts to the point $v(-\sqrt{-1})$, and its image exhausts the whole image of the sphere v, while the whole image of the first factor contracts to $v(-\sqrt{-1})$. Therefore, as $t \to \infty$, the images of the map $z \mapsto I(v \circ \phi_t)(z)$ converge to the constant disc at $(v(-\sqrt{-1}), v(-\sqrt{-1}))$, with a sphere

$$z \in S^2 \mapsto \big(v(-\sqrt{-1}), v(z)\big)$$

attached to the point. If we denote $w_t = I(v \circ \phi_t)$, it follows from our choice $V = \mathrm{pr}_1^* TN$ that the trivialization of $\ell_t^* V$, which is obtained by restricting the trivialization of $w_t^* V = (\mathrm{pr}_1 \circ I(v \circ \phi_t))^* TN$, coincides with the one induced by the frame of the fiber V at $v(-\sqrt{-1})$ for a sufficiently large t. Therefore, considering the linearized index of the family w_t for a large t, it follows from the explanation given in the above paragraph that the map $v \mapsto I(v \circ \phi_t) = w_t$ induces an isomorphism

$$\det(\mathrm{Index}\, D\overline{\partial}_{J_N}(v)) \cong \det(\mathrm{Index}\, D\overline{\partial}_{J_M}(w_t))$$

as oriented vector spaces. By flowing the orientation to $t = 0$ by the deformation ϕ_t, we have proven that the map $v \mapsto I(v)$ respects the orientations of the moduli spaces.

Now we compare the product $\cup_Q$ in (6-13) and the product on the quantum cohomology, presuming, for a while, that they can be calculated by the contribution from the interior of the moduli spaces only.

Definition 6.13 We define the equivalence relation $\sim$ on $\pi_2(N)$ by $\alpha \sim \alpha'$ if and only if $c_1(N)[\alpha] = c_1(N)[\alpha']$ and $\omega(\alpha) = \omega(\alpha')$.

For $\beta = [w\colon (D^2, \partial D^2) \to (N \times N, \Delta_N)] \in \Pi(\Delta_N)$, we set

$$\widetilde{\beta} = [(\mathrm{pr}_2 \circ w) \# (\mathrm{pr}_1 \circ w)\colon D^2 \cup \overline{D}^2 \to N] \in \pi_2(N)/\sim,$$

where $\overline{D}^2$ is the unit disc with the complex structure reversed and $D^2 \cup \overline{D}^2$ is the union of discs glued along boundaries. This defines a homomorphism

$$\psi\colon \Pi(\Delta_N) \to \pi_2(N)/\sim. \tag{6-14}$$

For $\alpha \in \pi_2(N)$, let $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \alpha)$ be the moduli space of pseudoholomorphic maps $v\colon S^2 \to N$ of a homotopy class α with three marked points (without bubble). For $\rho \in \pi_2(N)/\sim$, we put

$$\mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \rho) = \bigcup_{\alpha \in \rho} \mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \alpha).$$

For $\beta \in \Pi(\Delta_N)$, let $\mathcal{M}_3^{\mathrm{reg}}(J_{N\times N};\beta)$ be the moduli space of pseudoholomorphic maps $u\colon (D^2,\partial D^2) \to (N\times N,\Delta_N)$ of class β with three boundary marked points (without disc or sphere bubble). We denote by $\mathcal{M}_3^{\mathrm{main,reg}}(\beta)$ the subset that consists of elements in the main component. We put

$$\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho) = \bigcup_{\psi(\beta)=\rho} \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\beta).$$

Summing up the above construction, we have the following proposition. For a later purpose, we define the map $\mathring{\mathfrak{I}}$ by the inverse of I.

Proposition 6.14 *We have an isomorphism of spaces with Kuranishi structure:*

(6-15) $$\mathring{\mathfrak{I}}\colon \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho) \to \mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho).$$

Moreover, $\mathring{\mathfrak{I}}$ respects the orientations in the sense of Kuranishi structure.

Denote by $*$ the quantum cup product on the quantum cohomology $QH^*(N;\Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$.

For cycles P_0, P_1, P_2 in N such that $\dim \mathcal{M}_3^{\mathrm{sph}}(J_N;\rho) = \deg P_0 + \deg P_1 + \deg P_2$, we can take a multivalued perturbation, a *multisection*, of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)$ such that the intersection of its zero set and $(\mathrm{ev}_0\times\mathrm{ev}_1\times\mathrm{ev}_2)^{-1}(P_0\times P_1\times P_2)$ is a finite subset in $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)$, ie it does not contain elements with domains with at least two irreducible components. Counting these zeros with weights, we obtain the intersection number

$$(P_0\times P_1\times P_2)\cdot[\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)].$$

In other words, for homology classes $[P_0],[P_1],[P_2] \in H_*(N)$, we take cocycles a_0, a_1 and a_2 which represent the Poincaré duals of $[P_0]$, $[P_1]$ and $[P_2]$, respectively. We can take a multisection of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)$ such that the intersection of its zero set and the support of $\mathrm{ev}_0^* a_0 \cup \mathrm{ev}_1^* a_1 \cup \mathrm{ev}_2^* a_2$ is contained in $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)$. Since the zero set of the multivalued perturbation of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)$ is compact, $\mathrm{ev}_0^* a_0 \cup \mathrm{ev}_1^* a_1 \cup \mathrm{ev}_2^* a_2$ is regarded as a cocycle with a compact support. Using such a multivalued perturbation, we obtain $[\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)]$, which is a locally finite fundamental cycle. Thus we find that $(\mathrm{ev}_0^* a_0 \cup \mathrm{ev}_1^* a_1 \cup \mathrm{ev}_2^* a_2)[\mathcal{M}^{\mathrm{sph,reg}}(J_N;\rho)]$ makes sense.

The Poincaré pairing on cohomology is given by

(6-16) $$\langle a,b\rangle = (a\cup b)[N].$$

By definition we have

$$\begin{aligned}\langle a_0, a_1 * a_2\rangle &= \sum_{\rho\in\pi_2(N)/\sim} (\mathrm{ev}_0^* a_0 \cup \mathrm{ev}_1^* a_1 \cup \mathrm{ev}_2^* a_2)[\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)] T^{\omega(\rho)} e^{c_1(N)[\rho]}\\ &= \sum_{\rho\in\pi_2(N)/\sim} (\mathrm{ev}_0^* a_0 \cup \mathrm{ev}_1^* a_1 \cup \mathrm{ev}_2^* a_2)[\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)] T^{\omega(\rho)} e^{c_1(N)[\rho]}.\end{aligned}$$

From the assumption we made at the start of this subsection, the map $\mathfrak{m}_2$ is given by

$$\begin{aligned}\text{(6-17)}\qquad \sum_{\beta;\psi(\beta)=\rho} \mathfrak{m}_{2,\beta}(P_1, P_2) T^{\omega(\beta)} e^{\mu(\beta)/2}\\ = (\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho;P_1,P_2), \mathrm{ev}_0) T^{\omega(\beta)} e^{\mu(\beta)/2}.\end{aligned}$$

Here P_1 and P_2 are cycles, and

$$\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho;P_1,P_2) = (-1)^{\epsilon} \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho) \times_{N^2} (P_1\times P_2),$$

where $\epsilon = (\dim \Delta_N + 1)\deg P_1 = \deg P_1$; see (3-5). (We also assume that the right-hand side becomes a cycle.) These assumptions are removed later in Sections 6.3.3, 6.3.4 and 6.3.5.

Taking a homological intersection number with another cycle P_0, we have

$$\begin{aligned}&P_0 \cdot \big(\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho;P_1,P_2), \mathrm{ev}_0\big)\\ &\qquad =(-1)^{\epsilon} P_0 \cdot \big(\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho) \times_{(\mathrm{ev}_1,\mathrm{ev}_2)} (P_1\times P_2), \mathrm{ev}_0\big)\\ &\qquad =(-1)^{\epsilon} \big(\mathrm{ev}_0^* \mathrm{PD}[P_0] \cup (\mathrm{ev}_1,\mathrm{ev}_2)^* \mathrm{PD}[P_1\times P_2]\big)[\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho)],\end{aligned}$$

where $\mathrm{PD}[P_i]$ (resp. $\mathrm{PD}[P_j\times P_k]$) is the Poincaré dual of P_i in N (resp. $P_j\times P_k$ in $N\times N$). We adopt the convention that

$$\text{(6-18)}\qquad (\mathrm{PD}[P]\cup a)[N] = a[P] \quad \text{for } a\in H^*(N).$$

Since $\dim \Delta_N$ is even, (6-18) implies that

$$\mathrm{ev}_1^* \mathrm{PD}[P_1] \cup \mathrm{ev}_2^* \mathrm{PD}[P_2] = (-1)^{\deg P_1\cdot \deg P_2} (\mathrm{ev}_1,\mathrm{ev}_2)^* \mathrm{PD}[P_1\times P_2].$$

By identifying $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)$ and $\mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho)$ as spaces with oriented Kuranishi structures, we find that

$$\langle \mathrm{PD}[P_0], \mathrm{PD}[P_1] * \mathrm{PD}[P_2]\rangle = (-1)^{\deg P_1(\deg P_2+1)} \langle P_0, \mathfrak{m}_2(P_1,P_2)\rangle,$$

or equivalently,

(6-19) $$\langle \mathrm{PD}[P_1] * \mathrm{PD}[P_2], \mathrm{PD}[P_0]\rangle = (-1)^{\deg P_1(\deg P_2+1)}\langle \mathfrak{m}_2(P_1,P_2),P_0\rangle.$$

Here the right-hand side is the intersection product of P_0 and $\mathfrak{m}_2(P_1,P_2)$. Namely, we put

(6-20) $$\langle P,Q\rangle = P\cdot Q = (-1)^{\deg P\deg Q}\#(P\times_N Q) = \#(Q\times_N P).$$

Note that we use a different convention of the pairing on cycles from [9; 10], cf Definition 8.4.6 in [10], but the same as one in Definition 3.10.4 in Section 3.10.1 of [15]. Therefore we observe the following consistency between pairings on homology (6-20) and cohomology (6-16):

$$\langle P,Q\rangle = \langle \mathrm{PD}[P],\mathrm{PD}[Q]\rangle.$$

6.3.3 Proof of Theorem 1.9(2), II: The isomorphism as modules To complete the proof of Theorem 1.9(2), we need to remove the assumption (6-17), that is, that the product $\mathfrak{m}_2(P_1,P_2)$ is determined only on the part of the moduli space where there is no bubble. We study how our identification of the moduli spaces of pseudoholomorphic discs (attached to the diagonal Δ_N) and of pseudoholomorphic spheres (in N) extends to their compactifications for this purpose. To study this point, we define the isomorphism in Theorem 1.9(2) as $\Lambda_{0,\mathrm{nov}}$ *modules* more explicitly.

As discussed in the introduction, this isomorphism follows from the degeneration at the E_2–level of the spectral sequence of Theorem D [9]. The proof of this degeneration is based on the fact that the image of the differential is contained in the Poincaré dual to the kernel of the inclusion induced homomorphism $H(\Delta_N;\Lambda_{0,\mathrm{nov}})\to H(N\times N;\Lambda_{0,\mathrm{nov}})$, which is actually injective in our case. This fact (Theorem D (D.3) [9]) is proved by using the operator $\mathfrak{p}$ introduced in Section 3.8 of [9]. Therefore, to describe this isomorphism, we recall a part of the construction of this operator below.

Let $\beta\in\Pi(\Delta_N)=\pi_2(N\times N,\Delta_N)/\sim$. We consider $\mathcal{M}_{1;1}(J_{N\times N};\beta)$, the moduli space of bordered stable maps of genus zero with one interior and one boundary marked point in homotopy class β. Let $\mathrm{ev}_0\colon \mathcal{M}_{1;1}(J_{N\times N};\beta)\to\Delta_N$ be the evaluation map at the boundary marked point and $\mathrm{ev}_{\mathrm{int}}\colon \mathcal{M}_{1;1}(J_{N\times N};\beta)\to N\times N$ the evaluation map at the interior marked point. Let (P,f) be a smooth singular chain in Δ_N. We put

$$\mathcal{M}_{1;1}(J_{N\times N};\beta;P) = \mathcal{M}_{1;1}(J_{N\times N};\beta)\,{}_{\mathrm{ev}_0}\times_f P.$$

It has a Kuranishi structure. We take its multisection $\mathfrak{s}$ and a triangulation of its zero set $\mathcal{M}_{1;1}(J_{N\times N};\beta;P)^{\mathfrak{s}}$. Then $(\mathcal{M}_{1;1}(J_{N\times N};\beta;P)^{\mathfrak{s}},\mathrm{ev}_{\mathrm{int}})$ is a singular chain in $N\times N$, which is by definition $\mathfrak{p}_{1,\beta}(P)$; see Definition 3.8.23 of [9]. In our situation, where $\mathfrak{m}_0(1)=0$ for Δ_N in the chain level by Theorem 1.9(1), we have:

Lemma 6.15 *We identify N with Δ_N. Then for any singular chain $P \subset N$, we have*

$$(-1)^{n+1}\partial_{N\times N}(\mathfrak{p}_{1,\beta}(P)) + \sum_{\beta_1+\beta_2=\beta} \mathfrak{p}_{1,\beta_1}(\mathfrak{m}_{1,\beta_2}(P)) = 0. \tag{6-21}$$

Here $\partial_{N\times N}$ is the boundary operator in the singular chain complex of $N \times N$.

Remark 6.16 If $\beta = 0$, then $\mathfrak{p}_{1,0}$ is the identity map, and the second term in (6-21) is equal to $\mathfrak{m}_{1,0}(P)$. Recalling $\mathfrak{m}_{1,0}(P) = (-1)^n \partial_{\Delta_N}(P)$ in Definition 6.1, (6-21) turns out to be

$$(-1)^{n+1}\partial_{N\times N}(\mathfrak{p}_{1,0}(P)) + (-1)^n \mathfrak{p}_{1,0}(\partial_{\Delta_N}(P)) = 0. \tag{6-22}$$

When $\beta \neq 0$, (6-21) is equal to

$$\begin{aligned}(-1)^{n+1}\partial_{N\times N}(\mathfrak{p}_{1,\beta}(P)) &+ \mathfrak{p}_{1,\beta}(\mathfrak{m}_{1,0}(P)) \\ &+ \mathfrak{p}_{1,0}(\mathfrak{m}_{1,\beta}(P)) + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2\neq 0}} \mathfrak{p}_{1,\beta_1}(\mathfrak{m}_{1,\beta_2}(P)) = 0.\end{aligned}$$

Lemma 6.15 is a particular case of Theorem 3.8.9 (3.8.10.2) in [9]. See Remark 6.17 for the sign. We also note that $\mathfrak{p}_{1,0}(P) = P$; see (3.8.10.1) in [9]. We remark that even in the case when P is a singular cycle, $\mathfrak{m}_1(P)$ may not be zero. In other words, the identity map

$$(C(\Delta_N;\Lambda_{0,\mathrm{nov}}),\partial) \to (C(\Delta_N;\Lambda_{0,\mathrm{nov}}),\mathfrak{m}_1) \tag{6-23}$$

is *not* a chain map. We use the operator

$$\mathfrak{p}_{1,\beta}\colon C(\Delta_N;\Lambda_{0,\mathrm{nov}}) \to C(N\times N;\Lambda_{0,\mathrm{nov}})$$

to modify the identity map to obtain a chain map (6-23). Using the projection to the second factor, we define $p_2\colon N\times N \ni (x,y) \mapsto (y,y) \in \Delta_N$. We put

$$\overline{\mathfrak{p}}_{1,\beta} = p_{2*} \circ \mathfrak{p}_{1,\beta}.$$

Then by applying p_{2*} to (6-21), we obtain, for $\beta \neq 0$,

$$-\mathfrak{m}_{1,0}(\overline{\mathfrak{p}}_{1,\beta}(P)) + \overline{\mathfrak{p}}_{1,\beta}(\mathfrak{m}_{1,0}(P)) + \mathfrak{m}_{1,\beta}(P) + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1,\beta_2\neq 0}} \overline{\mathfrak{p}}_{1,\beta_1}(\mathfrak{m}_{1,\beta_2}(P)) = 0. \tag{6-24}$$

Remark 6.17 The sign in (6-21) looks slightly different from one in (3.8.10.2) in [9]. The sign for δ_M in [9; 10] was not specified, since it was not necessary there. Here we specify it as $\delta_M = (-1)^{n+1}\partial_M$. This sign is determined by considering the case when $\beta = 0$, which is nothing but (6-22). Another way to determine this sign is as follows. In the proof of (3.8.10.2) given in Section 3.8.3 of [9], we use the same argument in the proof of Theorem 3.5.11 (Theorem 6.2 in this article) where we define $\mathfrak{m}_{1,0} = (-1)^n \partial_L$ to get the A_∞ formula. The proof of Theorem 3.5.11 uses Proposition 8.5.1 in [10], which is the case where $\ell = \ell_1 = \ell_2 = 0$ in the formulas in Proposition 8.10.5 in [10].

On the other hand, in the case of (3.8.10.2), we use the 0^{th} *interior* marked point as the output evaluation point instead of the 0^{th} *boundary* marked point. Then the proof of (3.8.10.2) uses Proposition 8.10.4 instead of Proposition 8.10.5 in [10]. We can see that the difference of every corresponding sign appearing in Propositions 8.10.4 and 8.10.5 in [10] is exactly -1. Thus we find that $\delta_M = (-1)^{n+1}\partial_M$ in the formula (3.8.10.2) (and also (3.8.10.3)) of [9]. This difference arises from the positions of the factors corresponding to the 0^{th} *boundary* marked point and the 0^{th} *interior* marked point in the definitions of orientations on $\mathcal{M}_{(1,k),\ell}(\beta)$ and $\mathcal{M}_{k,(1,\ell)}(\beta)$, respectively. See the formulas given just before Definitions 8.10.1 and 8.10.2 in [10] for the notation $\mathcal{M}_{(1,k),\ell}(\beta)$ and $\mathcal{M}_{k,(1,\ell)}(\beta)$ respectively.

Definition 6.18 For each given singular chain P in N, we put

$$P(\beta) = \sum_{k=1}^{\infty} \sum_{\substack{\beta_1+\cdots+\beta_k=\beta \\ \beta_i \neq 0}} (-1)^k (\overline{\mathfrak{p}}_{1,\beta_1} \circ \cdots \circ \overline{\mathfrak{p}}_{1,\beta_k})(P),$$

regarding P as a chain in Δ_N. Then we define a chain $\mathcal{I}(P) \in C(N;\Lambda_{0,\mathrm{nov}})$ by

$$\mathcal{I}(P) = P + \sum_{\beta \neq 0} P(\beta) T^{\omega(\beta)} e^{\mu(\beta)/2}.$$

Lemma 6.19 $\mathcal{I}\colon (C(\Delta_N;\Lambda_{0,\mathrm{nov}}), \mathfrak{m}_{1,0}) \to (C(\Delta_N;\Lambda_{0,\mathrm{nov}}), \mathfrak{m}_1)$ *is a chain homotopy equivalence.*

Proof We can use (6-24) to show that $\mathcal{I}$ is a chain map as follows. We prove that $\mathfrak{m}_1 \circ \mathcal{I} - \mathcal{I} \circ \mathfrak{m}_{1,0} \equiv 0 \bmod T^{\omega(\beta)}$ by induction on $\omega(\beta)$. We assume that it holds modulo $T^{\omega(\beta)}$ and then study the terms of order $T^{\omega(\beta)}$. Those terms are sums of the form

$$\mathfrak{m}_{1,0}(P(\beta)) + \mathfrak{m}_{1,\beta}(P) + \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_i \neq 0}} \mathfrak{m}_{1,\beta_1}(P(\beta_2)) - (\mathfrak{m}_{1,0}P)(\beta) \tag{6-25}$$

for the given $\omega(\beta)$. By definition, (6-25) becomes

$$\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k(\mathfrak{m}_{1,0}\circ\overline{\mathfrak{p}}_{1,\beta_1}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k})(P) \tag{6-26}$$
$$+\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}\big((-1)^{k-1}(\mathfrak{m}_{1,\beta_1}\circ\overline{\mathfrak{p}}_{1,\beta_2}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k})(P)\big)-(\mathfrak{m}_{1,0}P)(\beta).$$

Using (6-24), we can show that (6-26) is equal to

$$\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k(\overline{\mathfrak{p}}_{1,\beta_1}\circ\mathfrak{m}_{1,0}\circ\overline{\mathfrak{p}}_{1,\beta_2}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k})(P) \tag{6-27}$$
$$+\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k(\overline{\mathfrak{p}}_{1,\beta_1}\circ\mathfrak{m}_{1,\beta_2}\circ\overline{\mathfrak{p}}_{1,\beta_3}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k})(P)$$
$$-\sum_{k=1,2,\dots}\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k(\overline{\mathfrak{p}}_{1,\beta_1}\circ\overline{\mathfrak{p}}_{1,\beta_2}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k}\circ\mathfrak{m}_{1,0})(P)$$
$$=\sum_{k=1,2,\dots}\sum_{\beta_1\neq 0}\overline{\mathfrak{p}}_{1,\beta_1}\circ\sum_{\substack{\beta_1+\dots+\beta_k=\beta\\ \beta_i\neq 0}}(-1)^k\big(\mathfrak{m}_{1,0}\circ\overline{\mathfrak{p}}_{1,\beta_2}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k}(P)$$
$$+\mathfrak{m}_{1,\beta_2}\circ\overline{\mathfrak{p}}_{1,\beta_3}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k}(P)-\overline{\mathfrak{p}}_{1,\beta_2}\circ\cdots\circ\overline{\mathfrak{p}}_{1,\beta_k}\circ\mathfrak{m}_{1,0}(P)\big).$$

Then (6-27) vanishes by the induction hypothesis.

On the other hand, $\mathcal{I}$ is the identity modulo $\Lambda^+_{0,\mathrm{nov}}$. The lemma follows by the standard spectral sequence argument. □

Thus we obtain an isomorphism as $\Lambda_{0,\mathrm{nov}}$–modules:

$$\mathcal{I}_\#\colon H(N;\Lambda_{0,\mathrm{nov}})\cong \mathrm{HF}(\Delta_N,\Delta_N;\Lambda_{0,\mathrm{nov}}).$$

In order to complete the proof of Theorem 1.9(2), we need to prove

$$\langle\mathcal{I}(P_1)\cup_Q\mathcal{I}(P_2),\mathcal{I}(P_0)\rangle=\langle\mathrm{PD}[P_1]*\mathrm{PD}[P_2],\mathrm{PD}[P_0]\rangle, \tag{6-28}$$

that is,

$$(-1)^{\deg P_1(\deg P_2+1)}\langle\mathfrak{m}_2(\mathcal{I}(P_1),\mathcal{I}(P_2)),\mathcal{I}(P_0)\rangle=\langle\mathrm{PD}[P_1]*\mathrm{PD}[P_2],\mathrm{PD}[P_0]\rangle. \tag{6-29}$$

For the orientation of the moduli spaces which define the operations in (6-29), see Sublemma 6.61 and (6-33)–(6-35) for the left-hand side and Definition 6.34 for the right-hand side. We will prove (6-29) in the next two subsections.

6.3.4 Proof of Theorem 1.9(2), III: Moduli space of stable maps with circle system To define the left-hand side of (6-29), we use the moduli spaces and multisections used in Lagrangian Floer theory, while we use other moduli spaces and multisections to define the right-hand side of (6-29). Recall that in Theorems 6.2 and 1.5, we took particular multisections to get the A_∞ structure. The point of the whole proof we will give here is to show that two multisections, one used for Lagrangian Floer theory and the other one more directly related to the quantum cup product, can be homotoped in the *moduli space of stable maps with circle system*, which will be introduced in Definition 6.29, so that those two multisections finally give the same answer. The purpose of this subsection is to define the moduli space of stable maps with circle system and to describe the topology and the Kuranishi structure in detail. In the next subsection, we will compare two moduli spaces and multisections to complete the proof by interpolating the moduli space of stable maps with circle system.

We consider a class $\rho \in \pi_2(N)/\sim$. Put

$$\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho) = \bigcup_{\alpha\in\rho} \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha),$$

where $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$ is the moduli space of stable maps of genus 0 with three marked points and of homotopy class α. Let $((\Sigma,(z_0,z_1,z_2)),v)$ be a representative of its element. We decompose $\Sigma = \bigcup \Sigma_a$ into irreducible components.

Definition 6.20 (1) Let $\Sigma_0 \subset \Sigma$ be the minimal connected union of irreducible components containing three marked points z_0, z_1 and z_2.

(2) An irreducible component Σ_a is said to be *type I* if it is contained in Σ_0. Otherwise, it is said to be *type II*.

(3) Let Σ_a be an irreducible component of type I. Let k_a be the number of its singular points in Σ_a which do not intersect irreducible components of $\Sigma \setminus \Sigma_0$. Let k'_a be the number of marked points on Σ_a. It is easy to see that $k_a + k'_a$ is either 2 or 3. (See Lemma 6.21 below or the proof.) We say that Σ_a is *type I-1* if $k_a + k'_a = 3$ and *type I-2* if $k_a + k'_a = 2$. We call these $k_a + k'_a$ points *the interior special points*.

Lemma 6.21 *Let k_a and k'_a be the numbers defined above. Then $k_a + k'_a$ is either 2 or 3. Moreover, there exists exactly one irreducible component of type I-1.*

Proof Consider the dual graph T of the prestable curve Σ_0. Note that T is a tree with three exterior edges with a finite number of interior vertices. Therefore, it follows that there is a unique 3–valent vertex, and all others are 2–valent. Since the number $k_a + k'_a$ is precisely the valence of the vertex associated to the component Σ_a, the first statement follows. Furthermore, since the dichotomy of $k_a + k'_a$ being 3 and 2 corresponds to the component Σ_a being of type I-1 and of type I-2, respectively, the second statement follows. □

Our next task is to relate a bordered stable map to $(N \times N, \Delta_N)$ of genus zero with three boundary marked points, to a stable map to N of genus 0 with three marked points. For this purpose, we introduce the notion of stable maps of genus 0 with circle systems, see Definition 6.29.

Firstly, we fix a terminology "circle" on the Riemann sphere. We call a subset C in $\mathbb{C}P^1$ a *circle* if it is the image of $\mathbb{R} \cup \{\infty\}$ by a projective linear transformation Φ as in complex analysis, ie $\Phi(\mathbb{R} \cup \{\infty\}) = C$. From now on, we only consider C with an orientation. The projective linear transformation Φ can be chosen so that the orientation coincides with the one on the boundary of the upper half space $\mathbb{H}$.

For a pseudoholomorphic disc w in $(N \times N, J_{N \times N})$ with boundary on the diagonal Δ_N, the map $\mathring{\mathfrak{I}}$ in Proposition 6.14 gives a pseudoholomorphic sphere $v = \mathring{\mathfrak{I}}(w)$ in (N, J_N). For three boundary marked points z_0, z_1 and z_2 of the domain of w, we have corresponding marked points, which we also denote by z_0, z_1 and z_2 by an abuse of notation, on the Riemann sphere, which is the domain of v. We also note that the boundary of the disc corresponds to the circle passing through z_0, z_1 and z_2 on the Riemann sphere. Thus, when describing the pseudoholomorphic sphere corresponding to a pseudoholomorphic disc, we regard it as a stable map of genus 0 with the image of the boundary of the disc, which is a circle on the Riemann sphere.

Remark 6.22 (1) For each genus-0 bordered stable map to $(N \times N, \Delta_N)$, we construct a genus-0 stable map to N as follows. The construction goes componentwise. For a disc component, we apply $\mathring{\mathfrak{I}}$ as we explained above. For a bubble tree w^{bt} of pseudoholomorphic spheres attached to a disc component at z^{int}, we attach $(\mathrm{pr}_1)_* w^{\mathrm{bt}}$ (resp. $(\mathrm{pr}_2)_* w^{\mathrm{bt}}$) to the lower hemisphere at $\overline{z^{\mathrm{int}}}$ (resp. the upper hemisphere at z^{int}). Here $(\mathrm{pr}_i)_* w^{\mathrm{bt}}$ is $\mathrm{pr}_i \circ w^{\mathrm{bt}}$ with each unstable component shrunk to a point. If $\mathrm{pr}_i \circ w^{\mathrm{bt}}$ is unstable, we do not attach it to the Riemann sphere.

(2) For $\beta \in \pi_2(N \times N, \Delta_N)$, pick a representative $w\colon (D^2, \partial D^2) \to (N \times N, \Delta_N)$. Although w is not necessarily pseudoholomorphic, we have $v\colon S^2 \to N$ in the same way as $\mathring{\mathfrak{I}}$. We call the class $[v] \in \pi_2(N)$ the *double* of the class $[w] \in \pi_2(N \times N, \Delta_N)$.

In preparation for the definition of stable maps of genus 0 with circle systems, we define *admissible systems of circles* in Definitions 6.23, 6.25 and 6.26.

Definition 6.23 includes the case of a moduli space representing various terms of (6-29), that is, a union of doubles of several discs. We glue them at a boundary marked point of one component and an interior or a boundary marked point with the other component.

Let Σ_a be an irreducible component of Σ, where $((\Sigma,(z_0,z_1,z_2)),v)$ is an element of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$. A *domain which bounds* C_a is a disc in $\mathbb{C}P^1$ whose boundary (together with orientation) is C_a. We decompose Σ_a into the union of two discs $D_a^{\pm}$ so that $\mathbb{C}P^1 = D_a^+ \cup D_a^-$, where $\partial D_a^+ = C_a$ as an oriented manifold. Then $\partial D_a^- = -C_a$ as an oriented manifold.

Now consider a component Σ_a of type I-2. We say an interior special point of Σ_a is *inward* if it is contained in the connected component of the closure of $\Sigma_0 \setminus \Sigma_a$ that contains the unique type I-1 component. Otherwise it is called *outward*. (An inward interior special point must necessarily be a singular point.) Note that each type I-2 component contains a unique inward interior marked point and a unique outward interior marked point.

Definition 6.23 An *admissible system of circles of type I* for $((\Sigma,(z_0,z_1,z_2)),v)$ is an assignment of C_a, which is a circle or an empty set (which may occur in case (2) below, see Example 6.24(2)), to each irreducible component Σ_a of type I, such that the following hold:

(1) If Σ_a is type I-1, C_a contains all three interior special points.

(2) Let Σ_a be type I-2. If C_a is not empty, we require the following two conditions:

- C_a contains the outward interior special point.
- The inward interior special point lies on the disc D_a^+, namely either on C_a or $\operatorname{Int} D_a^+$. If the inward interior special point p lies on C_a, the circle $C_{a'}$ on the adjacent component $\Sigma_{a'}$ contains p. Here $\Sigma_{a'}$ meets Σ_a at the node p.

(3) Denote by Σ_{a_0} the unique irreducible component of type I-1, and let C be the maximal connected union of all the C_a containing C_{a_0}. If C contains all z_i, we require the orientation of C to respect the cyclic order of (z_0,z_1,z_2). If some z_i is not on C, we instead consider the following point z_i' on C described below and require that the orientation of C respects the cyclic order of (z_0',z_1',z_2'): There exists a unique irreducible component Σ_a such that C is contained in a connected component of the closure of $\Sigma_0 \setminus \Sigma_a$, that z_i is contained in the other connected component or Σ_a, and that C intersects Σ_a (at the inward interior special point of Σ_a). Then the point z_i' is this inward interior special point of Σ_a.

Example 6.24 (1) Let us consider the admissible system of circles as in Figure 1. The left sphere is of type I-2 and the right sphere is of type I-1. The circle in the right sphere is C in Definition 6.23(3). This is the double of the configuration shown in Figure 2.

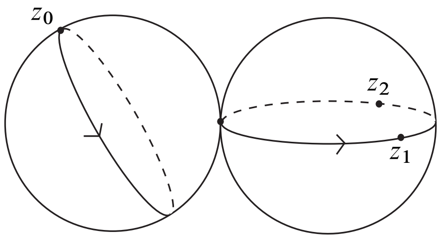

Figure 1: Illustration of Example 6.24(1)

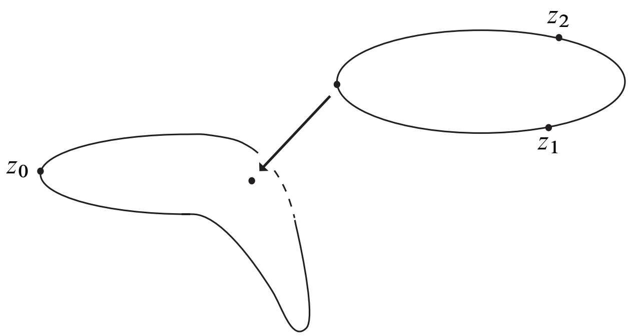

Figure 2: Illustration of Example 6.24(1): coincidence of second components

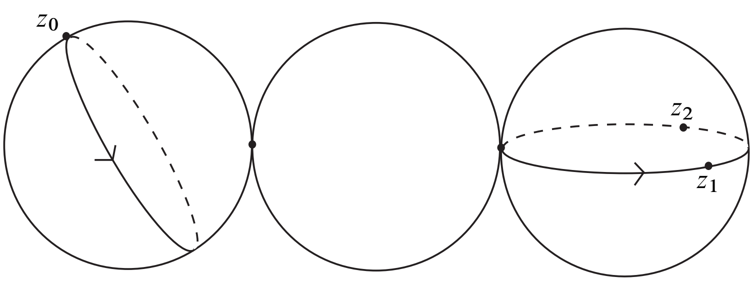

Figure 3: Illustration of Example 6.24(2)

The moduli space of such configurations is identified with the moduli space that is used to define

$$\langle \mathfrak{m}_{2,\beta_2}(P_1, P_2), \overline{\mathfrak{p}}_{1,\beta_1}(P_0)\rangle. \tag{6-30}$$

(2) Type I-2 components may not have circles. For example, see Figure 3.

Next we discuss the admissible system of circles on the irreducible components of type II. A *connected component of type II* of Σ is the closure of a connected component of $\Sigma \setminus \Sigma_0$. Each connected component of type II intersects Σ_0 at one point. We call this point the *root* of our connected component of type II.

We denote by Σ_ρ a connected component of type II and decompose it into the irreducible components:

$$\Sigma_\rho = \bigcup_{a \in I_\rho} \Sigma_a.$$

Then we consider a type II irreducible component Σ_a contained in a Σ_ρ. If Σ_a does not contain the root of Σ_ρ, we consider the connected component of the closure of $\Sigma_\rho \setminus \Sigma_a$ that contains the root of Σ_ρ. Then, there is a unique singular point of Σ_a contained therein. We call this singular point the *root* of Σ_a. Note that if Σ_a contains the root of Σ_ρ, it is, by definition, the root of Σ_a.

Definition 6.25 Let an admissible system of circles of type I on Σ be given. We define an *admissible system of circles of type II* on Σ_ρ to be a union

$$C_\rho = \bigcup_{a \in I_\rho} C_a$$

in which C_a is either a circle or an empty set, and which we require to satisfy the following:

1. If the root of Σ_ρ is not contained in our system of circles of type I, then all of the C_a are the empty set.
2. If C_ρ is nonempty, then it is connected and contains the root of Σ_ρ.
3. Let Σ_a be a type II irreducible component contained in Σ_ρ, and let Σ_b be the irreducible component of Σ that contains the root of Σ_a with $a \neq b$. If the root of Σ_a is contained in C_b, we require C_a to be nonempty.

Definition 6.26 An *admissible system of circles* on Σ is an admissible system of circles of type I together with that of type II on each connected component of type II.

Definition 6.27 Let $\Sigma = \bigcup_a \Sigma_a$ be the decomposition into irreducible components and $\{C_a\}$ the admissible system of circles. (Each C_a is either a circle or an empty set.) Let p be a node joining components Σ_a and Σ_b. We call p a *nonsmoothable* node if p lies in exactly one of C_a and C_b. Otherwise, we call p a *smoothable* node. That is, a node p is smoothable if and only if one of the following conditions holds: (1) $p \in C_a$ and $p \in C_b$, or (2) $p \notin C_a$ and $p \notin C_b$.

Remark 6.28 For a smoothable node p joining two components Σ_a and Σ_b, we can glue them in the following way. We call such a process the *smoothing* at the smoothable node p. If neither $C_a \subset \Sigma_a$ nor $C_b \subset \Sigma_b$ contains the node p, we can perform the gluing of stable maps at the node p. (In such a case, the admissibility of the circle system prohibits that both C_a and C_b are nonempty.) If both C_a and C_b contain the node p, we choose a complex coordinate z_a of Σ_a (resp. z_b of Σ_b) around p such that C_a (resp. C_b) is described as the real axis with the standard orientation. Here we give an orientation on the real axis by the positive direction. Gluing Σ_a and Σ_b by $z_a \cdot z_b = -t$ with $t \in [0, \infty)$, we obtain the gluing of stable maps such that C_a and C_b are glued to an oriented circle.

Definition 6.29 Let $\Sigma = \bigcup_a \Sigma_a$ be a prestable curve, ie a singular Riemann surface of genus 0 at worst with nodal singularities, z_0, z_1 and z_2 marked points on the smooth part of Σ, and $u\colon \Sigma \to N$ a holomorphic map, and let $C_a \subset \Sigma_a$ be either an oriented circle or an empty set. We call $\boldsymbol{x} = (\Sigma, z_0, z_1, z_2, \{C_a\}, u\colon \Sigma \to N)$ a *stable map of genus* 0 *with circle system* if the following conditions are satisfied:

(1) $\{C_a\}$ is an admissible system of circles in the sense of Definition 6.26.

(2) Let P be the set of nonsmoothable nodes on $(\Sigma, z_0, z_1, z_2, \{C_a\})$. For the closure Σ' of each connected component of $\Sigma \setminus P$, one of the following conditions holds:
   - (a) With circles forgotten, $u|_{\Sigma'}\colon \Sigma' \to N$ is still a stable map. Here we put marked points $(\{z_0, z_1, z_2\} \cup P) \cap \Sigma'$ on Σ'.
   - (b) The map u is nonconstant on some irreducible component of Σ'.

(3) The automorphism group $\mathrm{Aut}(\boldsymbol{x})$ is finite. Here we set $\mathrm{Aut}(\boldsymbol{x})$ to be the group of automorphisms ϕ of the singular Riemann surface (Σ, z_0, z_1, z_2) such that $u \circ \phi = u$ and $\phi(\bigcup C_a) = \bigcup C_a$.

Since we only consider stable maps of genus 0, we omit "of genus 0" from now on.

Remark 6.30 (1) If $C_a \neq \varnothing$, then C_a must contain a node or a marked point.

(2) Let Σ_a be an irreducible component, which becomes unstable after the circle system is forgotten; ie the map u is constant on Σ_a and the number of special points contained in Σ_a is less than three. Here a special point means a marked point or a node. In such a case, we find that the number of special points on Σ_a is exactly two and exactly one of them is on C_a. The type I-1 component contains three special points; hence it cannot be such a component. There are three possibilities, cases (i), (ii$+$) and (ii$-$) described in Definition 6.31 below.

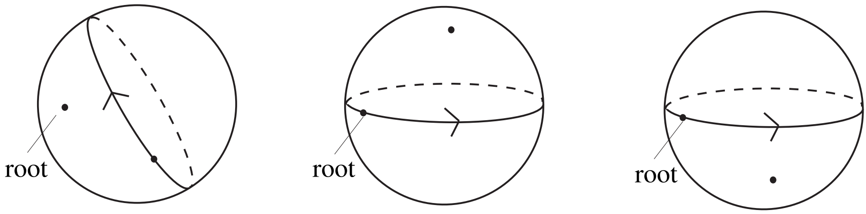

Figure 4: Illustration of Definition 6.31: left to right are cases (i), (ii$+$) and (ii$-$).

By an abuse of terminology, we call the inward special point of a type I-2 component Σ_a the root of Σ_a. (The definition of the root of a type II component is given just before Definition 6.25.)

Definition 6.31 Let Σ_a be an irreducible component which becomes unstable after the circle system is forgotten as we discussed in Remark 6.30(2). There are the following three cases (see Figure 4):

(i) The root is in $\operatorname{Int} D_a^+$ and another special point is on C_a.

(ii$+$) The root is on C_a and the other node is contained in $\operatorname{Int} D_a^+$.

(ii$-$) The root is on C_a and the other node is contained in $\operatorname{Int} D_a^-$.

Remark 6.32 Suppose that Σ_a is a component of case (i). The root is not on C_a, so Σ_a must be type I. Suppose that Σ_a is a component of case (ii). The node, which is different from the root, is not on C_a, so Σ_a must be type II.

Lemma 6.33 *Let $\boldsymbol{x} = (\Sigma, z_0, z_1, z_2, \{C_a\}, u\colon \Sigma \to N)$ be a stable map of genus* 0 *with circle system. If two adjacent components become unstable when forgetting the circle system in the sense of* Remark 6.30(2)*, both of them are of case* (i)*. Moreover, there cannot appear more than two consecutive components which become unstable after the circle system is forgotten.*

Proof Let Σ_{a_1} and Σ_{a_2} be adjacent components which become unstable after the circle system is forgotten. Without loss of generality, we assume that the root of Σ_{a_2} is attached to Σ_{a_1}. Then there are three cases:

(A) Σ_{a_1} and Σ_{a_2} are type I.

(B) Σ_{a_1} is type I and Σ_{a_2} is type II.

(C) Σ_{a_1} and Σ_{a_2} are type II.

By Remark 6.32, we find that in each case, we have the following:

(A) Σ_{a_1} and Σ_{a_2} belong to case (i).
(B) Σ_{a_1} belongs to case (i) and Σ_{a_2} belongs to case (ii).
(C) Σ_{a_1} and Σ_{a_2} belong to case (ii).

Consider case (B). Since Σ_{a_1} is type I, C_{a_1} must contain either one of z_0, z_1 or z_2, or a root of a component of type I. But the node at $\Sigma_{a_1} \cap \Sigma_{a_2}$ is the only special point on C_{a_1}, and Σ_{a_2} is type II. Hence case (B) cannot occur. Next consider case (C). In this case, Σ_{a_2} is type II, but its root is not on C_{a_1}. Hence case (C) cannot occur. Therefore the only remaining case is (A). Namely, both Σ_{a_1} and Σ_{a_2} are type I, hence case (i) in Definition 6.31.

Suppose that Σ_{a_1}, Σ_{a_2} and Σ_{a_3} are consecutive components which become unstable when forgetting the circle system. As we just showed, all of them are case (i). Note that the nodes $\Sigma_{a_1} \cap \Sigma_{a_2}$ and $\Sigma_{a_2} \cap \Sigma_{a_3}$ are nonsmoothable nodes. Thus the middle component Σ_{a_2} does not satisfy Condition (2) in Definition 6.29. Hence the proof. □

Definition 6.34 We denote by $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$ (resp. $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$) the moduli space consisting of stable maps (resp. stable maps with circle system) with three marked points $\vec{z} = (z_0, z_1, z_2)$ representing class α. We put

$$\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;P_1,P_2,P_0) = P_0 \times_{\mathrm{ev}_0} \big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times (P_1 \times P_2)\big),$$
$$\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0) = P_0 \times_{\mathrm{ev}_0} \big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times (P_1 \times P_2)\big).$$

As a space with oriented Kuranishi structure, we define

$$\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;P_1,P_2,P_0) = (-1)^{\deg P_1 \deg P_2} P_0 \times_{\mathrm{ev}_0} \big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times (P_1 \times P_2)\big).$$

To define the orientation on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0)$, we consider

$$\begin{aligned}\mathcal{M}_3^{\mathrm{sph,reg}}&(J_N;\alpha;P_1,P_2,P_0)\\ &= (-1)^{\deg P_1 \deg P_2} P_0 \times_{\mathrm{ev}_0} \big(\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\alpha)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times (P_1 \times P_2)\big)\\ &\subseteq \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;P_1,P_2,P_0).\end{aligned}$$

(See the sentence after (6-14) for the notation $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\alpha)$.) For any element in $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\alpha;P_1,P_2,P_0)$, there is a unique circle passing through z_0, z_1 and z_2 in this order. Hence $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\alpha;P_1,P_2,P_0)$ can be identified with a subset of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0)$. We denote this subset by $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0)$. We define an orientation on $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0)$ in such a way that this identification respects the orientations. We will explain in Remark 6.63(1) how to equip the whole space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C};P_1,P_2,P_0)$ with an orientation.

For $\rho \in \pi_2(N)/\sim$, we define $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$, $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ and $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C})$ in an obvious way.

Then we have

$$\begin{aligned}
&\langle \mathrm{PD}[P_1] * \mathrm{PD}[P_2], \mathrm{PD}[P_0]\rangle \\
&\qquad = (-1)^{\epsilon_1} \sum_{\rho} (\mathrm{ev}_0^* \mathrm{PD}[P_0] \cup \mathrm{ev}_1^* \mathrm{PD}[P_1] \cup \mathrm{ev}_2^* \mathrm{PD}[P_2])[\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)] \\
&\qquad = (-1)^{\epsilon_1+\epsilon_2} \sum_{\rho} (\mathrm{ev}_0^* \mathrm{PD}[P_0] \cup (\mathrm{ev}_1 \times \mathrm{ev}_2)^* \mathrm{PD}[P_1 \times P_2])[\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)] \\
&\qquad = (-1)^{\epsilon_2} \sum_{\rho} (\mathrm{ev}_0^* \mathrm{PD}[P_0])[\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho)_{(\mathrm{ev}_1,\mathrm{ev}_2)} \times (P_1 \times P_2)] \\
&\qquad = \sum_{\rho} (\mathrm{ev}_0^* \mathrm{PD}[P_0])[\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2)] \\
&\qquad = \sum_{\rho} (-1)^{\epsilon_1} P_0 \cdot [\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2)] \\
&\qquad = \sum_{\rho} \#\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0),
\end{aligned}$$

where $\epsilon_1 = \deg P_0(\deg P_1 + \deg P_2)$ and $\epsilon_2 = \deg P_1 \cdot \deg P_2$. For the second equality, we use that $\mathrm{PD}[P_1] \times \mathrm{PD}[P_2] = (-1)^{\deg P_1 \cdot \deg P_2} \mathrm{PD}[P_1 \times P_2]$ in $H^*(N \times N)$. The third and fifth equalities follow from our convention of the Poincaré dual in (6-18). The fourth equality is due to the definition of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2)$. The last equality follows from (6-20). In sum, we have

$$\langle \mathrm{PD}[P_1] * \mathrm{PD}[P_2], \mathrm{PD}[P_0]\rangle = \sum_{\rho} \#\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0). \tag{6-31}$$

Now we will put a topology on the moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ derived from the topology on the moduli spaces $\mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ of stable maps of genus 0 with $3+L$ marked points in Definition 6.48 and Proposition 6.49. Here L is a suitable positive integer explained later. To relate $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;C)$ to $\mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$, we will put L marked points on the source curve of elements of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;C)$. (This process of adding additional marked points is somewhat reminiscent of a similar process in the definition of stable map topology given in [16].)

We start with an elementary fact: for three distinct points p, q and r on $\mathbb{C}P^1$, the circle passing through p, q and r is characterized as the set of points $w \in \mathbb{C}P^1$ such that the cross ratio of p, q, r and w is either a real number or infinity.

The following lemma is clear.

Lemma 6.35 *Let $u^{(i)}, u\colon \Sigma^{(i)} \to N$ be pseudoholomorphic maps from prestable curves $\Sigma^{(i)}$ and Σ representing the class α. Let $w_1^{(i)}$, $w_2^{(i)}$, $w_3^{(i)}$ and $w_4^{(i)}$ be four distinct points on $\Sigma^{(i)}$ and w_1, w_2, w_3 and w_4 four distinct points on an irreducible component Σ_a of Σ, such that a sequence $(\Sigma^{(i)}, \vec{z}^{(i)} \cup \{w_1^{(i)}, \dots, w_4^{(i)}\}, u^{(i)})$ converges to $(\Sigma, \vec{z} \cup \{w_1, \dots, w_4\}, u)$ in $\mathcal{M}^{\mathrm{sph}}_{3+4}(J_N;\alpha)$. Then $w_1^{(i)}, \dots, w_4^{(i)}$ belong to an irreducible component of $\Sigma^{(i)}$ for sufficiently large i, and the cross ratio $[w_1^{(i)} : \cdots : w_4^{(i)}]$ converges to $[w_1 : \cdots : w_4]$ as i tends to $+\infty$.*

We consider a stratification of $\mathcal{M}^{\mathrm{sph}}_3(J_N;\alpha;\mathcal{C})$ by combinatorial types as follows. Recall that by circles we always mean oriented circles in $\mathbb{C}P^1$ as we stated in the second paragraph after the proof of Lemma 6.21.

Definition 6.36 The combinatorial type $\mathfrak{c}(\boldsymbol{x})$ of $\boldsymbol{x} \in \mathcal{M}^{\mathrm{sph}}_3(J_N;\alpha;\mathcal{C})$ is defined by the following data:

(1) The dual graph, whose vertices (resp. edges) correspond to irreducible components of the domain Σ (resp. nodes of Σ).

(2) The data that tells the irreducible components which contain z_0, z_1 and z_2, respectively.

(3) The homology class represented by u restricted to each irreducible component.

(4) For each irreducible component Σ_a, whether C_a is empty or not. If $C_a \neq \varnothing$, we include the data of the list of all nodes contained in $\operatorname{Int} D_a^+$ bounded by the oriented circle C_a and the list of all nodes on C_a. For each node p that is not the root of Σ_a, we include the data that determines whether p lies in the domain bounded by C_a or not. This data determines the side of C_a that p lies on. (Recall that C_a is an oriented circle, and that by "domain D_a^+ bounded by C_a" we involve the orientation together as we mentioned in the third paragraph after Remark 6.22. So the orientation of C_a is a part of data of the combinatorial type.)

For a node p of Σ, we denote by $\Sigma_{\mathrm{in},p}$ (resp. $\Sigma_{\mathrm{out},p}$) the component of Σ which contains p as an outward node (resp. the root node). The combinatorial type of p is defined by the following data:

(1) $C_{\mathrm{in},p}$ contains p or not.

(2) $C_{\mathrm{out},p}$ contains p or not.

Remark 6.37 The combinatorial type of a node p of Σ only depends on the components which contain p. The combinatorial data of $\mathfrak{c}(\boldsymbol{x})$ determine the combinatorial type of each node p, in particular, whether the node p is smoothable or not.

Lemma 6.38 *There are only finitely many combinatorial types in* $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$.

Proof This follows from Lemma 6.33 and the finiteness of combinatorial types of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha)$. ☐

Definition 6.39 Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be the combinatorial types of some elements in $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$. We define the partial order $\mathfrak{c}_1 \preceq \mathfrak{c}_2$ if and only if $\mathfrak{c}_2$ is obtained from $\mathfrak{c}_1$ by smoothing some of the smoothable nodes. See Remark 6.28 for the smoothing of a smoothable node.

We set

$$\mathcal{M}_3^{\mathrm{sph},\succeq\mathfrak{c}(\boldsymbol{x})}(J_N;\alpha;\mathcal{C}) = \{\boldsymbol{x}' \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C}) \mid \mathfrak{c}(\boldsymbol{x}') \succeq \mathfrak{c}(\boldsymbol{x})\}$$

and

$$\mathcal{M}_3^{\mathrm{sph},=\mathfrak{c}(\boldsymbol{x})}(J_N;\alpha;\mathcal{C}) = \{\boldsymbol{x}' \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C}) \mid \mathfrak{c}(\boldsymbol{x}') = \mathfrak{c}(\boldsymbol{x})\}.$$

Remark 6.40 (1) A combinatorial type $\mathfrak{c}$ determines the intersection pattern of the circle systems. For $\boldsymbol{x} \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$, consider the dual graph $\Gamma(\boldsymbol{x})$ of $\bigcup C_a$. Note that $\Gamma(\boldsymbol{x})$ is a subgraph of the dual graph of the domain of $\boldsymbol{x} \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$. Since the genus of the domain of $\boldsymbol{x}$ is 0, each connected component of $\Gamma(\boldsymbol{x})$ is a tree. Namely, we assign a vertex $v_a(\boldsymbol{x})$ to each nonempty circle C_a and an edge joining the vertices $v_{a_1}(\boldsymbol{x})$ and $v_{a_2}(\boldsymbol{x})$ corresponding to C_{a_1} and C_{a_2}, respectively, if they intersect at a node of $\Sigma(\boldsymbol{x})$. The graph $\Gamma(\boldsymbol{x})$ is determined by the combinatorial type $\mathfrak{c}(\boldsymbol{x})$, and we also denote it by $\Gamma(\mathfrak{c})$. The smoothing of a node, which is an intersection point of circles C_{a_1} and C_{a_2}, corresponds to the process of contracting the edge joining $v_{a_1}(\boldsymbol{x})$ and $v_{a_2}(\boldsymbol{x})$. Hence, if $\mathfrak{c}_1 \preceq \mathfrak{c}_2$, then $\Gamma(\mathfrak{c}_2)$ is obtained from $\Gamma(\mathfrak{c}_1)$ by contracting some of its edges. Therefore, we have a canonical one-to-one correspondence between connected components of $\Gamma(\mathfrak{c}_1)$ and $\Gamma(\mathfrak{c}_2)$. In particular, we find that

$$\#\pi_0(\Gamma(\mathfrak{c}_1)) = \#\pi_0(\Gamma(\mathfrak{c}_2)),$$

where $\#\pi_0(\Gamma(\mathfrak{c}))$ denotes the number of connected components of $\Gamma(\mathfrak{c})$.

(2) Each circle C_a in the admissible system of circles on $\boldsymbol{x}$ is oriented. If C_a intersects any other C_b in the circle system, we cut C_a at these intersection points to get a collection of oriented arcs. Recall that each connected component J of $\Gamma(\boldsymbol{x})$ is a tree. Hence the union of oriented circles corresponding to the vertices in J is regarded as an oriented Eulerian circuit, ie an oriented loop $\ell_J(\boldsymbol{x})$ which is a concatenation of the oriented arcs arising from C_a (a is a vertex in J). The oriented loop $\ell_J(\boldsymbol{x})$ is determined up to orientation-preserving reparametrization.

For a stable map with circle system, we can put appropriate additional marked points on the circles in such a way that the circles can be recovered from the additional marked points. More precisely, we consider the following conditions for the elements in $\mathcal{M}^{\mathrm{sph}}_{3+L}(J_N;\alpha)$:

Let $\widetilde{\boldsymbol{y}} \in \mathcal{M}^{\mathrm{sph}}_{3+L}(J_N;\alpha)$, and let $\vec{z}^{\,+} = \vec{z} \cup \{z_3,\dots,z_{2+L}\}$ be the marked points. Namely, $\widetilde{\boldsymbol{y}} = (\Sigma, \vec{z}^{\,+}, u\colon \Sigma \to N)$ is a stable map representing the class α. We consider the following.

Condition 6.41 For each irreducible component Σ_a of the domain Σ, we have that $\Sigma_a \cap \vec{z}^{\,+}$ is either empty or consists of at least three points. In the latter case, $\Sigma_a \cap \vec{z}^{\,+}$ lies on a unique circle C_a on Σ_a.

Note, however, the orientation of circles is not directly determined by Condition 6.41. For the circles $\{C_a\}$ in Condition 6.41, we can associate the dual graph $\Gamma(\widetilde{\boldsymbol{y}})$ in the same manner. Let $E(\widetilde{\boldsymbol{y}})$ be the set of edges of $\Gamma(\widetilde{\boldsymbol{y}})$.

For a fixed $\boldsymbol{x} \in \mathcal{M}^{\mathrm{sph}}_{3}(J_N;\alpha;\mathcal{C})$, we put additional marked points on the union of circles in the admissible system of circles to obtain $\widetilde{\boldsymbol{x}} \in \mathcal{M}^{\mathrm{sph}}_{3+L}(J_N;\alpha)$ such that Condition 6.41 is satisfied. We can find a neighborhood $V(\widetilde{\boldsymbol{x}})$ of $\widetilde{\boldsymbol{x}}$ in

$$\{\widetilde{\boldsymbol{y}} \in \mathcal{M}^{\mathrm{sph}}_{3+L}(J_N;\alpha) \mid \widetilde{\boldsymbol{y}} \text{ satisfies Condition 6.41.}\},$$

so that if $\widetilde{\boldsymbol{y}} \in V(\widetilde{\boldsymbol{x}})$, we can obtain $\Gamma(\widetilde{\boldsymbol{y}})$ from $\Gamma(\widetilde{\boldsymbol{x}})$ by contracting edges in E' and removing edges in E''. Here E' and E'' are disjoint subsets of $E(\widetilde{\boldsymbol{x}})$, which may possibly be empty. In particular, we have

$$\#\pi_0(\Gamma(\widetilde{\boldsymbol{x}})) \le \#\pi_0(\Gamma(\widetilde{\boldsymbol{y}})).$$

Condition 6.42 $\quad \widetilde{\boldsymbol{y}} \in V(\widetilde{\boldsymbol{x}}) \quad \text{and} \quad \#\pi_0(\Gamma(\widetilde{\boldsymbol{x}})) = \#\pi_0(\Gamma(\widetilde{\boldsymbol{y}})).$

Under Condition 6.42, we have a canonical one-to-one correspondence between connected components of $\Gamma(\widetilde{\boldsymbol{x}})$ and those of $\Gamma(\widetilde{\boldsymbol{y}})$. To prove this, it suffices to describe the way to determine the orientation of the circles $\{C_a\}$. We consider Condition 6.43 below for this purpose. Let $J(\widetilde{\boldsymbol{x}})$ be a connected component of $\Gamma(\widetilde{\boldsymbol{x}})$, and let $J(\widetilde{\boldsymbol{y}})$ be the corresponding connected component of $\Gamma(\widetilde{\boldsymbol{y}})$; ie $J(\widetilde{\boldsymbol{y}})$ is obtained from $J(\widetilde{\boldsymbol{x}})$ by contracting some edges.

To make clear that $\vec{z}^{\,+} = \vec{z} \cup \{z_3,\dots,z_{2+L}\}$ are marked points on $\Sigma(\widetilde{\boldsymbol{x}})$, we denote it by $\vec{z}^{\,+}(\widetilde{\boldsymbol{x}}) = (z_0(\widetilde{\boldsymbol{x}}),\dots,z_{2+L}(\widetilde{\boldsymbol{x}}))$. Let $\Sigma_{J(\widetilde{\boldsymbol{x}})}(\widetilde{\boldsymbol{x}})$ be the union of irreducible components which contain circles C_a corresponding to the vertices in $J(\boldsymbol{x})$. We can assign a cyclic order on

$$I_{J(\widetilde{\boldsymbol{x}})} = \{i \mid z_i(\widetilde{\boldsymbol{x}}) \in \Sigma_{J(\widetilde{\boldsymbol{x}})}(\widetilde{\boldsymbol{x}}),\ i = 0,1,2,3,\dots,2+L\}$$

using the oriented loop $\ell_{J(\tilde{\boldsymbol{x}})}(\tilde{\boldsymbol{x}})$ defined as in Remark 6.40(2). Namely, the oriented loop $\ell_{J(\tilde{\boldsymbol{x}})}(\tilde{\boldsymbol{x}})$ passes at z_i $(i \in I_{J(\tilde{\boldsymbol{x}})})$ in a manner compatible with the cyclic order on $I_{J(\tilde{\boldsymbol{x}})}$.

For $\tilde{\boldsymbol{y}}$ satisfying Condition 6.42, we find that $I_{J(\tilde{\boldsymbol{x}})} = I_{J(\tilde{\boldsymbol{y}})}$, which we denote by I_J.

Condition 6.43 Each circle $C_{a'}$ on the domain of $\tilde{\boldsymbol{y}}$ given in Condition 6.41 is equipped with an orientation with the following property. For each connected component J of $\Gamma(\tilde{\boldsymbol{x}})$, the cyclic order on I_J coming from the oriented loop $\ell_J(\tilde{\boldsymbol{x}})$ coincides with the one coming from the oriented loop $\ell_J(\tilde{\boldsymbol{y}})$ defined as in Remark 6.40(2) using the orientation on $C_{a'}$ for any vertex a' in $J(\tilde{\boldsymbol{y}})$.

Note that such an orientation on $C_{a'}$ is unique, if one exists. Under Condition 6.43, each circle C_a in Condition 6.41 is oriented in such a manner. This is the way to define the canonical one-to-one correspondence between connected components of $\Gamma(\tilde{\boldsymbol{x}})$ and those of $\Gamma(\tilde{\boldsymbol{y}})$.

Condition 6.44 The quadruple $(\Sigma, \vec{z}, \{C_a\}, u)$ defines a stable map with circle system $\mathcal{C} = \{C_a\}$.

Definition 6.45 We set

$$\mathcal{U}(\tilde{\boldsymbol{x}}) = \{\tilde{\boldsymbol{y}} \in V(\tilde{\boldsymbol{x}}) \mid \tilde{\boldsymbol{y}} \text{ satisfies Conditions 6.41–6.44 }\}.$$

We denote the natural map by

$$\pi^L_{\tilde{\boldsymbol{x}}} \colon \mathcal{U}(\tilde{\boldsymbol{x}}) \to \mathcal{M}^{\mathrm{sph}}_3(J_N;\alpha;\mathcal{C}), \quad (\Sigma, \vec{z}^{\,+}, u) \mapsto (\Sigma, \vec{z}, \{C_a\}, u).$$

Remark 6.46 Note that we can take the above set $V(\tilde{\boldsymbol{x}})$ for $\tilde{\boldsymbol{x}}$ in such a way that $\mathcal{M}^{\mathrm{sph},=\mathfrak{c}(\boldsymbol{x})}_3(J_N;\alpha;\mathcal{C})$ is contained in the image of $\pi^L_{\tilde{\boldsymbol{x}}} \colon \mathcal{U}(\tilde{\boldsymbol{x}}) \to \mathcal{M}^{\mathrm{sph}}_3(J_N;\alpha;\mathcal{C})$. Also note that we have

$$\pi^L_{\tilde{\boldsymbol{x}}}(\mathcal{U}(\tilde{\boldsymbol{x}})) \subset \mathcal{M}^{\mathrm{sph},\succeq\mathfrak{c}(\boldsymbol{x})}_3(J_N;\alpha;\mathcal{C}).$$

Summarizing the construction above, we have:

Lemma 6.47 *For any $\boldsymbol{x} \in \mathcal{M}^{\mathrm{sph}}_3(J_N;\alpha;\mathcal{C})$, there exist a positive integer L and $\tilde{\boldsymbol{x}} \in \mathcal{M}^{\mathrm{sph}}_{3+L}(J_N;\alpha)$ such that the above naturally defined map $\pi^L_{\tilde{\boldsymbol{x}}} \colon \mathcal{U}(\tilde{\boldsymbol{x}}) \to \mathcal{M}^{\mathrm{sph}}_3(J_N;\alpha;\mathcal{C})$ satisfies $\pi^L_{\tilde{\boldsymbol{x}}}(\tilde{\boldsymbol{x}}) = \boldsymbol{x}$.*

We equip $\mathcal{U}(\tilde{\boldsymbol{x}})$ with the subspace topology of $\mathcal{M}^{\mathrm{sph}}_{3+L}(J_N;\alpha)$.

Definition 6.48 For $U \subset \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$, we define U to be a neighborhood of $\boldsymbol{x}$ if and only if there exist a positive number L and $\widetilde{\boldsymbol{x}} \in \mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ as in Lemma 6.47 such that there exists a neighborhood $\widetilde{U} \subset \mathcal{U}(\widetilde{\boldsymbol{x}})$ of $\widetilde{\boldsymbol{x}}$ satisfying $\pi_{\widetilde{\boldsymbol{x}}}^{L}(\widetilde{U}) \subset U$. Let $\mathfrak{N}(\boldsymbol{x})$ be the collection of all neighborhoods of $\boldsymbol{x}$.

We can show the following:

Proposition 6.49 *The collection $\{\mathfrak{N}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})\}$ satisfies the axiom of the system of neighborhoods. Thus it defines a topology on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$.*

Proof Let U be a neighborhood of $\boldsymbol{x}$ and $\widetilde{U}$ a neighborhood of $\widetilde{\boldsymbol{x}}$ in $\mathcal{U}(\widetilde{\boldsymbol{x}})$ as in Definition 6.48. Take an open neighborhood $\widetilde{U}^{\circ}$ of $\widetilde{\boldsymbol{x}}$ in $\widetilde{U}$. For any $\widetilde{\boldsymbol{y}} \in \widetilde{U}^{\circ}$, there is a neighborhood $\widetilde{W}$ of $\widetilde{\boldsymbol{y}}$ in $\mathcal{U}(\widetilde{\boldsymbol{x}}) \subset \mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ such that $\widetilde{W}$ is contained in $\mathcal{U}(\widetilde{\boldsymbol{y}})$. Thus $\widetilde{W} \subset \mathcal{U}(\widetilde{\boldsymbol{x}}) \cap \mathcal{U}(\widetilde{\boldsymbol{y}})$. Hence we have that $\pi_{\widetilde{\boldsymbol{y}}}^{L}(\widetilde{W}) = \pi_{\widetilde{\boldsymbol{x}}}^{L}(\widetilde{W})$ is a neighborhood of $\boldsymbol{y}$. It remains to show that the definition of the neighborhood in Definition 6.48 is independent of the choice of $\widetilde{\boldsymbol{x}}$. This follows from the next lemma. □

Lemma 6.50 *Let $\widetilde{\boldsymbol{x}} \in \mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ and $\widetilde{\boldsymbol{x}}' \in \mathcal{M}_{3+L'}^{\mathrm{sph}}(J_N;\alpha)$ be as in Lemma 6.47 such that $\pi_{\widetilde{\boldsymbol{x}}}^{L}(\widetilde{\boldsymbol{x}}) = \pi_{\widetilde{\boldsymbol{x}}'}^{L'}(\widetilde{\boldsymbol{x}}') = \boldsymbol{x}$. Then for any neighborhood $\widetilde{U} \subset \mathcal{U}(\widetilde{\boldsymbol{x}})$ of $\widetilde{\boldsymbol{x}}$, there exists a neighborhood $\widetilde{U}' \subset \mathcal{U}(\widetilde{\boldsymbol{x}}')$ of $\widetilde{\boldsymbol{x}}'$ such that $\pi_{\widetilde{\boldsymbol{x}}'}^{L'}(\widetilde{U}') \subset \pi_{\widetilde{\boldsymbol{x}}}^{L}(\widetilde{U})$.*

Proof Let $\widetilde{W}' \subset \mathcal{U}(\widetilde{\boldsymbol{x}}')$ be a sufficiently small neighborhood of $\widetilde{\boldsymbol{x}}'$, which will be specified later. We will define a continuous mapping $\Phi_{\widetilde{\boldsymbol{x}}'}\colon \widetilde{W}' \to \mathcal{U}(\widetilde{\boldsymbol{x}}) \subset \mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ such that $\pi_{\widetilde{\boldsymbol{x}}}^{L} \circ \Phi_{\widetilde{\boldsymbol{x}}'} = \pi_{\widetilde{\boldsymbol{x}}'}^{L'}$. In other words, for $\widetilde{\boldsymbol{y}}' \in \widetilde{W}'$, we will find $\widetilde{\boldsymbol{y}} = \Phi_{\boldsymbol{x}'}(\widetilde{\boldsymbol{y}}') \in \mathcal{U}(\widetilde{\boldsymbol{x}})$ with $\pi_{\widetilde{\boldsymbol{x}}}^{L}(\widetilde{\boldsymbol{y}}) = \pi_{\widetilde{\boldsymbol{x}}'}^{L'}(\widetilde{\boldsymbol{y}}') = \boldsymbol{y}'$. Namely, we find L additional marked points on the circles on the domain $\Sigma(\boldsymbol{y}')$ of $\boldsymbol{y}'$. We define $\widetilde{\boldsymbol{y}}$ in the following steps. Firstly, for $\widetilde{\boldsymbol{y}}' \in \mathcal{U}(\widetilde{\boldsymbol{x}}')$, we define L mutually distinct marked points $w_j(\widetilde{\boldsymbol{y}}')$ $(j = 3, \dots, 2+L)$ on the union of circles $\bigcup C(\boldsymbol{y}')$ in the admissible circle systems of $\boldsymbol{y}'$. By the construction below, any $w_j(\widetilde{\boldsymbol{y}}')$ does not coincide with $z_0(\boldsymbol{y}')$, $z_1(\boldsymbol{y}')$ or $z_2(\boldsymbol{y}')$. Then there exists a neighborhood of $\widetilde{\boldsymbol{x}}'$ in $\mathcal{U}(\widetilde{\boldsymbol{x}}')$ such that $w_j(\widetilde{\boldsymbol{y}}')$ does not coincide with any nodes of $\boldsymbol{y}'$ and defines an element in $\mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$. Finally, we find a neighborhood $\widetilde{W}'$ of $\widetilde{\boldsymbol{x}}'$ in $\mathcal{U}(\widetilde{\boldsymbol{x}}')$ such that $\widetilde{\boldsymbol{y}}' \in \widetilde{W}'$ and $\widetilde{\boldsymbol{y}} \in \mathcal{U}(\widetilde{\boldsymbol{x}})$.

Denote by $\vec{z}^{\,+}(\widetilde{\boldsymbol{x}})$ (resp. $\vec{z}^{\,+}(\widetilde{\boldsymbol{x}}')$) the set of marked points in $\widetilde{\boldsymbol{x}}$, (resp. $\widetilde{\boldsymbol{x}}'$). For $j = 3, \dots, 2+L$, we need to find $w_j(\widetilde{\boldsymbol{y}}')$ on the admissible system of circles on $\Sigma(\boldsymbol{y}')$. Recall that $\boldsymbol{y}' = \pi_{\widetilde{\boldsymbol{x}}'}^{L'}(\widetilde{\boldsymbol{y}}')$.

Firstly, we pick three distinct points $z_{i_1(a)}(\widetilde{\boldsymbol{x}}')$, $z_{i_2(a)}(\widetilde{\boldsymbol{x}}')$ and $z_{i_3(a)}(\widetilde{\boldsymbol{x}}')$ on each circle $C(\boldsymbol{x})_a$ in the admissible circle system of $\boldsymbol{x}$. Let $C(\boldsymbol{x})_{a(j)} \subset \Sigma(\boldsymbol{x})_{a(j)}$ be a circle on an irreducible component $\Sigma(\boldsymbol{x})_{a(j)}$ of the domain of $\boldsymbol{x}$ such that $z_j(\widetilde{\boldsymbol{x}}') \in \vec{z}^{\,+}(\widetilde{\boldsymbol{x}}')$

is on $C(\boldsymbol{x})_{a(j)}$. Note that the graph $\Gamma(\widetilde{\boldsymbol{y}}')$ is obtained by contracting some edges of $\Gamma(\widetilde{\boldsymbol{x}}')$. Denote by $a'(j)$ the vertex of $\Gamma(\widetilde{\boldsymbol{y}}')$ which is the image of the vertex $a(j)$ of $\Gamma(\widetilde{\boldsymbol{x}}')$ by the contracting map. Then the component $\Sigma(\boldsymbol{y}')_{a'(j)}$ of the domain of $\boldsymbol{y}'$ contains $z_{i_1(a(j))}(\widetilde{\boldsymbol{y}}')$, $z_{i_2(a(j))}(\widetilde{\boldsymbol{y}}')$ and $z_{i_3(a(j))}(\widetilde{\boldsymbol{y}}')$. There is a unique circle passing through these three points, which is nothing but $C(\boldsymbol{y}')_{a'(j)} \subset \Sigma(\boldsymbol{y}')_{a'(j)}$ since $\boldsymbol{y}' = \pi^{L'}_{\widetilde{\boldsymbol{x}}'}(\widetilde{\boldsymbol{y}}')$. Then we define $w_j(\widetilde{\boldsymbol{y}}')$ on $C(\boldsymbol{y}')_{a'(j)} \subset \Sigma(\boldsymbol{y}')_{a'(j)}$ so that the cross ratio of $z_{i_1(a(j))}(\widetilde{\boldsymbol{x}}')$, $z_{i_2(a(j))}(\widetilde{\boldsymbol{x}}')$, $z_{i_3(a(j))}(\widetilde{\boldsymbol{x}}')$ and $z_j(\widetilde{\boldsymbol{x}})$ is equal to that of $z_{i_1(a(j))}(\widetilde{\boldsymbol{y}}')$, $z_{i_2(a(j))}(\widetilde{\boldsymbol{y}}')$, $z_{i_3(a(j))}(\widetilde{\boldsymbol{y}}')$ and $w_j(\widetilde{\boldsymbol{y}}')$. Here the cross ratio is taken on $\Sigma(\boldsymbol{x})_{a(j)}$ and $\Sigma(\boldsymbol{y}')_{a'(j)}$, which are both biholomorphic to $\mathbb{C}P^1$.

Note that $w_j(\widetilde{\boldsymbol{y}}')$ depends continuously on $\widetilde{\boldsymbol{y}}'$ in the following sense. Take a real projectively linear isomorphism $\psi_{\widetilde{\boldsymbol{y}}'}$ from $C(\boldsymbol{y}')_{a'(j)}$ to $\mathbb{R}P^1$ in $\mathbb{C}P^1$ such that $z_{i_1(a(j))}(\widetilde{\boldsymbol{y}}')$, $z_{i_2(a(j))}(\widetilde{\boldsymbol{y}}')$ and $z_{i_3(a(j))}(\widetilde{\boldsymbol{y}}')$ are sent to 0, 1 and ∞. Then $\psi_{\widetilde{\boldsymbol{y}}'}(w_j(\widetilde{\boldsymbol{y}}'))$ is continuous with respect to $\widetilde{\boldsymbol{y}}'$. Since $w_j(\widetilde{\boldsymbol{x}}') = z_j(\widetilde{\boldsymbol{x}})$ by the construction and $w_j(\widetilde{\boldsymbol{y}}')$ depends continuously on $\widetilde{\boldsymbol{y}}'$, if $\widetilde{\boldsymbol{y}}'$ is sufficiently close to $\widetilde{\boldsymbol{x}}'$ in $\mathcal{U}(\widetilde{\boldsymbol{x}}')$, the 0^{th}, 1^{st} and 2^{nd} marked points on $\boldsymbol{y}'$ and $w_j(\widetilde{\boldsymbol{y}}')$ $(j = 3, \dots, 2+L)$ are mutually distinct and do not coincide with nodes of $\Sigma(\boldsymbol{y}')$ on $\Sigma(\boldsymbol{y}')_{a'}$. Hence $\boldsymbol{y}'$, with marked points $z_0(\boldsymbol{y}'), z_1(\boldsymbol{y}'), z_2(\boldsymbol{y}'), w_3(\widetilde{\boldsymbol{y}}'), \dots, w_{2+L}(\widetilde{\boldsymbol{y}}')$, defines an element in $\mathcal{M}^{\text{sph}}_{3+L}(J_N;\alpha)$, which we denote by $\widetilde{\boldsymbol{y}}$. We take a neighborhood $\widetilde{W}'$ of $\widetilde{\boldsymbol{x}}'$ in $\mathcal{U}(\widetilde{\boldsymbol{x}}')$ in order that $\widetilde{\boldsymbol{y}} \in V(\widetilde{\boldsymbol{x}})$ for $\widetilde{\boldsymbol{y}}' \in \widetilde{W}'$. Note that the number of marked points of $\widetilde{\boldsymbol{y}}$ is either zero or at least three by the construction. Since $\pi^{L'}_{\widetilde{\boldsymbol{x}}'}(\widetilde{\boldsymbol{y}}') = \boldsymbol{y}'$ is a stable map with admissible system of circles, any $\widetilde{\boldsymbol{y}} \in \widetilde{W}'$ satisfies Conditions 6.41–6.44. We define $\Phi_{\widetilde{\boldsymbol{x}}'}\colon \widetilde{W}' \to \mathcal{U}(\widetilde{\boldsymbol{x}})$ by $\Phi_{\widetilde{\boldsymbol{x}}'}(\widetilde{\boldsymbol{y}}') = \widetilde{\boldsymbol{y}}$ constructed above. It is continuous, and $\Phi_{\widetilde{\boldsymbol{x}}'}(\widetilde{\boldsymbol{x}}') = \widetilde{\boldsymbol{x}}$. Hence, for any neighborhood $\widetilde{U}$ of $\widetilde{\boldsymbol{x}}$ in $\mathcal{U}(\widetilde{\boldsymbol{x}})$, there exists a neighborhood $\widetilde{U}'$ of $\widetilde{\boldsymbol{x}}'$ in $\mathcal{U}(\widetilde{\boldsymbol{x}}')$ such that $\Phi_{\widetilde{\boldsymbol{x}}'}(\widetilde{U}') \subset \mathcal{U}(\widetilde{\boldsymbol{x}})$. This implies that $\pi^{L'}_{\widetilde{\boldsymbol{x}}'}(\widetilde{U}') \subset \pi^{L}_{\widetilde{\boldsymbol{x}}}(\widetilde{U})$. □

When the combinatorial type $\mathfrak{c} = \mathfrak{c}(\boldsymbol{x})$ is fixed, we have mentioned in Remark 6.46 that $\mathcal{U}(\widetilde{\boldsymbol{x}})$ enjoys the following property:

$$\mathcal{M}^{\text{sph},=\mathfrak{c}(\boldsymbol{x})}_{3}(J_N;\alpha;\mathcal{C}) \subset \pi^{L}_{\widetilde{\boldsymbol{x}}}(\mathcal{U}(\widetilde{\boldsymbol{x}})). \tag{6-32}$$

We observe that $\mathcal{U}(\widetilde{\boldsymbol{x}})$ satisfies the second axiom of countability, since the moduli space $\mathcal{M}^{\text{sph}}_{3+L}(J_N;\alpha)$ does. We recall from Lemma 6.38 that $\mathcal{M}^{\text{sph}}_{3}(J_N;\alpha;\mathcal{C})$ carries only a finite number of combinatorial types. Combining these observations with (6-32), we obtain the following:

Proposition 6.51 *$\mathcal{M}^{\text{sph}}_{3}(J_N;\alpha;\mathcal{C})$ satisfies the second axiom of countability.*

Hence compactness is equivalent to sequential compactness, and the Hausdorff property is equivalent to the uniqueness of the limit of convergent sequences for moduli spaces of stable maps with circle system.

Proposition 6.52 *The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ is sequentially compact.*

Proof Let $\boldsymbol{x}^{(j)} = \big(\Sigma(\boldsymbol{x}^{(j)}), z_0^{(j)}, z_1^{(j)}, z_2^{(j)}, \{C_a^{(j)}\}, u^{(j)}\colon \Sigma(\boldsymbol{x}^{(j)}) \to N\big)$ be a sequence in $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$. Because there are a finite number of combinatorial types, we may assume that $\mathfrak{c}(\boldsymbol{x}^{(j)})$ are independent of j. First of all, we find a candidate of the limit of a subsequence of $\boldsymbol{x}^{(j)}$. In Step 1, we consider irreducible components in $\Sigma(\boldsymbol{x}^{(j)})$ explained in Remark 6.30(2). Since these components are not stable after forgetting the admissible system of circles, we put a point on each component so that these components become stable. We obtain $\boldsymbol{x}^{(j)+}$ at this stage. Because the moduli space of stable maps with marked points is compact, we can take a convergent subsequence. We denote its limit by $\boldsymbol{x}_\infty^+$. We will find an admissible system of circles on $\boldsymbol{x}_\infty^+$ and insert irreducible components explained in Remark 6.30(2), if necessary, to obtain a candidate of the limit of $\boldsymbol{x}^{(j)}$ in Steps 2 and 3. There are three cases, case (i), case (ii+) and case (ii−), in Definition 6.31. In Step 2, we deal with sequences $C_a^{(j)}$ of circles on $\Sigma(\boldsymbol{x}^{(j)+})_a$ which do not collapse to any node of $\Sigma(\boldsymbol{x}_\infty^+)$. In Step 3, we discuss when insertions of such irreducible components are necessary and explain how to perform insertions to obtain the candidate of the limit of $\boldsymbol{x}^{(j)}$. The detail follows.

Step 1 If $\boldsymbol{x}^{(j)}$ has irreducible components which become unstable after the circle system is forgotten, we put an additional marked point $z_a^{(j),++}$ for each such component $\Sigma(\boldsymbol{x}^{(j)})_a$. Such an irreducible component contains its root $p_{\mathrm{root},a}^{(j)}$, another special point $p_{\mathrm{out},a}^{(j)}$ and the circle $C_a^{(j)}$. We can take $z_a^{(j),++}$ for each j with the following property. For any j and j', there exists a biholomorphic map $\phi_{j',j}\colon \Sigma(\boldsymbol{x}^{(j)})_a \to \Sigma(\boldsymbol{x}^{(j')})_a$ such that $\phi_{j',j}$ sends $p_{\mathrm{root},a}^{(j)}$, $p_{\mathrm{out},a}^{(j)}$, $z_a^{(j),++}$ and $C_a^{(j)}$ to $p_{\mathrm{root},a}^{(j')}$, $p_{\mathrm{out},a}^{(j')}$, $z_a^{(j'),++}$ and $C_a^{(j')}$, respectively. Such components are determined by the combinatorial data, hence the number of these components are independent of j. If there are ℓ such components, we obtain a sequence $\boldsymbol{x}^{(j)+} \in \mathcal{M}_{3+\ell}^{\mathrm{sph}}(J_N;\alpha)$.

Step 2 Since $\mathcal{M}_{3+\ell}^{\mathrm{sph}}(J_N;\alpha)$ is compact, there is a convergent subsequence of $\boldsymbol{x}^{(j)+}$. We may assume that $\boldsymbol{x}^{(j)+}$ is convergent. Denote its limit by $\boldsymbol{x}_\infty^+ \in \mathcal{M}_{3+\ell}^{\mathrm{sph}}(J_N;\alpha)$. If the marked points $z_3,\dots,z_{2+\ell}$ are forgotten, $\Sigma(\boldsymbol{x}_\infty^+)$ is a prestable curve of genus 0 with three marked points z_0, z_1 and z_2. Hence we can define a unique irreducible component of type I-1 in the same way as in Definition 6.23(1).

Let $\{V_k\}_k$, with $V_{k+1} \subset V_k$, be a sequence of open neighborhoods of the set of nodes on the domain $\Sigma(\boldsymbol{x}_\infty^+)$ such that $\bigcap_k V_k$ is the set of nodes. There exists a positive integer $N(k)$ such that if $j \ge N(k)$, there is a holomorphic embedding

$$\phi_k^{(j)}\colon \Sigma(\boldsymbol{x}_\infty^+) \setminus V_k \to \Sigma(\boldsymbol{x}^{(j)+}).$$

For each component $\Sigma(\boldsymbol{x}_\infty^+)_a$, there is an irreducible component $\Sigma(\boldsymbol{x}^{(j)+})_{a'}$ such that $\phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_a \setminus V_k) \subset \Sigma(\boldsymbol{x}^{(j)+})_{a'}$.

From now on, we will take subsequences of $\{j\}$ successively and rename it $\{j\}$.

Step 2-1 If $C_{a'}^{(j)} \subset \Sigma(\boldsymbol{x}^{(j)+})_{a'}$ is empty for any j, we set $C_a \subset \Sigma(\boldsymbol{x}_\infty^+)_a$ to be an empty set.

Step 2-2 Consider the case where there is k such that $\phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_a \setminus V_k)$ intersects the circle $C_{a'}^{(j)}$ on $\Sigma(\boldsymbol{x}^{(j)+})_{a'}$ for any j.

We treat the following two cases separately.

Case 1 For any point $p \in \Sigma(\boldsymbol{x}_\infty^+)_a$ which is not a node, there is a neighborhood $U(p)$ of p such that $C_{a'}^{(j)}$ is not contained in $\phi_k^{(j)}(U(p))$ for any j.

Since $\Sigma(\boldsymbol{x}_\infty^+)_a \setminus V_k$ is compact, after taking a subsequence of j, we may assume that there are three distinct points $p_1, p_2, p_3 \in \Sigma(\boldsymbol{x}_\infty^+)_a \setminus V_k$ such that there are mutually disjoint neighborhoods $U(p_1)$, $U(p_2)$ and $U(p_3)$ with $\phi_k^{(j)}(U(p_i)) \cap C_{a'}^{(j)} \neq \varnothing$. We pick $p_i^{(j)} \in \phi_k^{(j)}(U(p_i)) \cap C_{a'}^{(j)}$. After taking a suitable subsequence of j, we may assume that $p_i^{(j)}$ converges to $p_i^{(\infty)}$ for $i = 1, 2, 3$. Then we take the circle passing through $p_1^{(\infty)}$, $p_2^{(\infty)}$ and $p_3^{(\infty)}$, which we denote by $C_a \subset \Sigma(\boldsymbol{x}_\infty^+)_a$. Since $C_{a'}^{(j)}$ are oriented and all $\boldsymbol{x}^{(j)}$ have the same combinatorial type, C_a is also canonically oriented. It is clear that the circle C_a is uniquely determined once the subsequence of j is taken as above. (For example, consider the cross ratios.)

Note that Case 1 is applied to each irreducible component treated in Step 1; hence an oriented circle is put on each such component.

Case 2 There is a point p on $\Sigma(\boldsymbol{x}_\infty^+)_a$ such that p is not a node and, for any neighborhood $U(p)$ of p, there exists a positive integer N such that if $j \geq N$, then $C_{a'}^{(j)}$ is contained in $\phi_k^{(j)}(U(p))$.

Since p as above is not a node, $C_{a'}^{(j)}$ must contain a unique special point by the admissibility of the circle system. (If there are at least two special points, then these points get closer as $j \to \infty$. Hence there should appear a new component attached at p in $\Sigma(\boldsymbol{x}_\infty^+)$.) In this case, we attach a new irreducible component at p as in case (i) in Definition 6.31. Note that the attached component contains a circle of type I.

Step 3 Let p be a node on $\Sigma(\boldsymbol{x}_\infty^+)$. Then either p is the limit of nodes $p^{(j)}$ on $\Sigma(\boldsymbol{x}^{(j)+})$, or p appears as a degeneration of $\Sigma(\boldsymbol{x}^{(j)+})$. We insert an irreducible component explained in Remark 6.30(2) as discussed in Cases 1-2 and 2-2 below.

Case 1 p is the limit of nodes $p^{(j)}$.

Case 1-1 If p and $p^{(j)}$ are either both smoothable or both nonsmoothable, we keep the node p as it is.

Case 1-2 If one of them is smoothable and the other is nonsmoothable, we insert a new irreducible component as in Remark 6.30(2) in the following way.

Firstly, we consider the case that $p^{(j)}$ are nonsmoothable but p is smoothable. In this case, we find that both components containing $p^{(j)}$ are of type I. We insert the component of case (i) in Definition 6.31 at p.

Next, we consider the case that $p^{(j)}$ are smoothable but p is nonsmoothable. If $p^{(j)}$ are smoothable nodes joining two components of type I in $\Sigma(\boldsymbol{x}^{(j)+})$, we insert the component of case (i) in Definition 6.31. Suppose that at least one of irreducible components containing $p^{(j)}$ is of type II. There are two possibilities, which we discuss separately. The first possibility is that both components contain nonempty circles in the admissible system. Let $\Sigma(\boldsymbol{x}^{(j)+})_a$ be the component such that $p^{(j)}$ is its outward node and $\Sigma(\boldsymbol{x}^{(j)+})_b$ the component such that $p^{(j)}$ is its root node. In this case, $C_a^{(j)}$ cannot collapse to p since $C_a^{(j)}$ must contain at least one special point other than $p^{(j)}$. Namely, if $\Sigma(\boldsymbol{x}^{(j)})_a$ is of type I, then $C_a^{(j)}$ must contain one of z_0, z_1, z_2 or a node of a tree of components of type I. If $\Sigma(\boldsymbol{x}^{(j)})_a$ is of type II, then $C_a^{(j)}$ must contain the root node of $\Sigma(\boldsymbol{x}^{(j)})_a$. Hence we consider the case that $C_b^{(j)}$ collapse to p. Recall that $D_b^{(j)+}$ denotes the domain in $\Sigma(\boldsymbol{x}^{(j)+})_b$ bounded by the oriented circle $C_b^{(j)}$. If $D_b^{(j)+}$ collapses to p, then we insert the component of case (ii$-$) in Definition 6.31. Otherwise, we insert the component of case (ii$+$) in Definition 6.31.

The second possibility is that only one of $\Sigma(\boldsymbol{x}^{(j)+})_a$ or $\Sigma(\boldsymbol{x}^{(j)+})_b$ contains a nonempty circle in the admissible system. Note that the component $\Sigma(\boldsymbol{x}^{(j)+})_a$ must contain $p^{(j)}$ as an outward node. It is impossible that $C_a^{(j)} = \varnothing$ and $C_b^{(j)} \neq \varnothing$. Hence $C_a^{(j)}$ is nonempty. Since $p^{(j)}$ is a smoothable node, $C_a^{(j)}$ does not contain $p^{(j)}$. Denote by $D_a^{(j)+}$ the domain in $\Sigma(\boldsymbol{x}^{(j)+})_a$ bounded by the oriented circle $C_a^{(j)}$. If $D_a^{(j)+}$ contains the node $p^{(j)}$, we insert the component of case (ii$+$) in Definition 6.31 at the node p. Otherwise, we insert the component of case (ii$-$) in Definition 6.31 at the node p.

Case 2 p appears as a degeneration of $\Sigma(\boldsymbol{x}^{(j)+})$.

Case 2-1 If p is smoothable, we keep the node p as it is.

Case 2-2 If p is nonsmoothable, we insert a new irreducible component explained in Remark 6.30(2) in a similar way to Case 1-2 above as follows.

Suppose that a sequence $\Sigma(\boldsymbol{x}^{(j)+})_a$ degenerates to a nodal curve with a node p, which is nonsmoothable. Let $\Sigma(\boldsymbol{x}^+_\infty)_{a_1}$ and $\Sigma(\boldsymbol{x}^+_\infty)_{a_2}$ be components containing the node p such that $\Sigma(\boldsymbol{x}^+_\infty)_{a_2}$ is farther from the component of type I-1 than $\Sigma(\boldsymbol{x}^+_\infty)_{a_1}$. Here the component of type I-1 of $\Sigma(\boldsymbol{x}^+_\infty)$ is defined in the beginning of Step 2.

There are two possibilities. The first possibility is that there exists a positive integer k such that $C_a^{(j)} \cap \phi_k^{(j)}(\Sigma(\boldsymbol{x}^+_\infty)_{a_1} \setminus V_k) = \varnothing$ for any sufficiently large $j > N(k)$. In this case, we find that $\Sigma(\boldsymbol{x}^{(j)+})_a$ is of type I. We insert the component of case (i) in

Definition 6.31 at the node p. The second possibility is that there exists a positive integer k such that $C_a^{(j)} \cap \phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_2} \setminus V_k) = \varnothing$ for any sufficiently large $j > N(k)$. If there exists a positive integer k' such that $D_a^{(j)+} \cap \phi_{k'}^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_2} \setminus V_{k'}) = \varnothing$, we insert the component of case (ii$-$) in Definition 6.31 at the node p. Here $D_a^{(j)+}$ is the domain in $\Sigma(\boldsymbol{x}^{(j)+})_a$ bounded by the oriented circle $C_a^{(j)}$. Otherwise, we insert the component of case (ii$+$) in Definition 6.31 at the node p.

Remark 6.53 Suppose that the adjacent components $\Sigma(\boldsymbol{x}_\infty^+)_{a_1}$ and $\Sigma(\boldsymbol{x}_\infty^+)_{a_2}$ contain circles C_{a_1} and C_{a_2}, respectively, which are put in Step 2-2, and suppose that $\Sigma(\boldsymbol{x}^{(j)})_{a_1'} = \Sigma(\boldsymbol{x}^{(j)})_{a_2'}$; ie this component degenerates to a nodal curve including $\Sigma(\boldsymbol{x}_\infty^+)_{a_1}$ and $\Sigma(\boldsymbol{x}_\infty^+)_{a_2}$. Since $C_{a_1'}^{(j)}$ is a circle which is connected, it passes through the neck region corresponding to the node between $\Sigma(\boldsymbol{x}_\infty^+)_{a_1}$ and $\Sigma(\boldsymbol{x}_\infty^+)_{a_2}$. Hence C_{a_1} and C_{a_2} pass through the node, which is smoothable in the sense of Definition 6.27. Case 2-2 is the case that one of C_{a_1} and C_{a_2} is a circle passing through the node and the other is empty.

After these processes, we obtain $\boldsymbol{x}''_\infty$ as the candidate of the limit of $\boldsymbol{x}^{(j)}$. By the construction, $\boldsymbol{x}''_\infty$ is equipped with an admissible system of circles.

Remark 6.54 By our construction, in particular, Step 3 Cases 1-2 and 2-2, we find

$$\mathfrak{c}(\boldsymbol{x}''_\infty) \preceq \mathfrak{c}(\boldsymbol{x}^{(j)}).$$

Now we show the following

Lemma 6.55 *There is a subsequence of $\boldsymbol{x}^{(j)}$ that converges to $\boldsymbol{x}''_\infty$ in $\mathcal{M}_3^{\mathrm{sph}}(J_N; \alpha; \mathcal{C})$.*

Proof We will add suitable marked points on $\boldsymbol{x}^{(j)}$ to obtain $\tilde{\boldsymbol{x}}^{(j)}$. For an irreducible component in $\boldsymbol{x}^{(j)}$, which becomes unstable when forgetting the circle, we added the marked point $z_a^{(j),++}$ in Step 1. Since such an irreducible component contains three special points, ie nodes or marked points and the holomorphic map is constant on the irreducible component, the limit $\boldsymbol{x}_\infty^+$ must contain an irreducible component of the same type. We add one more marked point on the circle on the component in $\boldsymbol{x}^{(j)}$ and $\boldsymbol{x}''_\infty$, respectively, so that the four special points on the component have the fixed cross ratio. (In total, we add two marked points on the circle in this case.)

We put additional marked points on other irreducible components as follows. If an irreducible component does not contain a circle in the admissible system, we do not put additional marked points. Then the remaining components are either those discussed in Step 2 or those discussed in Step 3. We deal with them separately.

For an irreducible component in Case 1 of Step 2-2, we have three marked points p_1, p_2 and p_3 on the circle C_a. We pick $p_i^{(j)} \in \phi_k^{(j)}(U(p_i)) \cap C_{a'}^{(j)}$, $(i = 1, 2, 3)$. (Here we added three marked points on the circle.)

For an irreducible component in Case 2 of Step 2-2, we have a special point q on the new attached irreducible component $\Sigma(\boldsymbol{x}''_\infty)_b$. For sufficiently large j, the circle $C_{a'}^{(j)}$ is contained in $\phi_k^{(j)}(U(p))$. By the admissibility of the circle system, there should be a special point $q^{(j)}$ on $C_{a'}^{(j)}$. We put two more marked points q_1 and q_2 on the circle C_b on $\Sigma(\boldsymbol{x}''_\infty)_b$. We choose $p^{(j)} \notin \phi_k^{(j)}(U(p))$ which converges to some $p' \in \Sigma(\boldsymbol{x}''_\infty)_a$. We choose $q_1^{(j)}, q_2^{(j)} \in C_{a'}^{(j)}$ in such a way that the cross ratio of $p^{(j)}$, $q^{(j)}$, $q_1^{(j)}$ and $q_2^{(j)}$ is equal to the one of $\overline{p}$, q, q_1 and q_2 for any j. Here $\overline{p}$ is the node, where we attach $\Sigma(\boldsymbol{x}''_\infty)_b$ to $\Sigma(\boldsymbol{x}''_\infty)_a$ in Case 2 of Step 2-2. (In this case, we add two marked points.)

For each newly inserted component $\Sigma(\boldsymbol{x}''_\infty)_c$ in Step 3, we add two additional marked points $q_1, q_2 \in C_c$ as follows. Firstly, we consider Case 1. Let $\Sigma(\boldsymbol{x}^+_\infty)_a$ and $\Sigma(\boldsymbol{x}^+_\infty)_b$ be the irreducible components of $\Sigma(\boldsymbol{x}^+_\infty)$ which intersect at p, and let $\Sigma(\boldsymbol{x}^{(j)+})_a$ and $\Sigma(\boldsymbol{x}^{(j)+})_b$ be the irreducible components of $\Sigma(\boldsymbol{x}^{(j)+})$ which intersect at $p^{(j)}$. Here we arrange that p (resp. $p^{(j)}$) is the root node of $\Sigma(\boldsymbol{x}^+_\infty)_b$ (resp. $\Sigma(\boldsymbol{x}^{(j)+})_b$). The new component $\Sigma(\boldsymbol{x}''_\infty)_c$ is inserted between $\Sigma(\boldsymbol{x}^+_\infty)_a$ and $\Sigma(\boldsymbol{x}^+_\infty)_b$. We denote by $\widetilde{p}$ the node of $\Sigma(\boldsymbol{x}''_\infty)_c$ which has the same combinatorial type as the node $p^{(j)}$ (see Definition 6.36 for the definition of the combinatorial type of a node) and by $\widetilde{p}'$ the other node of $\Sigma(\boldsymbol{x}''_\infty)_c$. Let $d = a$ or b such that $\widetilde{p}$ is the root of $\Sigma(\boldsymbol{x}''_\infty)_c$ if and only if $p^{(j)}$ is the root of $\Sigma(\boldsymbol{x}^{(j)+})_d$. We discuss the following two cases separately:

(a) $\widetilde{p} \in C_c$ and $\widetilde{p}' \notin C_c$,

(b) $\widetilde{p} \notin C_c$ and $\widetilde{p}' \in C_c$.

Pick and fix k. Then we have V_k and $\phi_k^{(j)}\colon \Sigma(\boldsymbol{x}^+_\infty) \setminus V_k \to \Sigma(\boldsymbol{x}^{(j)+})$ as in the beginning of Step 2. In case (a), we find that $p^{(j)} \in C_d^{(j)}$. Pick $q_1, q_2 \in C_c \setminus \{\widetilde{p}\}$. We choose points $q_1^{(j)}$ and $q_2^{(j)}$ on $C_d^{(j)}$ as follows. Pick $\widetilde{p}'^{(j)}$ on $\Sigma(\boldsymbol{x}^{(j)})_d \cap \phi_k^{(j)}(\Sigma(\boldsymbol{x}^+_\infty) \setminus V_k)$. Then we take $q_1^{(j)}, q_2^{(j)} \in C_d^{(j)}$ such that the cross ratios of $p^{(j)}$, $q_1^{(j)}$, $q_2^{(j)}$ and $\widetilde{p}'^{(j)}$ are equal to the one of $\widetilde{p}$, q_1, q_2 and $\widetilde{p}'$.

In case (b), we find that $p^{(j)} \notin C_d^{(j)}$. Pick $q_1, q_2 \in C_c \setminus \{\widetilde{p}'\}$. We choose points $q_1^{(j)}$ and $q_2^{(j)}$ on $C_d^{(j)}$ as follows. Pick $\widetilde{p}'^{(j)}$ on $C_d^{(j)} \cap \phi_k^{(j)}(\Sigma(\boldsymbol{x}^+_\infty) \setminus V_k)$. Then we take $q_1^{(j)}, q_2^{(j)} \in C_d^{(j)}$ such that the cross ratios of $p^{(j)}, q_1^{(j)}, q_2^{(j)}, \widetilde{p}'^{(j)}$ are equal to the one of $\widetilde{p}, q_1, q_2, \widetilde{p}'$.

Next we consider Case 2. Let $\Sigma(\boldsymbol{x}^+_\infty)_{a_1}$ and $\Sigma(\boldsymbol{x}^+_\infty)_{a_2}$ be the irreducible components which share the node p. Pick distinct three points q_1, q_2 and q_3 on the newly inserted components so that none of them are nodes. We note that the circle $C_{a'}^{(j)}$ intersects at least one of $\phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_i} \setminus V_k)$ $(i = 1, 2)$. From now on, we fix such a k and denote it by k_0. We may assume that $\phi_{k_0}^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_1} \setminus V_{k_0})$ intersects $C_{a'}^{(j)}$. (The other case is similar.) Then Case 2 in Step 2-2 is applied to $\Sigma(\boldsymbol{x}_\infty^+)_{a_2}$. Pick $p' \in C_a \cap \Sigma(\boldsymbol{x}_\infty^+)_{a_1} \setminus V_{k_0}$. For all sufficiently large j, we arrange the neck region $V_{\mathrm{neck},p}^{(j)}$ which is a connected component of the complement of $\phi_{k_0}^{(j)}(\Sigma(\boldsymbol{x}_\infty^+) \setminus V_{k_0})$ and degenerates to a neighborhood of the node p as follows. Pick and fix a suitable biholomorphic map $\varphi^{(j)}\colon \Sigma(\boldsymbol{x}^{(j)})_{a'} \to \mathbb{C}P^1$ such that

- $V_{\mathrm{neck},p}^{(j)}$ is mapped to an annulus $\{z \in \mathbb{C} \mid r^{(j)} < |z| < R^{(j)}\}$ for some $0 < r^{(j)} < \frac{1}{2}$ and $R^{(j)} > 1$,
- $(\varphi^{(j)})^{-1}(\{z \in \mathbb{C} \mid |z| < r^{(j)}\})$ contains $\phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_1} \setminus V_k)$.

By applying a dilation fixing 0 and ∞, we may assume that the circle $C_{a'}^{(j)}$ is tangent to the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Since $C_{a'}^{(j)} \cap \phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_1} \setminus V_k)$ is not empty for any k, we find that $r^{(j)}$ tends to 0. Similarly, since for each given k, the intersection of $C_{a'}^{(j)}$ and $\phi_k^{(j)}(\Sigma(\boldsymbol{x}_\infty^+)_{a_2} \setminus V_k)$ is empty for all sufficiently large j, the number $R^{(j)}$ tends to $+\infty$. Pick $p'^{(j)} \in C_{a'}^{(j)}$ such that $|\varphi^{(j)}(p'^{(j)})| < r^{(j)}$ and $\boldsymbol{x}_j^+$ with $p'^{(j)}$ added converges to $\boldsymbol{x}_\infty^+$ with p' added. We pick $q_1^{(j)}$, $q_2^{(j)}$ and $q_3^{(j)}$ on $C_{a'}^{(j)}$ such that $|\varphi^{(j)}(q_1^{(j)})| = 1$, $|\varphi^{(j)}(q_2^{(j)})| = \frac{1}{2}$ and the cross ratios of $p'^{(j)}$, $q_1^{(j)}$, $q_2^{(j)}$ and $q_3^{(j)}$ are the same as the cross ratio of p', q_1, q_2 and q_3.

After adding those new marked points, we obtain $\tilde{\boldsymbol{x}}^{(j)}$ for all sufficiently large j and $\tilde{\boldsymbol{x}}''_\infty$ in $\mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ for some L. By the choice of those points, we find that $\tilde{\boldsymbol{x}}^{(j)}$ converges to $\tilde{\boldsymbol{x}}''_\infty$ in $\mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$. By Definition 6.48, $\boldsymbol{x}^{(j)}$ converges to $\boldsymbol{x}''_\infty$ in $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$. □

This finishes the proof of Proposition 6.52. □

Now we have:

Theorem 6.56 *The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ is compact and Hausdorff.*

Proof Since $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ satisfies the second axiom of countability, compactness is implied by Proposition 6.52. Then the Hausdorff property follows in the same way as in Lemma 10.4 in [16]. □

Theorem 6.57 *The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ carries a Kuranishi structure.*

Proof We construct a Kuranishi structure on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ in the same way as the case of the moduli space of stable maps in [16]; see Parts 3 and 4 of [13] for more details.

Our strategy to construct Kuranishi structures on the moduli space of bordered stable maps with admissible system of circles is to reduce the construction to the one for moduli spaces of bordered stable maps with marked points by putting a suitable number of points on circles. The only points which we have to take care of are the following two points.

The first point to be taken care of is the way to deal with the admissible system of circles in terms of additional marked points on the domain curve. An element in $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ is a stable map with an admissible system of circles. For $\boldsymbol{x} \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$, we put suitable marked points on circles in the admissible system of circles to obtain an element $\tilde{\boldsymbol{x}} \in \mathcal{M}_{3+L}^{\mathrm{sph}}(J_N;\alpha)$ and

$$\pi_{\tilde{\boldsymbol{x}}}^{L}\colon \mathcal{U}(\tilde{\boldsymbol{x}}) \to \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C}), \quad (\Sigma, \vec{z}^{\,+}, u) \mapsto (\Sigma, \vec{z}, \{C_a\}, u).$$

(See Lemma 6.47.) Note that $\pi_{\tilde{\boldsymbol{x}}}^{L}$ is not injective. For $\boldsymbol{x} \in \mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$, we take a subspace of a Kuranishi neighborhood of $\tilde{\boldsymbol{x}} \in \mathcal{U}(\tilde{\boldsymbol{x}})$, as we explain in the next paragraph, to obtain a Kuranishi neighborhood of $\boldsymbol{x}$.

If an irreducible component $\Sigma_a \subset \Sigma$ contains a circle $C_a \neq \varnothing$, then C_a must contain at least one special point, ie a node or a marked point. Note that C_a is an oriented circle on Σ_a. If a holomorphic automorphism φ of Σ_a is of finite order preserving C_a and its orientation, and if φ fixes a point on C_a, then φ must be the identity. Hence the stabilizer of this component must be trivial. If the number of special points on C_a is less than three, we take the minimal number of marked points on C_a in such a way that the total number of special points is three. Let $p_1, \dots, p_c$ be nodes on C_a and $w_1, \dots, w_k$ marked points on C_a. Then, for each marked point w_j on C_a, we choose a short embedded arc A_{w_j} on Σ_a which is transversal to C_a at w_j. We may move the marked point w_j to w_j' on A_{w_j} such that $p_1, \dots, p_c, w_1', \dots, w_k'$ lie on a common circle. This last condition is expressed using the cross ratio, and these constraints cut out the set of such $w_1', \dots, w_k'$ transversally. Thus if we restrict $\pi_{\tilde{\boldsymbol{x}}}^{L}$ to the subset of $\mathcal{U}(\tilde{\boldsymbol{x}})$ such that the extra L marked points hit A_{w_j}, the restricted map is injective. (More precisely, a similar map on the Kuranishi neighborhood is injective.) Therefore, we can use this subset of a Kuranishi neighborhood of $\mathcal{U}(\tilde{\boldsymbol{x}})$ as a Kuranishi neighborhood of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$.

The second point to be taken care of is about the gluing construction. We use the gluing construction following Parts 3 and 4 of [13] at smoothable nodes. Let p be a smoothable node. If no circle in the admissible system of circles passes through p, we use the gluing construction as in the case of stable maps. Otherwise, we proceed as follows. Let Σ_a and Σ_b intersect at the node p. Then the circles C_a and C_b contain p. In order to perform gluing, we need a *coordinate at infinity*; see Definition 16.2 in [13]. For a stable map with an admissible system of circles, we use a coordinate at infinity adapted to the circle system as follows. We pick a complex local coordinate ξ on Σ_a (resp. η on Σ_b) around p such that C_a (resp. C_b) with the given orientation corresponds to the real line oriented from $-\infty$ to $+\infty$ in the ξ–plane (resp. the η–plane). For the gluing construction, we use the gluing parameter $T \in [T_0, \infty]$ for a sufficiently large $T_0 > 0$ such that the ξ–plane and η–plane are glued by $\xi \cdot \eta = -e^{-T}$. In this case the parameter to smooth this node is $[T_0, \infty)$. Therefore such a point corresponds to a boundary. (Or corner if there are more such points.) We remark that in the case when there is no circle on the node, the parameter to smooth this node is $[T_0, \infty) \times S^1$. Then the construction of a Kuranishi structure goes through as in Parts 3 and 4 of [13]. □

Once we have Kuranishi structures on $\mathcal{M}_3^{\mathrm{sph}}(J_N; \alpha)$ and $\mathcal{M}_3^{\mathrm{sph}}(J_N; \alpha; \mathcal{C})$, we also have Kuranishi structures on their fiber products with singular chains P_i.

For $\boldsymbol{x} = (\Sigma(\boldsymbol{x}), \vec{z}, \{C(\boldsymbol{x})_a\}, u\colon \Sigma(\boldsymbol{x}) \to N) \in \mathcal{M}_3^{\mathrm{sph}}(J_N; \alpha; \mathcal{C})$, we set

$$
\begin{aligned}
n(\boldsymbol{x}) &= \text{the number of irreducible components of } \Sigma(\boldsymbol{x}),\\
n_{\mathrm{I},\mathfrak{p}}(\boldsymbol{x}) &= \text{the number of irreducible components } \Sigma(\boldsymbol{x})_a \text{ of type I-2 in } \Sigma(\boldsymbol{x})\\
&\qquad\qquad \text{such that } C(\boldsymbol{x})_a \neq \varnothing \text{ does not contain the root node},\\
n_{\mathrm{I,bubble}}(\boldsymbol{x}) &= \text{the number of irreducible components } \Sigma(\boldsymbol{x})_a \text{ of type I-2 in } \Sigma(\boldsymbol{x})\\
&\qquad\qquad \text{such that } C(\boldsymbol{x})_a \text{ contains the root node},\\
n_{\mathrm{I},\varnothing}(\boldsymbol{x}) &= \text{the number of irreducible components } \Sigma(\boldsymbol{x})_a \text{ of type I-2 in } \Sigma(\boldsymbol{x})\\
&\qquad\qquad \text{such that } C(\boldsymbol{x})_a = \varnothing,\\
n_{\mathrm{II,circ}}(\boldsymbol{x}) &= \text{the number of irreducible components } \Sigma(\boldsymbol{x})_a \text{ of type II in } \Sigma(\boldsymbol{x})\\
&\qquad\qquad \text{such that } C(\boldsymbol{x})_a \text{ is nonempty},\\
n_{\mathrm{II},\varnothing}(\boldsymbol{x}) &= \text{the number of irreducible components } \Sigma(\boldsymbol{x})_a \text{ of type II}\\
&\qquad\qquad \text{such that } C(\boldsymbol{x})_a \text{ is empty}.
\end{aligned}
$$

Since these numbers depend only on the combinatorial type $\mathfrak{c}$, we also denote them by $n(\mathfrak{c})$, $n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})$, $n_{\mathrm{I,bubble}}(\mathfrak{c})$, $n_{\mathrm{I},\varnothing}(\mathfrak{c})$, $n_{\mathrm{II,circ}}(\mathfrak{c})$ and $n_{\mathrm{II},\varnothing}(\mathfrak{c})$. Note that

$$n(\mathfrak{c}) = 1 + n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c}) + n_{\mathrm{I,bubble}}(\mathfrak{c}) + n_{\mathrm{I},\varnothing}(\mathfrak{c}) + n_{\mathrm{II,circ}}(\mathfrak{c}) + n_{\mathrm{II},\varnothing}(\mathfrak{c})$$

because there always exists a unique irreducible component of type I-1. Then we find the following proposition. The proof is easy and so omitted.

Proposition 6.58 (1) *The virtual codimension* $\mathrm{vcd}(\mathfrak{c})$ *of the stratum with the combinatorial type* $\mathfrak{c}$ *is equal to*

$$2(n(\mathfrak{c})-1)-2n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})-n_{\mathrm{I,bubble}}(\mathfrak{c})-n_{\mathrm{II,circ}}(\mathfrak{c}) \\ = n_{\mathrm{I,bubble}}(\mathfrak{c})+2n_{\mathrm{I},\varnothing}(\mathfrak{c})+n_{\mathrm{II,circ}}(\mathfrak{c})+2n_{\mathrm{II},\varnothing}(\mathfrak{c}),$$

which is nonnegative.

(2) $\mathrm{vcd}(\mathfrak{c})=0$ *if and only if* $n(\mathfrak{c})=n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})+1$. *Namely, all irreducible components are of type I with nonempty circles and the circle on each component of type I-2 does not contain the root node.*

(3) $\mathrm{vcd}(\mathfrak{c})=1$ *if and only if either case* (A) $n(\mathfrak{c})=n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})+2$ *and* $n_{\mathrm{I,bubble}}(\mathfrak{c})=1$, *or case* (B) $n(\mathfrak{c})=n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})+2$ *and* $n_{\mathrm{II,circ}}(\mathfrak{c})=1$. *Namely, either all irreducible components are of type I with nonempty circles and there is exactly one irreducible component* Σ_a *of type I-2 such that the circle* C_a *contains the root node of* Σ_a, *or there is exactly one type II component* Σ_a *with* $C_a\neq\varnothing$, *all others are of type I with nonempty circles and the circle* C_a *on each irreducible component of type I-2 does not contain the root node.*

Proposition 6.58 describes combinatorial types $\mathfrak{c}$ such that the corresponding strata in $\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$ is codimension 1. There are two cases:

(A) $n(\mathfrak{c})=n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})+2$ and $n_{\mathrm{I,bubble}}(\mathfrak{c})=1$,

(B) $n(\mathfrak{c})=n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})+2$ and $n_{\mathrm{II,circ}}(\mathfrak{c})=1$.

Case (A) and case (B) are treated in different ways. Firstly, we consider case (A).

Note that the stable map is constant on the irreducible components explained in Remark 6.30(2). By our convention, we do not put obstruction bundles on these components. Therefore, we can identify the following two codimension-1 boundary components equipped with Kuranishi structures:

(1) A type I component splits into two irreducible components.

(2) A type I circle C_a meets an inward interior marked point, and an irreducible component of case (i) in Definition 6.31 is inserted at the node of the two irreducible components.

See Figure 5, which illustrates an example with $n_{\mathrm{I},\mathfrak{p}}(\mathfrak{c})=0$.

These two strata are glued to cancel codimension-1 boundaries. See Remark 6.63(2) for the cancellation with orientation. This is a key geometric idea to see the equality in Lemma 6.15. From now on, we denote by

$$\mathcal{M}_3^{\mathrm{sph}}(J_N;\alpha;\mathcal{C})$$

the moduli space with the codimension-1 boundaries of (1) and (2) identified as above.

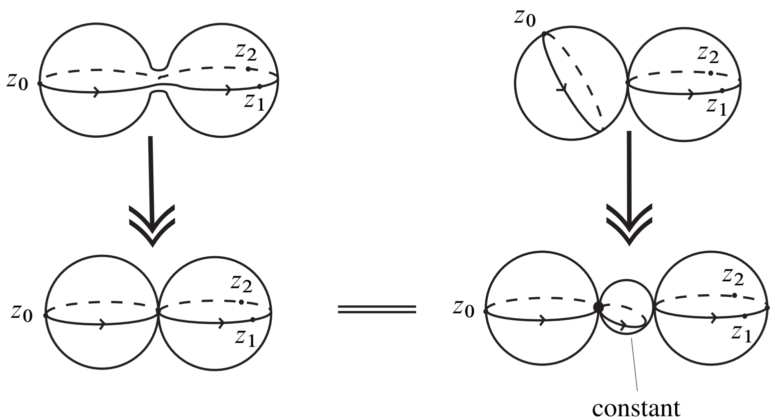

Figure 5

The remaining codimension-1 boundary components are in case (B), ie those with nonempty type II circles in the admissible system of circles, which correspond to the codimension-1 disc bubbling phenomenon in $\mathcal{M}_3^{\mathrm{main}}(J_{N\times N};\beta)$. We will study these codimension-1 boundary components in Lemma 6.69. See Remark 6.63(3), (4) and Section 6.3.1 for the cancellation with orientation in case (B).

6.3.5 Proof of Theorem 1.9(2), IV: Completion of the proof In this subsection we prove (6-29). First of all, we recall the following lemma, which is a well-known fact on the moduli space of pseudoholomorphic spheres which is used in the definition of quantum cup product [16]. Let $\rho \in \pi_2(N)/\sim$, where the equivalence relation $\sim$ was defined in Definition 6.13. For given ρ and cycles P_0, P_1 and P_2 in N, we defined $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ in Definition 6.34.

Lemma 6.59 *The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ carries a Kuranishi structure $\mathfrak{K}_0$ and a multisection $\mathfrak{s}_0$ such that*

$$\sum_{\rho}\#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)\big)^{\mathfrak{s}_0}T^{\omega(\rho)}e^{c_1(N)[\rho]}=\big\langle \mathrm{PD}[P_1]*\mathrm{PD}[P_2],\mathrm{PD}[P_0]\big\rangle.$$

The sum is over ρ for which the virtual dimension of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ is zero.

Now we will consider the moduli space used to define the left-hand side of (6-29). Let $\vec{\beta}=(\beta_1,\dots,\beta_k)$ such that $\beta_j\neq 0\in\Pi(\Delta_N)$. Set $\mathrm{length}(\vec{\beta})=k$. We define $\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta};P)$ by induction on $\mathrm{length}(\vec{\beta})$. Firstly, we consider the moduli space $\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta)$ of bordered stable maps representing the class β attached to $(N\times N,\Delta_N)$ with one interior marked point and one boundary marked point. Here, to specify the interior marked point as an output marked point, we use the notation $\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta)$ used in Section 8.10.2 of [10]. See the line just before Definition 8.10.2 in [10] where the orientation on $\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta)$ is given. We denote by z_1 the first (and only) boundary marked point. Then we define

$$\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta;P)=\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta)_{\mathrm{ev}_1}\times P.$$

This is a special case of Definition 8.10.2 in [10] with $k=1$ and $\ell=0$, and the sign is $(-1)^{\epsilon}=+1$ in this case.

When length$(\beta)=1$, ie $\vec{\beta}=(\beta_1)$, we set

$$\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta};P)=-\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta_1;P). \tag{6-33}$$

Here we reversed the orientation of $\mathcal{M}_{1,(1,0)}(J_{N\times N};\beta_1;P)$ so that it is compatible with Definition 6.18 in the case that $k=1$.

Suppose that the orientation of $\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta};P)$ is given for length$(\vec{\beta})\le k$. For $\vec{\beta}=(\beta_1,\dots,\beta_{k+1})$, we write $\vec{\beta}^-=(\beta_2,\dots,\beta_{k+1})$. Then we define

$$\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta};P)=-\mathcal{M}_{1,1}(J_{N\times N};\beta_1)_{\mathrm{ev}_1}\times_{p_2\circ\mathrm{ev}_{\mathrm{int}}}\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}^-;P). \tag{6-34}$$

Namely, we reversed the orientation so that it is consistent with Definition 6.18 for each positive integer $k=$ length$(\vec{\beta})$. We also denote by $\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta};P)$ the chain

$$p_2\circ\mathrm{ev}_{\mathrm{int}}\colon\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta};P)\to\Delta_N,$$

where $p_2(x,y)=(y,y)$. By an abuse of notation, we set $\mathcal{M}_{1,1}(J_{N\times N};\varnothing;P)=P$. From now on, $\vec{\beta}$ is either $\varnothing$ or $(\beta_1,\dots,\beta_k)$ with $\beta_j\neq 0$ for each $j=1,\dots,k$. For each P_0, P_1, P_2, $\vec{\beta}_0$, $\vec{\beta}_1$ and $\vec{\beta}_2$, we define

$$\begin{aligned}\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)&=\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_0;P_0)_{\mathrm{ev}_1}\\&\quad\times_{\mathrm{ev}_0}\mathcal{M}_3^{\mathrm{main}}\big(J_{N\times N};\beta';\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_1;P_1),\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_2;P_2)\big).\end{aligned} \tag{6-35}$$

Taking our Convention 8.2.1 (4) in [10] and the pairing (6-20) into account, the following is immediate from definition.

Lemma 6.60 *There exist a Kuranishi structure $\mathfrak{K}_1$ on*

$$\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$$

and a multisection $\mathfrak{s}_1$ with the following properties. We denote by n_β the sum of

$$\#\big(\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)\big)^{\mathfrak{s}_1}$$

over $(\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0)$ whose total sum is β. Then we have

$$\big\langle\mathfrak{m}_2(\mathcal{I}(P_1),\mathcal{I}(P_2)),\mathcal{I}(P_0)\big\rangle=\sum_{\beta}n_\beta T^{\omega(\beta)}e^{\mu(\beta)/2}. \tag{6-36}$$

Moreover, The multisection $\mathfrak{s}_1$ is invariant under the involution τ_* on each disc component with bubble trees of spheres.

The last statement follows from the fact that the multisection $\mathfrak{s}_1$ is constructed by induction on the energy of the bordered stable maps keeping the invariance under τ_*; see the explanation in Remark 6.9.

Consider the union of $\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$ over $(\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0)$ such that the total sum of $(\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0)$ is β whose double belongs to class $\rho\in\pi_2(N)/\sim$. (See Remark 6.22(2) for the double of β.) We glue them along virtual codimension-one strata appearing in case (A) in Proposition 6.58, and denote it by

$$\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0). \tag{6-37}$$

See Sublemma 6.61 for the description of codimension-one strata which we identify. Each $\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$ has Kuranishi structure in such a way that we can glue them to obtain a Kuranishi structure on $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$; see also Lemma 6.64. Namely, we have:

Sublemma 6.61 *The orientations of $\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$ are compatible, and $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$ has an oriented Kuranishi structure.*

Proof It is sufficient to see that two top-dimensional strata adjacent along a stratum of codimension 1 induce opposite orientations on the stratum of codimension 1.

First, consider the case that a transition of strata occurs in one of $\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_i;P_i)$ for $i=0,1,2$. It will suffice for us to check the compatibility of orientations inside $\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta};P)$ for a given $\vec{\beta}$. Let

$$\begin{aligned}\vec{\beta}&=(\beta_1,\dots,\beta_i,\beta_{i+1},\dots,\beta_k),\\ \vec{\beta}_{(1)}&=(\beta_1,\dots,\beta_i),\quad \vec{\beta}_{(2)}=(\beta_{i+1},\dots,\beta_k),\\ \vec{\beta}'&=(\beta_1,\dots,\beta_{i-1},\beta_i+\beta_{i+1},\beta_{i+2},\dots,\beta_k).\end{aligned}$$

We define

$$\mathcal{M}_2(J_{N\times N};\vec{\beta}_{(2)};P)=\mathcal{M}_2(J_{N\times N};\beta_{i+1};\mathcal{M}_{1,1}(J_{N\times N};(\beta_{i+2},\dots,\beta_k);P)).$$

By Proposition 8.10.4 in [10] with $\beta_1=0$, $k=k_2=1$ and $\ell_1=\ell_2=0$, (6-33) and the proof that $\mathfrak{p}_{1,0}\equiv i_!\mod\Lambda_{0,\mathrm{nov}}^+$ in page 739 thereof, we find that

$$\mathcal{M}_2(J_{N\times N};\vec{\beta}_{(2)};P)\subset(-1)^{\dim\Delta_N+1}\partial\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_{(2)};P).$$

By Proposition 8.10.4(2) in [10], we obtain

$$\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_{(1)};\mathcal{M}_2(J_{N\times N};\vec{\beta}_{(2)};P))\subset(-1)^{\dim\Delta_N+1}\partial\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta};P).$$

On the other hand, Proposition 8.10.4(1) [10] also implies that

$$\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_{(1)};\mathcal{M}_2(J_{N\times N};\vec{\beta}_{(2)};P))\subset(-1)^{\dim\Delta_N}\partial\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}';P).$$

Hence the orientations of $\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta};P)$ and $\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}';P)$ are compatible along $\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_{(1)};\mathcal{M}_2(J_{N\times N};\vec{\beta}_{(2)};P))$.

Next we consider the remaining case, ie a transition of strata involving $\mathcal{M}_3(J_{N\times N};\beta')$. For $\vec{\beta}_1=(\beta_{1,1},\dots,\beta_{1,k})$, we write $\vec{\beta}_1^-=(\beta_{1,2},\dots,\beta_{1,k})$. The moduli spaces

$$\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$$

and

$$\widehat{\mathcal{M}}(J_{N\times N};\beta'+\beta_{1,1};\vec{\beta}_1^-,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$$

are adjacent along a stratum of codimension 1:

$$\begin{aligned}(6\text{-}38)\quad & \mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_0;P_0)\\ &\quad {}_{\mathrm{ev}_1}\times_{\mathrm{ev}_0}\mathcal{M}_3\big(J_{N\times N};\beta';\mathcal{M}_2(J_{N\times N};\vec{\beta}_1;P_1),\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_2;P_2)\big).\end{aligned}$$

Instead of Proposition 8.10.4(1), (2), we use Proposition 8.5.1 in [10], and find that the orientations of

$$\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$$

and

$$\widehat{\mathcal{M}}(J_{N\times N};\beta'+\beta_{1,1};\vec{\beta}_1^-,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$$

are compatible along the stratum given in (6-38).

The same argument applies to $\mathcal{M}_{1,1}(J_{N\times N};\vec{\beta}_i;P_i)$ for $i=0,2$. Hence the orientations of the moduli spaces $\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$ are compatible with one another and so define an orientation on $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$.

Thus we can glue oriented Kuranishi structures on $\widehat{\mathcal{M}}(J_{N\times N};\beta';\vec{\beta}_1,\vec{\beta}_2,\vec{\beta}_0;P_1,P_2,P_0)$ to obtain an oriented Kuranishi structure on $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$. □

The map $\mathring{\mathfrak{I}}$ in (6-15) induces

$$(6\text{-}39)\qquad \mathfrak{I}^{\mathrm{reg}}\colon \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N};\rho;P_1,P_2,P_0)\to\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho;P_1,P_2,P_0),$$

where we recall

$$\begin{aligned}&\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho;P_1,P_2,P_0)\\ &\qquad=(-1)^{\deg P_1\cdot\deg P_2}P_0\times_{\mathrm{ev}_0}\big(\mathcal{M}_3^{\mathrm{sph,reg}}(J_N;\rho)_{(\mathrm{ev}_1,\mathrm{ev}_2)}\times_{N^2}(P_1\times P_2)\big)\end{aligned}$$

from Definition 6.34. We extend $\mathfrak{I}^{\mathrm{reg}}$ to a map

$$\mathfrak{I}\colon \widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)\to \mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0) \tag{6-40}$$

defined on the full moduli space as follows. Let $\bigl(p_i,(S_{i,j},(z_{i,j;0},z_{i,j;\mathrm{int}}),u_{i,j})_{j=1}^{k_i}\bigr)$ be an element of $\mathcal{M}_{1;1}(J_{N\times N};\vec{\beta}_i;P_i)$. Here $p_i\in|P_i|$ and $(S_{i,j},(z_{i,j;0},z_{i,j;\mathrm{int}}),u_{i,j})\in\mathcal{M}_{1,1}(J_{N\times N};\beta_{i,j})$ such that

$$f(p_i)=u_{i,1}(z_{i,1;0}),\ u_{i,1}(z_{i,1;\mathrm{int}})=u_{i,2}(z_{i,2;0}),\dots,u_{i,k_i-1}(z_{i,k_i-1;\mathrm{int}})=u_{i,k_i}(z_{i,k_i;0}),$$

where P_i is $(|P_i|,f)$, $|P_i|$ is a simplex, and $f\colon|P_i|\to N$ is a smooth map.

Suppose that $S_{i,j}$ is a disc component. Writing $u_{i,j}=(u_{i,j}^{+},u_{i,j}^{-})$, we obtain a map $\hat{u}_{i,j}\colon\Sigma_{i,j}\to N$ with $\Sigma_{i,j}$ a sphere, the double of $S_{i,j}$. For bubble trees, we goes as in Remark 6.22(1).

We denote by $C_{i,j}\subset\Sigma_{i,j}$ the circle along which we glued two copies of $S_{i,j}$. Then $\hat{u}_{i,j}$ is defined by gluing $u_{i,j}^{+}$ and $u_{i,j}^{-}\circ c$ along $C_{i,j}$ in $\Sigma_{i,j}$, where $c\colon\Sigma_{i,j}\to\Sigma_{i,j}$ is the conjugation with $C_{i,j}$ as its fixed point set.

We glue $(\Sigma_{i,j},u_{i,j})$ and $(\Sigma_{i,j+1},u_{i,j+1})$ at $z_{i,j;\mathrm{int}}$ and $z_{i,j+1;0}$. Here we identify $z_{i,j;\mathrm{int}}\in S_{i,j}$ as the corresponding point in $\Sigma_{i,j}$ such that it is in the disc bounding $C_{i,j}$. We thus obtain a configuration of a tree of spheres and a system of circles on it for each $i=1,2,0$. We glued them with the double of an element $\mathcal{M}_3^{\mathrm{main}}(J_{N\times N};\beta')$ in an obvious way. Thus we obtain the map (6-40). (In case some of the sphere component becomes unstable, we need to shrink it. See the proof of Lemma 6.62 below.)

It is easy to see that $\mathfrak{I}$ is surjective.

Lemma 6.62 *There is a subset $\mathcal{D}(\mathfrak{I})\subset\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$ of codimension at least 2 such that the map $\mathfrak{I}$ is an isomorphism outside $\mathcal{D}(\mathfrak{I})$.*

Proof We can easily check that the map $\mathfrak{I}$ fails to be an isomorphism only by the following reason. Let $((\Sigma^{\mathrm{dis}},(z_1,z_2,z_0)),v)$ be an element of $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$ and let Σ_i^{dis} be one of its irreducible sphere components. Suppose that Σ_i^{dis} is unstable. (Namely we assume that it has one or two singular points.) Then its automorphism group $\mathrm{Aut}(\Sigma_i^{\mathrm{dis}})$ will have positive dimension by definition of stability. (We require the elements of $\mathrm{Aut}(\Sigma_i^{\mathrm{dis}})$ to fix the singular point.) By restricting v to Σ_i^{dis}, we obtain $v_i=(v_i^{+},v_i^{-})$, where $v_i^{\pm}\colon\Sigma_i^{\mathrm{dis}}\to N$ are maps from the sphere domain Σ_i^{dis}. On the double (which represents $\mathfrak{I}((\Sigma^{\mathrm{dis}},(z_1,z_2,z_0)),v)$), the domain Σ^{dis} contains two sphere components Σ_i^{+} and Σ_i^{-} on which the maps v_i^{+} and v_i^{-} are defined respectively. We have two alternatives:

(1) If one of v_i^+ or v_i^- is a constant map, then this double itself is not a stable map. So we shrink the corresponding component Σ_i^+ or Σ_i^- to obtain a stable map. (This is actually a part of the construction used in the definition of $\mathfrak{I}$.)

(2) Suppose both v_i^+ and v_i^- are nonconstant and let $g \in \mathrm{Aut}(\Sigma_i^{\mathrm{dis}})$. Then the map $v_i^g = (v_i^+, v_i^- \circ g)$ defines an element of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ different from $v_i = (v_i^+, v_i^-)$ but is mapped to the same element under the map $\mathfrak{I}$.

We will denote by $\mathcal{D}(\mathfrak{I})$ the subset of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ which consists of $((\Sigma^{\mathrm{dis}}, (z_1, z_2, z_0)), v)$ with at least one unstable sphere component Σ_i^{dis}. This phenomenon occurs only at the stratum of codimension ≥ 2 because it occurs only when there exists a sphere bubble.

This finishes the proof. □

Remark 6.63 (1) We give the orientation on $\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ as follows: We recall from Proposition 6.14 that the map

$$\mathring{\mathfrak{I}}\colon \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho) \to \mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \rho)$$

is an orientation-preserving isomorphism between spaces with oriented Kuranishi structures. Taking Definitions 3.15 and 6.34 into account, we find that the map $\mathfrak{I}^{\mathrm{reg}}$ in (6-39) induces an isomorphism from $P_0 \times_{\mathrm{ev}_0} \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho; P_1, P_2)$ to $\mathcal{M}_3^{\mathrm{sph,reg}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ which is orientation-preserving if and only if we have $(-1)^{\deg P_1(\deg P_2+1)} = 1$. Since $P_i = \mathcal{M}_{1,1}(J_{N\times N}; \varnothing; P_i)$, the orientation of the fiber product $P_0 \times_{\mathrm{ev}_0} \mathcal{M}_3^{\mathrm{main,reg}}(J_{N\times N}; \rho; P_1, P_2)$ is the restriction of the orientation of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$. Recall that $\mathring{\mathfrak{I}}$ extends to $\mathfrak{I}$ in (6-40), which is an isomorphism outside $\mathcal{D}(\mathfrak{I})$ of codimension at least 2 (Lemma 6.62). Hence we can use $\mathfrak{I}$ to equip the moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ with an orientation in such a way that $\mathfrak{I}$ is orientation-preserving if and only if $(-1)^{\deg P_1(\deg P_2+1)} = 1$.

(2) For strata of virtual codimension 1, there are two cases: case (A) and case (B) in Proposition 6.58(3). We also explained that each stratum in case (A) arises in two ways of codimension-1 boundary of top-dimensional strata, ie phenomena (1) and (2); see Figure 5. Note that there is a canonical identification, ie inserting/forgetting the component of case (i) in Definition 6.31, between those arising from phenomenon (1) and those arising from phenomenon (2). The orientation of each stratum of top dimension in $\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ is defined using the orientation of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ as we just mentioned in Remark 6.63(1). Since the moduli space $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ is oriented, these two orientations are opposite under the above identification to give an orientation on the glued space with Kuranishi structure. As for the cancellation in case (B), see Section 6.3.1, Lemma 6.69 and the following items (3) and (4).

(3) We consider the involution τ_* applied to one of the disc components S of the fiber product factors appearing in a stratum of $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$, such that the double of S is type II. Then the orientation of the circle $C=\partial S$ embedded in the domain Σ of the corresponding sphere component is inverted under the operation on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ induced by τ_* under the map $\mathfrak{I}$.

(4) The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ is stratified according to the combinatorial types $\mathfrak{c}$, see Definition 6.36. When $\mathfrak{c}$ is fixed, there are finitely many type II components. There are involutions acting on these components by the reflection with respect to the circle of type II. Namely, the involution reverses the orientation of the circle of type II. We call a Kuranishi structure (resp. a multisection) on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ *invariant under the inversion of the orientation of circles of type II*, if the Kuranishi structure (resp. the multisection) restricted to each stratum corresponding to $\mathfrak{c}$ is invariant under the involution acting on each type II component in $\mathfrak{c}$. Note that these involutions are defined on the corresponding strata, not on the whole moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$.

Lemma 6.64 *The moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ carries a Kuranishi structure $\mathfrak{K}_2$ invariant under the inversion of the orientation of circles of type II. The Kuranishi structure can be canonically pulled back to a Kuranishi structure on the space $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$. Moreover, there is a multisection $\mathfrak{s}_2$ of the Kuranishi structure on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ with the following properties:*

(1) *The multisection $\mathfrak{s}_2$ is transversal to the zero section.*

(2) *The multisection $\mathfrak{s}_2$ is invariant under the inversion of the orientation of the circles of type II.*

(3) *The multisection $\mathfrak{s}_2$ does not vanish on $\mathcal{D}(\mathfrak{I})$.*

Proof Lemma 6.64 is clear from construction except the following points.

Firstly, we consider the point of the moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ such that one of the following two conditions is satisfied:

(1) A circle in type II component Σ_a hits the singular point of Σ_a other than the root thereof.

(2) A circle in type I-2 component Σ_a hits the singular point other than its outward interior special point.

We have to glue various strata meeting at such a point in $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$. We have already given such a construction during the proof of Theorem 6.57. By examining the way how the corresponding strata are glued in $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$, the gluing of corresponding strata of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ are performed in the same way. We like to note that the phenomenon spelled out in the proof of

Lemma 6.62 concerns *sphere* bubbles of the elements of $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$, while the phenomenon we are concerned with here arises from *disc* bubbles. Therefore, they do not interfere with each other.

Secondly, we need to make the choice of the obstruction bundle of the Kuranishi structure of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ in such a way that it is compatible with one of $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$. Lemma 6.62 describes the locus $\mathcal{D}(\mathfrak{I})$ where the map $\mathfrak{I}$ fails to be an isomorphism. Let $\nu_i=(\nu_i^+,\nu_i^-)$ be the sphere bubble as in the proof of Lemma 6.62. Then ν_i^+ (resp. ν_i^-) corresponds, via $\mathfrak{I}$, to a sphere bubble attached to a pseudoholomorphic sphere at a point in the lower hemisphere (resp. the upper hemisphere). In the construction of a Kuranishi structure on the moduli space of holomorphic spheres (or stable maps of genus 0), we take obstruction bundles in order to construct Kuranishi neighborhoods. Let $E(\nu_i^\pm)$ be a finite-dimensional subspace in $\Omega^{0,1}((\nu_i^\pm)^*TN)$ such that the linearization operator of the holomorphic curve equation at $\nu_i^\pm$ becomes surjective modulo $E(\nu_i^\pm)$. In order to extend $E(\nu_i^\pm)$ to a neighborhood of $\nu_i^\pm$, we used *obstruction bundle data* (introduced in Definition 17.7 in [13]), in particular, additional marked points $w_{i,j}^\pm$ and local transversals $\mathcal{D}_{i,j}^\pm$ to the image of $\nu_i^\pm$ at $\nu_i^\pm(w_{i,j}^\pm)$. For $\nu_i\colon \mathbb{C}P^1\to N\times N$, we regard $E(\nu_i^+)$ and $E(\nu_i^-)$ as subspaces of

$$\Omega^{0,1}((\nu_i)^*(TN\oplus 0))\cong\Omega^{0,1}((\nu_i^+)^*TN\oplus 0)$$

and

$$\Omega^{0,1}((\nu_i)^*(0\oplus TN))\cong\Omega^{0,1}(0\oplus(\nu_i^-)^*TN),$$

respectively. Note that the linearization operator of the pseudoholomorphic curve equation at ν_i is surjective modulo $E(\nu_i^+)\oplus E(\nu_i^-)\subset\Omega^{0,1}((\nu_i)^*(T(N\times N)))$. When we extend $E(\nu_i^+)$ (resp. $E(\nu_i^-)$) to a neighborhood of ν_i, we use $w_{i,j}^+$ and $\mathcal{D}_{i,j}^+\times N$ (resp. $w_{i,j}^-$ and $N\times\mathcal{D}_{i,j}^-$). Namely, we use the data $w_{i,j}^+$ and $\mathcal{D}_{i,j}^+$ and the data $w_{i,j}^-$ and $\mathcal{D}_{i,j}^-$ separately, not simultaneously.

The Kuranishi structure can be taken invariant under stratumwise involutions since the Kuranishi structure is constructed by induction on the energy, and we can keep the finite symmetries as in the explanation in Remark 6.9.

The existence of a multisection in the statement follows from general theory of Kuranishi structures once the following point is taken into account. By Lemma 6.62, there is a subset $\mathcal{D}(\mathfrak{I})$ of codimension at least 2 such that $\mathfrak{I}$ is an isomorphism outside $\mathcal{D}(\mathfrak{I})$. Since the expected dimension of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ is 0, we can choose an $\mathfrak{s}_2$ that does not vanish on $\mathcal{D}(\mathfrak{I})$. $\square$

Remark 6.65 Generally, note that we can pull back a Kuranishi structure $\mathfrak{K}$ on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ to a Kuranishi structure on $\widehat{\mathcal{M}}(J_{N\times N};\rho;P_1,P_2,P_0)$ in a canonical way if the next condition $(*)$ is satisfied. By construction of the Kuranishi structure, we take a sufficiently dense finite subset $\mathfrak{P} \subset \mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$, and for each $\mathfrak{x} \in \mathfrak{P}$, we take a finite-dimensional subspace $E_0(\mathfrak{x})$ of $\Omega^{0,1}(\Sigma_{\mathfrak{x}}, u_{\mathfrak{x}}^* TN) = C^\infty(\Sigma_{\mathfrak{x}}, \Lambda^{0,1} \otimes u_{\mathfrak{x}}^* TN)$, where $(\Sigma_{\mathfrak{x}}, \vec{z}_{\mathfrak{x}}^{\,+}, u_{\mathfrak{x}}\colon \Sigma_{\mathfrak{x}} \to N)$ is a stable map appearing in $\mathfrak{x}$. The subspace $E_0(\mathfrak{x})$ consists of smooth sections of compact support away from nodes. Moreover, the union of $E_0(\mathfrak{x})$ and the image of the linearized operator of the Cauchy–Riemann equation spans the space $\Omega^{0,1}(\Sigma_{\mathfrak{x}}, u_{\mathfrak{x}}^* TN)$; see Section 12 of [16]. Now we require the following:

(∗) The support of any element of $E_0(\mathfrak{x})$ does not intersect with the circles consisting $\mathfrak{x} \in \mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$.

We show that the condition (∗) implies that the Kuranishi structure $\mathfrak{K}$ can be pulled back to $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ below.

We recall the construction of Kuranishi structure on $\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ in Theorem 6.57 a bit more. For each $\mathfrak{x} \in \mathfrak{P}$, we take a sufficiently small closed neighborhood $U(\mathfrak{x})$ of $\mathfrak{x}$ in $\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$. Let $\mathfrak{y} \in \mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$. We consider $\mathfrak{P}(\mathfrak{y}) = \{\mathfrak{x} \in \mathfrak{P} \mid \mathfrak{y} \in U(\mathfrak{x})\}$. Using the complex linear part of parallel transport along minimal geodesics as in Definition 17.15, Lemma 18.6 and Definition 18.7 in [13], we transform a subspace $E_0(\mathfrak{x})$ with $\mathfrak{x} \in \mathfrak{P}(\mathfrak{y})$ to a subspace of $\Omega^{0,1}(\Sigma_{\mathfrak{y}}, u_{\mathfrak{y}}^* TN)$, where $(\Sigma_{\mathfrak{y}}, \vec{z}_{\mathfrak{y}}^{\,+}, u_{\mathfrak{y}}\colon \Sigma_{\mathfrak{y}} \to N)$ is a stable map appearing in $\mathfrak{y}$. We fix various data, such as obstruction bundle data on $\mathfrak{x}$, for our construction; see [13, Definition 17.7]. We define $E(\mathfrak{y}) \subset \Omega^{0,1}(\Sigma_{\mathfrak{y}}, u_{\mathfrak{y}}^* TN)$ as the sum of those subspaces for various $\mathfrak{x} \in \mathfrak{P}(\mathfrak{y})$. (We remark that this sum can be taken to be a direct sum [13, Lemma 18.8].)

By taking $U(\mathfrak{x})$ small, we may and will require that the supports of elements of $E(\mathfrak{y})$ are disjoint from the circles consisting $\mathfrak{y}$.

Now let $\tilde{\mathfrak{y}} \in \widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ with $\mathfrak{I}(\tilde{\mathfrak{y}}) = \mathfrak{y}$. Using the fact that the supports of elements of $E(\mathfrak{y})$ are disjoint from the circles consisting $\mathfrak{y}$, we can lift $E(\mathfrak{y})$ to a subspace $E(\tilde{\mathfrak{y}})$ of $\Omega^{0,1}(\Sigma_{\tilde{\mathfrak{y}}}, u_{\tilde{\mathfrak{y}}}^* TN)$, where $(\Sigma_{\tilde{\mathfrak{y}}}, \vec{z}_{\tilde{\mathfrak{y}}}^{\,+}, u_{\tilde{\mathfrak{y}}}\colon \Sigma_{\tilde{\mathfrak{y}}} \to N)$ is a stable map appearing in $\tilde{\mathfrak{y}}$. We use $E(\tilde{\mathfrak{y}})$ as the obstruction bundle to define the lift of our Kuranishi structure. See also Remark 6.70.

Let Z be a compact metrizable space, and let $\mathfrak{K}_0^Z$ and $\mathfrak{K}_1^Z$ be its Kuranishi structures with orientation. Let $\mathfrak{s}_0^Z$ and $\mathfrak{s}_1^Z$ be multisections of the Kuranishi structures $\mathfrak{K}_0^Z$ and $\mathfrak{K}_1^Z$, respectively. We say that $(\mathfrak{K}_0^Z, \mathfrak{s}_0^Z)$ is *homotopic* to $(\mathfrak{K}_1^Z, \mathfrak{s}_1^Z)$ if there exists an oriented Kuranishi structure $\mathfrak{K}^{Z\times[0,1]}$ on $Z \times [0,1]$, and its multisection $\mathfrak{s}^{Z\times[0,1]}$ which restricts to $(\mathfrak{K}_0^Z, \mathfrak{s}_0^Z)$ and $(\mathfrak{K}_1^Z, \mathfrak{s}_1^Z)$ at $Z \times \{0\}$ and $Z \times \{1\}$, respectively. We call such $(\mathfrak{K}^{Z\times[0,1]}, \mathfrak{s}^{Z\times[0,1]})$ a homotopy between $(\mathfrak{K}_0^Z, \mathfrak{s}_0^Z)$ and $(\mathfrak{K}_1^Z, \mathfrak{s}_1^Z)$.

The moduli space $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ is stratified according to combinatorial types. For $\boldsymbol{u} \in \widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$, we decompose the domain of $\boldsymbol{u}$ into disc components and sphere components. We define an *extended disc component* of $\boldsymbol{u}$ to be the union of a disc component $D_a(\boldsymbol{u})$ and all trees of spheres rooted on $D_a(\boldsymbol{u})$. We denote it by $\widehat{D}_a(\boldsymbol{u})$. An extended disc component is said to be of type I (resp. type II) if the corresponding component of $\mathfrak{I}(\boldsymbol{u})$, ie the double of $D_a(\boldsymbol{u})$, is of type I (resp. type II). The involution τ_* acts on each extended disc component. In particular, τ_* acting on an extended disc component $\widehat{D}_a(\boldsymbol{u})$ of type II is compatible with the inversion of the orientation of the circle on the component of $\mathfrak{I}(\boldsymbol{u})$ of type II, which is the double of the disc component $D_a(\boldsymbol{u})$; see Remark 6.63.

Lemma 6.66 *For the pull-back $\mathfrak{I}^*(\mathfrak{K}_2, \mathfrak{s}_2)$ and $(\mathfrak{K}_1, \mathfrak{s}_1)$, there is a homotopy $(\boldsymbol{K}, \boldsymbol{s})$ between them that is invariant under τ_* on each extended disc component of type II acting on the first factor of $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0) \times [0,1]$.*

Proof By Lemma 6.64, the moduli space $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$ has the pair $\mathfrak{I}^*(\mathfrak{K}_2, \mathfrak{s}_2)$ of Kuranishi structure and multisection, which are invariant under the inversion of the orientation of circles. We also have another such pair $(\mathfrak{K}_1, \mathfrak{s}_1)$. Then the standard theory of Kuranishi structure shows the existence of the desired homotopy. □

Lemma 6.67 *We have*

$$\sum_{\beta} n_\beta = \#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)\big)^{\mathfrak{s}_2}.$$

Here the sum is taken over the class $\beta \in \Pi(\Delta_N) = \pi_2(N\times N, \Delta_N)/\sim$ whose double belongs to class $\rho \in \pi_2(N)/\sim$, and the virtual dimension of $\mathcal{M}_3^{\mathrm{sph}}(J_N; \rho; \mathcal{C}; P_1, P_2, P_0)$ is zero.

Proof If the moduli space $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0) \times [0,1]$ had no codimension-1 boundary in the sense of Kuranishi structure, the existence of the homotopy $(\boldsymbol{K}, \boldsymbol{s})$ in Lemma 6.66 would immediately imply the conclusion. In reality, there does exist a codimension-1 boundary; however, it consists of elements with at least one component of type II. Since $\boldsymbol{s}$ is invariant under the action τ_* on the disc component of type II, the contribution from the boundary cancels as in the proof of unobstructedness of the diagonal in $(N\times N, -\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$ in Section 6.3.1. Hence the proof. □

Remark 6.68 Even though the dimension of a space Z with Kuranishi structure is 0, the codimension-1 boundary can be nonempty. This is because the dimension of a space with Kuranishi structure is *virtual* dimension. After taking a suitable multivalued perturbation, its zero set does not meet the codimension-1 boundary.

We now consider the forgetful map

$$\mathfrak{F}\colon \mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0) \to \mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0), \tag{6-41}$$

which is defined by forgetting all the circles in the admissible system of circles. We recall from Lemmas 6.59 and 6.64 that both moduli spaces $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ and $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ carry Kuranishi structures. We have also used multisections on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)$ and on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$, denoted by $\mathfrak{s}_0$ and $\mathfrak{s}_2$, respectively.

Lemma 6.69 *Let $(\mathfrak{K}_2,\mathfrak{s}_2)$ be a Kuranishi structure with multisection on $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ as in Lemma 6.64. Then $(\mathfrak{K}_2,\mathfrak{s}_2)$ is homotopic to the pull-back $\mathfrak{F}^*(\mathfrak{K}_0,\mathfrak{s}_0)$. Moreover, there is a homotopy between them which is invariant under the inversion of the orientation of the type II circles. In particular, we have*

$$\#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)\big)^{\mathfrak{s}_2} = \#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)\big)^{\mathfrak{s}_0}$$

if the virtual dimensions of the moduli spaces on both sides are zero.

Proof The existence of a homotopy between Kuranishi structures $\mathfrak{K}_2$ and $\mathfrak{F}^*\mathfrak{K}_0$ is again a consequence of the general theory once the following point is taken into account. Note that $\mathfrak{F}^*\mathfrak{s}_0$ and $\mathfrak{s}_2$ are invariant under the inversion of the orientation of circles of type II. Then we can take a homotopy which is also invariant under the inversion of the orientation of circles of type II.

We note that there are several components of $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ which are of codimension 0 or 1 and contracted by $\mathfrak{F}$. Note that we took $\mathfrak{s}_0$ in such a way that its zero set does not contain elements with domains of at least two irreducible components; see the paragraph right after Proposition 6.14. Hence the zeros of $\mathfrak{F}^*\mathfrak{s}_0$ are contained in the subset where $\mathfrak{F}$ gives an isomorphism, and we can count them with signed weights to obtain a rational number. Namely, we have

$$\#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)\big)^{\mathfrak{F}^*\mathfrak{s}_0} = \#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;P_1,P_2,P_0)\big)^{\mathfrak{s}_0}.$$

If the moduli space $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ had no codimension-1 boundary in the sense of Kuranishi structure, the existence of a homotopy between $(\mathfrak{K}_2,\mathfrak{s}_2)$ and $\mathfrak{F}^*(\mathfrak{K}_0,\mathfrak{s}_0)$ would immediately imply that

$$\#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)\big)^{\mathfrak{s}_2} = \#\big(\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)\big)^{\mathfrak{F}^*\mathfrak{s}_0},$$

which would complete the proof. However, $\mathcal{M}_3^{\mathrm{sph}}(J_N;\rho;\mathcal{C};P_1,P_2,P_0)$ does have a codimension-1 boundary, which consists of stable maps with admissible systems of circles containing at least one circle of type II. All the contributions from those components cancel out by the involution, which inverts the orientation of the circles of type II.

(This is a geometric way to see the vanishing of $\mathfrak{m}_0(1)$ in the chain level. We have already checked that it occurs *with sign* in Section 6.3.1. See also Remark 6.63(3), (4).) Hence the lemma. □

Remark 6.70 The pull-back Kuranishi structure $\mathfrak{F}^*(\mathfrak{K}_0, \mathfrak{s}_0)$ does not satisfy the condition $(*)$ appearing in Remark 6.65. In fact, the obstruction bundle of $\mathfrak{F}^*(\mathfrak{K}_0, \mathfrak{s}_0)$ is independent of the position of the circles. Therefore, we may not pull back $\mathfrak{F}^*(\mathfrak{K}_0, \mathfrak{s}_0)$ to a Kuranishi structure on $\widehat{\mathcal{M}}(J_{N\times N}; \rho; P_1, P_2, P_0)$, while we can pull back the Kuranishi structure $\mathfrak{K}_2$.

By Lemmas 6.59, 6.60, 6.67 and 6.69, the proof of Theorem 1.9(2) is complete. □

Proof of Corollary 1.10 Viewing N as a closed relatively spin Lagrangian submanifold of $(N \times N, -\mathrm{pr}_1^*\omega_N + \mathrm{pr}_2^*\omega_N)$, we can construct a filtered A_∞ structure on $H(N; \Lambda_{0,\mathrm{nov}}^{\mathbb{Q}})$ which is homotopy equivalent to the filtered A_∞ algebra given by Theorem 6.2. This is a consequence of Theorem W in [9]. See also Theorem A in [9]. Then (1) and (2) follow from Theorem 1.9. Assertion (3) follows from Theorem X in [9]. □

6.4 Calculation of Floer cohomology of $\mathbb{R}P^{2n+1}$

In this subsection, we apply the results proved in the previous sections to calculate Floer cohomology of real projective space of odd dimension. Since the case $\mathbb{R}P^1 \subset \mathbb{C}P^1$ is already discussed in Section 3.7.6 of [9], we consider $\mathbb{R}P^{2n+1}$ for $n > 0$. We note that $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ is monotone with minimal Maslov index $2n+2 > 2$ if $n > 0$. Therefore, by [17] and Section 2.4 of [9], Floer cohomology over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ is defined. In this case, we do not need to use the notion of Kuranishi structure and the technique of the virtual fundamental chain. From the proof of Corollary 1.6, we can take 0 as a bounding cochain. Hereafter, we omit the bounding cochain 0 from the notation. By [18] and Theorem D in [9], we have a spectral sequence converging to the Floer cohomology. Strictly speaking, in [18], the spectral sequence is constructed over $\mathbb{Z}_2$ coefficients. However, we can generalize his results to ones over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ coefficients in a straightforward way, as long as we take the orientation problem, which is a new and crucial point of this calculation, into account. Thus Oh's spectral sequence over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ is enough for our calculation of this example. (See Chapters 8 and 6 of [7] for a spectral sequence over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}$ in a more general setting.)

We use a relative spin structure in Proposition 3.14 when n is even and a spin structure when n is odd. We already check that $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ has two inequivalent relative spin structures. The next theorem applies to both of them.

Theorem 6.71 *Let n be any positive integer. Then the spectral sequence calculating $\mathrm{HF}(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n+1}; \Lambda_{0,\mathrm{nov}}^{\mathbb{Z}})$ has a unique nonzero differential*

$$d^{2n+1}\colon H^{2n+1}(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z} \to H^0(\mathbb{R}P^{2n+1}; \mathbb{Z}) \cong \mathbb{Z},$$

which is multiplication by ± 2. In particular, we have

$$\mathrm{HF}(\mathbb{R}P^{2n+1}, \mathbb{R}P^{2n+1}; \Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}) \cong (\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}}/2\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}})^{\oplus(n+1)}.$$

Remark 6.72 (1) Floer cohomology of $\mathbb{R}P^m$ over $\mathbb{Z}_2$ is calculated in [17] and is isomorphic to the ordinary cohomology. This fact also follows from Theorem 34.16 in [7], which implies that Floer cohomology of $\mathbb{R}P^m$ over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}_2}$ is isomorphic to the ordinary cohomology over $\Lambda_{0,\mathrm{nov}}^{\mathbb{Z}_2}$.

(2) Theorem 6.71 gives an example where Floer cohomology of the real point set is different from its ordinary cohomology. Therefore, it is necessary to use $\mathbb{Z}_2$ coefficients to study the Arnold–Givental conjecture; see Chapter 8 of [7].

Proof of Theorem 6.71 If $n \geq 1$, the set $\pi_2(\mathbb{C}P^{2n+1}, \mathbb{R}P^{2n+1})$ has exactly one element B_1 satisfying $\mu_{\mathbb{R}P^{2n+1}}(B_1) = 2n+2$, which is the minimal Maslov number of $\mathbb{R}P^{2n+1}$. By the monotonicity of $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$, a degree counting argument shows that only $\mathcal{M}_2(J; B_1)$, among the moduli spaces $\mathcal{M}_2(J; B)$ with $B \in \pi_2(\mathbb{C}P^{2n+1}, \mathbb{R}P^{2n+1})$, contributes to the differential of the spectral sequence. First of all, we note that τ induces an isomorphism modulo orientations:

$$\tau_*\colon \mathcal{M}_2(J; B_1) \to \mathcal{M}_2(J; B_1). \tag{6-42}$$

Later we examine whether τ_* preserves the orientation or not, after we specify relative spin structures.

Since $\omega[B_1]$ is the smallest positive symplectic area, $\mathcal{M}_2(J; B_1)$ has codimension-1 boundary corresponding to the strata consisting of elements which have a constant disc component with two marked points and a disc bubble. However, when we consider the evaluation map $\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathcal{M}_2(J; B_1) \to \mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}$, these strata are mapped to the diagonal set, whose codimension is bigger than 2. Thus we can define fundamental cycle over $\mathbb{Z}$ of $\mathrm{ev}(\mathcal{M}_2(J; B_1))$, which we denote by $[\mathrm{ev}(\mathcal{M}_2(J; B_1))]$. We also note

$$\dim \mathcal{M}_2(J; B_1) = 2n+2+2n+1+2-3 = 2\dim \mathbb{R}P^{2n+1}.$$

Lemma 6.73 *Consider the evaluation map* $\mathrm{ev}\colon \mathcal{M}_2(J; B_1) \to \mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}$. *Then we have*

$$[\mathrm{ev}(\mathcal{M}_2(J; B_1))] = \pm 2[\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}],$$

where $[\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}]$ is the fundamental cycle of $\mathbb{R}P^{2n+1} \times \mathbb{R}P^{2n+1}$.

Proof For any distinct two points $p, q \in \mathbb{C}P^{2n+1}$ there exists a holomorphic map $w\colon S^2 = \mathbb{C} \cup \{\infty\} \to \mathbb{C}P^{2n+1}$ of degree 1 such that $w(0) = p$ and $w(\infty) = q$, which is unique up to the action of $\mathbb{C} \setminus \{0\} \cong \mathrm{Aut}(\mathbb{C}P^1; 0, \infty)$. In case $p, q \in \mathbb{R}P^{2n+1}$, the uniqueness implies that $w(\bar{z}) = \tau w(cz)$ for some $c \in \mathbb{C} \setminus \{0\}$. Using this equality twice, we have $w(z) = w(|c|^2 z)$. In particular, we find that $|c| = 1$. Let a be a square root of c, and set $w'(z) = w(az)$. Since $\bar{a}c/a = 1$, we obtain $w'(\bar{z}) = \tau w'(z)$. (Note that w and w' define the same element in the moduli space $\mathcal{M}_2^{\mathrm{sph}}(J; [\mathbb{C}P^1])$.) Thus the restriction of w to the upper or lower half plane defines elements w_u or $w_l \in \mathcal{M}_2(J; B_1)$. Namely, there exist $w_u, w_l \in \mathcal{M}_2(J; B_1)$ such that $\mathrm{ev}(w_u) = \mathrm{ev}(w_l) = (p, q)$. Conversely, any such elements determine a degree-one curve by the reflection principle.

To complete the proof of Lemma 6.73, we have to show that the orientations of the evaluation map ev at w_u and w_l coincide. Note that $\tau_*(w_u) = w_l$ and $\tau_* \circ \mathrm{ev} = \mathrm{ev}$. Thus it suffices to show that τ_* in (6-42) preserves the orientation. First, we consider the case of $\mathbb{R}P^{4n+3}$. In this case, $\mathbb{R}P^{4n+3}$ is τ–relatively spin. Therefore, by Theorem 4.10, the map (6-42) is orientation-preserving because

$$\tfrac{1}{2}\mu_{\mathbb{R}P^{4n+3}}(B_1) + 2 = 2n + 4$$

is even. We next consider the case of $\mathbb{R}P^{4n+1}$. We pick its relative spin structure $[(V, \sigma)]$. By Theorem 4.10 again, the map

$$\tau_*\colon \mathcal{M}_2(J; B_1)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_2(J; B_1)^{[(V,\sigma)]}$$

is orientation-reversing because

$$\tfrac{1}{2}\mu_{\mathbb{R}P^{4n+1}}(B_1) + 2 = 2n + 3$$

is odd. On the other hand, by Proposition 3.14 we have $\tau^*[(V, \sigma)] \neq [(V, \sigma)]$. Let $\mathfrak{x}$ be the unique nonzero element of $H^2(\mathbb{C}P^{4n+1}, \mathbb{R}P^{4n+1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. It is easy to see that $\mathfrak{x}[B_1] \neq 0$. Then by Proposition 3.10, the identity induces an *orientation-reversing* isomorphism

$$\mathcal{M}_2(J; B_1)^{\tau^*[(V,\sigma)]} \to \mathcal{M}_2(J; B_1)^{[(V,\sigma)]}.$$

Therefore, we can find that

$$\tau_*\colon \mathcal{M}_2(J; B_1)^{[(V,\sigma)]} \to \mathcal{M}_2(J; B_1)^{[(V,\sigma)]}$$

is orientation-preserving. This completes the proof of Lemma 6.73. □

Then Lemma 6.73 and the definition of the differential d imply

$$d^{2n+1}(\mathrm{PD}([p])) = \big[\mathrm{ev}_0\big(\mathcal{M}_2(J; B_1)_{\mathrm{ev}_1} \times [p]\big)\big] = \pm 2\,\mathrm{PD}[\mathbb{R}P^{2n+1}],$$

which finishes the proof of Theorem 6.71. □

Remark 6.74 In Section 3.6.3 of [9], we introduced the notion of *weak unobstructedness* and *weak bounding cochains* using the homotopy unit of the filtered A_∞ algebra. We denote by $\mathcal{M}_{\mathrm{weak}}(L;\Lambda_{0,\mathrm{nov}})$ the set of all weak bounding cochains. We also defined the potential function $\mathfrak{PO}\colon \mathcal{M}_{\mathrm{weak}}(L;\Lambda_{0,\mathrm{nov}}) \to \Lambda^{+(0)}_{0,\mathrm{nov}}$, where $\Lambda^{+(0)}_{0,\mathrm{nov}}$ is the degree-zero part of $\Lambda^{+}_{0,\mathrm{nov}}$. Then the set of bounding cochains $\mathcal{M}(L;\Lambda_{0,\mathrm{nov}})$ is characterized by $\mathcal{M}(L;\Lambda_{0,\mathrm{nov}}) = \mathfrak{PO}^{-1}(0)$. About the value of the potential function, we have the following problem:

Problem 6.75 Let L be a relatively spin Lagrangian submanifold of a symplectic manifold M. We assume that L is weakly unobstructed and that the Floer cohomology $\mathrm{HF}((L,b),(L,b);\Lambda^{F}_{0,\mathrm{nov}})$ deformed by $b \in \mathcal{M}_{\mathrm{weak}}(L)$ does not vanish for some field F. In this situation, the question is whether $\mathfrak{PO}(b)$ is an eigenvalue of the operation

$$c \mapsto c \cup_Q c_1(M)\colon \mathrm{QH}(M;\Lambda^{F}_{0,\mathrm{nov}}) \to \mathrm{QH}(M;\Lambda^{F}_{0,\mathrm{nov}}).$$

Here $(\mathrm{QH}(M;\Lambda^{F}_{0,\mathrm{nov}}),\cup_Q)$ is the quantum cohomology ring of M over $\Lambda^{F}_{0,\mathrm{nov}}$.

Such statement was made by M Kontsevich in 2006 during a conference of homological mirror symmetry at Vienna. (According to some physicists this had been known to them before.) See also [1]. As we saw above, $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ for $n \geq 1$ is unobstructed. Since the minimal Maslov number is strictly greater than 2, we find that any $b \in H^1(\mathbb{R}P^{2n+1};F) \otimes \Lambda^{F}_{0,\mathrm{nov}}$ of total degree 1 is a bounding cochain; ie $\mathfrak{PO}(b) = 0$ by the dimension counting argument. On the other hand, Theorem 6.71 shows that the Floer cohomology does not vanish for $F = \mathbb{Z}_2$, and the eigenvalue is zero in the field $F = \mathbb{Z}_2$ because $c_1(\mathbb{C}P^{2n+1}) \equiv 0 \bmod 2$. Thus this is consistent with the problem. (If we take $F = \mathbb{Q}$, the eigenvalue is not zero in $\mathbb{Q}$. But Theorem 6.71 shows that the Floer cohomology over $\Lambda^{\mathbb{Q}}_{0,\mathrm{nov}}$ vanishes. So the assumption of the problem is not satisfied in this case.) Besides this, we prove this statement for the case of Lagrangian fibers of smooth toric manifolds in [15]. We do not have any counterexample to this statement at the time of writing this paper.

6.5 Wall crossing term in [5]

Let M be a 6–dimensional symplectic manifold and L its relatively spin Lagrangian submanifold. Suppose the Maslov index homomorphism $\mu_L\colon H_2(M,L;\mathbb{Z}) \to \mathbb{Z}$ is zero. In this situation, the first named author [5] introduced an invariant

$$\Psi_J\colon \mathcal{M}(L;\Lambda^{\mathbb{C}}_{0,\mathrm{nov}}) \to \Lambda^{+\mathbb{C}}_{0,\mathrm{nov}}.$$

In general, it depends on a compatible almost complex structure J, and the difference $\Psi_J - \Psi_{J'}$ is an element of $\Lambda^{+\mathbb{Q}}_{0,\mathrm{nov}}$.

Let us consider the case where $\tau\colon M \to M$ is an antisymplectic involution and $L = \operatorname{Fix}\tau$. We take the compatible almost complex structures J_0 and J_1 such that $\tau_* J_0 = -J_0$ and $\tau_* J_1 = -J_1$. Moreover, we assume that there exists a one-parameter family of compatible almost complex structures $\mathcal{J} = \{J_t \mid t \in [0,1]\}$ such that $\tau_* J_t = -J_t$. We will study the difference

$$\Psi_{J_1} - \Psi_{J_0} \tag{6-43}$$

below. Namely, we will study the wall crossing phenomenon by the method of this paper.

Let $\alpha \in H_2(M;\mathbb{Z})$. Denote by $\mathcal{M}_1(\alpha; J)$ the moduli space of J–holomorphic *spheres* with one interior marked point and of homology class α. We have an evaluation map $\mathrm{ev}\colon \mathcal{M}_1(\alpha; J) \to M$. We assume

$$\mathrm{ev}(\mathcal{M}_1(\alpha; J_0)) \cap L = \mathrm{ev}(\mathcal{M}_1(\alpha; J_1)) \cap L = \varnothing \tag{6-44}$$

for any $\alpha \neq 0$. Since the virtual dimension of $\mathcal{M}_1(\alpha; J)$ is 2, (6-44) holds in generic cases. The space

$$\mathcal{M}_1(\alpha; \mathcal{J}; L) = \bigcup_{t \in [0,1]} \{t\} \times (\mathcal{M}_1(\alpha; J_t)_{\mathrm{ev}} \times_M L) \tag{6-45}$$

has a Kuranishi structure of dimension 0, which is fibered on $[0,1]$. The assumption (6-44) implies that (6-45) has no boundary. Therefore, its virtual fundamental cycle is well defined and gives a rational number, which we denote by $\#\mathcal{M}_1(\alpha; \mathcal{J}; L)$. By Theorem 1.5 in [5], we have

$$\Psi_{J_1} - \Psi_{J_0} = \sum_{\alpha} \#\mathcal{M}_1(\alpha; \mathcal{J}; L) T^{\omega(\alpha)}.$$

The involution naturally induces a map $\tau\colon \mathcal{M}_1(\alpha; \mathcal{J}; L) \to \mathcal{M}_1(\tau_*\alpha; \mathcal{J}; L)$.

Lemma 6.76 *The map $\tau\colon \mathcal{M}_1(\alpha; \mathcal{J}; L) \to \mathcal{M}_1(\tau_*\alpha; \mathcal{J}; L)$ is orientation-preserving.*

Proof In the same manner as in the proof of Proposition 4.9, we can prove that $\tau\colon \mathcal{M}_1(\alpha; J) \to \mathcal{M}_1(\tau_*\alpha; J)$ is orientation-reversing. In fact, this case is similar to the case where $k = -1$, $\mu_L(\beta) = 2c_1(\alpha) = 0$ and $m = 1$ of Proposition 4.9. Note that τ also reverses the orientation on M if $\frac{1}{2}\dim_{\mathbb{R}} M$ is odd. Therefore, for any $t \in [0,1]$, τ respects the orientation on $\mathcal{M}_1(\alpha; J_t)_{\mathrm{ev}} \times_M L$ and that on $\mathcal{M}_1(\tau_*\alpha; J_t)_{\mathrm{ev}} \times_M L$. Hence the lemma. $\square$

Lemma 6.76 implies that, in the case of a τ fixed point, the cancellation of the wall crossing term via involution does *not* occur because of the sign. Namely, Formula (8.1) in the first author's paper [5] is wrong.

Appendix: Review of Kuranishi structure — orientation and group action

In this appendix, we briefly review the orientation on a space with Kuranishi structure and the notion of a group action on a space with Kuranishi structure for the readers convenience. For a more detailed explanation, we refer to Sections A1.1 of [10] and A1.3 of [10] for Sections A.1 and A.2, respectively.

A.1 Orientation

To define an orientation on a space with Kuranishi structure, we first recall the notion of its tangent bundle. Let $\mathcal{M}$ be a compact topological space, and let $\mathcal{M}$ have a Kuranishi structure. That is, $\mathcal{M}$ has a collection of a finite number of Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ for $p \in \mathcal{M}$ such that:

- (k-1) V_p is a finite-dimensional smooth manifold which may have boundaries or corners;
- (k-2) E_p is a finite-dimensional real vector space, and $\dim V_p - \dim E_p$ is independent of p;
- (k-3) Γ_α is a finite group acting smoothly and effectively on V_p, and linearly on E_p;
- (k-4) s_p, which is called a *Kuranishi map*, is a Γ_p–equivariant smooth section of the vector bundle $E_p \times V_p \to V_p$ called an *obstruction bundle*;
- (k-5) $\psi_p\colon s_p^{-1}(0)/\Gamma_p \to \mathcal{M}$ is a homeomorphism to its image;
- (k-6) $\bigcup_p \psi_p(s_p^{-1}(0)/\Gamma_p) = \mathcal{M}$;
- (k-7) the collection $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p\in\mathcal{M}}$ satisfies certain compatibility conditions under coordinate change.

See Definitions A1.3 and A1.5 in [10] for the precise definition and description of the *coordinate change* and the compatibility conditions in (k-7), respectively. We denote by $\mathcal{P}$ the finite set of $p \in \mathcal{M}$ above. By Lemma 6.3 in [16], we may assume that $\{(V_p, E_p, \Gamma_p, \psi_p, s_p)\}_{p\in P}$ is a *good coordinate system* in the sense of Definition 6.1 in [16]. In other words, there is a partial order $<$ on $\mathcal{P}$ such that the following conditions hold: Let $q < p$ $(p, q \in \mathcal{P})$ with $\psi_p(s_p^{-1}(0)/\Gamma_p) \cap \psi_q(s_q^{-1}(0)/\Gamma_q) \neq \varnothing$. Then there exist:

(gc-1) a Γ_q–invariant open subset V_{pq} of V_q such that
$$\psi_q^{-1}\big(\psi_p(s_p^{-1}(0)/\Gamma_p)\cap\psi_q(s_q^{-1}(0)/\Gamma_q)\big)\subset V_{pq}/\Gamma_q;$$
(gc-2) an injective group homomorphism $h_{pq}\colon \Gamma_q\to\Gamma_p$;

(gc-3) an h_{pq}–equivariant smooth embedding $\phi_{pq}\colon V_{pq}\to V_p$ such that the induced map $V_{pq}/\Gamma_q\to V_p/\Gamma_p$ is injective;

(gc-4) an h_{pq}–equivariant embedding $\widehat{\phi}_{pq}\colon E_q\times V_{pq}\to E_p\times V_p$ of vector bundles which covers ϕ_{pq} and satisfies
$$\widehat{\phi}_{pq}\circ s_q=s_p\circ\phi_{pq},\quad \psi_q=\psi_p\circ\underline{\phi}_{pq}.$$
Here $\underline{\phi}_{pq}\colon V_{pq}/\Gamma_q\to V_p/\Gamma_p$ is the map induced by ϕ_{pq}.

Moreover, if $r<q<p$ and $\psi_p(s_p^{-1}(0)/\Gamma_p)\cap\psi_q(s_q^{-1}(0)/\Gamma_q)\cap\psi_r(s_r^{-1}(0)/\Gamma_r)\neq\varnothing$, then there exists

(gc-5) $\gamma_{pqr}\in\Gamma_p$ such that
$$h_{pq}\circ h_{qr}=\gamma_{pqr}\cdot h_{pr}\cdot\gamma_{pqr}^{-1},\quad \phi_{pq}\circ\phi_{qr}=\gamma_{pqr}\cdot\phi_{pr},\quad \widehat{\phi}_{pq}\circ\widehat{\phi}_{qr}=\gamma_{pqr}\cdot\widehat{\phi}_{pr}.$$
Here the second and third equalities hold on $\phi_{qr}^{-1}(V_{pq})\cap V_{qr}\cap V_{pr}$ and on $E_r\times(\phi_{qr}^{-1}(V_{pq})\cap V_{qr}\cap V_{pr})$, respectively.

Now we identify a neighborhood of $\phi_{pq}(V_{pq})$ in V_p with a neighborhood of the zero section of the normal bundle $N_{V_{pq}}V_p\to V_{pq}$. Then the differential of the Kuranishi map s_p along the fiber direction defines an h_{pq}–equivariant bundle homomorphism
$$d_{\mathrm{fiber}}s_p\colon N_{V_{pq}}V_p\to E_p\times V_{pq}.$$
See Lemma A1.58 in [10] and also Theorems 13.2 and 19.5 in [13] for detail.

Definition A.1 We say that the space $\mathcal{M}$ with Kuranishi structure *has a tangent bundle* if $d_{\mathrm{fiber}}s_p$ induces a bundle isomorphism

$$N_{V_{pq}}V_p\cong\frac{E_p\times V_{pq}}{\widehat{\phi}_{pq}(E_q\times V_{pq})}\tag{A-1}$$

as Γ_q–equivariant bundles on a neighborhood of $V_{pq}\cap s_q^{-1}(0)$. (See also Chapter 2 of [13].)

Definition A.2 Let $\mathcal{M}$ be a space with Kuranishi structure with a tangent bundle. We say that the Kuranishi structure on $\mathcal{M}$ is *orientable* if there is a trivialization of
$$\Lambda^{\mathrm{top}}E_p^*\otimes\Lambda^{\mathrm{top}}TV_p$$
compatible with the isomorphism (A-1) and whose homotopy class is preserved by the Γ_p–action. The *orientation* is the choice of the homotopy class of such a trivialization.

Pick such a trivialization. Suppose that s_p is transverse to zero at p. Then we define an orientation on the zero locus $s_p^{-1}(0)$ of the Kuranishi map s_p, which may be assumed so that $p \in s_p^{-1}(0)$, by the equation

$$E_p \times T_p s_p^{-1}(0) = T_p V.$$

Since we pick a trivialization of $\Lambda^{\mathrm{top}} E_p^* \otimes \Lambda^{\mathrm{top}} T V_p$ as in Definition A.2, the above equality determines an orientation on $s_p^{-1}(0)$, and also on $s_p^{-1}(0)/\Gamma_p$. See Section 8.2 of [10] for a more detailed explanation of orientation on a space with Kuranishi structure.

A.2 Group action

Next we recall the definitions of a finite group action on a space with Kuranishi structure and its quotient space. In this paper, we used the $\mathbb{Z}_2$–action and its quotient space (in the proof of Theorem 1.5).

Let $\mathcal{M}$ be a compact topological space with Kuranishi structure. We first define the notion of an automorphism of Kuranishi structure.

Definition A.3 Let $\varphi\colon \mathcal{M} \to \mathcal{M}$ be a homeomorphism of $\mathcal{M}$. We say that it induces an *automorphism of Kuranishi structure* if the following holds: Let $p \in \mathcal{M}$ and $p' = \varphi(p)$. Then, for the Kuranishi neighborhoods $(V_p, E_p, \Gamma_p, \psi_p, s_p)$ and $(V_{p'}, E_{p'}, \Gamma_{p'}, \psi_{p'}, s_{p'})$ of p and p', respectively, there exist $\rho_p\colon \Gamma_p \to \Gamma_{p'}$, $\varphi_p\colon V_p \to V_{p'}$ and $\widehat{\varphi}_p\colon E_p \to E_{p'}$ such that:

- (au-1) ρ_p is an isomorphism of groups;
- (au-2) φ_p is a ρ_p–equivariant diffeomorphism;
- (au-3) $\widehat{\varphi}_p$ is a ρ_p–equivariant bundle isomorphism which covers φ_p;
- (au-4) $s_{p'} \circ \varphi_p = \widehat{\varphi}_p \circ s_p$;
- (au-5) $\psi_{p'} \circ \underline{\varphi}_p = \varphi \circ \psi_p$, where $\underline{\varphi}_p\colon s_p^{-1}(0)/\Gamma_p \to s_{p'}^{-1}(0)/\Gamma_{p'}$ is a homeomorphism induced by $\varphi_p|_{s_p^{-1}(0)}$.

We require that ρ_p, φ_p and $\widehat{\varphi}_p$ above satisfy the following compatibility conditions with the coordinate changes of Kuranishi structure: Let $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ and $q' \in \psi_{p'}(s_{p'}^{-1}(0)/\Gamma_{p'})$ such that $\varphi(q) = q'$. Let $(\widehat{\phi}_{pq}, \phi_{pq}, h_{pq})$ and $(\widehat{\phi}_{p'q'}, \phi_{p'q'}, h_{p'q'})$ be the coordinate changes. Then there exists $\gamma_{pqp'q'} \in \Gamma_{p'}$ such that the following conditions hold:

- (auc-1) $\rho_p \circ h_{pq} = \gamma_{pqp'q'} \cdot (h_{p'q'} \circ \rho_q) \cdot \gamma_{pqp'q'}^{-1}$;
- (auc-2) $\varphi_p \circ \phi_{pq} = \gamma_{pqp'q'} \cdot (\phi_{p'q'} \circ \varphi_q)$;
- (auc-3) $\widehat{\varphi}_p \circ \widehat{\phi}_{pq} = \gamma_{pqp'q'} \cdot (\widehat{\phi}_{p'q'} \circ \widehat{\varphi}_q)$.

Then we call $((\rho_p, \varphi_p, \widehat{\varphi}_p)_p; \varphi)$ an *automorphism of the Kuranishi structure*.

Remark A.4 Here $(\rho_p, \varphi_p, \widehat{\varphi}_p)_p$ are included as data of an automorphism.

Definition A.5 We say that an automorphism $((\rho_p, \varphi_p, \widehat{\varphi}_p)_p; \varphi)$ is *conjugate* to $((\rho'_p, \varphi'_p, \widehat{\varphi}'_p)_p; \varphi')$ if $\varphi = \varphi'$ and if there exists $\gamma_p \in \Gamma_{\varphi(p)}$ for each p such that:

(cj-1) $\rho'_p = \gamma_p \cdot \rho_p \cdot \gamma_p^{-1}$;

(cj-2) $\varphi'_p = \gamma_p \cdot \varphi_p$;

(cj-3) $\widehat{\varphi}'_p = \gamma_p \cdot \widehat{\varphi}_p$.

The *composition* of the two automorphisms is defined by the following formula:

$$\big((\rho^1_p, \varphi^1_p, \widehat{\varphi}^1_p)_p; \varphi^1\big) \circ \big((\rho^2_p, \varphi^2_p, \widehat{\varphi}^2_p)_p; \varphi^2\big) = \big((\rho^1_{\varphi^2(p)} \circ \rho^2_p, \varphi^1_{\varphi^2(p)} \circ \varphi^2_p, \widehat{\varphi}^1_{\varphi^2(p)} \circ \widehat{\varphi}^2_p)_p; \varphi^1 \circ \varphi^2\big).$$

Then we can easily check that the right-hand side also satisfies the compatibility conditions (auc-1)–(au-3). Moreover, we can find that the composition induces the composition of the conjugacy classes of automorphisms.

Definition A.6 An automorphism $((\rho_p, \varphi_p, \widehat{\varphi}_p)_p; \varphi)$ is *orientation-preserving* if it is compatible with the trivialization of $\Lambda^{\mathrm{top}} E_p^* \otimes \Lambda^{\mathrm{top}} T V_p$.

Let $\mathrm{Aut}(\mathcal{M})$ be the set of all conjugacy classes of the automorphisms of $\mathcal{M}$ and let $\mathrm{Aut}_0(\mathcal{M})$ be the set of all conjugacy classes of the orientation-preserving automorphisms of $\mathcal{M}$. Both of them become groups by composition.

Definition A.7 Let G be a finite group which acts on a compact space $\mathcal{M}$. Assume that $\mathcal{M}$ has a Kuranishi structure. We say that G *acts* on $\mathcal{M}$ (as a space with Kuranishi structure) if, for each $g \in G$, the homeomorphism $x \mapsto gx$, $\mathcal{M} \to \mathcal{M}$ induces an automorphism g_* of the Kuranishi structure, and the composition of g_* and h_* is conjugate to $(gh)_*$. In other words, an action of G to $\mathcal{M}$ is a group homomorphism $G \to \mathrm{Aut}(\mathcal{M})$.

An *involution* of a space with Kuranishi structure is a $\mathbb{Z}_2$ action.

Then we can show the following:

Lemma A.8 [10, Lemma A1.49] *If a finite group G acts on a space $\mathcal{M}$ with Kuranishi structure, then the quotient space $\mathcal{M}/G$ has a Kuranishi structure.*

If $\mathcal{M}$ has a tangent bundle and the action preserves it, then the quotient space has a tangent bundle. If $\mathcal{M}$ is oriented and the action preserves the orientation, then the quotient space has an orientation.

A.3 Invariant promotion

As we promised in the proof of Theorem 1.5, we explain how we adopt the obstruction theory developed in Sections 7.2.6–7.2.10 of [10] to promote a filtered $A_{n,K}$ structure to a filtered A_∞ structure keeping the symmetry (1-2). Since the modification is straightforward, we give the outline for readers' convenience. We use the same notation as in Section 6.1. We first note that in our geometric setup, the Lagrangian submanifold L is the fixed point set of the involution τ. So τ acts trivially on $C(L;\mathbb{Q})$ in Theorem 6.2. Moreover, the induced map τ_* given by Definition 4.2 is also trivial on the monoid $G(L)$. (See Remark 4.3.)

Let $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ be a monoid as in Section 6.1. We denote by β an element of G. Let R be a field containing $\mathbb{Q}$, and let $\bar{C}$ be a graded R–module. Put $C = \bar{C} \otimes \Lambda^R_{0,\mathrm{nov}}$. Following Section 7.2.6 of [10], we use the Hochschild cohomology to describe the obstruction to the promotion. In this article, we use $\bar{C}^e = \bar{C} \otimes R[e, e^{-1}]$ instead of $\bar{C}$ to encode the data of the Maslov index appearing in (1-2). Here e is the formal variable in $\Lambda^R_{0,\mathrm{nov}}$. Note that the promotion is made by induction on the partial order $<$ on the set $G \times \mathbb{Z}_{\geq 0}$, which is absolutely independent of the variable e. Thus the obstruction theory given in Sections 7.2.6–7.2.10 of [10] also works on $\bar{C}^e$. Let $\{\mathfrak{m}_{k,\beta}\}$ and $\{\mathfrak{f}_{k,\beta}\}$ be a filtered $A_{n,K}$ algebra structure (Definition 6.5) and a filtered $A_{n,K}$ homomorphism (Definition 6.6). They are R–linear maps from $B_k(\bar{C}[1])$ to $\bar{C}$. We naturally extend them as $R[e, e^{-1}]$ module homomorphisms and denote the extensions by the same symbols. We put

$$\begin{aligned} \mathfrak{m}^e_{k,\beta} &= \mathfrak{m}_{k,\beta} e^{\mathrm{pr}_2(\beta)/2} \colon B_k(\bar{C}^e[1]) \to \bar{C}^e[1], \\ \mathfrak{f}^e_{k,\beta} &= \mathfrak{f}_{k,\beta} e^{\mathrm{pr}_2(\beta)/2} \colon\ B_k(\bar{C}^e[1]) \to \bar{C}^e[1], \end{aligned} \tag{A-2}$$

where $\mathrm{pr}_2\colon G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z} \to 2\mathbb{Z}$ is the projection to the second factor. In the geometric situation, pr_2 is the Maslov index μ; see (6-3). For each K, we have a map

$$\mathrm{Op}\colon B_K \bar{C}^e[1] \to B_K \bar{C}^e[1] \tag{A-3}$$

defined by

$$\mathrm{Op}(a_1 x_1 \otimes \cdots \otimes a_K x_K) = (-1)^*(a_K^\dagger x_K \otimes \cdots \otimes a_1^\dagger x_1)$$

for $a_i = \sum_j c_j e^{\mu_j} \in R[e, e^{-1}]$ and $x_i \in \bar{C}$. Here

$$* = K + 1 + \sum_{1 \leq i < j \leq K} \deg' x_i \deg' x_j \tag{A-4}$$

and

$$a_i^\dagger = \sum_j c_j (-e)^{\mu_j}. \tag{A-5}$$

Obviously we have $\mathrm{Op}\circ\mathrm{Op}=\mathrm{id}$.

Definition A.9 An $R[e,e^{-1}]$ module homomorphism $\mathfrak{g}\in\mathrm{Hom}(B(\overline{C}^e_K[1]),\overline{C}^e[1])$ is called Op*–invariant* if $\mathfrak{g}\circ\mathrm{Op}=\mathrm{Op}\circ\mathfrak{g}$.

Then it is easy to check the following. Recall that $\tau_*\beta=\beta$ for $\beta\in G$.

Lemma A.10 *A filtered $A_{n,K}$ structure $\{\mathfrak{m}_{k,\beta}\}$ satisfies (1-2) if and only if $\mathfrak{m}^e_{k,\beta}$ defined by (A-2) is* Op*–invariant.*

Definition A.11 A filtered $A_{n,K}$ algebra $(C,\{\mathfrak{m}_{k,\beta}\})$ is called an Op*–invariant filtered $A_{n,K}$ algebra* if $\{\mathfrak{m}^e_{k,\beta}\}$ is Op–invariant. A filtered $A_{n,K}$ homomorphism $\{\mathfrak{f}_{k,\beta}\}$ is called Op*–invariant* if $\{\mathfrak{f}^e_{k,\beta}\}$ is Op–invariant. We define an Op*–invariant filtered $A_{n,K}$ homotopy equivalence* in a similar way.

The following is the precise statement of our invariant version of Theorem 6.7 (Theorem 7.2.72 in [10]) which is used in the proof of Theorem 1.5.

Theorem A.12 *Let C_1 be an* Op*–invariant filtered $A_{n,K}$ algebra and C_2 an* Op*–invariant filtered $A_{n',K'}$ algebra such that $(n,K)<(n',K')$. Let $\mathfrak{h}\colon C_1\to C_2$ be an* Op*–invariant filtered $A_{n,K}$ homomorphism. Suppose that $\mathfrak{h}$ is an* Op*–invariant filtered $A_{n,K}$ homotopy equivalence. Then there exist an* Op*–invariant filtered $A_{n',K'}$ algebra structure on C_1 extending the given* Op*–invariant filtered $A_{n,K}$ algebra structure and an* Op*–invariant filtered $A_{n',K'}$ homotopy equivalence $C_1\to C_2$ extending the given* Op*–invariant filtered $A_{n,K}$ homotopy equivalence $\mathfrak{h}$.*

Proof To prove this theorem, we mimic the obstruction theory to promote a filtered $A_{n,K}$ structure to a filtered A_∞ structure given in Sections 7.2.6–7.2.10 of [10].

Let $(C,\{\mathfrak{m}_{k,\beta}\})$ be a filtered $A_{n,K}$ algebra. As mentioned before, we first rewrite the obstruction theory by using $\overline{C}^e$ instead of $\overline{C}$. This is done by extending the coefficient ring R to $R[e,e^{-1}]$ and replacing $\mathfrak{m}_{k,\beta}$ by $\mathfrak{m}^e_{k,\beta}$ in (A-2) as follows: We put $\overline{\mathfrak{m}}_k=\mathfrak{m}_{k,\beta_0}$ for $\beta_0=(0,0)$. (Note that $\overline{\mathfrak{m}}_k=\mathfrak{m}^e_{k,\beta_0}$.) We naturally extend $\overline{\mathfrak{m}}_k$ to an $R[e,e^{-1}]$ module homomorphism which, by abuse of notation, we also write $\overline{\mathfrak{m}}_k\colon B_k(\overline{C}^e[1])\to\overline{C}^e[1]$.

As in the proof of Theorem 7.2.72 in [10], we may assume that $(n,K)<(n',K')=(n+1,K-1)$ or $(n,K)=(n,0)<(n',K')=(0,n+1)$ to consider the promotion. We consider an $R[e,e^{-1}]$–module $\mathrm{Hom}(B_{K'}\overline{C}^e[1],\overline{C}^e[1])$ and define the coboundary operator δ_1 on it by

$$\delta_1(\varphi)=\overline{\mathfrak{m}}_1\circ\varphi+(-1)^{\deg\varphi+1}\varphi\circ\widehat{\overline{\mathfrak{m}}}_1 \tag{A-6}$$

for $\varphi \in B_{K'}\bar{C}^e[1]$. Here $\widehat{\overline{\mathfrak{m}}}_1\colon B\bar{C}^e[1] \to B\bar{C}^e[1]$ is a coderivation induced by $\overline{\mathfrak{m}}_1$ on $\bar{C}^e$ as in (6-5). We denote the δ_1–cohomology by

$$H(\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1]), \delta_1) \tag{A-7}$$

and call it *Hochschild cohomology* of $\bar{C}^e$. We modify the definition of the obstruction class as follows. As in the proof of Lemma 7.2.74 in [10], we put $(\|\beta\|, k) = (n', K')$; see (6-7) for the definition of $\|\beta\|$. Then, replacing $\mathfrak{m}_{k,\beta}$ by $\mathfrak{m}^e_{k,\beta}$, we modify [10, (7.2.75)] so that

$$\sum_{\substack{\beta_1+\beta_2=\beta,\ k_1+k_2=k+1\\ (k_i,\beta_i)\neq(k,\beta)}} \sum_i (-1)^{\deg' \boldsymbol{x}_i^{(1)}} \mathfrak{m}^e_{k_2,\beta_2}\big(\boldsymbol{x}_i^{(1)}, \mathfrak{m}^e_{k_1,\beta_1}(\boldsymbol{x}_i^{(2)}), \boldsymbol{x}_i^{(3)}\big),$$

where $\boldsymbol{x} \in B_{K'}\bar{C}^e[1]$. Note that $e^{\mathrm{pr}_2(\beta_1)/2} e^{\mathrm{pr}_2(\beta_2)/2} = e^{\mathrm{pr}_2(\beta)/2}$. Then this defines an element

$$o^e_{K',\beta}(C) \in \mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])$$

which is a δ_1–cocycle. Thus we can define the *obstruction class*

$$[o^e_{K',\beta}(C)] \in H(\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1]), \delta_1) \tag{A-8}$$

for each $\|\beta\| = n'$. Under this modification, Lemma 7.2.74 in [10] also holds for $\bar{C}^e$.

Now we consider the Op–invariant version. The map Op defined by (A-3) acts on $B_{K'}\bar{C}^e[1]$ and $\bar{C}^e[1]$, and so on $\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])$ as involution. We decompose $\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])$ so that

$$\begin{aligned} &\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1]) \\ &\qquad = \mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])^{\mathrm{Op}} \oplus \mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])^{-\mathrm{Op}}, \end{aligned} \tag{A-9}$$

where $\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])^{\mathrm{Op}}$ and $\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])^{-\mathrm{Op}}$ are the Op–invariant part and the anti-Op–invariant part, respectively.

Suppose that $(C, \{\mathfrak{m}_{k,\beta}\})$ is an Op–invariant filtered $A_{n,K}$ algebra. By Lemma A.10, we have Op–invariant elements

$$\mathfrak{m}^e_{k,\beta} = \mathfrak{m}_{k,\beta} e^{\mathrm{pr}_2(\beta)/2} \in \mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])^{\mathrm{Op}}.$$

Note that the map Op and δ_1 defined by (A-6) commute. Therefore, if we use $\mathfrak{m}^e_{k,\beta}$, we can define an Op–*invariant Hochschild cohomology*

$$H(\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1]), \delta_1)^{\mathrm{Op}} := H(\mathrm{Hom}(B_{K'}\bar{C}^e[1], \bar{C}^e[1])^{\mathrm{Op}}, \delta_1). \tag{A-10}$$

Moreover, the construction of the obstruction class above yields the Op–invariant obstruction class

$$[o^e_{K',\beta}(C)]^{\mathrm{Op}} \in H(\mathrm{Hom}(B_{K'}\overline{C}^e[1], \overline{C}^e[1]), \delta_1)^{\mathrm{Op}}. \tag{A-11}$$

Then the following lemma is the Op–invariant version of Lemma 7.2.74 in [10] whose proof is straightforward.

Lemma A.13 *Let (n', K') be as above and C an* Op*–invariant filtered $A_{n,K}$ algebra. Then the obstruction classes*

$$[o^e_{K',\beta}(C)]^{\mathrm{Op}} \in H(\mathrm{Hom}(B_{K'}\overline{C}^e[1], \overline{C}^e[1]), \delta_1)^{\mathrm{Op}}$$

vanish for all β with $\|\beta\| = n'$ if and only if there exists an Op*–invariant filtered $A_{n',K'}$ structure extending the given* Op*–invariant filtered $A_{n,K}$ structure.*

Moreover, if $C \to C'$ is an Op*–invariant filtered $A_{n,K}$ homotopy equivalence, then $[o^e_{K',\beta}(C)]^{\mathrm{Op}}$ is mapped to $[o^e_{K',\beta}(C')]^{\mathrm{Op}}$ by the isomorphism*

$$H(\mathrm{Hom}(B_{K'}\overline{C}^e[1], \overline{C}^e[1]), \delta_1)^{\mathrm{Op}} \cong H(\mathrm{Hom}(B_{K'}\overline{C'}^e[1], \overline{C'}^e[1]), \delta_1)^{\mathrm{Op}}$$

induced by the Op*–invariant homotopy equivalence.*

Once Op–invariant obstruction theory is established, the rest of the proof is parallel to one in Section 7.2.6 of [10]. □

Similarly, using the Op–invariant obstruction theory for an Op–invariant filtered $A_{n,K}$ homomorphism, we can also show the Op–invariant version of Lemma 7.2.129 in [10] in a straightforward way.

Lemma A.14 *Let $(n, K) < (n', K')$, and let C_1, C_2, C'_1 and C'_2 be* Op*–invariant filtered $A_{n',K'}$ algebras. Let $\mathfrak{h}\colon C_1 \to C_2$ and $\mathfrak{h}'\colon C'_1 \to C'_2$ be* Op*–invariant filtered $A_{n',K'}$ homotopy equivalences. Let $\mathfrak{g}_{(1)}\colon C_1 \to C'_1$ be an* Op*–invariant filtered $A_{n,K}$ homomorphism and $\mathfrak{g}_{(2)}\colon C_2 \to C'_2$ an* Op*–invariant filtered $A_{n',K'}$ homomorphism. We assume that $\mathfrak{g}_{(2)} \circ \mathfrak{h}$ is* Op*–invariant $A_{n,K}$ homotopic to $\mathfrak{h}' \circ \mathfrak{g}_{(1)}$.*

Then there exists an Op*–invariant filtered $A_{n',K'}$ homomorphism $\mathfrak{g}^+_{(1)}\colon C_1 \to C'_1$ such that $\mathfrak{g}^+_{(1)}$ coincides with $\mathfrak{g}_{(1)}$ as an* Op*–invariant filtered $A_{n,K}$ homomorphism and that $\mathfrak{g}_{(2)} \circ \mathfrak{h}$ is* Op*–invariant filtered $A_{n',K'}$ homotopic to $\mathfrak{h}' \circ \mathfrak{g}^+_{(1)}$.*

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