Published for SISSA by Springer

Received: *May 3, 2013*
Accepted: *September 4, 2013*
Published: *September 30, 2013*

# The geometric algebra of Fierz identities in arbitrary dimensions and signatures

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Abstract: We use geometric algebra techniques to give a synthetic and computationally efficient approach to Fierz identities in arbitrary dimensions and signatures, thus generalizing previous work. Our approach leads to a formulation which displays the underlying real, complex or quaternionic structure in an explicit and conceptually clear manner and is amenable to implementation in various symbolic computation systems. We illustrate our methods and results with a few examples which display the basic features of the three classes of pin representations governing the structure of such identities in various dimensions and signatures.

Keywords: Flux compactifications, Differential and Algebraic Geometry, Classical Theories of Gravity, Supergravity Models

ArXiv ePrint: 1304.4403



doi:10.1007/JHEP09(2013)156

JHEP09(2013)156

## Contents

## 1 Introduction

Computations involving Fierz identities in curved backgrounds for various dimensions and signatures are a cumbersome ingredient of supergravity and string theories and their applications. As any student of the subject knows all too well, the very construction of such theories relies in crucial ways on such identities, whose expert manipulation is often essential for answering various questions.

So far, little progress appears to have been made in giving a conceptually unified and computationally efficient treatment of Fierz identities in various dimensions and signatures, though partial steps in this direction were taken from various perspectives. In the present paper, we initiate such a unified treatment by using concepts and techniques borrowed from a certain approach to spinors known as "geometric algebra". Employing such methods, we give a systematic and unified treatment of Fierz identities for form-valued pinor bilinears, which can be applied in curved backgrounds (including flux backgrounds) of any dimension and signature. We also show how various results which were obtained previously can be recovered quite efficiently through our approach.

As typical in the geometric algebra approach to pinors (see [1] for our formulation), we start by viewing a bundle $S$ of pinors over a pseudo-Riemannian manifold $(M, g)$ as a bundle of modules over the (real) Kähler-Atiyah bundle $(\wedge T^*M, \diamond)$ of $(M, g)$, with module structure specified by a morphism $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ of bundles of algebras. We give a uniform description of the image and kernel of this morphism using the *Schur bundle* $\Sigma_\gamma$ of $\gamma$, which we define as the commutant sub-bundle of the image of $\gamma$ inside the bundle of algebras $(\mathrm{End}(S), \circ)$. When $\gamma$ is irreducible (which is the case of interest in many applications), Schur's lemma implies that the Schur bundle is a bundle of simple associative algebras, thus — by the Frobenius theorem on the classification of such algebras — having fiber isomorphic with either of the algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. It turns out that the image of $\gamma$ equals the commutant sub-bundle $\mathrm{End}_{\Sigma_\gamma}(S)$ of $\Sigma_\gamma$ inside $(\mathrm{End}(S), \circ)$, while the kernel of $\gamma$ equals a (possibly vanishing) sub-bundle $\wedge^{-\gamma}T^*M$ of the Kähler-Atiyah bundle, whose complement in the latter is denoted by $\wedge^\gamma T^*M$ and whose construction we explain in the main text. The bundle $\wedge^\gamma T^*M$ plays the role of *effective domain* of definition of $\gamma$, allowing us to define a partial inverse $\gamma^{-1} : \mathrm{End}_{\Sigma_\gamma}(S) \to \wedge^\gamma T^*M$ of $\gamma$ which can be used to transport sections of $\mathrm{End}_{\Sigma_\gamma}(S)$ to inhomogeneous forms belonging to the subalgebra $\Omega^\gamma(M) \stackrel{\text{def.}}{=} \Gamma(M, \wedge^\gamma T^*M)$ of the Kähler-Atiyah algebra $(\Omega(M), \diamond)$ of $(M, g)$. This generalizes the 'dequantization' procedure which was used in [1] in a particular case. Using the partial inverse $\gamma^{-1}$ and the results of [2–6], we show how basic Fierz identities constraining differential forms constructed as bilinears in sections of $S$ admit a systematic formulation in terms of so-called *Fierz isomorphisms*, thereby providing algebraic constraints on certain systems of differential forms defined on $(M, g)$. We also show how the algebra of constraints on differential forms extracted in this manner can be formulated in a concise form which allows for easy analysis of their structural properties.

The paper is organized as follows. In section 2, we systematize the basic properties of pin bundles within the geometric algebra approach, taking the theory of representations of *real* Clifford algebras as a starting point. The discussion is organized into the normal, almost complex and quaternionic cases, according to the type of the corresponding Schur algebra. Section 3 discusses admissible bilinear forms on such bundles using the language and results of [5, 6]. Section 4 gives our treatment of basic Fierz identities for form-valued (s)pinor bilinears in the normal, almost complex and quaternionic case. Section 5 illustrates our treatment by considering three examples, one for each of the three cases mentioned above — while explaining how the more traditional treatment of the examples used to illustrate the almost complex and quaternionic cases can be recovered within our approach. Section 6 contains our conclusions while appendix A summarizes some properties of real (s)pinors in those dimensions of signatures which are of most direct physical interest.

**Notations.** We work within the smooth differential category, so all manifolds, vector bundles, maps, morphisms of bundles, differential forms etc. are taken to be smooth. We further assume that our smooth manifolds $M$ are connected, paracompact and Hausdorff.[1]

[1]As is well-known, this implies that $M$ is second countable, $\sigma$-compact and admits countable atlases. Furthermore, it implies that $M$ has finite Lebesgue covering dimension, equal to its usual dimension as a manifold. In particular, we have partitions of unity of finite multiplicity subordinate to any open cover and the smooth version of the Serre-Swan correspondence applies for finite rank vector bundles over $M$.

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If $V$ is an $\mathbb{R}$-vector bundle over $M$, we let $\Gamma(M,V)$ denote the space of smooth ($\mathcal{C}^\infty$) sections of $V$. We also let $\mathrm{End}(V) = \mathrm{Hom}(V,V) = V \otimes V^*$ denote the bundle of endomorphisms of $V$, where $V^* = \mathrm{Hom}(V,\mathcal{O}_\mathbb{R})$ is the dual vector bundle to $V$ while $\mathcal{O}_\mathbb{R}$ denotes the trivial $\mathbb{R}$-line bundle on $M$. The unital ring of smooth $\mathbb{R}$-valued functions defined on $M$ is denoted by $\mathcal{C}^\infty(M,\mathbb{R}) = \Gamma(M,\mathcal{O}_\mathbb{R})$. The tensor product of $\mathbb{R}$-vector bundles is denoted by $\otimes$, while the tensor product of $\mathcal{C}^\infty(M,\mathbb{R})$-modules is denoted by $\otimes_{\mathcal{C}^\infty(M,\mathbb{R})}$; hence $\Gamma(M, V_1 \otimes V_2) = \Gamma(M,V_1) \otimes_{\mathcal{C}^\infty(M,\mathbb{R})} \Gamma(M,V_2)$. The space of $\mathbb{R}$-valued smooth inhomogeneous globally-defined differential forms on $M$ is denoted by $\Omega(M) \stackrel{\text{def.}}{=} \Gamma(M, \wedge T^*M)$ and is a $\mathbb{Z}$-graded module over the commutative ring $\mathcal{C}^\infty(M,\mathbb{R})$. The fixed rank components of this graded module are denoted by $\Omega^k(M) = \Gamma(M, \wedge^k T^*M)$ ($k = 0 \ldots d$, where $d$ is the dimension of $M$).

The kernel and image of any $\mathbb{R}$-linear map $T : \Gamma(M,V_1) \to \Gamma(M,V_2)$ will be denoted by $\mathcal{K}(T)$ and $\mathcal{I}(T)$; these are $\mathbb{R}$-linear subspaces of $\Gamma(M,V_1)$ and $\Gamma(M,V_2)$, respectively. In the particular case when $T$ is $\mathcal{C}^\infty(M,\mathbb{R})$-linear (i.e. when it is a morphism of $\mathcal{C}^\infty(M,\mathbb{R})$-modules), the subspaces $\mathcal{K}(T)$ and $\mathcal{I}(T)$ are $\mathcal{C}^\infty(M,\mathbb{R})$-submodules of $\Gamma(M,V_1)$ and $\Gamma(M,V_2)$, respectively — even in those cases when $T$ is not induced by any bundle morphism from $V_1$ to $V_2$. We always denote a morphism $f : V_1 \to V_2$ of $\mathbb{R}$-vector bundles and the $\mathcal{C}^\infty(M,\mathbb{R})$-linear map $\Gamma(M,V_1) \to \Gamma(M,V_2)$ induced by it between the modules of sections through the same symbol. Because of this convention, we clarify that the notations $\mathcal{K}(f) \subset \Gamma(M,V_1)$ and $\mathcal{I}(f) \subset \Gamma(M,V_2)$ denote the kernel and the image of the corresponding map on sections $\Gamma(M,V_1) \xrightarrow{f} \Gamma(M,V_2)$, which in this case are $\mathcal{C}^\infty(M,\mathbb{R})$-submodules of $\Gamma(M,V_1)$ and $\Gamma(M,V_2)$, respectively. In general, there does *not* exist any sub-bundle $\ker f$ of $V_1$ such that $\mathcal{K}(f) = \Gamma(M, \ker f)$ nor any sub-bundle $\mathrm{im} f$ of $V_2$ such that $\mathcal{I}(f) = \Gamma(M, \mathrm{im} f)$ — though there exist sheaves $\ker f$ and $\mathrm{im} f$ with the corresponding properties.

Given a pseudo-Riemannian metric $g$ on $M$ of signature $(p,q)$, we let $(e_a)_{a=1\ldots d}$ (where $d = \dim M$) denote a local frame of $TM$, defined on some open subset $U$ of $M$. We let $(e^a)_{a=1\ldots d}$ be the dual local coframe (= local frame of $T^*M$), which satisfies $e^a(e_b) = \delta^a_b$ and $\hat{g}(e^a, e^b) = g^{ab}$, where $\hat{g}$ is the metric induced on the cotangent bundle and $(g^{ab})$ is the inverse of the matrix $(g_{ab})$. The contragradient frame $(e^a)^\#$ and contragradient coframe $(e_a)_\#$ are given by:

$$(e^a)^\# = g^{ab} e_b \ \ , \ \ (e_a)_\# = g_{ab} e^b \ \ ,$$

where the $\#$ subscript and superscript denote the (mutually inverse) musical isomorphisms between $TM$ and $T^*M$ given respectively by lowering and raising indices with the metric $g$. We set $e^{a_1 \ldots a_k} \stackrel{\text{def.}}{=} e^{a_1} \wedge \ldots \wedge e^{a_k}$ and $e_{a_1 \ldots a_k} \stackrel{\text{def.}}{=} e_{a_1} \wedge \ldots \wedge e_{a_k}$ for any $k = 0 \ldots d$. A general $\mathbb{R}$-valued inhomogeneous form $\omega \in \Omega(M)$ expands as:

$$\omega = \sum_{k=0}^{d} \omega^{(k)} =_U \sum_{k=0}^{d} \frac{1}{k!} \omega^{(k)}_{a_1 \ldots a_k} e^{a_1 \ldots a_k} \ \ , \tag{1.1}$$

where the symbol $=_U$ means that the equality holds only after restriction of $\omega$ to $U$ and where we used the expansion:

$$\omega^{(k)} =_U \frac{1}{k!}\omega^{(k)}_{a_1\ldots a_k} e^{a_1\ldots a_k} \quad . \tag{1.2}$$

The locally-defined smooth functions $\omega^{(k)}_{a_1\ldots a_k} \in \mathcal{C}^\infty(U,\mathbb{R})$ (the 'strict coefficient functions' of $\omega$) are completely antisymmetric in $a_1 \ldots a_k$. Given a pinor bundle on $M$ with underlying fiberwise representation $\gamma$ of the Clifford bundle of $T^*M$, the corresponding gamma 'matrices' in the coframe $e^a$ are denoted by $\gamma^a \stackrel{\text{def.}}{=} \gamma(e^a)$, while the gamma matrices in the contragradient coframe $(e_a)_\#$ are denoted by $\gamma_a \stackrel{\text{def.}}{=} \gamma((e_a)_\#) = g_{ab}\gamma^b$. We will occasionally assume that the frame $(e_a)$ is *pseudo-orthonormal* in the sense that $e_a$ satisfy:

$$g(e_a, e_b) \ (= g_{ab}) \ = \eta_{ab} \ ,$$

where $(\eta_{ab})$ is a diagonal matrix with $p$ diagonal entries equal to $+1$ and $q$ diagonal entries equal to $-1$.

## 2 Real (s)pin bundles over a pseudo-Riemannian manifold

Let $(M, g)$ be an *oriented* pseudo-Riemannian manifold of dimension $d = p + q$, where $p$ and $q$ are the numbers of positive and negative eigenvalues of the metric tensor $g$. Let $\nu$ be the (real) volume form of $(M, g)$.

### 2.1 Basics

We start by recalling the basics of our approach to "geometric algebra" and spin geometry, which is based on the theory of Kähler-Atiyah bundles. We refer the reader to [1] for a detailed discussion of this approach and for the derivation of some results which are used in the present paper.

**The Kähler-Atiyah algebra and Kähler-Atiyah bundle of $(M, g)$.** Recall that the Kähler-Atiyah bundle of $(M, g)$ is a bundle of $\mathbb{Z}_2$-graded associative and unital $\mathbb{R}$-algebras $(\wedge T^*M, \diamond)$ whose underlying vector bundle coincides with the exterior bundle of $M$ (endowed with its natural $\mathbb{Z}_2$-grading induced by rank, namely $\wedge T^*M = \wedge^{\text{ev}}T^*M \oplus \wedge^{\text{odd}}T^*M$) and whose fiberwise $\mathbb{R}$-bilinear, associative and unital multiplication $\diamond$ is the so-called *geometric product* of $(M, g)$ (see [1] for a detailed discussion). The fibers of the Kähler-Atiyah bundle are unital and associative algebras which are isomorphic with the real Clifford algebra $\mathrm{Cl}(p, q)$,[2] which we view as a $\mathbb{Z}_2$-graded algebra in the usual manner. Note that $(\wedge^{\text{ev}}T^*M, \diamond)$ is a bundle of unital subalgebras of the Kähler-Atiyah bundle, which we shall call the *even Kähler-Atiyah bundle of* $(M, g)$.

Let $\pi$ be the *main automorphism* (a.k.a. the *signature*, or *grading* automorphism), i.e. that involutive automorphism of the Kähler-Atiyah bundle which is uniquely determined

[2] Our convention is that the canonical generators $e_1, \ldots, e_{p+q}$ of $\mathrm{Cl}(p, q)$ satisfy $e_i^2 = +1$ for $i = 1, \ldots, p$ and $e_j^2 = -1$ for $j = p + 1, \ldots, p + q$. This is the same convention used in [1], to which we refer the reader for further details.



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by the property that it acts as minus the identity on all one-forms:

$$\pi(\omega) \stackrel{\text{def.}}{=} \sum_{k=0}^{d}(-1)^k\omega^{(k)} \ , \quad \forall \omega = \sum_{k=0}^{d}\omega^{(k)} \in \Omega(M) \ , \quad \text{where} \ \ \omega^{(k)} \in \Omega^k(M) \tag{2.1}$$

and $\tau$ be the *main anti-automorphism* (a.k.a. *reversion*), i.e. the involutive anti-automorphism of the Kähler-Atiyah bundle given by:

$$\tau(\omega) \stackrel{\text{def.}}{=} (-1)^{\frac{k(k-1)}{2}}\omega \ , \quad \forall \omega \in \Omega^k(M) \ . \tag{2.2}$$

It is the unique anti-automorphism of $(\wedge T^*M, \diamond)$ which acts trivially on all one-forms. The fact that the exterior product is recovered from the geometric product in the limit of infinite metric (through a trivial direct computation[3]) implies that $\tau$ and $\pi$ are also (anti-)automorphisms of the exterior bundle $(\wedge T^*M, \wedge)$ — see sections 3.2. and 3.3. of [1]. We have the relations:

$$\pi \circ \tau = \tau \circ \pi \ , \quad \pi \circ \pi = \tau \circ \tau = \mathrm{id}_{\Omega(M)} \ .$$

The (real) volume form $\nu = \mathrm{vol}_M \in \Omega^d(M)$ of $(M, g)$ satisfies the following properties (see table 1):

$$\nu \diamond \nu = (-1)^{q+\left[\frac{d}{2}\right]}1_M = \begin{cases} (-1)^{\frac{p-q}{2}}1_M \ , & \text{if } d = \text{even} \\ (-1)^{\frac{p-q-1}{2}}1_M \ , & \text{if } d = \text{odd} \end{cases} \tag{2.3}$$

and:

$$\nu \diamond \omega = \pi^{d-1}(\omega) \diamond \nu \ , \quad \forall \omega \in \Omega(M) \ . \tag{2.4}$$

Hence $\nu$ is central in the Kähler-Atiyah algebra when $d$ is odd and twisted central (i.e., we have $\nu \diamond \omega = \pi(\omega) \diamond \nu$) in the Kähler-Atiyah algebra when $d$ is even.

**Real pin bundles on $(M, g)$.** A *bundle of real pinors* on $(M, g)$ is a real vector bundle $S$ over $M$ endowed with a unital morphism of bundles of algebras $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ from the Kähler-Atiyah bundle of $(M, g)$ to the bundle of endomorphisms of $S$, i.e. a bundle of modules over the Kähler-Atiyah bundle of $(M, g)$. The map induced on global sections (which we denote by the same letter):

$$\gamma : (\Omega(M), \diamond) \to (\Gamma(M, \mathrm{End}(S)), \circ)$$

is a unital morphism of $\mathcal{C}^\infty(M, \mathbb{R})$-algebras from the Kähler-Atiyah algebra of $(M, g)$ to the algebra of globally-defined endomorphisms of $S$. For each point $x \in M$, the fiber $\gamma_x$ is a representation of the Clifford algebra $(\wedge T^*_x M, \diamond_x) \approx \mathrm{Cl}(p, q)$ in the $\mathbb{R}$-vector space $S_x$

[3]Namely, take $1 \leq a_1 < \ldots < a_k \leq d$. Then, on a sufficiently small open subset $U$ around any point, we have $e^{a_1} \diamond \ldots \diamond e^{a_k} = e^{a_1} \wedge \ldots \wedge e^{a_k}$ since $e^a$ is an orthonormal coframe. Since $\pi$ is an automorphism of the Kähler-Atiyah bundle, we have $\pi(e^{a_1} \diamond \ldots \diamond e^{a_k}) = \pi(e^{a_1}) \diamond \ldots \diamond \pi(e^{a_k})$. Using the fact that $\pi(e^a) = -e^a$ and the previous observation, this gives $\pi(e^{a_1} \wedge \ldots \wedge e^{a_k}) = \pi(e^{a_1} \diamond \ldots \diamond e^{a_k}) = (-1)^k e^{a_1} \diamond \ldots \diamond e^{a_k} = (-1)^k e^{a_1} \wedge \ldots \wedge e^{a_k} = \pi(e^{a_1}) \wedge \ldots \wedge \pi(e^{a_k})$. This implies that $\pi$ acts as an automorphism of the restricted exterior algebra $(\Omega(U), \wedge)$. Using a partition of unity, we find that $\pi$ is an automorphism of the full exterior algebra $(\Omega(M), \wedge)$. A similar argument shows that $\tau$ is an anti-automorphism of the exterior algebra.

| | $\nu \diamond \nu = +1$ | $\nu \diamond \nu = -1$ |
|---|---|---|
| $\nu$ is central | $\mathbf{1}(\mathbb{R}), \mathbf{5}(\mathbb{H})$ | $\mathbf{3}(\mathbb{C}), \mathbf{7}(\mathbb{C})$ |
| $\nu$ is not central | $\mathbf{0}(\mathbb{R}), \mathbf{4}(\mathbb{H})$ | $\mathbf{2}(\mathbb{R}), \mathbf{6}(\mathbb{H})$ |

**Table 1**. Properties of the (real) volume form $\nu$ according to the mod 8 reduction of $p-q$. At the intersection of each row and column, we indicate the values of $p-q$ (mod 8) for which the volume form $\nu$ has the corresponding properties. In parentheses, we also indicate the Schur algebra (see subsection 2.2) $\mathbb{S}$ for that value of $p-q$ (mod 8). The real Clifford algebra $\mathrm{Cl}(p,q)$ is non-simple iff. $p-q \equiv_8 1, 5$, which corresponds to the upper left cell of the table and is also indicated through the blue shading of that table cell. In the non-simple cases, there are two choices for $\gamma$, which are distinguished by the signature $\epsilon_\gamma = \pm 1$; these are also the only cases when $\gamma$ fails to be injective. Notice that $\nu$ is central iff. $d$ is odd. The green color indicates those values of $p-q$ (mod 8) for which a spin endomorphism can be defined (see next page).

(the fiber of $S$ at $x$). We say that $S$ is a real *pin* bundle if this representation is irreducible for each $x \in M$, i.e. when the fibers of $S$ are simple modules. Similarly, a *bundle of real spinors* of $(M,g)$ is a bundle of modules over the *even* Kähler-Atiyah bundle $(\wedge^{\mathrm{ev}} T^*M, \diamond)$ of $(M,g)$; it is called a *spin bundle* when its fibers are simple modules. Notice that any bundle of pinors is a bundle of spinors in a natural way.[4] From now on, we let $S$ be a pin bundle of $(M,g)$, so we assume that $\gamma$ is fiberwise irreducible.

**Spin projectors and spin bundles.** Giving a direct sum bundle decomposition $S = S_+ \oplus S_-$ amounts to giving a *product structure* on $S$, i.e. a bundle endomorphism $\mathcal{R} \in \Gamma(M, \mathrm{End}(S))$ such that $\mathcal{R} \circ \mathcal{R} = \mathrm{id}_S$. Indeed, $S_\pm$ determine $\mathcal{R}$ as that product structure whose eigenbundles associated with the eigenvalues $+1$ and $-1$ of $\mathcal{R}$ equal $S_+$ and $S_-$, while a product structure $\mathcal{R}$ determines $S_\pm$ as the sub-bundles associated with its two eigenvalues. We say that $\mathcal{R}$ is non-trivial if $S_+$ and $S_-$ are both non-zero, i.e. if $\mathcal{R}$ differs from $+\mathrm{id}_S$ as well as from $-\mathrm{id}_S$. It is easy to see that the restriction:

$$\gamma_{\mathrm{ev}} \stackrel{\mathrm{def.}}{=} \gamma|_{\wedge^{\mathrm{ev}} T^*M} : \wedge^{\mathrm{ev}} T^*M \to \mathrm{End}(S) \tag{2.5}$$

is reducible on the fibers as a morphism of bundles of algebras iff. there exists a nontrivial product structure on $S$ which lies in the commutant of $\gamma(\Omega^{\mathrm{ev}}(M))$, i.e. a globally-defined endomorphism $\mathcal{R} \in \Gamma(M, \mathrm{End}(S)) \setminus \{-\mathrm{id}_S, \mathrm{id}_S\}$ which satisfies:

$$\mathcal{R}^2 = \mathrm{id}_S \ \text{ and } \ [\mathcal{R}, \gamma(\omega)]_{-,\circ} = 0 \ , \ \ \forall \omega \in \Omega^{\mathrm{ev}}(M) \ . \tag{2.6}$$

Such an endomorphism (when defined) is called a *spin endomorphism*. When $\gamma_{\mathrm{ev}}$ is fiberwise reducible, we define the *spin projectors* determined by $\mathcal{R}$ to be the globally-defined endomorphisms:

$$\mathcal{P}_\pm^{\mathcal{R}} \stackrel{\mathrm{def.}}{=} \frac{1}{2}(\mathrm{id}_S \pm \mathcal{R}) \ ,$$

[4]Indeed, the restriction $\gamma_{\mathrm{ev}}$ of $\gamma$ to the sub-bundle $\wedge^{\mathrm{ev}} T^*M \subset \wedge T^*M$ makes any pinor bundle $(S,\gamma)$ into a spinor bundle $(S, \gamma_{\mathrm{ev}})$.

which are complementary idempotents in $\Gamma(M, \mathrm{End}(S))$. Then the eigen sub-bundles:

$$S^{\pm} \stackrel{\text{def.}}{=} \mathcal{P}^{\mathcal{R}}_{\pm}(S) \subset S$$

corresponding to the eigenvalues $+1$ and $-1$ of $\mathcal{R}$ are complementary in $S$:

$$S = S^+ \oplus S^-$$

and we have:

$$\gamma(\omega)(S^{\pm}) \subset S^{\pm} \ , \quad \forall \omega \in \Omega^{\mathrm{ev}}(M) \ .$$

This gives a nontrivial direct sum decomposition $\gamma_{\mathrm{ev}} = \gamma^+ \oplus \gamma^-$ of $\gamma_{\mathrm{ev}}$ into morphisms of bundles of algebras:

$$\gamma^{\pm} \stackrel{\text{def.}}{=} \gamma|^{\mathrm{End}(S^{\pm})}_{\wedge^{\mathrm{ev}} T^*M} : (\wedge^{\mathrm{ev}} T^*M, \diamond) \to (\mathrm{End}(S^{\pm}), \circ) \ .$$

Of course, the vector bundles $S^{\pm}$ are spin bundles with underlying morphisms given by $\gamma^{\pm}$. Such a nontrivial decomposition of $\gamma$ (and hence a spin endomorphism $\mathcal{R}$) exists iff. $p-q \equiv_8 0, 4, 6, 7$. In the Physics literature, the sections of $S^{\pm}$ are addressed by historically-motivated names. This terminology is summarized below:

- When $p - q \equiv_8 0$, the sections of $S^{\pm}$ are called *Majorana-Weyl spinors* (of positive and negative chirality).
- When $p - q \equiv_8 4$, the sections of $S^{\pm}$ are called *symplectic Majorana-Weyl spinors* (of positive and negative chirality).
- When $p - q \equiv_8 6$, the sections of $S^+$ are called *symplectic Majorana spinors.*
- When $p - q \equiv_8 7$, the sections of $S^+$ are called *Majorana spinors.*

**Local expressions.** Let $e_m$ be an oriented local pseudo-orthonormal frame of $(M, g)$. We set:

$$\gamma(e^m) \stackrel{\text{def.}}{=} \gamma^m \ , \quad \gamma_m = \eta_{mn}\gamma^n \tag{2.7}$$

For $A = (m_1, \ldots, m_k)$ with $1 \leq m_1 < \ldots < m_k \leq d$, we let $|A| \stackrel{\text{def.}}{=} k$ denote the *length* of $A$ and:

$$e^A \stackrel{\text{def.}}{=} e^{m_1} \wedge \ldots \wedge e^{m_k} \ , \quad \gamma^A \stackrel{\text{def.}}{=} \gamma^{m_1} \circ \ldots \circ \gamma^{m_k} \ , \quad \gamma_A \stackrel{\text{def.}}{=} \gamma_{m_1} \circ \ldots \circ \gamma_{m_k} \ . \tag{2.8}$$

Since $\gamma_m^{-1} = \gamma^m$ and $[\gamma_m, \gamma_n]_{+,\circ} = 2\eta_{mn}$, we have $\gamma_{m_k}^{-1} \circ \ldots \circ \gamma_{m_1}^{-1} = \gamma^{m_k} \circ \ldots \circ \gamma^{m_1}$, which gives:

$$\gamma_A^{-1} = (-1)^{\frac{|A|(|A|-1)}{2}} \gamma^A \ . \tag{2.9}$$

Also note the relation:

$$\gamma(\nu) = \gamma^{(d+1)} \stackrel{\text{def.}}{=} \gamma^1 \circ \ldots \circ \gamma^d \ . \tag{2.10}$$

### 2.2 The Schur bundle and algebra

As before, let $S$ be a real pin bundle with underlying morphism $\gamma$. We let:

$$N \stackrel{\text{def.}}{=} \mathrm{rk}_{\mathbb{R}} S \tag{2.11}$$

denote the rank of $S$.

**Definition.** Let $x$ be any point in $M$. The *Schur algebra* of $\gamma_x$ is the unital subalgebra $\Sigma_{\gamma,x}$ defined as the commutant of the image $\gamma_x(\wedge T_x^* M)$ inside the algebra $(\mathrm{End}(S_x), \circ_x)$:

$$\Sigma_{\gamma,x} \stackrel{\text{def.}}{=} \{T_x \in \mathrm{End}(S_x) \mid [T_x, \gamma_x(\omega_x)]_{-,\circ} = 0\,,\ \forall \omega_x \in \wedge T_x^* M\} \quad .$$

It is easy to see that the subset:

$$\Sigma_\gamma = \{(x, T_x) \mid x \in M\ ,\ T_x \in \Sigma_{\gamma,x}\} = \sqcup_{x\in M} \Sigma_{\gamma,x}$$

is a sub-bundle of unital algebras of the bundle of algebras $(\mathrm{End}(S), \circ)$, which we shall call the *Schur bundle* of $\gamma$. In particular, the isomorphism type (as a unital associative algebra) of the fiber $(\Sigma_{\gamma,x}, \circ_x)$ is independent of $x$ and is denoted by[5] $\mathbb{S}$, being called the *Schur algebra* of $\gamma$. Notice that the space $\Gamma(M, \Sigma_\gamma)$ of globally-defined smooth sections of the Schur bundle is a unital subalgebra of the $\mathcal{C}^\infty(M, \mathbb{R})$-algebra $(\Gamma(M, \mathrm{End}(S)), \circ)$, which coincides with the commutant of $\gamma(\Omega(M))$ inside $\Gamma(M, \mathrm{End}(S))$:

$$\Gamma(M, \Sigma_\gamma) = \{T \in \Gamma(M, \mathrm{End}(S)) \mid [T, \gamma(\omega)]_{-,\circ} = 0\,,\ \forall \omega \in \Omega(M)\} \quad .$$

Since $\gamma$ is irreducible on the fibers, Schur's lemma implies that $\mathbb{S}$ is a division algebra over $\mathbb{R}$ and hence (by the Frobenius theorem on classification of such algebras) the abstract fiber of $\Sigma_\gamma$ is isomorphic with $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Notice that $S$ becomes a bundle of left $\Sigma_\gamma$-modules if one defines left fiberwise multiplication with elements of $\Sigma_\gamma$ as evaluation of the corresponding $\mathbb{R}$-linear operator. On global sections, this gives:

$$C\xi \stackrel{\text{def.}}{=} C(\xi) \in \Gamma(M, S) \ \ ,\ \ \forall \xi \in \Gamma(M, S) \ \ ,\ \ \forall C \in \Gamma(M, \Sigma_\gamma) \quad .$$

As a consequence, the dual vector bundle $S^* = \mathrm{Hom}(S, \mathcal{O}_{\mathbb{R}})$ becomes a bundle of right $\Sigma_\gamma$-modules with external multiplication given by postcomposition, which on global sections gives:

$$\eta C \stackrel{\text{def.}}{=} \eta \circ C \in \Gamma(M, S^*) \ \ ,\ \ \forall \eta \in \Gamma(M, S^*) \ \ ,\ \ \forall C \in \Gamma(M, \Sigma_\gamma) \quad .$$

Finally, $\mathrm{End}(S) \approx S \otimes S^*$ becomes a bundle of $\Sigma_\gamma$-bimodules if one uses the module structures on $S$ and $S^*$, i.e. if one defines left and right multiplication with elements of $\Sigma_\gamma$ through composition from the left and right with the corresponding $\mathbb{R}$-linear operators. On global sections, this gives:

$$C_1 T C_2 \stackrel{\text{def.}}{=} C_1 \circ T \circ C_2 \ \ ,\ \ \forall T \in \Gamma(M, \mathrm{End}(S)) \ \ ,\ \ \forall C_1, C_2 \in \Gamma(M, \Sigma_\gamma) \quad .$$

[5] Since $\gamma$ is fiberwise irreducible, it turns out that $\mathbb{S}$ depends only on $p - q$ mod 8.

The fibers of $S$ are in fact *free* as left modules over the fibers of $\Sigma_\gamma$ (since the latter is a field or a skew-field), so we have an isomorphism of $\mathbb{R}$-vector bundles:

$$S \approx \Sigma_\gamma \otimes S_0 \ ,$$

where $S_0$ is an $\mathbb{R}$-vector bundle over $M$ whose rank we denote through:

$$\Delta \stackrel{\text{def.}}{=} \mathrm{rk}_{\mathbb{R}} S_0 = \mathrm{rk}_{\Sigma_\gamma} S \tag{2.12}$$

and call the *Schur rank* of $S$. We have the relation:

$$\mathrm{rk}_{\mathbb{R}} S = \mathrm{rk}_{\mathbb{R}} \Sigma_\gamma \ \mathrm{rk}_{\Sigma_\gamma} S \iff N = \Delta \dim_{\mathbb{R}} \mathbb{S} \ .$$

The bundle of bimodule endomorphisms of $S$ can be identified with the sub-bundle $\mathrm{End}_{\Sigma_\gamma}(S)$ of those endomorphisms of $S$ which commute with all elements of $\Sigma_\gamma$. On global sections, this gives:

$$\Gamma(M, \mathrm{End}_{\Sigma_\gamma}(S)) = \{T \in \Gamma(M, \mathrm{End}(S)) \mid [T, M]_{-,\circ} = 0\,, \\ \forall M \in \Gamma(M, \Sigma_\gamma)\} \subset \Gamma(M, \mathrm{End}(S)) \ , \tag{2.13}$$

which is a unital subalgebra of the $\mathcal{C}^\infty(M, \mathbb{R})$-algebra $(\Gamma(M, \mathrm{End}(S)), \circ)$. Notice the bundle isomorphism:

$$\mathrm{End}_{\Sigma_\gamma}(S) \approx \Sigma_\gamma \otimes \mathrm{End}(S_0) \ .$$

In the following (and especially in sections 4 and 5) we will sometimes denote $\mathrm{End}_{\Sigma_\gamma}(S)$ through $\mathrm{End}_{\mathbb{S}}(S)$; this is justified by the fact that left multiplication with elements of $\Sigma_\gamma$ induces an $\mathbb{S}$-module structure on each of the fibers of $S$.

### 2.3 The image and kernel of $\gamma$

**Twisted (anti-)selfdual forms.** When $p - q \equiv_8 0, 1, 4, 5$, we have $\nu \diamond \nu = +1$ and we can consider the sub-bundles $\wedge^\pm T^*M \subset \wedge T^*M$, whose spaces of smooth global sections:

$$\Omega^\pm(M) \stackrel{\text{def.}}{=} \{\omega \in \Omega(M) \mid \omega \diamond \nu = \pm\omega\} \subset \Omega(M)$$

are the $\mathcal{C}^\infty(M, \mathbb{R})$-modules of *twisted selfdual* and *twisted anti-selfdual* inhomogeneous differential forms of $(M, g)$, respectively (see [1] for details). We have the direct sum decompositions:

$$\wedge T^*M = \wedge^+ T^*M \oplus \wedge^- T^*M \ , \quad \Omega(M) = \Omega^+(M) \oplus \Omega^-(M) \ ,$$

which corresponds to the complementary projectors:

$$P_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(1 \pm \tilde{*}) \ , \quad \Omega^\pm(M) = P_\pm(\Omega(M)) \ ,$$

where $\tilde{*}$ is the twisted Hodge operator of [1]:

$$\tilde{*}\omega \stackrel{\text{def.}}{=} \omega \diamond \nu \ , \quad \forall \omega \in \Omega(M) \ . \tag{2.14}$$

The latter is related to the ordinary Hodge operator of $(M, g)$ though [1]:

$$\tilde{*} = * \circ \tau \ . \tag{2.15}$$

| | injective | non-injective |
|---|---|---|
| surjective | $\mathbf{0}(\mathbb{R}), \mathbf{2}(\mathbb{R})$ | $\mathbf{1}(\mathbb{R})$ |
| non-surjective | $\mathbf{3}(\mathbb{C}), \mathbf{7}(\mathbb{C}), \mathbf{4}(\mathbb{H}), \mathbf{6}(\mathbb{H})$ | $\mathbf{5}(\mathbb{H})$ |

**Table 2**. Fiberwise character of real pin representations $\gamma$. At the intersection of each row and column, we indicate the values of $p-q \pmod 8$ for which the map induced by $\gamma$ on each fiber of the Kähler-Atiyah algebra has the corresponding properties. In parentheses, we also indicate the Schur algebra $\mathbb{S}$ of $\gamma$ for that value of $p-q \pmod 8$. Note that $\gamma$ is fiberwise surjective exactly for the normal case, i.e. when the Schur algebra is isomorphic with $\mathbb{R}$. Also notice that $\gamma$ fails to be fiberwise injective precisely in the non-simple case $p-q \equiv_8 1, 5$, which we indicate through the blue shading of the corresponding table cells.

**Image, kernel and signature of $\gamma$.** Well-known facts from the representation theory of Clifford algebras imply the following:

**Proposition.**

1. The image of the bundle morphism $\gamma$ coincides with the sub-bundle $\mathrm{End}_{\Sigma_\gamma}(S)$. This gives:
$$\gamma(\Omega(M)) = \Gamma(M, \mathrm{End}_{\Sigma_\gamma}(S)) \quad .$$

2. $\gamma$ is fiberwise injective iff. $\mathrm{Cl}(p,q)$ is simple as an associative $\mathbb{R}$-algebra, i.e. iff. $p-q \not\equiv_8 1, 5$ (the so-called *simple case*).

3. When $\gamma$ fails to be fiberwise injective (i.e. when $p-q \equiv_8 1, 5$, the so-called *non-simple case*), we have:
$$\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S \quad ,$$
where the sign factor $\epsilon_\gamma \in \{-1, 1\}$ is called the *signature* of $\gamma$. The two choices for this sign factor lead to two inequivalent choices for $\gamma$. In this case, $\wedge^{\epsilon_\gamma} T^*M$ is a sub-bundle of algebras of the Kähler-Atiyah bundle of $(M, g)$ (see [1]). Moreover, the kernel of the bundle morphism $\gamma$ equals $\wedge^{-\epsilon_\gamma} T^*M$, which implies the following relation for the $\mathcal{C}^\infty(M, \mathbb{R})$-modules of global sections:
$$\mathcal{K}(\gamma) = \Omega^{-\epsilon_\gamma}(M) \quad .$$
Furthermore, the restriction of $\gamma$ to its so-called *effective domain* $\wedge^{\epsilon_\gamma} T^*M$ gives an isomorphism of bundles of algebras between $(\wedge^{\epsilon_\gamma} T^*M, \diamond)$ and $(\mathrm{End}_{\Sigma_\gamma}(S), \circ)$.

The fiberwise injectivity and surjectivity properties of $\gamma$ are summarized in table 2.

### 2.4 The effective domain and the partial inverse of $\gamma$

The following notation allows one to treat fiberwise injective and non-injective cases simultaneously:

$$\Omega^\gamma(M) \stackrel{\text{def.}}{=} \begin{cases} \Omega(M) \ , & \text{if } \ \gamma \ \text{ is fiberwise injective (simple case)} \, , \\ \Omega^{\epsilon_\gamma}(M) \ , & \text{if } \ \gamma \ \text{ is not fiberwise injective (non-simple case)} \end{cases} \tag{2.16}$$

| Type | $p-q$ mod 8 | $\mathbb{S}$ |
|---|---|---|
| normal | $\mathbf{0}, \mathbf{1}, \mathbf{2}$ | $\mathbb{R}$ |
| almost complex | $\mathbf{3}, \mathbf{7}$ | $\mathbb{C}$ |
| quaternionic | $\mathbf{4}, \mathbf{5}, \mathbf{6}$ | $\mathbb{H}$ |

**Table 3**. Type of the pin bundle of $(M, g)$ according to the mod 8 reduction of $p - q$. The pin bundle $S$ is called *normal*, *almost complex* or *quaternionic* depending on whether its Schur algebra is isomorphic with $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The non-simple sub-cases are indicated in blue, while those cases when a spin operator can be defined are indicated in green.

and:

$$\Omega^{-\gamma}(M) \stackrel{\text{def.}}{=} \begin{cases} 0 \ , & \text{if } \ \gamma \ \text{ is fiberwise injective (simple case)}\,, \\ \Omega^{-\epsilon_\gamma}(M) \ , & \text{if } \ \gamma \ \text{ is not fiberwise injective (non} - \text{simple case)}\ . \end{cases} \tag{2.17}$$

With this notation, we always have:

$$\mathcal{K}(\gamma) = \Omega^{-\gamma}(M) \ , \quad \Omega(M) = \Omega^{\gamma}(M) \oplus \Omega^{-\gamma}(M)$$

and the restriction of the (map on sections induced by) $\gamma$ to its *effective domain* $\Omega^{\gamma}(M)$ is injective. Since $\gamma(\Omega^{\gamma}(M)) = \Gamma(M, \mathrm{End}_{\Sigma_\gamma}(S))$, we can define the *partial inverse* $\gamma^{-1}$ : $\mathrm{End}_{\Sigma_\gamma}(S) \to \wedge^{\gamma} T^*M$ of $\gamma$ to be the (map of bundles which induces the) partial inverse of this restriction:

$$\gamma^{-1} \stackrel{\text{def.}}{=} \left(\gamma|_{\Omega^{\gamma}(M)}^{\Gamma(M,\mathrm{End}_{\Sigma_\gamma}(S))}\right)^{-1} : \Gamma(M, \mathrm{End}_{\Sigma_\gamma}(S)) \longrightarrow \Omega^{\gamma}(M) \ . \tag{2.18}$$

Notice the relations:

$$\gamma \circ \gamma^{-1} = \mathrm{id}_{\mathrm{End}_{\Sigma_\gamma}(S)} \ , \quad \gamma^{-1} \circ \gamma = P_\gamma|^{\wedge^{\gamma} T^*M} \stackrel{\text{def.}}{=} P_{\epsilon_\gamma}|^{\wedge^{\epsilon_\gamma} T^*M} \ ,$$

where in the right hand side we indicate explicitly the appropriate co-restriction.

**Local expressions.** Let $e^m$ be a pseudo-orthonormal local coframe of $M$ defined above an open subset $U \subset M$. For later reference, we define:

$$e_\gamma^m \stackrel{\text{def.}}{=} \gamma^{-1}(\gamma^m) = P_\gamma(e^m) \in \Omega^{\gamma}(U) \ , \quad \forall m = 1 \ldots d \ , \tag{2.19}$$

as well as:

$$e_\gamma^A \stackrel{\text{def.}}{=} \gamma^{-1}(\gamma^A) = \gamma^{-1}(\gamma^{m_1}) \diamond \ldots \diamond \gamma^{-1}(\gamma^{m_k}) = e_\gamma^{m_1} \diamond \ldots \diamond e_\gamma^{m_k} \in \Omega^{\gamma}(U) \tag{2.20}$$

for any increasingly-ordered $k$-uple $A = (m_1, \ldots, m_k)$.

### 2.5 Representation types

Recall that the Schur algebra $\mathbb{S}$ is isomorphic with $\mathbb{R}$ (the *normal case*), $\mathbb{C}$ (the *almost complex case*) or $\mathbb{H}$ (the *quaternionic case*), where the terminology in brackets is due to [2–4]. The results of [2–4] imply that the three types of real pin bundles occur according to the mod 8 reduction of the difference $p - q$ as shown in table 3 (see also table 4). In this section, we summarize some properties of the three types of real pin bundles.

| $\mathbb{S}$ | $p-q$ mod 8 | $\wedge T_x^* M \approx \mathrm{Cl}(p,q)$ | $\Delta$ | $N$ | Number of choices for $\gamma$ | $\gamma_x(\wedge T_x^* M)$ | Fiberwise injectivity of $\gamma$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{R}$ | $\mathbf{0}, \mathbf{2}$ | $\mathrm{Mat}(\Delta, \mathbb{R})$ | $2^{[\frac{d}{2}]} = 2^{\frac{d}{2}}$ | $2^{[\frac{d}{2}]}$ | 1 | $\mathrm{Mat}(\Delta, \mathbb{R})$ | injective |
| $\mathbb{H}$ | $\mathbf{4}, \mathbf{6}$ | $\mathrm{Mat}(\Delta, \mathbb{H})$ | $2^{[\frac{d}{2}]-1} = 2^{\frac{d}{2}-1}$ | $2^{[\frac{d}{2}]+1}$ | 1 | $\mathrm{Mat}(\Delta, \mathbb{H})$ | injective |
| $\mathbb{C}$ | $\mathbf{3}, \mathbf{7}$ | $\mathrm{Mat}(\Delta, \mathbb{C})$ | $2^{[\frac{d}{2}]} = 2^{\frac{d-1}{2}}$ | $2^{[\frac{d}{2}]+1}$ | 1 | $\mathrm{Mat}(\Delta, \mathbb{C})$ | injective |
| $\mathbb{H}$ | $\mathbf{5}$ | $\mathrm{Mat}(\Delta, \mathbb{H})^{\oplus 2}$ | $2^{[\frac{d}{2}]-1} = 2^{\frac{d-3}{2}}$ | $2^{[\frac{d}{2}]+1}$ | $2\ (\epsilon_\gamma = \pm 1)$ | $\mathrm{Mat}(\Delta, \mathbb{H})$ | non-injective |
| $\mathbb{R}$ | $\mathbf{1}$ | $\mathrm{Mat}(\Delta, \mathbb{R})^{\oplus 2}$ | $2^{[\frac{d}{2}]} = 2^{\frac{d-1}{2}}$ | $2^{[\frac{d}{2}]}$ | $2\ (\epsilon_\gamma = \pm 1)$ | $\mathrm{Mat}(\Delta, \mathbb{R})$ | non-injective |

**Table 4**. Summary of pin bundle types. The non-negative integer $N \stackrel{\text{def.}}{=} \mathrm{rk}_{\mathbb{R}} S$ is the real rank of $S$ while $\Delta \stackrel{\text{def.}}{=} \mathrm{rk}_{\Sigma_\gamma} S$ is the Schur rank of $S$. The non-simple cases are indicated by the blue shading of the corresponding table cells.

### 2.6 The normal case

This occurs when $\mathbb{S} \approx \mathbb{R}$, which happens for $p - q \equiv_8 0, 1, 2$. We have $N = \Delta = 2^{[\frac{d}{2}]}$ and the Schur bundle $\Sigma_\gamma \approx \mathcal{O}_{\mathbb{R}}$ is the trivial real line bundle generated by the identity section of $\mathrm{End}(S)$. This implies:

$$\Gamma(M, \Sigma_\gamma) = \mathcal{C}^\infty(M, \mathbb{R})\ \mathrm{id}_S = \{f\ \mathrm{id}_S \mid f \in \mathcal{C}^\infty(M, \mathbb{R})\} \approx \mathcal{C}^\infty(M, \mathbb{R}) \ ,$$

where $\approx$ denotes the obvious isomorphism of $\mathbb{R}$-algebras. We have $\nu \diamond \nu = +1$ for $p - q \equiv_8 0, 1$ and $\nu \diamond \nu = -1$ for $p - q \equiv_8 2$. Furthermore, $\nu$ is central in the Kähler-Atiyah algebra of $(M, g)$ iff. $p - q \equiv_8 1$.

#### 2.6.1 Injectivity and surjectivity

The morphism $\gamma$ is always fiberwise surjective. It is fiberwise injective for $p - q \equiv_8 0, 2$ (the *normal simple case*) but fails to be fiberwise injective when $p - q \equiv_8 1$ (the *normal non-simple case*). When $p - q \equiv_8 1$, we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, where $\epsilon_\gamma \in \{-1, +1\}$ is the signature of $\gamma$. The two choices of signature correspond to different fiberwise representations of the real Clifford algebra $\mathrm{Cl}(p, q)$ and of the pin group $\mathrm{Pin}(p, q) \subset \mathrm{Cl}(p, q)$, but induce equivalent representations of the spin group $\mathrm{Spin}(p, q) \subset \mathrm{Pin}(p, q)$.

#### 2.6.2 Spin projectors

In the normal case, the restriction $\gamma_{\mathrm{ev}}$ is fiberwise reducible iff. $p - q \equiv_8 0$, in which case spin projectors can be constructed from the product structure $\mathcal{R} = \gamma(\nu)$, which determines the eigen sub-bundles $S^\pm$ of *Majorana-Weyl spinors*, on which $\gamma(\nu)$ has eigenvalues $\pm 1$. Sections of $S^+$ or $S^-$ are called Majorana-Weyl spinors (of positive and negative chiralities, respectively). We have:

$$\Gamma(M, S^\pm) = \{\xi \in \Gamma(M, S) \mid \gamma(\nu)\xi = \pm\xi\} \ .$$

The situation is summarized in table 5.

| $p-q$ mod 8 | $\mathrm{Cl}(p,q)$ | $\gamma$ is injective | $\epsilon_\gamma$ | $\mathcal{R}$ (real spinors) | $\nu\diamond\nu$ | $\nu$ is central |
|---|---|---|---|---|---|---|
| **0** | simple | Yes | N/A | $\gamma(\nu)$ (Majorana-Weyl) | $+1$ | No |
| **1** | non-simple | No | $\pm 1$ | N/A | $+1$ | Yes |
| **2** | simple | Yes | N/A | N/A | $-1$ | No |

**Table 5**. Summary of subcases of the normal case.

### 2.7 The almost complex case

This case occurs when $\mathbb{S}\approx\mathbb{C}$, which happens for $p-q\equiv_8 3,7$. In this case, $d$ is odd and we have $N=2\Delta=2^{[\frac{d}{2}]+1}$. We also have $\nu\diamond\nu=-1$ and $\nu$ is always central in the Kähler-Atiyah algebra.

#### 2.7.1 Complex structures and the endomorphism $D$

There exist two complex structures on the bundle $S$ which lie in the commutant of the image of $\gamma$, i.e. two *globally-defined* endomorphisms $J\in\Gamma(M,\mathrm{End}(S))$ which satisfy:

$$J^2=-\mathrm{id}_S \quad \text{and} \quad [J,\gamma(\omega)]_{-,\circ}=0 \ , \quad \forall\omega\in\Omega(M) \ . \tag{2.21}$$

The two solutions $J$ of (2.21) differ by a sign factor:

$$J\to -J$$

and are given by:

$$J=\pm\gamma(\nu) \ .$$

In this case, the Schur bundle $\Sigma_\gamma$ is the trivial rank two real vector bundle spanned by $\mathrm{id}_S$ and by any of these two choices of $J$. In particular, we have:

$$\Gamma(M,\Sigma_\gamma)=\mathcal{C}^\infty(M,\mathbb{R})\mathrm{id}_S\oplus\mathcal{C}^\infty(M,\mathbb{R})J=\{f\mathrm{id}_S+gJ \mid f,g\in\mathcal{C}^\infty(M,\mathbb{R})\} \ .$$

Each of the two choices of $J$ determines an isomorphism of $\mathcal{C}^\infty(M,\mathbb{R})$-algebras from $\mathcal{C}^\infty(M,\mathbb{C})$ to $\Gamma(M,\Sigma_\gamma)$:

$$\varphi_J(f+ig)=f+gJ\in\Gamma(M,\Sigma_\gamma) \ , \quad \forall f,g\in\mathcal{C}^\infty(M,\mathbb{R}) \ ,$$

so $\varphi_J$ and $\varphi_{-J}$ are related through complex conjugation, which is an $\mathcal{C}^\infty(M,\mathbb{R})$-linear involutive automorphism of $\mathcal{C}^\infty(M,\mathbb{C})$:

$$\varphi_{-J}(u)=\varphi_J(\bar{u}) \ , \quad \forall u\in\mathcal{C}^\infty(M,\mathbb{C}) \ .$$

A choice for $J$ makes the Schur bundle into a trivial complex line bundle, the two opposite choices being related through complex conjugation. When viewing $S$ as a bundle of $\Sigma_\gamma$-modules, opposite choices for $J$ correspond to two choices of complex structure on $S$, which

| $p-q$ mod 8 | $\mathrm{Cl}(p,q)$ | $\gamma$ is injective | $\epsilon_\gamma$ | $D^2$ | $\mathcal{R}$ (real spinors) | $\nu \diamond \nu$ | $\nu$ is central |
|---|---|---|---|---|---|---|---|
| **3** | simple | Yes | N/A | $-\mathrm{id}_S$ | N/A | $-1$ | Yes |
| **7** | simple | Yes | N/A | $+\mathrm{id}_S$ | $D$ (Majorana) | $-1$ | Yes |

**Table 6**. Summary of subcases of the almost complex case. In this case, $\gamma(\nu)$ defines a complex structure $J$ on $S$ and we have $\mathrm{im}\gamma = \mathrm{End}_\mathbb{C}(S)$. We also have an endomorphism $D$ of $S$ which anticommutes with $J$ (thus giving a complex conjugation on $S$, when the latter is viewed as a complex vector bundle) and satisfies $[D, \gamma^m]_{+,\circ} = 0$. The two subcases $p-q \equiv_8 3$ and $p-q \equiv_8 7$ are distinguished by whether $D^2$ equals $-\mathrm{id}_S$ or $+\mathrm{id}_S$. In both cases, $\gamma$ can be viewed as an isomorphism of bundles of $\mathbb{R}$-algebras from the Kähler-Atiyah bundle $(\wedge T^*M, \diamond)$ to $(\mathrm{End}_\mathbb{C}(S), \circ)$, while its complexification $\gamma_\mathbb{C}$ gives an isomorphism of bundles of $\mathbb{C}$-algebras from the complexified Kähler-Atiyah bundle $(\wedge T^*_\mathbb{C} M, \diamond)$ to $(\mathrm{End}_\mathbb{C}(S), \circ)$. When $p-q \equiv_8 7$, $D$ is a real structure which can be used to identify $S$ with the complexification $(S_+)_\mathbb{C} \stackrel{\mathrm{def.}}{=} S_+ \otimes \mathcal{O}_\mathbb{C}$ of the real bundle $S_+ \subset S$ of *Majorana spinors*. When $p-q \equiv_8 3$, $D$ is a second complex structure on $S$, which anticommutes with the complex structure $J = \gamma(\nu)$. In that case, the operators $J, D$ and $J \circ D$ define a global quaternionic structure on $S$ — which, however, is not compatible with $\gamma$ since $D$ *anti*commutes with $\gamma^m$.

are again related through complex conjugation.[6] The results of [2] imply that there exists a globally-defined endomorphism $D \in \Gamma(M, \mathrm{End}(S))$ which satisfies:

$$D \circ \gamma(\omega) = \gamma(\pi(\omega)) \circ D \ , \quad \forall \omega \in \Omega(M) \ , \tag{2.22}$$

$$D^2 = (-1)^{\frac{p-q+1}{4}} \mathrm{id}_S = \begin{cases} -\mathrm{id}_S \,, & \text{if } p-q \equiv_8 3 \,, \\ +\mathrm{id}_S \,, & \text{if } p-q \equiv_8 7 \,, \end{cases} \tag{2.23}$$

$$[J, D]_{+,\circ} = 0 \tag{2.24}$$

and which is determined by these properties up to a sign ambiguity $D \to -D$.

### 2.7.2 Injectivity and surjectivity

In this case, $\gamma$ is always fiberwise injective but non-surjective. The image of $\gamma$ coincides with the sub-bundle $\mathrm{End}_\mathbb{C}(S) \stackrel{\mathrm{def.}}{=} \mathrm{End}_{\Sigma_\gamma}(S)$ of $\mathrm{End}(S)$, which in turn is the bundle of complex-linear endomorphisms of $S$, when the latter is viewed as a $\mathbb{C}$-vector bundle upon using the complex structure $J$. This sub-bundle of $\mathrm{End}(S)$ is isomorphic through $\gamma$ (as a bundle of $\mathbb{R}$-algebras) with the Kähler-Atiyah bundle of $(M, g)$. The situation is summarized in table 6.

### 2.7.3 Spin projectors

The restriction $\gamma_{\mathrm{ev}}$ is fiberwise reducible iff. $p-q \equiv_8 7$, so spin projectors can only be defined when this condition holds, which we assume to be the case for the remainder of this sub-subsection. When $p-q \equiv_8 7$, the spin endomorphism in (2.6) is given by

[6] The two choices for $J$ are exchanged when changing the orientation of $M$, since this maps the volume form $\nu$ into its opposite.

$\mathcal{R} = D$. Since $D$ and $J$ anticommute (see (2.24)), $D$ can be viewed as a real structure (complex conjugation operation) on the complex vector bundle obtained by endowing $S$ with the complex structure $J$. We then have $J(\xi) = i\xi$ for all $\xi \in \Gamma(M, S)$ and $\gamma(\omega)$ is an endomorphism of $S$ as a *complex* vector bundle:

$$\gamma(\omega) \in \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \ , \quad \forall \omega \in \Omega(M) \ ,$$

because $\gamma(\omega)$ commutes with $J$. Since $\gamma$ is fiberwise injective in this case, we can thus view $\gamma$ as an *isomorphism* of bundles of algebras from the *real* Kähler-Atiyah bundle of $(M, g)$ to the bundle $(\mathrm{End}_{\mathbb{C}}(S), \circ)$, where the latter is viewed as a bundle of $\mathbb{R}$-subalgebras of the bundle of $\mathbb{R}$-algebras $(\mathrm{End}(S), \circ)$:

$$\gamma : (\wedge T^*M, \diamond) \xrightarrow{\sim} (\mathrm{End}_{\mathbb{C}}(S), \circ) \ .$$

When this interpretation is used, we can denote the action of $D$ by an overline, defining the *complex conjugate* of a section of $S$ through:

$$\bar{\xi} \stackrel{\text{def.}}{=} D(\xi) \ , \quad \forall \xi \in \Gamma(M, S) \ .$$

The spin projectors $\mathcal{P}_{\pm} = \frac{1}{2}(\mathrm{id}_S \pm D)$ relate to taking the real and imaginary parts:

$$\xi_{\pm} \stackrel{\text{def.}}{=} \mathcal{P}_{\pm}(\xi) = \frac{1}{2}(\xi \pm \bar{\xi}) \ , \quad \forall \xi \in \Gamma(M, S) \ ,$$

with:

$$\mathrm{Re}\xi \stackrel{\text{def.}}{=} \xi_+ \ , \quad \mathrm{Im}\xi \stackrel{\text{def.}}{=} -J(\xi_-) \iff \xi_+ = \mathrm{Re}\xi \ , \quad \xi_- = J(\mathrm{Im}\xi) = i\mathrm{Im}\xi \ .$$

The $\mathbb{R}$-vector bundles $S_{\pm} \stackrel{\text{def.}}{=} \mathcal{P}_{\pm}(S)$ give the decomposition $S = S_+ \oplus S_-$, while the fact that $J$ anticommutes with $D$ implies:

$$J(S_{\pm}) = S_{\mp} \implies S = S_+ \oplus J(S_+) \ .$$

sections of $S_{\pm}$ can be characterized through:

$$\Gamma(M, S_{\pm}) = \{\xi \in \Gamma(M, S) \mid D(\xi) = \pm\xi \Leftrightarrow \bar{\xi} = \pm\xi\} \subset \Gamma(M, S)$$

and are called *real* and *purely imaginary*, respectively. In the physics literature, real sections of $S$ (i.e. sections $\xi$ of $S_+$) are also called *Majorana spinors*. Since $S$ can be identified with the complexification $(S_+)_{\mathbb{C}} \stackrel{\text{def.}}{=} S_+ \otimes \mathcal{O}_{\mathbb{C}}$ of $S_+$, this means that the bundle $S$ of real pinors can be viewed as the underlying $\mathbb{R}$-vector bundle of the complex vector bundle of complexified Majorana spinors. Since the latter are usually called complex spinors, it follows that real pinors in this case are simply complex spinors for which one has forgotten the complex structure $J$. We also have:

$$\Gamma(M, S_+) = \{\xi \in \Gamma(M, S) \mid \mathrm{Im}\xi = 0\} \ , \quad \Gamma(M, S_-) = \{\xi \in \Gamma(M, S) \mid \mathrm{Re}\xi = 0\} \ .$$

In fact, the restrictions $J_{\pm} \stackrel{\text{def.}}{=} J|_{S_{\pm}}^{S_{\mp}}$ give isomorphisms of $\mathbb{R}$-vector bundles:

$$J_{\pm} : S_{\pm} \xrightarrow{\sim} S_{\mp}$$

which satisfy $J_\mp \circ J_\pm = \mathrm{id}_{S^\pm}$, so any section $\xi \in \Gamma(M,S)$ decomposes *uniquely* as:

$$\xi = \xi_+ + \xi_- = \xi_R + i\xi_I$$
$$\text{with } \ \xi_\pm \in \Gamma(M,S_\pm)\,, \quad \xi_R = \mathrm{Re}\xi \in \Gamma(M,S_+)\,, \quad \xi_I = \mathrm{Im}\xi \in \Gamma(M,S_+) \ . \tag{2.25}$$

As mentioned above, this decomposition corresponds to a bundle isomorphism which identifies the complexification $(S_+)_\mathbb{C}$ with the complex vector bundle $S$, mapping sections of the former to sections of the latter via:

$$\Gamma(M,(S_+)_\mathbb{C}) \approx \Gamma(M,S_+) \oplus i\Gamma(M,S_+) \ni \xi_R + i\xi_I$$
$$\longrightarrow \xi = \xi_R + J(\xi_I) \in \Gamma(M,S)\,, \quad \forall \xi_R, \xi_I \in \Gamma(M,S_+) \ .$$

Since $D$ anticommutes with $J$, we have:

$$\bar{\xi} = \xi_R - J(\xi_I) = \xi_R - i\xi_I \iff \mathrm{Re}(\bar{\xi}) = \mathrm{Re}(\xi) \ , \quad \mathrm{Im}(\bar{\xi}) = -\mathrm{Im}(\xi) \ ,$$

so $D$ coincides with the complex conjugation induced by this presentation of $S$ as the complexification of $S_+$. We define the *complex conjugate* $\bar{T}$ of any endomorphism $T \in \Gamma(M,\mathrm{End}(S))$ through:

$$\bar{T} \stackrel{\text{def.}}{=} D \circ T \circ D \in \Gamma(M,\mathrm{End}(S)) \ , \quad \text{so } \ \bar{T}(\xi) = \overline{T(\bar{\xi})} \ , \quad \forall \xi \in \Gamma(M,S) \ ,$$

thus obtaining an antilinear involutive automorphism $T \to \bar{T}$ of $\Gamma(M,\mathrm{End}(S))$. We say that $T$ is *real* or *imaginary* if $\bar{T} = T$ or $\bar{T} = -T$, respectively. We have:

$$T(S_\pm) \subset \begin{cases} S_\pm\,, & \text{if } T = \text{real}\,, \\ S_\mp\,, & \text{if } T = \text{imaginary} \ . \end{cases}$$

For example, notice that $D$ is real while $J$ is imaginary. Also notice that the product of two endomorphisms is real when both of them are either real or imaginary and imaginary when one of them is real and the other is imaginary. Relation (2.22) takes the form:

$$\overline{\gamma(\omega)} = \gamma(\pi(\omega)) \ , \quad \forall \omega \in \Omega(M) \ ,$$

and hence:

$$\gamma(\omega) = \begin{cases} \text{real}\,, & \text{if } \omega \in \Omega^{\mathrm{ev}}(M)\,, \\ \text{imaginary}\,, & \text{if } \omega \in \Omega^{\mathrm{odd}}(M) \ . \end{cases} \tag{2.26}$$

**Local expressions for $p - q \equiv_8 7$.** Let $U \subset M$ be an open subset supporting both a local pseudo-orthonormal coframe $(e^a)_{a=1\ldots d}$ of $(M,g)$ and a local frame $(\epsilon_\alpha)_{\alpha=1\ldots\Delta}$ of $S_+$. Then $(\epsilon_\alpha, J(\epsilon_\alpha))$ is a local frame of $S$ above $U$. Setting $\gamma^a \stackrel{\text{def.}}{=} \gamma(e^a) \in \Gamma(U,\mathrm{End}(S))$, relation (2.26) shows that $\gamma^a$ are imaginary:

$$\overline{\gamma^a} = -\gamma^a \iff D \circ \gamma^a \circ D = -\gamma^a$$

while $\gamma^A \stackrel{\text{def.}}{=} \gamma^{a_1} \circ \ldots \circ \gamma^{a_k} \in \Gamma(U, \mathrm{End}(S))$ for an ordered index set $A = (a_1 \ldots a_k)$ are real or imaginary according to whether the length $k = |A|$ of $A$ is even or odd:

$$\gamma^A = \begin{cases} \text{real}\,, & \text{if } |A| = \text{even}\,, \\ \text{imaginary}\,, & \text{if } |A| = \text{odd}\,. \end{cases}$$

In particular, we have:

$$\gamma^a(S_\pm) = S_\mp \ \ , \ \ \gamma^A(S_\pm) \subset \begin{cases} S_\pm\,, & \text{if } |A| = \text{even}\,, \\ S_\mp\,, & \text{if } |A| = \text{odd}\,. \end{cases}$$

The matrix $\hat{\Gamma}^a = (\hat{\Gamma}^a_{i,j=1\ldots N})$ of $\gamma^a$ with respect to the local frame $(\epsilon_\alpha, J(\epsilon_\alpha))$ of the $\mathbb{R}$-vector bundle $S$ (a square matrix of size $N \times N$ with entries in $\mathcal{C}^\infty(U, \mathbb{R})$) is given by the expansions:

$$\begin{aligned} \gamma^a(\epsilon_\alpha) &= \sum_{\beta=1}^{\Delta} \gamma^a_{\beta\alpha}\epsilon_\beta = \sum_{\beta=1}^{\Delta} \left[\gamma^a_{R,\beta\alpha}\epsilon_\beta + \gamma^a_{I,\beta\alpha} J(\epsilon_\beta)\right] \ , \\ \gamma^a(J(\epsilon_\alpha)) &= J(\gamma^a(\epsilon_\alpha)) = \sum_{\beta=1}^{\Delta} \gamma^a_{\beta\alpha} J(\epsilon_\beta) = \sum_{\beta=1}^{\Delta} \left[\gamma^a_{R,\beta\alpha} J(\epsilon_\beta) - \gamma^a_{I,\beta\alpha}\epsilon_\beta\right] \ , \end{aligned}$$

where we decomposed the complex-valued functions $\gamma^a_{\beta\alpha} \in \mathcal{C}^\infty(U, \mathbb{C})$ into their real and imaginary parts:

$$\gamma^a_{\beta\alpha} = \gamma^a_{R,\beta\alpha} + i\gamma^a_{I,\beta\alpha} \ \ , \ \ \text{with} \ \ \gamma^a_{R,\beta\alpha}, \gamma^a_{I,\beta\alpha} \in \mathcal{C}^\infty(U, \mathbb{R})$$

and used the facts that $J$ and $\gamma^a$ commute and that the complex structure on $S$ is defined through $i\xi = J(\xi)$. It is convenient to encode the coefficients of the expansions above into the following square matrices of size $\Delta \times \Delta$ with entries in $\mathcal{C}^\infty(U, \mathbb{C})$:

$$\hat{\gamma}^a \stackrel{\text{def.}}{=} (\gamma^a_{\alpha\beta})_{\alpha,\beta=1\ldots\Delta} \ ,$$

which are called the *complex* gamma matrices defined by the local coframe $(e^a)_{a=1\ldots d}$ of $M$ with respect to the local frame $(\epsilon_\alpha)_{\alpha=1\ldots\Delta}$ of $S_+$. Since $(\epsilon_\alpha)_{\alpha=1\ldots\Delta}$ is also a local frame of the *complex* vector bundle $(S, J)$, these are just the gamma matrices of $e^a$ with respect to this local frame of $S$, when the latter is viewed as a *complex* vector bundle. With the notations above, the *real* gamma matrices defined by $(e^a)_{a=1\ldots d}$ with respect to the local frame $(\epsilon_\alpha, J(\epsilon_\alpha))_{\alpha=1\ldots\Delta}$ of $S$ are given by:

$$\hat{\Gamma}^a = \begin{bmatrix} \hat{\gamma}^a_R & \hat{\gamma}^a_I \\ -\hat{\gamma}^a_I & \hat{\gamma}^a_R \end{bmatrix} \ ,$$

where $\hat{\gamma}^a_R, \hat{\gamma}^a_I \in \mathrm{Mat}(\Delta, \mathcal{C}^\infty(U, \mathbb{R}))$ are the real and imaginary parts of $\hat{\gamma}^a$:

$$\hat{\gamma}^a = \hat{\gamma}^a_R + i\hat{\gamma}^a_I \ \ , \ \ \hat{\gamma}^a_R = (\hat{\gamma}^a_{R,\alpha\beta})_{\alpha,\beta=1\ldots\Delta} \ \ , \ \ \hat{\gamma}^a_I = (\hat{\gamma}^a_{I,\alpha\beta})_{\alpha,\beta=1\ldots\Delta} \ .$$



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A local section $\xi \in \Gamma(U, S)$ expands as:

$$\xi = \sum_{\alpha=1}^{\Delta} \xi^\alpha \epsilon_\alpha = \sum_{\alpha=1}^{\Delta} [\xi_R^\alpha \epsilon_\alpha + \xi_I^\alpha J(\epsilon_\alpha)] \quad , \tag{2.27}$$

where:

$$\xi^\alpha = \xi_R^\alpha + i\xi_I^\alpha \in \mathcal{C}^\infty(U, \mathbb{C}) \quad , \quad \text{with} \quad \xi_R^\alpha, \xi_I^\alpha \in \mathcal{C}^\infty(U, \mathbb{R}) \quad .$$

We let $\hat{\xi}$ denote the $\mathcal{C}^\infty(U, \mathbb{C})$-valued column matrix of size $\Delta$ with entries $\xi^\alpha$. Since $\gamma^A$ commute with $J$ (i.e. are complex-linear), we have $\gamma^A \xi = \sum_{\alpha=1}^{\Delta} (\gamma^A \xi)^\alpha \epsilon_\alpha$, the complex coefficient functions $(\gamma^A \xi)^\alpha \in \mathcal{C}^\infty(U, \mathbb{C})$ being given by the entries of the matrix $\hat{\gamma}^A \hat{\xi}$, where:

$$\hat{\gamma}^A = \widehat{\gamma^A} = \hat{\gamma}^{a_1} \circ \ldots \circ \hat{\gamma}^{a_k} \in \mathrm{Mat}(\Delta, \mathcal{C}^\infty(U, \mathbb{C})) \quad , \quad \forall \text{ ordered } \quad A = (a_1 \ldots a_k) \quad .$$

Notice that $\xi$ is real iff. the matrix $\hat{\xi}$ is real in the sense that its entries are real-valued functions, i.e. $\hat{\xi} \in \mathrm{Mat}(U, \mathcal{C}^\infty(U, \mathbb{R}))$. Furthermore, $\hat{\gamma}^A$ belongs to $\mathrm{Mat}(U, \mathcal{C}^\infty(U, \mathbb{R}))$ when $|A|$ is even and to $\mathrm{Mat}(U, \mathcal{C}^\infty(U, i\mathbb{R}))$ when $|A|$ is odd. In the physics literature, one often finds expressions written locally in terms of $\hat{\xi}$ and $\hat{\gamma}^a$; such expressions should be understood in the sense explained above.

### 2.8 The quaternionic case

This occurs for $\mathbb{S} \approx \mathbb{H}$, which happens for $p - q \equiv_8 4, 5, 6$. Then $N = 4\Delta = 2^{[\frac{d}{2}]+1}$ and $\Sigma_\gamma$ is a bundle of quaternion algebras (the topology of such bundles was studied in [7]). We have $\nu \diamond \nu = +1$ when $p - q \equiv_8 4, 5$ and $\nu \diamond \nu = -1$ for $p - q \equiv_8 6$. Furthermore, $\nu$ is central in the Kähler-Atiyah algebra iff. $p - q \equiv_8 5$. When $p - q \equiv_8 5$, we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, where $\epsilon_\gamma \in \{-1, 1\}$ is the signature of $\gamma$, which can only be defined in this subcase (known as the *quaternionic non-simple case*).

#### 2.8.1 The quaternionic structure of $S$

For any sufficiently small open subset $U \subset M$, there exist three local sections $J_j \in \Gamma(U, \Sigma_\gamma) \subset \Gamma(U, \mathrm{End}(S))$ $(j = 1 \ldots 3)$ lying in the commutant of the subset $\gamma(\Omega(U)) \subset \Gamma(U, \mathrm{End}(S))$, which satisfy the algebra of quaternion relations:

$$[J_i, \gamma(\omega)]_{-,\circ} = 0 \quad , \quad \forall \omega \in \Omega(U) \quad , \quad J_i \circ J_j = -\delta_{ij} \mathrm{id}_S + \sum_{k=1}^{3} \epsilon_{ijk} J_k \quad , \quad \forall i, j = 1 \ldots 3 \quad , \tag{2.28}$$

where $\epsilon_{ijk}$ is the totally anti-symmetric Levi-Civita symbol. In particular, we have $J_i^2 = -\mathrm{id}_S|_U$. Setting $J_0 \stackrel{\text{def.}}{=} \mathrm{id}_S|_U \in \Gamma(U, \mathrm{End}(S))$, we have:

$$\Gamma(U, \Sigma_\gamma) = \oplus_{\alpha=0}^{3} \mathcal{C}^\infty(U, \mathbb{R}) J_\alpha = \left\{ \sum_{\alpha=0}^{3} f_\alpha J_\alpha \;\middle|\; f_\alpha \in \mathcal{C}^\infty(U, \mathbb{R}) \right\} \quad .$$

In particular, the restriction $\Sigma_\gamma|_U$ is topologically trivial as a bundle of algebras. The group $\mathcal{C}^\infty(U, \mathrm{O}(3, \mathbb{R}))$ of $\mathrm{O}(3, \mathbb{R})$-valued functions defined on $U$ acts transitively on the space of solutions $J_i \in \Gamma(U, \mathrm{End}(S))$ to (2.28) through:

$$J_i \to \sum_{j=1}^{3} R_{ij} J_j \quad \text{where} \quad R = (R_{ij})_{i,j=1\ldots3} \in \mathcal{C}^\infty(U, \mathrm{O}(3, \mathbb{R})) \quad . \tag{2.29}$$



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| $p-q$ mod 8 | Cl$(p,q)$ | $\gamma$ is injective | $\epsilon_\gamma$ | $\mathcal{R}$ (real spinors) | $\nu \diamond \nu$ | $\nu$ is central |
|---|---|---|---|---|---|---|
| **4** | simple | Yes | N/A | $\gamma(\nu)$ (sympl. Majorana-Weyl) | $+1$ | No |
| **5** | non-simple | No | $\pm 1$ | N/A | $+1$ | Yes |
| **6** | simple | Yes | N/A | $\gamma(\nu) \circ J$ (sympl. Majorana) | $-1$ | No |

**Table 7**. Summary of subcases of the quaternionic case. $J$ denotes any of the complex structures induced on $S$ by the quaternionic structure.

The most general complex structure $J \in \Gamma(U, \Sigma_\gamma)$ on the restricted bundle $S|_U$ which lies in the commutant of $\gamma(\Omega(U))$ takes the form:

$$J = \sum_{i=1}^{3} f_i J_i \quad \text{where} \quad f_i \in \mathcal{C}^\infty(U, \mathbb{R}) \quad \text{with} \quad \sum_{i=1}^{3} (f_i)^2 = 1 \quad . \tag{2.30}$$

Equation (2.30) says that $J$ is a local section (defined above $U$) of the *twistor bundle* $\mathcal{U}_\gamma$ *of* $(S, \gamma)$, which is the $S^2$-sub-bundle of $\Sigma_\gamma$ consisting of the imaginary units of the fibers of $\Sigma_\gamma$:

$$\mathcal{U}_\gamma \stackrel{\text{def.}}{=} \{(x, J_x) \mid J_x \in \Sigma_{\gamma,x} \;\; \& \;\; J_x^2 = -1\} \subset \Sigma_\gamma \quad . \tag{2.31}$$

Let $\mathbf{j}_\alpha$ $(\alpha = 0\ldots3)$ be the canonical units of $\mathbb{H}$, where $\mathbf{j}_0 = 1$ and $\mathbf{j}_i$ $(i = 1\ldots3)$ are the canonical imaginary units. For any choice of solution $\vec{J} = (J_1, J_2, J_3)$ to (2.28) defined above $U$, the morphism of $\mathbb{R}$-vector bundles:

$$\varphi_{\vec{J}} : \mathbb{H} \otimes \mathcal{O}_\mathbb{R}|_U \to \Sigma_\gamma|_U$$

which acts on sections through:

$$\varphi_{\vec{J}}\left( \sum_{\alpha=0}^{3} f_\alpha \mathbf{j}_\alpha \right) \stackrel{\text{def.}}{=} \sum_{\alpha=0}^{3} f_\alpha J_\alpha$$

is an isomorphism of bundles of $\mathbb{R}$-algebras from $\mathbb{H} \otimes \mathcal{O}_U$ to the restricted Schur bundle $\Sigma_\gamma|_U$, which provides a local trivialization of $\Sigma_\gamma$ (as a bundle of $\mathbb{R}$-algebras) above $U$. Here, $\mathcal{O}_\mathbb{R}|_U$ stands for the restriction of $\mathcal{O}_\mathbb{R}$ to $U$.

### 2.8.2 Injectivity and surjectivity

The bundle morphism $\gamma$ is fiberwise injective iff. $p - q \equiv_8 4, 6$. It is always fiberwise non-surjective, with image equal to the sub-bundle $\mathrm{End}_\mathbb{H}(S) \stackrel{\text{def.}}{=} \mathrm{End}_{\Sigma_\gamma}(S)$ of $\mathrm{End}(S)$. This sub-bundle of $\mathrm{End}(S)$ is isomorphic through $\gamma$ (as a bundle of algebras) with the Kähler-Atiyah bundle of $(M, g)$ when $p - q \equiv_8 4, 6$ and with the sub-bundle $\wedge^{\epsilon_\gamma} T^*M$ when $p - q \equiv_8 5$. The situation is summarized in table 7.

### 2.8.3 Spin projectors

In the quaternionic case, the various possibilities for spin projectors are as follows:

- If $p - q \equiv_8 4$, then the restriction $\gamma_{\rm ev}$ is fiberwise reducible and we can use the spin projectors defined by the product structure $\mathcal{R} = \gamma(\nu)$, which lies in the commutant of $\Gamma(M, \Sigma_\gamma)$. Sections of the sub-bundles $S^\pm = \ker(\gamma(\nu) \mp \mathrm{id}_S) \subset S$ are called *symplectic Majorana-Weyl* spinors of positive and negative chiralities, respectively. They can be viewed as those sections $\xi$ of $S$ which satisfy the conditions:

$$\xi \in \Gamma(M, S_\pm) \implies \gamma(\nu)\xi = \pm\xi \quad .$$

  We have $S = S^+ \oplus S^-$, so any section $\xi \in \Gamma(M, S)$ decomposes uniquely as:

$$\xi = \xi^+ + \xi^- \quad \text{with} \quad \xi^\pm \in \Gamma(M, S^\pm) \quad .$$

  Since $J_\alpha$ commute with $\gamma(\nu)$, we have $J_\alpha(S^\pm) \subset S^\pm$ and hence the sub-bundles $S^\pm$ of symplectic Majorana-Weyl spinors inherit from $S$ the structure of bundles of free modules over the Schur bundle $\Sigma_\gamma$ — in particular, each of $S^\pm$ carries a quaternionic structure. The restricted morphism $\gamma_{\rm ev}$ can be viewed as an isomorphism of bundles of algebras from the even Kähler-Atiyah algebra of $M$ to the sub-bundle of algebras $\mathrm{End}_{\mathbb{H}}(S^+) \oplus \mathrm{End}_{\mathbb{H}}(S^-)$ of $(\mathrm{End}_{\mathbb{H}}(S), \circ)$:

$$\gamma_{\rm ev} : (\wedge^{\rm ev} T^* M, \diamond) \xrightarrow{\sim} (\mathrm{End}_{\mathbb{H}}(S^+), \circ) \oplus (\mathrm{End}_{\mathbb{H}}(S^-), \circ) \quad .$$

- If $p - q \equiv_8 5$ (the quaternionic non-simple case), then the restriction $\gamma_{\rm ev}$ is fiberwise irreducible, so spin projectors cannot be defined. This is also the only quaternionic non-simple subcase, i.e. the only quaternionic subcase for which $\gamma$ fails to be fiberwise injective.

- If $p - q \equiv_8 6$, then the restriction $\gamma_{\rm ev}$ is fiberwise reducible and we can define *symplectic Majorana spinors* using the product structure $\mathcal{R}^J = \gamma(\nu) \circ J$ induced by any *global* section $J \in \Gamma(M, \mathcal{U}_\gamma)$ of the twistor bundle $\mathcal{U}_\gamma$ of $(S, \gamma)$.

### 2.8.4 The biquaternion formalism

Real pinors in the quaternionic case can be described indirectly by first complexifying the real vector bundle $S$ and then recovering it from its complexification by using the appropriate real structure. This approach is common in the physics literature, though its precise relation with the direct construction of real pinors (on which we rely) is rarely clarified.

**The complexified pin bundle and its biquaternionic structure.** Recall that the algebra of biquaternions (or complexified quaternions) is the $\mathbb{C}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$, which is isomorphic with the $\mathbb{C}$-algebra $\mathrm{Mat}(2, \mathbb{C}) \approx \mathrm{Cl}_{\mathbb{C}}(2)$ through the map:

$$\mathbb{C} \otimes \mathbb{H} \ni w_0 \mathbf{j}_0 + w_1 \mathbf{j}_1 + w_2 \mathbf{j}_2 + w_3 \mathbf{j}_3 \longrightarrow \begin{bmatrix} w_0 + i w_1 & w_2 + i w_3 \\ -w_2 + i w_3 & w_0 - i w_1 \end{bmatrix} \in \mathrm{Mat}(2, \mathbb{C}) \quad ,$$

where $w_\alpha \in \mathbb{C}$. This well-known isomorphism maps the imaginary quaternion units into $i$ times the Pauli matrices:

$$\mathbf{j}_1 \to i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\boldsymbol{\sigma}_3 \ , \quad \mathbf{j}_2 \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\boldsymbol{\sigma}_2 \ , \quad \mathbf{j}_3 \to i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\boldsymbol{\sigma}_1 \ ,$$

where $i$ is, as usual, the imaginary unit of the field $\mathbb{C}$ of complex numbers. The *complexified Schur bundle*:

$$(\Sigma_\gamma)_\mathbb{C} \stackrel{\mathrm{def.}}{=} \Sigma_\gamma \otimes \mathcal{O}_\mathbb{C}$$

is a bundle of biquaternion algebras over $M$. We let $S_\mathbb{C} \stackrel{\mathrm{def.}}{=} S \otimes \mathcal{O}_\mathbb{C}$ be the *complexified pin bundle*, i.e. the complexification of the $\mathbb{R}$-vector bundle $S$, which is a bundle of free modules over the complexified Schur bundle and thus carries a biquaternionic structure. Notice that $S_\mathbb{C}$ is also a bundle of modules over the complexified Kähler-Atiyah bundle $\wedge T^*_\mathbb{C} M = \wedge T^*M \otimes \mathcal{O}_\mathbb{C}$ of $(M, g)$, through the morphism of bundles of algebras given by the complexification $\gamma_\mathbb{C}$ of $\gamma$, which commutes with the biquaternionic structure of $S_\mathbb{C}$. Using the $(\Sigma_\gamma)_\mathbb{C}$-module structure, left multiplication by the complex imaginary unit $i$ gives a globally-defined endomorphism $\mathfrak{J} \in \Gamma(M, \mathrm{End}(S_\mathbb{C}))$ whose action on sections is given by:

$$\mathfrak{J}(\xi) \stackrel{\mathrm{def.}}{=} i\xi \ , \quad \forall \xi \in \Gamma(M, S_\mathbb{C}) \ ,$$

while the complexifications of $J_k$ (which we denote $\mathfrak{J}_k$) define fiberwise $\mathbb{C}$-linear endomorphisms of $S_\mathbb{C}|_U$ which satisfy the quaternion relations. In particular, we have:

$$[\mathfrak{J}_k, \mathfrak{J}]_{-,\circ} = 0$$

and $\mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3$ form a local frame of the underlying $\mathbb{R}$-vector bundle of $(\Sigma_\gamma)_\mathbb{C}$ above $U$. In every fiber of $S_\mathbb{C}|_U$, the endomorphisms $\mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3$ represent the canonical basis $i, \mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$ of the biquaternion algebra, when the latter is viewed as an eight-dimensional associative algebra over the real numbers. The complexification $S_\mathbb{C}$ comes endowed with a natural real structure — a globally-defined endomorphism $\mathfrak{Z} \in \Gamma(M, \mathrm{End}(S_\mathbb{C}))$, which is $\mathbb{C}$-antilinear:

$$[\mathfrak{Z}, \mathfrak{J}]_{+,\circ} = 0 \ ,$$

and satisfies:

$$\begin{aligned} \Gamma(M, S) &= \{\xi \in \Gamma(M, S_\mathbb{C}) \mid \mathfrak{Z}(\xi) = \xi\} \ , \\ i\Gamma(M, S) &= \mathfrak{J}(\Gamma(M, S)) = \{\xi \in \Gamma(M, S_\mathbb{C}) \mid \mathfrak{Z}(\xi) = -\xi\} \ . \end{aligned}$$

We have:

$$[\mathfrak{Z}|_U, \mathfrak{J}_k]_{-,\circ} = 0 \ , \tag{2.32}$$

since the locally-defined endomorphisms $\mathfrak{J}_k$ are obtained through complexification.

**The product structure induced by $\mathfrak{J}_1$.** Since $\mathfrak{J}$ and $\mathfrak{J}_1$ commute and since both of them square to $-\mathrm{id}_{S_\mathbb{C}}|_U$, the opposite of their composition:

$$\mathfrak{R} \stackrel{\text{def.}}{=} -\mathfrak{J}\circ\mathfrak{J}_1 = -i\mathfrak{J}_1$$

is a $\mathbb{C}$-*linear* product structure on $S_\mathbb{C}|_U$ which anticommutes with $\mathfrak{Z}|_U$:

$$\mathfrak{R}^2 = +\mathrm{id}_{S_\mathbb{C}}|_U \ \ , \ \ [\mathfrak{R}, \mathfrak{Z}|_U]_{+,\circ} = 0$$

and hence the $\mathbb{C}$-linear endomorphisms of $S_\mathbb{C}|_U$ defined through:

$$\mathfrak{P}_\pm = \frac{1}{2}(\mathrm{id}_{S_\mathbb{C}}|_U \pm \mathfrak{R})$$

are complementary in $\mathrm{End}_\mathbb{C}(S_\mathbb{C}|_U)$. It follows that the complex sub-bundles $S_\mathbb{C}|_U^\pm$ of $S_\mathbb{C}|_U$ consisting of eigenvectors of $\mathfrak{R}$ corresponding to the eigenvalues $\pm 1$ give a direct sum decomposition:

$$S_\mathbb{C}|_U = S_\mathbb{C}|_U^+ \oplus S_\mathbb{C}|_U^- \ \ .$$

Notice the equalities:

$$\mathfrak{J}(S_\mathbb{C}|_U^\pm) = S_\mathbb{C}|_U^\pm \ \ , \ \ \mathfrak{Z}(S_\mathbb{C}|_U^\pm) = S_\mathbb{C}|_U^\mp$$

and the fact that $S_\mathbb{C}|_U^\pm$ are the eigen sub-bundles of $S_\mathbb{C}|_U$ determined by the eigenvalues $\pm i$ of the complexified endomorphism $\mathfrak{J}_1$. In particular, we have:

$$\Gamma(U, S_\mathbb{C}|_U^\pm) = \{\xi \in \Gamma(U, S_\mathbb{C}|_U) \mid \mathfrak{J}_1(\xi) = \pm i\xi\} \ \ .$$

Any $\xi \in \Gamma(U, S_\mathbb{C})$ decomposes uniquely as:

$$\xi = \xi^+ + \xi^- \ \ \text{with} \ \ \xi^\pm \in \Gamma(U, S_\mathbb{C}|_U^\pm)$$

and we have:

$$\mathfrak{J}_1(\xi)^\pm = \pm i\xi^\pm \ \ .$$

Since $\mathfrak{J}_2$ anticommutes with $\mathfrak{J}_1$, it induces an isomorphism of $\mathbb{C}$-vector bundles:

$$\mathfrak{J}_2|_{S_\mathbb{C}|_U^+}^{S_\mathbb{C}|_U^-} : S_\mathbb{C}|_U^+ \xrightarrow{\sim} S_\mathbb{C}|_U^- \ \ ,$$

which can be used to identify $S_\mathbb{C}|_U^-$ with $S_\mathbb{C}|_U^+$. Using this isomorphism, we find that any $\xi$ also decomposes uniquely as:

$$\xi = \xi^1 - \mathfrak{J}_2(\xi^2) \ \ , \ \ \text{where} \ \ \xi^1 \stackrel{\text{def.}}{=} \xi^+ \in \Gamma(U, S_\mathbb{C}|_U^+) \ \ \text{and} \ \ \xi^2 = \mathfrak{J}_2(\xi^-) \in \Gamma(U, S_\mathbb{C}|_U^+) \ \ .$$

We can thus describe $\xi$ through the column matrix:

$$\hat{\xi} \stackrel{\text{def.}}{=} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \in \mathrm{Mat}(2, 1; \Gamma(U, S_\mathbb{C}|_U^+)) \tag{2.33}$$



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and describe any endomorphism $T \in \Gamma(U, \mathrm{End}(S_{\mathbb{C}}|_U))$ through the matrix $\hat{T} \in \mathrm{Mat}(2,2;\Gamma(U,\mathrm{End}(S_{\mathbb{C}}|_U^+))$ defined by:

$$\widehat{T\xi} = \hat{T}\hat{\xi} \ , \quad \forall \xi \in \Gamma(U, S_{\mathbb{C}}) \ , \tag{2.34}$$

where juxtaposition in the right hand side stands for matrix multiplication. Since $\mathfrak{J}_1(\xi^\pm) = \pm i\xi^\pm$ and since $\mathfrak{J}_2$ is $\mathbb{C}$-linear, we have $\mathfrak{J}_1(\xi)^1 = i\xi^1$ and $\mathfrak{J}_1(\xi)^2 = -i\xi^2$. On the other hand, an easy computation gives:

$$\mathfrak{J}_2(\xi)^1 = \xi^2 \ , \quad \mathfrak{J}_2(\xi)^2 = -\xi^1 \implies \mathfrak{J}_3(\xi)^1 = i\xi^2 \ , \quad \mathfrak{J}_3(\xi)^2 = i\xi^1 \ ,$$

where the implication displayed follows from the relation $\mathfrak{J}_3 = \mathfrak{J}_1 \circ \mathfrak{J}_2$. In terms of description (2.34), this amounts to:

$$\widehat{\mathfrak{J}_1} = i\boldsymbol{\sigma}_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \ , \quad \widehat{\mathfrak{J}_2} = i\boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ , \quad \widehat{\mathfrak{J}_3} = i\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

and implies:

$$\hat{\mathfrak{R}} = \boldsymbol{\sigma}_3 \implies \hat{\mathfrak{P}}_\pm = \frac{1}{2}(1 \pm \boldsymbol{\sigma}_3) \ .$$

We have:

$$\begin{aligned} &\mathfrak{Z}(\xi)^1 = -\mathfrak{Z}_0(\xi^2) \ , \quad \mathfrak{Z}(\xi)^2 = \mathfrak{Z}_0(\xi^1) \\ &\qquad\qquad \iff \widehat{\mathfrak{Z}(\xi)} = -i\sigma_2(\mathfrak{Z}_0\xi) = \begin{bmatrix} 0 & -\mathfrak{Z}_0 \\ \mathfrak{Z}_0 & 0 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \ , \quad \forall \xi \in \Gamma(U, S_{\mathbb{C}}) \ , \end{aligned}$$

where $\mathfrak{Z}_0 = [\mathfrak{J}_2 \circ \mathfrak{Z}]\,|_{S_{\mathbb{C}}|_U^+}^{S_{\mathbb{C}}|_U^-} = [i\mathfrak{Z} \circ \mathfrak{J}_2]\,|_{S_{\mathbb{C}}|_U^+}^{S_{\mathbb{C}}|_U^+} \in \Gamma(U, \mathrm{End}_{\mathbb{R}}(S_{\mathbb{C}}|_U^+))$ is an antilinear endomorphism of $S_{\mathbb{C}}|_U^+$ which squares to $-\mathrm{id}_{S_{\mathbb{C}}|_U^+}$. Hence the $\mathfrak{Z}$-reality condition for sections of $S_{\mathbb{C}}$ takes the form:

$$\xi \in \Gamma(U, S) \iff \mathfrak{Z}(\xi) = \xi \iff \xi^1 = -\mathfrak{Z}_0(\xi^2) \iff \xi^2 = \mathfrak{Z}_0(\xi^1) \ ,$$

which can be viewed as a *'generalized symplectic Majorana condition'* (see the example in subsection 5.3) on the pair of sections (which are complex pinors) $\xi_1, \xi_2 \in \Gamma(U, S_{\mathbb{C}}|_U^+)$.

**The complex pin bundle and complex pinors of spin 1/2.** Local sections of $S_{\mathbb{C}}$ form a sheaf of left modules over the sheaf of sections of $(\Sigma_\gamma)_{\mathbb{C}}$. In particular, the space $\Gamma(U, S_{\mathbb{C}})$ is a left module over the non-commutative ring $\Gamma(U, (\Sigma_\gamma)_{\mathbb{C}})$ for any open subset $U$ of $M$ which supports a local frame of imaginary unit sections of $\Sigma_\gamma$. The discussion above shows that this module structure is given by:

$$\widehat{\mathfrak{f}\xi} = \begin{bmatrix} f_0 + if_1 & f_2 + if_3 \\ -f_2 + if_3 & f_0 - if_1 \end{bmatrix} \hat{\xi} \ , \quad \forall f \in \Gamma(U, (\Sigma_\gamma)_{\mathbb{C}}) \ , \quad \forall \xi \in \Gamma(U, S_{\mathbb{C}}) \ , \tag{2.35}$$

where we used the decomposition:

$$\mathfrak{f} = f_0 \mathrm{id}_{S_{\mathbb{C}}|_U} + f_1\mathfrak{J}_1 + f_2\mathfrak{J}_2 + f_3\mathfrak{J}_3 \in \Gamma(U, (\Sigma_\gamma)_{\mathbb{C}}) \ , \quad \text{with} \quad f_\alpha \in \mathcal{C}^\infty(U, \mathbb{C}) \ .$$

In equation (2.35), any complex-valued smooth function $g = g_R + i g_I \in \mathcal{C}^\infty(U, \mathbb{C})$ (with $g_{R,I} \in \mathcal{C}^\infty(U, \mathbb{R})$) acts on $\xi \in \Gamma(U, S_\mathbb{C})$ via the complex structure $\mathfrak{I}$:

$$g\xi = g_R \xi + g_I \mathfrak{I}(\xi) \;\; , \;\; \forall \xi \in \Gamma(U, S_\mathbb{C})$$

and we have $g\Gamma(U, S_\mathbb{C}|_U^+) \subset \Gamma(U, S_\mathbb{C}|_U^+)$.

Since the complexified operators $\gamma(\omega)$ commute with $\mathfrak{I}$ and $\mathfrak{J}_k$, we have:

$$[\gamma(\omega), \mathfrak{R}]_{-,\circ} = 0 \iff \gamma(\omega)(\Gamma(U, S_\mathbb{C}^\pm)) \subset \Gamma(U, S_\mathbb{C}^\pm) \;\text{ for }\; \omega \in \Omega(U)$$

and $\mathfrak{J}_2 \circ \gamma(\omega) \circ \mathfrak{J}_2^{-1} = \gamma(\omega)$. Defining $\gamma_\pm(\omega) \stackrel{\text{def.}}{=} \gamma(\omega)|_{S_\mathbb{C}|_U^\pm}^{S_\mathbb{C}|_U^\pm}$ and $\gamma_1(\omega) \stackrel{\text{def.}}{=} \gamma_+(\omega)$, $\gamma_2(\omega) \stackrel{\text{def.}}{=} \mathfrak{J}_2|_{S_\mathbb{C}|_U^-}^{S_\mathbb{C}|_U^+} \circ \gamma_-(\omega) \circ \left(\mathfrak{J}_2|_{S_\mathbb{C}|_U^-}^{S_\mathbb{C}|_U^+}\right)^{-1}$, the last relation implies $\gamma_2(\omega) = \gamma_1(\omega) \stackrel{\text{def.}}{=} \Gamma(\omega)$. We thus have a decomposition:

$$\gamma = \gamma_+ \oplus \gamma_- \iff \gamma = \Gamma \oplus \left[\left(\mathfrak{J}_2|_{S_\mathbb{C}|_U^-}^{S_\mathbb{C}|_U^+}\right)^{-1} \circ \Gamma \circ \mathfrak{J}_2|_{S_\mathbb{C}|_U^-}^{S_\mathbb{C}|_U^+}\right] \quad .$$

where:

$$\Gamma : (\wedge T^*U, \diamond) \to (\mathrm{End}_\mathbb{C}(S_\mathbb{C}|_U^+), \circ)$$

is a morphism of bundles of $\mathbb{R}$-algebras. The morphism $\Gamma$ complexifies to a morphism of bundles of $\mathbb{C}$-algebras:

$$\Gamma_\mathbb{C} : (\wedge T^*_\mathbb{C} U, \diamond) \to (\mathrm{End}_\mathbb{C}(S_\mathbb{C}|_U^+), \circ)$$

whose fibers are irreducible representations of the complexified Clifford algebra $\mathrm{Cl}_\mathbb{C}(p,q) \approx \mathrm{Cl}_\mathbb{C}(d)$. This complexified morphism $\Gamma_\mathbb{C}$ is always fiberwise surjective, being fiberwise injective iff. $d$ is even, i.e. iff. $p - q \equiv_8 4, 6$. Therefore, sections of $\Gamma(U, S_\mathbb{C}|_U^+)$ are locally-defined *complex pinors*[7] of spin 1/2. We have:

$$\widehat{\gamma(\omega)} = \begin{bmatrix} \Gamma(\omega) & 0 \\ 0 & \Gamma(\omega) \end{bmatrix} \;\; , \;\; \forall \omega \in \Omega(U) \quad .$$

and hence $(S_\mathbb{C}|_U, \gamma)$ is equivalent with the direct sum $(S_\mathbb{C}|_U^+, \Gamma) \oplus (S_\mathbb{C}|_U^+, \Gamma)$ as a bundle of modules over the Kähler-Atiyah bundle of $(M, g)$. Similarly, $(S_\mathbb{C}|_U, \gamma_\mathbb{C})$ is equivalent with $(S_\mathbb{C}|_U^+, \Gamma_\mathbb{C}) \oplus (S_\mathbb{C}|_U^+, \Gamma_\mathbb{C})$ as a bundle of modules over the complexified Kähler-Atiyah bundle of $(U, g|_U)$. In much of the supergravity literature, one constructs the bundle $S$ of real pinors by starting from such a direct sum of two copies of the bundle $S_\mathbb{C}$ of complex pinors and imposing the reality ('generalized symplectic Majorana') condition $\mathfrak{Z}(\xi) = \xi$. The discussion above shows that this construction can always be applied locally.

**Local expressions.** If $e^a$ is a local pseudo-orthonormal coframe of $(M, g)$ defined above $U$, we set $\gamma^a \stackrel{\text{def.}}{=} \gamma_\mathbb{C}(e^a) \in \Gamma(U, \mathrm{End}_\mathbb{C}(S_\mathbb{C}))$ and $\Gamma^a \stackrel{\text{def.}}{=} \Gamma(e^a) \in \Gamma(U, \mathrm{End}_\mathbb{C}(S_\mathbb{C}^+))$. We also set $\gamma^A = \gamma^{a_1} \circ \ldots \circ \gamma^{a_k} \in \Gamma(U, \mathrm{End}_\mathbb{C}(S_\mathbb{C}))$ and $\Gamma^A = \Gamma^{a_1} \circ \ldots \circ \Gamma^{a_k} \in \Gamma(U, \mathrm{End}_\mathbb{C}(S_\mathbb{C}^+))$ for any ordered index set $A = (a_1, \ldots, a_k)$. The relation above gives:

$$\widehat{\gamma}^a = \begin{bmatrix} \Gamma^a & 0 \\ 0 & \Gamma^a \end{bmatrix} \;\; , \;\; \widehat{\gamma}^A = \hat{\gamma}^{a_1} \circ \ldots \circ \hat{\gamma}^{a_k} = \begin{bmatrix} \Gamma^A & 0 \\ 0 & \Gamma^A \end{bmatrix} \quad .$$

[7] These are often called complex *spinors* in the physics literature, which is justified since any pinor is also a spinor.

| $\mathbb{S}$ | $p-q$ mod 8 | $\mathrm{Cl}(p,q)$ | $\mathcal{R}$ | Terminology for real spinors |
|---|---|---|---|---|
| $\mathbb{R}$ | **0** | simple | $\gamma(\nu)$ | Majorana-Weyl |
| $\mathbb{C}$ | **7** | simple | $D$ | Majorana |
| $\mathbb{H}$ | **4** | simple | $\gamma(\nu)$ | symplectic Majorana-Weyl |
| $\mathbb{H}$ | **6** | simple | $\gamma(\nu)\circ J$ | symplectic Majorana |

**Table 8**. The product structure $\mathcal{R}$ used in the construction of the spin projectors $\mathcal{P}^{\mathcal{R}}_{\pm} \stackrel{\text{def.}}{=} \frac{1}{2}(1\pm\mathcal{R})$ for those cases when they can be defined and the corresponding terminology for real spinors. When $p-q \equiv_8 6$, the locally-defined endomorphism $J \in \Gamma(U, \mathrm{End}(S))$ appearing in the expression for $\mathcal{R}$ is any of the complex structures associated with the quaternionic structure of $S$. Notice that $\mathrm{Cl}(p,q)$ is always simple as an $\mathbb{R}$-algebra (and hence $\gamma$ is fiberwise injective) in those cases when spin projectors can be defined.

### 2.9 Summary of spin projectors

The spin projectors in the three cases are summarized in table 8.

### 2.10 Relation to pin bundles over the complexified Kähler-Atiyah bundle of $(M,g)$

#### 2.10.1 General remarks

Let $T^*_{\mathbb{C}}M$ be the complexified cotangent bundle of $M$, endowed with the nondegenerate fiberwise $\mathbb{C}$-bilinear pairing induced by the complexification of $g$. The complexified exterior bundle $\wedge T^*_{\mathbb{C}}M$ carries a structure of bundle of algebras whose product (which we again denote by $\diamond$) is obtained by complexfiying the geometric product induced on $\wedge T^*M$ by $g$. The bundle $(\wedge T^*_{\mathbb{C}}M, \diamond)$ of unital associative algebras over $\mathbb{C}$ is called the *complexified Kähler-Atiyah bundle of* $(M,g)$; it coincides with the complexification of the real Kähler-Atiyah bundle as a bundle of algebras. Its fibers are isomorphic with $\mathrm{Cl}_{\mathbb{C}}(p,q) \approx \mathbb{C}\otimes_{\mathbb{R}} \mathrm{Cl}(p,q) \approx \mathrm{Cl}_{\mathbb{C}}(d,0)$, the complexification of the real Clifford algebra $\mathrm{Cl}(p,q)$. It is natural to consider *bundles of complex pinors*, i.e. bundles $S$ of modules over $(\wedge T^*_{\mathbb{C}}M, \diamond)$; these are $\mathbb{C}$-vector bundles $S$ over $M$ endowed with a morphism:

$$\gamma_{\mathbb{C}} : (\wedge T^*_{\mathbb{C}}M, \diamond) \to (\mathrm{End}_{\mathbb{C}}(S), \circ) \tag{2.36}$$

of bundles of $\mathbb{C}$-algebras. A $\mathbb{C}$-vector bundle $S$ over $M$ can be identified with the pair $(S, J)$, where $S$ is the underlying $\mathbb{R}$-vector bundle while $J \in \Gamma(M, \mathrm{End}_{\mathbb{R}}(S))$ is the globally-defined endomorphism given by multiplication with the imaginary unit in each fiber. This satisfies $J^2 = -\mathrm{id}_S$, being the complex structure on $S$ defining its original $\mathbb{C}$-vector bundle structure. The bundle $\mathrm{End}_{\mathbb{C}}(S)$ of complex-linear endomorphisms of $S$ identifies with the commutant of $J$ in $\mathrm{End}_{\mathbb{R}}(S)$, being a bundle of $\mathbb{R}$-subalgebras of the latter. On the other hand, the complexified Kähler-Atiyah bundle can be written $\wedge T^*_{\mathbb{C}}M = \mathcal{O}_{\mathbb{C}} \otimes \wedge T^*M$, where $\mathcal{O}_{\mathbb{C}}$ is the trivial complex line bundle on $M$. It is now easy to check that there exists a bijection between morphisms (2.36) of bundles of $\mathbb{C}$-algebras and morphisms:

$$\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}_{\mathbb{C}}(S), \circ) \tag{2.37}$$

of bundles of $\mathbb{R}$-algebras, where $\gamma_{\mathbb{C}}$ is recovered from $\gamma$ through *J-complexification*, an operation which takes the following form when applied to global sections:[8]

$$\gamma_{\mathbb{C}}((f+ig)\otimes\omega) = (f\mathrm{id}_S+gJ)\circ\gamma(\omega) = \gamma(\omega)\circ(f\mathrm{id}_S+gJ)~,~\forall\omega\in\Omega(M)~,~\forall f,g\in\mathcal{C}^\infty(M,\mathbb{R})~. \tag{2.38}$$

We can thus view a bundle of complex pinors as a pair $(S,J)$ where $J$ is a complex structure on $S$ and $S$ is a bundle of real pinors whose underlying morphism $\gamma$ has the property that its image lies in the commutant of $J$. It is quite obvious that fiberwise irreducibility of $\gamma$ implies fiberwise irreducibility (over $\mathbb{C}$) of $\gamma_{\mathbb{C}}$. Well-known results from the representation theory of complex Clifford algebras imply that the converse is also true, i.e. we have:

**Proposition.** Let $\gamma$, $\gamma_{\mathbb{C}}$ be related through (2.38) as above. Then $\gamma$ is fiberwise irreducible (over $\mathbb{R}$) iff. $\gamma_{\mathbb{C}}$ is fiberwise irreducible (over $\mathbb{C}$).
As in [1], it is convenient to consider the complex-valued volume form:

$$\nu_{\mathbb{C}} \stackrel{\text{def.}}{=} i^{q+\left[\frac{d}{2}\right]}\nu \tag{2.39}$$

When fiberwise irreducibility holds (i.e. when $(S,\gamma)$ is a real *pin* bundle and thus $(S,\gamma_{\mathbb{C}})$ is a complex pin bundle), the Schur algebra $\mathbb{S}$ must equal $\mathbb{C}$ or $\mathbb{H}$ (since $J$ belongs to the commutant of the image of $\gamma$ and thus $J$ is a section of the Schur bundle $\Sigma_\gamma$ of $\gamma$), so the real rank $N$ of $S$ equals $2^{\left[\frac{d}{2}\right]+1}$ and hence its complex rank equals $2^{\left[\frac{d}{2}\right]}$; this agrees with a well-known fact from the representation theory of complex Clifford algebras. Let us consider the two cases in turn:

### 2.10.2 The case $\mathbb{S}=\mathbb{C}$

In this case, we have two choices for $J$, namely $J=\pm\gamma(\nu)$, leading through (2.38) to the two $J$-complexifications:

$$\gamma^\pm_{\mathbb{C}}((f+ig)\otimes\omega) = (f\mathrm{id}_S\pm g\gamma(\nu))\circ\gamma(\omega)~~,~~\forall\omega\in\Omega(M)~~,~~\forall f,g\in\mathcal{C}^\infty(M,\mathbb{R})~~,$$

whose fiberwise representations are complex-conjugate to each other. They are distinguished by the property:

$$\gamma^\pm_{\mathbb{C}}(i\nu)=\mp\mathrm{id}_S \iff \gamma^\pm_{\mathbb{C}}(\nu_{\mathbb{C}})=\epsilon_{\gamma^\pm_{\mathbb{C}}}\mathrm{id}_S \quad ,$$

where we introduced the signature (see [1]) of $\gamma^\pm_{\mathbb{C}}$ (as morphisms of bundles of $\mathbb{C}$-algebras):

$$\epsilon_{\gamma^\pm_{\mathbb{C}}} \stackrel{\text{def.}}{=} \pm(-1)^{\frac{1}{2}(1+q+\left[\frac{d}{2}\right])} = \pm\begin{cases}(-1)^{q+1}\,, & \text{if } p-q\equiv_8 3\\ (-1)^q\,, & \text{if } p-q\equiv_8 7\end{cases}$$

and we used the fact that $\gamma_{\mathbb{C}}(\nu)=\gamma(\nu)=\pm J$ as well as the congruence:

$$1+q+\left[\frac{d}{2}\right]\equiv_4\begin{cases}2(q+1)\,, & \text{if } p-q\equiv_8 3\\ 2q\,, & \text{if } p-q\equiv_8 7\end{cases}\quad .$$

[8] Notice that $\Gamma(M,\wedge T^*_{\mathbb{C}}M)=\Omega_{\mathbb{C}}(M)=\mathcal{C}^\infty(M,\mathbb{C})\otimes_{\mathcal{C}^\infty(M,\mathbb{R})}\Omega(M)=\Omega(M)\otimes_{\mathbb{R}}\mathbb{C}$ is the $\mathcal{C}^\infty(M,\mathbb{C})$-module of complex-valued forms defined on $M$.

Notice that $(S, J = \gamma(\nu), \gamma_{\mathbb{C}}^{+})$ and $(S, J = -\gamma(\nu), \gamma_{\mathbb{C}}^{-})$ are inequivalent but mutually complex-conjugate complex pin bundles over $(M, g)$ — this corresponds to the well-known fact that the complexified Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(p, q) \approx \mathrm{Cl}_{\mathbb{C}}(d, 0)$ has two inequivalent (but mutually complex-conjugate) irreducible $\mathbb{C}$-representations when $d$ is odd.

#### 2.10.3 The case $\mathbb{S} = \mathbb{H}$

In this case, let us assume for simplicity that we are given a global section $J \in \Gamma(M, \mathcal{U}_\gamma)$ of the twistor bundle (2.31) of $(S, \gamma)$ (with slight adaptations of notations, the discussion generalizes when such a section is given only locally). We then have a corresponding morphism (acting on sections as in (2.38)) of bundles of $\mathbb{C}$-algebras, which we denote through $\gamma_J$. This satisfies:

$$\gamma_J(\nu_{\mathbb{C}}) = (-1)^{q+1} \begin{cases} \gamma(\nu)\,, & \text{if } p - q \equiv_8 4, 5 \\ \gamma(\nu) \circ J\,, & \text{if } p - q \equiv_8 6 \end{cases} \quad , \tag{2.40}$$

where we used the congruence:

$$q + \left[\frac{d}{2}\right] \equiv_4 \begin{cases} 2(q+1) \quad , & \text{if } p - q \equiv_8 4, 5 \\ 2q + 3\,, & \text{if } p - q \equiv_8 6 \end{cases} \quad .$$

It is convenient to distinguish the following subcases:

- When $p - q \equiv_8 5$ (so that $d$ is odd), we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, which gives $\gamma_J(\nu_{\mathbb{C}}) = \epsilon_{\gamma_J} \mathrm{id}_S$, where $\epsilon_{\gamma_J} = (-1)^{q+1} \epsilon_\gamma$. The two choices for the sign factor $\epsilon_{\gamma_J}$ correspond to the two inequivalent irreducible $\mathbb{C}$-representations of the complexified Clifford algebra $\mathrm{Cl}_{\mathbb{C}}(p, q) \approx \mathrm{Cl}_{\mathbb{C}}(d, 0)$ and lead to inequivalent complex pin bundles over $(M, g)$.

- When $p - q \equiv_8 4, 6$ (so that $d$ is even), equation (2.40) gives $\gamma_J(\nu_{\mathbb{C}}) = (-1)^{q+1}\mathcal{R}$, where $\mathcal{R}$ is the corresponding spin endomorphism, which obviously commutes with $J$. Hence $\gamma_J(\nu_{\mathbb{C}})$ is a globally-defined $\mathbb{C}$-linear endomorphism of $S$, when the latter is endowed with the complex structure induced by $J$ — it plays the role of a spin endomorphism acting on complex pinors given by sections of $S$.

## 3 Admissible bilinear pairings on the pin bundle

### 3.1 Basics

**The $\mathscr{B}$-transpose.** For any non-degenerate fiberwise bilinear pairing $\mathscr{B}$ on $S$, we let $T^t$ denote the transpose of $T \in \Gamma(M, \mathrm{End}(S))$ with respect to $\mathscr{B}$, which is defined through:

$$\mathscr{B}(T\xi, \xi') = \mathscr{B}(\xi, T^t\xi') \quad , \quad \forall \xi, \xi' \in \Gamma(M, S) \quad . \tag{3.1}$$

This operation satisfies $(T^t)^t = T$ and $(\mathrm{id}_S)^t = \mathrm{id}_S$. The operation $T \to T^t$ of taking the $\mathscr{B}$-transpose defines a $\mathcal{C}^\infty(M, \mathbb{R})$-linear anti-automorphism of the algebra $(\Gamma(M, \mathrm{End}(S)), \circ)$.

**Remark.** Consider a fiberwise non-degenerate bilinear pairing $\tilde{\mathscr{B}}$ on $S$ which is related to $\mathscr{B}$ through:

$$\tilde{\mathscr{B}} = \mathscr{B} \circ (\mathrm{id}_S \otimes A) \iff \tilde{\mathscr{B}}(\xi, \xi') = \mathscr{B}(\xi, A\xi') \ , \ \ \forall \xi, \xi' \in \Gamma(M, S) \ , \tag{3.2}$$

where $A \in \Gamma(M, \mathrm{Aut}(S))$ is a globally-defined automorphism of $S$. Then the transpose $T^{\tilde{t}}$ of a section $T \in \Gamma(M, \mathrm{End}(S))$ with respect to $\tilde{\mathscr{B}}$ is related to the transpose $T^t$ of $T$ with respect to $\mathscr{B}$ through:

$$T^{\tilde{t}} = A^{-t} \circ T^t \circ A^t \ \ . \tag{3.3}$$

Let us assume further that $\mathscr{B}(\xi, \xi') = \sigma \mathscr{B}(\xi', \xi)$ for all $\xi, \xi' \in \Gamma(M, S)$, with $\sigma \in \{-1, 1\}$ and that $A^t = \eta A$ for some $\eta \in \{-1, 1\}$. Then (3.2) implies $\tilde{\mathscr{B}}(\xi, \xi') = \tilde{\sigma} \tilde{\mathscr{B}}(\xi', \xi)$, with $\tilde{\sigma} = \eta\sigma$.

**Admissible pairings on $S$.** Recall from [5, 6] that a nondegenerate bilinear pairing $\mathscr{B}$ on the pin bundle $S$ is called *admissible* if it has the following properties:

1. $\mathscr{B}$ is either symmetric or skew-symmetric:

$$\mathscr{B}(\xi, \xi') = \sigma_{\mathscr{B}} \mathscr{B}(\xi', \xi) \ , \ \ \forall \xi, \xi' \in \Gamma(M, S) \ , \tag{3.4}$$

   where the sign factor $\sigma_{\mathscr{B}} = \pm 1$ is called the *symmetry* of $\mathscr{B}$;

2. For any $\omega \in \Omega(M)$, we have:

$$\gamma(\omega)^t = \gamma(\tau_{\mathscr{B}}(\omega)) \iff \mathscr{B}(\gamma(\omega)\xi, \xi') = \mathscr{B}(\xi, \gamma(\tau_{\mathscr{B}}(\omega))\xi') \ , \ \ \forall \xi, \xi' \in \Gamma(M, S) \ , \tag{3.5}$$

   where:

$$\tau_{\mathscr{B}} \stackrel{\text{def.}}{=} \tau \circ \pi^{\frac{1-\epsilon_{\mathscr{B}}}{2}} = \begin{cases} \tau \ , & \text{if } \epsilon_{\mathscr{B}} = +1 \\ \tau \circ \pi \ , & \text{if } \epsilon_{\mathscr{B}} = -1 \end{cases} \tag{3.6}$$

   is the *$\mathscr{B}$-modified reversion* (an anti-automorphism of the Kähler-Atiyah algebra of $(M, g)$) and the sign factor $\epsilon_{\mathscr{B}} \in \{-1, 1\}$ is called the *type* of $\mathscr{B}$;

3. If $p - q \equiv_8 0, 4, 6, 7$ (thus $S = S^+ \oplus S^-$ where $S^\pm \subset S$ are the real spin bundles), then $S^+$ and $S^-$ are either $\mathscr{B}$-orthogonal to each other or $\mathscr{B}$-isotropic. The *isotropy* of $\mathscr{B}$ is the sign factor $\iota_{\mathscr{B}} \in \{-1, 1\}$ defined through:

$$\iota_{\mathscr{B}} \stackrel{\text{def.}}{=} \begin{cases} +1 \ , & \text{if } \mathscr{B}(S^+, S^-) = 0 \\ -1 \ , & \text{if } \mathscr{B}(S^\pm, S^\pm) = 0 \end{cases} \ . \tag{3.7}$$

   When $p - q \not\equiv 0, 4, 6, 7$, then $\iota_{\mathscr{B}}$ is undefined.

When $p - q \equiv_8 0, 4, 6, 7$, we have $\iota_{\mathscr{B}} = +1$ iff. $(S^\pm)^\perp = S^\mp$, where $^\perp$ denotes the $\mathscr{B}$-orthogonal complement of a sub-bundle of $S$. This case can occur for any symmetry $\sigma_{\mathscr{B}}$ of $\mathscr{B}$. The case $\iota_{\mathscr{B}} = -1$ can occur only when $\sigma_{\mathscr{B}} = -1$, in which case $\mathscr{B}$ is a symplectic pairing on $S$. In this case, the condition $\iota_{\mathscr{B}} = -1$ implies that $S^+$ and $S^-$ are Lagrangian sub-bundles of the symplectic vector bundle $(S, \mathscr{B})$.



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**Remark.** Recall that (when $\gamma$ is reducible on the fibers) the spin projectors are given by $\mathcal{P}_\pm^\mathcal{R} = \frac{1}{2}(1 \pm \mathcal{R})$, where $\mathcal{R}$ is a product structure on $S$. It is easy to see that $\mathcal{R}$ satisfies the following relation, where $^t$ stands for the transpose with respect to an admissible pairing $\mathscr{B}$ on $S$:

$$\mathcal{R}^t = \iota_\mathscr{B}\mathcal{R} \iff (\mathcal{P}_\pm^\mathcal{R})^t = \mathcal{P}_{\pm\iota_\mathscr{B}}^\mathcal{R} \quad . \tag{3.8}$$

In fact, this relation can be used as an alternative definition of $\iota_\mathscr{B}$. In particular, an admissible pairing on $S$ has the property that $\mathcal{R}$ is either self-adjoint ($\iota_\mathscr{B} = +1$) or anti-selfadjoint ($\iota_\mathscr{B} = -1$) with respect to $\mathscr{B}$. For later reference, we also note that (3.5) implies the relation:

$$\gamma(\nu)^t = (-1)^{\left[\frac{d}{2}\right]}\epsilon_\mathscr{B}^d\gamma(\nu) \ , \tag{3.9}$$

where we noticed that $\tau(\nu) = (-1)^{\left[\frac{d}{2}\right]}\nu$ and $\pi^{\frac{1-\epsilon_\mathscr{B}}{2}}(\nu) = \epsilon_\mathscr{B}^d\nu$ .

**Local expressions.** Let $e^m$ be a pseudo-orthonormal local coframe of $(M, g)$ defined above an open subset $U \subset M$. Then property (3.5) amounts to:

$$(\gamma^m)^t = \epsilon_\mathscr{B}\gamma^m \iff \mathscr{B}(\gamma^m\xi, \xi') = \epsilon_\mathscr{B}\mathscr{B}(\xi, \gamma^m\xi') \ , \ \ \forall m = 1\dots d \ , \tag{3.10}$$

which in turn implies:

$$(\gamma^A)^t = \epsilon_\mathscr{B}^{|A|}(-1)^{\frac{|A|(|A|-1)}{2}}\gamma^A \tag{3.11}$$

and:

$$(\gamma^A)^{-t} = \epsilon_\mathscr{B}^{|A|}\gamma_A \iff \mathscr{B}((\gamma^A)^{-1}\xi, \xi') = \epsilon_\mathscr{B}^{|A|}\mathscr{B}(\xi, \gamma_A\xi') \ , \ \ \forall \xi, \xi' \in \Gamma(M, S) \ , \tag{3.12}$$

where $\epsilon_\mathscr{B}^{|A|} \stackrel{\text{def.}}{=} (\epsilon_\mathscr{B})^{|A|}$. If $(\varepsilon_i)_{i=1\dots N}$ is an arbitrary local frame of $S$ defined above $U$ (with dual local coframe of $S^*$ denoted by $(\varepsilon^i)_{i=1\dots N}$), then any $T \in \Gamma(M, \mathrm{End}(S))$ acts through:

$$T|_U\varepsilon_i = \sum_{j=1}^N T_{ji}\varepsilon^j = \sum_{j=1}^N \varepsilon^j(T\varepsilon_i)\epsilon_j$$

where $T_{ij} \stackrel{\text{def.}}{=} \varepsilon^i(T|_U\varepsilon_j) \in \mathcal{C}^\infty(U, \mathbb{R})$. This gives:

$$T(\xi) =_U \sum_{j=1}^N \varepsilon^j(T\xi|_U)\varepsilon_j \ , \ \ \forall \xi \in \Gamma(M, S) \ \text{ and } \ \mathrm{tr}(T|_U) = \sum_{i=1}^N T_{ii} = \sum_{i=1}^N \varepsilon^i(T|_U\varepsilon_i) \quad .$$

In particular, we have:

$$\gamma^A_{ij} \stackrel{\text{def.}}{=} \varepsilon^i(\gamma^A\varepsilon_j) \in \mathcal{C}^\infty(U, \mathbb{R}) \ \text{ and } \ \gamma^A(\varepsilon_i) = \sum_{j=1}^N (\gamma^A)_{ji}\varepsilon^j \ ,$$

with similar relations for $\gamma_A^{-1}$. The matrices $(\gamma^m_{ij})_{i,j=1\dots N}$ are the *gamma matrices* of $e^m$ (in fact, matrix-valued *functions*) with respect to the local frame $\varepsilon_i$ of $S$.

Let us give more detail on the admissible bilinear pairings $\mathscr{B}$ for each of the three types of real representations. If one drops the non-degeneracy assumptions, bilinear pairings on $S$ satisfying properties 1. − 3. above for any fixed $\sigma, \epsilon$ and $\iota$ form a free $\mathcal{C}^\infty(M, \mathbb{R})$-module



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whose rank depends on $p$ and $q$ (see [5, 6]). A convenient basis of this free module is provided by certain admissible bilinear pairings (determined up to multiplication with a nowhere vanishing function) which were constructed in loc. cit. Below, we give more detail on the properties of such admissible pairings.

### 3.2 Normal representation ($p - q \equiv_8 0, 1, 2$)

The spin projection exists only when $p - q \equiv_8 0$, with $\mathcal{R} = \gamma(\nu)$. The situation for admissible pairings is as follows.

**When $p - q \equiv_8 0, 2$, i.e. the normal simple case.** Up to multiplication by elements of $\mathcal{C}^\infty(M, \mathbb{R})$, there are two admissible pairings $\mathscr{B}_+$ and $\mathscr{B}_-$, which are distinguished by the value $\epsilon \in \{-1, 1\}$ of their type. Up to multiplication with a nowhere-vanishing function, we can take:[9]

$$\begin{aligned} \mathscr{B}_+ &= \mathscr{B}_- \circ (\mathrm{id}_S \otimes \gamma(\nu)) \\ \Leftrightarrow \mathscr{B}_- &= (-1)^{\frac{p-q}{2}} \mathscr{B}_+ \circ (\mathrm{id}_S \otimes \gamma(\nu)) = \begin{cases} +\mathscr{B}_+ \circ (\mathrm{id}_S \otimes \gamma(\nu))\,, & \text{if } p - q \equiv_8 0 \\ -\mathscr{B}_+ \circ (\mathrm{id}_S \otimes \gamma(\nu))\,, & \text{if } p - q \equiv_8 2 \end{cases} \end{aligned} \tag{3.13}$$

where we used that $d$ is even in this case and the relation $\gamma(\nu)^2 = (-1)^{q+\frac{d}{2}} \mathrm{id}_S = (-1)^{\frac{p-q}{2}} \mathrm{id}_S$ (which holds for $d$ =even). The symmetry $\sigma(\epsilon, d)$ of $\mathscr{B}_\epsilon$ is uniquely determined by $\epsilon$ and by the mod 8 reduction of $d$ according to the table:

| $d$ (mod 8) | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $\sigma(\epsilon, d)$ | $+1$ | $+\epsilon$ | $-1$ | $-\epsilon$ |

The isotropy type $\iota$ of $\mathscr{B}_\epsilon$ depends only on $p$ and $q$ and is as follows:

- When $p - q \equiv_8 0$, the restriction $\gamma_{\mathrm{ev}}$ is fiberwise reducible and $\iota = (-1)^{\frac{d}{2}} = \begin{cases} +1\,, & \text{if } d = 0, 4\,, \\ -1\,, & \text{if } d = 2, 6\,. \end{cases}$ Indeed, we have $\mathcal{R} = \gamma(\nu)$. Since $d$ is even for such $p, q$, $\mathcal{R}$ satisfies $\mathcal{R}^t = (-1)^{\frac{d}{2}} \mathcal{R}$ for any admissible pairing $\mathscr{B}$ on $S$ due to (3.9). Hence $\iota = (-1)^{\frac{d}{2}}$ (see (3.8)) for any admissible $\mathscr{B}$. Together with (3.13), this gives:
    1. When $d \equiv_8 0, 4$: $\mathscr{B}_+|_{S_\pm \otimes S_\mp} = \mathscr{B}_-|_{S_\pm \otimes S_\mp} = 0$ , $\mathscr{B}_+|_{S_\pm \otimes S_\pm} = \pm \mathscr{B}_-|_{S_\pm \otimes S_\pm}$
    2. When $d \equiv_8 2, 6$: $\mathscr{B}_+|_{S_\pm \otimes S_\pm} = \mathscr{B}_-|_{S_\pm \otimes S_\pm} = 0$ , $\mathscr{B}_+|_{S_\pm \otimes S_\mp} = \mp \mathscr{B}_-|_{S_\pm \otimes S_\mp}$, where we used the fact that $\gamma(\nu)|_{S_\pm} = \pm \mathrm{id}_{S^\pm}$.
- When $p - q \equiv_8 2$ , the restriction $\gamma_{\mathrm{ev}}$ is fiberwise irreducible, so $\iota$ is not defined.

The pairing $\mathscr{B}_+$ (which is uniquely determined up to multiplication by a nowhere-vanishing smooth function) will be called the *basic admissible pairing* and will also be denoted $\mathscr{B}_0$.

[9] Relation (3.13) implies $(\gamma^m)^{t_-} = \gamma(\nu)^{-t_+} \circ (\gamma^m)^{t_+} \circ \gamma(\nu)^{t_+}$, where $t_\pm$ denotes the transpose taken with respect to the pairing $\mathscr{B}_\pm$. Since $d$ is even in this case, it follows that $\gamma(\nu)$ anticommutes with all $\gamma^m$ and thus $(\gamma^m)^{t_-} = -(\gamma^m)^{t_+}$, so pairings related by (3.13) have opposite type, i.e. $\epsilon_- = -\epsilon_+$. Furthermore, we have $\gamma(\nu)^{t_+} = (-1)^{\frac{d}{2}} \gamma(\nu)$, which implies via relation (3.13) that the symmetries $\sigma_+$ and $\sigma_-$ are related through $\sigma_+ = (-1)^{\frac{d}{2}} \sigma_-$.

| $k$ | 1 | 2 | 3 |
|---|---|---|---|
| $\sigma_k/\sigma_0$ | $-(-1)^{\left[\frac{d}{2}\right]}$ | $+1$ | $(-1)^{\left[\frac{d}{2}\right]}$ |
| $\epsilon_k/\epsilon_0$ | $+1$ | $-1$ | $-1$ |
| $\iota_k/\iota_0$ | $-1$ | $+1$ | $-1$ |

**Table 9**. Characteristics of admissible bilinear forms in the almost complex case.

**When $p - q \equiv_8 1$, i.e. the normal non-simple case.** Then the admissible bilinear pairing $\mathscr{B}$ is unique up to multiplication by a nowhere vanishing smooth function[10] and $\gamma_{\rm ev}$ is fiberwise irreducible, so $\iota_{\mathscr{B}}$ is not defined. The values of $\sigma_{\mathscr{B}}$ and $\epsilon_{\mathscr{B}}$ are determined by the mod 8 reduction of $d$ as shown in the table below:

| $d$ (mod 8) | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $\sigma_{\mathscr{B}}$ | $+1$ | $-1$ | $-1$ | $+1$ |
| $\epsilon_{\mathscr{B}}$ | $+1$ | $-1$ | $+1$ | $-1$ |

### 3.3 Almost complex representation ($p - q \equiv_8 3, 7$)

In this case, there are four independent choices $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2$ and $\mathscr{B}_3$ for the nondegenerate admissible pairing, which we can take to be related through:

$$\mathscr{B}_1 = \mathscr{B}_0 \circ (\mathrm{id}_S \otimes J) \ , \quad \mathscr{B}_2 = \mathscr{B}_0 \circ (\mathrm{id}_S \otimes D) \ , \quad \mathscr{B}_3 = \mathscr{B}_0 \circ [\mathrm{id}_S \otimes (D \circ J)] \ . \tag{3.14}$$

Here, $\mathscr{B}_0$ (which we shall call the *basic admissible pairing*) is a particular choice of admissible pairing (determined up to multiplication by a nowhere-vanishing smooth real-valued function), with type $\epsilon_0 = -1$, whose symmetry $\sigma_0$ is given by:

| $d$ (mod 8) | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $\sigma_0$ | $+1$ | $-1$ | $-1$ | $+1$ |

and whose isotropy $\iota_0$ is as follows [6]:

- *When* $p - q \equiv_8 3$, the restriction $\gamma_{\rm ev}$ is fiberwise irreducible so $\iota_0$ is not defined.
- *When* $p - q \equiv_8 7$, the restriction $\gamma_{\rm ev}$ is fiberwise reducible and we have $\iota_0 = 1$, i.e. $\mathscr{B}_0|_{S_\pm \otimes S_\mp} = 0$.

The symmetry $\sigma_k$, type $\epsilon_k$ and isotropy $\iota_k$ (the latter being defined iff. $p - q \equiv_8 7$) of $\mathscr{B}_k$ for $k = 1 \ldots 3$ are related to those of $\mathscr{B}_0$ as shown in the table below.

[10] Recall that $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$ in this case, so $\mathscr{B} \circ (\mathrm{id}_S \otimes \gamma(\nu)) = \epsilon_\gamma \mathscr{B}$ is proportional to $\mathscr{B}$.



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**$\mathscr{B}_0$-symmetry properties of $J$ and $D$.** In all subcases of the almost complex case, the globally-defined endomorphisms $J = \gamma(\nu)$ and $D$ satisfy [2, 6]:

$$J^t = (-1)^{\frac{d(d+1)}{2}} J \ , \quad D^t = D \implies (D \circ J)^t = (-1)^{\frac{d(d-1)}{2}} D \circ J \ ,$$

where $^t$ denotes the transpose taken *with respect to the basic pairing* $\mathscr{B}_0$. In particular, $D$ is $\mathscr{B}_0$-selfadjoint. It is easy to check that these relations agree with the first row of table 9 if one uses the congruence:

$$\frac{d(d-1)}{2} \equiv_2 \left[\frac{d}{2}\right] \implies \frac{d(d+1)}{2} \equiv_2 1 + \left[\frac{d}{2}\right] \ .$$

Relations (2.23) and (3.3) imply:

$$D^{-t} = (-1)^{\frac{p-q+1}{4}} D \iff \mathscr{B}_0(D^{-1}\xi, \xi') = (-1)^{\frac{p-q+1}{4}} \mathscr{B}_0(\xi, D\xi') \ . \tag{3.15}$$

### 3.3.1 $\mathscr{B}_k$-symmetry properties

The symmetry properties of various endomorphisms of $S$ with respect to the other pairings $\mathscr{B}_k$ $(k = 1 \ldots 3)$ can be obtained using (3.14). Applying (3.3), we find that the transpositions $T^{t_k}$ of $T \in \Gamma(M, \mathrm{End}(S))$ with respect to the admissible pairings $\mathscr{B}_k$ $(k = 1 \ldots 3)$ are related to the transposition $T^t$ of $T$ with respect to $\mathscr{B}_0$ through:

$$\begin{aligned} T^{t_1} &= J^{-t} \circ T^t \circ J^t = -J \circ T^t \circ J \ , \\ T^{t_2} &= D^{-t} \circ T^t \circ D^t = (-1)^{\frac{p-q+1}{4}} D \circ T^t \circ D \ , \\ T^{t_3} &= D^{-t} \circ J^{-t} \circ T^t \circ J^t \circ D^t = -(-1)^{\frac{p-q+1}{4}} D \circ J \circ T^t \circ J \circ D \ . \end{aligned} \tag{3.16}$$

Relations (3.16) simplify further when $T$ is $\mathbb{C}$-linear or $\mathbb{C}$-antilinear, i.e. if $T$ commutes or anticommutes with $J$. In this case, we define:

$$\lambda_T \overset{\text{def.}}{=} \begin{cases} +1\,, & \text{if } [T,J]_{-,\circ} = 0\,, \\ -1\,, & \text{if } [T,J]_{+,\circ} = 0 \end{cases} \iff T \circ J = \lambda_T J \circ T$$

and find:

$$T^{t_1} = \lambda_T T^t \ , \quad T^{t_2} = (-1)^{\frac{p-q+1}{4}} D \circ T^t \circ D \ , \; T^{t_3} = -(-1)^{\frac{p-q+1}{4}} \lambda_T D \circ T^t \circ D \ ,$$

where we noticed that $T^t$ has the same (anti-)commutation properties with $D$ as $T$. Similarly, we find simplifications when $T$ commutes or anticommutes with $D$, in which case we define:

$$\delta_T \overset{\text{def.}}{=} \begin{cases} +1\,, & \text{if } [T,D]_{-,\circ} = 0 \\ -1\,, & \text{if } [T,D]_{+,\circ} = 0 \end{cases} \iff T \circ D = \delta_T D \circ T$$

and find:

$$T^{t_1} = -J \circ T^t \circ J \ , \; T^{t_2} = \delta_T T^t \ , \quad T^{t_3} = -\delta_T J \circ T^t \circ J \ ,$$

noticing now that $T^t$ has the same (anti-)commutation properties with $J$ as $T$. A case often encountered in applications is when $T$ is both $\mathbb{C}$-linear/antilinear and commutes/anticommutes with $D$. In this situation, both $\lambda_T$ and $\delta_T$ are defined and we find:

$$T^{t_1} = \lambda_T T^t \ , \quad T^{t_2} = \delta_T T^t \ , \quad T^{t_3} = \lambda_T \delta_T T^t \ .$$

**The $\mathscr{B}_\alpha$-transpose of $\gamma^A$.** A typical example of the last type mentioned in the previous paragraph is $T = \gamma^A$ for some ordered multi-index $A = (a_1, \ldots, a_k)$ of length $|A| = k$. Then $\lambda_{\gamma^A} = +1$ (since $\gamma^a$ commute with $J$) and $\delta_{\gamma^A} = (-1)^{|A|}$ (since $\gamma^a$ anti-commute with $D$). Since $\epsilon_0 = -1$, equation (3.11) becomes:

$$(\gamma^A)^t = (-1)^{\frac{|A|(|A|+1)}{2}} \gamma^A$$

and the relations above give:

$$\begin{aligned}
(\gamma^A)^{t_1} &= (-1)^{\frac{|A|(|A|+1)}{2}} \gamma^A \implies (\gamma^a)^{t_1} = -\gamma^a \ , \\
(\gamma^A)^{t_2} &= (-1)^{\frac{|A|(|A|-1)}{2}} \gamma^A \implies (\gamma^a)^{t_2} = +\gamma^a \ , \\
(\gamma^A)^{t_3} &= (-1)^{\frac{|A|(|A|-1)}{2}} \gamma^A \implies (\gamma^a)^{t_3} = +\gamma^a \ .
\end{aligned} \tag{3.17}$$

These identities agree with the second row of table 9. Relations (3.17) imply:

$$\begin{aligned}
\mathscr{B}_0(\xi, \gamma^A \xi') &= \sigma_0 (-1)^{\frac{|A|(|A|+1)}{2}} \mathscr{B}_0(\xi', \gamma^A \xi) \ , \\
\mathscr{B}_1(\xi, \gamma^A \xi') &= \sigma_1 (-1)^{\frac{|A|(|A|+1)}{2}} \mathscr{B}_1(\xi', \gamma^A \xi) \ , \\
\mathscr{B}_2(\xi, \gamma^A \xi') &= \sigma_2 (-1)^{\frac{|A|(|A|-1)}{2}} \mathscr{B}_2(\xi', \gamma^A \xi) \ , \\
\mathscr{B}_3(\xi, \gamma^A \xi') &= \sigma_3 (-1)^{\frac{|A|(|A|-1)}{2}} \mathscr{B}_3(\xi', \gamma^A \xi) \ .
\end{aligned} \tag{3.18}$$

### 3.3.2 The case $p - q \equiv_8 7$

Let us consider this situation in more detail, given that a spin projection can be defined in this case (leading to Majorana spinors). Viewing $S$ as a complex vector bundle (with complex structure given by $J$), recall that in this case $D$ is a real structure (complex conjugation) on $S$, which we also denote by an overline. For $p - q \equiv_8 7$, we always have $\iota_0 = 1$, which means that the basic admissible pairing $\mathscr{B}_0$ satisfies:

$$\mathscr{B}_0|_{S_\pm \otimes S_\mp} = 0 \implies \mathscr{B}_0(\xi_\pm, \xi'_\mp) = 0 \ , \quad \forall \xi, \xi' \in \Gamma(M, S) \tag{3.19}$$

and hence it is determined by its restrictions to $S_+ \otimes S_+$ and to $S_- \otimes S_-$. In turn, the second of these restrictions is determined by the first due to the first of identities (3.3), which imply:

$$\mathscr{B}_0(J\xi, J\xi') = (-1)^{\frac{d(d-1)}{2}} \mathscr{B}_0(\xi, \xi') \ , \quad \forall \xi, \xi' \in \Gamma(M, S)$$
$$\implies \mathscr{B}_0|_{S_- \otimes S_-} = (-1)^{\frac{d(d-1)}{2}} \mathscr{B}_0|_{S_+ \otimes S_+} \circ (J \otimes J)|_{S_- \otimes S_-}$$

Property (3.19) implies:

$$\begin{aligned}
\mathscr{B}_0(\xi, \xi') &= \mathscr{B}_0(\xi_+, \xi'_+) + \mathscr{B}_0(\xi_-, \xi'_-) \\
&= \mathscr{B}_0(\xi_R, \xi'_R) - (-1)^{\frac{d(d+1)}{2}} \mathscr{B}_0(\xi_I, \xi'_I) \ , \quad \forall \xi, \xi' \in \Gamma(M, S) \ ,
\end{aligned} \tag{3.20}$$

where we used decomposition (2.25) and the first of identities (3.3). Together with (2.7.3), this gives:

$$\mathscr{B}_0(\xi, \gamma(\omega)\xi') = \begin{cases} \mathscr{B}_0(\xi_R, \gamma(\omega)\xi'_R) - (-1)^{\frac{d(d+1)}{2}} \mathscr{B}_0(\xi_I, \gamma(\omega)\xi'_I) \ , & \text{if } \omega \in \Omega^{\mathrm{ev}}(M) \ , \\ \mathscr{B}_0(\xi_R, (J \circ \gamma(\omega))\xi'_I) + (-1)^{\frac{d(d+1)}{2}} \mathscr{B}_0(\xi_I, (J \circ \gamma(\omega))\xi'_R) \ , & \text{if } \omega \in \Omega^{\mathrm{odd}}(M) \ , \end{cases} \tag{3.21}$$

where we used (3.3) and the fact that $J$ and $\gamma(\omega)$ commute. On the other hand, we always have $\epsilon_0=-1$, which gives:

$$\gamma(\omega)^t=\gamma(\tau(\pi(\omega)))=\begin{cases}+\gamma(\tau(\omega))\,, & \text{if } \omega\in\Omega^{\mathrm{ev}}(M)\,,\\ -\gamma(\tau(\omega))\,, & \text{if } \omega\in\Omega^{\mathrm{odd}}(M)\,,\end{cases}$$

thereby implying:

$$\mathscr{B}_0(\xi,\gamma(\omega)\xi')=\sigma_0\mathscr{B}_0(\xi',\gamma((\tau\circ\pi)(\omega))\xi)\ \ ,\ \ \forall\xi,\xi'\in\Gamma(M,S)\ \ , \tag{3.22}$$

which in turn gives:

$$\mathscr{B}_0(\xi,\gamma^A\xi')=\sigma_0(-1)^{\frac{|A|(|A|+1)}{2}}\mathscr{B}_0(\xi',\gamma^A\xi)\ \ ,\ \ \forall\xi,\xi'\in\Gamma(M,S)\ \ . \tag{3.23}$$

The other admissible pairings (3.14) can be expressed as:

$$\begin{aligned}\mathscr{B}_1(\xi,\xi')\ &=\ \mathscr{B}_0(\xi,J\xi')=-\mathscr{B}_0(\xi_R,\xi'_I)-(-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,\xi'_R)\ \ ,\\ \mathscr{B}_2(\xi,\xi')\ &=\ \mathscr{B}_0(\xi,\overline{\xi'})=\mathscr{B}_0(\xi_R,\xi'_R)+(-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,\xi'_I)\ \ ,\\ \mathscr{B}_3(\xi,\xi')\ &=\ -\mathscr{B}_0(\xi,J(\overline{\xi'}))=-\mathscr{B}_0(\xi_R,\xi'_I)+(-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,\xi'_R)\ \ ,\end{aligned} \tag{3.24}$$

so in particular we have:

$$\begin{aligned}\mathscr{B}_1|_{S_\pm\otimes S_\pm}\ &=0\ \Longrightarrow\ \iota_1=-1\ \ ,\\ \mathscr{B}_2|_{S_\pm\otimes S_\mp}\ &=0\ \Longrightarrow\ \iota_2=+1\ \ ,\\ \mathscr{B}_3|_{S_\pm\otimes S_\pm}\ &=0\ \Longrightarrow\ \iota_3=-1\ \ ,\end{aligned}$$

which agrees with the third row of table 9. Also note the relations:

$$\begin{aligned}\mathscr{B}_0|_{S_+\otimes S_+}=\mathscr{B}_2|_{S_+\otimes S_+}\quad &,\quad \mathscr{B}_0|_{S_-\otimes S_-}=-\mathscr{B}_2|_{S_-\otimes S_-}\ \ ,\\ \mathscr{B}_1|_{S_+\otimes S_-}=\mathscr{B}_3|_{S_+\otimes S_-}\quad &,\quad \mathscr{B}_1|_{S_-\otimes S_+}=-\mathscr{B}_3|_{S_-\otimes S_+}\ \ .\end{aligned}$$

When combined with (3.20), identities (3.24) give the following expressions which we list for convenience of the reader:

$$\begin{aligned}\mathscr{B}_0(\xi_R,\xi'_R)\ &=\ \frac{1}{2}[\mathscr{B}_0(\xi,\xi')+\mathscr{B}_2(\xi,\xi')]\ ,\\ \mathscr{B}_0(\xi_I,\xi'_I)\ &=\ \frac{(-1)^{\frac{d(d+1)}{2}}}{2}[\mathscr{B}_2(\xi,\xi')-\mathscr{B}_0(\xi,\xi')]\,,\end{aligned} \tag{3.25}$$

$$\begin{aligned}\mathscr{B}_0(\xi_R,\xi'_I)\ &=\ -\frac{1}{2}[\mathscr{B}_1(\xi,\xi')+\mathscr{B}_3(\xi,\xi')]\ ,\\ \mathscr{B}_0(\xi_I,\xi'_R)\ &=\ \frac{(-1)^{\frac{d(d+1)}{2}}}{2}[\mathscr{B}_3(\xi,\xi')-\mathscr{B}_1(\xi,\xi')]\ .\end{aligned} \tag{3.26}$$

Relations (3.24) also imply the following relations which we list for completeness:

$$\mathscr{B}_0(\xi, J\circ\gamma(\omega)\xi') \\ = \begin{cases} -\mathscr{B}_0(\xi_R,\gamma(\omega)\xi'_I) - (-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,\gamma(\omega)\xi'_R), & \text{if } \omega\in\Omega^{\rm ev}(M)\,,\\ \mathscr{B}_0(\xi_R,(J\circ\gamma(\omega))\xi'_R) - (-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,(J\circ\gamma(\omega))\xi'_I), & \text{if } \omega\in\Omega^{\rm odd}(M)\,,\end{cases}$$

$$\mathscr{B}_0(\xi, D\circ\gamma(\omega)\xi') \\ = \begin{cases} \mathscr{B}_0(\xi_R,\gamma(\omega)\xi'_R) - (-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,\gamma(\omega)\xi'_I)\,, & \text{if } \omega\in\Omega^{\rm ev}(M)\,,\\ \mathscr{B}_0(\xi_R,(J\circ\gamma(\omega))\xi'_I) - (-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,(J\circ\gamma(\omega))\xi'_R)\,, & \text{if } \omega\in\Omega^{\rm odd}(M)\,,\end{cases}$$

$$\mathscr{B}_0(\xi, J\circ D\circ\gamma(\omega)\xi') \\ = \begin{cases} \mathscr{B}_0(\xi_R,\gamma(\omega)\xi'_I) - (-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,\gamma(\omega)\xi'_R), & \text{if } \omega\in\Omega^{\rm ev}(M)\,,\\ -\mathscr{B}_0(\xi_R,(J\circ\gamma(\omega))\xi'_R) - (-1)^{\frac{d(d+1)}{2}}\mathscr{B}_0(\xi_I,(J\circ\gamma(\omega))\xi'_I), & \text{if } \omega\in\Omega^{\rm odd}(M)\,.\end{cases}$$

**The case of Majorana spinors.** The expressions above simplify when $\xi,\xi'$ are Majorana spinors, i.e. sections of $S_+$ (that is, real sections of $S$ when the latter is viewed as the complexification of $S_+$), which means that $\xi_R=\xi$, $\xi'_R=\xi'$ and $\xi_I=\xi'_I=0$, i.e. $D(\xi)=\bar\xi=+\xi$ and $D(\xi')=\overline{\xi'}=+\xi'$. In this case, identities (3.21) become:

$$\mathscr{B}_0(\xi,\gamma(\omega)\xi')=0 \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ and } \ \omega\in\Omega^{\rm odd}(M) \tag{3.27}$$

while (3.24) give:

$$\mathscr{B}_1(\xi,\xi')=\mathscr{B}_3(\xi,\xi')=0 \ , \ \ \mathscr{B}_2(\xi,\xi')=\mathscr{B}_0(\xi,\xi') \ , \ \ \forall\xi,\xi'\in\Gamma(M,S_+) \ .$$

Notice that (3.25) and (2.7.3) imply the following relations, which generalize (3.27):

$$\begin{aligned}\mathscr{B}_0(\xi,\gamma(\omega)\xi') &= \mathscr{B}_2(\xi,\gamma(\omega)\xi')=0 \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ and } \ \omega\in\Omega^{\rm odd}(M) \ ,\\ \mathscr{B}_1(\xi,\gamma(\omega)\xi') &= \mathscr{B}_3(\xi,\gamma(\omega)\xi')=0 \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ and } \ \omega\in\Omega^{\rm ev}(M)\end{aligned} \tag{3.28}$$

and which can be summarized as:

$$\mathscr{B}_k(\xi,\gamma(\omega)\xi')=0 \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ and } \ \omega\in\Omega^{{\rm opar}(k)}(M) \ ,$$

where:

$${\rm opar}(k)\stackrel{\rm def.}{=}\begin{cases}{\rm ev}\,, & \text{if } k={\rm odd}\,,\\ {\rm odd}\,, & \text{if } k={\rm even}\,,\end{cases}$$

is the opposite of the parity of $k$. In particular, we have:

$$\mathscr{B}_k(\xi,\gamma^A\xi')=0 \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ unless } \ |A|\equiv_2 k \ . \tag{3.29}$$

We also have:

$$\begin{aligned}\mathscr{B}_0(\xi,\gamma(\omega)\xi') &= \mathscr{B}_2(\xi,\gamma(\omega)\xi') \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ and } \ \omega\in\Omega^{\rm ev}(M) \ ,\\ \mathscr{B}_1(\xi,\gamma(\omega)\xi') &= \mathscr{B}_3(\xi,\gamma(\omega)\xi') \ \text{ for } \ \xi,\xi'\in\Gamma(M,S_+) \ \text{ and } \ \omega\in\Omega^{\rm odd}(M) \ .\end{aligned} \tag{3.30}$$



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For a single Majorana spinor $\xi$, relations (3.18) imply the following properties, which are often encountered in applications:

$$\begin{aligned}
\mathscr{B}_0(\xi,\gamma^A\xi) &= 0 \;\text{ unless }\; (-1)^{\frac{|A|(|A|+1)}{2}} = \sigma_0 \;\;, \\
\mathscr{B}_1(\xi,\gamma^A\xi) &= 0 \;\text{ unless }\; (-1)^{\frac{|A|(|A|+1)}{2}} = \sigma_1 \;\;, \\
\mathscr{B}_2(\xi,\gamma^A\xi) &= 0 \;\text{ unless }\; (-1)^{\frac{|A|(|A|-1)}{2}} = \sigma_2 \;\;, \\
\mathscr{B}_3(\xi,\gamma^A\xi) &= 0 \;\text{ unless }\; (-1)^{\frac{|A|(|A|-1)}{2}} = \sigma_3
\end{aligned} \tag{3.31}$$

and which can be summarized as:

$$\mathscr{B}_k(\xi,\gamma^A\xi) = 0 \;\text{ unless }\; (-1)^{\frac{|A|(|A|-1)}{2}}\epsilon_k^{|A|} = \sigma_0 \;\;. \tag{3.32}$$

### 3.3.3 Local expressions

Consider a pseudo-orthonormal local coframe $(e^a)_{a=1\ldots d}$ of $(M,g)$ and a local frame $(\epsilon_\alpha)_{\alpha=1\ldots\Delta}$ of $S_+$, both supported above an open subset $U$ of $M$. Defining:

$$\mathscr{B}_{\alpha\beta} \stackrel{\text{def.}}{=} \mathscr{B}_0(\epsilon_\alpha,\epsilon_\beta) \in \mathcal{C}^\infty(U,\mathbb{R}) \;\;,\;\; \forall \alpha,\beta = 1\ldots\Delta \;\;,$$

we let $\hat{\mathscr{B}}$ denote the $\mathcal{C}^\infty(U,\mathbb{R})$ square matrix of dimension $\Delta$ with entries $\mathscr{B}_{\alpha\beta}$. The symmetry property of $\mathscr{B}$ implies:

$$\mathscr{B}_{\beta\alpha} = \sigma_0\mathscr{B}_{\alpha\beta} \;\;,\;\; \forall\alpha,\beta = 1\ldots\Delta \iff \hat{\mathscr{B}}^T = \sigma_0\hat{\mathscr{B}} \;\;,$$

where $^T$ denotes the ordinary transpose of matrices. If $\xi,\xi' \in \Gamma(M,S_+)$ expand as in (2.27) (with $\xi_I^\alpha = 0$, $\xi^\alpha = \xi_R^\alpha \in \mathcal{C}^\infty(U,\mathbb{R})$ and similarly for $\xi'$), we have:

$$\mathscr{B}_0(\xi,\xi') =_U \hat{\xi}^T\hat{\mathscr{B}}\hat{\xi}' \in \mathcal{C}^\infty(U,\mathbb{R}) \;\;.$$

**Remark.** As mentioned above, one can view the complex vector bundle $S$ as the compexification $S \approx S_+ \otimes \mathcal{O}_\mathbb{C}$ of the real vector bundle $S_+$, where $\mathcal{O}_\mathbb{C}$ is the trivial complex line bundle over $M$. With this interpretation, $D$ is the complex conjugation of this complexified bundle and we can consider the fiberwise $\mathbb{C}$-bilinear pairing $\beta$ on $S$ obtained by complexification of the restriction $\mathscr{B}_0|_{S_+\otimes S_+}$:

$$\begin{aligned}
\beta(\xi,\xi') \stackrel{\text{def.}}{=} \mathscr{B}_0(\xi_R,\xi'_R) - \mathscr{B}_0(\xi_I,\xi'_I) + i\left[\mathscr{B}_0(\xi_R,\xi'_I) + \mathscr{B}_0(\xi_I,\xi'_R)\right]& \\
\in \mathcal{C}^\infty(M,\mathbb{C}) \;\;,\;\; \forall \xi,\xi' \in \Gamma(M,S) \;\;,&
\end{aligned} \tag{3.33}$$

i.e. (using (3.24)):

$$\begin{aligned}
2\beta(\xi,\xi') = \;& \mathscr{B}_0(\xi,\xi') + \mathscr{B}_2(\xi,\xi') + (-1)^{\frac{d(d+1)}{2}}(\mathscr{B}_0(\xi,\xi') - \mathscr{B}_2(\xi,\xi')) \\
& -i\left[\mathscr{B}_1(\xi,\xi') + \mathscr{B}_3(\xi,\xi') + (-1)^{\frac{d(d+1)}{2}}(\mathscr{B}_1(\xi,\xi') - \mathscr{B}_3(\xi,\xi'))\right] \;\;.
\end{aligned}$$

This fiberwise $\mathbb{C}$-bilinear form satisfies:

$$\begin{aligned}
\beta(J\xi,\xi') = \beta(\xi,J\xi') = i\beta(\xi,\xi') \;\;,\;\; & \forall \xi,\xi' \in \Gamma(M,S) \;\;, \\
\beta(\xi,\xi') = \mathscr{B}_0(\xi,\xi') \;\;,\;\; & \forall \xi,\xi' \in \Gamma(M,S_+) \;\;.
\end{aligned}$$

Using the relations above, it is easy to check we have the following local expression for any pair of pinors $\xi, \xi' \in \Gamma(M, S)$:

$$\beta(\xi, \gamma^A \xi') = \hat{\xi}^T \hat{\gamma}^A \hat{\xi}' \quad . \tag{3.34}$$

We stress that the fiberwise $\mathbb{C}$-bilinear pairing $\beta$ on $S$ is often used in the physics literature instead of the admissible form $\mathscr{B}_0$, which is only fiberwise $\mathbb{R}$-bilinear. The admissible forms $\mathscr{B}_k$ can be reconstructed from $\beta$ as follows:

$$\begin{aligned}
\mathscr{B}_0(\xi, \xi') &= \beta(\xi_R, \xi'_R) - (-1)^{\frac{d(d+1)}{2}} \beta(\xi_I, \xi'_I) \ , \\
\mathscr{B}_1(\xi, \xi') &= -\beta(\xi_R, \xi'_I) - (-1)^{\frac{d(d+1)}{2}} \beta(\xi_I, \xi'_R) \ , \\
\mathscr{B}_2(\xi, \xi') &= \beta(\xi_R, \xi'_R) + (-1)^{\frac{d(d+1)}{2}} \beta(\xi_I, \xi'_I) \ , \\
\mathscr{B}_3(\xi, \xi') &= -\beta(\xi_R, \xi'_I) + (-1)^{\frac{d(d+1)}{2}} \beta(\xi_I, \xi'_R) \ .
\end{aligned}$$

### 3.4 Quaternionic representation ($p - q \equiv_8 4, 5, 6$)

In this case, spin projectors can only be defined when $p - q \equiv_8 4, 6$. In order to simplify notation, we shall assume that $M$ is contractible, so that all bundles under consideration are topologically trivial; in particular, $J_i$ are globally defined on $M$. Given that the discussion below is local, this condition can always be removed by replacing $M$ with a sufficiently small open subset $U$. The situation for admissible pairings is summarized below [2, 5, 6].

#### 3.4.1 The quaternionic simple case ($p - q \equiv_8 4, 6$)

When $p - q \equiv_8 4, 6$, the morphism $\gamma$ is fiberwise-injective and we have eight independent admissible pairings $\mathscr{B}^\epsilon_\alpha$ ($\epsilon = \pm 1,\ \alpha = 0 \dots 3$), which are given by [6]:

$$\mathscr{B}^\epsilon_k = \mathscr{B}^\epsilon_0 \circ (\mathrm{id}_S \otimes J_k) \ , \quad \forall k = 1 \dots 3 \ , \quad \forall \epsilon \in \{-1, +1\} \ , \tag{3.35}$$

where $\mathscr{B}^\epsilon_0$ are the two *fundamental admissible pairings* — which we can take to be related through:

$$\begin{aligned}
&\mathscr{B}^+_0 = \mathscr{B}^-_0 \circ (\mathrm{id}_S \otimes \gamma(\nu)) \qquad\qquad (3.36) \\
\Longleftrightarrow\ &\mathscr{B}^-_0 = (-1)^{\frac{p-q}{2}} \mathscr{B}^+_0 \circ (\mathrm{id}_S \otimes \gamma(\nu)) = \begin{cases} +\mathscr{B}^+_0 \circ (\mathrm{id}_S \otimes \gamma(\nu)) \ , & \text{if } p - q \equiv_8 4 \\ -\mathscr{B}^+_0 \circ (\mathrm{id}_S \otimes \gamma(\nu)) \ , & \text{if } p - q \equiv_8 6 \end{cases}
\end{aligned}$$

and have type $\epsilon$, symmetry $\sigma_0(\epsilon, d)$ given in the table below:

| $d$ (mod 8) | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| $\sigma_0(\epsilon, d)$ | $-1$ | $-\epsilon$ | $+1$ | $+\epsilon$ |

and isotropy given as follows:

- If $p-q \equiv_8 4$, then $\gamma_{\rm ev}$ is fiberwise reducible and $\gamma(\nu)^2 = \mathrm{id}_S$. Chiral projectors can be constructed from the product structure $\mathcal{R} = \gamma(\nu)$, which satisfies $\mathcal{R}^t = (-1)^{\frac{d}{2}}\mathcal{R}$ with respect to any admissible pairing $\mathscr{B}$ due to (3.9) since $d$ is even. Thus $\iota = (-1)^{\frac{d}{2}}$ (see (3.8)) for any admissible $\mathscr{B}$ and in particular $\iota_0^\pm = (-1)^{\frac{d}{2}}$. In this case, we have $E_k(S_\pm) = S_\pm$, where $S_\pm$ are the bundles of *symplectic Majorana-Weyl spinors* of positive and negative chiralities (which are defined as the eigenbundles of $\gamma(\nu)$ corresponding to the eigenvalues $\pm 1$ of the latter).

- If $p-q \equiv_8 6$, then $\gamma_{\rm ev}$ is fiberwise reducible and $\gamma(\nu)^2 = -\mathrm{id}_S$, so $\gamma(\nu)$ is a globally-defined complex structure on $S$. Spin projectors can be constructed using the product structure $\mathcal{R} = \gamma(\nu) \circ J$, where $J \in \Gamma(U, \mathcal{U}_\gamma)$ is any locally-defined complex structure associated with the quaternionic structure. The eigenbundles of $\mathcal{R}$ corresponding to the eigenvalues $\pm 1$ are denoted by $S_\pm$ and $S_+$ is the bundle of *symplectic Majorana spinors*. We have $\mathcal{R}^t = -(-1)^{\frac{d}{2}}\mathcal{R}$ where ${}^t$ denotes the $\mathscr{B}_0^\epsilon$-transpose and hence $\iota_0^\epsilon = -(-1)^{\frac{d}{2}}$ (independent of $\epsilon$).

The pairing $\mathscr{B}_0^+$ will also be denoted through $\mathscr{B}_0$:

$$\mathscr{B}_0 \stackrel{\rm def.}{=} \mathscr{B}_0^+$$

and will be called the *basic pairing*. The pairings $\mathscr{B}_k^\epsilon$ for $k = 1\ldots 3$ will be called *the derived pairings*. Notice that relations (3.35) and (3.36) imply:

$$\mathscr{B}_\alpha^\epsilon = \mathscr{B}_0 \circ (\mathrm{id}_S \otimes \gamma(\nu)^{\frac{1-\epsilon}{2}} \circ J_\alpha) \ , \quad \forall \alpha = 0\ldots 3 \ , \quad \forall \epsilon = \pm 1 \ , \tag{3.37}$$

so that it suffices to work with the basic pairing $\mathscr{B}_0$.

**Properties of derived pairings in the quaternionic simple case.** The type $\epsilon_k(\epsilon, d)$ and symmetry $\sigma_k(\epsilon, d)$ of $\mathscr{B}_k^\epsilon$ $(k = 1\ldots 3)$ satisfy:

$$\epsilon_k(\epsilon, d) = \epsilon_0(\epsilon, d) \ , \quad \sigma_k(\epsilon, d) = -\sigma_0(\epsilon, d)$$

while the isotropies of $\mathscr{B}_k^\epsilon$ are as follows:

- When $p-q \equiv_8 4$, we have:

$$\iota_k^\epsilon = \iota_0^\epsilon = (-1)^{\frac{d}{2}} \ , \quad \forall k = 1\ldots 3 \ , \quad \forall \epsilon \in \{-1, 1\} \ .$$

- When $p-q \equiv_8 6$, we have the following isotropies if we define spin projectors using the product structure $\mathcal{R} = \gamma(\nu) \circ J_1$:

| $k$ | 1 | 2 | 3 |
|---|---|---|---|
| $\iota_k^\pm / \iota_0^\pm$ | $+1$ | $-1$ | $-1$ |

Also notice the relations $J_1(S_\pm) = S_\pm$ and $J_2(S_\pm) = S_\mp$, $J_3(S_\pm) = S_\mp$.

### 3.4.2 The quaternionic non-simple case ($p - q \equiv_8 5$)

In this case, $\gamma$ is fiberwise non-injective and we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, where $\epsilon_\gamma \in \{-1, 1\}$ is the signature of $\gamma$. We have four admissible nondegenerate bilinear pairings (up to multiplication with a nowhere vanishing function), given by:

$$\mathscr{B}_k = \mathscr{B}_0 \circ (\mathrm{id}_S \otimes J_k) \ , \quad \forall k = 1 \ldots 3 \ , \tag{3.38}$$

where $\mathscr{B}_0$ is the *basic admissible pairing*, whose type $\epsilon_0(d)$ and symmetry $\sigma_0(d)$ are given in the table below:

| $d$ (mod 8) | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $\epsilon_0(d)$ | $+1$ | $-1$ | $+1$ | $-1$ |
| $\sigma_0(d)$ | $-1$ | $+1$ | $+1$ | $-1$ |

In the quaternionic non-simple case, the restricted bundle morphism $\gamma_{\mathrm{ev}}$ is fiberwise irreducible so the isotropy of admissible pairings on $S$ is not defined. The type $\epsilon_k(d)$ and symmetry $\sigma_k(d)$ of $\mathscr{B}_k$ ($k = 1 \ldots 3$) are given by:

$$\epsilon_k(d) = \epsilon_0(d) \ , \quad \sigma_k(d) = -\sigma_0(d) \ .$$

Note that relation (3.38) implies:

$$\mathscr{B}_\alpha = \mathscr{B}_0 \circ (\mathrm{id}_S \otimes J_\alpha) \ , \quad \forall \alpha = 0 \ldots 3 \ , \tag{3.39}$$

since $J_0 = \mathrm{id}_S$.

### 3.4.3 The $\mathscr{B}_0$-transpose of $J_\alpha$

In all sub-cases of the quaternionic case, the globally-defined endomorphisms $J_\alpha$ are $\mathscr{B}_0^\epsilon$-orthogonal (see [2, 6]) for $p - q \equiv_8 4, 6$ and $\mathscr{B}_0$-orthogonal for $p - q \equiv_8 5$:

$$(J_\alpha)^{-t} = J_\alpha \iff \mathscr{B}_0(J_\alpha^{-1}\xi, \xi') = \mathscr{B}_0(\xi, J_\alpha \xi') \ , \quad \forall \alpha = 0 \ldots 3 \ , \tag{3.40}$$

where ${}^t$ denotes the $\mathscr{B}_0^\epsilon$-transpose (for any $\epsilon = \pm 1$) when $p - q \equiv_8 4, 6$ or the $\mathscr{B}_0$-transpose, when $p - q \equiv_8 5$. Since $J_k^2 = -\mathrm{id}_S$, we have $J_k^{-1} = -J_k$ and hence:

$$J_k^t = -J_k \ , \quad \forall k = 1 \ldots 3 \ , \tag{3.41}$$

which implies:

$$J^t = -J \ , \quad \forall J \in \Gamma(U, \mathcal{U}_\gamma) \ .$$

### 3.4.4 Admissible pairings in the biquaternion formalism

Possibly replacing $M$ with a sufficiently small open subset, we assume for simplicity of notation that $M$ is contractible and hence $J_i$ are globally defined. Passing to the complexified bundle $S_\mathbb{C} = S \otimes \mathcal{O}_\mathbb{C}$, the bilinear pairings $\mathscr{B}_\alpha^\epsilon$ (for the quaternionic simple case) and $\mathscr{B}_\alpha$(for the quaternionic non-simple case) complexify to fiberwise $\mathbb{C}$-bilinear pairings $\beta_\alpha^\epsilon$ and $\beta_\alpha$ on $S_\mathbb{C}$, respectively. Relations (3.37) and (3.39) give:

$$\beta_\alpha^\epsilon = \mathscr{B}_0^\epsilon \circ (\mathrm{id}_{S_\mathbb{C}} \otimes \mathfrak{J}_\alpha) \ \ (\text{for } p-q \equiv_8 4,6) \ \ \text{and} \ \ \beta_\alpha = \mathscr{B}_0 \circ (\mathrm{id}_{S_\mathbb{C}} \otimes \mathfrak{J}_\alpha) \ \ (\text{for } p-q \equiv_8 5) \ .$$

Of course, the complexified basic pairing $\beta_0$ has the same symmetry $\sigma_0$ as $\mathscr{B}_0$. Using a $\beta_0$-orthogonal local frame of $S_\mathbb{C}$ when $\sigma_0 = +1$ and a $\beta_0$-symplectic (i.e. Darboux) local frame of $S_\mathbb{C}$ when $\sigma_0 = -1$, the various relations given above translate immediately into matrix identities familiar from the supergravity literature. We refer the reader to section 5 for an example of this.

# 4 Fierz identities for real pinors

## 4.1 Preparations

Given an admissible fiberwise bilinear pairing $\mathscr{B}$ on $S$, we define endomorphisms $E_{\xi,\xi'} \in \Gamma(M, \mathrm{End}(S)) \approx \mathrm{Hom}_{\mathcal{C}^\infty(M,\mathbb{R})}(\Gamma(M,S), \Gamma(M,S))$ through:

$$E_{\xi,\xi'}(\xi'') \stackrel{\text{def}}{=} \mathscr{B}(\xi'', \xi')\xi \ \ , \ \ \forall \xi, \xi' \in \Gamma(M,S) \ \ .$$

It is easy to check the identities:

$$E_{\xi_1,\xi_2} \circ E_{\xi_3,\xi_4} = \mathscr{B}(\xi_3, \xi_2) E_{\xi_1,\xi_4} \ \ , \ \ \forall \xi_1, \xi_2, \xi_3, \xi_4 \in \Gamma(M,S) \ \ , \tag{4.1}$$

as well as:

$$\mathrm{tr}(T \circ E_{\xi,\xi'}) = \mathscr{B}(T\xi, \xi') \ \ , \ \ \forall \xi, \xi' \in \Gamma(M,S) \ \ . \tag{4.2}$$

For later reference, note that the bundle $\mathrm{End}(S)$ is endowed with the natural nondegenerate and symmetric fiberwise bilinear pairing $\langle \, , \, \rangle$ whose action on sections is the $\mathcal{C}^\infty(M,\mathbb{R})$-bilinear map given by:

$$\langle A, B \rangle \stackrel{\text{def.}}{=} \mathrm{tr}(A \circ B) \in \mathcal{C}^\infty(M,\mathbb{R}) \ \ , \ \ \forall A, B \in \Gamma(M, \mathrm{End}(S)) \ \ . \tag{4.3}$$

Notice that this pairing does not depend on any choice of pairing of the bundle $S$ itself. Given $T \in \Gamma(M, \mathrm{End}(S))$, we define the operators of *left and right composition* with $T$ to be the following $\mathcal{C}^\infty(M,\mathbb{R})$-linear operators acting in $\Gamma(M, \mathrm{End}(S))$:

$$L_T(A) \stackrel{\text{def.}}{=} T \circ A \ \ , \ \ R_T(A) \stackrel{\text{def.}}{=} A \circ T \ \ , \ \ \forall A \in \Gamma(M, \mathrm{End}(S)) \ \ . \tag{4.4}$$

Cyclicity of the trace implies that $L_T$ and $R_T$ are adjoint with respect to $\langle \, , \, \rangle$:

$$\langle L_T(A), B \rangle = \langle A, R_T(B) \rangle \ \ , \ \ \forall T, A, B \in \Gamma(M, \mathrm{End}(S)) \ \ .$$

## 4.2 Fierz identities for the normal case

In this subsection, we consider the normal case, which corresponds to $p - q \equiv_8 0, 1, 2$. Since this was already discussed in [1] and [8] (see also [2–4, 9]), we shall be brief.

### 4.2.1 The completeness relation

We start from the local completeness relation (equation (2.7) of [3], see also [1, 2, 8]):

$$\sum_{A=\text{ordered}} (\gamma_A^{-1})_{jk}(\gamma_A)_{lm} = \frac{2^d}{N}\delta_{jm}\delta_{lk} \ , \tag{4.5}$$

where $A$ runs over increasingly-ordered tuples of length $|A| = 1\ldots d$. Multiplying both sides of (4.5) by $T_{kj}$ and summing over $j,k$ gives:

**Proposition.** We have the *completeness relation for the normal case*:

$$T =_U \frac{N}{2^d} \sum_{A=\text{ordered}} \text{tr}(\gamma_A^{-1} \circ T)\gamma_A \ , \quad \forall T \in \Gamma(M, \text{End}(S)) \ . \tag{4.6}$$

Setting $T = E_{\xi,\xi'}$ in this relation gives the expansion:

$$E_{\xi,\xi'} =_U \frac{N}{2^d} \sum_{A=\text{ordered}} \text{tr}(\gamma_A^{-1} \circ E_{\xi,\xi'})\gamma_A \ ,$$

which also takes the following form upon using identities (4.2) and (3.12):

$$E_{\xi,\xi'} =_U \frac{N}{2^d} \sum_{A=\text{ordered}} \epsilon_{\mathscr{B}}^{|A|} \mathscr{B}(\xi, \gamma^A \xi')\gamma_A \ . \tag{4.7}$$

### 4.2.2 The geometric Fierz identities

Relation (4.7) implies that the inhomogeneous differential forms:

$$\check{E}_{\xi,\xi'} \overset{\text{def.}}{=} \left(\gamma|_{\Omega^\gamma(M)}\right)^{-1}(E_{\xi,\xi'}) \in \Omega^\gamma(M)$$

have the expansion:

$$\check{E}_{\xi,\xi'} =_U \frac{N}{2^d} \sum_{A=\text{ordered}} \epsilon_{\mathscr{B}}^{|A|} \mathscr{B}(\xi, \gamma_A \xi') e_\gamma^A \ , \quad \forall \xi, \xi' \in \Gamma(M, S) \ , \tag{4.8}$$

where we used (2.20). We shall use the notation:

$$\check{E}_{\xi,\xi'} = \frac{N}{2^d} \sum_{k=0}^{d} \check{\boldsymbol{E}}^{(k)}_{\xi,\xi'} \ , \tag{4.9}$$

where:

$$\check{\boldsymbol{E}}^{(k)}_{\xi,\xi'} \overset{\text{def.}}{=}_U \frac{1}{k!} \epsilon_{\mathscr{B}}^k \mathscr{B}(\xi, \gamma_{a_1\ldots a_k}\xi') e_\gamma^{a_1\ldots a_k} \ \in \Omega^k(M) \cap \Omega^\gamma(M) \ .$$

The geometric Fierz identities amount to:

$$\check{E}_{\xi_1,\xi_2} \diamond \check{E}_{\xi_3,\xi_4} = \mathscr{B}(\xi_3, \xi_2)\check{E}_{\xi_1,\xi_4} \ , \quad \forall \xi_1, \xi_2, \xi_3, \xi_4 \in \Gamma(M, S) \ , \tag{4.10}$$

an equality which holds in $\Omega^\gamma(M)$. We refer the reader to [1, 8] for further details regarding this case.

### 4.3 Fierz identities for the almost complex case

This is the case $p - q \equiv_8 3, 7$ (which implies that $d = p + q$ is odd). In this situation, we always have $\epsilon_{\mathscr{B}_0} = -1$ (see section 3) and:

$$N = 2\Delta = 2^{[\frac{d}{2}]+1} \quad . \tag{4.11}$$

The Schur algebra $\mathbb{S}$ is isomorphic with the $\mathbb{R}$-algebra $\mathbb{C}$ of complex numbers, while the $\mathcal{C}^\infty(M, \mathbb{R})$-algebra $\Gamma(M, \Sigma_\gamma)$ is spanned by the operators $\mathrm{id}_S$ and $J \in \Gamma(M, \mathrm{End}(S))$, where $J$ is a complex structure on $S$ which lies in the commutant of the image of $\gamma$:

$$J^2 = -1 \ , \quad [J, \gamma_m]_{-,\circ} = 0 \ , \quad \forall m = 1 \ldots d \implies [J, \gamma_A]_{-,\circ} = 0 \ \text{ for all } A \quad . \tag{4.12}$$

We have:

$$J = \gamma(\nu) = \gamma^{(d+1)} \tag{4.13}$$

and $\nu \diamond \nu = -1$ (which, of course, implies the property $J^2 = -\mathrm{id}_S$ listed above). As shown in [2], there always exists a globally-defined endomorphism $D \in \Gamma(M, \mathrm{End}(S))$ which satisfies (2.22), (2.23) and (2.24). The space $\Gamma(M, S)$ admits a structure of $\mathcal{C}^\infty(M, \mathbb{C})$-module defined through:

$$(\alpha + i\beta)\xi \stackrel{\text{def.}}{=} \alpha\xi + \beta J(\xi) \ , \quad \forall \alpha, \beta \in \mathcal{C}^\infty(M, \mathbb{C}) \ , \quad \forall \xi \in \Gamma(M, S) \quad .$$

The module $\Gamma(M, \mathrm{End}_{\mathbb{C}}(S))$ of $\mathcal{C}^\infty(M, \mathbb{C})$-linear operators can be identified with the submodule of $\Gamma(M, \mathrm{End}(S))$ consisting of those $\mathcal{C}^\infty(M, \mathbb{R})$-linear operators which commute with $J$:

$$\Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \equiv \{T \in \Gamma(M, \mathrm{End}(S)) \mid [J, T]_{-,\circ} = 0\} \quad . \tag{4.14}$$

The map:

$$\gamma : \Omega(M) \longrightarrow \Gamma(M, \mathrm{End}(S))$$

is always injective, with image:

$$\gamma(\Omega(M)) = \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \quad .$$

Its partial inverse is given by:

$$\gamma^{-1} \stackrel{\text{def.}}{=} (\gamma|^{\Gamma(M, \mathrm{End}_{\mathbb{C}}(S))})^{-1} : \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \to \Omega(M) \quad . \tag{4.15}$$

#### 4.3.1 Preparations

Consider the following $\mathcal{C}^\infty(M, \mathbb{R})$-linear operator acting in $\Gamma(M, \mathrm{End}(S))$:

$$\mathcal{J}(T) \stackrel{\text{def.}}{=} J \circ T \circ J \ , \quad \forall T \in \Gamma(M, \mathrm{End}(S)) \quad .$$

The identity $J^2 = -\mathrm{id}_S$ implies:

$$\mathcal{J}^2 = \mathrm{id}_{\Gamma(M, \mathrm{End}(S))} \ ,$$

which shows that $\mathcal{J}$ is a product structure on the $\mathcal{C}^\infty(M,\mathbb{R})$-module $\Gamma(M,\mathrm{End}(S))$. On the other hand, direct computation using cyclicity of the trace shows that $\mathcal{J}$ is self-adjoint with respect to the natural pairing (4.3) on $\mathrm{End}(S)$:

$$\langle \mathcal{J}(A), B\rangle = \langle A, \mathcal{J}(B)\rangle \ , \ \ \forall A, B \in \Gamma(M,\mathrm{End}(S)) \ .$$

It follows that the operators:

$$\Pi_\pm \stackrel{\mathrm{def.}}{=} \frac{1}{2}(\mathrm{id}_{\Gamma(M,\mathrm{End}(S))} \pm \mathcal{J})$$

are complementary $\langle\,,\,\rangle$-orthoprojectors on the space $\Gamma(M,\mathrm{End}(S))$. In particular, we have:

$$\Pi_\pm^2 = \Pi_\pm \ , \ \ \Pi_+\circ\Pi_- = \Pi_-\circ\Pi_+ \ , \ \ \Pi_+ + \Pi_- = \mathrm{id}_{\Gamma(M,\mathrm{End}(S))}$$

and the submodules:

$$\begin{aligned}\Gamma(M,\mathrm{End}(S))^\pm &\stackrel{\mathrm{def.}}{=} \Pi_\pm(\Gamma(M,\mathrm{End}(S))) \\ &= \{T\in\Gamma(M,\mathrm{End}(S)) \mid \mathcal{J}(T) = \pm T\} \subset \Gamma(M,\mathrm{End}(S)) \qquad (4.16)\end{aligned}$$

provide a $\langle\,,\,\rangle$-orthogonal direct sum decomposition:

$$\Gamma(M,\mathrm{End}(S)) = \Gamma(M,\mathrm{End}(S))^+ \oplus \Gamma(M,\mathrm{End}(S))^- \ .$$

**Corollary.** Every $T\in\Gamma(M,\mathrm{End}(S))$ decomposes uniquely as:

$$T = T_+ + T_- \ \ \text{with} \ \ T_\pm = \Pi_\pm(T) = \frac{1}{2}(T\pm\mathcal{J}(T)) \in \Gamma(M,\mathrm{End}(S))^\pm \ . \qquad (4.17)$$

**Proposition.** We have:

$$\begin{aligned}\Gamma(M,\mathrm{End}(S))^- &= \{T\in\Gamma(M,\mathrm{End}(S)) | [J,T]_{-,\circ} = 0\} \equiv \Gamma(M,\mathrm{End}_{\mathbb{C}}(S)) \ , \qquad (4.18)\\ \Gamma(M,\mathrm{End}(S))^+ &= \{T\in\Gamma(M,\mathrm{End}(S)) | [J,T]_{+,\circ} = 0\} \ ,\end{aligned}$$

where, in the first relation, we used the identification (4.14).

*Proof.* It suffices to prove the first equality, since the second follows similarly. For this, consider the two inclusions in turn:

($\subset$) If $[J,T]_{-,\circ} = 0$ then direct computation shows that $\mathcal{J}(T) = -T$, so that $T\in\Gamma(M,\mathrm{End}(S))^-$ (see (4.16)).

($\supset$) If $T\in\Gamma(M,\mathrm{End}(S))^-$, the relation $\mathcal{J}(T) = -T$ (see (4.16)) reads:

$$J\circ T\circ J = -T \ ,$$

which implies $T\circ J = J\circ T$ upon composing with $J$ and using $J^2 = -\mathrm{id}_S$. Thus $[J,T]_{-,\circ} = 0$. □



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**Proposition.** We have:

$$L_D(\Gamma(M, \mathrm{End}(S))^{\pm}) = \Gamma(M, \mathrm{End}(S))^{\mp} \ , \tag{4.19}$$

where $L_D$ is the $\mathcal{C}^\infty(M, \mathbb{R})$-linear operator of left composition with $T$ (see (4.4)).

*Proof.* Relation (2.24) implies:

$$L_D \circ \Pi_{\pm} = \Pi_{\mp} \circ L_D \ ,$$

which immediately gives the conclusion. □

Since $\mathrm{id}_S \in \Gamma(M, \mathrm{End}(S))^-$ and $L_D(\mathrm{id}_S) = D$, the Proposition gives the following:

**Corollary.** We have:

$$D \in \Gamma(M, \mathrm{End}_{\mathbb{C}}(S))^+ \ ,$$

as well as:

**Corollary.** Every $T \in \Gamma(M, \mathrm{End}(S))$ decomposes uniquely as:

$$T = T_0 + D \circ T_1 \quad \text{with} \quad T_0, T_1 \in \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \ , \tag{4.20}$$

where we used the identification (4.14) .

*Proof.* Follows immediately from (4.17) and (4.19). □

**Proposition.** The following identities hold:

$$\mathrm{tr}(\gamma_A^{-1} \circ D^{-1} \circ T) = 0 \ , \quad \forall A, T \in \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \ . \tag{4.21}$$

*Proof.* Since $J$ commutes with $\gamma_A$ (see (4.12)), we have $\gamma_A \in \Gamma(M, \mathrm{End}(S))^-$. On the other hand, we have $D^{-1} \circ T \in L_D(\Gamma(M, \mathrm{End}_{\mathbb{C}}(S))) \equiv L_D(\Gamma(M, \mathrm{End}(S))^-) = \Gamma(M, \mathrm{End}(S))^+$ by (4.19). The conclusion now follows from the fact that $\Gamma(M, \mathrm{End}(S))^-$ and $\Gamma(M, \mathrm{End}(S))^+$ are mutually orthogonal with respect to the natural pairing (4.3). □

**Proposition.** For all $T \in \Gamma(M, \mathrm{End}(S))$ decomposed as in (4.20), we have:

$$\mathrm{tr}(\gamma_A^{-1} \circ T) = \mathrm{tr}(\gamma_A^{-1} \circ T_0) \quad \text{and} \quad \mathrm{tr}(\gamma_A^{-1} \circ D^{-1} \circ T) = \mathrm{tr}(\gamma_A^{-1} \circ T_1) \ . \tag{4.22}$$

*Proof.* Follows immediately from (4.21). □

Notice the relations:

$$D \circ T = \pi(T) \circ D \tag{4.23}$$

where $\pi$ is the parity signature acting in the $\mathbb{Z}_2$-graded $\mathcal{C}^\infty(M, \mathbb{R})$-module $\Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) = \gamma(\Omega(M))$ — the $\mathbb{Z}_2$-grading on the later being defined by transport from $\Omega(M)$ through the isomorphism $\gamma : \Omega(M) \xrightarrow{\sim} \Gamma(M, \mathrm{End}_{\mathbb{C}}(S))$. Consider the unital associative and commutative (but *not* graded-commutative) $\mathbb{Z}_2$-graded algebra $\mathbb{A} = \{\alpha + \beta e | \alpha, \beta \in \mathbb{R}\}$ generated over $\mathbb{R}$ by an odd element $\mathbf{e}$ which satisfies the single relation (cf. (2.23)):

$$\mathbf{e}^2 = (-1)^{\frac{p-q+1}{4}} 1 = \begin{cases} -1\,, & \text{if } p-q \equiv_8 3 \\ +1\,, & \text{if } p-q \equiv_8 7 \end{cases} .$$

Since $D$ satisfies (2.23), the results above show that $\Gamma(M, \mathrm{End}(S))$ is a *free* left $\mathbb{Z}_2$-graded module over the $\mathbb{Z}_2$-graded ring $\mathbb{A} \otimes_{\mathbb{R}} \mathcal{C}^\infty(M, \mathbb{R})$ of $\mathbb{A}$-valued smooth functions defined on $M$, where left multiplication with $\mathbf{e}$ is given by $L_D$:

$$\mathbf{e}T \stackrel{\text{def.}}{=} L_D(T) = D \circ T \ , \quad \forall T \in \Gamma(M, \mathrm{End}(S))$$

and the $\mathbb{Z}_2$-grading is given by the decomposition:

$$\Gamma(M, \mathrm{End}(S)) = \Gamma(M, \mathrm{End}(S))^{\mathrm{ev}} \oplus \Gamma(M, \mathrm{End}(S))^{\mathrm{odd}} \ ,$$

with:

$$\begin{aligned} \Gamma(M, \mathrm{End}(S))^{\mathrm{ev}} &\stackrel{\text{def.}}{=} \Gamma(M, \mathrm{End}_{\mathbb{C}}(S))^{\mathrm{ev}} \oplus L_D(\Gamma(M, \mathrm{End}_{\mathbb{C}}(S))^{\mathrm{odd}}) \ , \\ \Gamma(M, \mathrm{End}(S))^{\mathrm{odd}} &\stackrel{\text{def.}}{=} \Gamma(M, \mathrm{End}_{\mathbb{C}}(S))^{\mathrm{odd}} \oplus L_D(\Gamma(M, \mathrm{End}_{\mathbb{C}}(S))^{\mathrm{ev}}) \ . \end{aligned}$$

In fact, relation (4.23) implies that $(\Gamma(M, \mathrm{End}(S)), \circ)$ it is a unital $\mathbb{Z}_2$-graded $\mathbb{A} \otimes_{\mathbb{R}} \mathcal{C}^\infty(M, \mathbb{R})$-algebra with internal multiplication given by the composition $\circ$ of $\mathcal{C}^\infty(M, \mathbb{R})$-linear operators acting in $\Gamma(M, S)$. We have a unital isomorphism of $\mathbb{A} \otimes_{\mathbb{R}} \mathcal{C}^\infty(M, \mathbb{R})$-algebras (which maps $D$ into $\mathbf{e} \hat{\otimes}_{\mathbb{R}} \mathrm{id}_S$):

$$\Gamma(M, \mathrm{End}(S)) \approx \mathbb{A} \hat{\otimes}_{\mathbb{R}} \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \ ,$$

Here, $\Gamma(M, \mathrm{End}_{\mathbb{C}}(S))$ is viewed as a $\mathbb{Z}_2$-graded unital associative algebra with the $\mathbb{Z}_2$-grading induced via $\gamma$ from that of $(\Omega(M), \diamond)$ and $\hat{\otimes}_{\mathbb{R}}$ is the graded tensor product of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras. In particular, an $\mathbb{R}$-linear endomorphism $T \in \Gamma(M, \mathrm{End}(S))$ can be identified with:

$$T \equiv T_0 + \mathbf{e} \otimes T_1 \in \mathbb{A} \hat{\otimes}_{\mathbb{R}} \Gamma(M, \mathrm{End}_{\mathbb{C}}(S)) \ ,$$

where $T_0, T_1 \in \Gamma(M, \mathrm{End}_{\mathbb{C}}(S))$ are the components appearing in the decomposition (4.20).

### 4.3.2 The partial and full completeness relations

The two identities below (which hold above any open subset $U \subset M$ carrying a local orthonormal coframe $e^m$ of $M$ and a local frame $\varepsilon^i$ of $S$) follow immediately from the results proved in [2]:

$$\sum_{A=\text{ordered}} (\gamma_A^{-1})_{jk} (\gamma_A)_{lm} = \frac{2^d}{N} (\delta_{jm} \delta_{lk} - J_{jm} J_{lk}) \tag{4.24}$$

and:

$$\sum_{A=\text{ordered}} \left[ (\gamma_A^{-1})_{jk} (\gamma_A)_{lm} + \gamma_A^{-1} (D^{-1})_{jk} (D\gamma_A)_{lm} \right] = \frac{2^{d+1}}{N} \delta_{jm} \delta_{lk} \ . \tag{4.25}$$

These identities imply:

**Proposition.** We have the *partial completeness relation for the almost complex case*:

$$\frac{2^{d+1}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A \ , \quad \forall T\in\Gamma(M,\mathrm{End}_{\mathbb{C}}(S)) \ . \tag{4.26}$$

and the *full completeness relation for the almost complex case*:

$$\frac{2^{d+1}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\mathrm{ordered}} \left[\mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A + \mathrm{tr}(\gamma_A^{-1}\circ D^{-1}\circ T)D\circ\gamma_A\right] \ , \tag{4.27}$$
$$\forall T\in\Gamma(M,\mathrm{End}(S)) \ .$$

*Proof.* Multiplying both sides of (4.25) with $T_{jk}$ and summing over $j,k$ gives:

$$\frac{2^d}{\Delta}\Pi_-(T) = \frac{2^d}{N}(T-\mathcal{J}(T)) =_U \sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A \ , \quad \forall T\in\Gamma(M,\mathrm{End}(S)) \ , \tag{4.28}$$

which implies (4.26) for $T\in\Gamma(M,\mathrm{End}_{\mathbb{C}}(S))\equiv\Gamma(M,\mathrm{End}(S))^-$. On the other hand, multiplying both sides of (4.25) with $T_{jk}$ and summing over $j,k$ gives (4.27). □

**Corollary.** For any $T\in\Gamma(M,\mathrm{End}(S))$, we have:

$$\begin{aligned} T_0 &=_U \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A = \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T_0)\gamma_A \ , \\ T_1 &=_U \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ D^{-1}\circ T)\gamma_A = \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T_1)\gamma_A \ , \end{aligned} \tag{4.29}$$

where $T_0$ and $T_1$ are defined as in (4.20).

*Proof.* The equalities follow immediately from (4.27) and from the decomposition (4.20). In the second form of the expansions, we used (4.22). □

### 4.3.3 The Fierz identities

Relations (4.20) show that we can decompose $E_{\xi,\xi'}$ uniquely as:

$$E_{\xi,\xi'} = E^{(0)}_{\xi,\xi'} + D\circ E^{(1)}_{\xi,\xi'} \ ,$$

where:

$$\begin{aligned} E^{(0)}_{\xi,\xi'} &=_U \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ E_{\xi,\xi'})\gamma_A \in\Gamma(M,\mathrm{End}_{\mathbb{C}}(S)) \ , \\ E^{(1)}_{\xi,\xi'} &=_U \frac{\Delta}{2^d}\sum_{A=\mathrm{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ D^{-1}\circ E_{\xi,\xi'})\gamma_A \in\Gamma(M,\mathrm{End}_{\mathbb{C}}(S)) \end{aligned} \tag{4.30}$$

and (since $E^{(\alpha)}_{\xi,\xi'}\in\Gamma(M,\mathrm{End}_{\mathbb{C}}(S))$):

$$D\circ E^{(\alpha)}_{\xi,\xi'} = \pi(E^{(\alpha)}_{\xi,\xi'})\circ D \ , \quad \forall\alpha=0,1 \ , \tag{4.31}$$

where we used (4.23). Hence identity (4.1) takes the form:

$$\left(E^{(0)}_{\xi_1,\xi_2}+D\circ E^{(1)}_{\xi_1,\xi_2}\right)\circ\left(E^{(0)}_{\xi_3,\xi_4}+D\circ E^{(1)}_{\xi_3,\xi_4}\right)=\mathscr{B}_0(\xi_3,\xi_2)\left(E^{(0)}_{\xi_1,\xi_4}+D\circ E^{(1)}_{\xi_1,\xi_4}\right) \quad .$$

Using (4.31) and (2.23), this becomes:

$$\begin{aligned}E^{(0)}_{\xi_1,\xi_2}\circ E^{(0)}_{\xi_3,\xi_4}+(-1)^{\frac{p-q+1}{4}}\pi(E^{(1)}_{\xi_1,\xi_2})\circ E^{(1)}_{\xi_3,\xi_4}+D\circ\left(\pi(E^{(0)}_{\xi_1,\xi_2})\circ E^{(1)}_{\xi_3,\xi_4}+E^{(1)}_{\xi_1,\xi_2}\circ E^{(0)}_{\xi_3,\xi_4}\right)\\ =\mathscr{B}_0(\xi_3,\xi_2)\left(E^{(0)}_{\xi_1,\xi_4}+D\circ E^{(1)}_{\xi_1,\xi_4}\right)\end{aligned}$$

Separating components according to the decomposition (4.20) gives:

$$\begin{aligned}E^{(0)}_{\xi_1,\xi_2}\circ E^{(0)}_{\xi_3,\xi_4}+(-1)^{\frac{p-q+1}{4}}\pi(E^{(1)}_{\xi_1,\xi_2})\circ E^{(1)}_{\xi_3,\xi_4}&=\mathscr{B}_0(\xi_3,\xi_2)E^{(0)}_{\xi_1,\xi_4}\ ,\\ \pi(E^{(0)}_{\xi_1,\xi_2})\circ E^{(1)}_{\xi_3,\xi_4}+E^{(1)}_{\xi_1,\xi_2}\circ E^{(0)}_{\xi_3,\xi_4}&=\mathscr{B}_0(\xi_3,\xi_2)E^{(1)}_{\xi_1,\xi_4}\ .\end{aligned}\tag{4.32}$$

Using (4.2), (3.12), (3.15) and the fact that $\epsilon_0=-1$, we compute:

$$\begin{aligned}\mathrm{tr}(\gamma_A^{-1}\circ E_{\xi,\xi'})=\mathscr{B}_0(\gamma_A^{-1}\xi,\xi')&=(-1)^{|A|}\mathscr{B}_0(\xi,\gamma^A\xi')\ ,\\ \mathrm{tr}(\gamma_A^{-1}\circ D^{-1}\circ E_{\xi,\xi'})=\mathscr{B}_0((\gamma_A^{-1}\circ D^{-1})\xi,\xi')&=(-1)^{|A|}\mathscr{B}_0(D^{-1}\xi,\gamma^A\xi')=\\ &=(-1)^{\frac{p-q+1}{4}}(-1)^{|A|}\mathscr{B}_0(\xi,(D\circ\gamma^A)\xi')\ ,\end{aligned}$$

which allows us to rewrite (4.30) as:

$$\begin{aligned}E^{(0)}_{\xi,\xi'}&=_U\frac{\Delta}{2^d}\sum_{A=\text{ordered}}(-1)^{|A|}\mathscr{B}_0(\xi,\gamma^A\xi')\gamma_A\ ,\\ E^{(1)}_{\xi,\xi'}&=_U\frac{\Delta}{2^d}\sum_{A=\text{ordered}}(-1)^{\frac{p-q+1}{4}}(-1)^{|A|}\mathscr{B}_0(\xi,D\circ\gamma^A\xi')\gamma_A\ .\end{aligned}\tag{4.33}$$

#### 4.3.4 Geometric algebra formulation

Using the partial inverse (4.15), we define:

$$\check{E}^{(\alpha)}_{\xi,\xi'}\stackrel{\text{def.}}{=}\gamma^{-1}(E^{(\alpha)}_{\xi,\xi'})\ \in\Omega(M)\ ,\quad\forall\alpha=0,1\quad .$$

Applying $\gamma^{-1}$ shows that identities (4.32) are equivalent with the *geometric algebra form of the Fierz identities in the almost complex case*:

$$\begin{aligned}\check{E}^{(0)}_{\xi_1,\xi_2}\diamond\check{E}^{(0)}_{\xi_3,\xi_4}+(-1)^{\frac{p-q+1}{4}}\pi(\check{E}^{(1)}_{\xi_1,\xi_2})\diamond\check{E}^{(1)}_{\xi_3,\xi_4}&=\mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(0)}_{\xi_1,\xi_4}\ ,\\ \pi(\check{E}^{(0)}_{\xi_1,\xi_2})\diamond\check{E}^{(1)}_{\xi_3,\xi_4}+\check{E}^{(1)}_{\xi_1,\xi_2}\diamond\check{E}^{(0)}_{\xi_3,\xi_4}&=\mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(1)}_{\xi_1,\xi_4}\ ,\end{aligned}\tag{4.34}$$

while (4.33) give the expansions:

$$\begin{aligned}\check{E}^{(0)}_{\xi,\xi'}&=_U\frac{\Delta}{2^d}\sum_{A=\text{ordered}}(-1)^{|A|}\mathscr{B}_0(\xi,\gamma_A\xi')e^A\,,\\ \check{E}^{(1)}_{\xi,\xi'}&=_U\frac{\Delta}{2^d}\sum_{A=\text{ordered}}(-1)^{\frac{p-q+1}{4}}(-1)^{|A|}\mathscr{B}_0(\xi,(D\circ\gamma_A)\xi')e^A\,.\end{aligned}\tag{4.35}$$



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Later on, we shall also use the notations:

$$\check{\boldsymbol{E}}^{(0,k)}_{\xi,\xi'} \overset{\text{def.}}{=}_U \frac{1}{k!}(-1)^k \mathscr{B}_0(\xi, \gamma_{a_1\dots a_k}\xi')e^{a_1\dots a_k} \in \Omega^k(M)$$
$$\check{\boldsymbol{E}}^{(1,k)}_{\xi,\xi'} \overset{\text{def.}}{=}_U \frac{1}{k!}(-1)^{\frac{p-q+1}{4}}(-1)^k \mathscr{B}_0(\xi, (D\circ\gamma_{a_1\dots a_k})\xi')e^{a_1\dots a_k} \in \Omega^k(M) \ ,$$

so that:

$$\check{E}^{(\alpha)}_{\xi,\xi'} = \frac{\Delta}{2^d}\sum_{k=0}^{d} \check{\boldsymbol{E}}^{(\alpha,k)}_{\xi,\xi'} \ , \quad \forall \alpha = 0,1 \ . \tag{4.36}$$

Consider the unital and associative $\mathbb{Z}_2$-graded $\mathcal{C}^\infty(M,\mathbb{R})$-algebra defined through:

$$\Omega_{\mathbb{A}}(M) \overset{\text{def.}}{=} \mathbb{A}\hat{\otimes}_{\mathbb{R}}(\Omega(M),\diamond) \ ,$$

where the Kähler-Atiyah algebra of $M$ is viewed as a $\mathbb{Z}_2$-graded algebra. Denoting the composition of $\Omega_{\mathbb{A}}(M)$ again by $\diamond$ for simplicity of notation, equations (4.35) can be written in the equivalent form:

$$\check{E}_{\xi_1,\xi_2}\diamond\check{E}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3,\xi_2)\check{E}_{\xi_1,\xi_4} \ ,$$

where we defined:

$$\check{E}_{\xi,\xi'} \overset{\text{def.}}{=} \check{E}^{(0)}_{\xi,\xi'} + \mathbf{e}\otimes\check{E}^{(1)}_{\xi,\xi'} \in \Omega_{\mathbb{A}}(M) \ . \tag{4.37}$$

### 4.4 Fierz identities for the quaternionic case

In this case, the Schur algebra is isomorphic with the $\mathbb{R}$-algebra $\mathbb{H}$ of quaternions, being spanned over $\mathbb{R}$ by four linearly-independent elements $J_\alpha \in \Gamma(M, \mathrm{End}(S))$ ($\alpha = 0\dots 3$), which we can take to correspond to the quaternion units — where we once again (possibly replacing $M$ with a sufficiently small open subset — we assume for simplicity of notation that $M$ is contractible and $J_i$ are globally-defined. Hence $J_0 = \mathrm{id}_S$ while $J_1, J_2, J_3$ satisfy:

$$J_i\circ J_j = -\delta_{ij}J_0 + \epsilon_{ijk}J_k \ , \quad \forall i,j,k = 1\dots 3 \implies [J_i, J_j]_{+,\circ} = 0 \ . \tag{4.38}$$

This makes $S$ into a left $\mathbb{H}$-module through:

$$u\xi = \sum_{\alpha=0}^{3} u_\alpha J_\alpha(\xi) \ , \quad \forall u = \sum_{\alpha=0}^{3} u_\alpha \mathbf{j}_\alpha \in \mathbb{H} \ ,$$

where $u_\alpha \in \mathbb{R}$ and $\mathbf{j}_\alpha$ are quaternion units. The space $\Gamma(M, \mathrm{End}_{\mathbb{H}}(S))$ of globally-defined $\Sigma_\gamma$-linear endomorphisms of this module can be identified with the subspace of those $\mathcal{C}^\infty(M,\mathbb{R})$-linear operators acting in $\Gamma(M,S)$ which commute with $J_1, J_2$ and $J_3$:

$$\Gamma(M, \mathrm{End}_{\mathbb{H}}(S)) \equiv \{T\in\Gamma(M,\mathrm{End}(S)) \mid [T, J_i]_{-,\circ} = 0 \ , \quad \forall i = 1\dots 3\} \ . \tag{4.39}$$

### 4.4.1 Preparations

Consider the following $\mathcal{C}^\infty(M,\mathbb{R})$-linear operators $\mathcal{J}_\alpha$ acting in the space $\Gamma(M,\mathrm{End}(S))$:

$$\mathcal{J}_\alpha(T) \stackrel{\text{def.}}{=} J_\alpha \circ T \circ J_\alpha \ , \quad \forall i = 1\dots 3 \ .$$

We obviously have $\mathcal{J}_0 = \mathrm{id}_{\Gamma(M,\mathrm{End}(S))}$. Relations (4.38) imply:

$$\mathcal{J}_i^2 = \mathcal{J}_0 \ , \quad [\mathcal{J}_i, \mathcal{J}_j]_{-,\circ} = 0 \ , \quad \forall i \neq j$$

as well as:

$$\mathcal{J}_i \circ \mathcal{J}_j = \delta_{ij}\mathcal{J}_0 - |\epsilon_{ijk}|\mathcal{J}_k \quad \forall i,j = 1\dots 3 \ ,$$

i.e.:

$$\mathcal{J}_1 \circ \mathcal{J}_2 = \mathcal{J}_2 \circ \mathcal{J}_1 = -\mathcal{J}_3 \ , \quad \mathcal{J}_2 \circ \mathcal{J}_3 = \mathcal{J}_3 \circ \mathcal{J}_2 = -\mathcal{J}_1 \ , \quad \mathcal{J}_1 \circ \mathcal{J}_3 = \mathcal{J}_3 \circ \mathcal{J}_1 = -\mathcal{J}_2 \ .$$

Let us define:

$$\mathcal{J} \stackrel{\text{def.}}{=} \sum_{k=1}^{3} \mathcal{J}_k \ , \quad \mathcal{R} \stackrel{\text{def.}}{=} \frac{1}{2}(\mathcal{J}_0 + \mathcal{J}) \ .$$

Using the relations above, we compute:

$$\mathcal{J}^2 = 3\mathcal{J}_0 - 2\mathcal{J} \iff \mathcal{R}^2 = \mathcal{J}_0 \ .$$

This shows that $\mathcal{R}$ is a product structure on the $\mathcal{C}^\infty(M,\mathbb{R})$-module $\Gamma(M,\mathrm{End}(S))$. An easy computation using cyclicity of the trace shows that $\mathcal{J}$ is selfadjoint with respect to the pairing (4.3):

$$\langle \mathcal{J}(A), B\rangle = \langle A, \mathcal{J}(B)\rangle \ , \quad \forall A, B \in \Gamma(M,\mathrm{End}(S)) \ .$$

It follows that the $\mathcal{C}^\infty(M,\mathbb{R})$-linear operators $\Pi_\pm \in \mathrm{End}_{\mathcal{C}^\infty(M,\mathbb{R})}(\Gamma(M,\mathrm{End}(S)))$ defined through:

$$\Pi_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(\mathcal{J}_0 \pm \mathcal{R}) \iff \Pi_+ = \frac{1}{4}(3\mathcal{J}_0 + \mathcal{J}) \ \text{ and } \ \Pi_- = \frac{1}{4}(\mathcal{J}_0 - \mathcal{J})$$

are complementary $\langle\, ,\, \rangle$-orthoprojectors. In particular, we have:

$$\Pi_\pm^2 = \Pi_\pm \ , \quad \Pi_+ \circ \Pi_- = \Pi_- \circ \Pi_+ = 0 \ , \quad \Pi_+ + \Pi_- = \mathcal{J}_0$$

and the $\mathcal{C}^\infty(M,\mathbb{R})$-submodules:

$$\begin{aligned}\Gamma(M,\mathrm{End}(S))^\pm &\stackrel{\text{def.}}{=} \Pi_\pm(\Gamma(M,\mathrm{End}(S)) \\ &= \{T \in \Gamma(M,\mathrm{End}(S)) | \mathcal{R}(T) = \pm T\} \subset \Gamma(M,\mathrm{End}(S))\end{aligned}$$

give an $\langle\, ,\, \rangle$-orthogonal direct sum decomposition of the $\mathcal{C}^\infty(M,\mathbb{R})$-module $\Gamma(M,\mathrm{End}(S))$:

$$\Gamma(M,\mathrm{End}(S)) = \Gamma(M,\mathrm{End}(S))^+ \oplus \Gamma(M,\mathrm{End}(S))^- \ .$$



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Notice the characterizations:

$$\begin{aligned}
\Gamma(M,\mathrm{End}(S))^+ &= \{T\in\Gamma(M,\mathrm{End}(S))|\mathcal{J}(T)=T\}~,\\
\Gamma(M,\mathrm{End}(S))^- &= \{T\in\Gamma(M,\mathrm{End}(S))|\mathcal{J}(T)=-3T\}~.
\end{aligned}$$

Using identities (4.38), we compute:

$$\sum_{i=1}^{3} J_i\circ J_j\circ J_i = -J_j+\epsilon_{ijk}J_k\circ J_i~,~~\forall i,j=1\dots 3~,$$

a relation which implies:

$$\mathcal{J}(J_i)=J_i \iff J_i\in\Gamma(M,\mathrm{End}(S))^+~,~~\forall i=1\dots 3~. \tag{4.40}$$

For any $T\in\Gamma(M,\mathrm{End}(S))$, we define:

$$T_\pm \stackrel{\mathrm{def.}}{=} \Pi_\pm(T) \implies T=T_++T_-~.$$

Then:

$$T_+=\frac{1}{4}(3T+\mathcal{J}(T))~,~~T_-=\frac{1}{4}(T-\mathcal{J}(T))~. \tag{4.41}$$

**Proposition.** We have:

$$\Gamma(M,\mathrm{End}(S))^- = \{T\in\Gamma(M,\mathrm{End}(S))~|~[T,J_i]_{-,\circ}=0\,,~\forall i=1\dots 3~\}\equiv\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))~,$$

where we used the identification (4.39).

*Proof.* ($\subset$) Direct computation using (4.38) gives:

$$\begin{aligned}
J_i\circ T_- &= T_-\circ J_i\\
&=\frac{1}{4}([J_i,T]_{+,\circ}+\epsilon_{ijk}J_j\circ T\circ J_k)~,~~\forall T\in\Gamma(M,\mathrm{End}(S))~,~~\forall i,j=1\dots 3~,
\end{aligned}$$

which implies:

$$[J_i,T_-]_{-,\circ}=0~,~~\forall T\in\Gamma(M,\mathrm{End}(S))~,~~\forall i=1\dots 3~. \tag{4.42}$$

For $T\in\Gamma(M,\mathrm{End}(S))^-$, we have $T_-=T$ and (4.42) shows that $T$ commutes with all $J_i$.

($\supset$) Let $T\in\Gamma(M,\mathrm{End}(S))$ satisfy $[T,J_i]_{-,\circ}=0\,,~\forall i=1\dots 3$. Using these commutation relations as well as the identities $J_i^2=-\mathrm{id}_S$, equation (4.41) gives $T_-=T$, i.e. $T=\Pi_-(T)\in\Gamma(M,\mathrm{End}(S))^-$. $\square$



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**Proposition.** The $\mathcal{C}^\infty(M,\mathbb{R})$-submodules $L_{J_i}(\Gamma(M,\mathrm{End}(S)))$ are contained in $\Gamma(M,\mathrm{End}(S))^+$, have trivial pairwise intersection and are mutually orthogonal with respect to the pairing $\langle\,,\,\rangle$. Hence they give the $\langle\,,\,\rangle$-orthogonal direct sum decomposition:

$$\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))^+ = \oplus_{i=1}^3 L_{J_i}(\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))) ~,$$

where we have identified $\Gamma(M,\mathrm{End}_{\mathbb{H}}(S)) \equiv \Gamma(M,\mathrm{End}(S))^-$.

*Proof.* ($L_{J_i}(\Gamma(M,\mathrm{End}(S)))$ are contained in $\Gamma(M,\mathrm{End}(S))^+$). For any $T \in \Gamma(M,\mathrm{End}_{\mathbb{H}}(S))$, we have $[T,J_k]_{-,\circ} = 0\,,~\forall k = 1\ldots 3$, which gives:

$$\begin{aligned}(J_i\circ T)_- &= \frac{1}{4}\Big(J_i\circ T - \sum_{j=1}^3 J_j\circ J_i\circ T\circ J_j\Big)\\ &= \frac{1}{4}(J_i - \mathcal{J}(J_i))\circ T = \Pi_-(J_i)\circ T = 0 \implies J_i\circ T\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))^+\end{aligned}$$

where we used (4.40), which implies $\Pi_-(J_i) = 0$. Thus $L_{J_i}(\Gamma(M,\mathrm{End}(S)))$ are contained in $\Gamma(M,\mathrm{End}(S))^+$:

$$J_i(\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))) \subset \Gamma(M,\mathrm{End}_{\mathbb{H}}(S))^+ ~.$$

(trivial mutual intersection). Let $1\leq i\neq j\leq 3$. If $T\in L_{J_i}(\Gamma(M,\mathrm{End}_{\mathbb{H}}(S)))\cap L_{J_j}(\Gamma(M,\mathrm{End}_{\mathbb{H}}(S)))$, then $T = J_i\circ A = J_j\circ B$ for some $A,B\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))$. Composing with $J_i$ from the left and using $J_i^2 = -\mathrm{id}_S$ gives:

$$A = -J_i\circ J_j\circ B = \epsilon_{ijk}J_k\circ B\in\Gamma(M,\mathrm{End}(S))^+ ~,$$

where we used identities (4.38) (with $i\neq j$) as well as property (4.40). Since $A\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))\equiv\Gamma(M,\mathrm{End}(S))^-$, this implies $A\in\Gamma(M,\mathrm{End}(S))^-\cap\Gamma(M,\mathrm{End}(S))^+ = \{0\}$, where we used the fact that $\Gamma(M,\mathrm{End}(S))^-$ and $\Gamma(M,\mathrm{End}(S))^+$ are complementary submodules of $\Gamma(M,\mathrm{End}(S))$. It follows that $A = 0$ and hence $T = J_i\circ A = 0$. Therefore, we have:

$$L_{J_i}(\Gamma(M,\mathrm{End}(S)))\cap L_{J_j}(\Gamma(M,\mathrm{End}(S))) = \{0\}\,,\quad\forall\ 1\leq i\neq j\leq 3 ~.$$

($\langle\,,\,\rangle$-orthogonality). For any $A,B\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))$, we have $\mathrm{tr}(J_i\circ A\circ J_j\circ B) = \mathrm{tr}(J_i\circ J_j\circ A\circ B)$ since $J_j$ and $A$ commute. Using relations (4.38), we compute:

$$\mathrm{tr}(J_i\circ J_j\circ A\circ B) = \delta_{ij}\mathrm{tr}(A\circ B) - \epsilon_{ijk}\mathrm{tr}(J_k\circ A\circ B) ~. \tag{4.43}$$

On the other hand, the left hand side is symmetric in $i$ and $j$:

$$\mathrm{tr}(J_i\circ J_j\circ A\circ B) = \mathrm{tr}(A\circ B\circ J_i\circ J_j) = \mathrm{tr}(J_j\circ A\circ B\circ J_i) = \mathrm{tr}(J_j\circ J_i\circ A\circ B)$$

where we used that fact that $A$ and $B$ commute with $J_1, J_2$ and $J_3$ as well as cyclicity of the trace. Hence $\mathrm{tr}(J_i\circ J_j\circ A\circ B) = \frac{1}{2}(\mathrm{tr}(J_i\circ J_j\circ A\circ B)+\mathrm{tr}(J_j\circ J_i\circ A\circ B))$ and (4.43) gives:

$$\mathrm{tr}(J_i\circ J_j\circ A\circ B) = \delta_{ij}\mathrm{tr}(A\circ B) \implies \mathrm{tr}(J_i\circ A\circ J_j\circ B) = \delta_{ij}\mathrm{tr}(A\circ B) ~,$$

for all $A,B\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))$ and for all $i,j = 1\ldots 3$. This shows that the subspaces $L_{J_i}(\Gamma(M,\mathrm{End}(S)))$ are mutually orthogonal with respect to the pairing $\langle\,,\,\rangle$. $\square$

The following consequence of the proposition is obvious.

**Corollary.** Any $T \in \Gamma(M, \mathrm{End}(S))$ decomposes uniquely as:

$$T = \sum_{\alpha=0}^{3} J_\alpha \circ T_\alpha = T_0 + J_1 \circ T_1 + J_2 \circ T_2 + J_3 \circ T_3 \quad , \tag{4.44}$$

where $T_\alpha \in \Gamma(M, \mathrm{End}(S))^- \equiv \Gamma(M, \mathrm{End}_{\mathbb{H}}(S))\,,\ \forall \alpha = 0 \ldots 3$.

Hence $\mathrm{End}(S)$ is (as expected) a bundle of free modules over the Schur algebra, while $\Gamma(M, \mathrm{End}(S))$ is a free left module over the algebra $\Gamma(M, \Sigma_\gamma) \subset \Gamma(M, \mathrm{End}(S))$, where left multiplication with $u \in \Gamma(M, \Sigma_\gamma)$ is given by $L_u$:

$$uT \stackrel{\text{def.}}{=} L_u(T) = u \circ T \quad , \quad \forall T \in \Gamma(M, \mathrm{End}(S)) \quad .$$

In fact, $(\Gamma(M, \mathrm{End}(S)), \circ)$ is a left $\Gamma(M, \Sigma_\gamma)$-algebra whose internal multiplication is given by composition of $\mathbb{R}$-linear endomorphisms of $S$. We have a unital isomorphism of $\Gamma(M, \Sigma_\gamma)$-algebras:

$$\Gamma(M, \mathrm{End}(S)) \approx \Gamma(M, \Sigma_\gamma) \otimes_{\mathcal{C}^\infty(M,\mathbb{R})} \Gamma(M, \mathrm{End}_{\mathbb{H}}(S))$$

which maps $J_\alpha$ into $J_\alpha \otimes \mathrm{id}_S$. In particular, any endomorphism $T \in \Gamma(M, \mathrm{End}(S))$ can be identified with:

$$T \equiv \sum_{\alpha=0}^{3} J_\alpha \otimes T_\alpha \in \Gamma(M, \Sigma_\gamma) \otimes_{\mathcal{C}^\infty(M,\mathbb{R})} \Gamma(M, \mathrm{End}_{\mathbb{H}}(S)) \quad ,$$

where $T_\alpha \in \Gamma(M, \mathrm{End}_{\mathbb{H}}(S))$ are the components appearing in decomposition (4.44).

**Proposition.** We have:

$$\mathrm{tr}(J_i \circ T) = 0 \quad , \quad \forall T \in \Gamma(M, \mathrm{End}_{\mathbb{H}}(S)) \equiv \Gamma(M, \mathrm{End}(S))^- \quad , \quad \forall i = 1 \ldots 3 \quad . \tag{4.45}$$

*Proof.* The statement follows immediately from the fact that $J_i \in \Gamma(M, \mathrm{End}(S))^+$ (see (4.40)) together with the fact that $\Gamma(M, \mathrm{End}(S))^+$ and $\Gamma(M, \mathrm{End}_{\mathbb{H}}(S)) \equiv \Gamma(M, \mathrm{End}(S))^-$ are $\langle\,,\,\rangle$-orthogonal. The statement can also be proved directly through the following argument. If $T \in \Gamma(M, \mathrm{End}_{\mathbb{H}}(S))$, then $[J_i, T]_{-,\circ} = 0\,,\ \forall i = 1 \ldots 3$. Relation (4.38) implies:

$$J_i \circ J_j + J_j \circ J_i = -2\delta_{ij} \mathrm{id}_S \quad .$$

Multiplying the above with $T$ from the left and taking the trace gives:

$$\mathrm{tr}(T \circ J_i \circ J_j) + \mathrm{tr}(T \circ J_j \circ J_i) = -2\delta_{ij} \mathrm{tr}(T) \quad .$$

This implies:

$$\mathrm{tr}(T \circ J_i \circ J_j) = \mathrm{tr}(T \circ J_j \circ J_i) = -\delta_{ij} \mathrm{tr}(T) \quad , \quad \forall T \in \Gamma(M, \mathrm{End}_{\mathbb{H}}(S)) \quad , \tag{4.46}$$

where we noticed the identity $\mathrm{tr}(T \circ J_j \circ J_i) = \mathrm{tr}(J_i \circ T \circ J_j) = \mathrm{tr}(J_i \circ J_j \circ T)$, which follows from cyclicity of the trace and from the fact that $T$ commutes with $J_j$. Using (4.38), we compute:

$$\mathrm{tr}(T \circ J_i \circ J_j) = -\delta_{ij} \mathrm{tr}(T) + \epsilon_{ijk} \mathrm{tr}(T \circ J_k) \quad .$$

Hence $\epsilon_{ijk} \mathrm{tr}(T \circ J_k) = 0\,,\ \forall i, j \in 1 \ldots 3$, which implies $\mathrm{tr}(T \circ J_k) = \mathrm{tr}(J_k \circ T) = 0$. □



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#### 4.4.2 The partial and full completeness relations

The two identities below (which hold above any open subset $U$ of $M$ supporting a local pseudo-orthonormal coframe $e^m$ of $(M, g)$ and a local frame $\varepsilon^i$ of $S$) follow immediately from the results proved in [2]:

$$\sum_{A=\text{ordered}} (\gamma_A^{-1})_{jk}(\gamma_A)_{lm} = \frac{2^d}{N}\left(\delta_{jm}\delta_{lk} - \sum_{a=1}^{3}(J_a)_{jm}(J_a)_{lk}\right) \tag{4.47}$$

and

$$\sum_{A=\text{ordered}} (\gamma_A^{-1})_{jk}(\gamma_A)_{lm} - \sum_{a=1}^{3}(\gamma_A^{-1}\circ J_a)_{jk}(J_a\circ\gamma_A)_{lm} = \frac{2^{d+2}}{N}\delta_{jm}\delta_{lk} \quad . \tag{4.48}$$

**Proposition.** We have the *partial completeness relation for the quaternionic case*:

$$\frac{2^{d+2}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A \ , \quad \forall T\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S)) \tag{4.49}$$

and the *full completeness relation for the quaternionic case*:

$$\frac{2^{d+2}}{N}T = \frac{2^d}{\Delta}T =_U \sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T)\gamma_A - \sum_{k=1}^{3}\sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ J_k\circ T)J_k\circ\gamma_A \ , \tag{4.50}$$

for any $T\in\Gamma(M,\mathrm{End}(S))$.

*Proof.* Multiplying both sides of (4.47) with $T_{kj}$ and summing over $j, k$ gives the equivalent form:

$$\frac{2^d}{\Delta}\Pi_-(T) = \frac{2^d}{N}(T - \mathcal{J}(T)) =_U \sum_{A=\text{ordered}} \mathrm{tr}\left(\gamma_A^{-1}\circ T\right)\gamma_A \quad . \tag{4.51}$$

This gives (4.49) for $T\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))$. On the other hand, (4.48) implies (4.50) upon multiplying with the components $T_{kj}$ and summing over $j, k$. □

**Corollary.** For $\forall\alpha = 0\ldots 3$ we have the expansion:

$$\begin{aligned} T_\alpha &=_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ J_\alpha^{-1}\circ T)\gamma_A \\ &= \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1}\circ T_\alpha)\gamma_A \ , \quad \forall T\in\Gamma(M,\mathrm{End}(S)) \quad . \end{aligned} \tag{4.52}$$

*Proof.* Combine the previous proposition with the identity:

$$\mathrm{tr}(\gamma_A^{-1}\circ J_\alpha\circ T) = -\mathrm{tr}(\gamma_A^{-1}\circ T_\alpha) \ , \tag{4.53}$$

which follows from (4.44) upon using (4.38) and property (4.45). □

### 4.4.3 The Fierz identities

Relations (4.45) and (4.52) show that the operators $E_{\xi,\xi'} \in \Gamma(M, \mathrm{End}(S))$ have the unique decompositions:

$$E_{\xi,\xi'} = \sum_{\alpha=0}^{3} J_\alpha \circ E^{(\alpha)}_{\xi,\xi'} \ ,$$

where $E^{(\alpha)}_{\xi,\xi'} \in \Gamma(M, \mathrm{End}_{\mathbb{H}}(S))$ for all $\alpha = 0\ldots 3$ and:

$$E^{(\alpha)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} \mathrm{tr}(\gamma_A^{-1} \circ J_\alpha^{-1} \circ E_{\xi,\xi'})\gamma_A \ . \tag{4.54}$$

Relation (4.2) gives:

$$\mathrm{tr}(\gamma_A^{-1} \circ J_\alpha^{-1} \circ E_{\xi,\xi'}) = \mathscr{B}_0((\gamma_A^{-1} \circ J_\alpha^{-1})\xi, \xi') \ ,$$

which in turn implies:

$$E^{(\alpha)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} \mathscr{B}_0((\gamma_A^{-1} \circ J_\alpha^{-1})\xi, \xi')\gamma_A \ . \tag{4.55}$$

Thus:

$$E^{(0)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} \mathscr{B}_0(\gamma_A^{-1}\xi, \xi')\gamma_A \ , \tag{4.56}$$

$$E^{(i)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} \mathscr{B}_0((\gamma_A^{-1} \circ J_i^{-1})\xi, \xi')\gamma_A \ , \quad \forall i = 1\ldots 3 \tag{4.57}$$

which leads to the expansion:

$$E_{\xi,\xi'} =_U \sum_{\alpha=0}^{3} J_\alpha \circ E^{(\alpha)}_{\xi,\xi'} = \frac{\Delta}{2^d} \sum_{A=\text{ordered}} \sum_{\alpha=0}^{3} \mathscr{B}_0((\gamma_A^{-1} \circ J_\alpha^{-1})\xi, \xi') J_\alpha \circ \gamma_A \ . \tag{4.58}$$

Using (4.54) in (4.1) and relations (4.38), we find the *Fierz identities for the quaternionic case*:

$$E^{(0)}_{\xi_1,\xi_2} \circ E^{(0)}_{\xi_3,\xi_4} - \sum_{i=1}^{3} E^{(i)}_{\xi_1,\xi_2} \circ E^{(i)}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3, \xi_2) E^{(0)}_{\xi_1,\xi_4} \ , \tag{4.59}$$

$$E^{(0)}_{\xi_1,\xi_2} \circ E^{(i)}_{\xi_3,\xi_4} + E^{(i)}_{\xi_1,\xi_2} \circ E^{(0)}_{\xi_3,\xi_4} + \sum_{j,k=1}^{3} \epsilon_{ijk} E^{(j)}_{\xi_1,\xi_2} \circ E^{(k)}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3, \xi_2) E^{(i)}_{\xi_1,\xi_4} \quad (i = 1\ldots 3) \ .$$

Let us write (4.56) and (4.57) in an equivalent form which is more convenient in applications. Starting from relations (3.12) and (3.40), we compute:

$$\mathscr{B}_0((\gamma_A^{-1} \circ J_\alpha^{-1})\xi, \xi') = \epsilon_{\mathscr{B}_0}^{|A|} \mathscr{B}_0(J_\alpha^{-1}\xi, \gamma^A \xi') = \epsilon_{\mathscr{B}_0}^{|A|} \mathscr{B}_0(\xi, (J_\alpha \circ \gamma^A)\xi') \ , \quad \forall \alpha = 0\ldots 3 \ .$$

This allows us to write (4.55) as:

$$E^{(\alpha)}_{\xi,\xi'} =_U \frac{\Delta}{2^d} \sum_{A=\text{ordered}} \epsilon_{\mathscr{B}_0}^{|A|} \mathscr{B}_0(\xi, (J_\alpha \circ \gamma^A)\xi')\gamma_A \ , \quad \forall \alpha = 0\ldots 3 \ , \quad \forall \xi, \xi' \in \Gamma(M, S) \ .$$

Thus:

$$E^{(0)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,\gamma^A\xi')\gamma_A \ ,$$
$$E^{(i)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,(J_i\circ\gamma^A)\xi')\gamma_A \ , \quad \forall i=1\ldots 3 \tag{4.60}$$

and we have the expansion:

$$E_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}}\sum_{\alpha=0}^{3} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,(J_\alpha\circ\gamma^A)\xi')J_\alpha\circ\gamma_A \ .$$

### 4.4.4 Geometric algebra formulation

Since $E^{(\alpha)}_{\xi,\xi'}\in\Gamma(M,\mathrm{End}_{\mathbb{H}}(S))=\gamma(\Omega(M))$, applying $\gamma^{-1}$ to (4.59) gives:

$$\check{E}^{(0)}_{\xi_1,\xi_2}\diamond\check{E}^{(0)}_{\xi_3,\xi_4}-\sum_{i=1}^{3}\check{E}^{(i)}_{\xi_1,\xi_2}\diamond\check{E}^{(i)}_{\xi_3,\xi_4}=\mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(0)}_{\xi_1,\xi_4} \ ,$$
$$\check{E}^{(0)}_{\xi_1,\xi_2}\diamond\check{E}^{(i)}_{\xi_3,\xi_4}+\check{E}^{(i)}_{\xi_1,\xi_2}\diamond\check{E}^{(0)}_{\xi_3,\xi_4}+\sum_{j,k=1}^{3}\epsilon_{ijk}\check{E}^{(j)}_{\xi_1,\xi_2}\diamond\check{E}^{(k)}_{\xi_3,\xi_4}=\mathscr{B}_0(\xi_3,\xi_2)\check{E}^{(i)}_{\xi_1,\xi_4} \quad (i=1\ldots 3)\,,$$

while (4.60) gives the expansions:

$$\check{E}^{(\alpha)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,(J_\alpha\circ\gamma_A)\xi')e^A_\gamma \ , \quad \forall\alpha=0\ldots 3 \ , \quad \forall\xi,\xi'\in\Gamma(M,S) \ ,$$

i.e.:

$$\check{E}^{(0)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,\gamma_A\xi')e^A_\gamma \ , \quad \forall\xi,\xi'\in\Gamma(M,S) \ ,$$
$$\check{E}^{(i)}_{\xi,\xi'} =_U \frac{\Delta}{2^d}\sum_{A=\text{ordered}} \epsilon^{|A|}_{\mathscr{B}_0}\mathscr{B}_0(\xi,(J_i\circ\gamma_A)\xi')e^A_\gamma \ , \quad \forall i=1\ldots 3 \ .$$

Later on, we shall also use the notation:

$$\check{\boldsymbol{E}}^{(0,k)} \overset{\text{def.}}{=}_U \frac{1}{k!}\epsilon^k_{\mathscr{B}_0}\mathscr{B}_0(\xi,\gamma_{a_1\ldots a_k}\xi')e^{a_1\ldots a_k}_\gamma \ \in\Omega^k(M)\cap\Omega^\gamma(M)$$
$$\check{\boldsymbol{E}}^{(i,k)} \overset{\text{def.}}{=}_U \frac{1}{k!}\epsilon^k_{\mathscr{B}_0}\mathscr{B}_0(\xi,(J_i\circ\gamma_{a_1\ldots a_k})\xi')e^{a_1\ldots a_k}_\gamma \ \in\Omega^k(M)\cap\Omega^\gamma(M) \ ,$$

so that:

$$\check{E}^{(\alpha)}_{\xi,\xi'}=\frac{\Delta}{2^d}\sum_{k=0}^{d}\check{\boldsymbol{E}}^{(\alpha,k)}_{\xi,\xi'} \ .$$

Consider the $\mathcal{C}^\infty(M,\mathbb{R})$-module of $\Sigma_\gamma$-valued forms:

$$\Omega(M,\Sigma_\gamma)\overset{\text{def.}}{=}\Gamma(M,\wedge T^*M\otimes\Sigma_\gamma)\approx\Gamma(M,\Sigma_\gamma)\otimes_{\mathcal{C}^\infty(M,\mathbb{R})}\Omega(M) \ ,$$

which we endow with the noncommutative $\mathbb{Z}_2$-graded algebra structure (with product denoted once again by $\diamond$) induced through the tensor product of algebras from the algebra structures of $\Gamma(M,\Sigma_\gamma)$ and of $(\Omega(M),\diamond)$:

$$(\Omega(M,\Sigma_\gamma),\diamond)\overset{\text{def.}}{=}(\Gamma(M,\Sigma_\gamma),\circ)\hat{\otimes}_{\mathcal{C}^\infty(M,\mathbb{R})}(\Omega(M),\diamond) \ .$$

Thus

$$(u \otimes \omega) \diamond (v \otimes \eta) = (u \circ v) \otimes (\omega \diamond \eta) \quad \forall \omega, \eta \in \Omega(M)\,, \quad \forall u, v \in \Gamma(M, \Sigma_\gamma) \quad .$$

We define:

$$\check{E}_{\xi,\xi'} \stackrel{\text{def.}}{=} \sum_{\alpha=0}^{3} J_\alpha \otimes \check{E}^{(\alpha)}_{\xi,\xi'} \in \Omega(M, \Sigma_\gamma) \quad . \tag{4.61}$$

Then (4.61) is equivalent with:

$$\check{E}_{\xi_1,\xi_2} \diamond \check{E}_{\xi_3,\xi_4} = \mathscr{B}_0(\xi_3, \xi_2) \check{E}_{\xi_1,\xi_4} \quad ,$$

## 5 Examples

In this section, we illustrate our approach to Fierz identities with three examples belonging to the normal, real and quaternionic types. In previous work, we have already briefly exemplified the simple and non-simple normal cases of our formalism with two other applications — namely one real pinor in eight Euclidean dimensions [1] (simple normal type) and two real pinors in nine Euclidean dimensions [8] (non-simple normal type), giving some hints on the computer algebra implementation used to perform the computations. As in our previous work, the explicit form of the Fierz identities in the geometric algebra formulation can be extracted and analyzed by using a symbolic computation system such as `Ricci` [20].[11]

### 5.1 One real pinor in nine Euclidean dimensions (non-simple normal case)

Consider first a single real pinor on a nine-dimensional Riemannian manifold $(M, g)$, i.e. $p = d = 9$, $q = 0$. We have $p - q \equiv_8 1$, so this example belongs to the normal non-simple case, for which the properties of pinors were treated in subsections 2.6 and 3.2. This situation arises, for example, when using the cone or cylinder formalism [8] to study $\mathcal{N} = 1$ compactifications of M-theory on an eight-dimensional manifold by lifting the problem to the nine-dimensional metric cone or metric cylinder $(M, g)$ constructed over the compactification space.[12] The pin bundle $S$ is an $\mathbb{R}$-vector bundle of rank $N = 2^{\left[\frac{d}{2}\right]} = 16$. The real pin representation $\gamma : \wedge T^*M \to \mathrm{End}(S)$ of $(M, g)$ is fiberwise surjective but not fiberwise injective and we have $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, where $\epsilon_\gamma \in \{-1, +1\}$ is the signature of $\gamma$. We shall choose $\epsilon_\gamma = +1$. Since $d \equiv_8 1$, the discussion of subsection 3.2 shows that — up to multiplication by a nowhere-vanishing smooth function — there is only one admissible pairing $\mathscr{B}$ on $S$, which is symmetric ($\sigma_\mathscr{B} = +1$) and of type $\epsilon_\mathscr{B} = +1$. Upon multiplying with an appropriately-chosen nowhere vanishing function, one can assume without loss of generality that $\mathscr{B}$ is positive-definite and thus a scalar product on $S$. We will henceforth

[11] In contradistinction with the case studied in [8] (where direct computation is prohibitive), some of the examples discussed below can also be analyzed directly. However, we have used our implementation in order to speed up and check some of the computations. Given the space limitations and the focus of the present paper, we prefer to discuss the details of the implementation as well as other technical and mathematical aspects in a separate publication.

[12] Such a construction was alluded to — though not implemented — in [10].



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denote the corresponding norm by $|| \ ||$. The isotropy $\iota_{\mathscr{B}}$ is not defined, since no spin projection exists in this case.

Choosing the pin representation $\gamma$ with signature $\epsilon_\gamma = +1$, we realize the Kähler-Atiyah algebra $(\Omega^+(M), \diamond)$ of $(M, g)$ through the truncated model[13] $(\Omega^<(M), \blacklozenge_+)$, where $\Omega^<(M) = \oplus_{k=0}^4 \Omega^k(M)$. We are interested in pinor bilinears:

$$\check{\boldsymbol{E}}^{(k)} \stackrel{\text{def.}}{=} \frac{1}{k!}\mathscr{B}(\xi, \gamma_{a_1\ldots a_k}\xi')e^{a_1\ldots a_k} \in \Omega^k(M) \ , \quad \forall k \in \overline{0,9}\,, \tag{5.1}$$

where $\xi, \xi' \in \Gamma(M, S)$. From a single pinor $\xi \in \Gamma(M, S)$ (which we normalize through $||\xi|| = 1$) we can construct — up to twisted Hodge duality on $(M, g)$ — the following nontrivial bilinears (a scalar, a one-form and a four-form):

$$\mathscr{B}(\xi,\xi) = 1 \ , \quad V \stackrel{\text{def.}}{=} \check{\boldsymbol{E}}^{(1)} = \mathscr{B}(\xi, \gamma_a\xi)e^a \quad , \quad \Phi \stackrel{\text{def.}}{=} \check{\boldsymbol{E}}^{(4)} = \frac{1}{24}\mathscr{B}(\xi, \gamma_{a_1\ldots a_4}\xi)e^{a_1\ldots a_4}\,, \tag{5.2}$$

where we used identity (3.11) with $\epsilon_{\mathscr{B}} = +1$, which implies that $\mathscr{B}(\xi, \gamma^{a_1\ldots a_k}\xi)$ and thus $\check{\boldsymbol{E}}^{(k)}$ vanish unless $k(k-1) \equiv_4 0 \Leftrightarrow k \equiv_4 0, 1 \Leftrightarrow k = 0, 1, 4, 5, 8, 9$. We have also used the identity $\gamma(\nu) = \gamma^{(9)} = \gamma^1 \circ \ldots \circ \gamma^8 = \mathrm{id}_S$ (which holds since $\epsilon_\gamma = +1$), which implies the relations $\check{\boldsymbol{E}}^{(9-k)} = \tilde{\ast}\check{\boldsymbol{E}}^{(k)}$ for all $k = 0, 1, 4, 5, 8, 9$, where $\tilde{\ast}$ is the *twisted* Hodge operator. This means that the inhomogeneous differential form:

$$\check{\boldsymbol{E}} \stackrel{\text{def.}}{=} \sum_{k=0}^{9} \check{\boldsymbol{E}}^{(k)} \in \Omega(M)$$

is twisted selfdual and thus belongs to the effective domain of definition $\Omega^\gamma(M) = \Omega^+(M)$ of the morphism of $\mathcal{C}^\infty(M, \mathbb{R})$-algebras $\gamma : \Omega(M) \to \Gamma(M, \mathrm{End}(S))$. The lower truncation of $\check{\boldsymbol{E}}$ is the inhomogeneous differential form:

$$\check{\boldsymbol{E}}_< \stackrel{\text{def.}}{=} \sum_{k=0}^{4} \check{\boldsymbol{E}}^{(k)} = 1 + V + \Phi \in \Omega^<(M) \quad .$$

The truncated model of the Fierz algebra[14] admits a basis consisting of the single element:

$$\check{E}_< \stackrel{\text{def.}}{=} \frac{N}{2^d}\check{\boldsymbol{E}}_< = \frac{1}{32}(1 + V + \Phi) \quad .$$

We remind the reader of the relations:

$$\tilde{\ast}\omega = \omega \diamond \nu \ , \quad \ast\,\omega = \tau(\omega) \diamond \nu \ , \quad \forall \omega \in \Omega(M) \ ,$$

where $\ast$ is the ordinary Hodge operator. Since in this case the volume form $\nu$ is central (i.e. $\nu \diamond \omega = \omega \diamond \nu$ for any inhomogeneous differential form $\omega$) and since $\nu \diamond \nu = +1_M$, the

[13] See [1] for a detailed description of truncated models. Here, $\blacklozenge_+ : \Omega^<(M) \longrightarrow \Omega^<(M)$ denotes the reduced geometric product, defined through $\omega \blacklozenge_+ \eta = 2P_<(P_+(\omega) \diamond P_+(\eta))$, for any inhomogeneous forms $\omega$ and $\eta$, where $P_+(\omega) = \frac{1}{2}(\omega + \tilde{\ast}\omega)$ and $P_<(\omega) = \omega_<$.

[14] See [1] for the definition and properties of the Fierz algebra. This applies here since we are in the normal case.

truncated Fierz identity (where — for simplicity of notation — we write $\blacklozenge$ instead of $\blacklozenge_+$) reads (see [1]):

$$\check{E}_< \blacklozenge \check{E}_< = \frac{1}{2}\check{E}_< \quad \left( \iff \check{E} \diamond \check{E} = \check{E} \right)$$

and amounts to:

$$V \blacklozenge V + \Phi \blacklozenge \Phi + V \blacklozenge \Phi + \Phi \blacklozenge V = 15 + 14V + 14\Phi \quad . \tag{5.3}$$

Expanding the $\blacklozenge$-product into generalized products (see [1] for the definition of the latter), we find:

$$V \blacklozenge V = ||V||^2 \quad , \tag{5.4}$$
$$V \blacklozenge \Phi = \Phi \blacklozenge V = *(V \wedge \Phi) \quad , \tag{5.5}$$
$$\Phi \blacklozenge \Phi = ||\Phi||^2 + *(\Phi \wedge \Phi) - \Phi \bigtriangleup_2 \Phi \quad . \tag{5.6}$$

Equality (5.5) holds since $\iota_V \Phi$ vanishes.[15] Using relations (5.4)–(5.6) and separating (5.3) into its rank components gives:

$$||V||^2 + ||\Phi||^2 = 15 \quad , \quad *\,(\Phi \wedge \Phi) = 14V \quad , \quad 2 *(V \wedge \Phi) - \Phi \bigtriangleup_2 \Phi = 14\Phi \quad . \tag{5.7}$$

Let us define non-negative real-valued functions $a, c \in \mathcal{C}^\infty(M, \mathbb{R})$ through:

$$a \stackrel{\text{def.}}{=} ||V||^2 \quad , \quad c \stackrel{\text{def.}}{=} ||\Phi||^2 \quad . \tag{5.8}$$

Since any expression quadrilinear in $\xi$ expands into $\xi$-bilinears and since (5.2) and their Hodge duals generate the $\mathcal{C}^\infty(M, \mathbb{R})$-module of globally-defined inhomogeneous differential forms on $M$ which can be constructed as bilinears in $\xi$, there must exist functions $b, f, e \in \mathcal{C}^\infty(M, \mathbb{R})$ such that the following relations hold:

$$*\,(V \wedge \Phi) = b\Phi \quad , \tag{5.9}$$
$$*(\Phi \wedge \Phi) = fV \quad , \tag{5.10}$$
$$\Phi \bigtriangleup_2 \Phi = e\Phi \quad . \tag{5.11}$$

Using equations (5.8)–(5.11) in (5.7) gives:

$$a + c = 15 \quad , \quad f = 14 \quad , \quad 2b - e = 14 \quad . \tag{5.12}$$

In what follows we shall use associativity of the geometric product. Multiplying (5.4) with $\Phi$ from the right in the truncated model:

$$(V \blacklozenge V) \blacklozenge \Phi = ||V||^2 \Phi$$

and using (5.8) gives:

$$V \blacklozenge V \blacklozenge \Phi = a\Phi \quad . \tag{5.13}$$

[15] Indeed, the three-form $\iota_V \Phi$ should be a bilinear in $\xi$ due to the Fierz relations but — as shown above — one cannot construct any nontrivial three-form bilinear in $\xi$ in this example.

One the other hand, $\blacklozenge$-multiplying (5.5) with $V$ from the left:

$$V\blacklozenge(V\blacklozenge\Phi) = V\blacklozenge(*(V\wedge\Phi)) \ ,$$

and using (5.9) and then again (5.5) and (5.9) gives:

$$V\blacklozenge V\blacklozenge\Phi = bV\blacklozenge\Phi = b*(V\wedge\Phi) = b^2\Phi \ ,$$

which upon comparing with (5.13) lead to:

$$a = b^2 \ . \tag{5.14}$$

Finally, $\blacklozenge$-multiplying (5.5) with $\Phi$ from the right:

$$(V\blacklozenge\Phi)\blacklozenge\Phi = *(V\wedge\Phi)\blacklozenge\Phi$$

and comparing to the relation obtained through $\blacklozenge$-multiplication of (5.6) with $V$ from the left:

$$V\blacklozenge(\Phi\blacklozenge\Phi) = ||\Phi||^2 V + V\blacklozenge(*(\Phi\wedge\Phi)) - V\blacklozenge(\Phi\bigtriangleup_2\Phi)$$

gives, upon using (5.8)–(5.11):

$$bf = c \ , \quad bc = fa \ . \tag{5.15}$$

Relations (5.12)–(5.15) imply a second order equation for $b$:

$$b^2 + 14b - 15 = 0 \ ,$$

which has the solutions $b = 1$ and $b = -15$. The solution $b = 1$ further gives $a = 1$, $e = -12$ and $c = 14$, while the second solution $b = -15$ gives $c = -210$, which cannot be the case since $c$ is the square of the norm of $\Phi$ and hence must be non-negative. We conclude that the Fierz relations (5.3) are equivalent with the following system of conditions on the forms $V$ and $\Phi$:

$$\begin{gathered} ||V||^2 = 1 \ , \qquad ||\Phi||^2 = 14 \ , \\ \iota_V\Phi = 0 \ , \quad *(\Phi\wedge\Phi) = 14V \ , \quad *(V\wedge\Phi) = \Phi \ , \quad \Phi\bigtriangleup_2\Phi = -12\Phi \ . \end{gathered} \tag{5.16}$$

Via the cone formalism of [8], these relations provide a synthetic encoding of the Fierz identities used in [10, 11].

### 5.2 One Majorana spinor in seven Euclidean dimensions (almost complex case)

Consider a Riemannian seven-manifold $(M, g)$ ($d = p = 7$, $q = 0$). Since $p - q \equiv_8 7$, this belongs to the almost complex case, for which the properties of pinors were treated in subsections 2.7 and 3.3. The case of a single pinor (more precisely, that of a Majorana spinor) on a Riemannian seven-manifold arises, for example, in the well-studied case of $\mathcal{N} = 1$ compactifications of $M$-theory on 7-manifolds [12–14], which admit a geometric description through reductions of the structure group of $TM$ from $O(7)$ to $G_2$. In this case, the bundle morphism $\gamma$ is fiberwise injective but not fiberwise surjective, having image equal to $\mathrm{End}_{\mathbb{C}}(S) \subset \mathrm{End}(S)$. As an $\mathbb{R}$-vector bundle, $S$ has rank $\mathrm{rk}_{\mathbb{R}} S = N = 2^{\left[\frac{d}{2}\right]+1} = 16$, while as a $\mathbb{C}$-vector bundle (with complex structure given by $J = \gamma(\nu)$, where $\gamma(\nu) = \gamma^{(8)} = \gamma^1 \circ \ldots \circ \gamma^7$) it has rank $\mathrm{rk}_{\mathbb{C}} S = \Delta = 2^{\left[\frac{d}{2}\right]} = 8$. One has a spin endomorphism given by $\mathcal{R} = D$, which is a real structure (complex conjugation) on $S$ when the latter is viewed as a complex vector bundle. The $D$-real sections $\xi$ of $S$ (those satisfying $D(\xi) = \xi$) are *Majorana spinors*, while $D$-imaginary sections can be obtained by applying $J$ on such spinors. Since $S = S_+ \oplus S_- = S_+ \oplus J(S_+)$ (where $S_+$ is the sub-bundle of Majorana spinors), any section $\xi \in \Gamma(M, S)$ decomposes uniquely as $\xi = \xi_R + i\xi_I = \xi_R + J(\xi_I)$, with $\xi_R = \mathrm{Re}\xi \in \Gamma(M, S_+)$ and $\xi_I = \mathrm{Im}\xi \in \Gamma(M, S_+)$.

Up to rescaling by nowhere-vanishing smooth functions, there are four admissible bilinear pairings $\mathscr{B}_0, \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3$ on $S$ (see subsection 3.3) which can be used to construct form-valued pinor bilinears. Since all these pairings are related through the action of $J$ and $D$, we choose to work with the basic pairing $\mathscr{B}_0$, which is symmetric ($\sigma_0 = +1$) of type $\epsilon_0 = -1$ and isotropy $\iota_0 = +1$. In the following, we consider only the particular case when $\xi \in \Gamma(M, S_+)$ is a *Majorana spinor* — a case which was studied through less formal methods in [12–14].

**Form-valued bilinears constructed using $\gamma$ and $D$.** For $\xi \in \Gamma(M, S_+)$ a Majorana spinor, (3.21) reduces to:[16]

$$\mathscr{B}_0(\xi, \gamma_{a_1 \ldots a_k} \xi) = 0 \ \text{ unless } \ k = \text{even} \iff \check{\boldsymbol{E}}^{(0,k)} = 0 \ \text{ unless } \ k = \text{even} \ , \tag{5.17}$$

where $\check{\boldsymbol{E}}^{(0,k)}$ is defined as in (4.35)–(4.36). On the other hand, we have $(\gamma_a)^t = -\gamma_a$ since $\epsilon_0 = -1$. This implies:

$$(\gamma_{a_1 \ldots a_k})^t = (-1)^{\frac{k(k+1)}{2}} \gamma_{a_1 \ldots a_k} \ .$$

Together with the symmetry of $\mathscr{B}_0$, this shows $\check{\boldsymbol{E}}^{(0,k)}$ vanishes unless $(-1)^{\frac{k(k+1)}{2}} = \sigma_0 = +1$ (cf. (3.18)), i.e. unless $k(k+1) \equiv_4 0 \Leftrightarrow k = 0, 3, 4, 7$. Combining this with (5.17), we conclude that — for $\xi$ a Majorana spinor — the forms $\check{\boldsymbol{E}}^{(0,k)}$ vanish unless $k = 0, 4$. When $\xi$ is normalized through $\mathscr{B}_0(\xi, \xi) = 1$, this gives $\check{\boldsymbol{E}}^{(0,0)} = 1$ and the form-valued $\xi$-bilinear:

$$\phi \stackrel{\text{def.}}{=} \check{\boldsymbol{E}}^{(0,4)} = \frac{1}{24} \mathscr{B}_0(\xi, \gamma_{a_1 \ldots a_4} \xi) e^{a_1 \ldots a_4}$$

which in turn form the components of the first generator $\check{E}^{(0)}$ of the Fierz algebra (cf. expansions (4.35)):

$$\check{E}^{(0)} = \frac{1}{16}(1 + \phi) \ .$$

Notice that $\mathscr{B}_0(\xi, D \circ \gamma_{a_1 \ldots a_k} \xi)$ (which equals $\mathscr{B}_2(\xi, \gamma_{a_1 \ldots a_k} \xi)$ by identity (3.24)) coincides in this case with $(-1)^k \mathscr{B}_0(\xi, \gamma_{a_1 \ldots a_k} \xi)$ and hence leads to the same two bilinears written above. Using (4.35), this shows that the second generator of the Fierz algebra coincides with the first:

$$\check{E}^{(1)} = \check{E}^{(0)} \ .$$

[16]Indeed, the locally-defined endomorphisms $\gamma_a \in \Gamma(U, \mathrm{End}(S))$ are $D$-imaginary, so $\gamma_{a_1 \ldots a_k}$ is $D$-real or $D$-imaginary according to whether $k$ is even or odd. Since $\iota_0 = +1$ and $\xi$ is $D$-real, this implies that $\check{\boldsymbol{E}}^{(k)}$ vanishes unless $k$ is even (cf. subsection 3.3).

**The Fierz identities.** Using the observations above, we find that the Fierz identities (4.34) are mutually-equivalent in our case and amount to the following relation:

$$\check{E}^{(0)} \diamond \check{E}^{(0)} + \pi(\check{E}^{(0)}) \diamond \check{E}^{(0)} = \check{E}^{(0)} \quad .$$

Since $\check{E}^{(0)}$ has only components of even rank, we have $\pi(\check{E}^{(0)}) = \check{E}^{(0)}$ and hence the Fierz relations further reduce to:

$$2\check{E}^{(0)} \diamond \check{E}^{(0)} = \check{E}^{(0)} \quad \Longleftrightarrow \quad (1+\phi) \diamond (1+\phi) = 8(1+\phi) \quad . \tag{5.18}$$

Expanding the geometric product into generalized products, we find:

$$\phi \diamond \phi = ||\phi||^2 - \phi \bigtriangleup_2 \phi \ , \tag{5.19}$$

where we used the identity[17] $\phi \bigtriangleup_4 \phi = ||\phi||^2$. Substituting (5.19) into (5.18) and separarting into rank components shows that the Fierz identities are equivalent with the following two conditions:

$$||\phi||^2 = 7 \quad \text{and} \quad \phi \bigtriangleup_2 \phi = -6\phi \quad . \tag{5.20}$$

**Form-valued bilinears constructed using $\gamma$ and $J$ or $J \circ D$.** As usual in the almost complex case, one can construct further form-valued $\xi$-bilinears using both $\gamma$ and either $J$ or $J \circ D$. As we shall see below, these bilinears contain no new information in our example, but it is instructive to discuss them nonetheless.

Since $\xi$ is a Majorana spinor, the $\xi$-bilinears $\mathscr{B}_0(\xi, J \circ \gamma_{a_1 \ldots a_k}\xi)$ (which equal $\mathscr{B}_1(\xi, \gamma_{a_1 \ldots a_k}\xi)$ by virtue of (3.24)) give the same differential forms as $\mathscr{B}_0(\xi, J \circ D \circ \gamma_{a_1 \ldots a_k}\xi)$ (which equals $-\mathscr{B}_3(\xi, \gamma_{a_1 \ldots a_k}\xi)$), due to the 'selection rule':

$$\mathscr{B}_0(\xi, J \circ \gamma_{a_1 \ldots a_k}\xi) = -\mathscr{B}_0(\xi, J \circ D \circ \gamma_{a_1 \ldots a_k}\xi) = 0 \ \ \text{unless} \ \ k = \text{odd} \quad . \tag{5.21}$$

The first of identities (3.3) takes the form $J^t = J$ since $\frac{d(d+1)}{2} = 28$ is even in our case. Together with (3.11), this gives:

$$(J \circ \gamma_{a_1 \ldots a_k})^t = (-1)^{\frac{k(k+1)}{2}} J \circ \gamma_{a_1 \ldots a_k} \ , \tag{5.22}$$

where we used $\epsilon_0 = +1$ and the fact that $J$ commutes with $\gamma_a$. Relation (5.22) implies that the differential form $\frac{1}{k!}\mathscr{B}(\xi, J \circ \gamma_{a_1 \ldots a_k}\xi)e^{a_1 \ldots a_k}$ vanishes unless $k(k+1) \equiv_4 0 \Longleftrightarrow k \equiv_4 0, 3$. Together with (5.21), this shows that the nontrivial homogeneous form-valued bilinears constructed using $J$ and $\gamma$ have ranks $k = 3$ or $k = 7$ and can thus be written as follows, up to multiplication by elements of $\mathcal{C}^\infty(M, \mathbb{R})$:

$$\psi \stackrel{\text{def.}}{=} \frac{1}{3!}\mathscr{B}_0(\xi, J \circ \gamma_{a_1 \ldots a_3}\xi)e^{a_1 \ldots a_3} \ ,$$
$$\eta \stackrel{\text{def.}}{=} \frac{1}{7!}\mathscr{B}_0(\xi, J \circ \gamma_{a_1 \ldots a_7}\xi)e^{a_1 \ldots a_7} \quad .$$

Using the case $d = 7$ of the well-known formula:

$$\gamma_{a_1 \ldots a_k} = \frac{(-1)^{\frac{k(k-1)}{2}}}{(d-k)!}\epsilon_{a_1 \ldots a_k}{}^{a_{k+1} \ldots a_d}\gamma_{a_{k+1} \ldots a_d}\gamma^{(d+1)} \tag{5.23}$$

[17]One has $\omega \bigtriangleup_k \omega = ||\omega||^2$ for all $\omega \in \Omega^k(M)$ and any $k = 0 \ldots d$.

and the fact that $J = \gamma^{(8)}$, we find:

$$\gamma_{a_1\ldots a_3} = -\frac{1}{4!}\epsilon_{a_1\ldots a_3}{}^{b_1\ldots b_4}\gamma_{b_1\ldots b_4}J \ ,$$
$$\gamma_{a_1\ldots a_7} = -\epsilon_{a_1\ldots a_7}\gamma^{(8)} = -\epsilon_{a_1\ldots a_7}J \ .$$

Since $J$ commutes with $\gamma^a$ and $J^2 = -\mathrm{id}_S$, substitution of these relations into (5.23) gives:

$$\psi = \frac{1}{4!}\mathscr{B}_0(\xi, \gamma_{b_1\ldots b_4}\xi)\epsilon_{a_1\ldots a_3}{}^{b_1\ldots b_4}e^{a_1\ldots a_3} \implies \psi = \tilde{*}\phi = *\phi \ ,$$
$$\eta = \epsilon_{a_1\ldots a_7}\mathscr{B}_0(\xi,\xi)e^{a_1\ldots a_7} \implies \eta = \nu \ ,$$

where we noticed that $\tilde{*}\phi = *\phi$ since $\mathrm{rk}\phi = 4$ implies $\tau(\phi) = \phi$. This shows that the $\xi$-bilinears constructed using $\gamma$ and $J$ or $J \circ D$ contain no information beyond that already contained in $\phi$.

**Some identities for $\phi$ and $\psi = *\phi$.** The identity $||*\omega|| = ||\omega||$ (which holds for any $\omega \in \Omega(M)$), implies that the norm of $\psi$ equals that of $\phi$:

$$||\psi|| = ||\phi|| \ .$$

Since $d = 7$ and $q = 0$ in our example, we have:

$$* \circ * = (-1)^q\pi^{d-1} = +\mathrm{id}_{\Omega(M)} \ \text{ and } \ \nu \diamond \nu = (-1)^{q+\left[\frac{d}{2}\right]} = -1_M \ .$$

This implies that relation $\psi = *\phi = \phi \diamond \nu$ is equivalent with $\phi = *\psi = -\psi \diamond \nu$. Since $\mathrm{rk}\phi = 4$, we have $\pi(\phi) = \phi$, which implies $\nu \diamond \phi = \phi \diamond \nu$. Using these observations, we compute: $\psi\diamond\psi = (*\phi)\diamond(*\phi) = \phi\diamond\nu\diamond\phi\diamond\nu = \phi\diamond\phi\diamond\nu\diamond\nu = -\phi\diamond\phi$, which gives $\psi\diamond\psi = -\phi\diamond\phi$. Combining this with (5.19) and the generalized product expansion:

$$\psi \diamond \psi = -||\psi||^2 + \psi \bigtriangleup_1 \psi \tag{5.24}$$

gives:

$$\psi \bigtriangleup_1 \psi = \phi \bigtriangleup_2 \phi \ . \tag{5.25}$$

Similarly, we have $\psi \diamond \phi = -(\psi \diamond \psi) \diamond \nu = ||\psi||^2\nu - *(\psi \bigtriangleup_1 \psi)$, where we noticed that $\tilde{*}(\psi \bigtriangleup_1 \psi) = *(\psi \bigtriangleup_1 \psi)$ since $(\psi \bigtriangleup_1 \psi)$ has rank 4. Comparing this with the generalized product expansion:

$$\psi \diamond \phi = \psi \wedge \phi + \psi \bigtriangleup_1 \phi - \psi \bigtriangleup_2 \phi - \psi \bigtriangleup_3 \phi \tag{5.26}$$

and separating ranks gives:

$$\psi \wedge \phi = ||\psi||^2\nu \ , \quad \psi \bigtriangleup_1 \phi = 0 \ , \quad \psi \bigtriangleup_2 \phi = *(\psi \bigtriangleup_1 \psi) \ , \quad \psi \bigtriangleup_3 \phi = 0 \ . \tag{5.27}$$

Combining (5.25) and (5.27) with the Fierz identities (5.18) gives the following system of relations for the four-form $\phi$ and its Hodge dual $\psi = *\phi$:

$$||\phi||^2 = ||\psi||^2 = 7 \qquad , \psi \bigtriangleup_1 \psi = \phi \bigtriangleup_2 \phi = -6\phi \ , \tag{5.28}$$
$$\psi \wedge \phi = 7\nu \qquad , \psi \bigtriangleup_1 \phi = \psi \bigtriangleup_3 \phi = 0 \ , \qquad \psi \bigtriangleup_2 \phi = -6\psi \ . \tag{5.29}$$



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**Comparison with the literature.** As discussed at the end of subsection 3.3, the spinor bilinears can also be written using the fiberwise $\mathbb{C}$-bilinear paring $\beta$ on $S$ obtained by complexification of the restriction $\mathscr{B}_0|_{S_+\otimes S_+}$ (cf. (3.33)):

$$\beta(\xi,\xi') = \mathscr{B}_0(\xi,\xi') - i\mathscr{B}_0(\xi, J\xi') \ , \quad \forall \xi,\xi' \in \Gamma(M,S) \ .$$

For $J\xi = i\xi$, this gives:[18]

$$\begin{aligned}
\beta(\xi,\xi) = \mathscr{B}_0(\xi,\xi) \quad &\Longleftrightarrow \quad \hat{\xi}^T\hat{\xi} = 1 \ , \\
\beta(\xi,\gamma_{a_1\ldots a_3}\circ J\xi) = \mathscr{B}_0(\xi,\gamma_{a_1\ldots a_3}\circ J\xi) \quad &\Longleftrightarrow \quad i\hat{\xi}^T\hat{\gamma}_{a_1\ldots a_3}\hat{\xi} = \psi_{a_1\ldots a_3} \ , \\
\beta(\xi,\gamma_{a_1\ldots a_4}\xi) = \mathscr{B}_0(\xi,\gamma_{a_1\ldots a_4}\xi) \quad &\Longleftrightarrow \quad \hat{\xi}^T\hat{\gamma}_{a_1\ldots a_4}\hat{\xi} = \phi_{a_1\ldots a_4} \ , \\
\beta(\xi,\gamma_{a_1\ldots a_7}\circ J\xi) = \mathscr{B}_0(\xi,\gamma_{a_1\ldots a_7}\circ J\xi) \quad &\Longleftrightarrow \quad i\hat{\xi}^T\hat{\gamma}_{a_1\ldots a_7}\hat{\xi} = \eta_{a_1\ldots a_7} \ .
\end{aligned}$$

The authors of [12] use the bilinears $\varpi \stackrel{\text{def.}}{=} -\psi, \varphi \stackrel{\text{def.}}{=} -\phi$ and $\epsilon \stackrel{\text{def.}}{=} -\eta$, i.e.:

$$\begin{aligned}
\varpi_{a_1\ldots a_3} &= -i\hat{\xi}^T\hat{\gamma}_{a_1\ldots a_3}\hat{\xi} = -i\beta(\xi,\gamma_{a_1\ldots a_3}\xi) \ , \\
\varphi_{a_1\ldots a_4} &= -\hat{\xi}^T\hat{\gamma}_{a_1\ldots a_4}\hat{\xi} = -\beta(\xi,\gamma_{a_1\ldots a_4}\xi) \ , \\
\epsilon_{a_1\ldots a_7} &= -i\hat{\xi}^T\hat{\gamma}^{a_1\ldots a_7}\hat{\xi} = -i\beta(\xi,\gamma_{a_1\ldots a_7}\xi) \ .
\end{aligned}$$

The minus sign in the correspondence $\phi \leftrightarrow -\varphi$, $\psi \leftrightarrow -\varpi$, $\eta \leftrightarrow -\epsilon$ arises from the fact that [12] use the complex structure $J' = -\gamma(\nu) = -J$ on the bundle $S$, which is the conjugate of the complex structure $J = +\gamma(\nu)$ on $S$ used in the present paper. This translates into the relation $J'\gamma(\nu) = +\mathrm{id}_S$ and implies $J'\xi = -i\xi$ if multiplication by $i$ is defined using $J$. Notice the Hodge duality relation:

$$\varpi_{a_1\ldots a_3} = \beta\left(\xi, \frac{1}{4!}\epsilon_{a_1\ldots a_3}{}^{a_4\ldots a_7}\gamma_{a_4\ldots a_7}\gamma(\nu)\circ J\xi\right) = -\frac{1}{4!}\beta(\xi,\epsilon_{a_1\ldots a_3}{}^{a_4\ldots a_7}\gamma_{a_4\ldots a_7}\xi) \stackrel{\text{def.}}{=} *(\varphi_{a_4\ldots a_7}) \ .$$

The Fierz identities listed in [13, page 4]:

$$\begin{aligned}
\varpi_{abe}\varpi^{ecd} &= -\varphi_{ab}{}^{cd} + 2\delta^{cd}_{ab} \ , \\
\varpi^{abf}\varphi_{fcde} &= -6\delta^{[a}_{[c}\varpi^{b]}{}_{de]} \ , \\
\varphi_{abcg}\varphi^{defg} &= 6\delta^{abc}_{def} - 9\varphi_{[ab}{}^{[de}\delta^{f]}_{c]} - \varpi_{abc}\varpi^{def}
\end{aligned} \tag{5.30}$$

imply the relations:

$$\begin{aligned}
\varpi_{abe}\varpi^{ecd} = -\varphi_{ab}{}^{cd} + 2\delta^{cd}_{ab} \quad &\Longrightarrow \quad \varpi \bigtriangleup_1 \varpi = -6\varphi, \\
\varpi^{bfa}\varphi_{fade} = 4\varpi_{bde} \quad &\Longrightarrow \quad \varpi \bigtriangleup_2 \varphi = 6\varpi \ , \\
\varphi_{abcg}\varphi^{decg} = 42\delta^{ab}_{de} - 2\varphi_{ab}{}^{de} \quad &\Longrightarrow \quad \varphi \bigtriangleup_2 \varphi = -6\varphi \ ,
\end{aligned} \tag{5.31}$$

which indeed agree with (5.28)–(5.29).

[18] The expressions in the right hand side hold in a local frame $(\epsilon_\alpha, J(\epsilon_\alpha))$ of $S$ defined above an open subset $U$ of $M$, where $(\epsilon_\alpha)_{\alpha=1,\ldots,\Delta}$ is a local frame of $S_+$ — see subsection 3.3 for details.

### 5.3 One real pinor in five dimensions with metric signature $(p,q) = (1,4)$ (quaternionic case)

Consider one real pinor on a psedo-Riemannian five-manifold $(M,g)$ with the mostly minus signature $(p,q) = (1,4)$. We have $p-q \equiv_8 5$, which places this example in the quaternionic non-simple case (see sections 2.8, 3.4). This situation occurs in the study of (gauged) $\mathcal{N}=1$ supergravity in five-dimensions (see, for example, [15–18]). The morphism $\gamma : \wedge T^*M \to \mathrm{End}(S)$ is neither injective nor surjective, having image equal to $\mathrm{End}_{\mathbb{H}}(S)$. It satisfies $\gamma(\nu) = \epsilon_\gamma \mathrm{id}_S$, where $\epsilon_\gamma \in \{-1,1\}$ is the signature of $\gamma$. We choose to work with $\epsilon_\gamma = +1$, i.e. $\gamma(\nu) = \mathrm{id}_S$, and with the *basic admissible pairing* $\mathscr{B}_0$ out of the four independent admissible pairings ($\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{B}_3$) which exist in this case. Notice that $\mathscr{B}_0$ is symmetric ($\sigma_0 = +1$) and has type $\epsilon_0 = +1$. The restriction $\gamma_{\mathrm{ev}}$ is fiberwise irreducible, so chiral projectors cannot be defined in this case.

Choosing the pin representation $\gamma$ with signature $\epsilon_\gamma = +1$, we realize the Kähler-Atiyah algebra $(\Omega^+(M), \diamond)$ through the truncated model $(\Omega^<(M), \blacklozenge_+)$, where $\Omega^<(M) \stackrel{\mathrm{def.}}{=} \oplus_{k=0}^2 \Omega^k(M)$. In the case where there is only one globally-defined pinor $\xi \in \Gamma(M,S)$, we are interested in the following pinor bilinears:

$$\begin{aligned}
\check{\boldsymbol{E}}^{(0,k)} &\stackrel{\mathrm{def.}}{=} \frac{1}{k!}\mathscr{B}_0(\xi, \gamma_{a_1\ldots a_k}\xi')e^{a_1\ldots a_k} \in \Omega^k(M) \ , \quad \forall k \in \overline{0,2} \ , \\
\check{\boldsymbol{E}}^{(i,p)} &\stackrel{\mathrm{def.}}{=} \frac{1}{p!}\mathscr{B}_0(\xi, J_i \circ \gamma_{a_1\ldots a_p}\xi')e^{a_1\ldots a_p} \in \Omega^p(M) \ , \quad \forall p \in \overline{0,2}\,, \ \forall i \in \overline{1,3} \ .
\end{aligned}$$

**Form-valued bilinears constructed using only $\boldsymbol{\gamma}$.** The symmetry $\sigma_0 = 1$ of $\mathscr{B}_0$ implies:

$$\mathscr{B}_0(\xi, \gamma_A \xi) = \mathscr{B}_0(\gamma_A\xi, \xi) \ .$$

Since $\mathscr{B}_0$ has type $\epsilon_0 = 1$, we also have:

$$\mathscr{B}_0(\gamma_A\xi, \xi) = (-1)^{\frac{|A|(|A|-1)}{2}}\mathscr{B}_0(\xi, \gamma_A\xi) \ ,$$

which further implies that $\check{\boldsymbol{E}}^{(0,k)} = \frac{1}{k!}\mathscr{B}_0(\xi, \gamma_{a_1\ldots a_k}\xi)e^{a_1\ldots a_k}$ vanishes unless $k$ satisfies $k(k-1) \equiv_4 0$. The latter holds when $k = 0,1,4,5$, which gives the non-trivial pinor bilinears:

$$\begin{aligned}
f &= \mathscr{B}_0(\xi,\xi)\,, & V &= \mathscr{B}_0(\xi, \gamma_a\xi)e^a\,, \\
W &= \frac{1}{4!}\mathscr{B}_0(\xi, \gamma_{a_1\ldots a_4}\xi)e^{a_1\ldots a_4}\,, & \nu &= \frac{1}{5!}\mathscr{B}_0(\xi, \gamma_{a_1\ldots a_5}\xi)e^{a_1\ldots a_5} \ .
\end{aligned} \tag{5.32}$$

**Form-valued bilinears constructed using $\boldsymbol{\gamma}$ and $\boldsymbol{J_i}$.** Since $\sigma_0 = +1$ while $J_i$ commutes with $\gamma_a$, we have:

$$\mathscr{B}_0(\xi, J_i \circ \gamma_A\xi) = \mathscr{B}_0(J_i \circ \gamma_A\xi, \xi) = \mathscr{B}_0(\gamma_A \circ J_i\xi, \xi) \ .$$

Using the fact that the $\mathscr{B}_0$-transpose of $J_i$ is given by $J_i^t = -J_i$, we find:

$$\mathscr{B}_0(\gamma_A \circ J_i\xi, \xi) = (-1)^{\frac{|A|(|A|-1)}{2}+1}\mathscr{B}_0(\xi, J_i \circ \gamma_A\xi) \ .$$

The last two identities imply that the bilinears $\check{\boldsymbol{E}}^{(i,p)}$ vanish unless $p(p-1)+2 \equiv_4 0$, i.e unless $p=2,3$. This gives the nontrivial bilinears:

$$\Phi^i = \frac{1}{2!}\mathscr{B}_0(\xi, J^i \circ \gamma_{a_1 a_2}\xi)e^{a_1 a_2} \quad , \quad \Psi^i = \frac{1}{3!}\mathscr{B}_0(\xi, J^i \circ \gamma_{a_1 \dots a_3}\xi)e^{a_1 \dots a_3} \quad , \quad \forall i = 1, \dots, 3 \quad .$$

Identity (5.23) and the relation $\tilde{*} = * \circ \tau$ imply that the form-valued bilinears of rank greater than two can be written as:

$$\nu = *f = \tilde{*}f \quad , \quad W = *V = \tilde{*}V \quad , \quad \Psi^i = -*\Phi^i = \tilde{*}\Phi^i$$

and are thus dual to the scalar, vector and two-forms respectively. As expected, this means that the non-truncated inhomogeneous forms:

$$\check{E}^{(0)} = \frac{1}{2^4}(f + V + W + \nu) \quad \text{and} \quad \check{E}^{(i)} = \frac{1}{2^4}(\Phi^i + \Psi^i)$$

are *twisted selfdual*. The factor of $\frac{1}{2^4}$ comes from the prefactor $\frac{\Delta}{2^d}$ in the general expressions for the generators of the Fierz algebra, where we have substituted $\Delta = 2$ and $d = 5$.

Since for the truncated model we only need the form bilinears of rank less than or equal to 2:

$$\begin{aligned} f &= \mathscr{B}_0(\xi,\xi)\,, \qquad V = \mathscr{B}_0(\xi,\gamma_m\xi)e^m\,, \\ \Phi^i &= \mathscr{B}_0(\xi, J^i \circ \gamma_{mn}\xi)e^{mn}\,, \ \forall i = 1, \dots, 3\,, \ \forall m,n = 1, \dots, 5, \end{aligned} \tag{5.33}$$

we build the truncated generators of the Fierz algebra as:

$$\check{E}^{(0)}_{<} = \frac{1}{2^4}(f + V) \quad \text{and} \quad \check{E}^{(i)}_{<} = \frac{1}{2^4}\Phi^i \ . \tag{5.34}$$

**Analysis of Fierz relations for the truncated algebra $(\boldsymbol{\Omega}^{<}(\boldsymbol{M}), \blacklozenge_+)$.** In what follows we shall omit the subscript '+' of $\blacklozenge_+$. Using the expression [1]:

$$\check{E}^{(0)} = 2P_+(\check{E}^{(0)}_{<}) \quad \text{and} \quad \check{E}^{(i)} = 2P_+(\check{E}^{(i)}_{<})$$

and the relation $P_+(\omega \blacklozenge \eta) = P_+(\omega) \diamond P_+(\eta)$ (where $P_+ = \frac{1}{2}(1 + \tilde{*})$), the Fierz identities for the truncated generators become:

$$\begin{aligned} \check{E}^{(0)}_{<} \blacklozenge \check{E}^{(0)}_{<} - \sum_i \check{E}^{(i)}_{<} \blacklozenge \check{E}^{(i)}_{<} &= \frac{1}{2}\mathscr{B}_0(\xi,\xi)\check{E}^{(0)}_{<} \quad , \quad \forall i = 1 \dots 3 \ , \\ \check{E}^{(0)}_{<} \blacklozenge \check{E}^{(i)}_{<} + \check{E}^{(i)}_{<} \blacklozenge \check{E}^{(0)}_{<} + \sum_{j,k=1}^{3} \epsilon_{ijk} \check{E}^{(j)}_{<} \blacklozenge \check{E}^{(k)}_{<} &= \frac{1}{2}\mathscr{B}_0(\xi,\xi)\check{E}^{(i)}_{<} \ , \end{aligned}$$

i.e.:

$$\begin{aligned} (f + V) \blacklozenge (f + V) - \sum_i \Phi^i \blacklozenge \Phi^i &= 8f(f + V) \ , \\ (f + V) \blacklozenge \Phi^i + \Phi^i \blacklozenge (f + V) + \sum_{j,k} \epsilon_{ijk} \Phi^j \blacklozenge \Phi^k &= 8f\Phi^i \ . \end{aligned} \tag{5.35}$$

The latter simplify to:

$$V \blacklozenge V - \sum_i \Phi^i \blacklozenge \Phi^i \,=\, 7f^2 + 6fV \ , \tag{5.36}$$

$$V \blacklozenge \Phi^i + \Phi^i \blacklozenge V + \sum_{j,k} \epsilon_{ijk} \Phi^j \blacklozenge \Phi^k \,=\, 6f\Phi^i \ . \tag{5.37}$$

The various terms on the left hand side expand as:

$$\begin{aligned} V \blacklozenge V \,&=\, ||V||^2 \ , \\ \sum_i \Phi^i \blacklozenge \Phi^i \,&=\, \sum_i \left(\tilde{*}(\Phi^i \wedge \Phi^i) - ||\Phi^i||^2\right) \ , \\ V \blacklozenge \Phi^i + \Phi^i \blacklozenge V \,&=\, 2\tilde{*}(V \wedge \Phi^i) \ , \\ \sum_{j,k} \epsilon_{ijk} \Phi^j \blacklozenge \Phi^k \,&=\, \sum_{j,k} \epsilon_{ijk} \left(\tilde{*}(\Phi^j \wedge \Phi^k) - \Phi^j \bigtriangleup_1 \Phi^k - \Phi^j \bigtriangleup_2 \Phi^k\right) \ . \end{aligned} \tag{5.38}$$

We have $\iota_V \Phi^i = 0$ since $V$ is the only non-trivial pinor bilinear which can be constructed at rank one and $\iota_V \Phi^i$ is a rank one bilinear, which must therefore be proportional to $V$:

$$\iota_V \Phi^i = c_1 V \ \Rightarrow \ \iota_V \iota_V \Phi^i = c_1 ||V||^2$$

for some constant $c_1$. However, we have:

$$\iota_V \iota_V \Phi^i = \iota_{V \wedge V} \Phi^i = 0$$

since $V \wedge V = 0$, which implies that $c_1 ||V||^2 = 0$ and thus $c_1 = 0$, which means that $\iota_V \Phi^i$ vanishes. This further leads to:

$$V \blacklozenge \Phi^i = \Phi^i \blacklozenge V = \tilde{*}(V \wedge \Phi^i) \ . \tag{5.39}$$

Let us introduce the following notations:

$$\begin{gathered} ||V||^2 = a \ , \qquad \sum_i ||\Phi^i||^2 = c \ , \qquad \sum_i \tilde{*}(\Phi^i \wedge \Phi^i) = eV \ , \qquad \tilde{*}(V \wedge \Phi^i) = b\Phi^i \, , \\ \sum_{j,k} \epsilon_{ijk} \tilde{*}(\Phi^j \wedge \Phi^k) = w^i V \ , \quad \sum_{j,k} \epsilon_{ijk} \Phi^j \bigtriangleup_1 \Phi^k = u\Phi^i \quad , \ \sum_{j,k} \epsilon_{ijk} \Phi^j \bigtriangleup_2 \Phi^k = l^i \, , \end{gathered} \tag{5.40}$$

where $a, b, c, e, u, l^i, w^i$ are non-negative smooth real-valued functions on $M$. Using relations (5.38) in (5.36)–(5.37), we find:

$$a + c = 7f^2 \ , \quad e = -6f \ , \quad 2b - u = 6f \ , \quad w^i = l^i = 0 \ . \tag{5.41}$$

With (5.39)–(5.41), the system (5.38) simplifies to:

$$V \blacklozenge V \,=\, ||V||^2 = a \ , \tag{5.42}$$

$$\sum_i \Phi^i \blacklozenge \Phi^i \,=\, -6fV - c \ , \tag{5.43}$$

$$V \blacklozenge \Phi^i = \Phi^i \blacklozenge V \,=\, b\Phi^i \ , \tag{5.44}$$

$$\sum_{j,k} \epsilon_{ijk} \Phi^j \blacklozenge \Phi^k \,=\, -(6f - 2b)\Phi^i \ . \tag{5.45}$$



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In what follows we use associativity of the reduced geometric product $\blacklozenge$. First notice that $\blacklozenge$-multipling (5.44) with $\Phi^i$ from the right and summing the result over $i = 1 \ldots 3$ gives:

$$\sum_i (V\blacklozenge\Phi^i)\blacklozenge\Phi^i = b\sum_i \Phi^i\blacklozenge\Phi^i \ ,$$

which upon using (5.43) leads to:

$$\sum_i V\blacklozenge\Phi^i\blacklozenge\Phi^i = -6bfV - 6bc \ . \tag{5.46}$$

Furthermore, $\blacklozenge$-multiplying (5.43) with $V$ from the left gives:

$$V\blacklozenge\left(\sum_i \Phi^i\blacklozenge\Phi^i\right) = -6fV\blacklozenge V - 6cV$$

which takes the following form upon using (5.42):

$$V\blacklozenge\sum_i \Phi^i\blacklozenge\Phi^i = -6fa - 6cV \ . \tag{5.47}$$

Subtracting (5.47) from (5.46) gives:

$$c = 6bf \ , \quad a = b^2 \ . \tag{5.48}$$

From (5.48) and the first relation in (5.41), we find a second order equation for $b$:

$$b^2 + 6bf - 7f^2 = 0 \ ,$$

which has the solutions $b = f$ and $b = -7f$. The solution $b = f$ further gives $a = f^2$, $c = 6f^2$ and $u = -4f$, while the second solution $b = -7f$ gives $c = -42f^2$, which cannot hold since $c = \sum_i ||\Phi^i||^2$ must be non-negative. We conclude that the Fierz relations (5.36)–(5.37) are equivalent with the following system of conditions on the forms $V$ and $\Phi^i$:

$$\begin{aligned}
&||V||^2 = f^2 \ , \quad \sum_{i=1}^{3} ||\Phi^i||^2 = 6f^2 \ , \quad \iota_V\Phi^i = 0 \ , \\
&\tilde{*}(V\wedge\Phi^i) = f\Phi^i \implies \iota_V\tilde{*}\Phi^i = f\Phi^i \iff \iota_V * \Phi^i = -f\Phi^i \ , \\
&\sum_{i=1}^{3}\tilde{*}(\Phi^i\wedge\Phi^i) = -6fV \implies \sum_{i=1}^{3}(\Phi^i\wedge\Phi^i) = -6f\tilde{*}V = -6f*V \ , \\
&\sum_{j,k}\epsilon_{ijk}\tilde{*}(\Phi^j\wedge\Phi^k) = 0 \implies \Phi^j\wedge\Phi^k = 0 \ , \ \forall j,k = 1\ldots 3 \ , \ j\neq k \ , \\
&\sum_{j,k}\epsilon_{ijk}\Phi^j \vartriangle_2 \Phi^k = 0 \implies \iota_{\Phi^j}\Phi^k = 0 \ , \ \forall j,k = 1\ldots 3 \ , \ j\neq k \ , \\
&\sum_{j,k}\epsilon_{ijk}\Phi^j \vartriangle_1 \Phi^k = -4f\Phi^i \implies \Phi^j \vartriangle_1 \Phi^k = -2f\epsilon^{ijk}\Phi^i \ , \ \forall i,j,k = 1\ldots 3 \ .
\end{aligned} \tag{5.49}$$

To arrive at the equations above, we made use of the definitions:

$$\tilde{*}\omega = \omega\diamond\nu \ , \quad *\,\omega = \tau(\omega)\diamond\nu = \iota_\omega\nu$$

and of the properties $\tilde{*} \circ \tilde{*} = * \circ * = \mathrm{id}_{\Omega(M)}$ as well as of the following identities (see [19]) which hold for homogeneous forms $\omega \in \Omega^r(M)$ and $\eta \in \Omega^s(M)$:

$$\begin{aligned} \omega \wedge *\eta &= (-1)^{r(s-1)} * \iota_{\tau(\omega)}\eta \ , \quad \text{when} \quad r \leq s \ , \\ \iota_\omega(*\eta) &= (-1)^{rs} * (\tau(\omega) \wedge \eta) \ , \quad \text{when} \quad r+s \leq d \ . \end{aligned} \tag{5.50}$$

**Comparison with the literature.** Using one 'generalized symplectic-Majorana spinor' (see below), which appears in the literature as a pair of spinors of real dimension $N = 2^{[\frac{d}{2}]} = 4$, the authors of [15–18] construct a scalar, a vector and three two-forms as spinor bilinears. Specifically, writing the symplectic Majorana spinor as the spinor-pair $(\xi_1, \xi_2)$, these bilinears are written in [18] as:

$$s = i\bar{\xi}_j \xi^j \ , \quad v_m = i\bar{\xi}_j \Gamma_m \xi^j \ , \quad \phi^r_{mn} = \bar{\xi}_j (\boldsymbol{\sigma}^r)_k{}^j \Gamma_{mn} \xi^k \ ,$$

for all $j, k \in 1, 2\,,\ \ m, n \in 0, \ldots, 4\,,\ \ r \in 1, \ldots, 3$ and where $\boldsymbol{\sigma}^r$ are the Pauli matrices.

Using the *biquaternion formalism* of subsection 2.8 for the case $p = 1,\ q = 4$, we have $(\Gamma^4)^2 = -\mathrm{id}_{S^+_\mathbb{C}}$ and $\Gamma^4$ is a complex structure on $S^+_\mathbb{C}$. Furthermore, one can choose $\Gamma^4$ such that $[\mathfrak{Z}_0, \Gamma^4]_{-,\circ} = 0$, where the endomorphism $\mathfrak{Z}_0 \in \Gamma(M, \mathrm{End}_\mathbb{R}(S^+_\mathbb{C}))$ was defined in subsection 2.8, and hence the endomorphism $\rho \in \Gamma(M, \mathrm{End}_\mathbb{C}(S_\mathbb{C}))$, which has the matrix form:

$$\hat{\rho} \stackrel{\text{def.}}{=} i\boldsymbol{\sigma}_2 \Gamma^4 = \begin{bmatrix} 0 & \Gamma^4 \\ -\Gamma^4 & 0 \end{bmatrix}$$

satisfies $[\mathfrak{Z}, \rho]_{-,\circ} = 0$. Recall the notation of subsection 2.8: $\mathfrak{Z}$ is a natural real structure on $S_\mathbb{C}$, $\mathfrak{R}$ is a $\mathbb{C}$-linear product structure anticommuting with $\mathfrak{Z}$ and $\mathfrak{J}_2$ is the complexification of the endomorphism $J_2 \in \Gamma(U, \mathrm{End}_\mathbb{C}(S_\mathbb{C}))$. It is easy to check that $\rho$ satisfies:

$$\rho^2 = \mathrm{id}_{S_\mathbb{C}} \ , \quad [\rho, \mathfrak{R}]_{+,\circ} = 0 \ , \quad [\rho, \mathfrak{J}_2]_{-,\circ} = 0 \ .$$

We can thus define another real structure $\ ^*$ on $S$ through:

$$\xi^* \stackrel{\text{def.}}{=} \rho \circ \mathfrak{Z}(\xi) \ ,$$

since the properties listed above insure that $\ ^*$ is antilinear and that it squares to $+\mathrm{id}_{S_\mathbb{C}}$. We have:

$$\xi^* = \Gamma^4 \circ \mathfrak{Z}_0(\xi) \iff \mathfrak{Z}_0(\xi) = -\Gamma^4(\xi^*) \ , \quad \forall \xi \in \Gamma(M, S^+_\mathbb{C}) \ .$$

Condition $[\mathfrak{Z}, \rho]_{-,\circ} = 0$ is equivalent with $(\Gamma^4)^* = \Gamma^4$, i.e. $(\Gamma^4 \circ T^*)^* = \Gamma^4 \circ T$ for any $T \in \Gamma(M, \mathrm{End}(S^+_\mathbb{C}))$. Since $\mathfrak{Z}$ anticommutes with $\mathfrak{R}$ and commutes with $\mathfrak{J}_2$, it follows that the real structure $\ ^*$ commutes with $\mathfrak{R}$ and with $\mathfrak{J}_2$ and hence it satisfies:

$$(\xi^*)^1 = (\xi^1)^* \ , \quad (\xi^*)^2 = (\xi^2)^* \implies \widehat{\xi^*} = \begin{bmatrix} (\xi^1)^* \\ (\xi^2)^* \end{bmatrix} \ , \quad \forall \xi \in \Gamma(M, S) \ .$$

The restriction of $\ ^*$ to $S_+$ is a real structure on $S^+_\mathbb{C}$. It follows that $\mathfrak{Z}$ can be expressed as:

$$\mathfrak{Z}(\xi) = \rho(\xi^*) \implies \widehat{\mathfrak{Z}(\xi)} = \begin{bmatrix} 0 & \Gamma_4 \\ -\Gamma_4 & 0 \end{bmatrix} \begin{bmatrix} (\xi^1)^* \\ (\xi^2)^* \end{bmatrix} \ .$$

The reality condition $\xi \in \Gamma(M, S) \Longleftrightarrow \mathfrak{Z}(\xi) = \xi$ can be written as:

$$\rho(\xi^*) = \xi \iff (\xi^1)^* = \Gamma^4 \xi^2 \iff (\xi^2)^* = -\Gamma^4 \xi^1 \quad . \tag{5.51}$$

Condition (5.51) is sometimes called the '(generalized) symplectic Majorana condition' in the literature on supergravities in five dimensions (see, for example, [15–18]). In those references, one also chooses the morphism $\Gamma : \wedge T^*M \to \mathrm{End}_{\mathbb{C}}(S^+_{\mathbb{C}})$ such that $\Gamma^0, \ldots, \Gamma^3$ are purely imaginary with respect to the complex conjugation $^*$ on $S^+_{\mathbb{C}}$, i.e. :

$$(\Gamma^0)^* = -\Gamma^0 \ , \ (\Gamma^1)^* = -\Gamma^1 \ , \ (\Gamma^2)^* = -\Gamma^2 \ , \ (\Gamma^3)^* = -\Gamma^3 \ , \ (\Gamma^4)^* = +\Gamma^4 \ .$$

The matrices of $\Gamma^m$ with respect to a local frame of $S^+_{\mathbb{C}}$ are the (complex) 'gamma matrices' used in [15–18].

The 'generalized symplectic Majorana' pinors $(\xi_1, \xi_2)$ therefore correspond to those denoted by $(\xi_+, \xi_-)$ in [15–18], where $\xi_\pm \in \Gamma(M, S^\pm_{\mathbb{C}})$. Using the correspondence between the endomorphisms $J_k \in \Gamma(U, \mathrm{End}_{\mathbb{C}}(S_{\mathbb{C}}))$ and the Pauli matrices:

$$J_1 \to i\boldsymbol{\sigma}^3 \ , \ J_2 \to i\boldsymbol{\sigma}^2 \ , \ J_3 \to i\boldsymbol{\sigma}^1 \ ,$$

the fact that [15–18] defines the Majorana conjugate (using the charge conjugation matrix $\mathcal{C} = i\Gamma^0\Gamma^4$) through the relation $\bar{\xi}^i = \xi^{iT}\mathcal{C}$ and expressing $\mathscr{B}_0(\xi, \xi')$ as $\xi^T\mathcal{C}\xi' = \bar{\xi}\xi'$, we can re-write the definitions of $s, v$ and $\phi^r$ as:

$$\begin{aligned}
s &= i\mathscr{B}(\hat{\xi}, \hat{\xi}) = is' \ , \\
v &= i\mathscr{B}_0(\hat{\xi}, \Gamma_m \hat{\xi}) e^m = iv' \ , \\
\phi^1 &= -i\mathscr{B}(\hat{\xi}, \hat{J}_3 \otimes \Gamma_{mn} \hat{\xi}) e^{mn} = -i\phi'^3 \ , \\
\phi^2 &= i\mathscr{B}(\hat{\xi}, \hat{J}_2 \otimes \Gamma_{mn} \hat{\xi}) e^{mn} = i\phi'^2 \ , \\
\phi^3 &= -i\mathscr{B}(\hat{\xi}, \hat{J}_1 \otimes \Gamma_{mn} \hat{\xi}) e^{mn} = -i\phi'^1
\end{aligned} \tag{5.52}$$

for all $m, n \in \overline{1, 5}$. For future reference, we identify the form-valued bilinears corresponding to our notations by decorating them with primes.

One can compare (5.49) with the Fierz identities given in [15, 16, 18], in particular with those of [17, page 35], which read:

$$\begin{aligned}
v_a v^a = s^2 \quad , \quad v^a {\phi^r}_{ab} = 0 \quad , \quad v^a (*\phi^r)_{abc} &= -s{\phi^r}_{bc} \ , \\
{\phi^r}_a{}^c {\phi^s}_{cb} &= -\delta^{rs}(\eta_{ab}s^2 - v_a v_b) - \epsilon^{rst} s {\phi^t}_{ab} \ , \\
{\phi^r}_{[ab} {\phi^s}_{cd]} &= -\frac{1}{4} s \delta^{rs} \epsilon_{abcde} v^e \quad .
\end{aligned}$$

The latter can be written as follows in terms of the redefined forms $s', v'$ and $\phi'^r$:

$$\begin{aligned}
v'_a v'^a = -s'^2 \quad &\Longrightarrow \quad ||v'||^2 = -s'^2 \ , \\
v'^a {\phi'^r}_{ab} = 0 \quad &\Longrightarrow \quad \iota_{v'} \Phi'^r = 0 \ , \\
v'^a (*\phi'^r)_{abc} = -s' {\phi'^r}_{bc} \quad &\Longrightarrow \quad \iota_{v'} * \Phi'^r = -s'\phi'^r \ ,
\end{aligned} \tag{5.53}$$

$$\phi'^r{}_a{}^c\phi'^s{}_{cb} = -\delta^{rs}(\eta_{ab}s'^2 - v'_a v'_b) + \epsilon^{rst}s'\phi'^t{}_{ab} \implies \phi'^r \bigtriangleup_1 \phi'^s = -2\epsilon^{rst}s'\phi'^t ~,$$

$$||\Phi'^r||^2 = 6s'^2 ~,$$

$$\phi'^r{}_{[ab}\phi'^s{}_{cd]} = -\frac{1}{4}s'\delta^{rs}\epsilon_{abcde}v'^e \implies \phi'^r \wedge \phi'^s = -2s'\delta^{rs}(*v') ~.$$

These agree with (5.49) (which was derived through geometric algebra techniques) upon identifying $f \leftrightarrow s'$, $V \leftrightarrow v'$ and $\Phi^i \leftrightarrow \phi'^i$.

## 6 Conclusions and further directions

We gave a geometric algebra reformulation of Fierz identities for form-valued pinor bilinears in arbitrary dimensions and signatures, which displays the underlying Schur algebra structure in a conceptually transparent manner. This approach to Fierz identities uncovers the underlying reason for phenomena which occur in various applications and allows for a unified and efficient treatment of Fierz identities in various problems of interest in supergravity and string theory. In particular, this formulation opens the way for unified studies of flux vacua and of other questions arising in string and M-theory compactifications. Our approach is highly-amenable to implementation in various symbolic computation systems such as `Ricci`[20] (and we have carried out such implementations as illustrated in our previous papers [1, 8] and briefly touched upon in section 5). Using this implementation, we illustrated our approach by recovering certain well-known results within our formalism. Further applications of our methods will be discussed in forthcoming publications.


## Acknowledgments

This work was supported by the CNCS projects PN-II-RU-TE (contract number 77/2010), PN-II-ID-PCE (contract numbers 50/2011 and 121/2011) and PN 09 37 01 02 / 2009. The work of C.I.L. is also supported by the Research Center Program of IBS (Institute for Basic Science) in Korea (grant CA1205-01). E.M.B. thanks the Center for Geometry and Physics, Institute for Basic Science and Pohang University of Science and Technology (POSTECH), Korea and especially Jae-Suk Park for providing excellent working conditions during her visit in the final stages of preparation for this work. I.A.C. acknowledges the student scholarship from the Dinu Patriciu Foundation "Open Horizons", which supported part of her studies.


## A Systematics of pin bundles for Riemannian and Lorentzian manifolds of dimension up to eleven

In table 10 and table 11, we systematize the properties of real pinors for pseudo-Riemannian manifolds $(M, g)$ of dimension up to 11 and such that $g$ has Euclidean or Minkowskian signature.

For the name of (s)pinors we used here the following abbreviations: M=Majorana, MW=Majorana-Weyl, SM=symplectic Majorana, SMW=symplectic Majorana-Weyl, DM=double Majorana.



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| $d$ | $d$ mod 8 | $\mathbb{S}$ | $\Delta$ | N | $\mathrm{Cl}(p,q)$ | Irrep. image | No. of $\mathbb{R}-$irreps. | Injective ? | Chirality operator $\mathcal{R}$ | $\gamma^{(d+1)}$ | Name of pinors (spinors) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $\mathbb{R}$ | 1 | 1 | $\mathrm{Mat}(1,\mathbb{R})^{\oplus 2}$ | $\mathrm{Mat}(1,\mathbb{R})$ | 2 | no | N/A | $\pm\mathbb{1}$ | M |
| 2 | 2 | $\mathbb{R}$ | 2 | 2 | $\mathrm{Mat}(2,\mathbb{R})$ | $\mathrm{Mat}(2,\mathbb{R})$ | 1 | yes | N/A | $\gamma^{(3)}$ | M |
| 3 | 3 | $\mathbb{C}$ | 2 | 4 | $\mathrm{Mat}(2,\mathbb{C})$ | $\mathrm{Mat}(2,\mathbb{C})$ | 1 | yes | N/A | $\pm J$ | M |
| 4 | 4 | $\mathbb{H}$ | 2 | 8 | $\mathrm{Mat}(2,\mathbb{H})$ | $\mathrm{Mat}(2,\mathbb{H})$ | 1 | yes | $\gamma(\nu)$ | $\gamma^{(5)}$ | SM (SMW) |
| 5 | 5 | $\mathbb{H}$ | 2 | 8 | $\mathrm{Mat}(2,\mathbb{H})^{\oplus 2}$ | $\mathrm{Mat}(2,\mathbb{H})$ | 2 | no | N/A | $\pm\mathbb{1}$ | SM |
| 6 | 6 | $\mathbb{H}$ | 4 | 16 | $\mathrm{Mat}(4,\mathbb{H})$ | $\mathrm{Mat}(4,\mathbb{H})$ | 1 | yes | $\gamma(\nu)\circ J$ | $\gamma^{(7)}$ | DM (M) |
| 7 | 7 | $\mathbb{C}$ | 8 | 16 | $\mathrm{Mat}(8,\mathbb{C})$ | $\mathrm{Mat}(8,\mathbb{C})$ | 1 | yes | $D$ | $\pm J$ | DM (M) |
| 8 | 0 | $\mathbb{R}$ | 16 | 16 | $\mathrm{Mat}(16,\mathbb{R})$ | $\mathrm{Mat}(16,\mathbb{R})$ | 1 | yes | $\gamma(\nu)$ | $\gamma^{(9)}$ | M (MW) |
| 9 | 1 | $\mathbb{R}$ | 16 | 16 | $\mathrm{Mat}(16,\mathbb{R})^{\oplus 2}$ | $\mathrm{Mat}(16,\mathbb{R})$ | 2 | no | N/A | $\pm\mathbb{1}$ | M |
| 10 | 2 | $\mathbb{R}$ | 32 | 32 | $\mathrm{Mat}(32,\mathbb{R})$ | $\mathrm{Mat}(32,\mathbb{R})$ | 1 | yes | N/A | $\gamma^{(11)}$ | M |

**Table 10**. Clifford algebras, representations and character of (s)pinors for Riemannian manifolds. In this case, one has $q=0$ and $d=p$.

| $d$ | $d-2$ mod 8 | $\mathbb{S}$ | $\Delta$ | N | $\mathrm{Cl}(p,q)$ | Irrep. image | No. of $\mathbb{R}-$irreps. | Injectve ? | Chirality operator $\mathcal{R}$ | $\gamma^{(d+1)}$ | Name of pinors (spinors) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | $\mathbb{C}$ | 1 | 2 | $\mathrm{Mat}(1,\mathbb{C})$ | $\mathrm{Mat}(1,\mathbb{C})$ | 1 | yes | $D$ | $\pm J$ | DM (M) |
| 2 | 0 | $\mathbb{R}$ | 2 | 2 | $\mathrm{Mat}(2,\mathbb{R})$ | $\mathrm{Mat}(2,\mathbb{R})$ | 1 | yes | $\gamma(\nu)$ | $\gamma^{(3)}$ | M (MW) |
| 3 | 1 | $\mathbb{R}$ | 2 | 2 | $\mathrm{Mat}(2,\mathbb{R})^{\oplus 2}$ | $\mathrm{Mat}(2,\mathbb{R})$ | 2 | no | N/A | $\pm\mathbb{1}$ | M |
| 4 | 2 | $\mathbb{R}$ | 4 | 4 | $\mathrm{Mat}(4,\mathbb{R})$ | $\mathrm{Mat}(4,\mathbb{R})$ | 1 | yes | N/A | $\gamma^{(5)}$ | M |
| 5 | 3 | $\mathbb{C}$ | 4 | 8 | $\mathrm{Mat}(4,\mathbb{C})$ | $\mathrm{Mat}(4,\mathbb{C})$ | 1 | yes | N/A | $\pm J$ | SM |
| 6 | 4 | $\mathbb{H}$ | 4 | 16 | $\mathrm{Mat}(4,\mathbb{H})$ | $\mathrm{Mat}(4,\mathbb{H})$ | 1 | yes | $\gamma(\nu)\circ J$ | $\gamma^{(7)}$ | SM (SMW) |
| 7 | 5 | $\mathbb{H}$ | 4 | 16 | $\mathrm{Mat}(4,\mathbb{H})^{\oplus 2}$ | $\mathrm{Mat}(4,\mathbb{H})$ | 2 | no | N/A | $\pm\mathbb{1}$ | SM |
| 8 | 6 | $\mathbb{H}$ | 8 | 32 | $\mathrm{Mat}(8,\mathbb{H})$ | $\mathrm{Mat}(8,\mathbb{H})$ | 1 | yes | $\gamma(\nu)\circ J$ | $\gamma^{(9)}$ | DM (M) |
| 9 | 7 | $\mathbb{C}$ | 16 | 32 | $\mathrm{Mat}(16,\mathbb{C})$ | $\mathrm{Mat}(16,\mathbb{C})$ | 1 | yes | $D$ | $\pm J$ | DM (M) |
| 10 | 0 | $\mathbb{R}$ | 32 | 32 | $\mathrm{Mat}(32,\mathbb{R})$ | $\mathrm{Mat}(32,\mathbb{R})$ | 1 | yes | $\gamma(\nu)$ | $\gamma^{(11)}$ | M (MW) |
| 11 | 1 | $\mathbb{R}$ | 32 | 32 | $\mathrm{Mat}(32,\mathbb{R})^{\oplus 2}$ | $\mathrm{Mat}(32,\mathbb{C})$ | 2 | no | N/A | $\pm\mathbb{1}$ | M |

**Table 11**. Clifford algebras, representations and character of (s)pinors for Lorentzian manifolds. In this case, one has $p=d-1$, $q=1$ and $p-q=d-2$.

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