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BIRKHOFF SPECTRUM FOR PIECEWISE MONOTONE INTERVAL MAPS

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ABSTRACT. For piecewise monotone, piecewise continuous interval maps we look at Birkhoff spectra for regular potential functions. This means considering the Hausdorff dimension of the set of points for which the Birkhoff average of the potential takes a fixed value. In the uniformly hyperbolic case we obtain complete results, in the case with parabolic behaviour we are able to describe the part of the sets where the lower Lyapunov exponent is positive. In addition we give some lower bounds on the full spectrum in this case. This is an extension of work of Hofbauer on the entropy and Lyapunov spectra.

1. INTRODUCTION

Consider a dynamical system (X, T) and a potential function $\varphi : X \to \mathbb{R}$. For any point $x \in X$ we can define the Birkhoff average of φ at x by

$$A(x, \varphi) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x)$$

wherever the limit exists. A natural question arises: given a potential function, what is the set of points with its Birkhoff average taking a prescribed value:

$$L_a := \{x \in X; A(x, \varphi) = a\}.$$

Usually one describes the size of this set in terms of either topological entropy or Hausdorff dimension, the functions $a \to h_{\text{top}} L_a$ and $a \to \dim_H L_a$ are called entropy and dimension Birkhoff spectra of the potential φ.

This kind of problems has been studied for quite some time, possibly starting with the work of Besicovitch and Knichal, [B, K] on the frequency of digits in dyadic expansions which was subsequently expanded by Eggleston and Billingsley, [E],[Bi]. In dynamical systems the Birkhoff spectrum was

initially investigated by Pesin and Weiss ([PW], Feng-Lau-Wu [FLW] and Olsen, [O]). Several other authors investigated the case where Hausdorff dimension is replaced by topological entropy including Fan, Feng and Wu, Takens and Verbitskiy and Feng and Huang [FFW, TV, FH] .

Looking at the definition one does not expect the Birkhoff spectrum to be any nicer than a measurable function. However, it turned out that for a Hölder continuous potential function on a $C^{1+\beta}$ conformal repeller the entropy Birkhoff spectrum is the Legendre transform of a suitable pressure function, thus real analytic and concave.The dimension Birkhoff spectrum is more complicated, the exception being the Lyapunov spectrum, that is the Birkhoff spectrum for the potential $\log |T'|$, and the cases where the Lyapunov spectrum is trivial. In these cases the dimension Birkhoff spectrum can be deduced from the entropy Birkhoff spectrum as was done by Pesin and Weiss and by Weiss, [PW, W]. In the general situation, however, the dimension Birkhoff spectrum formula for conformal repellers was considered by Feng, Lau and Wu [FLW] where they gave a conditional variational principle. Barreira and Saussol considered the case of Hölder continuous potential functions where they gave a formula in terms of pressure and its derivative with which they showed the spectrum is analytic but in general not concave.

Naturally the results for one dimensional repellers were afterwards generalised for more general classes of systems, in particular ones with non-uniformly hyperbolic behaviour. We should mention here several works including, [N, Oli, KS, GR, IT, GPR] for the Lyapunov spectrum and [C, JJOP, CTa, IJ1, IJ2] for the Birkhoff spectrum. Of special interest for us is the paper [H2] by Franz Hofbauer. In this paper he constructs the Lyapunov spectrum and entropy Birkhoff spectrum for piecewise monotone maps of the interval.

The aim of our paper is to extend Hofbauer's result by constructing the dimension Birkhoff spectrum for piecewise monotone maps. We are not able to completely solve the problem, we are only able to describe the intersection of the sets L_a with the set D_0 of points with positive lower Lyapunov exponent. The formulation of our results is rather complicated and depends on the notation yet to be introduced, so we postpone it till section 3. Before that, we describe in some detail results from the theory of piecewise monotone maps (section 2) and of the Birkhoff spectra (section 3) which we will need.

2. Piecewise monotone maps

The theory of piecewise monotone maps in the setting of this paper was created by Hofbauer, see in particular [H1] for the general theory and [H2] for selected facts, relevant for multifractal formalism. In this section we will define the objects of our study and state, without proofs, the properties we are going to use.

We consider a piecewise monotonic map, $T : [0,1] \to [0,1]$ based on subintervals of the form $C_1 = [0 = c_1, c_2], C_2 = [c_2, c_3], \ldots, C_n = [c_n, c_{n+1} = 1]$. We assume that T is continuous and monotone on each open interval (c_{i-1}, c_i) but not necessarily continuous at the endpoints c_i. We will denote by Z the grand orbit of the set $\{c_1, \ldots, c_{n+1}\}$, it is a countable set.

Systems of this type can exhibit several unwelcome types of behaviour, including (but not restricted to) intervals of fixed points. To avoid those we will only consider the restriction of the full system to a fixed set $A \subset [0,1]$ with following properties:

- A is closed,
- A is completely invariant, i.e. $A = T^{-1}(T(A)) \supset T(A)$,
- $T|_A$ is topologically transitive,
- the partition $(C_i)_1^n$ is generating,
- the topological entropy $h_{\text{top}}(T|_A)$ is positive.

The dynamic on $A \setminus Z$ is topologically conjugated to dynamic of the shift map on a subset of the full (one-sided) shift space with n symbols. The sets C_i are the first level cylinders. In general this dynamics is not of a subshift of finite type.

We call a function $f : [0,1] \to \mathbb{R}$ *regular* if at each point $x \in [0,1)$ the right limit $f_+(x) = \lim_{y \to x^+} f(y)$ exists, at each point $x \in (0,1]$ the left limit $f_-(x) = \lim_{y \to x^-} f(y)$ exists, and if for all $x \in (0,1)$, $f(x)$ is between $f_-(x)$ and $f_+(x)$. We will also assume that the function $\phi = \log |T'|$ exists everywhere except on a countable set and can be extended to a function which is regular on the whole of $[0,1]$. Note that this means that the derivative can never be 0 or approach 0.

Given $x \in [0,1]$, we denote the *Birkhoff average* of f at x by

$$A(x, f) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} f \circ T^i(x)$$

if the limit exists. We denote

$$L_a = \{x \in A; A(x, f) = a\}.$$

We will call $A(x, \phi)$ the *Lyapunov exponent* and denote it by $\chi(x) = A(x, \phi)$. We will also define the lower Lyapunov exponent

$$\underline{\chi}(x) = \liminf_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \phi \circ T^i(x).$$

The set of points with lower Lyapunov exponent strictly larger than α will be denoted D_α. In particular

(2.1) $$D_0 = \{x : \underline{\chi}(x) = 0\}.$$

For an invariant measure μ we will denote the Lyapunov exponent of the measure by

$$\lambda(\mu) = \int \phi d\mu.$$

It is shown by Hofbauer and Raith in [HR] that for an T-invariant ergodic measure with positive measure theoretic entropy, $h(\mu)$,

$$\dim_H \mu = \frac{h(\mu)}{\lambda(\mu)}.$$

We need to present in a few words the definition of topological pressure we are going to use. This definition comes from [H2]. We first define a *Markov set* as a nonempty closed set $B \subset [0, 1]$ for which there exists a partition $\mathcal{Y}$ with respect to which T is piecewise monotone and such that $T(B \cap Y) \subset B$ and that either $T(B \cap Y_1) \cap Y_2 = \emptyset$ or $T(B \cap Y_1) \supset B \cap Y_2$ for all $Y, Y_1, Y_2 \in \mathcal{Y}$.

If the domain of T is Markov then we can work with the whole system but if the domain is not Markov this is difficult. However in the non-Markov case we work with Markov subsystems. Thus we define by $\mathcal{M}(A)$ the set of all Markov subsets of A. The following Proposition gives two important properties of $\mathcal{M}(A)$ which allow us to work with an increasing sequence of Markov subsystems. The first part of the proposition follows from Lemma 7 in [H1] (the proof of Proposition 1 in [H1] explains why the condition necessary for Lemma 7 is always met in our setting). The second part is Lemma 8 in [H1].

Proposition 2.1. *If A is topologically transitive, completely invariant and has positive entropy then $\mathcal{M}(A)$ is nonempty. Moreover, for every two Markov sets $B_1, B_2 \in \mathcal{M}(A)$ on which T is topologically transitive we can find $B_3 \in \mathcal{M}(A), B_3 \supset B_1 \cup B_2$ on which T is also topologically transitive.*

The pressure of a potential f on A is then defined as

$$P(f) = \sup_{B \in \mathcal{M}(A)} \sup_{h \in C([0,1]); h \leq f} P(T|_B, h),$$

where $P(T|_B, h)$ is the usual topological pressure of a continuous potential on a subshift of finite type. Observe that Proposition 2.1 implies that the supremum in the definition of pressure can be replaced by a limit. In particular it imples that there exists a sequence $(B_i)_i; B_i \in M(A)$ such that $P(T|_{B_i}, h)$ form an increasing sequence converging to P uniformly on compact sets. This is what allows us to find the Markov set B in the proof of Lemma 4.4.

An important result of Hofbauer and Urbański [HU] shows that this definition of pressure agrees with the construction of conformal measures. Denote

$$C_{i_1 i_2 \ldots i_k} = C_{i_1} \cap T^{-1}(C_{i_2}) \cap \ldots \cap T^{-k+1}(C_{i_k}),$$

those sets will be called kth level *cylinders*. We call a probabilistic measure μ supported on A *conformal* for potential f if for every kth level cylinder $C_{i_1 i_2 \ldots i_k}$ we have

$$\mu(T(C_{i_1 i_2 \ldots i_k})) = \int_{C_{i_1 i_2 \ldots i_k}} e^{P(f) - f} d\mu.$$

As the partition is generating, this formula can be iterated:

$$\mu(T^k(C_{i_1 i_2 \ldots i_k})) = \int_{C_{i_1 i_2 \ldots i_k}} e^{kP(f) - S_k f} d\mu,$$

where $S_k f(x) = f(x) + f(T(x)) + \ldots + f(T^{k-1}(x))$.

The following is [HU, Lemma 7].

Proposition 2.2. *Let A be topologically transitive, completely invariant and with positive entropy. Then for every regular potential there exists a conformal measure, positive on open sets.*

3. Thermodynamical formalism and Birkhoff spectra

3.1. **Conformal repellers and smooth expansive maps.** Before stating the main result we will remind the related results of Barreira and Saussol and Feng, Lau and Wu calculating the Hausdorff dimension Birkhoff spectrum for interval expanding maps and then we will proceed to Johansson, Jordan, Pollicott and Öberg's generalisation to the nonuniformly hyperbolic situation.

Let S be a uniformly hyperbolic $C^{1+\beta}$ map on the interval I. Let $f : I \rightarrow \mathbb{R}$ be a continuous potential. Let $H = \{a \in \mathbb{R}; \exists x \in I \; A(x, f) = a\}$, it is a closed interval. We denote

$$P(q, a, \delta) := P(q(f - a) - \delta\phi).$$

For $a \in H$ we will set

$$\delta_0(a) = \sup\{\delta : \inf\{P(q, a, \delta) : q \in \mathbb{R}\} > 0\}.$$

A version of the results of [BS] and [FLW] is the following. Note that [FLW] gives the proof of the conditional variational principle in the general case where as [BS] gives the qquality to $\delta_0(a)$ in the Hölder case for the interior of H. We include a proof showing $\delta_0(a)$ is equal to the value given in the conditional variational principle in the continuous case and include the endpoints of H.

Proposition 3.1. *Let S be a $C^{1+\beta}$ expanding map on the interval, as above. For $a \in H$*

$$(3.1) \quad \dim_H L_a = \max\left\{\frac{h(\mu)}{\lambda(\mu)} : \int f d\mu = a \text{ and } \mu \text{ } S\text{-invariant}\right\} = \delta_0(a).$$

Proof. We will outline the proof. To show the first equality in (3.1), as done in [FLW], the idea is to adapt the ideas of the proof the classical variational principle, this works for any continuous potential.

To show the second equality let μ be an S-invariant measure with $\int f d\mu = a$. By the variational principle we have that if $\delta < \frac{h(\mu)}{\lambda(\mu)}$ then for any $q \in \mathbb{R}$,

$$P(q, a, \delta) \geq 0 - \delta\lambda(\mu) + h(\mu) > 0$$

and thus $\delta \leq \delta_0(a)$.

We now suppose

$$\delta > \max\left\{\frac{h(\mu)}{\lambda(\mu)} : \int f d\mu = a \text{ and } \mu \text{ } S\text{-invariant}\right\}.$$

If the function $q \mapsto P(q, a, \delta)$ has a minimum q_0 then there must be equilibrium states μ_1, μ_2 at this point such that $\int f d\mu_1 \leq a$ and $\int f d\mu_2 \geq a$ (it may well be the case that $\mu_1 = \mu_2$). By taking a suitable convex combination we can find an equilibrium state μ with $\int f d\mu = a$. This gives

$$P(q_0, a, \delta) = h(\mu) + q_0\left(\int f d\mu - a\right) - \lambda(\mu)\delta = h(\mu) - \lambda(\mu)\delta < 0$$

and so $\delta \geq \delta_0(a)$.

In the case where $q \mapsto P(q, a, \delta)$ does not have a minimum then there must be a sequence of equilibrium states μ_n and values q_n such that

$$\lim_{n\to\infty} h(\mu_n) + q_n\left(\int f d\mu_n - a\right) - \lambda(\mu_n)\delta = \inf_{q\in\mathbb{R}}\{P(q, a, \delta)\}.$$

By continuity we can assume the sequence q_n diverges to $\pm\infty$. If $\lim_{n\to\infty} q_n = \infty$ then we must have $\limsup_{n\to\infty} \int f d\mu_n \leq a$ and thus

$$\lim_{n\to\infty} h(\mu_n) + q_n \left(\int f d\mu_n - a \right) - \lambda(\mu_n)\delta \leq h(\mu_n) - \delta\lambda(\mu_n) < 0.$$

Similarly if $\lim_{n\to\infty} q_n = -\infty$ then we must have $\liminf_{n\to\infty} \int f d\mu_n \geq a$ and thus

$$\lim_{n\to\infty} h(\mu_n) + q_n \left(\int f d\mu_n - a \right) - \lambda(\mu_n)\delta \leq h(\mu_n) - \delta\lambda(\mu_n) < 0.$$

Thus $\delta \geq \delta_0(a)$.

So we can conclude that

$$\max\left\{ \frac{h(\mu)}{\lambda(\mu)} : \int f d\mu = a \text{ and } \mu \text{ } S\text{-invariant} \right\} = \delta_0(a).$$

$\square$

Note that in [BS] by assuming more regularity on the function f and working in the interior of H stronger results can be deduced. In particular for $a \in \text{int(H)}$ we have $\lim_{q\to-\infty} P(q, a, \delta) = \lim_{q\to\infty} P(q, a, \delta) = \infty$ for all δ. Hence, there exists $P_{\min}(\delta) = \inf_q P(q, a, \delta)$. Moreover when f is Hölder continuous $(q, a, \delta) \to P(q, a, \delta)$ will be analytic on $\mathbb{R} \times \text{int(H)} \times \mathbb{R}$. Since this is the uniformly hyperbolic situation ϕ is bounded away from zero and so is $\partial P(q, a, \delta)/\partial\delta$. Hence for $a \in \text{int(H)}$ one can find (q_0, δ_0) such that $\delta_0(a) = \delta_0$ and $P(q_0, a, \delta_0) = 0$. Using this in [BS] it is shown that if f is Hölder then $a \to \dim_H L_a$ is analytic on int(H).

In the nonuniformly hyperbolic situation the picture is more complicated. Let now S be $C^{1+\beta}$ expansive Markov map, but we allow the existence of parabolic fixed points $(p_1, \ldots, p_\ell)$. We take a continuous function $f : [0, 1] \to [0, 1]$ as before and let H_p be the convex hull of $\{f(p_1), \ldots, f(p_\ell)\}$. In this case it is shown in [JJOP] that if $a \in H_p$ then $\dim_H L_a = 1$ and if $a \in (H \backslash H_p)$ then as before

$$\dim_H L_a = \max\left\{ \frac{h(\mu)}{\lambda(\mu)} : \int f d\mu = a \text{ and } \mu \text{ } S\text{-invariant} \right\} = \delta_0(a)$$

where $\delta_0(a)$ is defined as in (3.2). However in H_p it will not always be the case that $\delta_0(a) = 1$. We will show that $\delta_0(a)$ is lower-semi continuous whereas in [IJ2] an example is given where $a \to \dim_H L_a$ is not lower-semi continuous. It turns out that $\delta_0(a)$ will give the hyperbolic dimension of L_a (dimension of points with positive local Lyapunov exponent), but not always the full Hausdorff dimension.

3.2. **The main result.** We now turn back to our more general setting. Observe that because of the existence of the set Z with completely erratic dynamic (and hence, erratic behaviour of Birkhoff averages) it is not a good idea to define the spectrum interval H as in the previous subsection. We will thus use a different definition, which in the cases considered in the previous subsection was equivalent but which works better in our generality.

Denote by H the set of all $a \in \mathbb{R}$ for which there exists a sequence of non-atomic ergodic measures μ_n, each supported on a Markov subset of A, such that $\int f d\mu_n \to a$. This is equivalent to the definition of H given in [H2]. By Proposition 2.1, H is a closed interval. While it is not known whether $A(x, f) \in H$ for all $x \in A$, Hofbauer proves in [H2] that for all $a \notin H$ $h_{\text{top}} L_a = 0$, and hence $\dim_H L_a \cap D_0 = 0$. In what follows we will restrict our attention to $a \in H$.

The set H can be divided into two subsets. Let H_p be the set of all $a \in H$ such that we can not only choose a sequence μ_n as in the definition of H, but in addition the measures μ_n can be chosen as to satisfy $\int \phi d\mu_n \to 0$. Then $H_h = H \setminus H_p$. The sets H_p and H_h will be called *parabolic* and *hyperbolic*.

By Lemma 4.3, which will be proved in the next section, in this more general setting for $a \in H$ it still makes sense to define

(3.2) $\qquad \delta_0(a) = \sup\{\delta : \inf\{P(q(f - a) - \delta\phi) : q \in \mathbb{R}\} > 0\}.$

We will also define

$$\dim_{\text{hyp}}(A) = \sup\{\dim M : \ M \text{ is Markov and } M \subseteq A\}.$$

It can be also shown that this is both the supremum of dimensions of hyperbolic ergodic measures supported on Markov subsets of A and the supremum of $\delta_0(a)$ where $a \in H$.

Theorem 3.2. *Let T be a piecewise monotone map.*

   (1) If $a \in H$ then $\dim_H(L_a \cap D_0) = \delta_0(a)$.
   (2) For all $a \in int(H)$

$$\delta_0(a) = \sup\left\{\frac{h(\mu)}{\lambda(\mu)} : \mu \ T\text{-ergodic and } \int f d\mu = a\right\}$$

   (3) If $a \in H_p$ then $\dim_H(L_a) \geq \dim_{hyp}(A)$. Moreover if $a \in int(\mathrm{H_p})$ then $\dim_H(L_a \cap D_0) = \dim_{hyp}(A)$.
   (4) The function $a \to \dim_H(L_a \cap D_0)$ is continuous on H_h and lower semi-continuous on H.

Note that we are using the convention that $\dim_H \emptyset = -\infty$ and that if an endpoint a of H is also in H_p then it is possible that $\delta_0(a) = -\infty$ (in

this case L_a will be non-empty and have dimension at least equal to the hyperbolic dimension but $L_a \cap D_0 = \emptyset$).

The rest of the paper will be structured as follows. In section 4 we will prove the result on the dimension of $L_a \cap D_0$, in section 5 we will prove the results for end points and for H_p and in section 6 we will prove the continuity properties of the spectrum. Finally in section 7 we will discuss some examples.

4. Dimension of $L_a \cap D_0$

4.1. **Preparation – exceptional sets.** We would like to work with symbolic dynamics and then apply the results to the geometry of some invariant sets. The symbolic dynamics is not that complicated, we are working with closed invariant positive entropy subsets of a full shift. But the interplay between dynamics and geometry is rather involved.

Observe that if a partition (C_i) satisfies our assumptions, any subpartition of (C_i) will also satisfy them. That is, the choice of partition is not determined by the map T. It will be convenient for us to make use of this phenomenon, in particular of the fact that we can freely demand that our partition consists of sets of arbitrarily small diameter. So, let $\mathcal{Z}$ be some partition satisfying our assumptions. Denote by Z_0 the partition points of $\mathcal{Z}$ and let $Z_k = T^{-1}(Z_{k-1})$. The elements of partition $\mathcal{Z}^k$ generated by points $Z_0 \cup \ldots \cup Z_{k-1}$ will be called k-level cylinders.

If both endpoints of a k level cylinder I belong to $Z_1 \cup \ldots \cup Z_{k-1}$ then $T(I)$ is a $k-1$ level cylinder. However, if one or both of the endpoints of I belongs to Z_0 then $T(I)$ is in general only a subset of some $k-1$ level cylinder. We would like to get some bounds on how small can $|T^k(I)|$ be.

For the upper bound we will work with full system (A, T) using part of the statement of [H2, Lemma 14, 15]. Given $x \in [0,1]$, let $\hat{Z}_k(x)$ be the k level cylinder containing x. Let μ be any probabilistic measure on $[0,1]$, positive on all cylinders (of any level) intersecting A. Given $d > 0$ let

$$N_d(\mu, Z) = \bigcup_{k=0}^{\infty} Z_k \cup \{x \in A; \limsup_{k \to \infty} \mu(T^k(\hat{Z}_k(x))) < d\}.$$

Denote also

$$\mathrm{var}_{\mathcal{Z}} f = \sup_{Z \in \mathcal{Z}} \sup_{x_1, x_2 \in Z} |f(x_1) - f(x_2)|.$$

We have

Lemma 4.1. *For every $\alpha > \mathrm{var}_{\mathcal{Z}}\,\phi$ and for μ as above we have*

$$\lim_{d \to 0} \dim_H(N_d(\mu, Z) \cap D_\alpha) = 0$$

For the lower bound we need to work with Markov subsets. Symbolically they are just subshifts of finite type. We will need the following lemma.

Lemma 4.2. *Let $(B, \mathcal{Z})$ be a Markov set. Then there exists $K > 0$ such that for every k for every kth level cylinder I with nonempty intersection with B we have*

$$|T^k I| > K.$$

Assume in addition that $T|B$ is topologically transitive. Then for every $x \in B$ there exist $r > 0$ and $l > 0$ such that for every k for every kth level cylinder I there exists interval $J \subset T^k(I)$ and an integer $j \le l$ such that $T^j(J) = B(x, r)$ and T^j is continuous and monotone on J.

Proof. Consider any kth level cylinder I contained in a first level cylinder J_0 and intersecting B. We have $T(I) \subset I' \subset J_1$, where I' is a $k - 1$ level cylinder and J_1 is a first level cylinder. We have $I_1 \cap B \supset T(I) \cap B \ne \emptyset$, hence $J_1 \cap B = T(J_0 \cap B)$ by the definition of Markov set. Hence, the set $I' \cap B = B \cap T(I)$. Following the same reasoning, $T^k(I)$ is contained in some first level cylinder J_2 and $J_2 \cap B = B \cap T^k(I)$. Then we can set K as the minimum (over all first level cylinders J) of $|\mathrm{span}(J \cap B)|$ and the first part of the assertion is proved.

The second part follows easily because every open interval of a given size centred at a point in A contains a (symbolic) cylinder of a bounded level. □

4.2. Upper bound.
For convenience from now on we will denote

$$P(a, q, \delta) = P(q(f - a) - \delta\phi)$$

We will obtain the upper bound in the dimension formula with the use of conformal measures. Let a and δ be such that $\inf_q P(a, q, \delta) \le 0$. Fix $\alpha > 0$ and $\varepsilon > 0$. Consider $\mathcal{Y}$, a subdivision of $\mathcal{Z}$ such that $\mathrm{var}_{\mathcal{Y}}\,\phi < \alpha$. As for arbitrarily small α we can always find such $\mathcal{Y}$, we need only to prove that

$$\dim_H((L_a \setminus N_d(\mu, \mathcal{Y})) \cap D_\alpha) \le \delta_0 + \gamma$$

for arbitrarily small γ. Indeed, as the Hausdorff dimension of $N_d \cap D_\alpha$ will go to 0 together with d, we will get an upper bound for the Hausdorff dimension of $L_a \cap D_\alpha$, we will then pass with α to 0 to obtain the assertion.

Consider q_0 such that $P(a, q_0, \delta_0) = p \in [0, \varepsilon)$. Let μ be a $q_0(f-a)-\delta_0\phi-$ p-conformal measure. Choose $d > 0$. Denote by $G_d(n)$ the set of cylinders $I \in \mathcal{Y}^n$ for which $\mu(T^n(I)) > d$. Choose a small ε and denote

$$X_{a,n,\varepsilon} = \{I \in G_d(n); a - \varepsilon < A_N(x, f) < a + \varepsilon, \ A_n(x, \phi) > \alpha - \varepsilon \ \forall x \in I\}.$$

Note that we have

$$|I| \le e^{n(\alpha - \varepsilon - \mathrm{var}_{\mathcal{Y}} \phi)}$$

for all $I \in X_{a,n,\varepsilon}$. Fixing $n \in N$, we have

$$1 = \sum_{I \in \mathcal{Y}^n} \mu(I) \ge \sum_{I \in G_d(n)} \mu(I) \ge \sum_{I \in X_{a,n,\varepsilon}} \mu(I).$$

Substituting the definition of μ, we get

$$\mu(I) \ge \inf_{x \in I} \exp(-q_0 S_n(f-a)+np)|(T^n)'|^{\delta_0}(x) \cdot \mu(T^n(I)) \ge e^{-n\varepsilon|q_0|+np} e^{-n\delta_0 \, \mathrm{var}_{\mathcal{Y}}} |I|^{\delta_0} d.$$

Hence, for any γ such that

$$\gamma > \frac{\varepsilon(|q_0| + 1) + \delta_0 \, \mathrm{var}_{\mathcal{Y}} \phi}{\alpha - \varepsilon - \mathrm{var}_{\mathcal{Y}} \phi}$$

we have

$$\sum_n \sum_{I \in X_{a,n,\varepsilon}} |I|^{\delta_0+\gamma} \le d^{-1} \sum_n e^{-n(\varepsilon-p)} < \infty,$$

which implies that

$$\dim_H((L_a \setminus N_d(\mu, \mathcal{Y})) \cap D_\alpha) \le \delta_0 + \gamma.$$

As ε and $\mathrm{var}_{\mathcal{Y}}$ can be chosen arbitrarily small, γ also can be arbitrarily small. We are done.

4.3. **Lower bound.** We start with the following lemma

Lemma 4.3. *For all $\delta \in \mathbb{R}$ and $a \in int(H)$,*

$$\lim_{q \to -\infty} P(q(f-a) - \delta\phi) = \lim_{q \to \infty} P(q(f-a) - \delta\phi) = \infty.$$

Proof. Since $a \in int\, H$ there exist $a_1 < a < a_2$ such that $a_1, a_2 \in H$. Thus there exist non-atomic hyperbolic measures μ_1, μ_2 such that $\int f \mathrm{d}\mu_1 = a_1$ and $\int f \mathrm{d}\mu_2 = a_2$. Thus

$$P(q(f-a) - \delta\phi) \ge q(a_1 - a) + h(\mu) - \delta \int \phi \mathrm{d}\mu$$

and since ϕ is bounded we have that $\lim_{-q \to \infty} P(q(f-a) - \delta\phi) = \infty$. We also have that

$$P(q(f-a) - \delta\phi) \geq q(a_2 - a) + h(\mu) - \delta \int \phi d\mu$$

and so $\lim_{q \to \infty} P(q(f-a) - \delta\phi) = \infty$. $\square$

We can now proceed with the lower bound for Theorem 3.2, starting from the interior of the spectrum. We will fix a in the interior of H and we will choose δ such that

$$\gamma := \inf_{q \in \mathbb{R}} \{P(q(f-a) - \delta\phi)\} > 0.$$

We show that

$$\dim_H(L_a \cap D_0) \geq \delta$$

via the following key lemma.

Lemma 4.4. *If $a \in int(H)$ and $\delta > 0$ is such that*

$$\gamma := \inf_{q \in \mathbb{R}} \{P(q(f-a) - \delta\phi)\} > 0$$

then there exists an ergodic measure μ such that $\int f d\mu = a$, $\lambda(\mu) > \gamma/2$, and $\frac{h(\mu)}{\lambda(\mu)} \geq \delta$.

Proof. We start by finding c and an interval $I = [x_0, x_1]$ which contains c such that $P(c(f-a) - \delta\phi) = \gamma$, where $P(x_0(f-a) - \delta\phi) > 2\gamma$ and $P(x_1(f-a) - \delta\phi) > 2\gamma$. We can do this by Lemma 4.3. Since f and ϕ are regular we can find step functions $\tilde{f}$ and $\tilde{\phi}$ such that

(1) $|f - \tilde{f}| < \frac{\gamma}{2(x_1 - x_0)}$,
(2) For all $q \in I$,

$$|q(f-a) - \delta\phi - (q(\tilde{f}-a) - \delta\tilde{\phi})| \leq \frac{\gamma}{4}.$$

Thus we will have that by Lemma 6 of [H2]

$$
\begin{aligned}
P(c(\tilde{f}-a) - \delta\tilde{\phi}) &< 5\gamma/4 \\
P(x_0(\tilde{f}-a) - \delta\tilde{\phi}) &> 7\gamma/4 \\
P(x_1(\tilde{f}-a) - \delta\tilde{\phi}) &> 7\gamma/4 \\
\forall q \in I, P(q(\tilde{f}-a) - \delta\tilde{\phi}) &> 3\gamma/4.
\end{aligned}
$$

Thus we can find a Markov set B on which T is topologically transitive such that $\tilde{f}$ and $\tilde{\phi}$ are constant on all elements of the partition for B and where

$$\begin{aligned}
P_B(c(\tilde{f}-a)-\delta\tilde{\phi}) &< 5\gamma/4 \\
P_B(x_0(\tilde{f}-a)-\delta\tilde{\phi}) &> 7\gamma/4 \\
P_B(x_1(\tilde{f}-a)-\delta\tilde{\phi}) &> 7\gamma/4 \\
\forall q \in I, P_B(q(\tilde{f}-a)-\delta\tilde{\phi}) &> 3\gamma/4.
\end{aligned}$$

We have that the function $q(\tilde{f}-a)-\delta\tilde{\phi})$ when restricted to B will have unique ergodic, non-atomic equilibrium states for all $q \in \mathbb{R}$. Moreover $q \to P_B(q(\tilde{f}-a)-\delta\tilde{\phi})$ is analytic and the derivative with at the point q will be given by $\int \tilde{f}\mathrm{d}\mu_q - a$. We also know that the measures μ_q will vary weak-*-continuously with q. By the mean value theorem we can see that

$$\left\{\int \tilde{f}\mathrm{d}\mu_q : q \in I\right\} \supseteq \left[a - \frac{\gamma}{2(x_1-x_0)}, a + \frac{\gamma}{2(x_1-x_0)}\right].$$

Thus we know that there exist $q_0, q_1 \in I$ such that $\int f\mathrm{d}\mu_{q_0} < a$ and $\int f\mathrm{d}\mu_{q_1} > a$. Since f is discontinuous at at most countably many points, by weak-*-continuity the function $q \to \int f\mathrm{d}\mu_q$ will vary continuously. Thus there exists $q_2 \in I$ such that $\int f\mathrm{d}\mu_{q_2} = a$.

We have, since μ_{q_2} is an equilibrium state,

$$h(\mu_{q_2}) + \int (q_2(\tilde{f}-a)-\delta\tilde{\phi})\mathrm{d}\mu_{q_2} = P_B(q_2(\tilde{f}-a)-\delta\tilde{\phi}) > 3\gamma/4$$

Thus since

$$|q_2(f-a)-\delta\phi-(q_2(\tilde{f}-a)-\delta\tilde{\phi})| < \frac{\gamma}{4}$$

we know that

$$h(\mu_{q_2}) + \int (q_2(f-a)-\delta\phi)\mathrm{d}\mu_{q_2} > \gamma/2.$$

Since $\int f\mathrm{d}\mu_{q_2} = a$ we can conclude that $h(\mu_{q_2}) - \delta\int \phi\mathrm{d}\mu_2 > 0$ and since $0 < h(\mu_{q_2}) \le \lambda(\mu_{q_2})$ we have that $\lambda(\mu_{q_2}) > \gamma/2$. Hence,

$$\frac{h(\mu_{q_2})}{\lambda(\mu_{q_2})} \ge \delta.$$

$\square$

By combining the Birkhoff ergodic theorem and the fact proved in [HR] that in this setting the dimension of an ergodic measure with positive entropy is given by the entropy divided by the Lyapunov exponent we can conclude that $\dim_H(L_a \cap D_0) \ge \delta_0(a)$ for any $a \in \mathrm{int}(H)$. Combining this with our upper bound shows that for $a \in \mathrm{int}(H)$, $\dim_H L_a \cap D_0 = \delta_0(a)$. So for part one of Theorem 3.2 the part remaining to be proven is the lower

bound for the endpoints, it will be done in the next section. Part 2 of the Theorem is an immediate corollary of what we have just done

Corollary 4.5. *For $a \in int(H)$ we have that*

$$\delta_0(a) = \sup\left\{\frac{h(\mu)}{\lambda(\mu)} : \mu \ T \ \text{ergodic and} \ \int f d\mu = a\right\}.$$

Proof. This is proved by combining Lemma 4.4 and the fact that the dimension of any ergodic measure with positive entropy which has integral a will give a lower bound for $\dim_H L_a \cap D_0$.   □

5. THE LOWER BOUNDS FOR THE PARABOLIC PART AND THE END POINTS

Limiting measures. The lower bound at the endpoints of the spectrum will be obtained with the use of the following theorem which will also be used for our lower bound for the dimension of L_a when $a \in H_p$

Theorem 5.1. *Let $\{(B_i, \mathcal{Y}_i)\}$ be a sequence of Markov sets contained in the topologically transitive completely invariant set $A \subset [0,1]$. For each i let μ_i be a non-atomic ergodic measure supported on B_i with Hausdorff dimension d_i, Lyapunov exponent $\lambda_i > 0$, entropy h_i, and let $a_i = \int f d\mu_i$. Assume that $a_i \to a$. Then there exists a measure μ supported on A such that*

$$\dim_H \mu \geq \limsup d_i$$

and for a μ-typical point x we have $A(x, f) = a$ and $\underline{\chi}(x) = \liminf \lambda_i$.

Proof. The proof will use the w-measures technique and will be a slightly modified version of [GR, Proposition 9].

By passing to a subsequence we might assume that the sequence $\{d_i\}$ converges to a limit, let us denote it by d. By Proposition 2.1 we can freely assume that $T|B_i$ is topologically transitive and $B_i \cap B_{i+1} \neq \emptyset$ for all i. We choose a fast decreasing sequence $\varepsilon_i \searrow 0$ such that ε_i is sufficiently small compared with λ_i and h_i.

Our first step is preparation of our Markov sets. The dynamics $T|_{B_i}$ is a subshift of finite type, we can choose a subpartition $\mathcal{Z}_i$ such that $T|_{B_i}$ is a subshift of finite type in the new partition as well and that $\text{var}_{\mathcal{Z}_i} f, \text{var}_{\mathcal{Z}_i} \phi < \varepsilon_i$. Let $x_i \in B_i \cap B_{i+1}$ and let K_i, r_i, l_i be as given by Lemma 4.2 as applied to $(B_i, \mathcal{Z}_i, x_i)$. We assume that the partition $\mathcal{Z}_{i+1}$ is so detailed with respect to $\mathcal{Z}_i$ that the ball $B(x_i, r_i)$ contains a first level cylinder $J_{i+1} \in \mathcal{Z}_{i+1}$. Denote $L = \sup |T'|$.

We will construct inductively a fast increasing sequence $\{m_i\}$ and will, also inductively, construct μ. Let us begin the induction by considering the cylinders of level $m_1 + 1$ for $(T, B_1, \mathcal{Z}_1)$. Denote by $\mathcal{I}_1$ the family of those cylinders $I \in \mathcal{Z}_1^{m_1+1}$ for which for all $x \in I \cap B_1$ we have

$$(5.1) \qquad \left| \sum_{k=0}^{m_1-1} f \circ T^k(x) - m_1 a_1 \right| < 2m_1 \varepsilon_1,$$

$$(5.2) \qquad \left| \sum_{k=0}^{m_1-1} \phi \circ T^k(x) - m_1 \lambda_1 \right| < 2m_1 \varepsilon_1,$$

and

$$\left| \log \mu_1(I) + m_1 h_1 \right| < 2m_1 \varepsilon_1.$$

By the Birkhoff Theorem and Shannon-McMillan-Breiman Theorem, for m_1 big enough we have

$$\sum_{I \in \mathcal{C}_1} \mu_1(I) > 1 - \varepsilon_1.$$

By our assumption about $\mathrm{var}_{\mathcal{Z}_1} f, \mathrm{var}_{\mathcal{Z}_1} \phi$, (5.1) and (5.2) imply

$$\left| \sum_{k=0}^{m_1-1} f \circ T^k(x) - m_1 a_1 \right| < 3m_1 \varepsilon_1,$$

and

$$\left| \sum_{k=0}^{m_1-1} \phi \circ T^k(x) - m_1 \lambda_1 \right| < 3m_1 \varepsilon_1,$$

for all $x \in I$. For each of those intervals Lemma 4.2 gives us some j which can take only l_1 different values. We have thus for some $j_1 \leq l_1$ at least $\frac{1}{l_1} \exp(m_1(h_1 - 3\varepsilon_1))$ intervals I of length at least $L^{-l_1} \exp(m_1(-\lambda_1 - 3\varepsilon_1))|J_2| > \exp(m_1(-\lambda_1 - 4\varepsilon_1))$ such that $T^{m_1+j_1}(I) = J_2$ and that for all $x \in I$

$$\left| \sum_{k=0}^{m_1+j_1-1} f \circ T^k(x) - m_1 a_1 \right| < 3m_1 \varepsilon_1 + j_1 \sup |f| < 4(m_1 + j_1)\varepsilon_1,$$

and

$$\left| \sum_{k=0}^{m_1+j_1-1} \phi \circ T^k(x) - m_1 \lambda_1 \right| < 3m_1 \varepsilon_1 + j_1 \sup |\phi| < 4(m_1 + j_1)\varepsilon_1.$$

We equidistribute the measure μ over all $I \in \mathcal{C}_1$.

Now we are ready for the inductive step. For m_i we have at least $\exp(m_i(h_i - 4\varepsilon_i))$ intervals I with following properties:

$$\exp(m_i(-\lambda_i - 4\varepsilon_i)) \leq \operatorname{diam} I,$$

$$m_i(a_i - 4\varepsilon_i) \leq m_i A_{m_i+j_i}(x, f) \leq m_i(a_i + 4\varepsilon_i),$$

$$\exp(m_i(-h_i - 3\varepsilon_i)) \leq \mu(I) \leq \exp(m_i(-h_i + 3\varepsilon_i)),$$

$$T^{m_i+j_i}(I) = J_{i+1}.$$

Applying Birkhoff ergodic theorem, Shannon-McMillan-Breiman theorem and Egorov's theorem we can find in J_{i+1} a family $\mathcal{D}_{i+1}$ of $(m_{i+1} - m_i - j_i + 1)$ level cylinders for $(T, B_{i+1}, \mathcal{Z}_{i+1})$ for which the following are true:

   i) For any $k = 1, \ldots, m_{i+1} - m_i - j_i$ for any $x \in I \cap B_{i+1}, I \in \mathcal{D}_{i+1}$

$$|A_k(x, f) - ka_{i+1}| \leq m_i\varepsilon_i + 2k\varepsilon_{i+1},$$

   ii) For any $k = 1, \ldots, m_{i+1} - m_i - j_i$ for any $x \in I \cap B_{i+1}, I \in \mathcal{D}_{i+1}$

$$|A_k(x, \phi) - k\lambda_{i+1}| \leq m_i\varepsilon_i + 2k\varepsilon_{i+1},$$

   iii) For any $k = 1, \ldots, m_{i+1} - m_i - j_i$ for any $x \in I \cap B_{i+1}, I \in \mathcal{D}_{i+1}$, the k level cylinder J of $(T, B_{i+1}, \mathcal{Z}_{i+1})$ containing x satisfies

$$|\log \mu_{i+1}(J) + kh_{i+1}| \leq m_i\varepsilon_i + 2k\varepsilon_{i+1}.$$

If m_{i+1} is sufficiently big (depending on m_i, ε_i, ε_{i+1}) then

$$\mu_{i+1} \bigcup_{I \in \mathcal{D}_{i+1}} I \geq (1 - \varepsilon_{i+1})\mu_{i+1}(J_{i+1}).$$

By our assumption about $\operatorname{var}_{\mathcal{Z}_1} f, \operatorname{var}_{\mathcal{Z}_1} \phi$, points i) and ii) imply that for all $x \in I, I \in \mathcal{D}_{i+1}, k = 1, \ldots, m_{i+1} - m_i - j_i$

$$(5.3) \qquad\qquad |A_k(x, f) - ka_{i+1}| \leq m_i\varepsilon_i + 3k\varepsilon_{i+1},$$

and

$$(5.4) \qquad\qquad |A_k(x, \phi) - k\lambda_{i+1}| \leq m_i\varepsilon_i + 3k\varepsilon_{i+1},$$

Like in the first step, we then find for each of those cylinders the time j_{i+1} given by Lemma 4.2 and we choose some of them with the same j_{i+1}, at

least $1/l_{i+1}$ of the total number. In the chosen cylinders we find subintervals which are moved by $T^{m_{i+1}-m_i-j_i+j_{i+1}}$ onto J_{i+2}.

We denote by $\mathcal{E}_{i+1}$ the set of those chosen subintervals J and let

$$\mathcal{C}_{i+1} = \{T_I^{-m_i-j_i}(J); J \in \mathcal{E}_{i+1}, I \in \mathcal{C}_i\}.$$

We equidistribute μ on $\mathcal{C}_{i+1}$.

Note that, provided m_{i+1} is sufficiently big,

$$m_{i+1}\varepsilon_{i+1} > 4m_i\varepsilon_i + j_{i+1}\sup(|f| + |\phi|)$$

and hence (5.3), (5.4) and iii) give us back the inductive assumption for the step $i + 1$.

This way, we have constructed a measure μ with following properties:

a) For any $l = m_i + k < m_{i+1}$ for μ-almost every x we have

$$|l \cdot A_l(x, f) - m_i a_i - k a_{i+1}| \leq 4m_i\varepsilon_i + 4k\varepsilon_{i+1},$$

b) For any $l = m_i + k < m_{i+1}$ for μ-almost every x we have

$$|l \cdot A_l(x, \phi) - m_i\lambda_i - k\lambda_{i+1}| \leq 4m_i\varepsilon_i + 4k\varepsilon_{i+1},$$

c) For any $l = m_i+k < m_{i+1}$, for any interval I of length $\exp(m_i(-\lambda_i - 4\varepsilon_i) + k(-\lambda_{i+1} - 4\varepsilon_{i+1}))$ we have

$$\mu(I) \leq 2(1 - \varepsilon_{i+1})^{-2}\exp(m_i(-h_i + 4\varepsilon_i) + k(-h_{i+1} + 4\varepsilon_{i+1})).$$

The points a), b) follow from (5.3), (5.4). The point c) follows from iii) because, by (5.4), the interval of this length can intersect at most two of the sets $T_C^{-m_i}(G)$ of positive measure μ, where G is a k level cylinder of $(T, B_{i+1}, \mathcal{Z}_{i+1})$. Point a) implies a μ-typical point has Birkhoff average of f equal to a. Point b) implies, by Frostman's lemma, that the Hausdorff dimension of μ is at least d. $\qquad\square$

5.1. **Endpoints of H.** Let us now take $a = \max H$ (the case $a = \min H$ is analogous), assume that $\delta_0(a) \neq -\infty$ and thus we can assume that $P(a, q, \delta) \geq \gamma > 0$ for some δ and all $q \in \mathbb{R}$. As $a = \max H$, $P(a, q, \delta)$ is nonincreasing and

$$\lim_{q\to\infty} \frac{d}{dq}P(a, q, \delta) = 0.$$

Moreover, as $\lim_{q\to\infty} P(a, q, \delta) > -\infty$, we have

$$\limsup_{q\to\infty} q\frac{d}{dq}P(a, q, \delta) = 0.$$

Let us choose some sequence $q_i \to \infty$ with $q_i \frac{d}{dq} P(a, q, \delta)_{|q=q_i} > -\gamma/2$. Let

$$a_i = a + \frac{d}{dq} P(a, q, \delta)_{|q=q_i},$$

assume that $a_i \in H$. Then, as $P(a', q, \delta) = q(a' - a) + P(a, q, \delta)$, we have

$$P(a_i, q_i, \delta) \quad > \quad \frac{1}{2}\gamma,$$

$$\frac{d}{dq} P(a_i, q, \delta)_{|q=q_i} \quad = \quad 0.$$

By convexity of P, q_i is the minimum point of $P(a_i, \cdot, \delta)$. We can thus apply Lemma 4.4 to construct an ergodic measure μ_i such that

$$\lambda(\mu_i) \quad > \quad \gamma_2,$$

$$\int f d\mu_i \quad = \quad a_i,$$

$$\frac{h(\mu_i)}{\lambda(\mu_i)} \quad \geq \quad \delta.$$

Applying Theorem 5.1 we can construct a set of points with Birkhoff average a and this set has Hausdorff dimension δ. Moreover, as all the measures μ_i have Lyapunov exponents uniformly bounded away from zero, all those points are in $D_{\gamma/2}$. Hence,

$$\dim_H L_a \cap D_0 \geq \delta.$$

5.2. **The full spectrum in H_p.** a sequence $\{a_i\} \in H$ converging to a and a sequence of invariant measures $\{\mu_i\}$ such that $\int f d\mu_i = a_i$ and $\lambda(\mu_i) \to 0$.

We use the following preliminary result

Lemma 5.2. *Let μ be a T-invariant prob measure with support contained on a Markov set M. We can find a sequence of measures μ_n each supported on M such that each μ_n is ergodic and non-atomic, μ is a weak* limit of $\{\mu_n\}$ and $\lim_{n\to\infty} h(\mu_n) = h(\mu)$.*

Proof. Let η be the measure of maximal entropy (Parry measure for M). We let $\nu_n = (1 - n^{-1})\mu_n + n^{-1}\eta$. Note that ν_n is T-inv and fully supported on M. Thus we can apply Lemma 4.2 from [O] to find a sequence of Gibbs measures (so non-atomic and ergodic) converging to ν_n weak* and in terms of entropy. The result now easily follows. $\square$

We can now deduce that H_p is an interval

Lemma 5.3. H_p *is a (possibly empty or trivial) closed interval.*

Proof. The fact that H_p is a closed set is immediate from the definition. To see it is an interval we will show that if $a_1, a_2 \in H_p$ and $p \in (0,1)$ then $pa_1 + (1-p)a_2 \in H_p$.

To see this note that if μ and ν are both supported on subsets of Markov sets then we can find a Markov set M such that $p\mu + (1-p)\nu$ support is contained on M. We can then apply Lemma to approximate the measure weak* and in terms of entropy. From this it easily follows that if $a_1, a_2 \in H_p$ and $p \in (0,1)$ then $pa_1 + (1-p)a_2 \in H_p$. $\qquad\square$

Our lower bound for the dimension of L_a when $a \in H_p$ can be deduced from the following lemma which allows us to make use of the w-measure approach from the first part of the section.

Lemma 5.4. *For $a \in H_p$ there exist $\{a_i\} \in H, a_i \to a$ and ergodic measures $\{\nu_i\}$ such that $\int f d\mu_i = a_i$, $\lambda(\nu_i) \to 0$, $\lambda(\nu_i) > 0$, $\mathrm{supp}(\nu_i) \subset M_i$ where M_i is Markov and $\dim_H \nu_i \to \dim_{\mathrm{hyp}}(A)$.*

Proof. We will let $s = \dim_{\mathrm{hyp}}(A)$. Let $\varepsilon > 0$ and let ν be a ergodic measure with Markov support such that $h(\nu)/\lambda(\nu) > s - \varepsilon$ and $\int f d\nu = b$ for some $b \in H$. We can find μ an ergodic measure with support contained in a Markov set such that

$$\left| \int f d\mu - a \right| \le \varepsilon$$

and

$$\lambda(\mu) \le s^{-1}\varepsilon^2 \lambda(\mu).$$

Thus by Proposition 2.1 $(1-\varepsilon)\mu + \varepsilon\nu$ will be an invariant measure with support contained in a Markov set and with no atomic part. Moreover

$$h((1-\varepsilon)\mu + \varepsilon\nu) \ge \varepsilon h(\nu) \ge (s - 2\varepsilon)\varepsilon\lambda(\nu) + s(1-\varepsilon)\lambda(\mu)$$

Thus if we let $\eta = (1-\varepsilon)\mu + \varepsilon\nu$ we have an invariant measure with support contained in a Markov set with

$$\left| \int f d\eta - a \right| \le \varepsilon|b - a|$$

and

$$h(\eta)/\lambda(\eta) > s - 2\varepsilon.$$

We can now apply Lemma 5.2 to find a non-atomic ergodic measure $\tilde{\nu}$ with

$$\left| \int f d\tilde{\nu} - a \right| \le 2\varepsilon|b - a|$$

and

$$h(\tilde{\nu})/\lambda(\tilde{\nu}) > s - 4\varepsilon.$$

The result follows by letting $\varepsilon \to 0$.   □

We can now obtain the lower bound in H_p: applying Theorem 5.1 to the sequence of measures $\{\nu_i\}$ given by Lemma 5.4 we get $\dim_H L_a \geq \dim_{\mathrm{hyp}} A$. We finish the section by dealing with the case $\dim_H L_a \cap D_0$ when $a \in \mathrm{int(H)}$.

Corollary 5.5. *If $a \in \mathrm{int(H)}$ then $\delta_0(a) = \dim_{\mathrm{hyp}}(A)$ and thus $\dim_H L_a \cap D_0 = \dim_{\mathrm{hyp}}(A)$.*

Proof. Let $\delta = \dim_{\mathrm{hyp}}(A) - \varepsilon$ for some $\varepsilon > 0$. By Lemma 5.4 and Proposition 2.1 we can find non-atomic ergodic measures μ and ν supported on the same Markov set such that $\int f d\mu > a$, $\int f d\nu < a$ and $\min\{\frac{h(\mu)}{\lambda(\mu)}, \frac{h(\nu)}{\lambda(\nu)}\} > \delta$. Thus for $q \geq 0$ we have that

$$\int q(f - a) + \delta\phi \mathrm{d}\mu > 0$$

and for $q < 0$ we have that

$$\int q(f - a) + \delta\phi \mathrm{d}\nu > 0.$$

Thus by the variational principle (Lemma 8 [H2]) $P(q(f - a) - \delta\phi) > 0$ for all $q \in \mathbb{R}$. Since $a \in \mathrm{int(H)}$ we also know that $\lim_{q \to \pm\infty} P(q(f-a)-\delta\phi) = \infty$ and so $\delta_0(a) > \delta$. The result now follows.   □

6. FURTHER PROPERTIES OF THE BIRKHOFF SPECTRUM

The final step of the proof of Theorem 3.2 is to prove part 4 of the statement, that is to obtain the results on the continuity properties of the spectrum.

Proposition 6.1. *The function $a \to \dim_H L_a \cap D_0$ is lower semicontinuous for each point $x \in H_p$ and continuous for each $x \in H_h$.*

Proof. To prove the lower semicontinuity at a it is enough to prove that if $a_i \to a$, $\delta_i \to \delta$, and $\inf_q P(a_i, q, \delta_i) \leq 0$ then $\inf_q P(a, q, \delta + \varepsilon) \leq 0$ for all $\varepsilon > 0$.

Assume for simplicity that $a_i \nearrow a$ (the case $a_i \searrow a$ is done analogously, and by choosing a subsequence we can always make one of those cases to happen). We can also freely assume that a_i are in the interior of H. Fix some $\varepsilon > 0$, then $\delta + \varepsilon > \delta_i$, and hence also $\inf_q P(a_i, q, \delta + \varepsilon) \leq 0$, for sufficiently large i.

Consider the set

$$\Delta_i = \{q; P(a_i, q, \delta + \varepsilon) \leq 0\}$$

and divide it into two subsets $\Delta_i^+ = \Delta \cap (0, \infty)$ and $\Delta_i^- = \Delta \cap (-\infty, 0]$. As $P(a', q, \delta) = P(a, q, \delta) + q \cdot (a - a')$, the sequence $\{\Delta_i^+\}$ is decreasing and the sequence $\{\Delta_i^-\}$ is increasing. Moreover, as a_i is in the interior of H, Δ_i is always a compact set. Thus, Δ_i^- is compact as well.

Our assumption is that all the sets Δ_i are nonempty. If Δ_ℓ^+ is nonempty for some ℓ, we can take some $q \in \Delta_\ell^-$ and this q will also belong to all Δ_i^- for $i > \ell$. Hence,

$$(6.1) \qquad P(a, q, \delta + \varepsilon) = \lim P(a_i, q, \delta + \varepsilon) \leq 0.$$

On the other hand, if Δ_i^+ is nonempty for all i then Δ_i^- must be nonempty, also for all i. Thus, $\{\Delta_i^-\}$ forms a decreasing family of nonempty compact sets, hence it has nonempty intersection, and for any $q \in \bigcap \Delta_i^-$ inequality (6.1) holds as well.

Consider now $a \in H \setminus H_p$. To prove upper semicontinuity at a it is enough to prove that if $a_i \to a$, $\delta_i \to \delta$, and $\inf_q P(a_i, q, \delta_i) > 0$ then $\inf_q P(a, q, \delta - \varepsilon) > 0$ for all $\varepsilon > 0$.

As $a \notin H_p$, there exists $\varepsilon' > 0$ such that for every measure μ satisfying $\int f d\mu \in (a - \varepsilon', a + \varepsilon')$ we have $\lambda(\mu) > \varepsilon'$. Naturally, $a_i \in (a - \varepsilon', a + \varepsilon')$ and $\delta_i > \delta - \varepsilon/2$ for i large enough.

By Lemma 4.4, for all those i we can find an ergodic measure μ_i with $\int f d\mu_i = a_i$ and $h(\mu_i)/\lambda(\mu_i) \geq \delta_i$. Moreover, by the above, $\lambda(\mu_i) > \varepsilon'$. Thus, for every $q \in \mathbb{R}$ we have

$$P(a_i, q, \delta - \varepsilon) \geq h(\mu_i) + \int q(f - a_i) d\mu_i - (\delta - \varepsilon)\lambda(\mu_i) = \left(\frac{h(\mu_i)}{\lambda(\mu_i)} - \delta + \varepsilon\right) \lambda(\mu_i) \geq \frac{1}{2}\varepsilon\varepsilon'.$$

Thus,

$$P(a, q, \delta - \varepsilon) = \lim_i P(a_i, q, \delta - \varepsilon) \geq \frac{1}{2}\varepsilon\varepsilon'.$$

$\square$

We know that in general $a \to \dim_H L_a \cap D_0$ will not be concave but we can present a result which shows that $a \to \dim_H L_a \cap D_0$ is piecewise monotone with at most two pieces.

Proposition 6.2. *For any $a_1 < a_2 < a_3$,*

$$\dim_H L_{a_2} \cap D_0 \geq \min(\dim_H L_{a_1} \cap D_0, \dim_H L_{a_3} \cap D_0)$$

Proof. Fix $\delta < \min(\dim_H L_{a_1} \cap D_0, \dim_H L_{a_3} \cap D_0)$. We have for any $q \in \mathbb{R}$, $P(a_1, q, \delta) > 0$ and $P(a_3, q, \delta) > 0$. Moreover

$$P(a_2, q, \delta) = P(a_3, q, \delta) + q(a_2 - a_3) = P(a_1, q, \delta) + q(a_2 - a_1)$$

and since $a_2 - a_3$ and $a_2 - a_1$ have opposing signs it follows that $P(a_2, q, \delta) > 0$. Thus $\dim_H L_{a_2} > \delta$ and the proof follows.

$\square$

7. EXAMPLES

One example where the Birkhoff spectra has already been studied but gives a nice illustration of the hyperbolic spectra is the Manneville-Pomeau map. Let $0 < \gamma < 1$ and $T : [0, 1] \to [0, 1]$ be defined by $T(x) = x + x^{1+\gamma}$ mod 1 if $x \in [0, 1)$ and $T(1) = 1$. In this case we can take $A = [0, 1]$ and for $f : [0, 1] \to \mathbb{R}$ we have that $H_p = \{f(0)\}$.

In this setting $H = [a_{\min}, a_{\max}]$ where

$$a_{\min} = \min \left\{ \int f \mathrm{d}\mu : \mu \ T - \mathrm{inv} \right\}$$

and

$$a_{\max} = \max \left\{ \int f \mathrm{d}\mu : \mu \ T - \mathrm{inv} \right\}.$$

We also have $H_p = \{f(0)\}$. In [JJOP] it is shown that $a \to \dim_H L_a$ is continuous on $[a_{\min}, f(0))$ and $[f(0), a_{\max}]$. In Theorem 6.3 in [IJ2] it is shown it is possible to have a discontinuity at $f(0)$ where the dimension jumps up (so the spectrum is upper-semi continuous but not lower semi-continuous). Our results show that in this discontinuous case if we instead consider the hyperbolic spectrum $a \to \dim_H X_a \cap D_0$ then the spectrum is lower semi continuous and so $\dim_H L_{f(0)} \cap D_0 = \liminf_{a \to f(0)} \dim_H L_a < \limsup_{a \to f(0)} \dim_H L_a = 1$. (Note that it can be the case when $f(0) = a_{\min}$ that $L_a \cap D_0 = \emptyset$.

Another example where our results can be applied is the map $T : [0, 1] \to [0, 1]$ given by $T(x) = x + \gamma x^{1+\varepsilon} \mod 1$ where $\gamma > 1$ and $\varepsilon \in (0, 1)$. These maps are conjugate to β-expansions but have similar parabolic behaviour to the Manneville-Pomeau maps. Their thermodynamic properties (in particular uniqueness of certain equilibrium states) are studied in [CT]. Our results can be applied to these maps and again $H_p = \{f(0)\}$ and the hyperbolic dimension is 1. This means we can fully compute the spectrum both $a \to \dim_H L_a$ and $a \to \dim_H L_a \cap D_0$.

References

[BS]   L. Barreira and B. Saussol, Variational principles and mixed multifractal spectra, *Trans. Amer. Math. Soc.*, 353, no 10, 3919–3944, 2001.

[B]   A. Besicovitch, On the sum of digits of real numbers represented in the dyadic system, *Math. Ann.*, 110, no 1, 321–330.1935.

[Bi]   P. Billingsley, Hausdorff dimension in probability theory, *Illinois J. Math.*4 1960 187–209.

[C]   Y. Chung, Birkhoff spectra for one-dimensional maps with some hyperbolicity, *Stoch. Dyn.* 10 (2010), no. 1, 53–75.

[CTa]   Y. Chung and H. Takahasi, Multifractal formalism for Benedicks-Carleson quadratic maps, *Ergodic Theory Dynam. Systems* 34 (2014), no. 4, 1116–1141.

[CT]   V. Climenhaga and D. Thompson, Equilibrium states beyond specification and the Bowen property, *J. Lond. Math. Soc.* (2) 87 (2013), no. 2, 401–427.

[E]   H. Eggleston, The fractional dimension of a set defined by decimal properties ,*Quart. J. Math., Oxford Ser.* 20, (1949). 31–36.

[FFW]   A-H Fan, D-J Feng and J. Wu Recurrence, dimension and entropy, *J. London Math. Soc.* (2) 64 (2001), no. 1, 229–244.

[FH]   D-J Feng and W. Huang Lyapunov spectrum of asymptotically sub-additive potentials, *Comm. Math. Phys.* 297 (2010), no. 1, 1?43.

[FLW]   D-J. Feng, K-S. Lau and J. Wu, Ergodic limits on the conformal repellers, *Adv. Math.*, 169, no.1, 58–91,2002.

[GPR]   K. Gelfert, F. Przytycki and M. Rams, Lyapunov spectrum for multimodal maps, *Ergodic Theory Dynam. Systems* 36 (2016), no. 5, 1441–1493.

[GR]   K. Gelfert and M. Rams , The Lyapunov spectrum of some parabolic systems, *Ergodic Theory Dynam. Systems*, 29, no. 3, 919–940,2009.

[H1]   F. Hofbauer, Local dimension for piecewise monotonic maps on the interval, *Ergodic Theory Dynam. Systems*, 15, no 6, 1119–1142,1995.

[H2]   F. Hofbauer, Multifractal spectra of Birkhoff averages for a piecewise monotone interval map, *Fund. Math.*, 208,no 2, 95–121,2010.

[HR]   F. Hofbauer and P. Raith, The Hausdorff dimension of an ergodic invariant measure for a piecewise monotonic map of the interval, *Canad. Math. Bull* 35 (1992), no. 1, 84–98.

[HU]   F. Hofbauer,and M. Urbański Fractal properties of invariant subsets for piecewise monotonic maps on the interval, *Trans. Amer. Math. Soc.* 343 (1994), no. 2, 659–673.

[IJ1]   G. Iommi and T. Jordan, Multifractal analysis of Birkhoff averages for countable Markov maps, *Ergodic Theory Dynam. Systems* 35 (2015), no. 8, 2559–2586.

[IJ2]   G. Iommi and T. Jordan, Multifractal analysis for quotients of Birkhoff sums for countable Markov maps, *Int. Math. Res. Not.* 2015, no. 2, 460–498.

[IT]   G. Iommi and M. Todd, Dimension theory for multimodal maps, *Ann. Henri Poincaré*12 (2011), no. 3, 591–620.

[JJOP] A. Johansson, T. Jordan, A. Öberg, M. Pollicott, Multifractal analysis of non-uniformly hyperbolic systems, *Israel J. Math.* 177 (2010), 125–144.

[KS]   M. Kesseböhmer and B. Stratmann, A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates, J. Reine Angew. Math. 605 (2007), 133–163.

[K]   V. Knichal, Dyadische Entwicklungen und Hausdorffsches Mass, *Mem. Soc. Roy . Sci . Boheme , Cl . Sci . Nr . XIV* (1933) .

[N]   K. Nakaishi. Multifractal formalism for some parabolic maps, *Ergod. Th. Dynam. Sys.* 20, 843–857, 2000

[Oli]   E. Olivier, Structure multifractale d'une dynamique non expansive définie sur un ensemble de Cantor (French) [Multifractal structure of a nonexpanding dynamics defined on a Cantor set], *C. R. Acad. Sci. Paris Ser. I Math.* 331 (2000), no. 8, 605?610.

[O]   L. Olsen, Multifractal analysis of divergence points of deformed measure theoretical Birkhoff averages, *J. Math. Pures Appl*, (9) 82 (2003), no. 12, 1591–1649.

[PW]   Y. Pesin and H. Weiss, The multifractal analysis of Birkhoff averages and large deviations,*Global analysis of dynamical systems,* 419–431, Inst. Phys., Bristol, 2001.

[TV]   F. Takens and E. Verbitskiy, On the variational principle for the topological entropy of certain non-compact sets, *Ergodic Theory Dynam. Systems* 23 (2003), no. 1, 317–348.

[W]   H. Weiss, The Lyapunov spectrum for conformal expanding maps and axiom-A surface diffeomorphisms, *J. Statist. Phys.* 95 (1999), no. 3-4, 615?-632.

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