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FIR FILTERING OF DISCONTINUOUS SIGNALS: A RANDOM-STRATIFIED SAMPLING APPROACH

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ABSTRACT

This paper presents a novel approach, based on random stratified sampling (StSa) technique, to estimate the output of a finite impulse response (FIR) filter when the input signal is either a piecewise-continuous function having first-derivative discontinuities (FDDs), or a piecewise-discontinuous function, i.e. having zero-derivative discontinuities (ZDDs). The proposed approach investigates the implications of such discontinuities on the output signal and its statistical properties. Mainly, we devise mathematical expressions for the variance of the StSa estimator in the two cases above, along with other minor special cases. It is found that the uniform convergence rate of the estimator, in the FDDs case, is N^{-3}, where N is the number of random samples. However, the variance in the ZDDs case is adversely affected by the existence of discontinuities. We prove that it converges more slowly with a uniform rate of N^{-2}.

Index Terms— Random sampling, FIR filter, zero- and first-derivative discontinuities, uniform convergence rate, stratification.

1. INTRODUCTION

In classical digital signal processing (DSP), where uniform sampling is utilised, aliasing problem in Fourier transform (FT) arises when the sampling frequency is less than the Nyquist rate [1], [2]. Random sampling and digital alias-free signal processing (DASP) have been addressed by many researchers as a promising approach to mitigate this issue [3]–[7]. Different DASP-based applications have been introduced in the literature [8]–[11], as well as filtering of randomly sampled signals [12]–[16]. However, no publications investigated the use of StSa-based sampling technique in the filtering of bandlimited signals except [15]. Nonetheless, the work in [15] does not include any cases for discontinuous-time signals: neither such discontinuities exist in the first-derivative (FD), nor the zero-derivative (ZD).

Practically, discontinuities in the FD or the ZD of input signals occur so frequently in everyday applications. For instance, clipping or rectifying of continuous-time signals leads to discontinuities in one or more orders of the derivative [17]. Other examples include transient signals, power cut and digital data. Moreover, many window functions used in filtering also have ZDDs or, at least, FDDs. To name few: Rectangular, Triangular, Hamming, Tukey and Exponential (Poisson) are all examples of discontinuous windows.

In real-life applications, the impulse responses of FIR filters are normally continuous, though, any mathematical function, including discontinuous ones, can be theoretically used as an impulse response, let alone the above-mentioned discontinuous windows, which can be used as filters by themselves.

In this paper, we explore different cases where an input signal, window function or impulse response of an FIR filter may have such discontinuities. The next section introduces the StSa technique. The core of this paper is presented in section 3, while section 4 illustrates some numerical examples. A brief conclusion is provided at the end.

2. STRATIFIED SAMPLING TECHNIQUE

The random StSa technique depends on the notion of stratification of an observation window. A time interval, $[0, T]$, is divided into an N number of sub-intervals called strata. Then, an arbitrary sample is taken per each stratum. The function used to calculate the strata borders (limits) is detailed in [18]. However, one of its simplest forms is to allow all the strata to have equal widths, i.e. T/N, across the whole observation window, as shown in Fig.1.

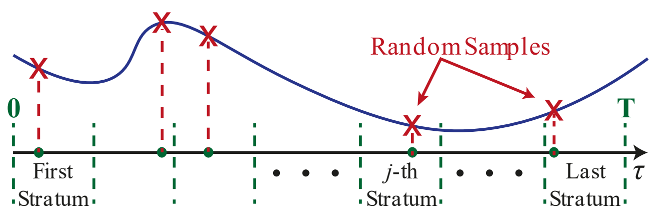


Fig. 1. In StSa, one sample is randomly acquired per each stratum.

In [18], an estimator based on StSa technique was used to estimate the Fourier transform (FT) of a randomly sampled signal. The estimator was proven to be unbiased, and its variance or mean squared error (MSE) uniformly converges at a rate of N^{-3}. Yet, a fundamental assumption had been made to apply the Taylor series expansion of the input signal accurately. The assumption emphasised that the input signal should have a continuous first-derivative across the observation window, which is a necessary and sufficient condition for expanding functions using Taylor series.

In contrast, this work has the following main contributions: a) introducing the application of StSa on FIR filtering analytically and numerically, b) studying the effects of having jump discontinuities in both first- and zero-order derivatives of the input signal on the filter estimator, c) devising mathematical expressions for the estimator's variance in different cases, and (d) finding the convergence rates of the variance in those cases. Other minor contributions are presented as special cases for the variance when the jump discontinuity exists at specific locations within a stratum.

3. FILTERING WITH DISCONTINUITIES

Consider an FIR filter with an impulse response $h(t)$, input signal $x(t)$, window function $w(t)$ and output signal $y(t)$. Thus, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)w(\tau)h(t-\tau)d\tau. \quad (1)$$

We define the integrated function inside the integral in (1) as $g(t,\tau)$,

$$g(t,\tau) = x(\tau)w(\tau)h(t-\tau). \quad (2)$$

The analysis in this section applies for the case where the function $g(t,\tau)$ and its FD, $g'(t,\tau)$, are both bounded. Also, $g(t,\tau)$ may have an up to one FDD (or ZDD) in any stratum. If it exists, it will be at time instant τ_{D_j}.Table I lists the notations used in this paper.

3.1. First-Derivative Discontinuities (FDDs)

Firstly, we assume that the integrated function $g(t,\tau)$ is piecewise continuous and having FDDs over the observation window, $[0,T]$. Furthermore, $g(t,\tau)$ is randomly sampled using StSa technique, where there are N strata and one random sample is acquired per each stratum at time instant τ_j, as depicted in Fig. 2.

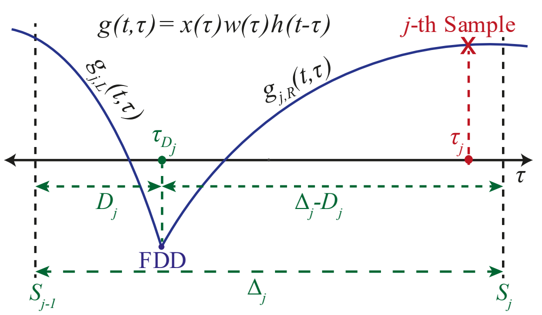


Fig. 2. An example of a stratum where $g(t,\tau)$ has an FDD at time instant τ_{D_j}, showing the left- and right-hand pieces $g_{j,L}(t,\tau)$ and $g_{j,R}(t,\tau)$, respectively, alongside other required notations.

It is possible to rewrite $g(t,\tau)$ as a summation of N sub-functions:

$$g(t,\tau) = \sum_{j=1}^{N} g_j(t,\tau)\, , \tau \in [0,T]. \quad (3)$$

Each sub-function, $g_j(t,\tau)$, comprises two pieces: left-hand, $g_{j,L}(t,\tau)$ and right-hand, $g_{j,R}(t,\tau)$, as defined here

$$g_j(t,\tau) = \begin{cases} g_{j,L}(t,\tau), & S_{j-1} \le \tau \le \tau_{D_j} \\ g_{j,R}(t,\tau), & \tau_{D_j} \le \tau \le S_j \end{cases}. \quad (4)$$

TABLE I: Notations as per the j-th stratum (Fig. 2).

Δ_j	The time width of the j-th stratum.
S_{j-1}	The start time of the j-th stratum.
S_j	The end time of the j-th stratum.
τ_{D_j}	The time instant at which $g(t,\tau)$ may have an FDD or ZDD.
$A_{j,L}$	The j-th stratum left subinterval, i.e. $A_{j,L} = [S_{j-1}, \tau_{D_j}]$.
$A_{j,R}$	The j-th stratum right subinterval, i.e. $A_{j,R} = [\tau_{D_j}, S_j]$.
D_j	The time width of $g_{j,L}(t,\tau)$.
$\Delta_j - D_j$	The time width of $g_{j,R}(t,\tau)$.
τ_j	The time instant of the j-th random sample. We assume that τ_j is a uniformly distributed random variable over the stratum width, i.e. its PDF, $p_\tau(\tau_j)$, is $1/\Delta_j$ within the j-th stratum and zero elsewhere.
K_j	The ratio D_j/Δ_j. Therefore $0 \le K_j \le 1$.

Let Z_j be the estimator of the area under the curve of $g(t,\tau)$ within the sub-interval of the j-th stratum, hence

$$Z_j = g(t,\tau_j)\Delta_j = g_j(t,\tau_j)\Delta_j. \quad (5)$$

The expected value of Z_j is

$$\mathrm{E}[Z_j] = \int_{-\infty}^{\infty} g_j(t,\tau)\Delta_j\, p_\tau(\tau)d\tau, \quad (6a)$$

$$\mathrm{E}[Z_j] = \int_{A_{j,L}} g_{j,L}(t,\tau)d\tau + \int_{A_{j,R}} g_{j,R}(t,\tau)d\tau. \quad (6b)$$

Define $e_j = Z_j - E[Z_j]$, thus, we have from (5) and (6b)

$$e_j = g_j(t,\tau_j)\Delta_j - \left(\int_{A_{j,L}} g_{j,L}(t,\tau)d\tau + \int_{A_{j,R}} g_{j,R}(t,\tau)d\tau\right). \quad (7)$$

Using Taylor series to expand $g_{j,L}(t,\tau)$ and $g_{j,R}(t,\tau)$ about τ_{D_j},

$$e_j = g_j(t,\tau_j)\Delta_j - \Big(\int_{A_{j,L}} \Big(g_{j,L}\left(t,\tau_{D_j}\right) + \left(\tau - \tau_{D_j}\right) \times g'_{j,L}\left(t,\tau_{D_j}\right) + o\left(\left|\tau - \tau_{D_j}\right|\right)\Big) d\tau + \int_{A_{j,R}} \Big(g_{j,R}\left(t,\tau_{D_j}\right) + \left(\tau - \tau_{D_j}\right) g'_{j,R}\left(t,\tau_{D_j}\right) + o\left(\left|\tau - \tau_{D_j}\right|\right)\Big) d\tau\Big), \quad (8a)$$

$$e_j = g_j(t,\tau_j)\Delta_j - \Big(g_{j,L}\left(t,\tau_{D_j}\right) D_j - \tfrac{1}{2} g'_{j,L}\left(t,\tau_{D_j}\right) D_j^{\,2} + o\left(D_j^{\,2}\right) + g_{j,R}\left(t,\tau_{D_j}\right)(\Delta_j - D_j) + \tfrac{1}{2} g'_{j,R}\left(t,\tau_{D_j}\right) \times (\Delta_j - D_j)^2 + o\left((\Delta_j - D_j)^2\right)\Big). \quad (8b)$$

Note that both $o\left(D_j^{\,2}\right)$ and $o\left((\Delta_j - D_j)^2\right) \in o\left(\Delta_j^{\,2}\right)$ since $0 \le D_j \le \Delta_j$ and $0 \le (\Delta_j - D_j) \le \Delta_j$ for all $j \in \{1,2,3,\dots,N\}$, so

$$e_j = g_j(t,\tau_j)\Delta_j - \Big(\Big(g_{j,L}\left(t,\tau_{D_j}\right) - g_{j,R}\left(t,\tau_{D_j}\right)\Big) D_j + g_{j,R}\left(t,\tau_{D_j}\right)\Delta_j - M_j\,\Delta_j^{\,2} + o\left(\Delta_j^{\,2}\right)\Big), \quad (9)$$

where $M_j = \frac{1}{2} g'_{j,L}\left(t,\tau_{D_j}\right) K_j^{\,2} - \frac{1}{2} g'_{j,R}\left(t,\tau_{D_j}\right)\left(1 - K_j\right)^2$.

Remark that $g_j(t,\tau_j)$ is a piecewise continuous function, i.e. $g_{j,L}\left(t,\tau_{D_j}\right) = g_{j,R}\left(t,\tau_{D_j}\right) = g_j\left(t,\tau_{D_j}\right)$. Thus

$$e_j = \left(g_j(t,\tau_j) - g_{j,R}\left(t,\tau_{D_j}\right)\right)\Delta_j + M_j\Delta_j^{\,2} - o\left(\Delta_j^{\,2}\right). \quad (10)$$

Equation (10) is applicable for the whole stratum, and it can be split into two terms, $e_{j,L}$ and $e_{j,R}$, to match $A_{j,L}$ and $A_{j,R}$ boundaries,

$$e_j = \begin{cases} e_{j,L}, & \tau_j \in A_{j,L} \\ e_{j,R}, & \tau_j \in A_{j,R} \end{cases}, \quad (11)$$

where $e_{j,L} = \left(g_{j,L}(t,\tau_j) - g_{j,L}\left(t,\tau_{D_j}\right)\right)\Delta_j + M_j\,\Delta_j^{\,2} - o\left(\Delta_j^{\,2}\right)$,

and $e_{j,R} = \left(g_{j,R}(t,\tau_j) - g_{j,R}\left(t,\tau_{D_j}\right)\right)\Delta_j + M_j\,\Delta_j^{\,2} - o\left(\Delta_j^{\,2}\right)$.

Now, the variance of the estimator for the j-th stratum, $\mathrm{Var}[Z_j]$, is calculated by finding the second moment of the error term in (11),

$$\mathrm{Var}[Z_j] = \int_{-\infty}^{\infty} |e_j|^2 p_\tau(\tau) d\tau. \quad (12)$$

For paper size limitation, we only show here the final answer[1],

$$\mathrm{Var}[Z_j] = \left(\frac{1}{3}\left|g'_{j,L}\left(t,\tau_{D_j}\right)\right|^2 K_j^3 + \frac{1}{3}\left|g'_{j,R}\left(t,\tau_{D_j}\right)\right|^2 \times (1-K_j)^3 + |M_j|^2 - \mathrm{Re}\left[K_j^2 (M_j)^* g'_{j,L}\left(t,\tau_{D_j}\right) - (1-K_j)^2 (M_j)^* g'_{j,R}\left(t,\tau_{D_j}\right)\right]\right)\Delta_j^4 + o(\Delta_j^4), \quad (13)$$

where $(M_j)^*$ is the complex conjugate of M_j.

3.1.1. Some Special Cases

1. If both $g'_{j,L}\left(t,\tau_{D_j}\right)$ and $g'_{j,R}\left(t,\tau_{D_j}\right)$ are real-valued, then

$$\mathrm{Var}[Z_j] = \frac{1}{12}\left({g'_{j,L}}^2\left(t,\tau_{D_j}\right) l_j + g'_{j,L}\left(t,\tau_{D_j}\right) g'_{j,R}\left(t,\tau_{D_j}\right) \times m_j + {g'_{j,R}}^2\left(t,\tau_{D_j}\right) r_j\right)\Delta_j^4 + o(\Delta_j^4). \quad (14)$$

Here $l_j = (-3K_j^4 + 4K_j^3)$, $m_j = (6K_j^4 - 12K_j^3 + 6K_j^2)$, and $r_j = (-3K_j^4 + 8K_j^3 - 6K_j^2 + 1)$. Note that the sum of these three scalers (weights) at any given K_j ratio is precisely one.

2. If $K_j = 0$, then (14) reduces to

$$\mathrm{Var}[Z_j] = \frac{1}{12}{g'_{j,R}}^2\left(t,\tau_{D_j}\right)\Delta_j^4 + o(\Delta_j^4) = \frac{1}{12}{g'_j}^2\left(t,\tau_{D_j}\right) \times \Delta_j^4 + o(\Delta_j^4). \quad (15)$$

3. If $K_j = 1$, then (14) also reduces to (15), with L instead of R.

4. If $K_j = \frac{1}{2}$, i.e. τ_{D_j} is exactly at the centre of the j-th stratum, then

$$\mathrm{Var}[Z_j] = \left(\frac{5}{192}{g'_{j,L}}^2\left(t,\tau_{D_j}\right) + \frac{1}{32}g'_{j,L}\left(t,\tau_{D_j}\right) g'_{j,R}\left(t,\tau_{D_j}\right) + \frac{5}{192}{g'_{j,R}}^2\left(t,\tau_{D_j}\right)\right)\Delta_j^4 + o(\Delta_j^4). \quad (16)$$

5. If $g'_{j,L}\left(t,\tau_{D_j}\right) = g'_{j,R}\left(t,\tau_{D_j}\right) = g'_j\left(t,\tau_{D_j}\right)$, i.e. no FDD, then

$$\mathrm{Var}[Z_j] = \frac{1}{12}{g'_j}^2\left(t,\tau_{D_j}\right)\Delta_j^4 + o(\Delta_j^4). \quad (17)$$

Note that (15) and (17) match (4.18) in [18], as all reflect the case where there are no FDDs in the j-th stratum.

3.1.2. Variance of the whole N-strata estimator

So far, we have calculated, in (14), the variance related to the j-th stratum only. However, for the whole N-strata estimator, $\hat{y}(t)$, we need to add up the N variances as they are i.i.d random variables,

$$\mathrm{Var}[\hat{y}(t)] = \sum_{j=1}^{N}\left(\frac{1}{12}\left({g'_{j,L}}^2\left(t,\tau_{D_j}\right) l_j + g'_{j,L}\left(t,\tau_{D_j}\right) \times g'_{j,R}\left(t,\tau_{D_j}\right) m_j + {g'_{j,R}}^2\left(t,\tau_{D_j}\right) r_j\right)\Delta_j^4 + o(\Delta_j^4)\right). \quad (18)$$

Recall that for equal strata widths, we have $\Delta_j = \frac{T}{N} \equiv \Delta$,

$$\mathrm{Var}[\hat{y}(t)] = \frac{T^3}{12N^3}\sum_{j=1}^{N}\left({g'_{j,L}}^2\left(t,\tau_{D_j}\right) l_j + g'_{j,L}\left(t,\tau_{D_j}\right) \times g'_{j,R}\left(t,\tau_{D_j}\right) m_j + {g'_{j,R}}^2\left(t,\tau_{D_j}\right) r_j\right)\Delta + o\left(\frac{1}{N^3}\right). \quad (19)$$

It is evident from (19) that the uniform convergence rate of $\mathrm{Var}[\hat{y}(t)]$ is N^{-3}. Moreover, we can use Riemann integration technique to estimate the variance by taking the limit as $N \to \infty$,

$$\lim_{N\to\infty} N^3(\mathrm{Var}[\hat{y}(t)]) = \lim_{N\to\infty}\left(\frac{T^3}{12}\sum_{j=1}^{N}\left({g'_{j,L}}^2\left(t,\tau_{D_j}\right) l_j + g'_{j,L}\left(t,\tau_{D_j}\right) g'_{j,R}\left(t,\tau_{D_j}\right) m_j + {g'_{j,R}}^2\left(t,\tau_{D_j}\right) r_j\right)\Delta\right). \quad (20)$$

The FD of the integrated function, $g'(t,\tau)$, has either a limited number of discontinuities, say M, or an infinite number of discontinuities. In the first case, $g'(t,\tau)$ comprises M+1 pieces, and since it is bounded by assumption; equation (20) can be calculated by adding up $M+1$ integral terms,

$$\lim_{N\to\infty} N^3(\mathrm{Var}[\hat{y}(t)]) = \frac{T^3}{12}\sum_{k=1}^{M+1}\int_{T_{k-1}}^{T_k} {g'}^2(t,\tau)d\tau. \quad (21)$$

where T_k is the k-th time instant at which the FD is discontinuous except for the limits of the observation window, where we define $T_0 = 0$ and $T_{M+1} = T$.

Note that as the number of samples approaches infinity, i.e. $N \to \infty \equiv \Delta \to 0$, and if $g(t,\tau)$ has only M FDDs, then all the scalers l_j, m_j and r_j are either zeros or ones, as a result of K_j values being zeros or ones, except for a negligible number of scalers compared to infinity. In this case, either the j-th left- or right-hand derivative terms in the summation of (20) will be cancelled out, and the remaining terms will always reduce to ${g'}^2(t,\tau)$, as shown in (21).

For the second case, where $g(t,\tau)$ is everywhere continuous but nowhere differentiable, then the variance of the estimator diverges. And so, our estimator can't be used in this case.

3.2. Zero-Derivative Discontinuities (ZDDs)

If $g_j(t,\tau_j)$ is discontinuous at point τ_{D_j}, so as the derivative, then

$$g_{j,L}\left(t,\tau_{D_j}\right) \neq g_{j,R}\left(t,\tau_{D_j}\right), \text{and } g'_{j,L}\left(t,\tau_{D_j}\right) \neq g'_{j,R}\left(t,\tau_{D_j}\right). \quad (22)$$

In this case, the estimator error for the j-th stratum, (11), is no longer valid. Instead, we introduce here a tweaked version of it,

$$e_j = \left\{\left(g_{j,L}(t,\tau_j) - g_{j,R}\left(t,\tau_{D_j}\right)\right)\Delta_j - C_j D_j + M_j\Delta_j^2 - o(\Delta_j^2), \text{for } \tau_j \in A_{j,L}\right\}, \text{and } e_j = \left\{\left(g_{j,R}(t,\tau_j) - g_{j,R}\left(t,\tau_{D_j}\right)\right)\Delta_j - C_j D_j + M_j\Delta_j^2 - o(\Delta_j^2), \text{ for } \tau_j \in A_{j,R}\right\}, \quad (23)$$

where C_j is the difference between the left- and right-hand pieces of the integrated function at the jump discontinuity of the j-th stratum,

$$C_j = g_{j,L}\left(t,\tau_{D_j}\right) - g_{j,R}\left(t,\tau_{D_j}\right). \quad (24)$$

Using Taylor series expansion about τ_{D_j}, we get

$$e_j = \left\{\left(\left(\tau_j - \tau_{D_j}\right)g'_{j,L}\left(t,\tau_{D_j}\right) + o\left(\left|\tau_j - \tau_{D_j}\right|\right) + C_j(1-K_j)\right)\Delta_j + M_j\Delta_j^2 - o(\Delta_j^2), \text{for } \tau_j \in A_{j,L}\right\}, \text{ and}$$
$$e_j = \left\{\left(\left(\tau_j - \tau_{D_j}\right)g'_{j,R}\left(t,\tau_{D_j}\right) + o\left(\left|\tau_j - \tau_{D_j}\right|\right) - C_j K_j\right)\Delta_j + M_j\Delta_j^2 - o(\Delta_j^2), \text{ for } \tau_j \in A_{j,R}\right\}, \quad (25)$$

Recall that $K_j = D_j/\Delta_j$, and $\left|\tau_j - \tau_{D_j}\right| \leq \Delta_j$, therefore $o\left(\left|\tau_j - \tau_{D_j}\right|\right) = o(\Delta_j)$. As $N \to \infty$, we have $\Delta_j \to 0$, and so, the least power terms of the error expression in (25) will be dominating over all other higher power terms. Hence, it can be reduced to

[1] The full mathematical derivation is included in a much more comprehensive and revised journal paper to be submitted soon.

$$e_j = \begin{cases} \left(\left(\tau_j - \tau_{D_j}\right) g'_{j,L}\left(t, \tau_{D_j}\right) + o(\Delta_j) + C_j(1 - K_j)\right)\Delta_j, & \tau_j \in A_{j,L} \\ \left(\left(\tau_j - \tau_{D_j}\right) g'_{j,R}\left(t, \tau_{D_j}\right) + o(\Delta_j) - C_j K_j\right)\Delta_j, & \tau_j \in A_{j,R} \end{cases} \quad (26)$$

Now, the expected value of the error for one stratum is

$$\mathrm{Var}[Z_j] = \Delta_j \int_{S_{j-1}}^{\tau_{D_j}} \left|\left(\tau - \tau_{D_j}\right) g'_{j,L}\left(t, \tau_{D_j}\right) + o(\Delta_j) + C_j(1 - K_j)\right|^2 d\tau + \Delta_j \int_{\tau_{D_j}}^{S_j} \left|\left(\tau - \tau_{D_j}\right) g'_{j,R}\left(t, \tau_{D_j}\right) + o(\Delta_j) - C_j K_j\right|^2 d\tau. \quad (27)$$

For paper size limitation, we provide the worked-out result for the N strata estimator's variance, assuming there is a limited number of ZDDs, i.e. there are only M nonzero C_j's. Thus, we have

$$\mathrm{Var}[\hat{y}(t)] = \left(\frac{T^3}{12N^3} \sum_{\substack{n=1 \\ n \notin \mathrm{I}_M}}^{N} \left(f\left(K_n, t, \tau_{D_n}\right)\right)\Delta_n\right) + o\left(\frac{1}{N^3}\right) + \frac{T^2}{12N^2} \sum_{j \in \mathrm{I}_M} K_j(1 - K_j)\left|C_j\right|^2, \quad (28a)$$

$$\lim_{N \to \infty} N^2(\mathrm{Var}[\hat{y}(t)]) = \frac{T^2}{12}\left(\sum_{j \in \mathrm{I}_M} K_j(1 - K_j)\left|C_j\right|^2\right), \quad (28b)$$

where $\mathrm{I}_M = \{i_1, i_2, i_3, \ldots, i_M\}$ is a set of integers representing the indices of the strata where there are ZDDs, and $f\left(K_n, t, \tau_{D_n}\right)$ is some function that is calculated from the integral of (27), and the subscript n, used in (28a), is to distinguish between the two summations.

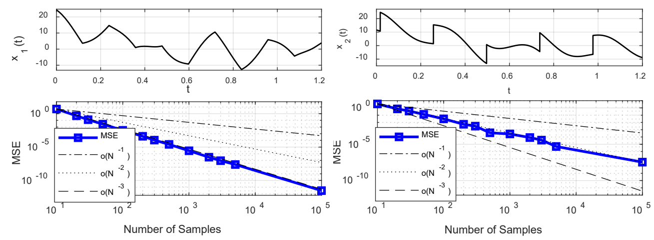
Note that (28b) is obtained by multiplying (28a) by N^2 and taking the limit as $N \to \infty$. Hence, the first two parts of the RHS of (28a) will, then, be zero, while the third part is a summation of exactly M numbers. Therefore, the variance of the estimator, in this case, will be converging at a uniform rate of N^{-2}, as shown in (28b). This rate will "increase" to N^{-3} when all C_j's are precisely zeros, i.e. no function discontinuities at all, which is the same conclusion for the case discussed in the previous section.

Finally, one may note that if the ratios $\{K_j,\ j \in \mathrm{I}_M\}$ are only zeros or ones, which is highly unlikely to happen all the times as N approaching infinity, then the third part of the RHS of (28a) will always be zero, and so the convergence rate will be N^{-3} again.

4. NUMERICAL RESULTS

Assume we have two functions $x_1(t) = 4\pi sinc(2t) + 2\sin(2\pi \times 1.7t) + 5\cos(2\pi \times 2.9t) + 7\ sawtooth(2\pi \times 4.167(t - 0.118), ,0.5)$ and $x_2(t) = 4\pi sinc(2t) + 2\sin(2\pi \times 1.7t) + 5\cos(2\pi \times 2.9t) + 7 sawtooth(2\pi \times 4.167(t - 0.017), 0)$. The first one has no ZDDs but ten FDDs within an observation window of [0,1.2] sec. Whereas $x_2(t)$ has five ZDDs and five FDDs within this window. We have integrated them numerically based on the random StSa technique and compared the results with their actual integral values. Their MSE uniform convergence rates are in $o(N^{-3})$ and $o(N^{-2})$, respectively, as shown in Fig. 3.

In another example, we have designed a lowpass filter with a cutoff frequency of 20kHz and used it to filter uniformly and randomly StSa-based sampled versions of three almost similar input signals having: 1) no ZDDs and no FDDs, 2) no ZDDs and many FDDs and 3) many ZDDs and many FDDs. The spectra of the three input signals include fixed FT components (two *sincs* and one *sinusoid*) within the bandwidth of the filter, and arbitrary FT components (one *sinc* and three *sinusoids*) spanning from just above the bandwidth of the filter up to 65.536kHz. Fig. 4 shows the filter inputs and outputs, where the random signals are nearly matching the uniform ones even though the average random sampling frequency is only 65.536kHz. While we would need, for uniform sampling, at least 131.072kHz to avoid aliasing problem. Note that the spectrum errors of StSa estimator in Fig. 4 (d) is larger than those in (b) and (c) although the number of samples is equal for the three estimators; this is because the input signal in (d) has many ZDDs and many FDDs, i.e. the estimator variance converges with a slower rate of N^{-2}, while the other two signals have no ZDDs, and so, the estimators converge more quickly at a rate of N^{-3}.



(a) (b)

Fig. 3. The variance of the integrated functions (a) $x_1(t)$ with a convergence rate of $o(N^{-3})$, and (b) $x_2(t)$ with a rate of $o(N^{-2})$.

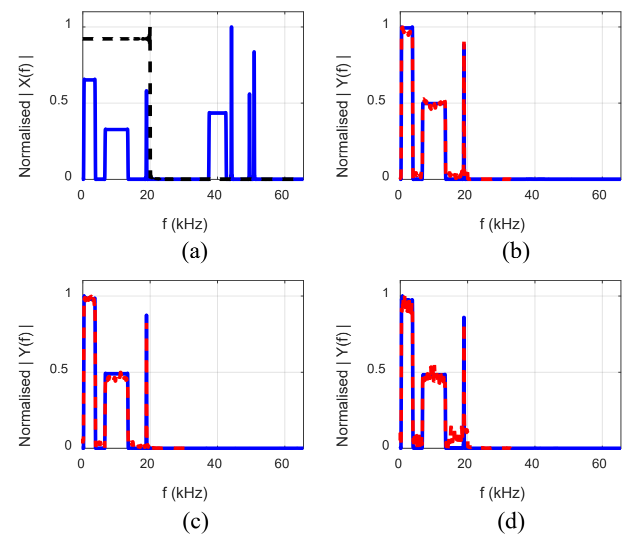


Fig. 4. The spectra of (a) the filter impulse response (dashed black) and one of the uniform input signals, and {(b), (c) and (d)} the filter outputs of the uniform (blue) and random StSa-based (dashed red) signals for {1), 2) and 3)} cases, respectively.

5. CONCLUSION

We introduce an implementation of StSa-based random sampling technique in digital filtering of discontinuous-time signals. The implications of having discontinuities in the first- or zero- and first-order derivatives of the filter's integrated function are investigated. In the first case, where the discontinuities only occur in the FD, the rate of uniform convergence of the estimator is proven to be N^{-3}. Whereas, the estimator is converging more slowly at a rate of N^{-2} for the second case, where there are ZDDs. The demonstrated numerical examples emphasise our analytical findings.

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