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CONTENTS

ABSTRACT

of

ADJUSTMENT METHODS FOR PLANIMETRIC OBSERVATIONS AND COORDINATES IN SURVEY NETWORKS

Correction and adjustment of observed angles and sides in geodetic networks are necessary for the purpose of the correct location of coordinated points. Correction equations are usually in the form of linear overdetermined (observation) or underdetermined (condition) equations which are solved by the least squares theorem.

The introduction of the electro-magnetic methods of linear measurement requires the adjustment of sides as well as the adjustment of angles necessary in classical triangulation nets.

For the simultaneous adjustment of angles and sides two new sets of conditions have been introduced - (i) the area misfit condition, where the area obtained from distances and angles has to satisfy a special condition, and (ii) that the sum of the projections of the three sides of a triangle on the coordinate axes have to satisfy a zero condition.

A special study has been made into the adjustment of the braced geodetic quadrilateral, as being one of the most favourable figures from the adjustment point of view. The different apices of this quadrilateral have been investigated to allow a choice to be made as to which of them shall be introduced in an angle misfit condition during adjustment. This gives rise to the conclusion that all apices will introduce the same corrections for all practical purposes. Since observed angles and sides are different physical quantities, the question of relative weighting has been given special attention and recommendations have been made in the light of various theoretical and practical investigations.

The Systematic relaxation method for adjusting survey nets has been theoretically derived by Professors Southwell and

Black based on the minimum strain-energy conserved in an elastic frame-work at the position of equilibrium. Using this theory mechanical analogues have been designed and constructed for the first time to carry out the adjustment of the triangulation net directly from field observations, without the necessity of forming and solving a set of linear equations. An analogue for the adjustment of angles was found to be excessively complicated mechanically and difficult to use in practice. Through the use of a direction adjustment method these limitations have been overcome. The final model constructed achieved comparable results to those obtained numerically by a least squares solution. Suggestions for a more highly developed version are made and the situations favourable to mechanical analogue computations are discussed.

The use of an electrical analogue as suggested by Su using a D.C. circuit has been thoroughly investigated from both the theoretical and practical points of view, which showed that such a solution will be simple only for certain limited cases. As an alternative the possibilities of the more general purpose electronic analogue computer have been investigated via the solution of correction equations. Various examples have been solved on this computer which proved to have several advantages over other computation methods.

Finally equivalent problems have been solved numerically on a digital computer for a comparison of the relative merits of analogue and digital methods for the particular case of adjustment of geodetic networks. The relative merits of these solutions are discussed in the light of different problems and circumstances.

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INTRODUCTION

With the advent of electro-magnetic distance measuring equipment, such as the Geodimeter and Tellurometer, completely new observation methods have been made available to the surveyor who can now accurately observe both angles and sides, or sides only as well as the classical angular observations to fix a framework of precisely positioned points. This has had its effect on the traditional methods of computing and adjusting triangulation nets, which had to be heavily modified to cope with the new form of observation data. These radical alterations have been made in contributions by Gale [38], Lilly [63], Murphy [71], and Rainsford [79]. In particular, these made possible the simultaneous adjustment of observed sides and angles, but the formulae and procedures devised for this purpose were found to require a great increase in computational work, which led others such as Biesheuvel [7], and Thorntom-Smith [115], to prefer quite separate, successive adjustment of the two different types of measured data.

In the first chapter of this thesis, further investigations were carried out to provide less complicated formulae, and procedures for simultaneous adjustments using new geometric conditions. Practical experience with these new formulae show that the corrections to the sides obtained from the simultaneous adjustment of sides and angles, are virtually identical to those obtained from the adjustment of sides of figures with pre-adjusted angles.

In the field of survey adjustment, numerical methods have been always preferred, especially for triangulation adjustments.

Southwell and Black [10], [12], introduced the way for solving these problems physically by utilising what were basically structural analogues, but no practical method of achieving this mechanically was devised, and so their investigations merely led to yet another numerical method.

The possibilities of having a cheap and portable mechanical system of adjustment, capable of being operated by relatively untrained personnel is an attractive one and to this Jerie has devised a mechanical analogue system for trilateration only. The investigation of mechanical analogue methods has been taken some stages further, which led to the construction of mechanical analogues for adjusting angular quantities, which after considerable modifications proved to give accurate results using quite simple operational procedures. These investigations are reported in chapter II.

Continuing the investigation into the physical representation of survey problems led to consideration of the electrical analogues proposed by Su [103], [104], [105], [106], and Speart [96]. Chapter III reports the investigations made to bring these to the point where they could be used in a really practical way for the solution of survey problems. This was achieved using a powerful electronic analogue computer, which has been applied to the solution of survey problems for the first time.

At a time where electronic digital computers are coming to be applied in all fields of human activity, it may be thought that physical methods of computation and adjustment are unworthy of consideration. However, fashions change and as technology advances, methods thought to be only of academic interest suddenly become important again. So it is quite necessary that a further study of physical analogue methods be made, if only to see the extent of their limitations.

The final chapter therefore includes some discussion of the many different methods of computation and adjustment of survey networks which are available.

It should be noted that, neither the theoretical justification of the least squares method, nor a discussion of the necessity and merits of a simultaneous adjustment have been made in this thesis, as they are widely known already.

Chapter I

NUMERICAL MODELS OF ADJUSTMENT

1.1 ADJUSTMENT OF TRIANGULATION NETS

1.1.1. The Adjustment by The Rigorous Least Squares Method

The use of the rigorous least squares solution for the adjustment of geodetic triangulation nets has been known over 170 years, having been devised by the French Mathematician Legendre in 1806, and further developed by the German Gauss some twenty years later. Since Gauss introduced his mechanical procedure most triangulation nets have been solved by this mechanical procedure using manual calculations with the aid of tables before even the advent of the desk calculators. The advantages of this mechanical method are the checks on the calculation work which make inevitable mistakes easily detectable.

1.1.2. Least Squares in the Matrix Form

Although the Gauss mechanical procedure does not require the use of matrix algebra, it has been found that the use of this mathematical tool has the advantage of setting out the mathematical formulae in the most concise form. The least squares concept is applicable only when the normal law of errors distribution is accepted for the problem to be solved by this way; this is the case with independent observations made in the field.

The equation of probability is given by

$$P = K^n . e^{-[c^2v^2]} \qquad \text{..........(1.1)}$$

$[c^2v^2]$ is a positive quadratic which may be rewritten in the form

$$[c^2v^2] = c_1^2.v_1^2 + 2.c_1.c_2.v_1.v_2 + c_2^2.v_2^2 + \ldots + c_n^2.v_n^2 \qquad \text{...(1.2)}$$

where c^2 is the precision index, v being the error of observations.

On account of the correlation freedom, the products $c_1.c_2$, where c_1 is different from c_2, will be zero.

The quadratic form will be in this case,

$$[c^2v^2] = c_1^2.v_1^2 + c_2^2.v_2^2 + \ldots\ldots + c_n^2.v_n^2 \qquad \text{..........(1.3)}$$

Using p instead of c^2 to represent the conventional weights knowing that p is inversely proportional to the standard error, which is inversely proportional to the precision index c, then we obtain:

$$[pv^2] = p_{11}v_1^2 + p_{22}v_2^2 + \ldots\ldots\ldots\ldots + p_{nn}v_n^2 \quad \ldots.(1.4)$$

Using the matrix notation,

$$[pv^2] = V'pV \quad \ldots\ldots\ldots(1.5)$$

where p is a symmetrical positive matrix of p = p', and for all values of V $V'pV \geq 0$

1.1.3. The Formation and Solution of Normal Equations

1.1.3.1. Observation Equations

Observation equations are usually given in the following linear relations.

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1m}x_m + l_1 = V_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2m}x_m + l_2 = V_2 \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots + a_{nm}x_m + l_n = V_n \end{array} \quad \ldots(1.6)$$

$V_1 \ldots. V_n$ are the independent observed quantities.

$a_1 \ldots. a_n$, $l_1 \ldots. l_n$ are known quantities.

$x_1 \ldots. x_n$ are the unknowns to be obtained from the solution.

Due to the fact that observed quantities could not be absolute quantities, the solution will be for the most probable values and not for the true values of x.

For any assumed set of values x's, let $F_1 \ldots. F_n$ be the values of the left hand side of the equations (1.6). If $v_1 \ldots. v_n$ are the residuals,

$$- k_1 = a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1m}x_m \text{, etc.}$$

then $v_1 = F_1 - V_1 \quad \ldots\ldots\ldots\ldots \quad v_n = F_n - V_n$

also $k_1 = l_1 - V_1 \quad \ldots\ldots\ldots\ldots \quad k_n = l_n - V_n$

Substituting these values equation (1.6) will be:

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m + k_1 = v_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m + k_2 = v_2 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m + k_n = v_n \end{matrix} \quad \ldots\ldots\ldots(1.7)$$

n is always bigger than m.

and in matrix notation [9]; we obtain:

$$\underset{nm}{A}\ \underset{m1}{X} = \underset{n1}{K} + \underset{n1}{V} \quad \ldots\ldots\ldots(1.8)$$

The solution of the equation (1.8) is given by:

$$\underset{m1}{X} = \underset{mn}{A^{-1}}\ \underset{n1}{K} + \underset{mn}{A^{-1}}\ \underset{n1}{V} \quad \ldots\ldots\ldots(1.9)$$

Substituting (1.9) into (1.8) we obtain:

$$\underset{n1}{K} + \underset{n1}{V} = \underset{nn}{A^{\circ}}\ \underset{n1}{K} + \underset{nn}{A^{\circ}}\ \underset{n1}{V} \quad \ldots\ldots\ldots(1.10)$$

where, $$\underset{nn}{A^{\circ}} = \underset{nm}{A}\ \underset{mn}{A^{-1}} = \underset{nm}{A}\ (\underset{mn}{A'}\ \underset{nm}{A})^{-1}\underset{mn}{A'} \quad \ldots\ldots\ldots(1.11)$$

A° is thus an extraordinary unit matrix and it is always a singular matrix, I is the unit matrix or the identity matrix

$$\underset{mm}{I} = \underset{mn}{A^{-1}}\ \underset{nm}{A} = (\underset{mn}{A'}\ \underset{nm}{A})^{-1}\underset{mn}{A'}\ \underset{nm}{A}$$

Let $$\underset{mn}{A^{-1}}\ \underset{n1}{V} = \underset{m1}{M}$$

Equation (1.10) will be:

$$\underset{n1}{V} = (\underset{nn}{A^{\circ}} - \underset{nn}{I}).\underset{n1}{K} + \underset{nn}{A^{\circ}}\ \underset{n1}{V} \quad \ldots\ldots\ldots(1.12)$$

$$= (\underset{nn}{A^{\circ}} - \underset{nn}{I}).\underset{n1}{K} + \underset{nm}{A}\ \underset{mn}{A^{-1}}\ \underset{n1}{V}$$

$$= (\underset{nn}{A^{\circ}} - \underset{nn}{I}).\underset{n1}{K} + \underset{nm}{A}\ \underset{m1}{M} \quad \ldots\ldots\ldots(1.13)$$

As the solution required must satisfy the least squares concept (i.e. $[v^2]$ = minimum for the same precision of observations),

$$\underset{1n}{V'}\ \underset{n1}{V} = \underset{1n}{K'}(\underset{nn}{A^{\circ}} - \underset{nn}{I}).\underset{n1}{K} + 2\,\underset{1m}{M'}\ \underset{mn}{A'}(\underset{nn}{A^{\circ}} - \underset{nn}{I}).\underset{n1}{K} + \underset{1m}{M'}\ \underset{mm}{A}\ \underset{m1}{M} \quad \ldots(1.14)$$

where $\underset{1m}{M'}\ \underset{mm}{A}\ \underset{m1}{M} \geq 0$

and $\underset{1m}{M'}\ \underset{mn}{A'}(\underset{nn}{A^{\circ}} - \underset{nn}{I}).\underset{n1}{K} = 0$

for all real values of the elements.

Therefore V'V is minimum when M = 0, in which case the solution of equation (1.8) will be:

$$\underset{ml}{X} = \underset{mn}{A^{-1}}.\underset{nl}{K} \qquad \text{........(1.15)}$$

For n bigger than m $\qquad \underset{mn}{A^{-1}} = (\underset{mn}{A'}\ \underset{nm}{A})^{-1}\underset{mn}{A'} \qquad$ ....(1.16)

The mechanical solution by Gauss when using manual desk calculators, starts by the formation of the normal equations. The formation is simply obtained by forming A'A for the coefficients of the unknowns, and A'K for the absolute terms.

Thus $\qquad \underset{nm}{A}\ \underset{ml}{X} - \underset{nl}{K} = 0 \qquad$ ....(1.17)

and $\qquad \underset{mn}{A'}\ \underset{nm}{A}\ \underset{ml}{X} - \underset{mnnl}{A'K} = 0 \qquad$ ....(1.18)

which are the normal equations required for least squares solution of the observation equations. The formation will appear in the following mechanical way:

$$\begin{aligned}
&[a_{c1}a_{c1}]x_1 + [a_{c1}a_{c2}]x_2 + \ldots\ldots\ldots + [a_{c1}a_{cm}]x_m + [a_{c1}k_1] = 0\\
&[a_{c2}a_{c1}]x_1 + [a_{c2}a_{c2}]x_2 + \ldots\ldots\ldots + [a_{c2}a_{cm}]x_m + [a_{c2}k_2] = 0\\
&\ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots\\
&[a_{cm}a_{c1}]x_1 + [a_{cm}a_{c2}]x_2 + \ldots\ldots\ldots + [a_{cm}a_{cm}]x_m + [a_{cm}k_n] = 0
\end{aligned} \qquad \text{...(1.19)}$$

Solution of the above normal equation is given by:

$$\underset{ml}{X} = (\underset{mn}{A'}\ \underset{nm}{A})^{-1}\ \underset{mn}{A'}\ \underset{nl}{K} \qquad \text{........(1.20)}$$

The mechanical solution which transfers the above equation is described in detail in Raisford [77].

1.1.3.2. <u>Condition Equations</u>

Following the same notation for n and m, the condition equations have the following form, in which m is bigger than n.

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1m}x_m + q_1 = 0\\
&a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2m}x_m + q_2 = 0\\
&\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots\ldots \quad \ldots\ldots \quad \ldots \quad \ldots\\
&a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots + a_{nm}x_m + q_n = 0
\end{aligned} \qquad \text{..........(1.21)}$$

from m observation equations as;

$$x_1 = O_1 + v_1$$
$$x_2 = O_2 + v_2 \qquad \text{.........(1.22)}$$
$$\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot$$
$$x_m = O_m + v_m$$

Substituting (1.22) into (1.21) we obtain:

$$a_{11}v_1 + a_{12}v_2 + \cdots\cdots + a_{1m}v_m + k_1 = 0$$
$$a_{21}v_1 + a_{22}v_2 + \cdots\cdots + a_{2m}v_m + k_2 = 0 \qquad \text{.........(1.23)}$$
$$\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdot\cdot$$
$$a_{n1}v_1 + a_{n2}v_2 + \cdots\cdots + a_{nm}v_m + k_n = 0$$

which in matrix form according to [9], will be:

$$\underset{nm}{A}\,\underset{m1}{V} = \underset{n1}{K} \qquad \text{.........(1.24)}$$

As can be seen from the condition equations (1.23), the most probable values will be for the system of corrections having $[v^2]$ a minimum when the same precision is used for observations. Solution of the condition equations by the method of correlatives will be:

$$\underset{mn}{A^{-1}}\,\underset{nm}{A}\,\underset{m1}{V} = \underset{mn}{A^{-1}}\,\underset{n1}{K} \qquad \text{.........(1.25)}$$

According to (1.11),

$$\underset{mm}{A^{o}}\,\underset{m1}{V} = \underset{mn}{A^{-1}}\,\underset{n1}{K}$$

However, since the extraordinary unit matrix is singular the solution could not be obtained in the usual way. Solution of equation (1.24) is usually given by:

$$\underset{m1}{V} = \underset{mn}{A^{-1}}\,\underset{n1}{K} + (\underset{mm}{A^{o}} - \underset{mm}{I}).\underset{m1}{M} \qquad \text{.........(1.26)}$$

M is any arbitrary column matrix with m elements.

therefore: $$\underset{1mm1}{V'V} = \underset{1nnm}{K'A'^{-1}}\,\underset{mn}{A^{-1}}\,\underset{n1}{K} + \underset{1m}{M'}(\underset{mm}{A^{o}} - \underset{mm}{I})(\underset{mm}{A^{o}} - \underset{mm}{I}).\underset{m1}{M}$$
$$+ 2.\underset{1nnm}{K'A'^{-1}}(\underset{mm}{A^{o}} - \underset{mm}{I}).\underset{m1}{M} \qquad \text{.........(1.27)}$$

where $\underset{1m}{M'}(\underset{mm}{A^{o}} - \underset{mm}{I})(\underset{mm}{A^{o}} - \underset{mm}{I}).\underset{m1}{M} \geq 0$

and $\underset{1n}{K'}\,\underset{nm}{A'}^{-1}(\underset{mm}{\Lambda^{o}} - \underset{mm}{I}).\underset{ml}{M} = 0$

for all real values of the elements.

Therefore the minimum value of V'V is obtained when M = 0.

Hence the solution of the equation (1.24) is

$$\underset{ml}{V} = \underset{mn}{A}^{-1}.\underset{nl}{K}$$

To obtain the reciprocal of the rectangular matrix A^{-1} in the solut on the correlates are used:

$$\underset{ml}{V} = \underset{mn}{A'}\,\underset{nl}{C} \qquad ....(1.28)$$

where $$\underset{nl}{C} = (\underset{nmmn}{A\ A'})^{-1}\underset{nl}{K} \qquad ....(1.29)$$

from (1.28) and (1.29) we obtain:

$$\underset{ml}{V} = \underset{mn}{A'}(\underset{nmmn}{A\ A'})^{-1}\underset{nl}{K} \qquad ....(1.30)$$

In a similar way to the mechanical solution used by Gauss, the first step in forming the normal equation is to calculate $(\underset{nmmn}{A\ A'})$, and then the absolute term K is used to obtain C without the need to multiply by any coefficients. The unknowns V will be obtained by substituting in equation (1.28) in the usual way. The mechanical method of formation and solution of the normal equations which transfers the above equations is described in details in Raisford [77].

1.1.4. Number of Condition Equations in Triangulation

The conditions required in the case of free nets, where only two datum points are held fixed, fall under three different categories:

1- Triangle conditions.

2- Centre conditions.

3- Side conditions.

These conditions are the figural conditions and are always used for the adjustment of any triangulation net. In order to avoid ill-conditioning the number of conditions is controlled by the following formulae:

For observed angles : = N - 2S + 3.

For observed directions: = D - 3S + S_u + 4.

N is the total number of angles observed.

D .. .. directions observed.

S .. .. stations occupied or not.

S_u.. ... . unoccupied stations, at which no observations have been taken.

In the case of n base-lines previously fixed, additional conditions equal to (n-1) are found necessary. The general formula which is to be used as check for the number of the condition equations is that "the number of conditions be equal to the number of variables minus the number of independent unknowns".

1.2. ADJUSTMENT OF TRILATERATION NETS

In this technique sides are measured instead of observing angles in the classical way.

1.2.1. Methods of Adjustment

Adjustment may be carried out on the spheroid, but in order to avoid unnecessary complications, adjustment is normally carried out on the projection plane, by applying a scale factor to lengths and an arc to chord correction to angles.

The analytical adjustment is traditionally carried out in in two different ways:

1- Adjustment by variation of coordinates,(observation equations).

2- Adjustment by satisfying a special condition,(condition equations). which may be:

a- Figural condition,

b- Bearing condition,

c- Position condition.

1.2.1.1. Adjustment by Variation of Coordinates

In this method approximate coordinates of new stations are obtained by calculation from the observed sides, or from observed directions. From these approximate coordinates which are rounded off to the nearest whole unit, preliminary bearings and lengths are computed. Differential changes give the required increaments to these approximate coordinates, using the principle that the sum of the squares of the residuals is minimum. Observation equations for the indirect observed quantities (coordinates) will be obtained for every line in the the following way:

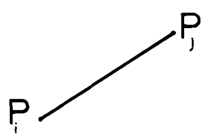


Figure 1.1

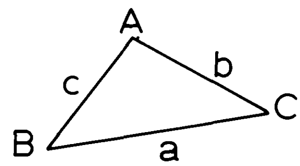


Figure 1.2

Let $P_iP_j = D$; figure 1.1

$$= \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

$$dD = \frac{1}{2.D}[2(x_j - x_i)(dx_j - dx_i) + 2(y_j - y_i)(dy_j - dy_i)]$$

$$= \frac{x_j - x_i}{D}(dx_j - dx_i) + \frac{y_j - y_i}{D}(dy_j - dy_i)$$

If $dD = v_{ij}$, $(x_j - x_i)/D = \cos A_{ij}$, and $(y_j - y_i)/D = \sin A_{ij}$

the correction equation will have the final form of:

$$v_{ij} = (dx_j - dx_i).\cos A_{ij} + (dy_j - dy_i)\sin_{ij} + D_o - D_c \quad ..(1.31)$$

where, D_c is the plane computed distance from the approximate coordinates.

D_o is the observed distance reduced to the projection plane.

A similar equation will be formed for every observed side of the trilateration net. The adjustment is obtained by solving these observation equations in the normal way. For fixed stations, dx and dy will be zero in the above equations.

1.2.1.2a. <u>Adjustment by Figural Condition</u>

Figural conditions are used for the adjustment of doubly braced quadrilateral and centered polygons, and these require that the angles of the figure be computed. These geometrical figures may be viewed as being composed of several individual triangles.

If a single triangle ABC, figure 1.2 is considered, then using the cosine formula:

$$a^2 = b^2 + c^2 - 2.b.c.\cos A \qquad \ldots\ldots\ldots\ldots(1.32)$$

and partially differentiating equation (1.32) we obtain:

$$a.da = b.db + c.dc - (b.dc + c.db)\cos A + c.b.\sin A.dA \quad ..(1.33)$$

and,

$$- b.c.\sin A.dA = b.db + c.dc - (c.db + b.dc)\cos A - a.da..(1.34)$$

$$= (c - b.\cos A).dc + (b - c.\cos A).db - a.da \ (1.35)$$

However,

$$(c - b.\cos A) = c - \frac{c^2 + b^2 - a^2}{2.c} = \frac{2.c^2 - c^2 - b^2 + a^2}{2.c}$$

$$= \frac{a}{a} . \frac{c^2 + a^2 - b^2}{2.c} = a.\cos B \quad .....(1.36)$$

similarly; $(b - c.\cos A) = a.\cos C$ ....(1.37)

Substituting (1.36) and (1.37) into (1.35) we obtain:

$$- dA = \frac{a}{b.c} . \frac{\cos B}{\sin A} . dc + \frac{a}{b.c} . \frac{\cos C}{\sin A} . db - \frac{a}{b.c} . \frac{da}{\sin A}$$

$$..........(1.38)$$

but, $\frac{a}{b} = \frac{\sin A}{\sin B}$, and $\frac{a}{c} = \frac{\sin A}{\sin C}$ ....(1.39)

therefore,

$$dA = - \frac{dc}{c}.\cot B - \frac{db}{b}.\cot C + \frac{da}{c}.\mathrm{cosec} B \ ....(1.40)$$

The last term in equation (1.40) could be written as $\frac{da}{b}.\mathrm{cosec} C$ instead of $\frac{da}{c}.\mathrm{cosec} B$.

The condition used for figural adjustment is found when angle A occurs in a doubly braced quadrilateral, figure 1.3, in which the component triangles are:

triangle ABC = (1), triangle BCD = (2),

triangle CDA = (3) and triangle DAB = (4).

The condition is obtained by requiring that:

$$A_1 + A_3 = A_4.$$

In satisfying this condition, any discrepancy found will be distributed by employment of the Lagrange multiplier* to give

* For an equation of the form $a_1x_1 + a_2x_2 + \ldots = k$, the Lagrange multiplier is $= \frac{a_i}{\Sigma a^2}$.

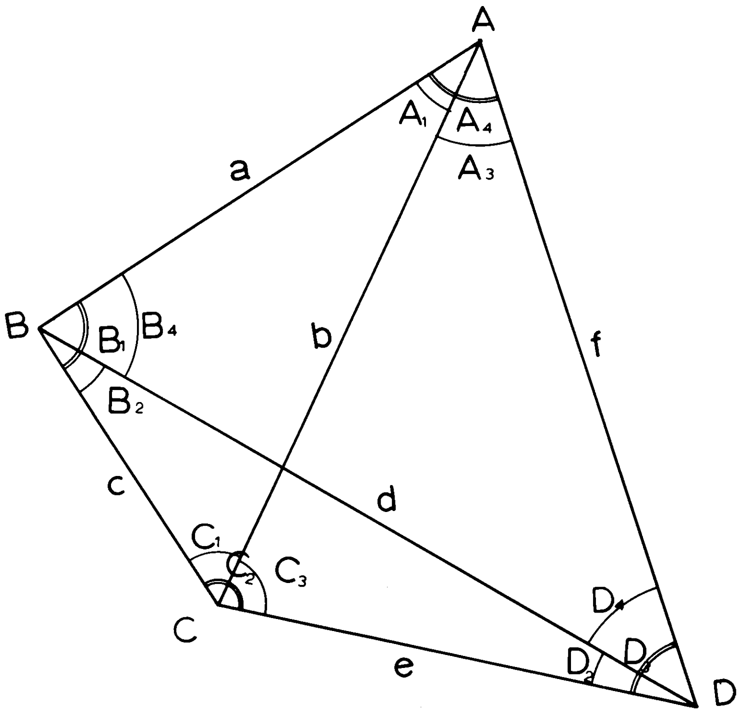


Figure 1.3

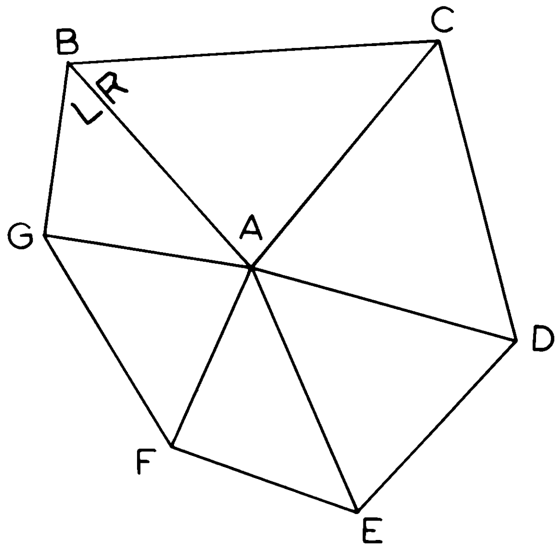


Figure 1.4

corrections necessary for the self consistency required.

In the case of a centered polygon, figure 1.4 the figural condition is given by $dA = \Sigma A - 360^{\circ}$, which will provide the absolute term for the condition equation which has to be solved.

The number of figural conditions required for the adjustment of any net is given by: $L - 2S + 3$ ......(1.41) where L is the number of lines, and S the number of the stations. Corrections obtained for sides by using equation (1.40), always come in special pattern. For the centered polygon, the coefficients of the adjustments to the peripheral lines are all of the same sign and positive, while those of the radial lines from the centre-point are all of the same sign and negative, provided that none of B_R, B_L, and $(B_R + B_L)$ is greater than $180°$.

1.2.1.2b. <u>Adjustment by Bearing Condition</u>

This condition [2] is used when the trilateration net is extended between two sides with known lengths and bearings. This is normally employed when the figural condition could not be used.

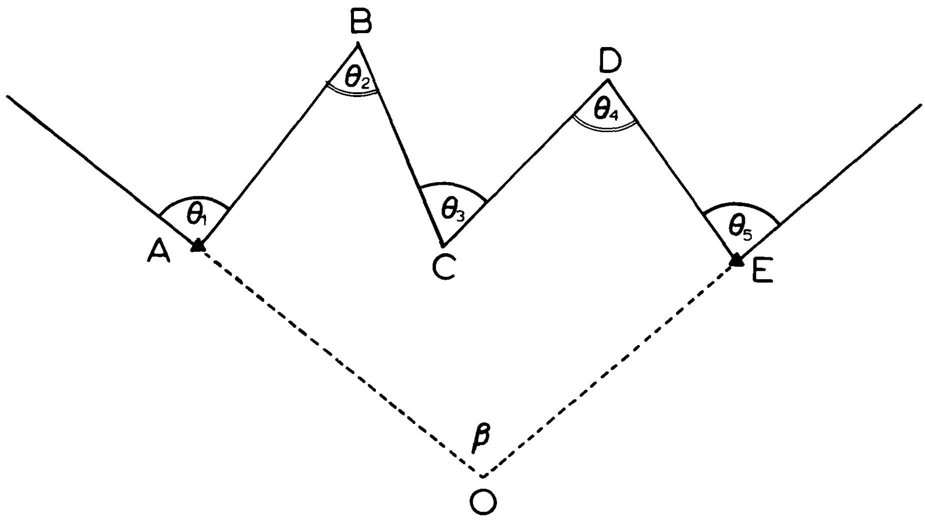


Figure 1.5

In figure 1.5 only the two bearings OA and OE are fixed. The trilateration net ABCDE is extended be between these bearings. The relationship between angular and linear changes given in (1.40) is used to satisfy the condition of fixed bearings. If the angle between these two fixed bearings is given by β, therefore the condition will be:

$$(\theta_1 + \theta_3 + \ldots.) - (\theta_2 + \theta_4 + \ldots.) + \Sigma d\theta = \beta \quad \ldots.(1.42)$$

1.2.1.2c. <u>Position Condition</u>

This condition is used when the trilateration net is extended between two fixed stations. As a figural condition could not be used, where the net is not a closed one, this condition is used.

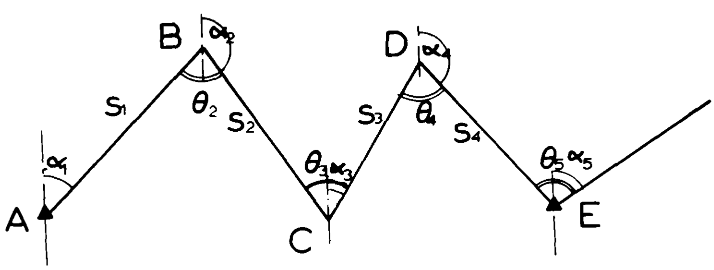


Figure 1.6

In figure 1.6 the trilateration net is extended between the two fixed positions A and E. The condition in this case is the sum of the projections on the coordinate system axes of all observed lines between the two fixed stations have to be equal to the difference between the corresponding coordinates of the two fixed stations.

Thus if $X_E - X_A = \Delta X$, and $Y_E - Y_A = \Delta Y$

$$\begin{aligned} &\text{therefore} \quad s_1.\sin\alpha_1 + s_2.\sin\alpha_2 + \ldots\ldots = \Delta Y \\ &\text{and} \quad s_1.\cos\alpha_1 + s_2.\cos\alpha_2 + \ldots\ldots = \Delta X \end{aligned} \qquad \ldots\ldots\ldots\ldots(1.43)$$

Differentiating with respect to both angles and sides we obtain the required corrections for the purpose of keeping the positions of the fixed points unchanged. In this case differentiation has to be made with respect to angles and not to bearings to be able to use the relationship given by (1.40). The relationships between bearings and angles are as follows:

$$\alpha_1 = \theta_1 + \text{constant},$$

$$\alpha_2 = \theta_1 - \theta_2 + \text{constant} \quad \text{etc..}$$

therefore

$$d\alpha_1 = d\theta_1,$$

$$d\alpha_2 = d\theta_1 - d\theta_2, \qquad \text{etc....}$$

Differentiating equation (1.43) we have:

$$s_1.\cos\alpha_1.d\alpha_1 + s_2.\cos\alpha_2.d\alpha_2 + \ldots\ldots + \sin\alpha_1.ds_1 + \sin\alpha_2.ds_2 + \ldots. + \delta Y = 0 ,$$

and,

$$-(s_1.\sin\alpha_1.d\alpha_1 + s_2.\sin\alpha_2.d\alpha_2 + \ldots\ldots) + (\cos\alpha_1.ds_1 + \cos\alpha_2.ds_2 + \ldots.) + \delta X = 0. \qquad \ldots\ldots(1.44)$$

Substituting $d\theta$ for $d\alpha$ equation (1.44) will be:

$$d\theta_1(s_1.\cos\alpha_1 + s_2.\cos\alpha_2 + \ldots.) - d\theta_2(s_2.\cos\alpha_2 + s_3.\cos\alpha_3 + \ldots.) + d\theta_3(s_3.\cos\alpha_3 + s_4\cos\alpha_4 + \ldots) - .. + .. + \delta Y = 0$$

$$\ldots\ldots.(1.45)$$

$$-d\theta_1(s_1.\sin\alpha_1 + s_2.\sin\alpha_2 + \ldots) + d\theta_2(s_2.\sin\alpha_2 + s_3.\sin\alpha_3 + - d\theta_3(s_3.\sin\alpha_3 + s_4.\sin\alpha_4 + ..) + .. - .. + \delta X = 0$$

1.2.2. Discussion

In the following pharagraphs, special consideration will be given to the following points:

1- Number, selection and accuracy of computed angles, for figural condition.

2- Characteristics of figural condition.

3- Adjustment by bearing and position conditions of pure trilateration.

4- Weights applicable to trilateration nets.

1.2.2.1. Computed Angles

In order to form the figural condition for adjusting the doubly braced quadrilateral, figure 1.3, a particular apex has to be chosen. Different points of view have been given, as to how this may be selected. Murphy and Thornton-Smith, [70], and [73] recommend the use of the apex of the triangle of the smallest area, while Tarczy-Hornoch and L. Hovanyi [109] recommend the use of the apex of the triangle of the largest area. Thus using figure 1.3, the figural condition for the doubly braced quadrilateral is either $C_1 + C_3 = C_2$, according to Murphy and Thornton-Smith, or $A_1 + A_3 = A_4$, according to Hornoch and Hovanyi.

Four triangles of the doubly braced quadrilateral have to be solved so that this condition may be formed. Furthermore to obtain accurate corrections, angles should be accurately computed. This is because the absolute terms in the condition equations are linear functions of the computed angles.

Calculation of all angles to obtain this condition is a very laborious task. Therefore, if another figural condition can be found which will eliminate this calculation it will be of considerable advantage. In choosing another geometric condition it would also help if one could avoid having to decide which is the most favourable apex for the adjustment, e.g. either that contained in the triangle of the small area or in the triangle of the large area. Suggestions along these lines are discussed in detail later in section 1.5.

1.2.2.2. Characteristics of Figural Conditions

Figural conditions are associated with the adjustment of the doubly braced quadrilateral and the centered polygon.

In the doubly braced quadrilateral figure 1.3, the figural condition is obtained by measuring any arbitrary redundant. Stations A, B, C, and D are defined in the quadrilateral with one diagonal, but as soon as the other diagonal is measured, a figural condition will be established.

In the case of the centered polygon figure 1.4, stations A, B, C, D, E, F, and G are defined without measuring GF, but with the measurement of this side, the figural condition will be established, exactly as in the quadrilateral. In both cases the figural condition will allow consistency to be achieved with the observed quantities. However it is worth mentioning a difference that exists between these two figural conditions.

In the doubly braced quadrilateral the condition is obtained by equating $(A_1 + A_3)$ and A_4(figure 1.3). Both these quantities are obtained by computation from the observed quantities. In the centered polygon, the condition is obtained by equating ΣA and $360°$. ΣA is obtained by computation from the observed quantities, while $360°$ is a known geometrical fact.

In figure 1.7 a centered polygon ABCO having the two figural conditions mentioned above. In this case:

(a) $A_1 + A_3 = A_4$,

or (b) $\Sigma O = 360°$.

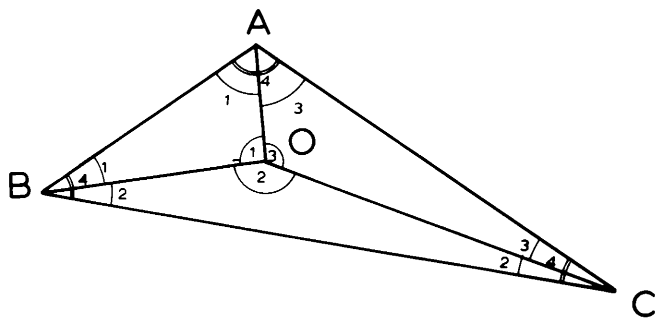


Figure 1.7

In (a) all three angles at A are obtained by computation from observed sides. In (b) ΣO is obtained by computation from the sides, while $360°$ is the known geometrical condition.

To differentiate between these two characteristics the doubly braced quadrilateral, and any other figure, such as figure 1.7 will be called a self-checking figure.

1.2.2.3. Adjustment Using Bearing and Position Conditions

In order to derive bearing and position conditions, figures 1.5, and 1.6, it is necessary to observe all sides forming complete triangles. The number of triangles involved, depends on the length of the trilateration net between the two fixed bearings, or the two fixed positions, so that the number of unknowns will be equal to (2n + 1), where n is the number of triangles in the net. For three triangles the number of unknowns will be equal to seven, for four triangles this number will be nine, etc..At the same time, it should be noticed that, figural conditions can not be formed for such a net. Solution of one condition equation to obtain large number of unknowns especially when it requires that every side of triangles be observed, seems to be unwarranted. The solution given for these two conditions, is a theoretical one, but practically, it will be a waste of efforts as the estimate for the coefficient of correlation will be of the order

$$\gamma = \frac{1}{\sqrt{(2n+1)}}$$

The same could be said about figural condition for adjusting a centered polygon.

As it will be shown in 1.3 the aim of simultaneous observation of angles and sides is to greatly increase the consistency of adjusted nets, with the little increase of efforts in the field.

1.2.2.4. Weights Applicable to Trilateration

Weights are applicable to the above mentioned methods of adjustment where condition equations have been employed.

An essential requirement for adjusting any observations, is to have an idea of the observation errors likely to be present, the effect of these observation errors being incorporating by applying weights for the relavant parts of the adjustment.

In trilateration the observation errors used for obtaining weights are normally related to the distance (L) observed. However different systems of weighting have been recommended which are inversely proportional to:

1) L^2 [71],
2) L [115],
3) The mean square error for every observed distance [8].
4) $a^2 + b^2.L^2$ [20],

where a = 1 cm. in the Geodimeter, and
= 1 in. in the Tellurometer.
b = 1/200,000 = meteorological uncertainty.

5) Unity [68], [79].

In practice one finds it difficult to give preference to one system over another, For a relative weight proportional to $1/L^2$ it is important to quote a statement made by Murphy [73] viz "The position is still not clear, but it is known that the probable error of an observed side is a function of the length of line and as an interim approximation for lines of a medium distance, this may be taken as a direct proportion", in which case the weight being $(1/p.e)^2$ will equal to $1/L^2$. Furthermore, when tapes or wires were used to measure sides or bases, it was easier to evaluate the reliability of measurement, since most of the factors affecting this were known or could be established. In measuring sides by this way the relationship between the length of the measured side and its weight is found to be proportional to $1/L^2$. For sides measured by electronic measuring equipments, there will be considerable uncertainty as to the meteorological conditions between each pair of end stations and this can have a vital effect on the precision obtained. Thus the probable error of the instrument used does not necessarily relate to the probable error of the measured line. The instrumental probable error is evaluated under standard conditions, which may or may not exist in the field. Uniformity of weather conditions along a long line is very rare, if not impossible, even when measuring over a desert or the sea. Such uncertainty makes almost any assumption for weighting liable to objection. Even if meteorological stations can be used between the two ends for more accurate estimation (and this is not very practicable), this will not eliminate the effect of the as yet known internal electrical and mechanical errors in a very complicated instrument such as a Geodimeter or Tellurometer.

All the above points should be kept in mind before using any relative weight proportional to the length. In this case, a short measured side may be less reliable than a longer side due to special meteorological conditions or defects in the particular instrument.

To obtain the mean square error of observation for every line is a very laborious job. When accuracy is required, the error of observation should be obtained for every instrument as each will have a different systematic error. Also meteorological conditions affect our observations affecting the relative weights to be used. The result of all these considerations is that there is no specific evidence to give an estimation of observational error, in which case weights may be chosen according to the following criteria:

1- That they result in a minimal amount of additional computation.
2- That they correspond to the requirements of least squares theory.
3- That they give good results when compared with the results obtained by using other weights, when the latter are theoretically approved.

So, using the same weight (equal to unity) for all observed sides seems to be a reasonable suggestion, satisfying the three points given above.

As to whether the same reliability of observations of different lengths will be obtained in practice depends a great deal on the experience of the observing team, and on approximately similar meteorological conditions prevailing while observations are being taken. It should be mentioned here that the pattern of observations, and thus the mean square errors, may be misleading to those undertaking the computing. If the observations are made over a short period, they have the tendency to give uniform readings with a small value for the mean square error. On the other hand observations made over a long period and during different meteorological conditions will have the tendency to give readings of a greater spread with a larger value for the mean square error. The small value of the mean square error obtained in the first case does not necessarily indicate a high absolute precision, while the large value for the mean square error in the second case does not necessarily indicate a low absolute precision. This sort of experience is in fact neglected by the computing people, when the mean square error is taken as a measure for the relaibility of observations.

In very special cases, i.e. under similar meteorological conditions, the use of the number of observations as a measure for the reliability of observations follows the assumption of a unit weight for each observation. However; similar meteorological conditions can be reasonably assumed when sides of approximately equal length are being measured over fairly uniform terrain conditions. In such a case the number of observations of each side can be used as a weight to be given to each particular side during the adjustment of the net. For example the centered polygon figure 1.8, in which all sides are nearly equal, may be given different weights depending on the number of measurements of each side.

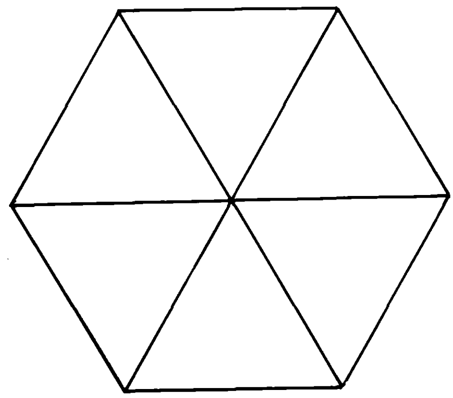


Figure 1.8

It has to be stated here that the assumption of having similar weights for adjusting a trilateration net is chosen to facilitate the computation, depending on the considerations given above, and on the conclusions made by Rainsford in [79]. This is so, especially in the normal cases, when sides of a net fall under the particular class of being first, second, or tertiary order.

1.2.3. Comparison Between Triangulation and Trilateration Nets.

Consider a geodetic doubly braced quadrilateral in which 8 angles are observed, and another similar quadrilateral of which the six sides are measured.

The adjustment of the first requires the solution of four condition equations, one of which is the side condition. The latter quadrilateral will be adjusted by using the side condition only. This means that a better agreement between corrections and real errors will be obtained from a least squares solution when adjusting the first (because the estimate of the coefficient of correlation which is equal to $\sqrt{n_c/n}$ will be greater in this case) than that obtained from the adjustment of the second quadrilateral.

On the other hand if accumulation of errors is being considered a comparison of the two is rather more favourable to trilateration. All errors in scale, azimuth and position increase with the length of the triangulation chain, while in the case of trilateration the scale error at the last line of the chain is the same as that of the first line, i.e. scale errors do not accumulate. The other two errors do accumulate, but the position error is due to the swing in the azimuth. Thus a short trilateration chain, or long one where Laplace azimuths are used, might give results as good as those obtained by the adjustment of triangulation net. However in general it is apparent that trilateration gives a weak shape, but has a better scale.

1.3. MIXED FIGURES (HYBRID OBSERVATIONS)

These are geodetic figures in which both sides and angles are observed.

1.3.1. Effect of Observations on the Nets

Before the advent of the electronic measuring equipment triangulation nets were formed between two base lines, or extended from one base line only. The change in the shape of the net due to the errors accumulated along this net is similar to the change of the shape of an elastic body subtended between two fixed points, or supported at one side only as a cantilever. In this case the elastic body will have a deformed shape different to its original shape, e.g. the straight line between two fixed ends will no longer be straight. Original shape can be re-obtained by introducing an infinite number of supports between the fixed ends. The same could be said about a net of triangulation, i.e. the effect of accumulated errors for scale will be reduced if the number of bases is increased.

A distinction must be introduced between the old type of base, which was very difficult to measure and so infrequent that it usually was left unadjusted as an absolute quantity, and the new measured side lengths, which can be introduced comparatively easily, and frequently and can be subjected to an adjustment, just as measured angles are.

1.3.2. Adjustment of Both Shape and Size Simultaneously

Another way of increasing the consistency of adjusted net is to adjust both shape and size. This is the aim of the current technique of observing both angles and sides in a net, taking account of the fact that scale errors cannot accumulate to any thing like the same extent where more sides are measured. This means that the number of the redundants will be doubled or tripled, e.g. a triangle will be solved for three condition equations, rather than the simple angle condition used in normal triangulation. Also the doubly braced geodetic quadrilateral requires nine condition equations instead of the well-known four conditions required in the adjustment of triangulation net.

Solution by this method requires a great deal of extra work for a given figure or net, and should be used only in primary work or where high accuracy is all important. This is especially so when the net to be adjusted is a large one, for the solution by the least squares is known to be a quadratic function of the number of the equations to be solved.

The main problem to be considered for the simultaneous adjustment of both sides and angles is the selection of suitable relative weights for such adjustment.

1.3.2.1. Relative Weights for Angles and Sides

In the case of pure triangulation or pure trilateration, it is easy to use relative weights once one has decided to follow a particular rule. The observed quantities here are similar in that they have been obtained by using the same set of equipment and probably a standard technique. In mixed figures totally different principles are being used for the measurement of the different observed quantities, and this dissimilarity requires different weights to be given to both angles and sides when they are to be adjusted as a single problem by the method of the least squares.

Various suggestions have been made by different contributors, e.g. Murphy, Lilly, Biesheuvel, and Rainsford.

1- Murphy [71], gives a unit weight to all angular observations while giving a different weight to each side according to its length. In this case he assumes that the mean square error of a linear measurement is proportional to the length of the side. This assumption is entirely based on convenience (as Murphy himself said [71]) and not necessarily upon the characteristics of the electronic distance measuring apparatus itself.

   For this assumption $(\Delta a)'' = \frac{v_a}{a} x 206265$

   where, v_a is the linear correction and $(\Delta a)''$ is the equivalent angular correction.

   Minimizing the expression $[(\Delta a)^2+(\Delta b)^2+(\Delta c)^2+v_A^2+v_B^2+v_C^2]$

where v_A is the correction to the observed angles, means that angular corrections are all of the same weight, i.e. unity. The linear corrections which are presented in the form of angular corrections are obtained by having a weight inversely proportional to the square of the length of the side measured.

2- Lilly [63] suggests that the concept "Sum of the Squares of the Residuals" has to be replaced by the concept "Sum of the Squares of the Reduced Residuals", where the reduced residual is defined as the ratio of the residual to the probable error. This is equivalent to the weighted squares of the residuals, having regard to the fact that the weights here are dimensioned quantities instead of being pure numbers. Thus when the dimensional residuals are divided by the weights the least squares solution can be applied to dimensionless quantities. This explanation is given to justify the use of the least square concept to the problems of residuals of different dimensions. In this case if v is the residual, r is the probable error and i = 1, 2, ...n, therefore,

$$\frac{(v_i)^2}{(r_i)^2} = \frac{(x - x_1)^2}{r_1^2} + \frac{(x - x_2)^2}{r_2^2} + \ldots\ldots + \frac{(x - x_n)^2}{r_n^2}$$

For $\frac{d}{dx} \cdot \frac{(v_i)^2}{r_i^2} = 0,$ we have:

$$x = \frac{\Sigma(x_i/r_i^2)}{\Sigma(1/r_i^2)} = \frac{\Sigma(x_i/kr_i^2)}{\Sigma(1/r_i^2)} = \frac{\Sigma(p_i \cdot x_i)}{\Sigma(p_i^2)}$$

In this case k is an arbitrary constant.

Lilly's solution is completed by choosing suitable probable errors. He proposed that this equals 0.6" for all observed angles, and 1 part in 200,000 for the observed linear measurements.

3- Biesheuvel [7] prefers that the residual of linear measurement should be released from its dimension before it is combined with the angular residuals in the solution. For example when v_q is the linear residual, then the amount to be obtained by the least squares solution is $v_s = \frac{v_q}{s}$, where s is the side observed. The correction then will be $v_s.s = v_q$ for the linear measurements, and $v_r = v_\varphi.s$ for chord measurement, (figure 1.9). In this case both v_s and v_φ will be of the same angular dimension. This consideration is commonly used in adjusting traverses, but he recommends against the use of this method in a combined net unless the mean square error is known, or a reasonable estimate of it can be made, otherwise the combination will be futile.

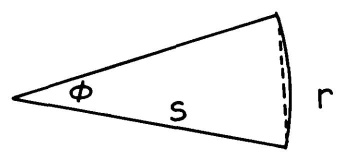


Figure 1.9

4- Rainsford[79] carried out a comprehensive study using the information for the Ridgeway and Caithness bases, with different criteria being considered in turn. These involved the adjustment of:

a- Angles only - all of uniform weight.

b- Lengths only - all of uniform weight.

c - Combined angles and lengths - weight of a side inversely proportional to the square of the length.

d- Combined angles and lengths - weight of a side is inversely proportional to the length.

e- Combined angles and lengths - Angles and sides are of uniform weight.

After comparing the various results from a practical point of view comes to the following conclusions:

A- There is little (if any) evidence that the error in Tellurometer observations varies with the length of line, unless the lines considered are longer than say 35 to 40 miles. It would then be a matter of uncertainty in the meteorological conditions rather than systematic error in the Tellurometer observations.

B- The introduction of observed lengths (even quite a large proportion) does not appear to distort the general adjustment appreciably, considered in relation to the adjustment of the angle observations only.

C- Good results will be obtained by weighting angle observations and Tellurometer observations uniformly in the same solution, i.e. if the weight of an angle is unity the weight of a Tellurometer length is also unity.

The suggestions of both Murphy and Biesheuvel are basically the same. In both cases the combination of the two dissimilar quantities is allowed by having the linear correction of a side divided by the length. The angle corrections are obtained directly from the correction equations. This division by the length of the side introduces in both cases a relative weight = $1/L^2$, which suffers from the shortage of a strong evidence of being so.

Lilly solves the problem of combination of the two dissimilar quantities by assuming a probable error for both angles and sides. The choice of 0.6" as a probable error for an observed angle is obtained by special studies in the field. Gale [38] who did the same studies proposed 0.85", which seems to be quite different from that of Lilly's assumption. Both Lilly and Gale agree on having one part in 200,000 as a probable error for an observed side. However, although this solution seems to be a practical one but the uncertainty of the probable error to be given to angles together with the extra computational work necessary does not give this solution much priority.

Rainsford's conclusions seem to be more likely acceptable, as a solution according to his suggestions requires less computational work, beside giving good practical results.

1.3.3. Methods of Adjustment of Hybrid System

Contributors to the problem of adjustment of hybrid observations, e.g. Murphy [71], Tarczy-Hornoch [110], Lilly [63], Biesheuvel [8], and Thornton-Smith [115],[116], use different methods, but basically they can be classified under the following:

1) Simultaneous adjustment by formation of angle and side

conditions assigning weights according to one of the systems suggested above, and using normal equations to obtain direct corrections to the observed quantities, e.g. the solution given by Murphy [71] for this problem is obtained from the following condition equations:

a) Angle condition

$$A + B + C + \theta = 180^\circ + \varepsilon \qquad \text{........(1.46)}$$

θ is the angular correction necessary to close the triangle ABC, and therefore θ will be dispersed by applying corrections v_A, v_B, and v_C to the observed angles. It follows that:

$$v_A + v_B + v_C = \theta \qquad \text{.......(1.47)}$$

b) Length conditions

When the application of the sine formula to the Legendre triangle ABC does not satisfy the rule exactly, then

$$\frac{a}{\sin(A-\varepsilon/3)} \cdot \frac{\sin(B-\varepsilon/3)}{b} = 1 - k \qquad \text{........(1.48)}$$

$$= F(a,b,A,B)$$

where k is the small divergence produced by the errors in the observed quantities.

Differentiating the left hand side of equation (1.48), with respect to a, b, A, and B and choosing the linear corrections v_a, v_b, v_c, and the angular corrections v_A, v_B, and v_C so that they satisfy the sine rule, then

$$k = \frac{1.\sin(B-\varepsilon/3)}{b.\sin(A-\varepsilon/3)}.v_a - \frac{a\ .\sin(B-\varepsilon/3)}{b^2.\sin(A-\varepsilon/3)}.v_b$$

$$\frac{a.\sin(B-\varepsilon/3).\cos(A-\varepsilon/3)}{b.\sin^2(A-\varepsilon/3)}.v_A + \frac{a.\cos(B-\varepsilon/3)}{b.\sin(A-\varepsilon/3)}.v_B \text{..(1.49)}$$

$$= \frac{v_a}{a} - \frac{v_b}{b} - v_A.\cot A + v_B.\cot B \qquad \text{.......(1.50)}$$

Substituting $(v_A)'' = v_A/206265$,

and $(\Delta a)'' = 206265.v_a/a$

therefore, $206265\ k_1 = (\Delta a)'' - (v_A)''\cot A - (\Delta b)''$

$$+ (v_B)''\cot B \qquad \text{..........(1.51)}$$

also, $206265\ k_2 = (\Delta a)'' - (v_A)''\cot A - (\Delta c)'' + (v_C)''\cot C$

......(1.52)

Similar equations to (1.47), (1.51) and (1.52) have to be obtained and solved for each triangle.

2) Thornton-Smith [115], [116] has two main considerations [(a) and (b) below] in his solution to the problem. Both of these disregard the question of a simultaneous combination at all, and give a solution for the case when both angles and sides are observed. Thornton-Smith says that to combine both angles and sides in one solution, i.e. to solve nine condition equations for the doubly braced quadrilateral "is a herculean task", from the computational point of view. Furthermore, he feels that it would be futile to combine the two in the first order geodetic work unless the two kinds of measurements are each capable of the same first order precision. So his opinion is that there is no need for such a combination. However, if both are observed in one figure one of two possibilities exists:

(a) The adjustment should be carried out for angles first, then using the adjusted angles, the sides are corrected which means in fact that the shape be adjusted first and then the size or scale.

(b) The most probable shape should be obtained from the adjustment of observed angles only by the least squares method. Next another value for the most probable shape is obtained by deriving the angles from a separate adjustment of the sides. In this case we have two shapes where the angles satisfy the geometric condition and two sets of residuals. The final adopted shape will be the weighted mean of the two values of each individual angle in the figure, where the weights are the sum of the squares of residuals in each case. In other words the required angles are obtained by distributing the difference between the two sets of residuals according to the assigned weights, to produce one final set of residuals. The angles obtained satisfy both the angle and sides conditions for each tringle.

In these two ways Thornton-Smith avoids using relative weights at all, simply by going to the separate adjustment of each net.

1.3.4. Discussion

The number of conditions to be satisfied for a trilateration net is very small when compared with the number of conditions to be satisfied for the adjustment of the classical triangulation net(section 1.2.3.). This will affect the value of the whole operation. The main aim of the current technique of observing sides and angles is to increase the consistency of results. This means that observation of both sides and angles will be more useful when the adjustment of both is carried out simultaneously. The idea of adjusting angles and sides separately requires much more investigation and research. It has been stated by Thornton-Smith [115] that the combination of angles and sides in one solution would be futile unless both observations are capable of giving the same precision. This statement can be discussed in two parts.

(a) First of all most investigations lead to the conclusion that the angular and linear accuracies are about equal, e.g. Rainsford's opinion [79] is that both types of observation are capable of giving the same precision for first order work. Therefore both sides and angles are given uniform weights irrespective of the length measured. Also after a series of practical investigations, Gale [38] proposed a probable error for an observed angles of 0.85", and for sides one part in 200,000. If we consider that a probable error of 1" represents one part in 206,265, we can see that precision proposed for each is very similar.

However, precision can be increased if special precautions are taken, e.g. the U.S. Coast and Geodetic Survey carried out a project for the U.S. Air Force in Florida in connection with the missile range [90]. In this project an accuracy of 1:400,000 was required between Cape Kennedy and nine ballistic

Camera sites. Such an accuracy has not been achieved before over a large net. However the project was carried out by measuring sides using Mk2 Geodimeter, and angles using Wild T3 Theodolite. Special procedures and precautions were followed. The simultaneous adjustment of sides and angles was carried out by the variation of coordinates method. The Geodimeter lengths being introduced as observation equations with weights equal to those for the directions, i.e. a correction of 1:206000 (arc 1") to a length was considered the same as the correction of one second to a direction. So for ultra-high precision nets the weighting follows the same lines as with Rainsford and Gale.

Also for the connection between Britain and France across the English Channel 1963, satisfactory results were obtained by using the Tellurometer MRA2 for the direct measurement of 103 kms. between Dunnose and Beachy Head. In this case the final mean value obtained from 12 measurements, each combrising 36 sets of fine readings differed from the distance obtained by triangulation by one part in a million. Again this backs up the other example quoted.

Lilly [62] and Konecny [55] carried out complete investigations and comparisons for sides derived from existing triangulation nets and the length of the same sides measured directly by means of electro-magnetic waves. An explanation has been always given, whenever a discrepancy exists, e.g. due to the deformation of the existing triangulation nets, some angular errors, etc...

The above investigations were mainly carried out for first order work. For second order work, an investigation has been made by Kelsey [51] for a net measured by the Ordnance Survey in South Scotland. In this investigation the Tellurometer distances were compared with the distances computed from existing coordinates derived by triangulation. The maximum discrepancy was 0.2 metre in a line of 17.7 kms, i.e. one part in 88,000. This check showed that the second order control can be supplied by Tellurometer traverses to an accuracy comparable with that of triangulation.

(b) However the assumption of quite different precision for the two types of observation, would not necessarily demand successive separate adjustment. The simultaneous adjustment will take much more computation, but it is not futile to undertake this, as Thornton-Smith has suggested, because of the increased consistency which results as has been shown by Konecny [20].

The current technique of measuring sides, is especially useful in cases when observations of angles are very difficult or impossible, e.g. in the case of fog, or conditions of poor visibility. In such cases, the observed net will be a mixed one, i.e. some angles and some sides are observed. However, for the adjustment of the general case, it is not necessary to have an angle condition, provided that a method such as Murphy's (already discussed) is suitably modified. In this case the angle condition will be ommitted.

In the rest of this chapter, further investigations into the possible use of the area covered by a net as a condition for mixed adjustment are reported. Also the author's point of view about apices of the doubly braced quadrilateral will be given. Furthermore, a new method of forming condition equations for the mixed adjustment will be derived and modified.

1.4. AREA COVERED BY A SURVEY NET

Survey nets consist of individual geometrical figures, e.g. a triangle, a quadrilateral, a centered polygon. The basic individual unit is the triangle. Each geometric figure has a geometric condition to satisfy, e.g. 180° for a triangle, 360° for a quadrilateral, etc.. When only sides are observed, such a geometric condition will not be found to be satisfied. For example, computed angles satisfy always this condition, and thus eliminate the possibility of using this condition for adjustment. Computed angles are commonly used by contributors, e.g. Murphy

Tarczy-Hornoch, Thornton-Smith, etc... to satisfy the figural, bearing and position conditions for trilateration adjustment. For observed sides, three sides will form a unique triangle, and with n observations of the same sides n triangles each slightly different to the other will be obtained. Whatever number of observations are made the figures lack the condition to adjust them, a situation remedied as soon as any redundant is known.

In triangulation, the angle condition cannot be used for the adjustment of a triangle unless the three angles of the triangle are observed. It should be mentioned here that the adjustment of this triangle will not have any specific meaning without additional knowledge of the scale. In other words four parameters have to be known for the adjustment. This can more obviously be understood, when mechanical computers for adjusting these nets are to be put to work. It should also be kept in mind that the scale always exists whenever the adjustment of angles only is being discussed, but the fourth parameter(the side) is left unadjusted. This does not affect the number of conditions mentioned in <u>1.1.4.</u> and <u>1.2.1.2a</u>.

However with one angle and three sides observed or with two angles and two sides observed, i.e. again with the minimum four measurements used for adjustment there is only one geometric condition that could be satisfied. This can be an angle or side condition, but whatever is adopted it can be viewed in the general case as having to satisfy an area condition. As this requires at least four out of the six elements of the triangle to be observed, it follows that an area condition can only be used in a triangle in adjusting hybrid observations. However area conditions can also be used for adjustment of pure trilateration of larger figures with self-checking properties, e.g. a doubly braced quadrilateral or similar figure. The adjustment of the quadrilateral using an area condition also avoid difficulties of choosing the apex to be used for obtaining a figural condition (see <u>1.2.2.1</u>). Furthermore, it is shown that adjustment by this condition avoids the necessity for any extra computation.

1.4.1. Coefficients of Corrections and Absolute Terms in Condition Equations

The order of the absolute term in the condition equation is important factor in obtaining accurate results. The absolute term normally used with an angular misfit condition is much increased when an area misfit condition is applied with consequent favourable effects on the accuracy of the solution.

Size of the figure may be defined as the area covered by the sides of this figure. Errors of observations appear in the area calculation as a multiple of those errors by other quantities. The same errors appear in the angle calculation in a totally different way, as follows:

1) Error in the calculated area

Consider figure 1.10, and let

a_o = the observed side a

Δ_o = the calculated area from observed sides a_o.

da = the error of observation of side a_o.

$d\Delta$ = the error in the calculated area.

a = the corrected side = $(a_o + da)$

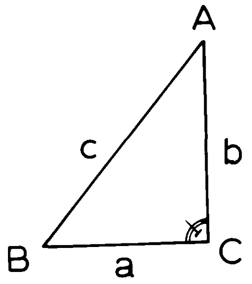


Figure 1.10

for a = b we have:

$$d\Delta + \Delta_o = \frac{1}{2}.a^2 = \frac{1}{2}(a_o^2 + da^2 + 2.a_o.da) \quad ....(1.53)$$

neglecting da^2, therefore,

$$d\Delta + \Delta_o = \frac{1}{2}(a_o^2 + 2.a_o.da) \quad ......(1.54)$$

but $\Delta_o = \frac{1}{2}.a_o^2$ ......(1.55)

therefore $d\Delta = a_o.da$ ......(1.56)

in the case when $a \neq b$

$$d\Delta = \frac{1}{2}(a_o.db + b_o.da) \quad ....(1.57)$$

2) Errors in the calculated angles

The errors in the calculated angles which result from errors in the observed sides b and c in figure 1.10 can be given by:

$$\cos A = \frac{b_o}{c_o} \qquad \text{........(1.58)}$$

and $$\cos(A + dA) = \frac{b_o + db}{c_o + dc} \qquad \text{........(1.59)}$$

for dA small, $$\cos A - dA.\sin A = \frac{\frac{b_o}{c_o}}{1 + \frac{dc}{c_o}} + \frac{\frac{db}{c_o}}{1 + \frac{dc}{c_o}}$$

therefore $$dA.\sin A = \frac{b_o}{c_o} - \frac{\frac{b_o}{c_o}}{1 + \frac{dc}{c_o}} - \frac{\frac{db}{c_o}}{1 + \frac{dc}{c_o}}$$

as $\frac{dc}{c_o}$ is very small, therefore the denominator could be taken as unity, and

$$dA.\sin A = \frac{b_o}{c_o} - \frac{b_o}{c_o} - \frac{db}{c_o}$$

for $\sin A = \frac{a}{c_o}$, therefore equation (1.59) will be:

$$dA = - \frac{db}{c_o} \cdot \frac{c_o}{a_o} = \frac{db}{a_o} \qquad \text{.......(1.60)}$$

From equation (1.60) it can be seen that the error in dA is a function of the linear error divided by the length, whereas the error in the area, as given by equation (1.56) is a function of the linear error multiplied by the length.

Since the problem of adjustment is to obtain corrections for the measured sides, it is more suitable to get these corrections by using the condition which has larger coefficients and absolute terms [71], i.e. the area misfit condition.

1.4.2 Area Covering Condition for Pure Trilateration

Deriviation of the condition equation is made for a doubly braced quadrilateral, where all sides are observed.

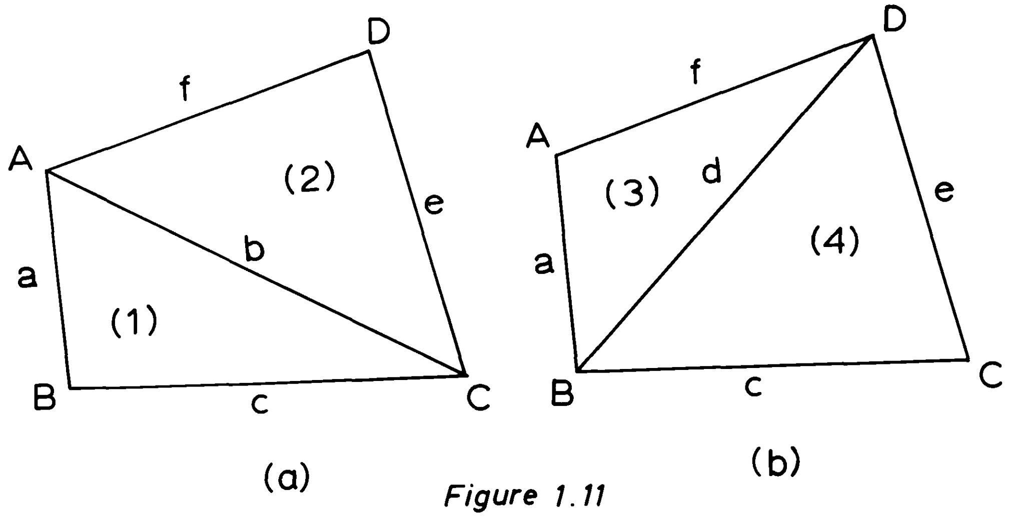


Figure 1.11

Consider quadrilateral ABCD, figure 1.3 in which a, b, c, d, e and f are observed sides. The area covered by this quadrilateral may be obtained by either the sum of the areas of the two triangles ABC and ADC figure 1.11a, or by the sum of the areas of the two triangles ABD and BCD figure 1.11b.

In fact there will be a difference in the area when calculated from figure 1.11a or from figure 1.11b. Stations A, B, C and D are obtained by direct observation of sides a, c, e, f and either b or d. When both diagonals b and d are observed, the quadrilateral will be given further rigidity, but due to accidental or observation errors adjustment of observed quantities is necessary to provide consistency to the calculated stations. This requires that figures 1.11a and 1.11b must coincide with each other exactly. Stations A, B, C and D must have the same values no matter how they are computed. In this case the area covered by both must be exactly the same, angle DAB in figure 1.11b will then be equal to the sum of the two angles CAD and CAB in figure 1.11a, whether A is the apex of the triangle of the largest or smallest area. Discussion about which one is to be chosen does not apply here, as the area of the whole figure is considered.

1.4.2.1. <u>Derivation of the Condition Equation in Pure Trilateration</u>

For the quadrilateral given in figure 1.11, there will be only one condition due to the fact that there is only one surplus observation produced by the measurement of the extra diagonal.

Considering an individual triangle (1) in the doubly braced quadrilateral given in figure 1.3, the area in this triangle with the three measured sides is given by:

$$\Delta = \sqrt{s(s-a)(s-c)} \qquad \text{......(1.61)}$$

where $s = \frac{a + b + c}{2}$, and Δ = the area if the triangle ABC.

Due to the large figures found when calculating area, the use of logarithmic tables is inevitable.

From equation (1.61) we have:

$$2\log.\Delta = \log.s + \log.(s-a) + \log.(s-b) + \log.(s-c) \quad ......(1.62)$$

Differentiating equation (1.62) with respect to a, b, and c,

$$\frac{d\Delta}{\Delta} = \frac{1}{2}\left[\frac{ds}{s} + \frac{d(s-a)}{(s-a)} + \frac{d(s-b)}{(s-b)} + \frac{d(s-c)}{(s-c)}\right]$$

$$= \frac{1}{2}\left[\frac{2}{2}\,\frac{da+db+dc}{a+b+c} + \frac{2}{2}\,\frac{db+dc-da}{b+c-a} + \frac{2}{2}\,\frac{da+dc-db}{a+c-b} + \frac{2}{2}\,\frac{da+db-dc}{a+b-c}\right] \quad ....(1.63)$$

where, $(s-a) = \frac{1}{2}(b+c-a)$, $(s-b) = \frac{1}{2}(a+c-b)$, and $(s-c) = \frac{1}{2}(a+b-c)$

Rearranging terms:

$$d\Delta = \frac{\Delta}{2}\Big[\left(\frac{1}{a+b+c} + \frac{1}{b+c-a} + \frac{1}{a+c-b} + \frac{1}{a+b-c}\right)da$$
$$+ \left(\frac{1}{a+b+c} + \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{a+c-b}\right)db$$
$$\left(\frac{1}{a+b+c} + \frac{1}{a+c-b} + \frac{1}{b+c-a} + \frac{1}{a+b-c}\right)dc\Big] \quad ....(1.64)$$

let;

$$\frac{\Delta/2}{a+b+c} = \lambda_o, \quad \frac{\Delta/2}{b+c-a} = \lambda_1, \quad \frac{\Delta/2}{a+c-b} = \lambda_2, \text{ and } \quad \frac{\Delta/2}{a+b-c} = \lambda_3 \quad ..(1.65)$$

Substituting (1.65) into (1.64) equation (1.64) will be:

$$d\Delta = [(\lambda_o+\lambda_2+\lambda_3-\lambda_1)da + (\lambda_o+\lambda_1+\lambda_3-\lambda_2)db + (\lambda_o+\lambda_1+\lambda_2-\lambda_3)dc] \quad ....(1.66)$$

Equation (1.66) gives the change in the area of a triangle due to the change in the observed sides. In other words, corrections are obtained by adjusting observed sides a, b, and c of the triangle ABC by using the change in the area of this triangle as given by equation (1.66). For the other three triangles of the quadrilateral similar three equations can be obtained.

The geometry of the adjusted figure requires that:

$$\Delta_1 + d\Delta_1 + \Delta_2 + d\Delta_2 = \Delta_3 + d\Delta_3 + \Delta_4 + d\Delta_4$$

where, Δ_1, Δ_2, Δ_3, and Δ_4 are the area of the triangles ABC, ACD ABD, and BCD successively figure 1.11.

and $(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4) + (d\Delta_1 + d\Delta_2 - d\Delta_3 - d\Delta_4) = 0 \quad ..(1.67)$

the absolute term in the equation $k = (\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)$

let λ^1, λ^2, λ^3, and λ^4 refer to triangles (1), (2), (3), and (4) successively, figure 1.11, therefore

	λ^1	λ^2	λ^3	λ^4
λ_o	$\frac{\Delta_1/2}{a+b+c}$	$\frac{\Delta_2/2}{b+e+f}$	$\frac{\Delta_3/2}{c+d+e}$	$\frac{\Delta_4/2}{a+d+f}$
λ_1	$\frac{\Delta_1/2}{b+c-a}$	$\frac{\Delta_2/2}{e+f-b}$	$\frac{\Delta_3/2}{d+e-c}$	$\frac{\Delta_4/2}{d+f-a}$
λ_2	$\frac{\Delta_1/2}{a+c-b}$	$\frac{\Delta_2/2}{b+f-e}$	$\frac{\Delta_3/2}{c+e-d}$	$\frac{\Delta_4/2}{a+f-d}$
λ_3	$\frac{\Delta_1/2}{a+b-c}$	$\frac{\Delta_2/2}{b+e-f}$	$\frac{\Delta_3/2}{c+d-e}$	$\frac{\Delta_4/2}{a+d-f}$

table 1.1

Using table 1.1 to form the condition equation for adjusting the quadrilateral figure 1.3, therefore,

$$(\lambda_o^1-\lambda_1^1+\lambda_2^1+\lambda_3^1-\lambda_o^4+\lambda_1^4-\lambda_2^4-\lambda_3^4)v_a + (\lambda_o^1+\lambda_1^1-\lambda_2^1+\lambda_3^1+\lambda_o^2-\lambda_1^2+\lambda_2^2+\lambda_3^2)v_b$$

$$+ (\lambda_o^1+\lambda_1^1+\lambda_2^1-\lambda_3^1-\lambda_o^3-\lambda_1^3-\lambda_2^3+\lambda_3^3)v_c + (-\lambda_o^3-\lambda_1^3+\lambda_2^3-\lambda_3^3-\lambda_o^4-\lambda_1^4+\lambda_2^4-\lambda_3^4)v_d$$

$$+ (\lambda_o^2+\lambda_1^2-\lambda_2^2+\lambda_3^2-\lambda_o^3-\lambda_1^3-\lambda_2^3+\lambda_3^3)v_e + (\lambda_o^2+\lambda_1^2+\lambda_2^2-\lambda_3^2-\lambda_o^4-\lambda_1^4-\lambda_2^4+\lambda_3^4)v_f$$

$$+ k = 0 \qquad ....(1.68)$$

Equation (1.68) is used to adjust the example given by Murphy [70], figure 1.3. The results of this adjustment obtained by this equation together with those obtained by using Murphy's solution are shown below,

Correction	Due to equation (1.68)	Due to Murphy
v_a	- 0.093	- 0.096
v_b	+ 0.132	+ 0.137
v_c	- 0.178	- 0.185
v_d	+ 0.199	+ 0.207
v_e	- 0.153	- 0.159
v_f	- 0.073	- 0.076

table 1.2

The two solutions are practically identical. The maximum difference for any side (in this case d) amounts to 0.008, while the mimimum difference is = 0.003, with an average difference of

0.005 for side b.

It could be said that the amount of work here is greater than that when solving by Murphy's solution, but it should be noticed that most of the work is ordinary arithmetic addition and subtraction, which are applied to values obtained from logarithmic tables.

1.4.2.1.1. Modification of Equation (1.64)

Equation (1.64) could be modified, to reduce the amount of work involved in the following way:

Consider the coefficient of da in equation (1.64),

$$= \frac{\Delta}{2}\left[\frac{1}{a+b+c} - \frac{1}{b+c-a} + \frac{1}{a+c-b} + \frac{1}{a+b-c}\right]$$

$$= \frac{\Delta}{2}\left[\frac{-4.a^3 + 4.b^2.a + 4.c^2.a}{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}\right]$$

$$= \frac{\Delta}{2}\left[\frac{4a(b^2+c^2-a^2)}{16s(s-a)(s-b)(s-c)}\right] = \frac{a}{8\Delta}[b^2+c^2-a^2] \quad .....(1.69)$$

But $b^2 + c^2 - a^2 = 2.b.c.\cos A$

therefore equation (1.69) will be:

$$= \frac{1}{4\Delta}\, a.b.c.\cos A \qquad ....(1.70)$$

The coefficients of db and dc can be obtained in the same way,

coefficient of db $= \frac{1}{4\Delta}\, a.b.c.\cos B$, also

.... dc $= \frac{1}{4\Delta}\, a.b.c.\cos C$

Substituting in equation (1.64), we have:

$$d\Delta = \frac{a.b.c.}{4\Delta}(\cos A.da + \cos B.db + \cos C.dc) \quad ....(1.71)$$

but we have $2\Delta = a.b.\sin C$,

therefore, $d\Delta = \frac{c}{2\sin C}(\cos A.da + \cos B.db + \cos C.dc) \quad ...(1.72)$

Although equations (1.71) and (1.72) appear in more convenient form than equation (1.64) the amount of computation included does not differ much.

1.4.3. Use of the Area Condition for Adjuting Observed Sides And Angles

The use of the area coverage condition above has been for adjusting sides only. It is however even more convenient and effective when both angles and sides are observed, or in other words when one redundant or more is availabe for the adjustment of each triangle.

Due to errors of observation, a different area will be obtained for the same triangle when using the different observed elements of this triangle, e.g. a triangle ABC (figure 1.2) of which a, b, c and A are observed gives the following geometric condition:

$$\Delta_1 = \Delta_2 \qquad \text{........(1.73)}$$

Due to errors of observations this condition will have the form:

$$\Delta_1 + d\Delta_1 = \Delta_2 + d\Delta_2 \qquad \text{........(1.74)}$$

where $\Delta_1 = \sqrt{s(s-a)(s-b)(s-c)}$, ...(1.75)

and $\Delta_2 = \frac{1}{2}.b.c.\sin A$ ....(1.76)

Equation (1.74) can be solved by the same procedure followed in solving equation (1.68), noting that angle A is to be adjusted.

1.4.3.1. Use of the Logarithmic Difference in the Formation of the Required Condition Equation

It has been found that the most suitable way to reduce the amount of computation in solving this equation, due to the large figures arising from the area consideration, is the following Using equation (1.75) in forming equation (1.74), we have:

$$\Delta_1 + d\Delta_1 =$$

$$= \sqrt{s+\frac{da+db+dc}{2})(\underline{s-a}+\frac{db+dc-da}{2})(\underline{s-b}+\frac{da+dc-db}{2})(\underline{s-c}+\frac{da+db-dc}{2})} \qquad \text{..(1.77)}$$

if δ is the log. difference, therefore;

from equation (1.75) we have:

$$\log.\Delta_1 = \frac{1}{2}[\log.s + \log.(s-a) + \log.(s-b) + \log.(s-c)] \qquad \text{........(1.78)}$$

and from equation (1.77) we have:

$$\begin{aligned} \log.\Delta_1 + \delta_{\Delta 1}d\Delta_1 = \tfrac{1}{2}[&\log.s + \tfrac{1}{2}\delta_s(da+db+dc) + \log.(s-a) \\ &+ \tfrac{1}{2}\delta_{(s-a)}(db+dc-da) + \log.(s-b) \\ &+ \tfrac{1}{2}\delta_{(s-b)}(da+dc-db) + \log.(s-c) \\ &+ \tfrac{1}{2}\delta_{(s-c)}(da+db-dc)] \end{aligned} \qquad \text{......(1.79)}$$

Subtracting (1.78) from (1.79) we have:

$$\delta_{\Delta 1} d\Delta_1 = \tfrac{1}{4}[\delta_s(da+db+dc) + \delta_{(s-a)}(db+dc-da) + \delta_{(s-b)}(da+dc-db) + \delta_{(s-c)}(da+db-dc)] \quad ....(1.80)$$

Rearranging terms equation (1.80) will become:

$$\delta_{\Delta 1} d\Delta_1 = \tfrac{1}{4}[(\delta_s-\delta_{(s-a)}+\delta_{(s-b)}+\delta_{(s-c)})da + (\delta_s+\delta_{(s-a)}-\delta_{(s-b)}+\delta_{(s-c)})db + (\delta_s+\delta_{(s-a)}+\delta_{(s-b)}-\delta_{(s-c)})dc] \quad ....(1.81)$$

Following the same procedure, a similar equation can be obtained for Δ_2 using the sides b, and c together with angle A.

Thus using equation (1.76) to form equation (1.74) we have:

$$\Delta_2 + d\Delta_2 = \tfrac{1}{2}(b+db)(c+dc)\sin(A+dA) \quad ....(1.82)$$

From equation (1.76) we have:

$$\log.\Delta_2 = \log.b + \log.c + \log.\sin A - \log.2 \quad ......(1.83)$$

also from equation (1.82) we have:

$$\log(\Delta_2+d\Delta_2) = \log.(b+db) + \log.(c+dc) + \log.\sin(A+dA) - \log.2 \quad ....(1.84)$$

Using the log. difference δ we have:

$$\log.\Delta_1 + \delta_{\Delta 1} d\Delta_1 = \log.b + \delta_b db + \log.c + \delta_c dc + \log.\sin A + \delta_A^{\sin} dA - \log.2 \quad ....(1.85)$$

Subtracting equation (1.83) from equation (1.85) we have:

$$\delta_{\Delta 2} d\Delta_2 = \delta_b.db + \delta_c.dc + \delta_A^{\sin}.dA \quad ....(1.86)$$

Equation (1.74) satisfies the geometric condition for the triangle figure 1.2 and gives,

$$\log.\Delta_1 + \delta_{\Delta 1}.d\Delta_1 = \log.\Delta_2 + \delta_{\Delta 2}.d\Delta_2$$

and hence $\log.\Delta_1 - \log.\Delta_2 = \delta_{\Delta 2}.d\Delta_2 - \delta_{\Delta 1}.d\Delta_1$ ..(1.87)

Subtracting equation (1.86) from equation (1.81) we have:

$$\delta_{\Delta 1}.d\Delta_1 - \delta_{\Delta 2}.d\Delta_2 = \tfrac{1}{4}[(\delta_s-\delta_{(s-a)}+\delta_{(s-b)}+\delta_{(s-c)})da + (\delta_s+\delta_{(s-a)}-\delta_{(s-b)}+\delta_{(s-c)})db + (\delta_s+\delta_{(s-a)}+\delta_{(s-b)}-\delta_{(s-c)})dc] - \delta_b.db - \delta_c.dc - \delta_A^{\sin}.dA \quad ......(1.88)$$

Substituting from equation (1.87) equation (1.88) will become:

$$\tfrac{1}{4}[\delta_s-\delta_{(s-a)}+\delta_{(s-b)}+\delta_{(s-c)})da + (\delta_s+\delta_{(s-a)}-\delta_{(s-b)}+\delta_{(s-c)}-4\cdot\delta_b)db$$

$$+ \ (\delta_s+\delta_{(s-a)}+\delta_{(s-b)}-\delta_{(s-c)}-4\cdot\delta_c)dc - \delta_A^{\sin}.dA +$$

$$+ \log.\Delta_1 - \log.\Delta_2 = 0 \qquad ....(1.89)$$

Equation (1.89) gives a condition for the area coverage of a triangle with one redundant. For two redundants, there will be one more condition equation and one further if there is a third.

It can be seen that the use of this method is of great advantage as it reduces the amount of computational work considerably. The logarithmic difference required can be obtained easily from log. tables for the values required.

Example: a triangle of

a =	69 847.62	feet	(observed)
b =	94 277.10	....	....
c =	102 017.34	....	....
A_o =	75° 13' 21".60		....
A_c =	75 13 20.07		(computed)

Using the area condition equation (1.89) the following is obtained:

$$dA_o + 1.48\ da + 2.45\ db - 3.28\ dc + 1.50 = 0 \quad ..(1.90)$$

For the sake of comparison an angle condition is used for adjusting this triangle, in which case,

$$A_o + dA_o = A_c + dA_c \qquad ......(1.91)$$

gives the following equation,

$$dA_o + 1.48\ da + 2.47\ db - 3.29\ dc + 1.53 = 0 \quad ...(1.92)$$

Comparison between equations (1.90) and (1.92) shows that they are virtually identical.

The main advatage in using equation (1.89) is to use the difference $(\Delta_1 - \Delta_2)$ in obtaining the absolute term, while all other coefficients of corrections are obtained from standard mathematical tables without any additional work. It has been found however that the same procedure would not produce the same advantages when applied to the adjustment of sides of a doubly braced quadrilateral. In the case of the quadrilateral as in equation (1.67) the geometric condition is:

$$\Delta_1 + d\Delta_1 + \Delta_2 + d\Delta_2 = \Delta_3 + d\Delta_3 + \Delta_4 + d\Delta_4,$$

which requires that;

$$(\Delta_1 + \Delta_2) + d(\Delta_1 + \Delta_2) = (\Delta_3 + \Delta_4) + d(\Delta_3 + \Delta_4)$$

to be able to apply the log. difference, which would be:

$$\log.(\Delta_1+\Delta_2) + \delta_{(\Delta 1+\Delta 2)}d(\Delta_1+\Delta_2) = \log.(\Delta_3+\Delta_4) + \delta_{(\Delta 3+\Delta 4)}d(\Delta_3+\Delta_4)$$

However due to the fact that,

$$\Delta_1+d\Delta_1=\sqrt{(s+\frac{da+db+dc}{2})(\underline{s-a}+\frac{db+dc-da}{2})(\underline{s-b}+\frac{da+dc-db}{2})(\underline{s-c}+\frac{da+db-dc}{2})}$$

$$\Delta_2+d\Delta_2=\sqrt{(s+\frac{db+de+df}{2})(\underline{s-b}+\frac{de+df-db}{2})(\underline{s-e}+\frac{db+df-de}{2})(\underline{s-f}+\frac{db+de-df}{2})}$$

The quantity$[\log(\Delta_1+\Delta_2) + \delta_{(\Delta_1+\Delta_2)}d(\Delta_1+\Delta_2)]$ cannot be obtained without great effort as it requires using the log. difference of the sum of two quantities, which does not exist in mathematical tables. Hence the simplification intended by using equation (1.89) cannot be reached.

The alternative solution given by (1.68) can be obtained from the equation $(\Delta_1+\Delta_2) - (\Delta_3+\Delta_4) = (d\Delta_3+d\Delta_4) - (d\Delta_1+d\Delta_2)$

This has the advantage of giving the corrections directly, without the necessity of going to tables to obtain the log. differences.

1.5. CHARACTERISTICS OF AN APEX

The question of selecting which apex should be used for the adjustment of the doubly braced quadrilateral raises some interesting points. From [73] and due to Murphy, the following is quoted:"In practice the condition equation having the largest coefficients should take priority of selection in the adjustment and so it follows that the condition equation should be formed with respect to the apex of the triangle of smallest area. This conclusion is exactly the opposite to the view expressed by Dr. Tárczy-Hornoch and the former recommendation made by Murphy and Thornton-Smith, while being correct for the majority of figures, cannot now be adopted as a general rule".

To find out which of these two consideration is acceptable a special study of the problem has been carried out in the following.

1.5.1. Mathematical Considerations

Consider angle A, obtained by calculation from observed sides a, b, and c in the triangle ABC figure 1.2, in this case we have:

$$\cos A = \frac{b^2 + c^2 - a^2}{2.b.c} \qquad \text{........(1.93)}$$

Differentiating with respect to a, b, and c we have:

$$- \sin A.\frac{\partial A}{\partial a} = - \frac{a}{b.c},$$

$$- \sin A.\frac{\partial A}{\partial b} = \frac{a}{b.c}.\cos C \qquad \text{........(1.94)}$$

$$- \sin A.\frac{\partial A}{\partial c} = \frac{a}{b.c}.\cos B$$

denoting $\frac{\partial A}{\partial a} + \frac{\partial A}{\partial b} + \frac{\partial A}{\partial c} = dA_{a,b,c}$ ....(1.95)

Therefore from (1.94) and (1.95) we have:

$$dA_{a,b,c} = - \frac{a}{2.\text{area}}(\cos B + \cos C - 1) \qquad \text{........(1.96)}$$

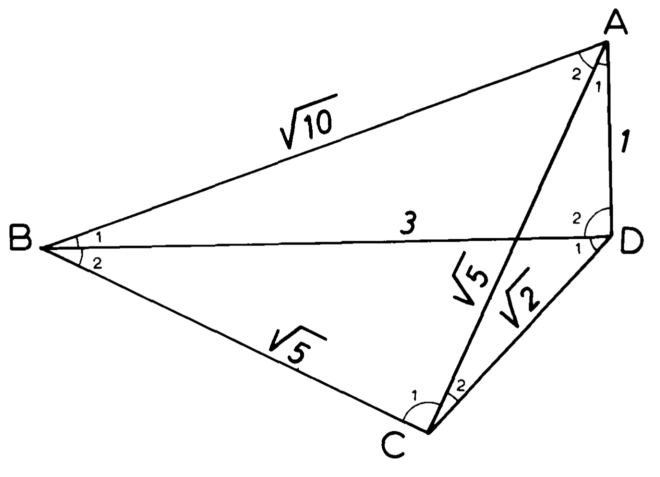


Figure 1.12

Table 1.3 gives the change in the different angles of the doubly braced quadrilateral ABCD, figure 1.12, with respect to the change of the sides a, b, c, d, e, and f. It also shows the area of each component triangle of the quadrilateral ABCD.

triangle		area sq. units	angle	cos	change in angle according to equation 1.96 (rad)
1	ABD	3	B_1	0.959 607	+ 0.2279
			D_2	0.000 000	- 0.2792
			A	0.316 228	+ 0.0513
					Σ= 0.0000
2	BCD	3	B_2	0.894 428	+ 0.2871
			C	- 0.316 228	- 0.6015
			D_1	0.707 107	+ 0.3144
					Σ= 0.0000
3	ABC	5	B	0.707 107	+ 0.1310
			C_1	0.000 000	- 0.2620
			A_2	0.707 107	+ 0.1310
					Σ= 0.0000
4	ACD	1	C_2	0.948 684	+ 0.8127
			D	- 0.707 107	- 1.8853
			A_1	0.894 428	+ 1.0726
					Σ= 0.0000

table 1.3

Table 1.4 gives the difference between the compound angle and the sum of its two individual component angles as computed from the known sides given in six figures. This table also gives the difference between the change in the compound angle and the sum of the changes in its two individual component angles, due to equation(1.96). In both cases the difference at every apex is expressed as a percentage of the largest difference found at a particular apex, in this case apex D.

Angle	° (calculated)	'	"	Change in angle due to equation 1.96 (rads.)
A_1	26	35	09.2	+ 1.0726
A_2	44	58	56.5	+ 0.1310
(A_1+A_2)	71	34	05.7	+ 1.2036
A	71	33	51.1	- 0.0513
Difference	00	00	14.6	+ 1.1526
Diff. as o/oage	59.8			60

Angle	° ' " (calculated)	Change in angle due to equation 1.96 (rads.)
B_1	18 26 08.5	+ 0.2279
B2	26 33 19.4	+ 0.2871
(B_1+B_2)	44 59 27.9	+ 0.5150
B	44 59 22.8	- 0.1310
Difference	00 00 05.1	+ 0.3840
Diff. as o/oage	20.9	20
C_1	90 01 40.4	- 0.2620
C_2	18 27 05.9	+ 0.8127
(C_1+C_2)	108 28 46.3	+ 0.5507
C	108 28 31.7	+ 0.6015
Difference	00 00 14.6	+ 1.1522
Diff. as o/oage	59.8	60
D_1	44 58 09.0	+ 0.3144
D_2	90 00 00.0	- 0.2792
(D_1+D_2)	134 58 09.0	+ 0.0352
D	134 57 44.6	+ 1.8853
Difference	00 00 24.4	+ 1.9205
Diff. as o/oage	100	100

table 1.4

Inspection of the set of results in tables 1.3 and 1.4 shows that:

(1) A special geometric condition is always satisfied, as shown by the zero sum of the effects of the three sides of the triangle on the three angles of this triangle, as shown in table 1.3. This condition is explained in the following: Consider equation (1.96), where,

$$- dA_{a,b,c} = \frac{a}{2.\text{area}}(\cos C + \cos B - 1) \quad \ldots\ldots\ldots\ldots(I),$$

$$- dB_{a,b,c} = \frac{b}{2.\text{area}}(\cos A + \cos C - 1) \quad \ldots\ldots\ldots\ldots(II)$$

$$- dC_{a,b,c} = \frac{c}{2.\text{area}}(\cos A + \cos B - 1) \quad \ldots\ldots\ldots\ldots(III)$$

Adding (I), (II) and (III), we have:

$$- (dA_{a,b,c} + dB_{a,b,c} + dC_{a,b,c}) = \frac{1}{2.\text{area}}(a.\cos B + a.\cos C + a + b.\cos A + b.\cos C + b + c.\cos A + c.\cos B + c)$$

$$= \frac{1}{2.\text{area}}\left[\frac{a(a^2+b^2-c^2)}{2.a.b} + \frac{a(a^2+c^2-b^2)}{2.a.c} - a + \frac{b(b^2+c^2-a^2)}{2.b.c} + \frac{b(b^2+a^2-c^2)}{2.b.a} - b + \frac{c(c^2+b^2-a^2)}{2.c.b} + \frac{c(c^2+a^2-b^2)}{2.c.a} - c\right]$$

$$= \left[\frac{b^2+a^2-c^2+c^2+a^2-b^2-2a^2}{2a} + \frac{a^2+b^2-c^2+c^2+b^2-a^2-2b^2}{2b} + \frac{a^2+c^2-b^2+b^2+c^2-a^2-2c^2}{2c}\right] = [0+0+0] = 0$$

(2) Comparing columns 2 and 3 in table 1.4, it may be noticed clearly that the relationship between the discrepancies at the different apices of the quadrilateral is the same, whether this discrepancy is due to computation of angles, or due to the change in the angle with respect to the change in the sides of a triangle.

(3) The sum of the smallest and largest differences is equal to the sum of the other two values. In other words the sum of differences concerned at two opposite apices equals the sum of the differences at the other two opposite apices.

(4) Largest area is associated with the smallest apex, and the smallest area with the largest apex (table 1.5).

(5) Largest difference between the compound angle and its two individual components (and hence the 100 percent) is associated with the large angle, and the smallest with the small angle (table 1.5)

Triangle	1	2	3	4
Area (unit)^2	3	3	5	1
Apex	A	C	B	D
Angle	71° 33' 51"	108° 28' 32"	44° 59' 23"	134° 57'45"
Diff. as o/oage	60	60	20	100

table 1.5

1.5.2. Effect of the Shape of Different Figures

Figure 1.13 is obtained by calculation of sides from coordinates. Using equation (1.96), and the calculated angles at the different apices, results are shown in table 1.6

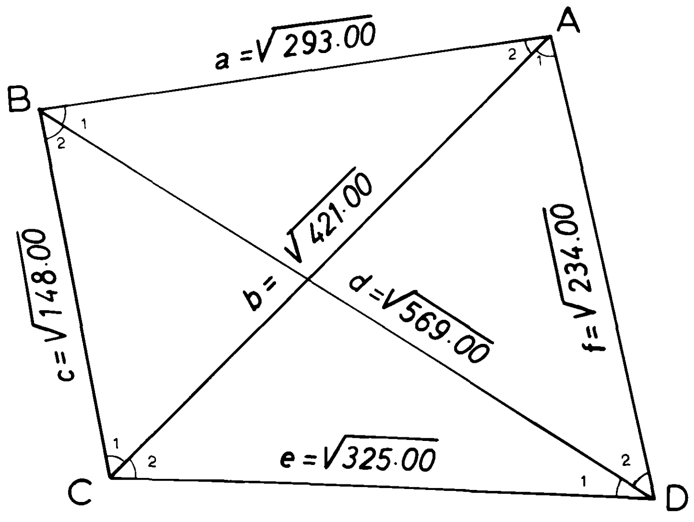


Figure 1.13

Angle	°	'	" (calculated)	$dA_{a,b,c}$ etc.. (radians)	Δ of triangle with apex at angle...*
A_1	58	17	12.16	+ 0.0039	
A_2	36	19	02.64	+ 0.0233	
(A_1+A_2)	94	36	14.80	+ 0.0272	103.5 unit^2
A	94	36	09.84	- 0.0428	
Difference	00	00	04.96	+ 0.0700	
Diff. as..		81.60		80.10	
B_1	39	43	58.00	+ 0.0224	
B_2	47	30	51.42	+ 0.0296	
(B_1+B_2)	87	14	49.42	+ 0.0520	104 unit^2
B	87	14	43.34	- 0.0354	
Difference	00	00	06.08	+ 0.0874	
Diff. as..		100		100	
C_1	56	26	13.28	+ 0.0120	
C_2	46	12	18.04	+ 0.0128	107 unit^2
(C_1+C_2)	102	38	31.32	+ 0.0248	
C	102	38	25.58	- 0.0605	
Difference	00	00	05.74	+ 0.0853	
Diff. as..		94.40		97.6	
D_1	29	50	42.40	+ 0.0309	
D_2	54	39	52.20	+ 0.0204	133.5 unit^2
(D_1+D_2)	75	30	34.60	+ 0.0513	
D	75	30	29.42	- 0.0167	
Difference	00	00	05.18	+ 0.0680	
Diff. as..		85.20		77.80	

* Δ is the area

table 1.6

Some other remarks may be made here:

1) The largest and smallest angles (apices) are not associated with the smallest and largest area repectively. It may be noted that this is quite experience opposite to (4) given in 1.5.1

2) The largest difference between the compound angle and its two individual components is not associated with the largest angle, nor the smallest difference with the smallest angle, which is contrary to point (5) given 1.5.1.

3) There is still a constancy of the percentage difference between the sum of the individual angles and the compound angle in any figure, taken as a percent of the largest difference of the all angles of the figure.

1.5.3. Application to the Doubly Braced Geodetic Quadrilateral

Equation (1.96) has also been applied to the observed geodetic quadrilateral figure 1.3 used by Murphy and Thornton-Smith, [70], in which all sides have been observed. In order to show the effect of the adjustment on the percentage ratios, the observed and adjusted sides have been used. Angles are obtained by calculation from both observed and adjusted sides, then comparison is made for both cases. Observed sides are used in equation (1.96) to give $dA_{a,b,c}$.

For the quadrilateral ABCD, figure 1.3 we have the following:

side	observed	adjusted
a	69 847.62	69 847.52
b	83 587.77	83 587.91
c	44 679.24	44 679.06
d	102 017.34	102 017.55
e	65 824.23	65 824.07
f	94 277.10	94 277.02

table 1.7

Calculation of the relationship between the misfits at each apex.

Angle	$(dA_{a,b,c}$etc..) x10^6. radians	° ' " calculated from observed sides	° ' " calculated from adjusted sides
A	+ 6.708	32 18 19.65	32 18 18.87
A	+ 3.395	42 55 01.95	42 55 01.55
(A +A)	+10.103	75 13 21.60	75 13 20.42
A	- 3.179	75 13 19.57	75 13 20.60
Difference	+13.282	- 02.03	-00.18
Diff. as..	33	33.7	36
B	+23.072	27 42 19.78	27 42 16.68
B	- 0.068	63 19 26.50	63 19 25.87
(B +B)	+23.004	91 01 46.38	91 01 42.55
B	-10.575	91 01 41.21	91 01 42.99
Difference	+33.579	+05.17	-00.44
Diff. as..	84	85.8	88
C	+ 3.867	56 39 59.13	56 39 58.14
C	- 4.122	77 14 02.85	77 14 02.50
(C +C)	- 0.255	133 54 01.98	133 54 00.64
C	-40.070	133 53 55.96	133 54 01.14
Difference	+39.815	+06.02	-00.50
Diff. as..	100	100	100
D	+16.997	18 23 44.25	18 23 42.17
D	+ 3.247	41 27 13.93	41 27 13.53
(D +D)	+20.244	59 50 58.18	59 50 55.70
D	+ 0.727	59 50 55.20	59 50 55.95
Difference	+19.517	+02.98	-00.25
Diff. as..	49	49.5	50

table 1.8

It should be noted that area of the triangle with apex at A is the largest area, while that with the apex at C is the smallest area. Also the largest percentage is associated with the largest angle but the smallest is not associated with the smallest angle. As $dA_{a,b,c}$ is derived from the observed sides, the percentages resulting from observed sides are very close to those obtained by $dA_{a,b,c}$. Although the difference between the compound angle and the sum of its two individual components obtained by adjusted sides are negative (opposite to the signs of the corresponding differences in the case of observed sides) the percentages of this difference is still the same.

1.5.4 Triangles with Different Area and Shape

Consider equation (1.96),

$$dA_{a,b,c} = - \frac{a}{2.\text{area}}(\cos B + \cos C - 1)$$

In this equation, dA represents the rate of change in angle A with respect to the change in sides a, b, and c. For simplicity a, b, and c are considered to have the same change.

The maximum and minimum rate of change in angle A with respect to the changes in sides a, b, and c will be obtained at the following cases:

(1) The rate of change will be maximum (infinity when the tangent to the curve is parallel to the dA axis) when the area of the triangle is infinitesimally small (infinity rate of change when the area of the triangle is equal to zero). A maximum value can also be obtained when any of the three angles of the triangle has its cosine = $\pm$ 1, (i.e. the angle is 0° or 180°), which means that the triangle will be a straight line, or in other words, when $a = b \pm c$. This can be proved by the following:-

Let Δ be the area of the triangle ABC figure 1.2. Differentiating equation (1.61) with respect to a, b, and c and for the maximum and minimum areas equation (1.97) has to be satisfied.

$$- a^3 - b^3 - c^3 + b^2 a + b^2 c + c^2 a + c^2 b + a^2 b + a^2 c = 0$$

i.e.

$$a^2(b + c - a) + b^2(a + c - b) + c^2(a + b - c) = 0 \quad \text{...........(1.97)}$$

Equation (1.97) can only be satisfied if a, b, and c are zero, which gives a minimum area, and the triangle in this is reduced to a point. Also equation (1.97) will be zero if each quantity between brackest is zero, which can be achieved only if a = b = c = 0.

Also if we consider equation (1.76), where the area is a function of b, c, and angle A, then differentiating with respect to b, c, and A, the area of the triangle can be maximum or minimum if,

$$b.\sin A + c.\sin A + b.c.\cos A = 0 \qquad \text{.........(1.98)}$$

The left hand side of the equation can be zero if,

$$-\cot A = \frac{b + c}{b.c} \qquad \text{.........(1.99)}$$

It can be seen that equation (1.99) can only be valid if both sides are zero, i.e. $b = c = 0$.

In fact b, and c must always be positive or zero, and thus the quantity (b+c)/bc should be always positive and less than unity if neither of the two sides is unity, and zero if b and c are zero. Thus only a minimum area for any triangle can be obtained, and hence only the maximum rate of change in the angle A with respect to the changes in the sides a, b, and c can be considered through the area of this triangle.

However, zero and infinity areas for a triangle are not practical considerations, especially if we have a real triangle with three known sides. The above discussion shows that the area of the triangle does not have much to do with the selection of the apex and hence with the condition equation for the adjustment of the doubly braced quadrilateral. Besides, there are cases where the area remains constant and the effect on the quantity dA is only due to the quantity a(cosB + cosC - 1), e.g. triangles having the same base and height, and different shapes.

(2) On the other hand the equation (1.96) will be zero (i.e. having the tangent to the curve parallel to the da axis) when the quantity (cosB + cosC - 1) is zero. This will be obtained when $A = B = C = 60°$. In this case (cosB + cosC - 1) will be equal to $(0.5 + 0.5 - 1) = 0$. That is to say, a minimum rate of the change in the angle A with respect to the changes in the sides a, b, and c will be obtained for the equilateral triangle, or when the three angles of the

triangle are about to be equal. The minimum rate of change does not depend only on equal angles but also on the sum of the cosines of the two angles B and C, or in other words, on the shape of the triangle.

For the angles B and C in the quantity (cosB + cosC - 1) let the following be considered:

$$\cos B = 1 - \frac{B^2}{2} \qquad \text{........(1.100)}$$

B is in radians, (expression 1.100 is due to Taylor's series), therefore:

$$\cos B + \cos C - 1 = 1 - \frac{B^2}{2} + 1 - \frac{C^2}{2} - 1 = 0$$

i.e. $$B^2 + C^2 = 2 \qquad \text{........(1.101)}$$

Equation (1.101) can be satisfied by any point on the circle of radius $\sqrt{2}$, figure 1.14. One of the solutions will be

$$B = C = 1r \simeq 60^\circ.$$

Also a maximum rate of change in angle A with respect to the change in the three sides of triangle ABC, can be obtained by considering the quantity (cosB + cosC - 1). In this case cosB = cosC = 0, i.e. B = C = 90°. Maximum rate of change can also be obtained if cosB = cosC = 1, in this case B = C = 0°, and the area will be minimum or zero.

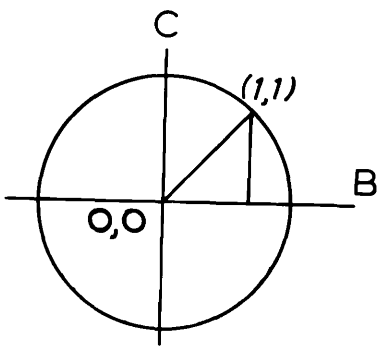


Figure 1.14

Therefore, the maximum and minimum rate of change in the angle A can be obtained by considering the quantity(cosB + cosC -1) rather than by considering the area of the triangle. For each triangle in the doubly braced quadrilateral dA is supposed to be affected by the shape of this triangle. The quantity dA of a compound angle will be different to the sum of the two individual components of dA.

For the different quadrilaterals considered it has been shown that the difference between the rate of change in the considered angle with respect to the change of the different sides of a compound angle and the sum of the differences of its two individual components have special relationships for all apices of the quadrilateral. These special percentages are found to be the same when the misfit between every individual compound angle and its two components are treated in the same way, i.e. both sides of the selected condition equation have this special relationship. It follows from the previous discussion that variation in the differences of the rate of change will be expected from the same individual compound angle when it has two different individual components.

As for the selection of the condition equation to be used, according to Murphy and Thornton-Smith [70] angle C, figure 1.3, should be used. This does have a percentage of 100, and is associated with the triangle of the smallest area. Sometimes, however it is not, as shown in the different cases tested above. Tarczy-Hornoch and Hovanyi [109], recommended angle A for the adjustment of the same figure. This has the smallest precentage (33) and is associated with the triangle of the largest area in this case.

However, it has been shown that the same compound angle associated with the special area may have different percentage according to the shape of its two components, i.e. the percentage obtained is affected by the shape (cosB + cosC - 1) in each triangle and not by the area of this triangle.

1.5.5 Effect of the Misfit in the Different Apices on the Adjusted Figures

Using the misfit in the angles of the different apices in the condition equation suggested by Murphy, the following results are obtained.

corrections	misfit at				final actual values
	A	B	C	D	
$-v_a$	0.09626	0.09741	0.09760	0.09640	0.10
$+v_b$	0.13668	0.13831	0.13858	0.13690	0.14
$-v_c$	0.18500	0.18722	0.18758	0.18528	0.19
$+v_d$	0.20701	0.20949	0.20989	0.20732	0.21
$-v_e$	0.15849	0.16039	0.16070	0.15873	0.16
$-v_f$	0.07555	0.07646	0.07660	0.07338	0.08

table 1.9

Results shown in table 1.9, hardly show any practical difference, even though the corrections are of the order 0.01. It should also be noticed that corrections for sides using apex C are the largest corrections, while corrections for sides using apex A are the smallest. If we consider Σv^2, the minimum value will be obtained from the corrections due to the misfit at apex A. However the value Σv^2 obtained using the other apices is for all practical purposes identical.

1.6 NEW CONDITIONS FOR ADJUSTING HYBRID OBSERVATIONS

A possible solution for this problem has been given in 1.4.3., as a further application to the use of the area condition for adjusting trilateration problems. In this section, observed angles and sides will be adjusted simultaneously assuming the same accuracy of observations in both.

1.6.1. Errors and Corrections

Before applying any geometric condition, the distinction between errors and corrections is stated to be:-

(1) Errors, or errors of observations, are quantities beyond any investigation or adjustment's reach. They occur even with the most accurate tools, following the best known methods of observations. So it is impossible to avoid them during observations, or to find them during calculation and adjustment.

(2) Corrections, these are quantities obtained by satisfying a special geometric condition, which tend to disperse the misfit between figures formed by the observed quantities and requirements of these geometric conditions. These corrections have nothing to do with the real errors, but it could be said that, applying the least squares solution to these problems will produce corrections as near as possible to these errors [78].

For adjusting a quadrilateral whose sides have been observed, the corrections obtained come out in a special pattern (the quadrilateral here is treated as a special case of the centered figure, and for this see sec. 1.2.1.2a). This special pattern of signs does not give much chance for the validity of the statement about the relationship between the corrections and the errors if the problem is solved by the least squares method. The least squares method in this case supposes a different sign for the two diagonals, while no one can say that errors follow any rule except the rule of normal distribution. However, the sign convention here is only due to vector analysis. Besides, the adjustment of pure trilateration is found to give a weak solution from the estimate of the coefficient of correlation point of view (see 1.2.3.). For this, angles have to be included in accurate work, rather than observing sides only.

1.6.2. Condition Equations for Adjusting Hybrid Observations

Using equation (1.96) for obtaining the change in the three angles of a triangle with respect to the change in the three sides we have:

$$dA_{a,b,c} + dB_{a,b,c} + dC_{a,b,c} =$$
$$\frac{1}{2.\text{area}} \, [a(\cos B + \cos C - 1) + b(\cos A + \cos C - 1) + c(\cos A + \cos B - 1)\,] = 0$$

Rearranging terms we have:

$$dA_{a,b,c} + dB_{a,b,c} + dC_{a,b,c} =$$
$$\frac{1}{2.\text{area}} \, [a.\cos B + b.\cos A - c) + (a.\cos C + c.\cos A - b) + (b.\cos C + c.\cos B - a)] = 0 \quad ..(1.102)$$

Equation (1.102) has been proved in 1.5.1. In this equation it can be seen that each quantity within the inner brackets represents the geometric condition for a side, that the sum of the projections of the two sides of a triangle on the third side must be zero. Therefore for the whole triangle the three quantities between the three inside brackets must be zero.

Applying the geometric condition given in equation (1.102) means that we will be satisfying:-

(1) A direct geometric condition for each side, e.g.

$$a.\cos B + b.\cos A - c = 0, \quad \text{and}$$

(2) A geometric condition that the sum of the changes in the three angles with respect to the change in the three sides has to be zero. That is, the sum of the three angles of the triangle is always 180°. This is exactly the angle condition, but in the form of the sum of the derivatives of the three angles instead of the sum of the three angles themselves.

In order to obtain the corrections using equation (1.102), its individual components will be considered separately to give a condition for each side, thus:

$$b.\cos C + c.\cos B - a = 0 \qquad \text{........(1.103a)}$$

$$a.\cos C + c.\cos A - b = 0 \qquad \text{........(1.103b)}$$

$$a.\cos B + b.\cos A - c = 0 \qquad \text{........(1.103c)}$$

which gives a set of linked geometric conditions for angles and sides.

For any triangle in which all sides and angles have been observed equation (1.103) can be taken as the geometric conditions which have to be satisfied by the adjustment of each triangle in the net. Being observed quantities, sides and angles will always contain errors of observations, so corrections have to be introduce in these equations. To allow for the corrections of the five observed quantities in (1.103c) we have:

$$(a + \delta a).\cos(B + \delta B) + (b + \delta b).\cos(A + \delta A) - (c + \delta c) = 0 \qquad \text{.........(1.104c)}$$

Expanding this equation results in:

$$(a + \delta a)(\cos B.\cos\delta B - \sin B.\sin\delta B) +$$
$$(b + \delta b)(\cos A.\cos\delta A - \sin A.\sin\delta A) - (c + \delta c) = 0 \quad \text{..(1.105c)}$$

If we consider the quantities δB, and δA as being of small dimensions, then $\cos\delta B = 1$, and $\cos\delta A = 1$. Also, $\sin\delta A = \delta A$ and $\sin\delta B = \delta B$, where δA and δB are in radians. Furthermore we may neglect the product of small quantities δb and δA (i.e $\delta b.\delta A = 0$), and of δa and δB (i.e. $\delta a.\delta B = 0$), so that:

$$(a + \delta a)(\cos B - \delta B.\sin B) + (b + \delta b)(\cos A - \delta A.\sin A) - (c + \delta c) = 0 \qquad ....(1.106c)$$

Therefore,

$$(a.\cos B + b.\cos A - c) + (\cos B.\delta a + \cos A.\delta b - \delta c) - a.\sin B.\delta B - b.\sin A.\delta A = 0 \qquad ....(1.107c)$$

To convert δB and δA to angular measure, these must be multiplied by sin 1", which has the advantage of providing coefficients of the same rank as the coefficients of δa, δb, and δc in the same equation.

Substituting k_c for (a.cosB + b.cosA - c), the final form of equation (1.107c) will be:

$$\cos B.\delta a + \cos A.\delta b - \delta c - a.\sin B.\sin 1''.\delta B - b.\sin A.\sin 1''.\delta A + k_c = 0 \qquad ....(1.108c)$$

The corresponding set of condition equations for the doubly braced quadrilateral figure 1.15 will be:

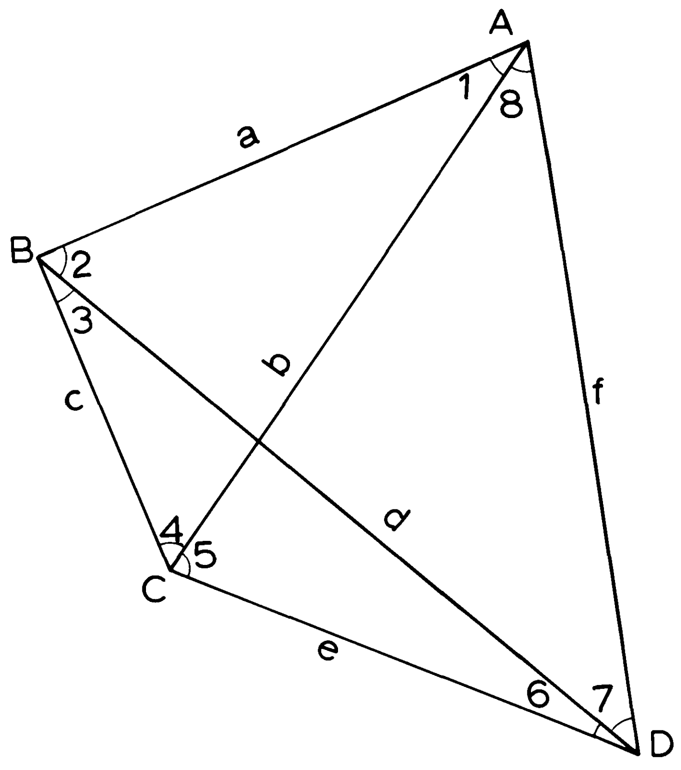


Figure 1.15

$$\left.\begin{aligned}
&\cos 1.\delta a + \cos 4.\delta c - \delta b - a.\sin 1.\sin 1''.\delta 1 - \\
&\qquad - c.\sin 4.\sin 1''.\delta 4 + k_{b1} = 0 \quad \ldots\ldots \\
&\cos 1.\delta b + \cos(2+3).\delta c - \delta a - b.\sin 1.\sin 1''.\delta 1 - \\
&\qquad - c.\sin(2+3).\sin 1''.(\delta 2+\delta 3) + k_{a1} = 0 \quad \ldots\ldots \\
&\cos(2+3).\delta a + \cos 4.\delta b - \delta c - b.\sin 4.\sin 1''.\delta 4 - \\
&\qquad - a.\sin(2+3).\sin 1''.(\delta 2+\delta 3) + k_{c1} = 0 \quad \ldots\ldots
\end{aligned}\right\} \ldots(1.109\text{-}1)$$

$$\left.\begin{aligned}
&\cos 3.\delta c + \cos 6.\delta e - \delta d - c.\sin 3.\sin 1''.\delta 3 - \\
&\qquad - e.\sin 6.\sin 1''.\delta 6 + k_{d2} = 0 \quad \ldots\ldots \\
&\cos 3.\delta d + \cos(4+5).\delta e - \delta c - d.\sin 3.\sin 1''.\delta 3 - \\
&\qquad - e.\sin(4+5).\sin 1''.(\delta 4+\delta 5) + k_{c2} = 0 \quad \ldots\ldots \\
&\cos(4+5).\delta c + \cos 6.\delta d - \delta e - d.\sin 6.\sin 1''.\delta 6 - \\
&\qquad - c.\sin(4+5).\sin 1''.(\delta 4+\delta 5) + k_{e2} = 0 \quad \ldots\ldots
\end{aligned}\right\} \ldots(1.109\text{-}2)$$

$$\left.\begin{aligned}
&\cos 5.\delta e + \cos 8.\delta f - \delta b - e.\sin 5.\sin 1''.\delta 5 - \\
&\qquad - f.\sin 8.\sin 1''.\delta 8 + k_{b3} = 0 \quad \ldots\ldots \\
&\cos 5.\delta b + \cos(6+7).\delta f - \delta e - b.\sin 5.\sin 1''.\delta 5 - \\
&\qquad - f.\sin(6+7).\sin 1''.(\delta 6+\delta 7) + k_{e3} = 0 \quad \ldots\ldots \\
&\cos(6+7).\delta e + \cos 8.\delta b - \delta f - b.\sin 8.\sin 1''.\delta 8 - \\
&\qquad - e.\sin(6+7).\sin 1''.(\delta 6+\delta 7) + k_{f3} = 0 \quad \ldots\ldots
\end{aligned}\right\} \ldots(1.109\text{-}3)$$

$$\left.\begin{aligned}
&\cos 7.\delta f + \cos 2.\delta a - \delta d - f.\sin 7.\sin 1''.\delta 7 - \\
&\qquad - a.\sin 2.\sin 1''.\delta 2 + k_{d4} = 0 \quad \ldots\ldots \\
&\cos 7.\delta d + \cos(1+8).\delta a - \delta f - d.\sin 7.\sin 1''.\delta 7 - \\
&\qquad - a.\sin(1+8).\sin 1''.(\delta 1+\delta 8) + k_{f4} = 0 \quad \ldots\ldots \\
&\cos 2.\delta d + \cos(1+8).\delta f - \delta a - d.\sin 2.\sin 1''.\delta 2 - \\
&\qquad - f.\sin(1+8).\sin 1''.(\delta 1+\delta 8) + k_{a4} = 0 \quad \ldots\ldots
\end{aligned}\right\} \ldots(1.109\text{-}4)$$

where the component triangles are:

triangle ABC = (1), triangle BCD = (2)

triangle CDA = (3), and triangle DAB = (4)

An advantage of using these condition equations (1.109) over other sets, such as Murphy's [71], is that the coefficients of angle corrections are the same in each equation. For example in triangle (1), figure 1.15, we have:

$$\frac{c}{\sin 1} = \frac{a}{\sin 4}$$

i.e. $c.\sin 4 = a.\sin 1$

where $c.\sin 4$ is the coefficient of $\delta 4$ and $a.\sin 1$ is the coefficient of $\delta 1$ in the first equation in the set (1.109-1). Thus the amount of computation involved in making use of these coefficients will be greatly reduced.

1.6.3. Relative Weights Applied with Condition Equations (1.109)

The fundamental requirement of the least squares method is that the sum of the squares of the residuals shall be a minimum, (or if weights are used, the sum of the weighted squares of the residuals shall be a minimum). It is also known that only similar quantities can be summed up in this way. The agreement between the solution of equations (1.109) and the concept of the least squares is given by the following:-

(1) To accept the principle of adjusting the dissimilar quantities in a single least squares solution, the explanation given by Lilly [63] is very good, as it reduces the dissimilarity by simply dividing each quantity by an error of the same dimensions. In this case there is no need to use "the sum of the weighted squares of the residuals" as a concept of least squares, as the "sum of the squares of the reduced residuals" will be more suitable.

(2) Different probable errors can be assumed, but the one to be preferred is that requiring less computation, on the condition that results obtained by this solution must be just as good from practical point of view as results obtained by other assumptions such that of Lilly [63].

The assumption of the same relative weight for angles and sides depends on the fact that the same accuracy is achieved in measuring sides and angles in every type of net, i.e. in primary nets the probable error is assumed to be 1" in every angle and 1 ft. for each side, or one part in 206 265. Similar assumption may be made for secondary nets, (see sec. <u>1.3.4.</u>)

(3) The solution by least squares method will then give undimensioned corrections to angles and sides, which need to be multiplied by 1" for angles and 1 ft. for sides to give practical corrections. In fact this will not require any extra work.

Corrections obtained from the new set of condition equations are examined in the following section.

1.6.4. Examples

Table 1.10 gives all the data obtained from the field with reference to figure 1.14.

Plane observed angles				Plane observed sides	
	°	'	"		
1	32	18	19.05	a	69 847.62 ft.
2	63	19	25.20	b	83 587.77 ...
3	27	42	17.73	c	44 679.24 ...
4	56	39	56.65	d	102 017.34 ...
5	77	14	02.48	e	65 824.23 ...
6	18	23	44.13	f	94 277.10 ...
7	41	27	12.40		
8	42	55	01.88		
2+3	91	01	43.36		
4+5	133	53	59.64		
6+7	59	50	56.78		
8+1	75	13	12.10		

table 1.10

The adjustment of triangle (4) of the quadrilateral, figure 1.14, gives the following results in table 1.11, which also compares them with those obtained by Murphy for the same problem.

Corrections	from equation (1.109)	Due to Murphy
δf	- 0.12 ft.	- 0.11 ft.
δd	+ 0.33 ...	+ 0.36 ...
δa	- 0.32 ...	- 0.17 ...
$\delta 2$	+ 0.43"	+ 0.39"
$\delta(8+1)$	+ 0.35"	+ 0.08"
$\delta 7$	+ 0.52"	+ 0.83"

table 1.11

Applying the set of conditions newly obtained to adjust the whole quadrilateral, figure 1.14, results obtained and compared with those obtained by Murphy, Lilly, and Thornton-Smith are given in table 1.12.

Correction	Present solution	Murphy's solution	Thornton-Smith's sol.	Lilly's solution
δa	- 0.44	- 0.31	- 0.39	- 0.45
δb	+ 0.16	+ 0.24	+ 0.21	+ 0.18
δc	+ 0.10	+ 0.03	+ 0.07	+ 0.12
δd	+ 0.31	+ 0.31	+ 0.33	+ 0.35
δe	- 0.08	- 0.05	- 0.04	- 0.07
δf	- 0.15	- 0.12	- 0.12	- 0.13
δ"1	+ 0.58	+ 0.27	+ 0.45	+ 0.66
δ"2	+ 0.46	+ 0.54	+ 0.48	+ 0.42
δ"3	+ 0.07	+ 0.03	+ 0.09	+ 0.08
δ"4	- 0.16	- 0.08	- 0.08	- 0.22
δ"5	- 0.50	- 0.56	- 0.54	- 0.49
δ"6	- 0.87	- 1.05	- 0.98	- 0.87
δ"7	+ 0.29	+ 0.53	+ 0.37	+ 0.24
δ"8	- 0.03	- 0.05	0.00	- 0.02

table 1.12

<u>Comparison of Results</u>

i) These results when compared with those of Murphy can be seen to be slightly different. They are however identical to those of Thornton-Smith and Lilly. Thornton-Smith obtained his by using the same basic information, first adjusting angles and then using the difference between the side values derived from the adjusted angles and the observed sides to obtain the corrections to the sides. This avoids the problem of selecting relative weights. Lilly used his method of simultaneous adjustment already discussed (sec. 1.3.2.1.), allocating a probable error 0.6 secs. for an angle, and 1:200 000 for sides.

ii) For all the triangles, the corrections for each angle are summed to see if they satisfy the usual angle condition (180°) which has not been mentioned in the new solution. In each case, there is only negligible departure from this.

iii) When the adjusted sides and angles are applied to the basic conditions,

$$a.\cos B + b.\cos A - c = 0,$$

$$a.\cos C + c.\cos A - b = 0,$$

$$b.\cos C + c.\cos B - a = 0.$$

For triangle (1) of the quadrilateral in the figure, the following results are obtained:

	present sol.	Murphy's sol.
a.cos1 + c.cos4 - b =	- 0.01	+ 0.07
b.cos1 + c.cos(2+3) - a =	0.00	0.00
a.cos(2+3) + b.cos4 - c =	+ 0.02	+ 0.02

iv) The present corrections have been obtained directly in the required dimensions, whereas with the other solutions extra work is required to reach the final answers.

v) In triangle ABD which has been adjusted twice, four sets of results are given in table 1.13 for comparison;

Corrections	Adjustment of triangle		Adjustment of doubly braced quadrilateral	
	(1) present	(2) Murphy	(3) present	(4) Murphy
δf ft.	- 0.12	- 0.12	- 0.15	- 0.13
δd ...	+ 0.34	+ 0.36	+ 0.32	+ 0.32
δa ...	- 0.34	- 0.57	- 0.44	- 0.32
δ"2	+ 0.43	+ 0.39	+ 0.46	+ 0.55
δ"(8+1)	+ 0.35	+ 0.08	+ 0.54	+ 0.21
δ"7	+ 0.52	+ 0.83	+ 0.30	+ 0.54

table 1.13

The agreement between the set of results given by (1) and those given by either (3) or (4) is much better than the agreement between these and the set of results given in (2). In particular the corrections to the angles is triangle ABD given in (2) are quite different to those obtained when solving the doubly braced quadrilateral. The corrections to side a are quite different too.

Although the results obtained by satisfying the geometric condition of a triangle should be different to those obtained by satisfying the geometric condition for the quadrilateral, this difference should not be very large. The fact that the same element is observed once only does not allow two different corrections to be applied, especially if this difference is too large. The set of equations which provides two corrections for the same element which are only slightly different should be preferred.

It could be seen that corrections given in column (1) show better agreement to those given in columns (3) and (4) than that given in column (2). This agreement between corrections obtained for the same elements either by solving the triangle or the quadrilateral when using the set of conditions derived here leaves no doubt that they are more satisfactory for the adjustment of triangles when both angles and sides have been observed.

If we use the estimate for the coefficient of correlation between the system of real errors and the least squares corrections,

$$\gamma = \sqrt{n_c/n}$$

where n = number of observed quantities,

n_c = number of condition equations.

For a triangle with all sides and angles observed,

$$\gamma_t = \sqrt{3/6} = 0.7$$

For a doubly braced quadrilateral with all sides and angles observed we have $\gamma_q = \sqrt{9/14} = 0.8$.

It is clear that the corrections obtained by adjusting a quadrilateral should be nearer to the real errors, but if the adjustment of a triangle gives closely similar corrections as shown above, it indicates that it may not be necessary to form quadrilaterals. The difference in the coefficient of correlation which equals 0.10 has much more effect in Murphy's solution, which gives priority to the use of the doubly braced quadrilaterals. But in this new set of conditions there is no real advantage in using quadrilaterals, as triangles will give similar results, as well being simpler and easier to adjust. Also from the economy point of view, it would be more economic to observe triangles instead of spending time and money in forming doubly braced quadrilaterals for slightly more consistent results.

1.6.5. Computation of the Side Corrections in the Successive Adjustment Suggested by Thornton-Smith

Thornton-Smith finds that solution of nine equations for the adjustment of a doubly braced quadrilateral represents a herculean task [116]. To avoid this enormous labour, he offers an alternative for the solution of this problem. In his solution adjusted angles are used to obtain the required corrections to observed sides.

Mention has been made in 1.6.3. above of the identity of the results obtained using the present solution with those obtained by Thornton-Smith using his method. This identity created the possibility of deriving a formula for adjusting sides in the case of pre-adjusted shape.

1.6.5.1. Condition Equations for Adjusting Sides of the Doubly Braced Quadrilateral of Pre-adjusted Shape

For the purpose of simplification, adjusted angles obtained from a classical triangulation problem are used. For example, the doubly braced quadrilateral, figure 1.14, is adjusted by solving four condition equations. The corrected angles obtained are then inserted in equations (1.109).

The condition equations to be solved are those which have been used in (1.109), with the modification that as previously adjusted angles have been used, the corrections $\delta 1, \delta 2, \ldots, \delta 8$ are zero. Equation (1.108) will now have the form:

$$\delta a.\cos B + \delta b.\cos A - \delta c + k_c' = 0 \qquad \ldots\ldots(1.110c)$$

where $k_c' = a.\cos B + b.\cos A - c$ ...etc.

The equations derived for the doubly braced quadrilateral, figure 1.14, corresponding to this case will be:

$$\cos 1.\delta a + \cos 4.\delta c - \delta b + k'_{b1} = 0 \qquad \ldots\ldots(1.111\text{-}1.1)$$

$$\cos 1.\delta b + \cos(2+3).\delta c - \delta a + k'_{a1} = 0 \qquad \ldots\ldots(1.111\text{-}1.2)$$

$$\cos(2+3).\delta a + \cos 4.\delta b - \delta c + k'_{c1} = 0 \qquad \ldots\ldots(1.111\text{-}1.3)$$

$$\cos 3.\delta c + \cos 6.\delta e - \delta d + k'_{d2} = 0 \qquad \ldots\ldots(1.111\text{-}2.4)$$

$$\cos(4+5).\delta e + \cos 3.\delta e - \delta c + k'_{c2} = 0 \qquad \ldots\ldots(1.111\text{-}2.5)$$

$$\cos(4+5).\delta c + \cos 6.\delta d - \delta e + k'_{e2} = 0 \qquad \ldots\ldots(1.111\text{-}2.6)$$

$$\cos 5.\delta e + \cos 8.\delta f - \delta b + k'_{b3} = 0 \qquad \ldots\ldots(1.111\text{-}3.7)$$

$$\cos(6+7).\delta f + \cos 5.\delta b - \delta e + k'_{e3} = 0 \qquad \ldots\ldots(1.111\text{-}3.8)$$

$$\cos(6+7).\delta e + \cos 8.\delta b - \delta f + k'_{f3} = 0 \qquad \ldots\ldots(1.111\text{-}3.9)$$

$$\cos 7.\delta f + \cos 2.\delta a - \delta d + k'_{d4} = 0 \qquad \ldots\ldots(1.111\text{-}4.10)$$

$$\cos(8+1).\delta a + \cos 7.\delta d - \delta f + k'_{f4} = 0 \qquad \ldots\ldots(1.111\text{-}4.11)$$

$$\cos(8+1).\delta f + \cos 2.\delta d - \delta a + k'_{a4} = 0 \qquad \ldots\ldots(1.111\text{-}4.12)$$

The number of equations given in (1.111) is twelve, and should be reduced to only five equations. This is because in the quadrilateral of pre-adjusted shape with six measurements, only one of these is required to give the size, the other five being redundants, and thus five conditions only required.

In order to reduce the number of equations in (1.111) to the required number, it should be noticed that the unit error δd affects both triangles (2) and (4), thus by adding equations (1.111-2.4) and (1.111-4.10) the effect of unit error δd on the whole quadrilateral will be obtained. This could be done for all the other sides, which leads to six equations instead of the given twelve, as can be seen in figure 1.16.

These six equations will be:

$$\cos 1.\delta b + \cos(2+3).\delta c + \cos 2.\delta d + \cos(1+8).\delta f - 2.\delta a + (k'_{a1}+k'_{a4})=0$$

$$\cos 1.\delta a + \cos 4.\delta c + \cos 5.\delta e + \cos 8.\delta f - 2.\delta b + (k'_{b1}+k'_{b3})=0$$

$$\cos 6.\delta b + \cos(2+3).\delta a + \cos 3.\delta d + \cos(4+5).\delta e - 2.\delta c + (k'_{c1}+k'_{c2})=0$$

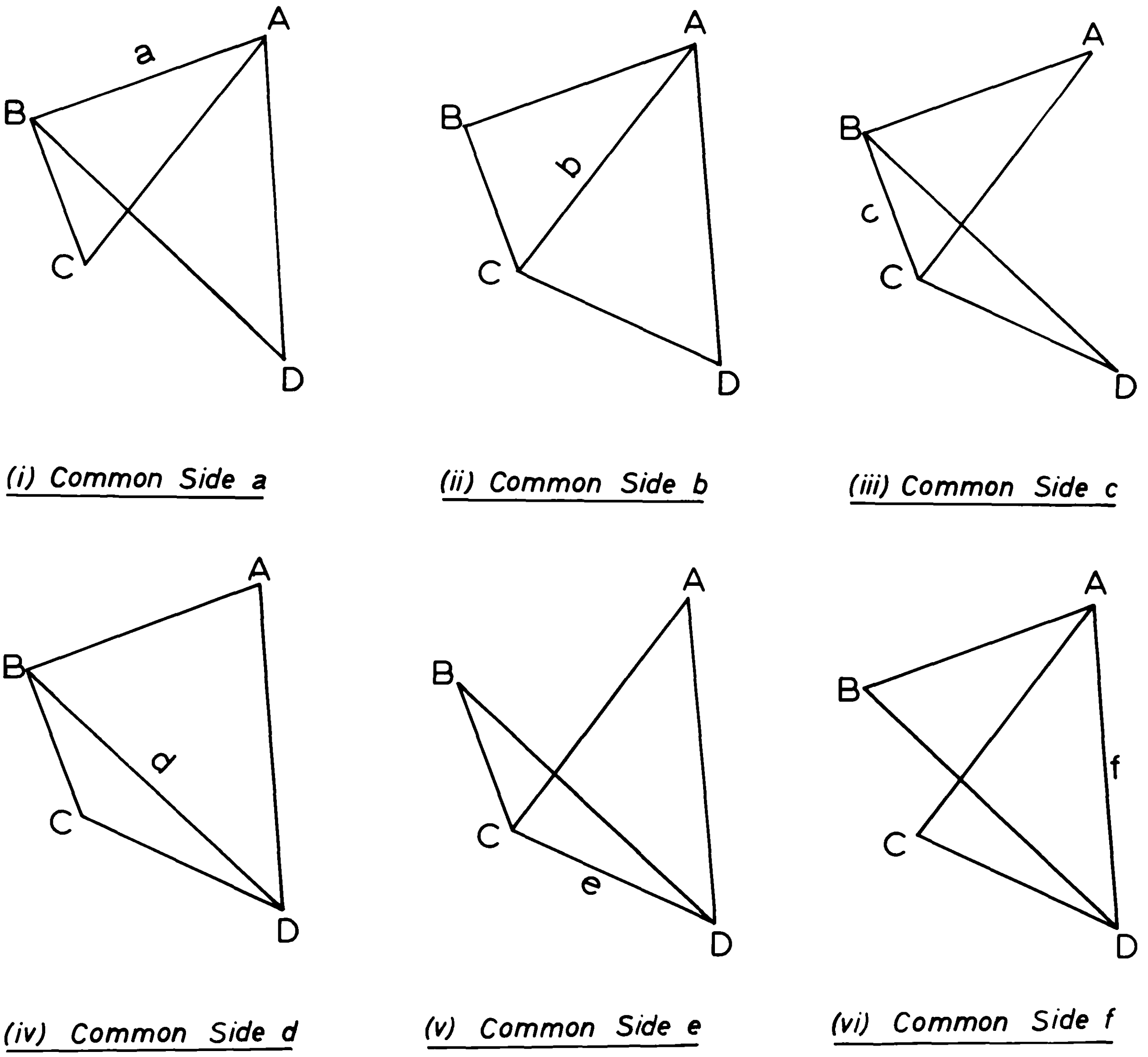
$$\cos 3.\delta c + \cos 6.\delta e + \cos 7.\delta f + \cos 2.\delta a - 2.\delta d + (k'_{d2}+k'_{d4})=0$$

$$\cos 6.\delta d + \cos(4+5).\delta c + \cos 5.\delta b + \cos(6+7).\delta f - 2.\delta e + (k'_{e2}+k'_{e3})=0$$

$$\cos 8.\delta b + \cos(6+7).\delta e + \cos 7.\delta d + \cos(8+1).\delta a - 2.\delta f + (k'_{f3}+k'_{f4})=0$$

$$\ldots\ldots(1.112)$$

The five required equations could be selected as, the two diagonals



(i) Common Side a *(ii) Common Side b* *(iii) Common Side c*

(iv) Common Side d *(v) Common Side e* *(vi) Common Side f*

Figure 1.16

and any other three of the other four.

The solution of the problem is obtained here, with the five equations for the common sides a, b, c, d, and f. Results given below in table 1.14, which also gives the comparable figures obtained from the original solution of equation (1.109) and those obtained by Thornton-Smith.

Corrections	From equation (1.112)	From equation (1.109)	By Thornton-Smith sol.
δa	- 0.45	- 0.44	- 0.45
δb	+ 0.15	+ 0.16	+ 0.18
δc	+ 0.13	+ 0.10	+ 0.12
δd	+ 0.33	+ 0.31	+ 0.35
δe	- 0.09	- 0.08	- 0.08
δf	- 0.15	- 0.15	- 0.13

table 1.14

It can be seen that when previously adjusted angles are used for the additional operation in the newly developed solution, corrections are obtained which are virtually the same as those resulting from a combined adjustment of sides and angles. There is also a close agreement between results obtained by this new solution and these of Thornton-Smith, with the difference that corrections here are obtained in an easy direct way.

1.6.5.2. Condition Equations for Adjusting Angles of the Doubly Braced Quadrilateral of Pre-Adjusted Size

The use of the adjusted sides for the adjustment of observed angles, i.e. adjustment of the pre-adjusted size, does not give the same advantages, as it requires the solution of the same number of equations as unknowns. In this case eight equations have to be solved, so that the number of equations is not reduced much from the nine equations which have to be solved in the simultaneous adjustment of the problem for angles and sides. Besides, the solution is based on the use of the adjusted sides which is known to be a rather weak solution as shown in the comparison 1.16.

Thus using equations (1.109) for the quadrilateral figure 1.14, for the adjusted sides and observed angles, we have the the following twelve equations:

$$a.\sin 1.\sin 1''.\delta 1 + c.\sin 4.\sin 1''.\delta 4 - k''_{b1} = 0 \quad ..(1.113\text{-}1.1)$$

$$b.\sin 1.\sin 1''.\delta 1 + c.\sin(2+3).\sin 1''.(\delta 2+\delta 3) - k''_{a1} = 0 \quad ..(1.113\text{-}1.2)$$

$$b.\sin 4.\sin 1''.\delta 4 + a.\sin(2+3).\sin 1''.(\delta 2+\delta 3) - k''_{c1} = 0 \quad ..(1.113\text{-}1.3)$$

$$c.\sin 3.\sin 1''.\delta 3 + e.\sin 6.\sin 1''.\delta 6 - k''_{d2} = 0 \quad ..(1.113\text{-}2.4)$$

$$d.\sin 3.\sin 1''.\delta 3 + e.\sin(4+5).\sin 1''.(\delta 4+\delta 5) - k''_{c2} = 0 \quad ..(1.113\text{-}2.5)$$

$$d.\sin 6.\sin 1''.\delta 6 + c.\sin(4+5).\sin 1''.(\delta 4+\delta 5) - k''_{e2} = 0 \quad ..(1.113\text{-}2.6)$$

$$e.\sin 5.\sin 1''.\delta 5 + f.\sin 8.\sin 1''.\delta 8 - k''_{b3} = 0 \quad ..(1.113\text{-}3.7)$$

$$b.\sin 5.\sin 1''.\delta 5 + f.\sin(6+7).\sin 1''.(\delta 6+\delta 7) - k''_{e3} = 0 \quad ..(1.113\text{-}3.8)$$

$$b.\sin 8.\sin 1''.\delta 8 + e.\sin(6+7).\sin 1''.(\delta 6+\delta 7) - k''_{f3} = 0 \quad ..(1.113\text{-}3.9)$$

$$f.\sin 7.\sin 1''.\delta 7 + a.\sin 2.\sin 1''.\delta 2 - k''_{d4} = 0 \quad ..(1.113\text{-}4.10)$$

$d.\sin 7.\sin 1''.\delta 7 + a.\sin(1+8).\sin 1''.(\delta 1+\delta 8) - k''_{f4} = 0$ ..(1.113-4.11)

$d.\sin 2.\sin 1''.\delta 1 + f.\sin(1+8).\sin 1''.(\delta 1+\delta 8) - k''_{a4} = 0$ ..(1.113-4.12)

Coefficients of $\delta 1$, $\delta 2$, ..., $\delta 8$ are the same as before in (1.109), because the effect of the adjusted sides is negligible. Only k''_{b1}, ..., k''_{a4} have different values to those previously obtained in (1.109). Eight equations out of the twelve given in (1.113) have to be solved. Equations 2,3,5,6,8,9,11, and 12 are chosen because they give a diagonal matrix representing observations of the angles in anti-clockwise direction, as follows:

$$\left.\begin{array}{l} \alpha_1(\delta 1 + \delta 2 + \delta 3) - k''_{a1} = 0 \quad ..... \\ \alpha_2(\delta 2 + \delta 3 + \delta 4) - k''_{c1} = 0 \quad ..... \\ \alpha_3(\delta 3 + \delta 4 + \delta 5) - k''_{c2} = 0 \quad ..... \\ \alpha_4(\delta 4 + \delta 5 + \delta 6) - k''_{e2} = 0 \quad ..... \\ \alpha_5(\delta 5 + \delta 6 + \delta 7) - k''_{e3} = 0 \quad ..... \\ \alpha_6(\delta 6 + \delta 7 + \delta 8) - k''_{f3} = 0 \quad ..... \\ \alpha_7(\delta 7 + \delta 8 + \delta 1) - k''_{f4} = 0 \quad ..... \\ \alpha_8(\delta 8 + \delta 1 + \delta 2) - k''_{a4} = 0 \quad ..... \end{array}\right\} ..(1.114)$$

α is the coefficient of the corrections in each respective equations, e.g. $\alpha_1 = a.\sin 1''.\sin 1 = c.\sin 1''.\sin 4$ in equation (1.113-1.1). The absolute term k" is taken as the difference between the sum of the projections of two sides of each triangle on the third and the length of the third side. For triangles (1), (2), (3), and (4) common sides a, c, e, and f are taken as the respective third sides.

The results of the angle corrections obtained by solving equation (1.114) compared with those obtained by solving (1.109) is given in table 1.15.

table 1.15

Corrections	Due to (1.109)	Due to (1.114)
$\delta''1$	+ 0.58	- 0.15
$\delta''2$	+ 0.47	+ 0.65
$\delta''3$	+ 0.07	- 1.04
$\delta''4$	- 0.17	+ 1.52
$\delta''5$	- 0.51	- 0.04
$\delta''6$	- 0.87	- 1.93
$\delta''7$	+ 0.30	+ 1.14
$\delta''8$	- 0.04	- 0.33

The corrections obtained are totally different to each other which does not allow the use of the set of conditions given in (1.114). These different results would appear to be due to the use of the adjusted sides obtained from a previous separate trilateration adjustment for the quadrilateral.

1.6.5.3. Comparison Between Methods of Adjustments

A comparison between all the possible solutions including those which could be obtained by using these newly developed methods is shown in table 1.16.

Let: I- Adjustment of observed angles in the quadrilateral with one known side left unadjusted, (the classical method of triangulation adjustment).

II- Adjustment of observed sides in the quadrilateral (a pure trilateration problem).

III- Simultaneous adjustment of I and II, using equations (1.109).

IV- Adjustment of size for the pre-adjusted shape, using equations (1.112).

V- Adjustment of shape for the pre-adjusted size, using equations (1.114).

For the doubly braced quadrilateral we have:

Method	No. of redundant observations	No. of unknowns	Estimate for coefficient of correlation
I	4	8	0.7
II	1	6	0.4
III	9	14	0.8
IV	5	6	0.9
V	8	8	1.0

table 1.16

It should be mentioned that, for I, II, and III the estimate for the coefficient of correlation is true, as observed quantities are free correlated quantities. For IV, and V adjusted quantities are correlated quantities, hence the estimate for the coefficient of correlation will be affected by the estimate for the coefficients of correlation previously obtained for I and II. Thus if:

0.7 I = estimate for the coefficient of correlation in I,

and 0.4 II= .... II,

therefore,

0.9x0.7 I = 0.63 I = estimate for the coefficient of correlation in IV,

and

1x0.4 II = 0.4 II = estimate for the coefficient of correlation in V.

From the above comparison, methods of adjustment can be re-arranged in the following order of merit from the point of view of the estimate for the coefficient of correlation:

1) III-for angles and sides.
2) I-for angles only.
3) IV-for sides only, using equation (1.112).
4) II-for sides, when sides are observed only.
5) V-this solution does not have any real advantage as has been explained already.

Equation (1.109) is recommended for the solution of III, on the following grounds:-

(i) Corrections for angles are very close to those obtained by solution I, i.e. the introduction of sides does not distort the original solution by I, (see Rainsford, sec. 1.3.2.1.)

(ii) Corrections to sides are close to those obtained by IV.

(iii) It allows the easy deriviation of a further method (IV), which simplifies the task of solving nine condition equations for the quadrilateral, if an electronic digital computer is not available for the purpose.

Chapter II

STRUCTURAL AND MECHANICAL MODELS OF ADJUSTMENT

2. INTRODUCTION

Mathematics is the logic language, by which different physical phenomena in different fields in science are explained. In Surveying some problems may be mathematically represented in forms, which are popular in some other fields such as mechanics, electricity, and structures. In particular, observations in surveying are liable to errors, causing deformation in the nets and errors in the positions. Corrections obtained by the least squares method are found to be the solution nearest to the real errors [78], i.e. the corrected figure by this method will have the sum of the squares of residuals minimum. In mechanics, and structures, an elastic loaded frame reaches the equilibrium position at the state of minimum strain energy. That is to say the configuration which gives the least strain energy. Also in electricity, the balance of an electrical net is obtained with the least amount of energy conserved.

2.1. ADJUSTMENT OF TRILATERATION BY GRAPHICAL METHODS

A graphical method may be introduced to help obtain the coefficients for the correction equations, but the corrections themselves will be obtained by the usual way of solving the normal equations.

For trilateration Thornton-Smith [113] uses the calculated coefficient of one unknown to construct a graphical figure which will give the remaining coefficients. This is done in the following way:

For the quadrilateral ABCD figure 1.3, the ratio between the different coefficients is given by:-

$$k_d : -k_a : -k_c = \sin B_1 : \sin B_2 : \sin B_4$$

$$k_b : -k_c : -k_e = \sin C_2 : \sin C_3 : \sin C_1$$

$$k_d : -k_e : -k_f = \sin D_3 : \sin D_4 : \sin D_2$$

and to start with, one of these coeffieients has to be calculated.

The original quadrilateral ABCD is then drawn to the right hand side so that an additional figure may be constructed to the left.

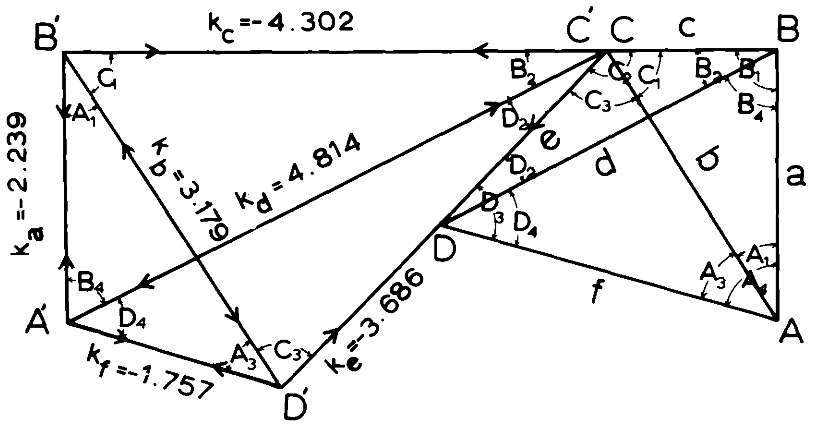


Figure 2.1

Corner C is selected, in the same way as apex was selected for the numerical solution of the problem [70]. BC' is produced towards B'. A line C'A' parallel to BD is drawn from C'. A' is fixed after scaling on the length C'A' equal to the computed coefficient.

$$k_d = \frac{1}{c} \cdot \sin B = 4.814$$

B' and D' will be fixed by drawing parallels to AB and DA from A'. Lastly the values of the coefficients other than k_d are obtained by scaling off the values of the other sides.
Plotting of the vector diagram is given in figure 2.1

The graphical plotting of the vector diagram is new to surveying but it has been used for some time in the theory of structures under the name of the force polygon, to obtain the axial forces in the pin-jointed elastic frameworks. However in the survey application by Thornton-Smith instead of using unit coefficient to decide the scale, the actual value is used which will give the required coefficients directly. Furthermore the sign convention followed in a survey networks is similar to that used in the theory of structures. In the latter, equilibrium at each joint results from having equal and opposite forces acting at this joint. This means that tension (or compression) must exist in the diagonal with positive (or negative) signs, while the forces in the other two members meeting and flanking the diagonal at the same joint have opposite signs, in this case compression (or tension) with the appropriate negative (or positive) signs. In Thornton-Smith's method the sign convention is explained by his statement " At each of the four corners the directions on a diagonal must be balanced by the opposite directions on the two sides flanking it".

2.2. SURVEY NETWORKS AND STRUCTURAL ANALOGY

The analogy between an observed survey net and a pin-jointed or rigidly-jointed elastic frame-structure, is used for the adjustment, when the latter structure has redundant members with an initial lack of fit.

From the point of view of the analogy between survey nets and structural problems, connections and joints can be divided into the following:-

a- Pin-Jointed elastic frameworks: in which the frame rigidity is obtained by deciding the length of the component members. Rotation of the involved sides around hinges is allowed so that no moments resulting from the structure can exist. At the same time movement of the hinges is restricted by the chosen elastic properties. This case is similar and directly analogous to trilateration networks in surveying, where small changes in the angles due to the errors or corrections applied to the observed sides are allowed.

b- Rigidly-Jointed elastic frameworks: where the angles of the joints are in a fixed configuration, i.e. each angle might rotate as a whole but has a fixed size. These joints are designed to take moments. This case is similar and directly analogous to a survey net in which both angles and sides have been observed. The angles are adjusted first to give the required shape, then the sides are adjusted. Corrections to the sides are in this case allowed only on condition that they will not affect the corrected angles.

Linkwitz in [64] has suggested that angles can be treated in an exactly similar way to the way sides are treated in his thesis. A full investigation has been made to find out whether such a possibility exists.

To establish the analogy and hence to use structural methods for adjusting triangulation nets the following comparison is made,

	Mathematical Model	Structural Model
1	β is the angle observed	β is the length of an elastic arc.
2	$\delta\beta$ is the error of observation of angle β	$\delta\beta$ is the change in the length of the elastic arc.
3	P_i any station in the survey net.	P_i corresponding nodal point in the framework.
4	p is the weight of observation.	f is the elasticity coefficient.
5	A triangulation net with just necessary angles and one side to allow the net to be defined in the field.	An elastic framework to a given scale without any redundancy, i.e. statically determinate structure.
6	A triangulation net with r-excess observed angles, which gives r condition equations.	An elastic statically indeterminate framework with r-times indeterminacy.

table 2.1

Applying the analogy given in table 2.1 to a doubly braced quadrilateral the following comparison can be made:-

(1) In surveying a quadrilateral figure 2.2a is obtained by the intersection of rays produced from each station P_i applying the angles 1, 2, ..., 8 after determination of scale. Four angles and the scale are necessary to define this figure, while the other four angles are redundants.

(2) In the corresponding structure there are two possibilities:-

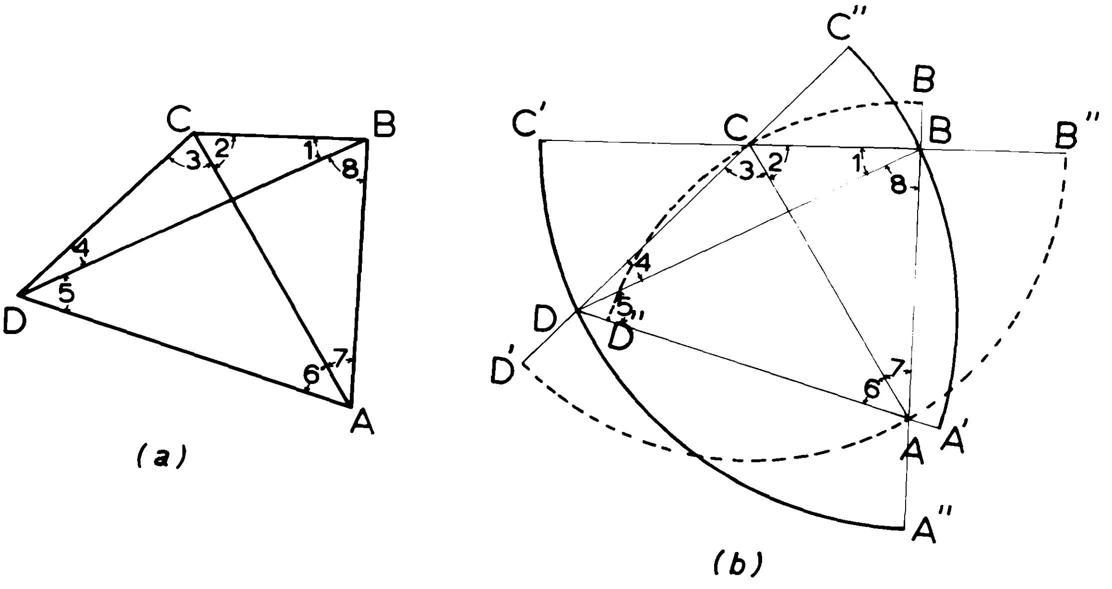


Figure 2.2

i- Circular elastic arcs represent the angles which are provided with strictly straight sides (members) which are heavy enough to resist deformation and sufficiently light to move freely without affecting the positions of the points. For this particular figure, due to the different length of sides, purely circular arcs cannot be constructed between the different nodal points, e.g. in figure 2.2b if diagonal BD is known and can decide the scale, arc 1 can be constructed between D and C' to represent angle 1. Again arc 4 is constructed between B and C" to represent angle 4. From B and D two arcs 8 and 5 are constructed to give the arcs through A' and A" respectively. C and A are thus obtained by the intersection of the corresponding radii from B and D.

For a purely structural and physical point of view it is impossible to have all of these arcs in one plane without intersection, as can be seen from the diagram of the situation for a single point B. To overcome this difficulty the connection of the four elastic arcs for nodal point B may be made in two planes figure 2.3. In figure 2.3a a plan of the connection at the nodal point B

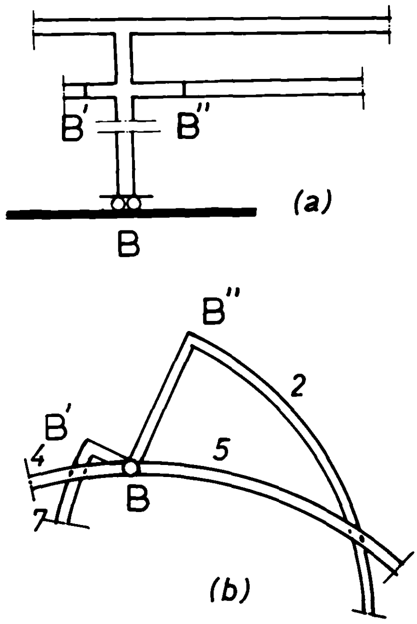


Figure 2.3

is shown, and in figure 2.3b the elevation of this connection is shown. In figure 2.3a the top plane gives the plane of the two elastic arcs 4 and 5, while the other plane gives that the of the two rigid arms BB' and BB". The elastic change of arcs 1, 2, ...,8, which have to be considered for the adjustment requires the free nodal point B to move while the spindle is kept vertical. This cannot occur unless the two planes in figure 2.3a coincide which appears to be a physical possibility without the elastic arcs being intersected, as shown in figure 2.3.

However, construction of such as that in figure 2.2 has to be subjected to thrust only, in order to have axial deformation and angular change. Construction of such kind do exist either as arches or rings, and the illustrated figure does not correspond to either type.

ii- The second possibility is of a rigid arc constructed at each nodal point (joint). This case is previously given in (b) above where joints should be rigidly connected and deformation of sides are allowed only.

In the case where an analogy is possible a direct application of the theory of strain energy is involved.

2.2.1. <u>Basic Strain-Energy</u>

Work and energy relations are found in many fields of science and in the field of structural mechanics, which has adapted many of these relations and concepts to determine the slope and deflections of elastic members.

For elastic structures, external work is done on the structure by physically applied external forces or moments which results in an equal amount of potential energy being stored through the action of the internal forces and the elastic strain mechanics of the structure. The loads are assumed to be slowly applied so that the dynamic influence may be omitted.

The work done by the constantly applied force is equal to the magnitude of the force multiplied by the distance through which the point of application of the force moves in the direction of the force. The work is positive if the displacement is in the direction of the force, and negative if the displacement is opposite to the direction of the force.

To explain the relationship between work done and potential energy a simple example figure 2.4 may be given.

L is the length of the member used,

A is the cross-sectional area of the member.

E is the modulus of elasticity of the material of the member.

P is the applied load.

$\Delta = \frac{PL}{AE}$

The load is initially applied at B, and the resisting force is slowly built up to the value of P, which is reached at B'. By this time the load is fully applied to the member, and the member is elongated by Δ.

The work done is $= P.\frac{\Delta}{2}$, but the loss in potential energy is given by $P.\Delta$, i.e. it is twice the work done on the member. It appears therefore that a discrepancy exists until it is required that half the loss in potential energy is utilised in doing work. The strain-energy stored in the member equals the net change in potential energy, i.e. the internal energy is equal to the external work done.

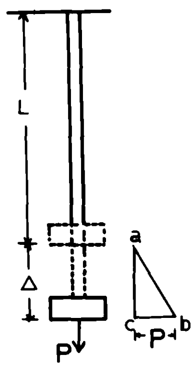


Figure 2.4

The triangle abc figure 2.4 shows that, the force in the elastic member is directly proportional to the deformation, and that the area of the triangle represents the strain-energy = $P.\frac{\Delta}{2} = P^2.\frac{L}{2AE}$ stored internally in the member.

A pin-jointed elastic framework with redundant., forced in to its appropriate position will produce various elements of strain in the different members of this structure. This is due to the lack of fit due to the redundant members being longer or shorter than necessary. A problem of such character could be solved by Castigliano's theorems [56]. The linear displacements in the various bars of the analogic frame-structure correspond to the corrections to the relevant surveying lines.

Professor Southwell [91] has extended the use of Castigliano's theorems to many problems in science and engineering, including the adjustment of level nets. Professor Black has extended this to the adjustment of directions [10], and it is apparent that it can be extended still further.

2.2.2. Use of Castigliano's Theorems in Trilateration

Castigliano's second theorem states that "The stress-distribution resulting from given forces, applied to a body initially in a state of ease, can be deduced from the conditions of equilibrium combined with the conditions for a minimum value of U"[95].

Using figure 2.5 we have:

$$\delta_1 = \frac{\partial U}{\partial P_1}$$

which means that the deflection at a particular point 1 is equal to the rate of change of total energy"for all members" with respect to the force P_1. The use of the partial derivative means

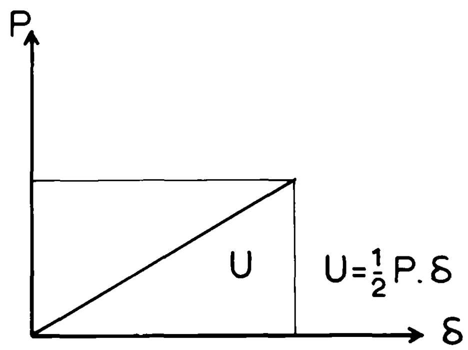


Figure 2.5

that all other forces acting on the structure are assumed to remain constant while P_1 is varied by small amount. The purely linear relationship between P and δ shown in figure 2.5 represents Hooke's law which states that "within the limits of elasticity the strain produced by a stress of any one kind is proportional to the stress producing it".

The use of this theorem for trilateration adjustment is given by Leung Kui-Wai, [56].

Suppose that λ_r is the misfit in the redundant member r, i.e. the difference between calculated and inserted sides. With reference to figure 2.6 according to Castigliano,

$$\lambda_r = \frac{\partial U}{\partial S_r} \quad ....(2.1)$$

where S_r is the axial force in the redundant member r.

U is the strain-energy in the whole structure $= \Sigma \frac{S_i^2 . l_i}{2A_i E_i}$ ....(2.2)

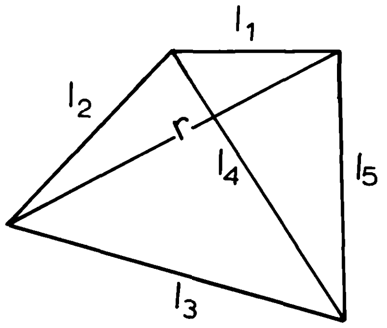


Figure 2.6

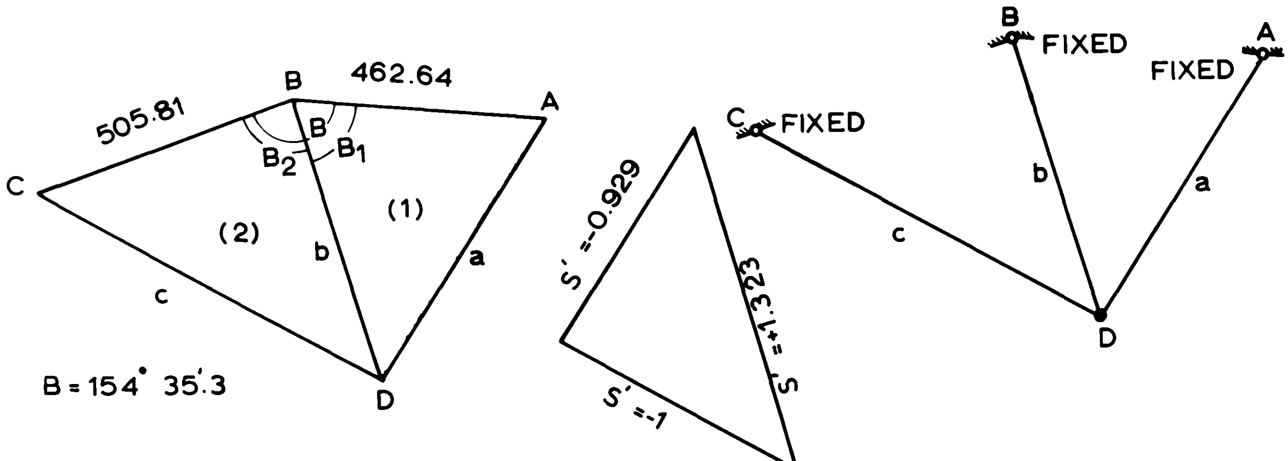
S_i is the axial force in the member i, where i = 1,...,6

l_i is the length of the member i.

$A_i E_i$ is the rigidity of the side i, (A_i is the cross-sectional area of the member i, and E_i is the elastic modulus of the material).

$$\text{therefore} \quad \frac{\partial U}{\partial S_r} = \Sigma \frac{S_i . l_i}{A_i . E_i} \frac{\partial S_i}{\partial S_r} = \Sigma \frac{S_i . l_i}{A_i . E_i} . s_i' = S_r . \Sigma \frac{(s_i')^2 . l_i}{A_i . E_i} = \lambda_r \quad ....(2.3)$$

s_i' is the force in any member introduced by a unit force in the redundant member r, and obtained from the force polygon figure 2.7



(a) SURVEY NET (b) FORCE DIAGRAM (c) ANALOGIC FRAME

Figure 2.7

For the use in trilateration adjustment the above formula (2.3) can be used in the following way:
Force in side c due to misfit of λ_c is given by

$$S_c = \frac{A_i.E_i.\lambda_c}{\Sigma(s_i')^2.l_i} \qquad ....(2.4)$$

hence,

$$v_a = \frac{S_a.l_a}{A_a.E_a} = S_c.s_a'.l_a, \qquad v_b = \frac{S_b.l_b}{A_b.E_b} = S_c.s_b'.l_b,$$

$$\text{and} \quad v_c = \frac{S_c.l_c}{A_c.E_c} = S_c.l_c \qquad ....(2.5)$$

A tensile force is reckoned positive and a compressive force negative. If the length of the bar c is too long, then it needs to be shortened and hence compressive force is required, thus the negative sign is given.

For more complicated problems, when more than one redundant exists, the effect of unit force in each redundant should be considered. A set of normal linear equations of the same number as the redundants will exist to determine the actual forces in the redundant sides. From these the displacements and hence the corrections may be applied. The adjustment of trilateration nets and error analysis according to the theory of elastic system is given by Professor Linkwitz [64] in his doctoral thesis.

2.2.3. Systematic Relaxation of Constraints

Mechanical analogue for survey adjustment problems may be based on the systematic relaxation of constraints method adopted for solving linear equations which is a Seidelian iterative method of computation. For an explanation of this one may quote Prof. Southwell [95] . "The method of Systematic Relaxation of Constraints was devised for the determination of stresses in frameworks- that is in elastic structures having the characteristic that a strained configuration can be specified by attaching values to a finite number of co-ordinates. Recently it has been extended to continuous systems (e.g. beams) on the understanding that a finite number of co-ordinates will define a configuration for practical purposes, though not from a mathematical standpoint. So far the power of the method has been exhibited only in relation to elastic problems: in these its results appear to converge rapidly, judged by a few examples of which the exact solutions were known".

That convergence takes place in this method may not be obvious but it must be remembered that in any problem of equilibrium we are concerned with a configuration of minimum energy. At every step in the relaxation process, if positive work is done on the relaxed constraint, the total energy of the system (i.e. the strain-energy stored in the framework plus potential energy of the external forces) will be reduced. Therefore the system must tend towards the required configuration of equilibrium, in which this total energy has its minimum value. The required configuration can be approached as closely as possible depending on the accuracy needed. The approach to the required configuration can be accelerated by using "block relaxation" in which case any number of points can move together as a rigid body [81].

The advantages of using this method in solving structural problems and certain problems of adjustment in surveying have been given by Southwell [95] for the following:

(a) It obviates entirely the necessity of solving simultaneous equations which is the main object to existing methods.
(b) It is simple to apply and involves only a few standard numerical processes, easy to grasp and readily checked.
(c) Its complexity is not dependent on the order of the redundancy and the time required for a solution, although it increases with the number of joints in the framework, does not increase rapidly as it does when simultaneous equations are involved.
(d) The joint displacements are calculated simultaneously with the action of the members.
(e) The physical meaning of each process is clear, and the order of the approximation can be judged at every stage, when this is deemed sufficient the solution can be stopped.

2.2.3.1. Dorivation of the Method for Structural Calculation

Consider figure 2.8 where an external force $\bar{X}$ is acting at joint A. To achieve equilibrium $\bar{\bar{X}} = \bar{X} + X = 0$ .....(2.6)

where, X is the internal force (exerted by the framework)

$\bar{\bar{X}}$ is the residual force

= 0 , at equilibrium.

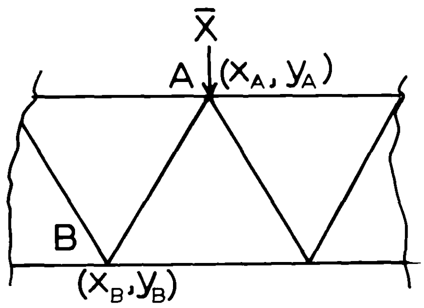


Figure 2.8

For the general case, A and B can be given coordinates (x_A, y_A) and (x_B, y_B). The force which $\bar{X}$ exerts on the joint A in the direction of x has the component,

$$\bar{X}_{AB}.\frac{x_B - x_A}{l_{AB}} = X_{AB}(x_B - x_A) \qquad \text{......(2.7)}$$

If B, C, ..., M are joints connected to A by members, then equation (2.6) will be:

$$\bar{X}_A + X_{AB}.(x_B - x_A) + X_{AC}.(x_C - x_A) + \ldots + X_{AM}.(x_M - x_A) = 0 \qquad \text{.....(2.8)}$$

Also, let u and v be the component displacements of A in the directions x and y respectively under the load, therefore the fractional extension of the member AB will be:

$$e_{AB} = \frac{1}{l_{AB}^2}[(x_B - x_A)(u_B - u_A) + (y_B - y_A)(v_B - v_A)] \quad \text{..(2.9)}$$

However the strain-energy is equal to $X.\frac{e_{AB}}{2}$.

If U is the strain-energy and V is the potential energy, then

$$U = \frac{1}{2}\sum_B^M \left\{R_{AB}[(x_B - x_A)(u_B - u_A) + (y_B - y_A)(v_B - v_A)]^2\right\} \qquad \text{.......(2.10)}$$

and $V = \text{constant} - \Sigma_j(\bar{X}_A.u_A + \bar{Y}_A.v_A)$ by definition ....(2.11)

Initially when all joints are held fixed $\bar{\bar{X}} = \bar{X}$, but when one joint or all joints in the case of "block relaxation", is relaxed, force $\bar{X}$ will be equal and opposite to the force exerted by the constraint upon the framework, by Castigliano's first theorem,

where $X_A = -\frac{\partial U}{\partial u_A}$, and $\bar{X}_A = -\frac{\partial V}{\partial u_A}$

Substituting in (2.6) which is the condition of equilibrium, we have:

$$\bar{\bar{X}}_A = - \frac{\partial}{\partial u_A}(U + V) = 0 \quad \ldots\ldots\ldots\ldots\ldots(2.12)$$

which is also a condition of the minimum energy.

Since the potential energy is a linear function of the same displacements with constant coefficients therefore:

$$\begin{aligned} \frac{\partial \bar{\bar{X}}_A}{\partial u_A} &= \frac{\partial X_A}{\partial u_A} = - \frac{\partial^2 U}{\partial u^2} \\ \frac{\partial \bar{\bar{X}}_A}{\partial v_A} &= \frac{\partial X_A}{\partial v_A} = - \frac{\partial^2 U}{\partial v_A . \partial u_A} \\ \frac{\partial \bar{\bar{X}}_A}{\partial u_B} &= \frac{\partial X_A}{\partial u_B} = - \frac{\partial^2 U}{\partial u_B . \partial u_A} \end{aligned} \quad \ldots\ldots\ldots(2.13)$$

Since U is a quadratic function of the displacements u and v, therefore,

$$\begin{aligned} \frac{\partial^2 U}{\partial u_A^2} &= \Sigma_A[R_{AB}(x_B - x_A)^2] = \Sigma_A[(x,x)_{AB}] \\ \frac{\partial^2 U}{\partial v_A . \partial u_A} &= \Sigma_A[R_{AB}(x_B - x_A)(y_B - y_A)] = \Sigma_A[(x,y)_{AB}] \\ \frac{\partial^2 U}{\partial u_B . \partial u_A} &= - R_{AB}(x_B - x_A) = - [(x,x)_{AB}] \end{aligned}$$

and therefore equation (2.12) will be:

$$\bar{\bar{X}}_A = \bar{X}_A - u_A . \frac{\partial^2 U}{\partial u_A^2} - v_A . \frac{\partial^2 U}{\partial v_A . \partial u_A} - u_B . \frac{\partial^2 U}{\partial u_B . \partial u_A} - \quad \ldots\ldots\ldots(2.14)$$

Corresponding expressions can be derived for the residual force at B.

According to this equation residual force $\bar{\bar{X}}$ will be brought to zero by imposing displacement,

$$\Delta u_A = \bar{\bar{X}}_A / \frac{\partial^2 U}{\partial u_A^2} \quad \ldots\ldots\ldots(2.15)$$

Because the strain-energy is necessarily a positive quantity, the coefficients of u_A^2 , u_B^2 , etc.. in U will be always positive, and so will be such differential coefficient as $\partial^2 U/\partial u_A^2$.

Hence according to equation (2.15) any residual force can be brought to zero by the imposing displacement having the same direction and sense, which is the basis of the relaxation method.

The greatest possible decrease for a displacement of a given type will be obtained by bringing the corresponding force to zero. Usually we have,

$$(U + V) = - \frac{1}{2}.\bar{\bar{X}}_A.\Delta u = - \frac{1}{2}.\bar{\bar{X}}_A/(\frac{\partial^2 U}{\partial u_A^2}) \quad \ldots.(2.16)$$

Relaxation can be always continued until (U+V) has been brought to its absolute minimum, that all residual forces are negligible.

2.2.3.2. Derivation of the Method for Survey Problems

For different problems when it is required to minimize the value of some quadratic function Q (or U) of parameters x, y, .., these parameters could be treated as displacements [10], and procedure is carried exactly as before. Forces are fictitious, energy and work done are imaginary or virtual.

For problems of adjustment in surveying,

let φ stand for an observed quantity,

α stand for the calculted value of φ from some approximate, (x,y) coordinates.

As before, due to the errors of observations,

$$v = \varphi - \alpha = f(x,y) \quad \ldots\ldots\ldots.(2.17)$$

while the least squares solution requires that Σpv^2 = minimum. Therefore the problem is to solve for v, such that:

$$2Q = \Sigma pv^2 = \text{minimum} \quad \ldots\ldots\ldots.(2.18)$$

2Q being the quadratic function mentioned before. It is termed here total energy (virtual) since it is analogous to that term as used in structural problems.

For the adjustment of any net specified by n independent parameters x_1, x_2, ..., x_n, y_1, ..., y_n , the problem is to find displacements that make 2Q minimum. x,y are the displacements corresponding to u and v in the structural problem. Consider the fictitious forces,

$$X_1 = - \frac{\partial Q}{\partial x1} \quad \text{and} \quad X_2 = - \frac{\partial Q}{\partial x2} \quad \ldots\ldots\ldots(2.19)$$

The change in the residual forces X due to the relaxation of δx is given by:-

$$\delta X_1 = \frac{\partial X_1}{\partial x_1}.\delta x_1 = - \frac{\partial^2 Q}{\partial x_1^2}.\delta x = - \Sigma p(\frac{\partial v}{\partial x_1})^2.\delta x_1$$

$$\delta X_2 = \frac{\partial X_2}{\partial x_1}.\delta x = - \frac{\partial^2 Q}{\partial x_1.\partial x_2}.\delta x_1 = - \Sigma p(\frac{\partial v}{\partial x_1})(\frac{\partial v}{\partial x_2}).\delta x_1 \quad ..(2.20)$$

....

The values $\frac{\partial^2 Q}{\partial x_1^2}$, $\frac{\partial^2 Q}{\partial x_1.\partial x_2}$, .. etc. are called the "Influence coefficients" normally used to distribute the residual forces X to the required accuracy, where residual forces X become negligible.

2.2.3.2.1. Application of Systematic Relaxation to Directions Adjustment in Triangulation Networks

Professor Southwell [91] applied this method to many engineering problems and to the adjustment of levels in surveying. At this time trilateration was not known as an alternative method to triangulation, more recently it has been shown by many contributors, e.g. Kui-Wai [56], Linkwitz [64], that the adjustment of a trilateration net is simply the problem of achieving equilibrium of a pin-jointed elastic framework, under the effect of a redundant member of non-appropriate length. Professor Black [10] applied this method to a more difficult problem, to the adjustment of triangulation nets. His adjustment is made to the observed directions. Observation equations in this case are:

$$v_{AB} = \varphi_{AB} + r_A - \alpha_{AB} \quad ....(2.21)$$

.... etc.

where φ_{AB} is the observed bearing AB,

r_A is the bearing of the zero of the horizontal circle of the theodolite at station A,

α_{AB} is the calculated bearing AB, from the known approximate coordinates (x_A, y_A) and (x_B, y_B).

The minimum energy solution requires that $2Q = \Sigma pv^2$ = minimum.

therefore, $2Q_{AB} = \Sigma p_{AB}.v_{AB}^2 = \Sigma p_{AB}(\varphi_{AB} + r_A - \alpha_{AB})^2$ = minimum

...........(2.22)

Q will be a quadratic function of the parameters x, y when we consider

$$\tan\alpha_{AB} = \frac{y_B - y_A}{x_B - x_A} \quad ....(2.23)$$

where, x_A, y_A, x_B, and y_B are the independent parameters. Forces acting at point A may be resolved into three components as follows:-

$$X_A = -\frac{\partial Q}{\partial x_A} = \Sigma_A(p_{AB}\cdot v_{AB}\cdot\frac{\partial\alpha_{AB}}{\partial x_A} + p_{BA}\cdot v_{BA}\cdot\frac{\partial\alpha_{BA}}{\partial x_A})$$

$$Y_A = -\frac{\partial Q}{\partial y} = \Sigma_A(p_{AB}\cdot v_{AB}\cdot\frac{\partial\alpha_{AB}}{\partial y_A} + p_{BA}\cdot v_{BA}\cdot\frac{\partial\alpha_{BA}}{\partial y_A}) \quad ......(2.24)$$

$$R_A = -\frac{\partial Q}{\partial r_A} = -\Sigma\, p_{AB}\cdot v_{AB}$$

X and Y are components of the force acting in the direction of the displacements x, and y. R is another force exerted by the energy conserved causing rotation of all rays at this special station A. Similar rotations are caused at every other station in the net. Thus residual forces X and Y are obtained which cause displacements of the zero of the theodolite circle and at the same time residual force R is obtained causing the rotation of the circle itself. From equation (2.24) the influence coefficients are:-

$$\frac{\partial X_A}{\partial x_A} = -\Sigma_A[p_{AB}(\frac{\partial\alpha_{AB}}{\partial x_A})^2 + p_{BA}(\frac{\partial\alpha_{BA}}{\partial x_A})^2]$$

$$\frac{\partial X_A}{\partial r_A} = \Sigma_A p_{AB}\ \frac{\partial\alpha_{AB}}{\partial x_A} = \frac{\partial R_A}{\partial x_A} \quad ....(2.25)$$

$$\frac{\partial R_A}{\partial r_A} = -\Sigma_A p_{AB}.$$

Professor Black illustrated this by solving a quadrilateral for which all bearings are observed from both ends of each line.

In this application all the information necessary for accurate mathematical solution was supplied. The results obtained were as accurate as those obtained by the normal least squares solution. The snag is that solution by this method was very complicated and more difficult to surveyors with no background in structural theory to follow. The complications and difficulties arise through the use of the bearing of the zero of the horizontal circle of the theodolite, (usually called station adjustment), but the method shows a complete agreement with the classical methods of adjustment.

The advantages of using systematic relaxation appear more clearly when a mechanical analogue takes care of such complications as will be shown later (sec. 2.3) or when the station adjustment is avoided by using angles instead of directions, also discussed in 2.2.3.2.2.

2.2.3.2.2. Application of Systematic Relaxation to Angles Adjustment in Triangulation Networks.

It has already been mentioned that the difficulties in using systematic relaxation method for directions adjustment in triangulation networks are due to the introduction of the station adjustment. To simplify the use of this method the possibility of adjusting angles has been investigated in the following.

In this case observation equations are formed for the difference between the observed and calculated angles. This bears no relation to any geometric condition of the triangle. Therefore there is no need to go beyond considering the effect of the residual forces at each station.

In triangle ABC figure 1.2 the three angles will be considered as three elastic units in a structural problem, connected to each other by the sides of the triangle.

To understand the necessity of station adjustment, consider figures 2.9 and 2.10.

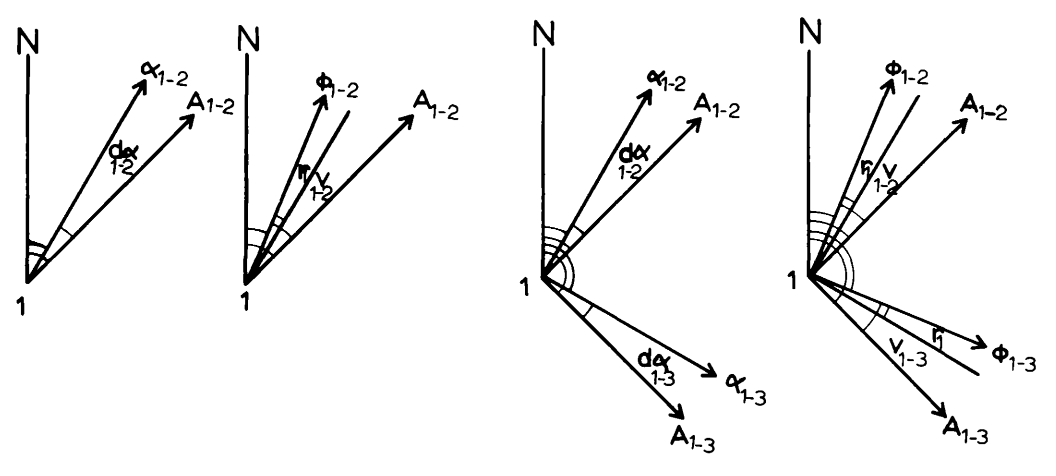


Figure 2.9 *Figure 2.10*

Figure 2.9 shows $d\alpha_{1-2}$ the difference between the calculated bearing α_{1-2} and the adjusted bearing A_{1-2} for the direction 1-2. It also shows v_{1-2} the correction applied to the observed direction φ_{1-2} and r_1 the constant station correction (bearing of the zero of the horizontal circle of the theodolite), and applied to all directions observed from station 1. It is more understandable if we notice that r is associated with every set-up of the theodolite, i.e. it will have different value each time the instrument is set-up. However it must be given a constant value when all directions are obtained in one set-up of the theodolite. The correction equation will be in the following form:-

$$v_{1-2} = \alpha_{1-2} + d\alpha_{1-2} - (\varphi_{1-2} + r_1) \quad \ldots\ldots\ldots(2.26)$$

The constant quantity r_1 will be reduced if one direction is subtracted from another. Thus for the adjustment of <u>angles</u> these constant values will not appear in the adjustment process.

Using the same principle of minimum strain-energy for the angles adjustment we have $2Q = \Sigma pv^2 = \text{minimum} \quad \ldots.(2.18)$

Referring to figure 2.10, the calculated angle θ obtained from the two calculated directions is $\theta_{213} = \alpha_{1-3} - \alpha_{1-2}, \ldots.(2.27)$

the observed angle β obtained from the two observed directions is

$$\beta_{213} = \varphi_{1-3} - \varphi_{1-2} \text{ , and} \quad \ldots\ldots\ldots(2.28)$$

the adjusted angle obtained from the two adjusted directions is

$$A_{213} = A_{1-3} - A_{1-2} \quad \ldots\ldots\ldots(2.29)$$

Therefore,

$$\begin{aligned} v_{213} = v_{1-3} - v_{1-2} &= [(\alpha_{1-3} + d\alpha_{1-3}) - (\alpha_{1-2} + d\alpha_{1-2})] \\ &\quad - [(\varphi_{1-3} + r_1) - (\varphi_{1-2} + r_1)] \\ &= (\alpha_{1-3} - \alpha_{1-2}) - (\varphi_{1-3} - \varphi_{1-2}) + \\ &\quad (d\alpha_{1-3} - d\alpha_{1-2}) - (r_1 - r_1) \quad \ldots.(2.30) \end{aligned}$$

$(r_1 - r_1) = 0$, where r is constant at a single station.

For deriviation of forces and influence coefficients, equation (2.30) will be kept in the form,

$$v_{213} = (v_{1-3} - v_{1-2}) = (d\alpha_{1-3} - d\alpha_{1-2}) + \theta_{213} - \beta_{213} \ldots(2.31)$$

As the triangle is the unit in any triangulation net, in which angles are to be corrected, demonstration and derivation of formulae will be devoted to a triangle. Consider the triangle CDE figure 2.11, The force acting at joint C due to contradiction (θ-β) (or in other words due to v), according to equation (2.18) is given in the following:

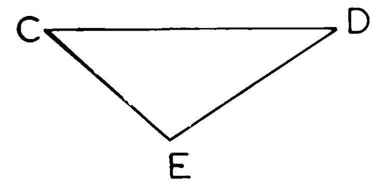


Figure 2.11

$$X_C = -\Sigma\frac{\partial Q}{\partial x_C} = [(v_{CE} - v_{CD})(\frac{\partial\alpha_{CE}}{\partial x_C} - \frac{\partial\alpha_{CD}}{\partial x_C}) + (v_{DC} - v_{DE})(\frac{\partial\alpha_{DC}}{\partial x_C})$$

$$+ (v_{ED} - v_{EC})(-\frac{\partial\alpha_{EC}}{\partial x_C})] \qquad ....(2.32)$$

Similarly the forces acting at D and E are:

$$X_D = -\Sigma\frac{\partial Q}{\partial x_D} = [(v_{DC} - v_{DE})(\frac{\partial\alpha_{DC}}{\partial x_D} - \frac{\partial\alpha_{DE}}{\partial x_D}) + (v_{ED} - v_{EC})(\frac{\partial\alpha_{ED}}{\partial x_D})$$

$$+ (v_{CE} - v_{CD})(-\frac{\partial\alpha_{CD}}{\partial x_D})] \qquad ....(2.33)$$

and,

$$X_E = -\Sigma\frac{\partial Q}{\partial x_E} = [(v_{ED} - v_{EC})(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}) + (v_{CE} - v_{CD})(\frac{\partial\alpha_{CE}}{\partial x_E})$$

$$+ (v_{DC} - v_{DE})(-\frac{\partial\alpha_{DE}}{\partial x_E})] \qquad ....(2.34)$$

Since the triangle is a unit figure, equilibrium should be considere for this unit. Equilibrium is obtained when the three following conditions are satisfied:

$$\Sigma X = 0, \qquad \Sigma Y = 0, \qquad \text{and } \Sigma M = 0.$$

where ΣM are the moments applied at the joints E, C, and D. Since in this case being considered M does not exist, equilibrium is obtained by satisfying the two conditions $\Sigma X = 0$, and $\Sigma Y = 0$. To check the stability of the triangle under the considered forces X_C, X_D, and X_E, their sum must satisfy the equilibrium conditions. Adding equations (2.32), (2.33) and (2.34) we have:

$$[(v_{CE} - v_{CD})(\frac{\partial\alpha_{CE}}{\partial x_C} - \frac{\partial\alpha_{CD}}{\partial x_C}) + (v_{DC} - v_{DE})(\frac{\partial\alpha_{DC}}{\partial x_C}) + (v_{ED} - v_{EC})(-\frac{\partial\alpha_{EC}}{\partial x_C})]$$

$$+[(v_{DC} - v_{DE})(\frac{\partial\alpha_{DC}}{\partial x_D} - \frac{\partial\alpha_{DE}}{\partial x_D}) + (v_{ED} - v_{EC})(\frac{\partial\alpha_{ED}}{\partial x_D}) + (v_{CE} - v_{CD})(-\frac{\partial\alpha_{CD}}{\partial x_D})]$$

$$+[(v_{ED} - v_{EC})(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}) + (v_{CE} - v_{CD})(\frac{\partial\alpha_{CE}}{\partial x_E}) + (v_{DC} - v_{DE})(-\frac{\partial\alpha_{DE}}{\partial x_E})]$$

$$= \quad ? \qquad ....(2.35)$$

but we have:

$$\frac{\partial\alpha_{CE}}{\partial x_E} = \frac{\partial\alpha_{EC}}{\partial x_E} = -\frac{\partial\alpha_{CE}}{\partial x_C} = -\frac{\partial\alpha_{EC}}{\partial x_C}$$

$$\frac{\partial\alpha_{DC}}{\partial x_C} = \frac{\partial\alpha_{CD}}{\partial x_C} = -\frac{\partial\alpha_{DC}}{\partial x_D} = -\frac{\partial\alpha_{CD}}{\partial x_D} \quad \ldots\ldots\ldots(2.36)$$

$$\frac{\partial\alpha_{ED}}{\partial x_D} = \frac{\partial\alpha_{DE}}{\partial x_D} = -\frac{\partial\alpha_{ED}}{\partial x_E} = -\frac{\partial\alpha_{DE}}{\partial x_E}$$

Substituting from equation (2.36) into equation (2.35) and re-arranging terms we have;

$$\frac{\partial\alpha_{CE}}{\partial x_E}(- v_{CE} + v_{CD} + v_{ED} - v_{EC} + v_{CE} - v_{CD} - v_{ED} + v_{EC}) +$$

$$\frac{\partial\alpha_{DC}}{\partial x_C}(- v_{CE} + v_{CD} + v_{DC} - v_{DE} - v_{DC} + v_{DE} + v_{CE} - v_{CD}) +$$

$$\frac{\partial\alpha_{ED}}{\partial x_E}(- v_{DC} + v_{DE} + v_{ED} - v_{EC} - v_{ED} + v_{EC} + v_{DC} - v_{DE}) = 0 \quad \ldots\ldots\ldots(2.37)$$

Equation (2.37) shows that a triangle worked on by the forces X will be stable, and it should be kept stable when sucessive relaxation is being considered.

If point E is to be fixed, coordinates x_E, y_E, will be liable to corrections, but the effect of a unit relaxation (Influence coefficients) of the coordinates of point E on the forces calculated before has to be derived first.

Referring to equations (2.32), (2.33) and (2.34) the effect of a unit relaxation of coordinates of point E is obtained simply by partially differentiating these quantities with respect to x_E, y_E. The influence coefficient of unit relaxation δx_E on the force at C is:

$$\frac{\partial X_C}{\partial x_E} = \Sigma\frac{\partial^2 Q}{\partial x_E.\partial x_C} = -\left[\left(\frac{\partial\alpha_{CE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{CE}}{\partial x_C} - \frac{\partial\alpha_{CD}}{\partial x_C}\right) + \left(-\frac{\partial\alpha_{DE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{DC}}{\partial x_C}\right) + \left(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}\right)\left(-\frac{\partial\alpha_{EC}}{\partial x_C}\right)\right] \quad \ldots\ldots\ldots(2.38)$$

At D the effect is:

$$\frac{\partial X_D}{\partial x_E} = \Sigma\frac{\partial^2 Q}{\partial x_E.\partial x_D} = -\left[\left(-\frac{\partial\alpha_{DE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{DC}}{\partial x_D} - \frac{\partial\alpha_{DE}}{\partial x_D}\right) + \left(\frac{\partial\alpha_{ED}}{\partial x_E}\right)\left(\frac{\partial\alpha_{ED}}{\partial x_D}\right) + \left(\frac{\partial\alpha_{CE}}{\partial x_E}\right)\left(-\frac{\partial\alpha_{CD}}{\partial x_D}\right)\right] \quad \ldots\ldots\ldots(2.39)$$

and at E it is:

$$\frac{\partial X_E}{\partial x_E} = \Sigma\frac{\partial^2 Q}{\partial x_E^2} = - [(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E})^2 + (\frac{\partial\alpha_{CE}}{\partial x_E})^2 + (-\frac{\partial\alpha_{DE}}{\partial x_E})^2] \quad ..(2.40)$$

The above deriviation is obtained for forces in one direction only. The force in the perpendicular direction Y should be obtained in a similar way. This could be easily obtained by rewriting equations (2.32), (2.33), (2.34), (2.38), (2.39), and (2.40) with respect of y instead of x. This will give the following set of values for Y_C, and ∂Y_C, ... etc.

$$Y_C = -\Sigma\frac{\partial Q}{\partial y_C} = [(v_{CE} - v_{CD})(\frac{\partial\alpha_{CE}}{\partial y_C} - \frac{\partial\alpha_{CD}}{\partial y_C}) + (v_{DC} - v_{DE})(\frac{\partial\alpha_{DC}}{\partial y_C}) + (v_{ED} - v_{EC})(-\frac{\partial\alpha_{EC}}{\partial y_C})] \quad ....(2.41)$$

$$Y_D = -\Sigma\frac{\partial Q}{\partial y_D} = [(v_{DC} - v_{DE})(\frac{\partial\alpha_{DC}}{\partial y_D} - \frac{\partial\alpha_{DE}}{\partial y_D}) + (v_{ED} - v_{EC})(\frac{\partial\alpha_{ED}}{\partial y_D}) + (v_{CE} - v_{CD})(-\frac{\partial\alpha_{CD}}{\partial y_D})] \quad ....(2.42)$$

$$Y_E = -\Sigma\frac{\partial Q}{\partial y_E} = [(v_{ED} - v_{EC})(\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E}) + (v_{CE} - v_{CD})(\frac{\partial\alpha_{CE}}{\partial y_E}) + (v_{DC} - v_{DE})(-\frac{\partial\alpha_{DE}}{\partial y_E})] \quad ....(2.43)$$

Accordingly the effect of a unit relaxation δy_E on these forces will be:

$$\frac{\partial Y_C}{\partial y_E} = \Sigma\frac{\partial^2 Q}{\partial y_E \cdot \partial y_C} = - [(\frac{\partial\alpha_{CE}}{\partial y_E})(\frac{\partial\alpha_{CE}}{\partial y_C} - \frac{\partial\alpha_{CD}}{\partial y_C}) + (-\frac{\partial\alpha_{DE}}{\partial y_E})(\frac{\partial\alpha_{DC}}{\partial y_C}) + (\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E})(-\frac{\partial\alpha_{EC}}{\partial y_C})], \quad ....(2.44)$$

$$\frac{\partial Y_D}{\partial y_E} = \Sigma\frac{\partial^2 Q}{\partial y_E \cdot \partial y_D} = - [(-\frac{\partial\alpha_{DE}}{\partial y_E})(\frac{\partial\alpha_{DC}}{\partial y_D} - \frac{\partial\alpha_{DE}}{\partial y_D}) + (\frac{\partial\alpha_{CE}}{\partial y_E})(-\frac{\partial\alpha_{CD}}{\partial y_D}) + (\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E})(\frac{\partial\alpha_{ED}}{\partial y_D})] \quad ....(2.45)$$

and,

$$\frac{\partial Y_E}{\partial y_E} = \Sigma\frac{\partial^2 Q}{\partial y_E^2} = - [(\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E})^2 + (\frac{\partial\alpha_{CE}}{\partial y_E})^2 + (\frac{\partial\alpha_{DE}}{\partial y_E})^2] \quad ....(2.46)$$

Also the effect of unit relaxation δx_E on force Y, and δy_E on force

X will be as follows:

$$\frac{\partial X_C}{\partial y_E} = \Sigma\frac{\partial^2 Q}{\partial y_E \cdot \partial x_C} = -\left[\left(\frac{\partial\alpha_{CE}}{\partial y_E}\right)\left(\frac{\partial\alpha_{CE}}{\partial x_C} - \frac{\partial\alpha_{CD}}{\partial x_C}\right) + \left(-\frac{\partial\alpha_{DE}}{\partial y_E}\right)\left(\frac{\partial\alpha_{DC}}{\partial x_C}\right) + \left(\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E}\right)\left(-\frac{\partial\alpha_{EC}}{\partial x_C}\right)\right] \quad \ldots\ldots\ldots(2.47)$$

$$\frac{\partial X_D}{\partial y_E} = \Sigma\frac{\partial^2 Q}{\partial y_E \cdot \partial x_D} = -\left[\left(-\frac{\partial\alpha_{DE}}{\partial y_E}\right)\left(\frac{\partial\alpha_{DC}}{\partial x_D} - \frac{\partial\alpha_{DE}}{\partial x_D}\right) + \left(\frac{\partial\alpha_{ED}}{\partial y_E}\right)\left(\frac{\partial\alpha_{ED}}{\partial x_D}\right) + \left(\frac{\partial\alpha_{CE}}{\partial y_E}\right)\left(-\frac{\partial\alpha_{CD}}{\partial x_D}\right)\right] \quad \ldots\ldots\ldots(2.48)$$

$$\frac{\partial X_E}{\partial y_E} = \Sigma\frac{\partial^2 Q}{\partial y_E \cdot \partial x_E} = -\left[\left(\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E}\right)\left(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}\right) + \left(\frac{\partial\alpha_{CE}}{\partial y_E}\right)\left(\frac{\partial\alpha_{CE}}{\partial x_E}\right) + \left(-\frac{\partial\alpha_{DE}}{\partial y_E}\right)\left(-\frac{\partial\alpha_{DE}}{\partial x_E}\right)\right] \quad \ldots\ldots\ldots(2.49)$$

$$\frac{\partial Y_C}{\partial x_E} = \Sigma\frac{\partial^2 Q}{\partial x_E \cdot \partial y_C} = -\left[\left(\frac{\partial\alpha_{CE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{CE}}{\partial y_C} - \frac{\partial\alpha_{CD}}{\partial y_C}\right) + \left(-\frac{\partial\alpha_{DE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{DC}}{\partial y_C}\right) + \left(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}\right)\left(-\frac{\partial\alpha_{EC}}{\partial y_C}\right)\right] \quad \ldots\ldots\ldots(2.50)$$

$$\frac{\partial Y_D}{\partial x_E} = \Sigma\frac{\partial^2 Q}{\partial x_E \cdot \partial y_D} = -\left[\left(-\frac{\partial\alpha_{DE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{DC}}{\partial y_D} - \frac{\partial\alpha_{DE}}{\partial y_D}\right) + \left(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}\right)\left(\frac{\partial\alpha_{ED}}{\partial y_D}\right) + \left(\frac{\partial\alpha_{CE}}{\partial x_E}\right)\left(-\frac{\partial\alpha_{CD}}{\partial y_D}\right)\right] \quad \ldots\ldots\ldots(2.51)$$

and,

$$\frac{\partial Y_E}{\partial x_E} = \Sigma\frac{\partial^2 Q}{\partial x_E \cdot \partial y_E} = -\left[\left(\frac{\partial\alpha_{ED}}{\partial x_E} - \frac{\partial\alpha_{EC}}{\partial x_E}\right)\left(\frac{\partial\alpha_{ED}}{\partial y_E} - \frac{\partial\alpha_{EC}}{\partial y_E}\right) + \left(\frac{\partial\alpha_{CE}}{\partial x_E}\right)\left(\frac{\partial\alpha_{CE}}{\partial y_E}\right) + \left(-\frac{\partial\alpha_{DE}}{\partial x_E}\right)\left(-\frac{\partial\alpha_{DE}}{\partial y_E}\right)\right] \quad \ldots\ldots\ldots(2.52)$$

Although mathematical details for the solution given in equations (2.32) to (2.52) show that solution by this method may be a lengthy one, it will be very simple after $\partial\alpha/\partial x$, and $\partial\alpha/\partial y$ are obtained for each station considered. Then initial forces and influence coefficients are obtained by simple substitution of these values in the equations given before.

From the influence coefficients, a table of standard operations can be made out showing the effects on all forces of unit relaxations δx , δy ,... etc. The initial values of the forces are entered on a relaxation table. Usually the largest forces, e.g. X_i is relaxed first, Then we apply a relaxation δx_i sufficient to reduce this approximately to zero. The effects of this on all the other forces can be found from the table of the standard operations and entered on the relaxation table. The largest remaining force can then be picked out and eliminated in the same way, and by continuing this process until the forces have all been reduced to negligible size the solution is obtained. A table of this type has been made for a specific example given later.

2.2.3.2.2.1. Accuracy of Results Obtained by Using Systematic Relaxation Method for Angles Adjustment

It is very important from the calculation point of view, to check the accuracy of the given method against the results obtained by the least squares solution. Also, as this method of solution is given as alternative solution to that given by Prof. Black [10], the accuracy of results should be also checked against his results for the same problem.

(a) To check this, triangle CDE in figure 7.19 given by Rainsford in "Survey Adjustment and Least Squares" is solved by the method given above and by the normal least squares method. The observation equations are given by:

$$- 2.12\ \delta x_E - 2.76\ \delta y_E - 9.80 = 0$$

$$+ 4.68\ \delta x_E + 2.17\ \delta y_E + 7.85 = 0$$

$$- 2.56\ \delta x_E + 0.59\ \delta y_E + 1.60 = 0$$

The solution by the least squares gives the following results:

$\delta x_E = - 0.117$, and $\delta y_E = - 3.419$

Using the same data systematic relaxation method [equations (2.32) to (2.52)] gives the following results:

$\delta x = - 0.117$, and $\delta y = - 3.430$

(b) To check the accuracy of results obtained by this new method against those obtained by Prof. Black in Empire Survey Review vol. 4, p 406 [10], the same problem is used. A doubly braced quadrilateral for which observations are obtained for all directions from both ends of each line. In this problem two stations are fixed and two stations are to be fixed. Tables 2.2 and 2.3 give the influence coefficients and the systematic relaxation for this example using the new method for angular adjustment.

	X_N	X_S	X_R	X_T	Y_N	Y_S	Y_R	Y_T
δx_R	-12425	-11025	15549	8082	5025	- 9318	2033	2260
δy_R	- 9832	6050	2033	1749	- 2924	- 2615	5953	- 413
δx_T	-10858	- 9727	8082	12503	- 9824	5920	1749	2155
δy	3314	- 7730	2260	2155	- 2368	- 1096	- 413	3837

table 2.2 - Influence Coefficients

	X_N	X_S	X_R	X_T	Y_N	Y_S	Y_R	Y_T
I.F.*	- 5331	- 5923	9525	1729	6881	-10658	1717	2060
$-0.77x_R$**	9567	8489	-11972	- 6223	- 3869	7175	-1565	-1740
$0.44x_T$	- 4777	- 4280	3556	5501	- 4322	2604	769	- 948
$-0.37y_T$	- 1226	2860	- 836	- 797	876	405	152	-1420
$-0.18y_R$	1769	- 1089	- 366	- 314	526	470	-1072	74
Sum***	2	52	7	- 104	92	- 4	1	- 49

* I.F. are the values of initial forces obtained from equations (2.32),(2.33),(2.34),(2.41),(2.42), and (2.43)

** Values entered in lines 3, 4, 5, and 6 are the necessary displacements and their effect on the I.F. obtained from table 2.2.

*** These are the sum of all rows in the table, which give the residual forces neglected at the end of the solution

table 2.3 - Relaxation Operations

The results given by Prof. Black are:

$\delta x_R = -0.77$ $\delta y_R = -0.18$

$\delta x_T = 0.44$ $\delta y_T = -0.37$

and by this newly derived method the results for the same problem are identical as can be seen by the left hand column of table 2.3.

2.2.3.2.2.2. Comparison of the Two Methods

The main purpose, as mentioned before, is to eliminate the complications found in the method, created by Prof. Black [10] due to the use of the station correction necessary when adjusting directions. Solution by this newly derived method gives the same results while avoiding the use of the forces due to rotations of stations. In Prof. Black's method it is essential for adjusting directions by the systematic relaxation method to consider the effect of each linear displacement on the force R, as well as finding its effect on forces X and Y. At the same time the effect of the small rotations for the purpose of reducing force R on the two linear forces X and Y have to be calculated and entered into tables. In fact this is a most difficult operation to follow and excute. The number of tables necessary for his method of solution is large, so that it takes a long time to reach the right entries for the different tables, even when the solution is clearly set up in tabular form. On the other hand, using the systematic relaxation for angles adjustment as given by this new method, requires much less knowlege and skill. There is only one way of computation, X, Y, $\partial X/\partial x$, ... etc. are calculated in similar way, as shown in the solved examples.

The second way by which the difficulties of using systematic relaxation method may be overcome is to construct a mechanical analogue which can take over all the mathematical differentiations and substitutions. It should also be able to carry out the calculations arising from rotation of stations, which proves to be the main difficulty in using systematic relaxation method for adjustment of directions. If an analogue of this type can be devised it would offer a very practical solution of such problems.

2.3. SURVEY NETWORKS AND MECHANICAL ANALOGY

Dr. Jerie [46], [47], has applied the systematic relaxation method making use of the theory of least energy conserved mechanically to different problems in surveying and photogrammetry. His mechanical analogue computers for the block adjustment of planimetric coordinates and heights in photogrammetry are well established, and have been used in many photogrammetric organisations all over the world. It is only quite recently that electronic computers in these organisations have had much effect upon the use of these mechanical analogue computers and still many are in every day use.

In the surveying field mechanical computers for trilateration adjustment have been constructed by Dr. Jerie and showed some advantages over the use of the electronic computers especially in the field and in the detection of gross errors.

From the calculation point of view the main advantage of using these mechanical analogues is to avoid calculation of the residual forces and influence coefficients required in the mathematical solution by systematic relaxations. Instead of the latter contradictions or difference between observed and calculated quantities are obtained and introduced to a mechanically constructed elastic system which obeys Hooke's Law. Adjustment is reached when the elastic analogue reaches the equilibrium position with least energy conserved in the system.

At the start of computation, approximate values for the coordinates of the new points to be fixed are obtained. Improvements to these values are sought through the adjustment. The approximate coordinates could be obtained by the equipment in which case there would be no need for their calculation. However it speeds the adjustment to first calculate the approximate values as this reduces the number of iterations required.

The practical use of mechanical analogues does not require a high level of scientific or mathematical knowledge, as the most important part in adjusting the survey problems, which consists mainly of setting and choosing the condition equations or observation equations is avoided so the personnel required to execute the operation need be familiar or skilled only in operation with a hand calculating machine to a prescribed routine.

2.3.1. Different Aspects for the Construction of a Mechanical Analogue for Triangulation

The construction of the mechanical analogue for angle adjustment has been a dream for sometime ago, but not realised because of the mechanical complications encountered in the construction, which were also expected to produce results of rather low precision.

Professors Southwell and Black who discussed the generalisation of the systematic relaxation method to solving linear equations did not make any suggestions as to the construction of a possible mechanical analogue. Prof. Southwell did suggest a mechanical analogue for the adjustment of levelling but this was not constructed.

Purely theoretical work can exaggerate difficulties, and lead to the conclusion that a solution is impracticable. It must be recognised that the mechanical construction of delicate equipment can be very difficult and may need much refinement and experimentation over a long period before the desired accuracy is achieved. It is worth mentioning therefore that although the basic idea was established two years ago, its practical construction and improvements have been carried out continuously ever since. Probably some further improvements may still be necessary before the mechanical analogue computer is perfect, but it has reached the stage of working fairly satisfactorily to the precision required.

A big difficulty has been the expense and the time required to have basic pieces and frequent modifications made by a number of different outside firms since the facilities were not available in the University.

2.3.2. Mechanical Interpretation of the Formulae for the Adjustment of Triangulation by Variation of Coordinates Method

The essential conditions for the construction of a suitably dimensioned elastic model for the computation and solution of special problems are:

(1) The mechanical system must give the same mathematical relationship between the different quantities to be calculated.

(2) It must be possible to distort elastically the mechanical components representing the quantities to be adjusted.

Other essential conditions which are specific to the construction of this analogue will be given in due course.

2.3.2.1. Mathematical Relationship Between Different Quantities Represented in the Mechanical Analogue

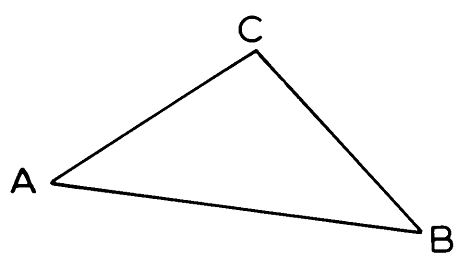


Figure 2.12

A triangle ABC is fully defined when three quantities are known. Usually there should be a base line and two angles. In figure 2.12 if base line is BC, the triangle will be determined when angles B and C are observed. If angle A is obsrved also, observation in this case has a degree of freedom, which gives one condition for the adjustment of the observations. Corrections for these observations may be obtained by using any of the well-known methods of adjustment, that currently favoured being the variation of coordinates method. Using this method, both Σv^2 of the mathematical model, and Σv^2 of the mechanical model will be functions of the same parameters. In figure 2.12 if the observed direction of AB is φ_{AB} and the calculated direction from approximate coordinates is α_{AB}, then the correction to this observed direction will be given by:

$$v_{AB} = -(r_A + \varphi_{AB}) + d\alpha_{AB} + \alpha_{AB} \qquad \text{........(2.53)}$$

(see figure 2.9, where A and B stand for 1, and 2 in the figure).

The relationship between α the calculated direction and the parameters (x,y) is given by:

$$\tan\alpha = \frac{y_B - y_A}{x_B - x_A} \qquad \text{.........(2.54)}$$

To obtain $d\alpha$ with respect to each of the four parameters we have:

$$\partial\alpha_{AB} = \frac{y_B - y_A}{l_{AB}^2}.\partial x_A = \frac{r_1}{l_{AB}}.\partial x_A \qquad \text{.........(2.55)}$$

$$\partial\alpha_{AB} = -\frac{y_B - y_A}{l_{AB}^2}.\partial x_B = -\frac{r_1}{l_{AB}}.\partial x_B \qquad \text{..........(2.56)}$$

$$\partial\alpha_{AB} = -\frac{x_B - x_A}{l_{AB}^2}.\partial y_A = -\frac{q_1}{l_{AB}}.\partial y_A \qquad \text{.........(2.57)}$$

$$\partial\alpha_{AB} = \frac{x_B - x_A}{l_{AB}^2}.\partial y_B = \frac{q_1}{l_{AB}}.\partial y_B \qquad \text{.........(2.58)}$$

where,

$$r_1 = \frac{y_B - y_A}{l_{AB}}, \quad \text{and} \quad q_1 = \frac{x_B - x_A}{l_{AB}}$$

Adding equations (2.55), (2.56), (2.57) and (2.58) we have:

$$d\alpha_{AB} = \frac{r_1}{l_{AB}}.dx_A - \frac{r_1}{l_{AB}}.dx_B - \frac{q_1}{l_{AB}}.dy_A + \frac{q_1}{l_{AB}}.dy_B . \text{(2.59)}$$

It should be noted that if,

$\sin\alpha_{AB} = r_1$, $\cos\alpha_{AB} = q_1$, ...etc.

therefore $\sin\alpha_{BA} = - r_1$, and $\cos\alpha_{BA} = - q_1$. .etc.

Following the same procedure we have:

$$d\alpha_{AC} = \frac{r_2}{l_{AC}}.dx_A - \frac{r_2}{l_{AC}}.dx_C - \frac{q_2}{l_{AC}}.dy_A + \frac{q_2}{l_{AC}}.dy_C . \text{(2.60)}$$

Subtracting (2.60) from (2.59) we have:

$$d\theta_A = \left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right).dx_A - \frac{r_1}{l_{AB}}.dx_B + \frac{r_2}{l_{AC}}.dx_C$$

$$- \left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right).dy_A + \frac{q_1}{l_{AB}}.dy_B - \frac{q_2}{l_{AC}}.dy_C \text{.........(2.61)}$$

Similarly we have

$$d\theta_B = \left(\frac{r_3}{l_{BC}} - \frac{r_1}{l_{AB}}\right).dx_B - \frac{r_3}{l_{BC}}.dx_C - \frac{r_1}{l_{AB}}.dx_A$$

$$- \left(\frac{q_3}{l_{BC}} + \frac{q_1}{l_{AB}}\right).dy_B + \frac{q_3}{l_{BC}}.dy_C + \frac{q_1}{l_{AB}}.dy_A \text{ ....(2.62)}$$

and,

$$d\theta_C = -\left(\frac{r_2}{l_{AC}} - \frac{r_3}{l_{BC}}\right).dx_C + \frac{r_2}{l_{AC}}.dx_A - \frac{r_3}{l_{BC}}.dx_B$$

$$+ \left(\frac{q_2}{l_{AC}} - \frac{q_3}{l_{BC}}\right).dy_C - \frac{q_2}{l_{AC}}.dy_A + \frac{q_3}{l_{BC}}.dy_B \quad \ldots\ldots\ldots(2.63)$$

If BC is the base line, with the two stations B and C fixed, dx_B, dx_C, dy_B, and dy_C will be zero, so that equations (2.61),(2.62), and (2.63) will be reduced to the following:

$$d\theta_A = \left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right).dx_A - \left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right).dy_A \quad \ldots\ldots\ldots(2.64)$$

$$d\theta_B = -\frac{r_1}{l_{AB}}.dx_A + \frac{q_1}{l_{AB}}.dy_A \quad \ldots\ldots\ldots(2.65)$$

$$d\theta_C = +\frac{r_2}{l_{AC}}.dx_A - \frac{q_2}{l_{AC}}.dy_A \quad \ldots\ldots\ldots(2.66)$$

From equations (2.64),(2.65) and (2.66) it could be seen that the change in angle A is minus the sum of the changes in the two directions AB and AC.

Substituting the equations (2.64), (2.65) and (2.66) into equation (2.30) we have:

$$v_A = (v_{AB} - v_{AC}) = d\theta_A - k_A \quad \ldots\ldots\ldots(2.67)$$

where, $k_A = (\alpha_{AB} - \alpha_{AC}) - (\varphi_{AB} - \varphi_{AC}) = \theta_A - \beta_A$

Thus v_A is obtained and v_B and v_C may be derived similarly.

Correction equations for the problem given by figure 2.12 will be given by the matrix:

$$\begin{vmatrix} v_A \\ v_B \\ v_C \end{vmatrix} = \begin{vmatrix} \frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}} & -\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right) \\ -\frac{r_1}{l_{AB}} & +\frac{q_1}{l_{AB}} \\ +\frac{r_2}{l_{AC}} & -\frac{q_2}{l_{AC}} \end{vmatrix} \begin{vmatrix} dx_A \\ dy_A \end{vmatrix} - \begin{vmatrix} k_A \\ k_B \\ k_C \end{vmatrix}$$

Components of normal equations are then given by:

$$A'A = \left|\begin{array}{c:c} (\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}})(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}) & -(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}})(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}) \\ \frac{r_1}{l_{AB}}.\frac{r_1}{l_{AB}} & -\frac{r_1}{l_{AB}}.\frac{q_1}{l_{AB}} \\ \frac{r_2}{l_{AC}}.\frac{r_2}{l_{AC}} & -\frac{r_2}{l_{AC}}.\frac{q_2}{l_{AC}} \\ \hdashline -(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}})(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}) & (\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}})(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}) \\ -\frac{q_1}{l_{AB}}.\frac{r_1}{l_{AB}} & \frac{q_1}{l_{AB}}.\frac{q_1}{l_{AB}} \\ \frac{q_2}{l_{AC}}\ \frac{r_2}{l_{AC}} & \frac{q_2}{l_{AC}}\ \frac{q_2}{l_{AC}} \end{array}\right|$$

$$K = \left|\begin{array}{c} -(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}).k_A \\ +\frac{r_1}{l_{AB}}.k_B \\ -\frac{r_2}{l_{AC}}.k_C \\ \hdashline +(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}).k_A \\ -\frac{q_1}{l_{AB}}.k_B \\ +\frac{q_2}{l_{AC}}.k_C \end{array}\right| \qquad \text{..........(2.69)}$$

2.3.2.2. Mechanical Relationships Between Different Quantities Represented in the Mechanical Analogue

For the same mechanical problem, since stations B and C are fixed, there are two fixed supports at B and C, with station A allowed to move. Also using the superposition theorem, we have the following.

In figure 2.13a moment $M_C = 1$ is equivalent to an effective force at A $= 1/l_{AC}$ and a reaction at B $= 1/l_{BC}$. Similarly in figure 2.13b $M_B = 1$, which is equivalent to an effective force equal to $1/l_{AB}$ at A, and a reaction force = $1/l_{BC}$ at B. For $M_A = 1$, two effective forces will be acting at A equal to $1/l_{AB}$ $1/l_{AC}$ in different sense to those produced by M_B and M_C for equilibrium reasons.

The problem will be solved by finding out the displacements δx and δy of the station A which is necessary to reduce the strain-energy to minimum, or in other words to find out the displacements which satisfy the equilibrium conditions, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$. The change in angle θ between any two sides is the result of:

(1) A deformation in the elastic unit representing the angle due to the moment applied.

(2) A movement of the free end of the station A.

Expressing this change in the mathematical form using equation (2.9) to give the change due to the translation of the free station A, we have:

$$v_A = -k_A + \left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right).dx_A - \left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right).dy_A$$

$$v_B = -k_B - \frac{r_1}{l_{AB}}.dx_A + \frac{q_1}{l_{AB}}.dy_A \qquad \text{.........(2.70)}$$

$$v_C = -k_C + \frac{r_2}{l_{AC}}.dx_A - \frac{q_2}{l_{AC}}.dy_A$$

If f is the stiffness of the elastic angle, therefore:

$$f.M = v_A \qquad \text{..........(2.71)}$$

and, $$M = g.v_A$$

when g(the weight) is taken unity,

therefore, $$M = v_A$$

and acting forces as a result as this moment will be equal to M/L. To obtain the components of these forces in the directions of coordinates x and y, equation (2.7) is used, thus we have:

$$\sum_1^3 X = \Sigma\left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right).v_A$$

$$\text{.........(2.72)}$$

$$\sum_1^3 Y = \Sigma-\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right).v_A$$

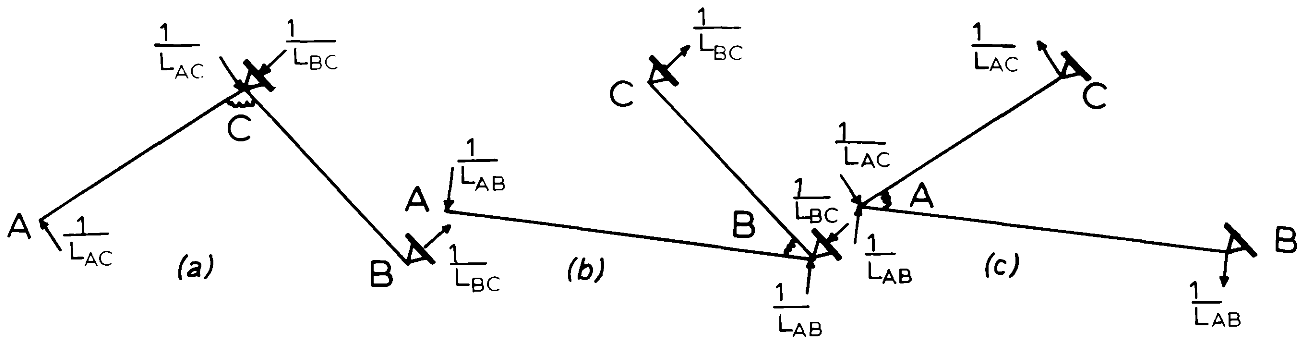


Figure 2.13

Substituting equations (2.70) into equations (2.72) we have:

$$\Sigma X = \left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right)\left[-k_A + \left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right).dx_A - \left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right).dy_A\right]$$

$$- \frac{r_1}{l_{AB}}\left[-k_B - \frac{r_1}{l_{AB}}.dx_A + \frac{q_1}{l_{AB}}.dy_A\right] +$$

$$+ \frac{r_2}{l_{AC}}\left[-k_C + \frac{r_2}{l_{AC}}.dx_A - \frac{q_2}{l_{AC}}.dy_A\right] \qquad ....(2.73)$$

$$=\left[\left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right)\left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right) + \frac{r_1}{l_{AB}}.\frac{r_1}{l_{AB}} + \frac{r_2}{l_{AC}}.\frac{r_2}{l_{AC}}\right].dx_A$$

$$-\left[\left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right)\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right) + \frac{r_1}{l_{AB}}.\frac{q_1}{l_{AB}} + \frac{r_2}{l_{AC}}.\frac{q_2}{l_{AC}}\right].dy_A$$

$$-\left[\left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right).k_A - \frac{r_1}{l_{AB}}.k_B + \frac{r_2}{l_{AC}}.k_C\right] \qquad ....(2.74)$$

Similarly for the other component of forces we have:

$$\Sigma Y = -\left[\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right)\left(\frac{r_1}{l_{AB}} - \frac{r_2}{l_{AC}}\right) + \frac{q_1}{l_{AB}}.\frac{r_1}{l_{AB}} + \frac{q_2}{l_{AC}}.\frac{r_2}{l_{AC}}\right].dx_A$$

$$+\left[\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right)\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right) + \frac{q_1}{l_{AB}}.\frac{q_1}{l_{AB}} + \frac{q_2}{l_{AC}}.\frac{q_2}{l_{AC}}\right].dy_A$$

$$+\left[\left(\frac{q_1}{l_{AB}} - \frac{q_2}{l_{AC}}\right).k_A - \frac{q_1}{l_{AB}}.k_B + \frac{q_2}{l_{AC}}.k_C\right] \qquad ....(2.75)$$

Equations (2.74) and (2.75) when put in a matrix form will show that they are identical to equation (2.69). For larger nets the same procedure can be followed, to show that mechanical system is capable of giving the same solution given by the mathematical forms. Identity between equations (2.74) and (2.75) and equation (2.69) shows that the normal equations in the mathematical solution correspond to the equilibrium conditions in the mechanical solution, i.e. correspond to $\Sigma X = 0$ and $\Sigma Y = 0$.

2.3.3. Physical Representation of Angles

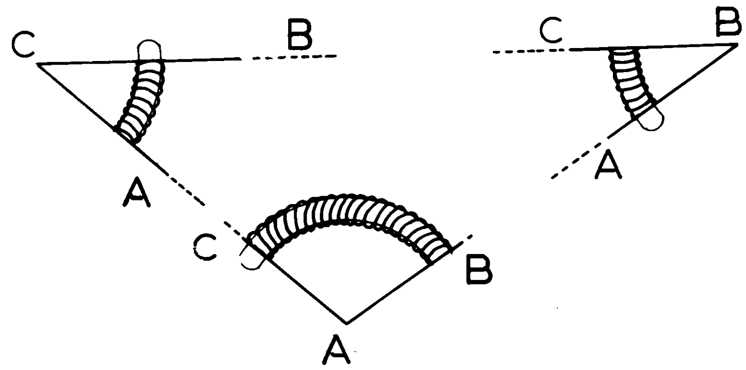


Figure 2.14

The derivation of the above equations takes into account the different readings of a certain bearing taken at both ends of a line, e.g. $v_{AB} \neq v_{BA} \pm 180^\circ$. The physical representation of this consideration may be understood by the diagram given in figure 2.14.

If we consider forces acting at the joints A, B, and C, the effect of the sum of these forces will be obtained by assembling the pieces of this figure. When this representation is assembled the six component sides will represent the three sides of the triangle concerned. This will be the only way of representing the different readings for the bearings at the two ends of a single line. The idea of the mechanical analogue is to find a way of constructing a model representing this assembly, and make this in such a way that it can produce an adjustment according to the law of energy conserved.

2.3.3.1. Joints of the Mechanical Network for Triangulation Adjustment

To represent an angle in the mechanical network, the size of this angle should have the value obtained in the field. As observations obtained from the field are always adjustable, the way to represent them physically is to introduce them in an elastic form which accepts deformation.

To elastically restrict the size of an angle, produces a partial rigidity and conserves energy. Differential deformation is required therefore to allow displacements necessary for equilibrium position. To allow displacements in a net where strictly straight sides are necessary, joints should be very flexible.

The partial rigidity produced depends on the stiffness of the elastic units used to represent the angles, but this rigidity must not be too great or the sides will not be kept straight when a bending moment is applied.

Flexibility combined with smooth movement and rotation which release the excess energy conserved in the elastic joints at different stations is the main object of the construction. When such joints are constructed the resulting deformation in the springs will give the corrections required for the adjustment. So the mechanical features should not stop the nodal points from moving to the required positions in order to reach the state of the minimum strain-energy conserved.

2.3.4. Mechanical Components for Constructing the Analogue for Adjustment of Triangulation Nets

In the field survey it is essential that:

(1) A theodolite having sufficient accuracy for the work in hand should be employed.

(2) New stations must be connected to existing coordinated points.

The result of the theodolite observations will be a set of readings which will contain errors. The next stage is to adjust these readings by solving a set of linear equations for the different observed quantities. As shown before these equations will be solved for the purpose of distributing the discrepancy between observed and calculated quantities.

The mechanical adjustment by the analogue duplicates these procedures, ensuring the distribution of the discrepancy according to the least squares (least energy) theorem. The linear equations solved in the mathematical method are those obtained from the observations in the field. To avoid having to introduce these linear equations to any computing system, mechanical analogue should be assembled in such a way that it is identical to the observed net. This requires that directions have to be represented physically by straight mechanical members in such a way that these will not be affected by the procedure of adjustment, (i.e. they must be kept straight, as in the initial assembly). The angles are represented by the intersection of the individual members representing the directions, which are connected by suitable elastic components.

Existing and new stations have to be located physically and analytically in a coordinated system. In the former case it is plotted on a sheet of graph or other sqaure paper. The mechanical members representing the directions have to be set in the correct directions. In this case a circular disc which represents the horizontal circle of the theodolite is used. The disc will not be expected to give exactly the same readings as those obtained by the theodolite, but it will do so accurately enough for the purpose.

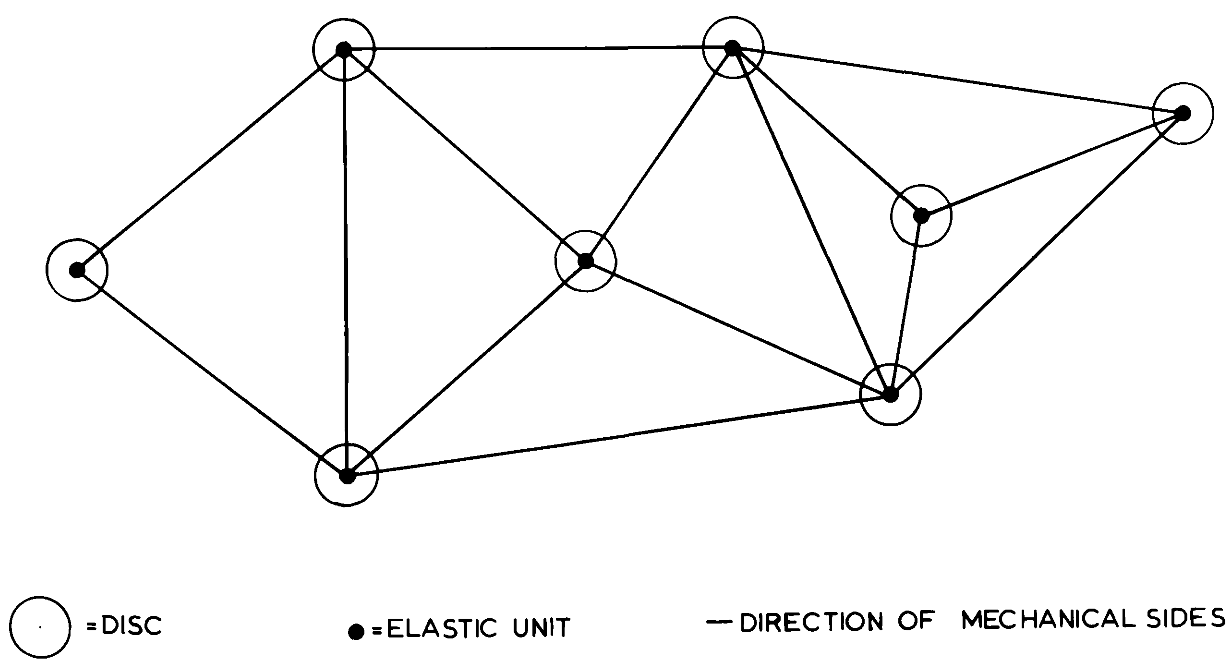


Figure 2.15

The individual components will be discussed in more detail later, but this introduction will suffice for the moment. A typical net of the type described is shown diagrammatically in figure 2.15

2.3.5. Mechanical Representation of Angles and Directions

Adjustment of angles and directions are known to be two alternative methods for adjusting triangulation nets. Using mathematical solutions the difference between using either of these methods is insignificant, as there is hardly any difference in results. The main difference is that the adjustment using angles is generally regarded as being quicker in practice as it involves less computation. Using the mechanical analogue the construction of the joints when using one method will be totally different to that required for the other method. Both methods have their advantages and disadvantages.

The main advantage of using the angles method of adjustment is that it avoids the station corrections which seem to require extra computation. The main snag is the complication in the construction of these joints, which adds extra weight at every station, and so caused difficulties in movement of the appropriate elastic units unless undue force is applied. Another disadvantage is the complication of the mechanical features required in using two different sides to represent one direction as mentioned in 2.3.3. This is really so complicated as to be impractical.

The main advantage in using the direction method of adjustment is the ease of construction of these joints as compared with those required for angles. The circular disc acts as an adjusting device which can be subjected to all forces acting at one joint. Another advantage is that the use of the two mechanical members in the analogue gives an exact representation of the two rays observed at each end of a single line in the field.

Both possibilities for adjustment have been constructed to see if these apparent advantages and disadvantages are realised in practice.

Whether angles or directions are to be adjusted a mechanical analogue simulating the triangulation net must have the following:

(1) Control Points. (2) Elastic Units.

2.3.5.1. Control Points

Examples of these may be primary stations acting as control for secondary nets, or Laplace stations for primary nets. These points are fixed mechanically while translation of the other stations is allowed. The rapidity with which the final values are reached will depend to a considerable extent on the number of the control points and on their location and distribution.

2.3.5.2. Elastic Units

The type of elastic unit to be inserted in the joint constructed is very important. The size of the proposed analogue must be limited both because of weight and the sheer difficulty of operating a physically large device. In addition, with a large system of some weight, elastic system of considerable size would be necessary, which would make it very difficult to insert small angles or small values of the directions.

The use of the different types of elastic units is discussed in the following:

(i) Flat Clock Springs: The first possibility is the use of a flat clock spring, the centre of which is attached to a spindle representing the point and the free end attached to one direction(the side). The direction of each side has to be set using this clock spring, but there are considerable difficulties in producing graduated clock springs for this purpose, and even further mechanical difficulties occur in making connection of the flat clock spring to the sides. A circular disc may be used for setting, but its connection and relation to the spiral clock spring is very complicated. All the other directions observed at a point will also have to be represented by other clock springs acting about the same vertical spindle and thus pose still further difficulties

Because of these difficulties flat clock springs do not appear to offer a practical solution for the representation of angles in a mechanical analogue.

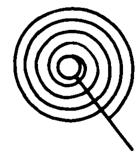
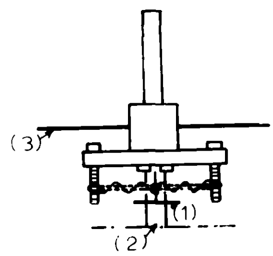


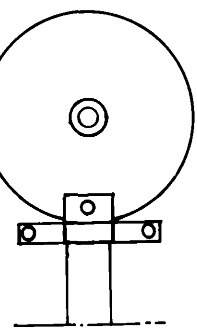
Figure 2.16

(ii) Coil Springs: The familiar coil spring with a constant diameter has been considered both as a tension and as a compression spring. Figure 2.17 shows how such a coil spring may be connected to side a and to the circular disc. The action of the spring will be due to the rotation of side a towards the vertical screws.



(a) ELEV. OF THE CONNECTION

(1) = SIDE a (2) = SPINDLE (3) = HORIZ. DISC



(b) PLAN OF THE CONNECTION

Figure 2.17

(a) Tension Springs:

Tension springs can be inserted directly as shown in figure 2.18. There is no need for any extra wire to keep the spring in position, as the connection of the initial assembly of an analogue requires preliminary tensioning of the spring to give it the spring action wanted. The disadvantage of using such a spring is that when side a moves towards one direction such additional stretching could result that one of the springs reaches the critical point of elasticity and does not act as an elastic unit in the analogue. The power obtained from such a spring depends on its stiffness together with the limit to which this spring can be initially pulled, which depends in turn on the length of the spring, and again on the size

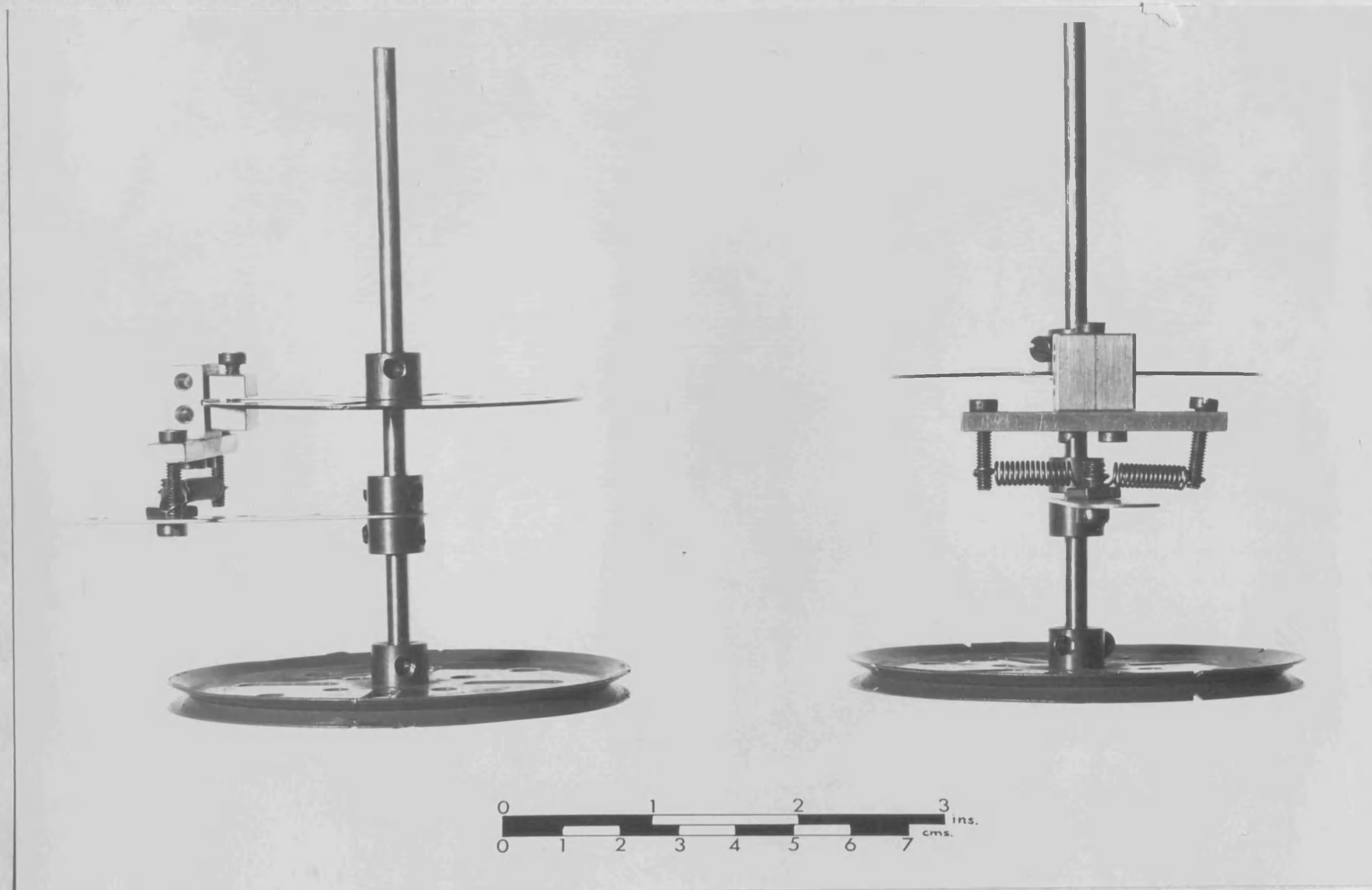


figure 2.18

of the joint. This special type of springs has been used in the experimental analogue for angles adjustment, where the size of the joint was large, and long springs were used.

(b) Compression Springs

In figure 2.17 springs may be compression springs, but in this case a special wire, figure 2.19, has to be used to ensure that the compressed springs act in the direction of rotation of the circular disc. This will not cause any extra difficulty provided the design is such that the wire itself is not compressed or interfers with the forces applied, as shown in figure 2.19, when the size of the joint is small. On the other hand the difficulty of connection will be increased if the size of the joint is large, as the wire required for guiding the long compression spring has to be stiff and exactly circular throughout its length.

The compression spring does allow the full strength of each spring to be used. Also such springs are usually constructed to allow full compression without loosing any of its strength. Compression springs are used in the final construction of the mechanical analogue for the direction adjustment.

(iii) Flat Steel Springs: This is a piece of specially tempered steel which would appear to require a simple small connection and to offer the facility for adjusting numereous directions at a single point. This has been tried experimentally, e.g. in figures 2.20 and 2.20a the rotation of the direction a is determined elastically by the two flat steel springs.

The connection is relatively small when compared with that of figure 2.17, but certain difficulties prevent the successful use of this type. These difficulties are:

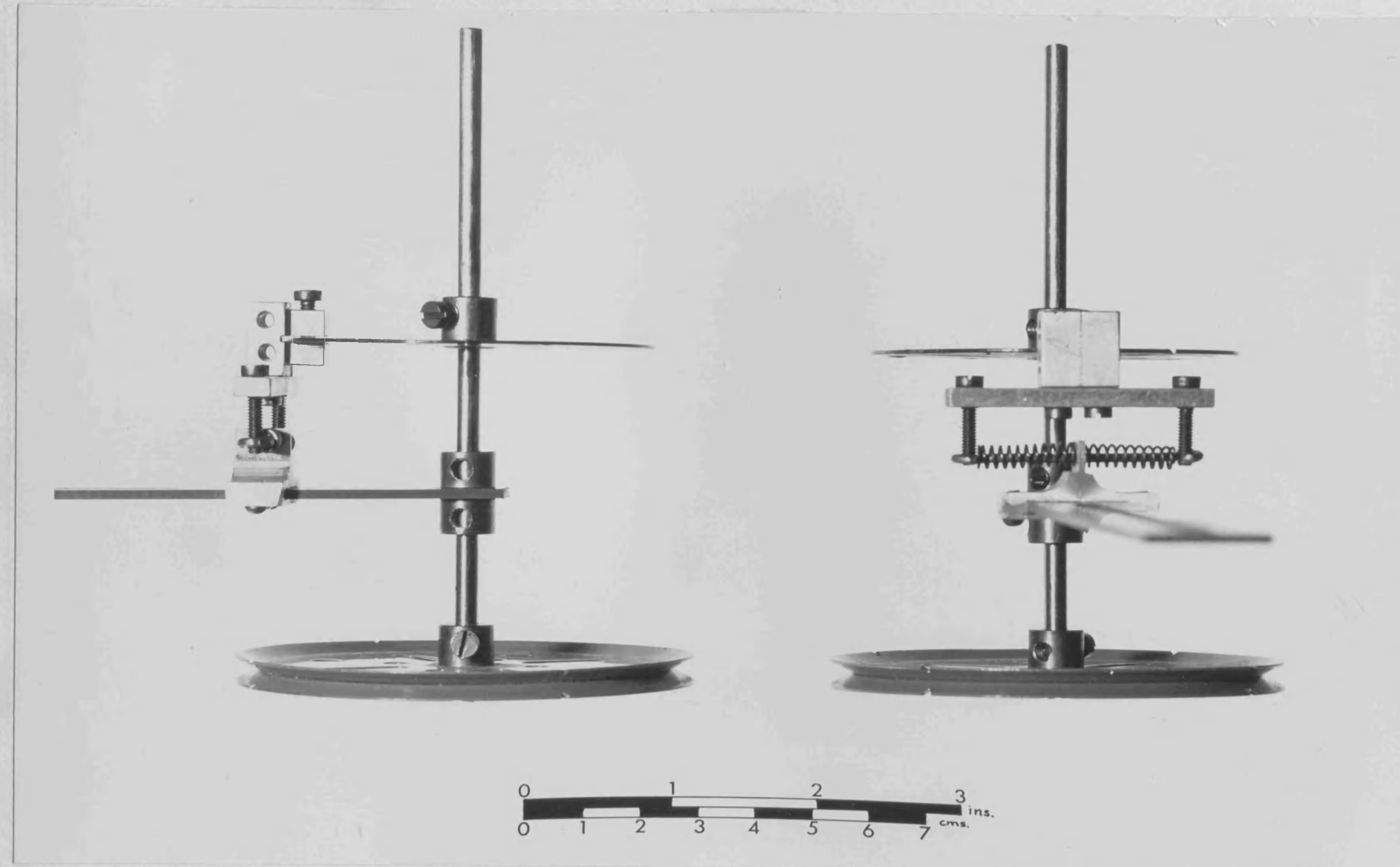


figure 2.19

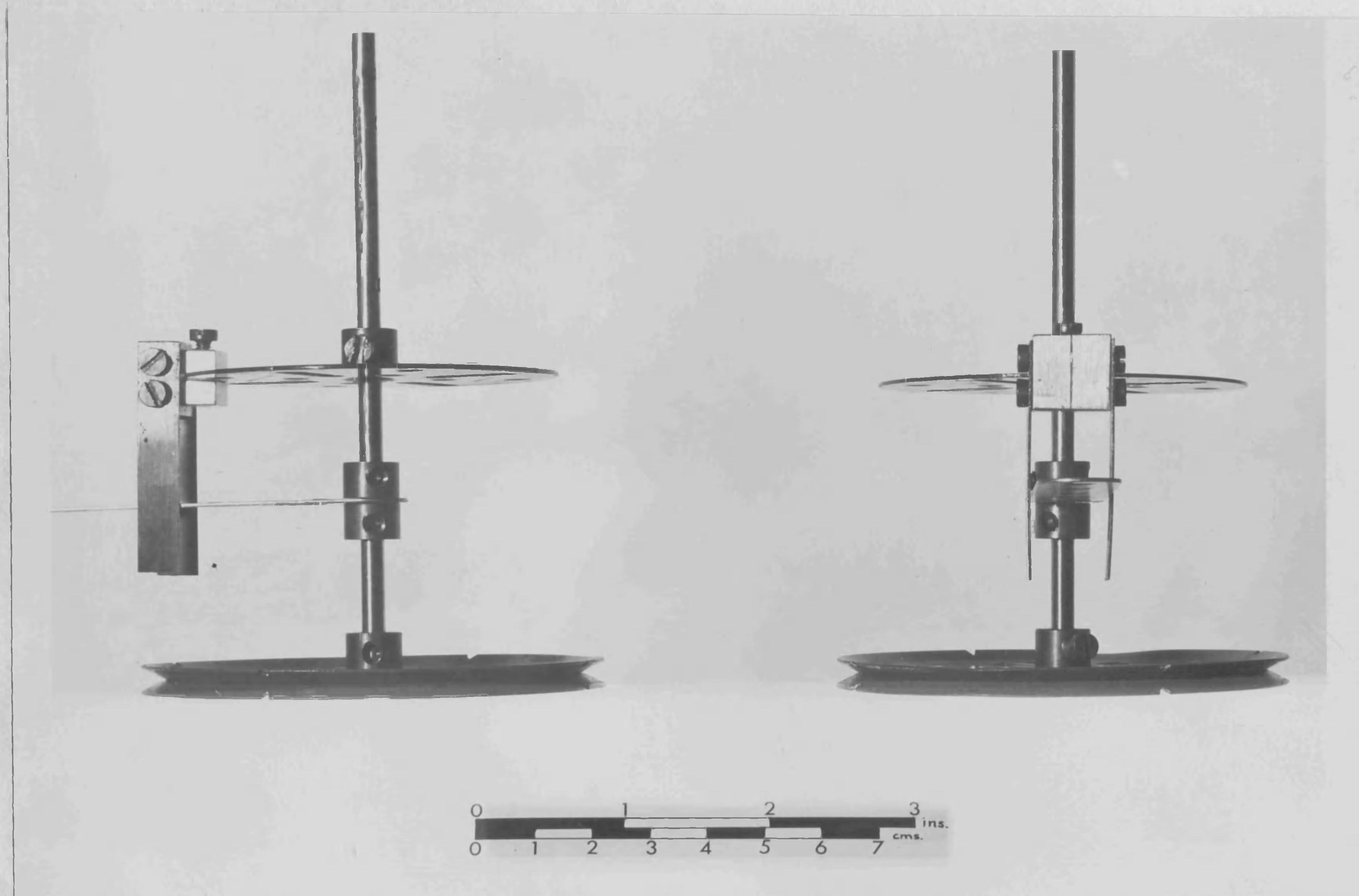
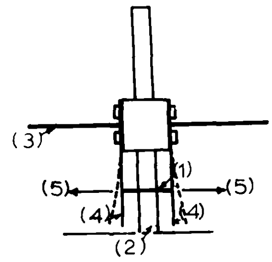


figure 2.20a

(1) Since it is mounted at one end of each side, this type of spring does not have enough elastic power to translate the joint at the other end of the side.

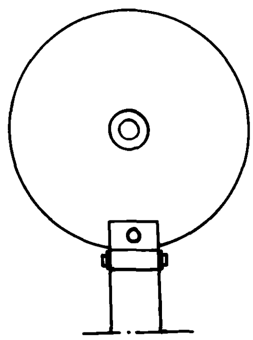
(2) The elastic effect of this type of spring on the side is greatly affected by the vertical position of the side against the spring. Since the sides have to be connected at different levels, the elastic effect will not be equal.

The first difficulty might be overcome perhaps by utilising some special type of steel made up for the purpose.



(a) ELEV. OF THE CONNECTION

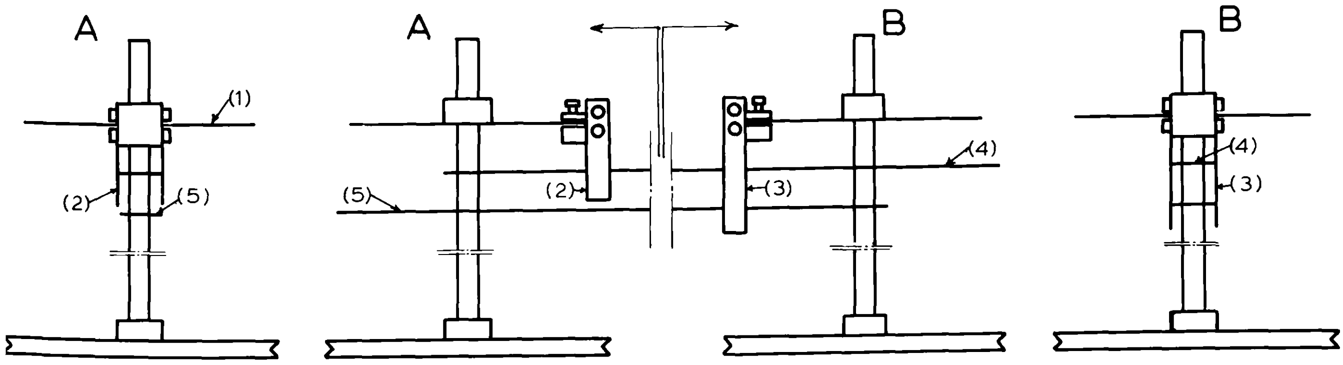
(1) = SIDE a (2) = SPINDLE (3) = HORIZ. DISC



(b) PLAN OF THE CONNECTION

(4) = STEEL SPRING
(5) = DIRECTION OF ACTION OF THE SPRING

Figure 2.20



(b) SECTION (a) ELEVATION (b) SECTION

1) HORIZONTAL DISC, (2) SHORT SPRING FOR SIDE 1, (3) LONG SPRING FOR SIDE 2, (4) SIDE 1, (5) SIDE 2

Figure 2.21

However it appears impossible to avoid the second difficulty for this particular design of the analogue. As a result, one side of the two representing the two directions obtained at both ends of a line, will be elastically affected at both ends, while the other one will be affected at one end only. This can be seen infigure 2.21.

2.4. EXPERIMENTAL ANALOGUE FOR ADJUSTMENT OF ANGLES

Bearing all the above considerations in mind a mechanical analogue has been constructed. This is shown in figures 2.22 and 2.22a.

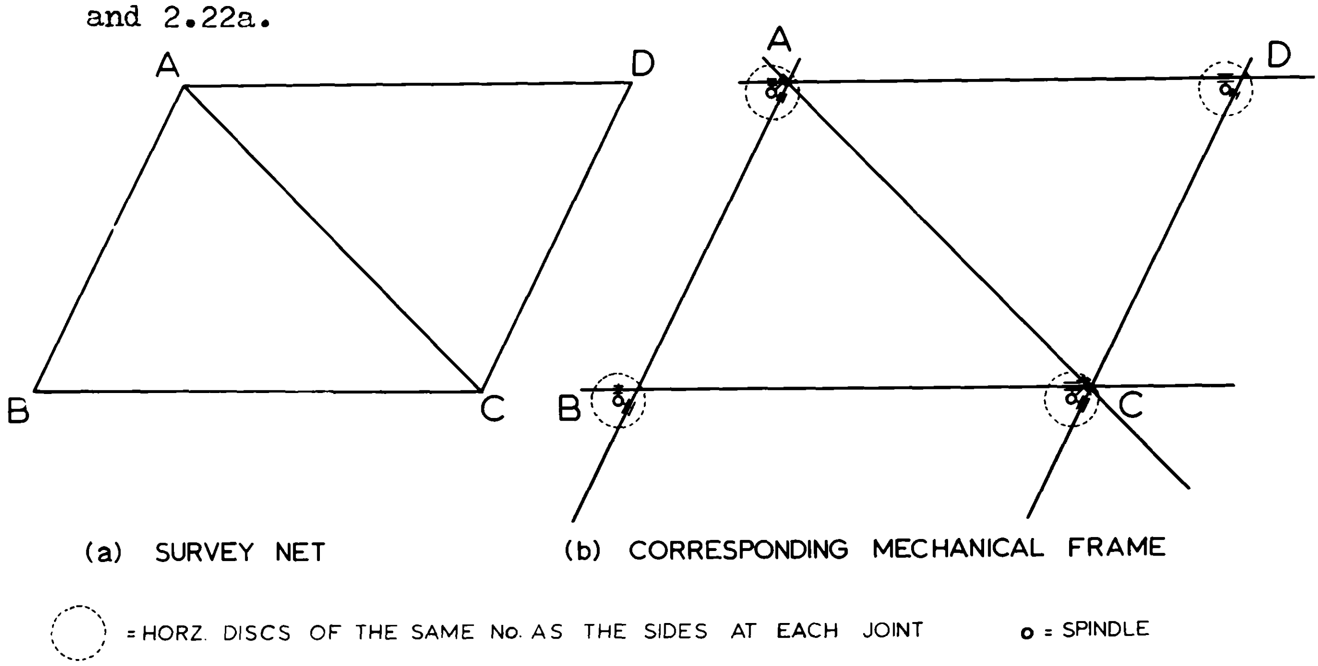


Figure 2.22

Turning to the individual components we have:

(1) Short sleeves, figure 2.23, which can hold the sides clamped or allow completely free movement, in the direction of observation. Rollers are used to assist this movement. The sleeves are provided with collars to make a connection with the corresponding vertical spindle when the tensile force is applied.

(ii) Two horizontal discs, figure 2.24, or a disc and an arm which are provided with collars connecting them to the corresponding spindles. Again this connection should have a clamp against slipping.

(iii) Two spindles, figure 2.25 of two different diameters, one which could be inserted exactly inside the other. Insertion in this case should allow free rotation, with minimum friction,

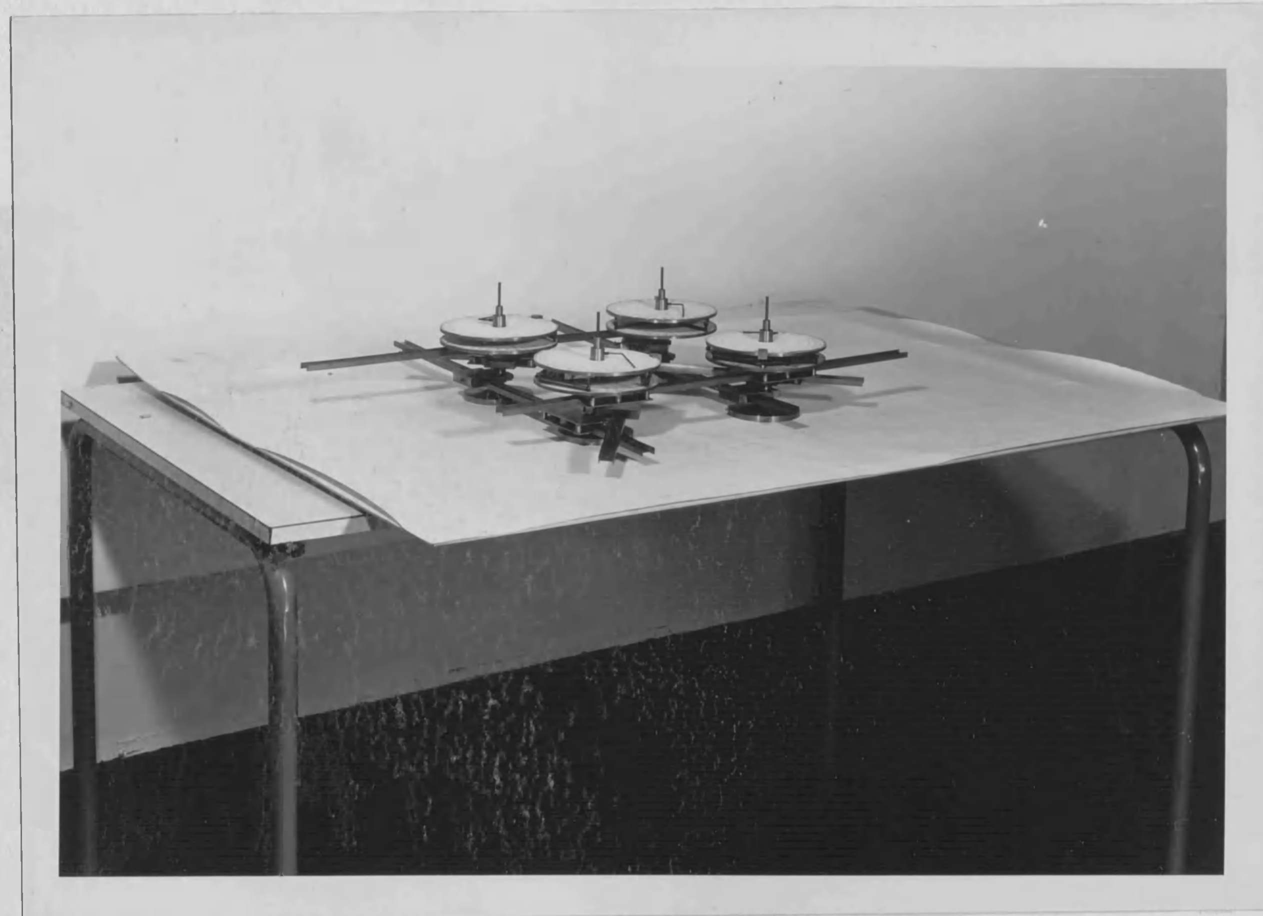


figure 2.22a

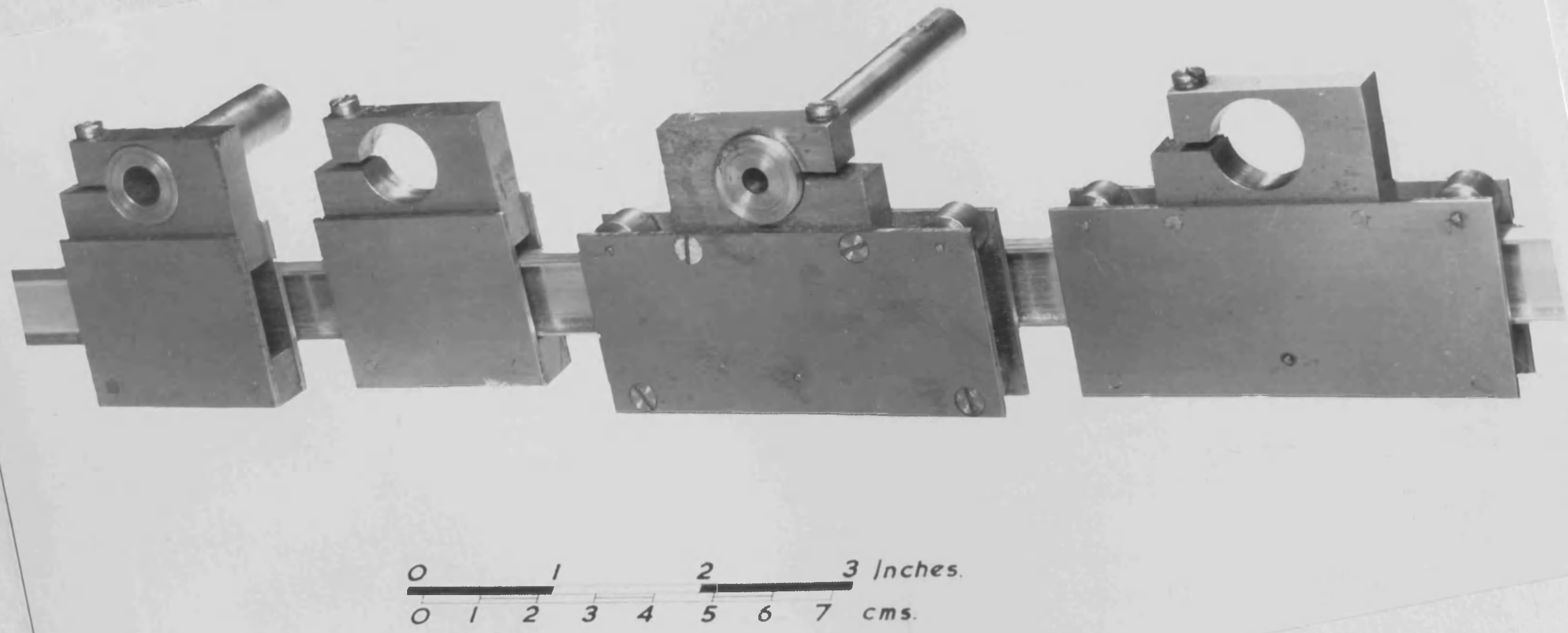


figure 2.23

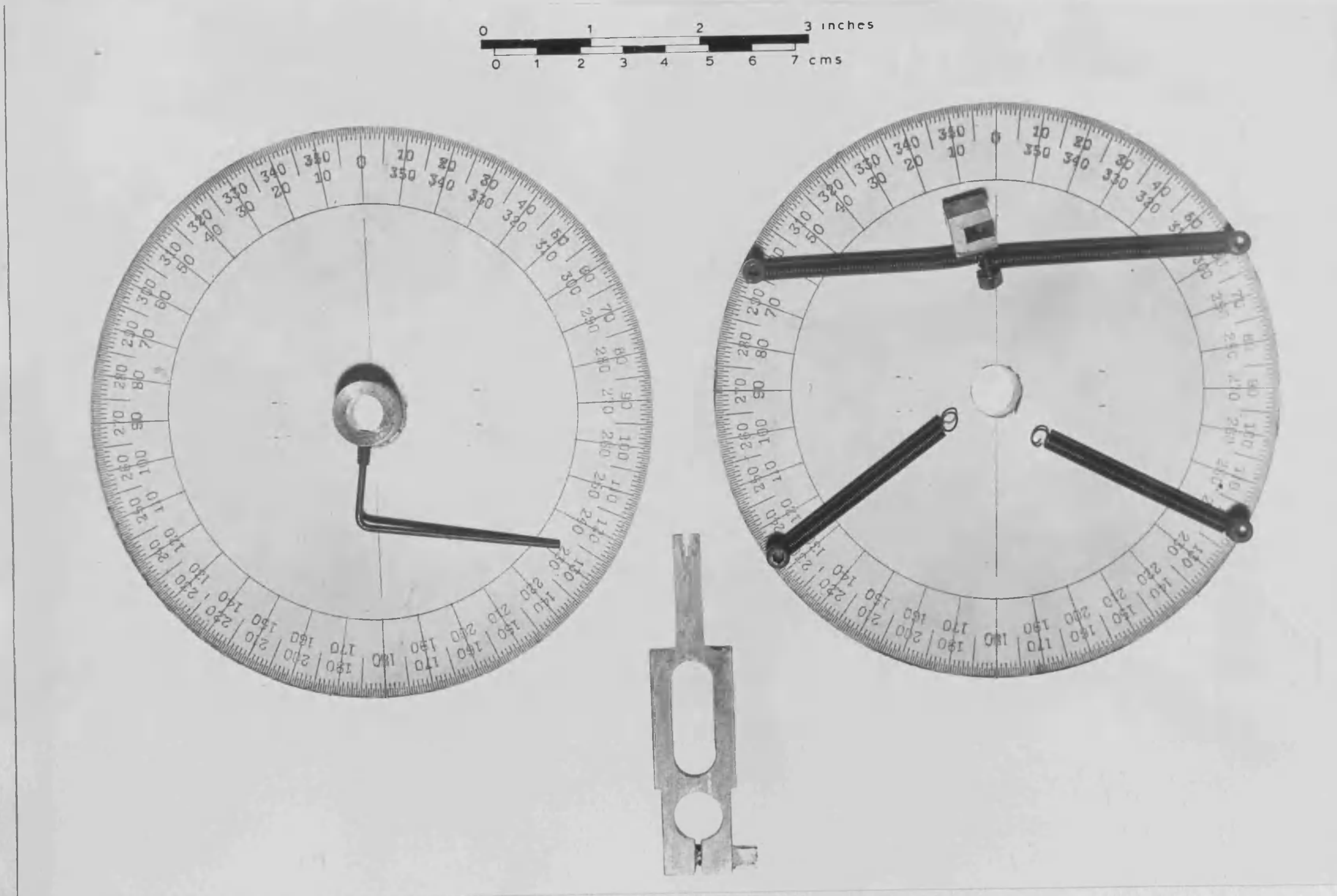


figure 2.24

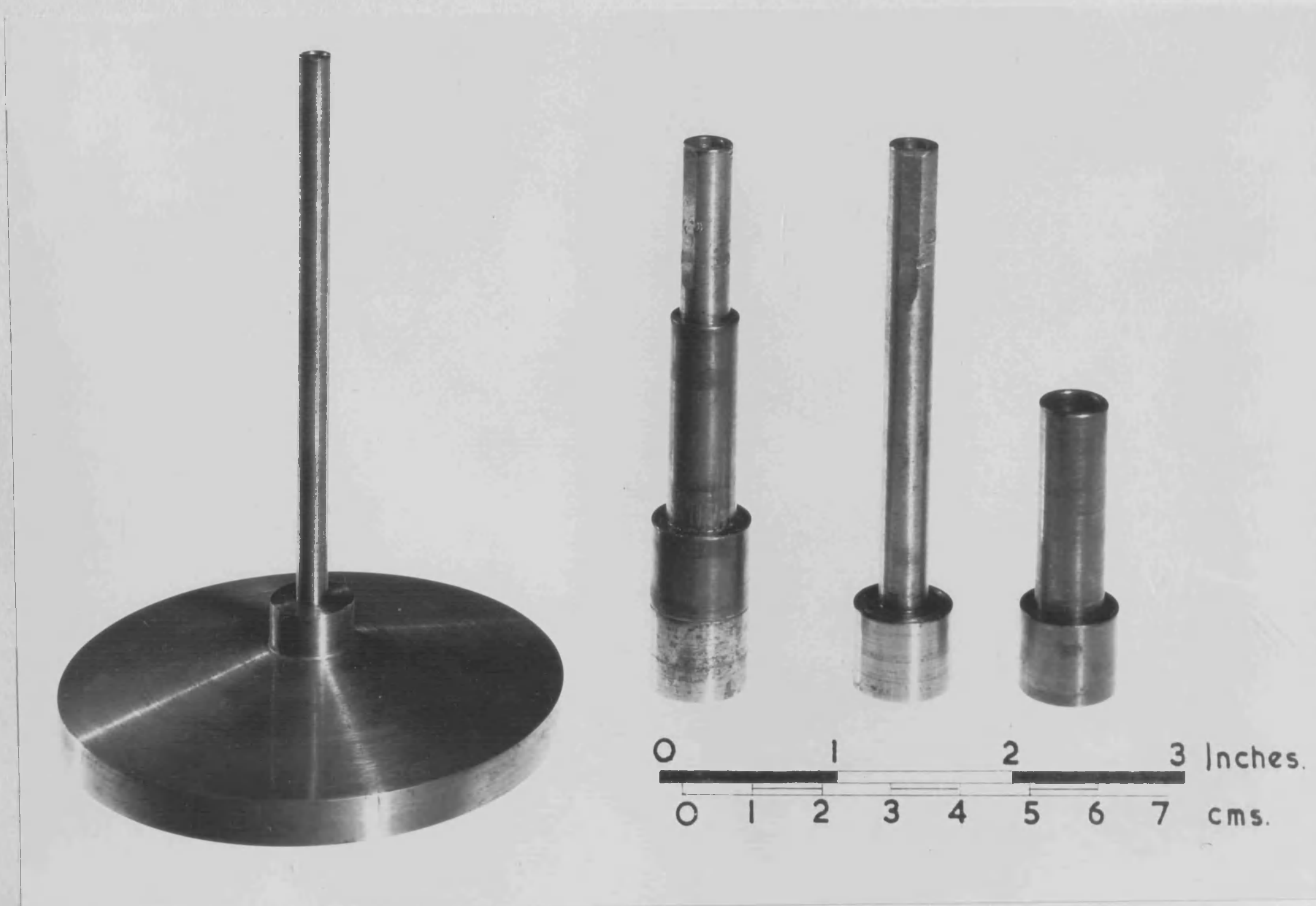


figure 2.25

i.e. the force applied should be exerted fully on the two sides.

(iv) Two tension springs, figure 2.26, connected to the two discs. A proper connection of these springs is given in figure 2.27.

(v) Two direction arms (sides), figure 2.28, which can be inserted inside the sleeves. These should not allow any appreciable elastic deformation, so the sides used are of the width that can resist any lateral bending. The thickness of the sides is of less importance, but this should be chosen to suit the clearance of the sleeves. The sides should be of minimum length necessary to avoid adding extra weights to the joints.

(vi) The stud, figure 2.25, plays a major part in the construction of the elastic joint, as it is the pivot to which all mentioned parts are connected and so to represent mechanically the station in the triangulation net. It is made up of (a) A circular base-plate which is necessary to prevent the joint from being turned over or tilted when forces or moments are applied at the joint. This should also have a smooth undersurface to prevent frictional forces from becoming significant against the working forces.

The accuracy of results depends to a great deal on such achievement, and so the undersurface has been made hollow. The frictional forces also depend on the weight of the joint itself. (b) The spindle, which is attached rigidly to the base-plate and has an external diameter equal to the inner diameter of the inner collar allowing a free rotation of one around the other. (c) Long pin of hard steel, which is introduced into the hollow spindle to fix it to the base board in the case of the control points and to mark the position found after the adjustment in the case of the new stations to be fixed.

(vii) The setting discs, figure 2.24, these are the horizontal discs mentioned before, and are used as setting protractors.

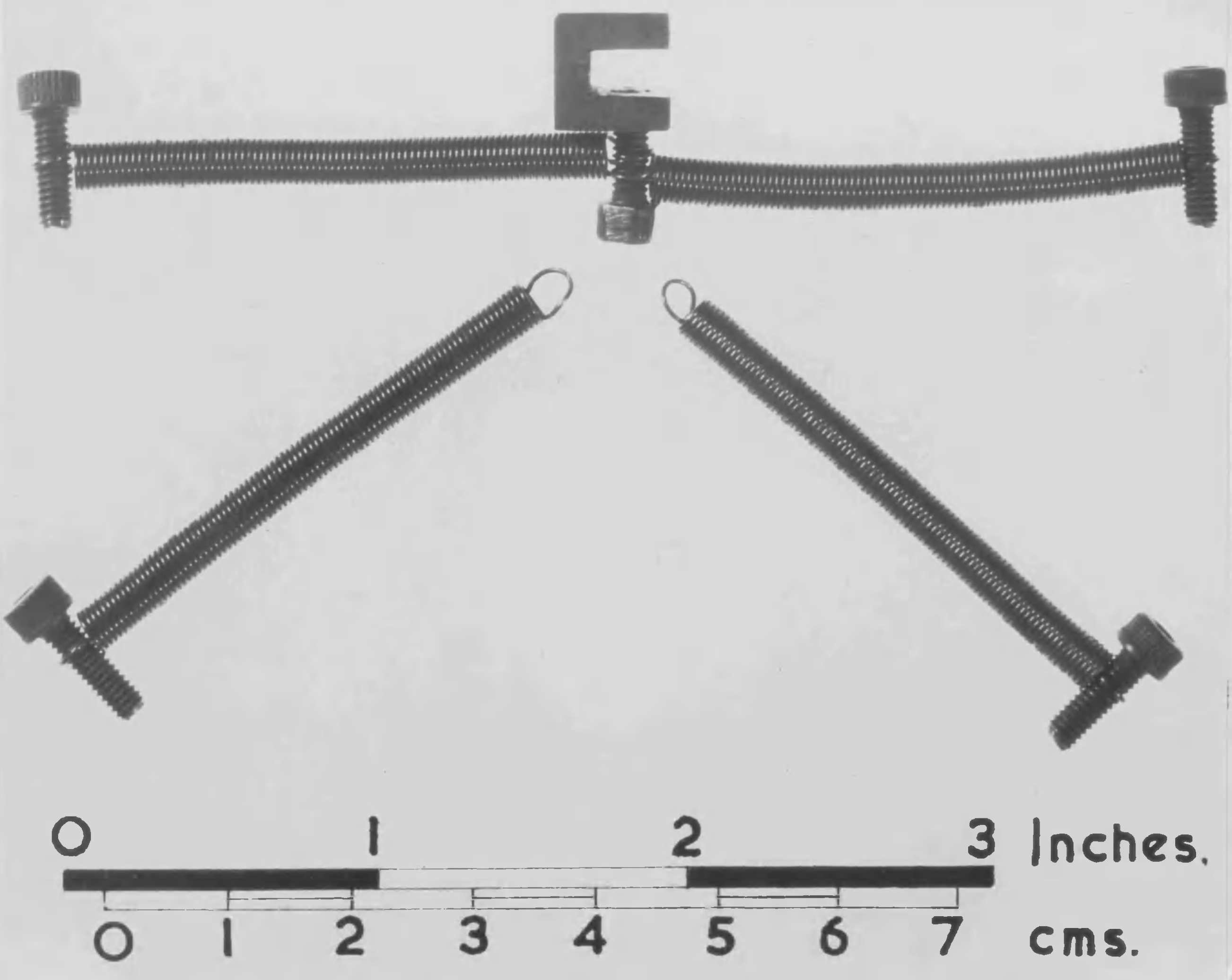


figure 2.26

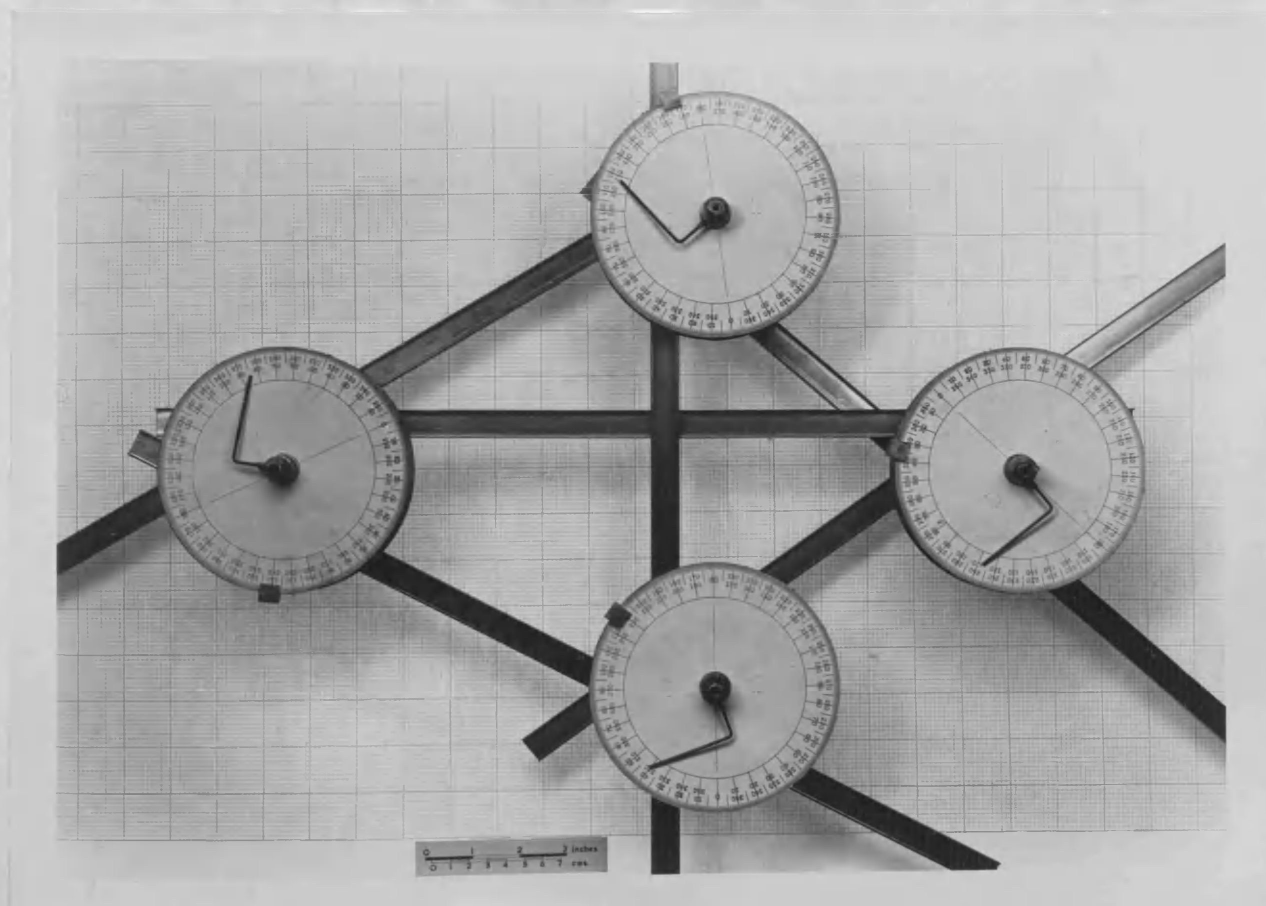


figure 2.28

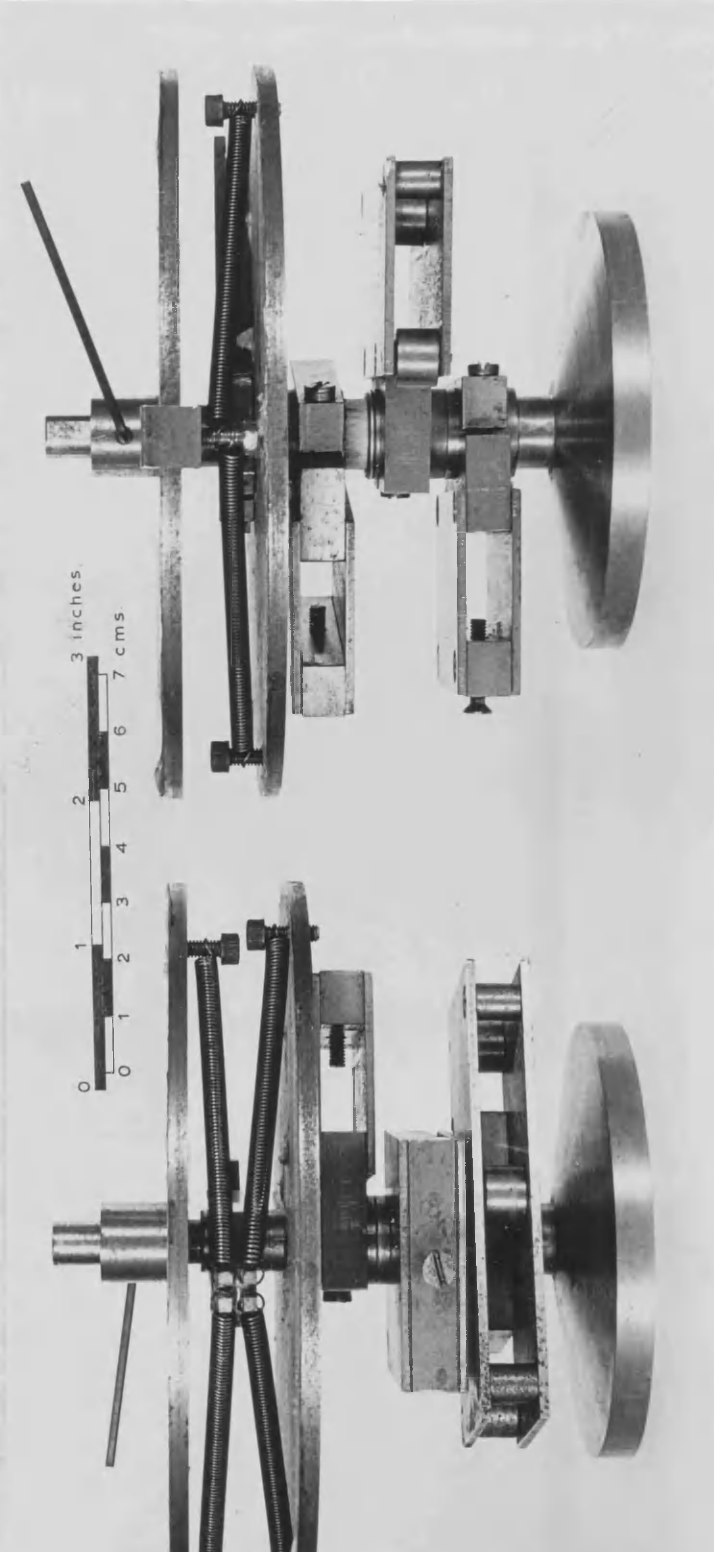


figure 2.27

A small channel shaped metal piece with a clamping screw can be set at any position on the circumference of the disc as indicated by a fine mark which is set against the required angular value. The two tension springs are connected to this piece as shown in the figure.

In figure 2.27, the flexible joint is assembled to show the connections. Beside the above mentioned components, there are clamping screws necessary for inserting the springs, and tightening the connections between spindles, discs and sleeves. The spindle height is made so that it allowes the assembly of the components necessary for adjusting three angles at one station.

It must be said here that the construction of the experimental model is much too heavy, so that the results obtained were perhaps a little less accurate than they could be, but most of all, the weight and friction resulted in more iterations and computations than would be necessary in a fully developed version.

2.4.1. Weights Applied to Observed Angles

The accuracy of observations in the field normally affects the weights used for the adjustments of these observations, but the weights here will be considered unity. The validity of this consideration is based on the conclusions drawn in 1.3.2.1 as this has been theoretically accepted when the adjustment of the combined net is considered, in which case it is also justified for adjusting angles only. Computers prefer to use the same weight for simplicity and for the reason that the observational equipment is now so improved and the accuracies normally achieved are so high that slight difference could be easily neglected. So all angles will be given the same weight and hence the springs used will be of the same stiffness and length. Even if different weights were accepted it would be necessary to give the angles unit weight for the following mechanical reasons:

(1) The space which can be allowed for the springs used is very restricted, which prevents different lengths of the same spring being used to represent these different weights.

(2) Springs of equal length, but of different stiffness, can be considered, but it was very difficult in practice to obtain sufficient reliability in springs which are normally available. It is also very difficult to obtain a sufficient range of springs of varying stiffness to correspond to all the weights which may be encountered.

2.4.2. Working Procedure

2.4.2.1. Preparing the Working Surface

The computation is carried out on a plane horizontal wooden surface such as plane chip board. This must allow the pins to be fixed with a minimum of effort, when marking the new stations after each iteration, or fixing the control points on the working board. For fixing or marking points a light hammer can be used. The area of this working surface must be large enough (say, $1.5m^2$) to allow the adjustment of different problems of varying area. However this area should not be so large that it might be difficult to shake the board as this is necessary to release any forces that may not be working due to the friction between the board and the base plates. The weight of the board must allow this vibration and not be deformed by it.

2.4.2.2. Reference Grid

Rectangular graph paper is used to provide a reference grid for the necessary control points and the computation which follows, so that the accuracy of this representation will be that of the graph paper itself. Normally millimetre graph paper is used, and in practice because of the pin hole size and the mechanical limitations no attempt has been made to ~~seek~~ scale off values better than this. If one millimetre represents one kilometre, (i.e. 1 : 1000,000) then any fraction of the kilometre will be neglected in this first assembly. However with each iteration the plotting scale of correction is progressively enlarged (e.g. 1:1000, 1:100, 1:10, or even 1:1). So there is a linear reference scale S_L, which should be fairly small for plotting the initial position of the control points. The correction scale (S_C for linear values) will be that of the reference scale S_L at the first assembly but will become progressively larger as each iteration is made. S_A is the angular scale and will be equal to S_C/S_L in each case.

2.4.2.3. Choice of Linear Reference Scale (S_L) of the Net

A small scale has the advantage of reducing the weight to be translated under the effect of the lateral forces acting at the different stations. A larger scale means longer rays, and consequently longer mechanical pieces. As the new points to be fixed by the mechanical analogue have to be translated by lateral forces acting at different stations, the scale used affects these forces considerably. For instance, if M is the moment acting at one station of the mechanical side, the other station will be translated due to a force = M/L, where L is the length of the side. As has been seen M is introduced by the tension applied between the two horizontal discs due to the contradiction used, and the elastic unit used will be of the same stiffness irrespective of scale the moment applied will be the same = unity (say). Hence the force acting and causing the translation of the different joints will be inversely proportional to its distance from the point of application of the moment in consideration.

The ideal case is to reduce the length of the sides to a minimum. The choice of the minimum length used is restricted by the size of either the base plate or horizontal discs used. So the necessary length of a side connected to two horizontal discs can not be less than $(2r + \delta)$, for the shortest side in the net, where r is the radius of the horizontal disc and δ is the maximum displacement caused by the different iterations required for the adjustment. Practice and experience with the experimental analogue show that the minimum length of a side used should not be less than 3r.

2.4.2.4. Setting up the Zero Assembly

The zero assembly or initial assembly represents the observed quantities at scale S_L and as already mentioned this will be the same for the zero and every other assembly. The angular values represent the observed angles in the mechanical net and can only be set to $\pm 0.5^\circ$ in the zero assembly. This is due to the fact that the disc will not be graduated to finer than 0.5° unless a disc of much larger diameter is used for the assembly, which is not desirable.

Setting the zero assembly results in the approximate location of the new points at the intersections of the sides. The coordinates of these points of intersection are obtained from the graph paper, and normally will be checked against the more accurate preliminary values obtained by calculation. The latter are preferred in order to save some iterations during the process of adjustment.

As the location of the points of intersection depends mainly on the direction of different rods, the difference of the height of **both** ends of the same rod will not affect this location provided that the spindles remain vertical. Sometimes due to difficulties in mechanical setting, it is advised to have the sides slightly inclined to the horizontal direction.

After assembling the dirction rods, and connecting the different joints using the original observations, the working surface together with the zero assembly is slightly shaken. To achieve this properly, the surface of the board is beaten by hand in a rapid drumming action. So releasing any strain that might be created by the assembly of the mechanical analogue.

The approximate positions of the stations are then marked by inserting the long steel pins through the hollow spindles and pricking through the graph paper. The assembly is then disconnected to mark up these positions and scale off coordinates.

2.4.2.5. Computation of the Angle Discrepancies

The scaled coordinates are used to calculate the directions (α), as for example from figure 2.10,

$$\tan\alpha_{1-2} = \frac{y_2 - y_1}{x_2 - x_1}$$

and angle $\theta_1 = \alpha_{1-3} - \alpha_{1-2}$

So the difference between the observed and computed values is obtained.

2.4.2.6. Choosing the Angular and Linear Correction Scales

This is the angular scale at which the discrepancies are introduced to the analogue. In determining this scale, it must be kept in mind that to obtain high accuracy using a mechanical analogue based on the method of systematic relaxation, the contradictions (discrepancies) must be introduced at an appropriate scale, which is neither too large or too small. If the contradictions are too large, the translation of joints will be fairly large too, this means that large values of stress and strain are introduced. In this case the relationship between stress and strain of the elastic material may not be a linear one, which means that there will not be a linear relationship between contradictions introduced and the resultant translation of the joints. If the scale is too small the contradictions introduced will not give appreciable translation to the joints.

The angular scale will of course be changed for each successive setting of the mechanical analgue. For example, in the first assembly an angle 156° 23' 30" will be set as 156° 30', the angular scale is 1:1. When a difference (say +5") is calculated between the observed and computed angles as given above, then this difference and those for the other angles will be inserted at a larger angular scale(say, 1":30', i.e.1:1800) and the new setting of the angle will be given the value of 156° 30' + (5x1800) = 156° 30'+2° 30' = 159°. The introduction of the difference to the mechanical analogue in this way is known as the contradictions. In practice when 1" was represented by 30' for the first iteration, then 1":1° (1:3600), 1":2° (1:7200), are found to be convenient for the successive iterations without any excessive strain being produced in the springs.

The relationship between linear scale S_L, angular correction scale S_A, and the linear correction scale S_C can be explained in the following,

The larger the linear scale S_L, the larger will be the amount of translation of joints. Also for the effect of the angular correction scale, the amount of linear translation of joints will be directly proportional to the quantity introduced in the form of contradiction, hence the linear correction scale.

If S_C is the linear correction scale used to convert the scaled units to the actual units, therefore $S_C = S_L.S_A$. S_L will be fixed from the start of the computation process, and S_C will be varied according to the change in S_A.

2.4.2.7. Computation of the First Correction to the Observed Angles

Having decided the angular correction scale S_A and hence the linear correction scale S_C, the mechanical analogue is re-assembled to give the first correction to the approximate coordinates. The same procedure is followed as before, but with the contradictions introduced. The working surface is again shaken properly to overcome any resistance to the forces acting on the mechanical pieces. When the analogue has reached the position of least energy conserved (when it has ceased to move) the new positions are pricked with the long steel pins. Then the coordinates differences between the initial points obtained from the zero assembly and those obtained by the new assembly are measured and given the symbols $\delta x_1, \delta x_2, \delta y_1$, δy_2, etc.... δx, and δy will be positive if the joint translates in the positive direction of x and y axes, and negative if it translates in the negative direction. The corrected direction will be obtained from the formula,

$$\tan\alpha_{12} = \frac{(y_2 + \Delta y_2) - (y_1 + \Delta y_1)}{(x_2 + \Delta x_2) - (x_1 + \Delta x_1)}$$

where $\Delta x = S_C.\delta x$, and $\Delta y = S_C.\delta y$.
From the new values of α_{12} and α_{13} the new values of the angle θ_1 will be obtained.

2.4.2.8. The Iteration Process

Usually the steps given by 2.4.2.6, and 2.4.2.7 with the new contradictions are repeated to reach the final solution. Due to mechanical and graphical limitations the final adjusted values cannot be obtained from a single iteration. The improvement obtained after each iteration should be appreciable, practical experience on the analogue shows that normally the largest discrepancies are reduced after the first iteration to a fifth of their original values. This could reach a tenth of their original values in certain cases.

The iteration process should be ended when no further improvement can be obtained, i.e. when the angular values obtained are the same after two successive iterations at least to within the order of corrections appropriate to these observations.

The number of iterations necessary also depends very much on the number of control points and the size of the problem to be adjusted. Mechanical limitations of friction and weight are usually the main reasons for having to increase the number of iterations necessary with larger problems.

The number of each iteration being given in different colour for each successive set up.

2.4.3. Practical Examples

Results using the experimental mechanical analogue for adjusting angles of different examples are given below.

2.4.3.1. Adjustment of an Equilateral Triangle

The simplest problem to be solved is to find the corrections to coordinates of a triangle, such that the corrected coordinates would satisfy the angle condition, $\Sigma\beta_i - 180° = 0$.

The least squares solution for such a problem is found by using Largange multiplier, for the correction equation,

$$a_1.\delta\theta_1 + a_2.\delta\theta_2 + a_3.\delta\theta_3 - k = 0 \qquad \text{........(2.76)}$$

the correction will be:

$$\delta\theta = \frac{a_i.k}{\Sigma a_i^2}$$

where $\frac{k}{\Sigma a_i^2}$ is the multiplier.

The solution of the triangle, where $a_1 = a_2 = a_3 = 1$ is

$$\delta\theta_1 = \delta\theta_2 = \delta\theta_3 = \frac{k}{3}$$

Another way of obtaining the corrections for angles is by mathematically adjusting the coordinates by variation of coordinates method and then computing the angles.

Using the mechanical analogue, the base line AB is fixed by the two fixed stations A and B, the three observed angles are set on the horizontal discs, and the initial positions of points are obtained. The contradictions between the observed and calculated values of the three angles are introduced to the appropriate joints. The joint C will then move to its equilibrium position, so that the corrections to its approximate coordinates are obtained. The three observed angles are:

$$A = B = C = 59^\circ \; 59' \; 59''.00$$

The base line is given by the fixed coordinates A(0.00, 0.00), and B(50,000.00, 0.00), approximate coordinates of the third station C is given by (25,000, 43,301).

Using the least squares method for the solution of this triangle and applying equation (2.76) we have:

$$(179^\circ\ 59'\ 57'' + \delta\theta_1 + \delta\theta_2 + \delta\theta_3) - 180^\circ = 0$$

and, $\delta\theta_1 + \delta\theta_2 + \delta\theta_3 - 3'' = 0$

therefore $\delta\theta_1 = \delta\theta_2 = \delta\theta_3 = \frac{3}{3} = 1''$

The solution of the same triangle by the mechanical analogue yields the following coordinates (25,000.00, 43,301.31), the corrected angles being $A = B = 60^\circ\ 00'\ 0''.05$, and $C = 59^\circ\ 59'\ 59''.90$. These results are obtained with a minimum linear scale S_L allowed by the horizontal discs (AB= 7.5 ins.)

Since the mechanical triangle satisfies the angle condition, $\Sigma\beta_i = 180^\circ$ exactly the amount of computation can be reduced, since only two angles of the triangle need be computed, the third one being obtained by subtraction of the sum of these from 180°.

As the analogue used is only an experimental model, the required solution was obtained after the third iteration process which is a very lengthy calculation when compared with the easy and quick way of the least squares solution.

2.4.3.2. Adjustment of a Parallelogram Figure with One Diagonal

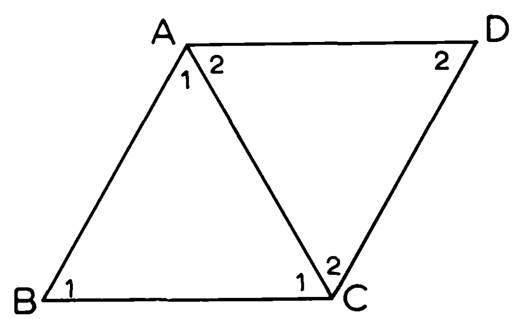


Figure 2.29

The purpose is to find out the change in the corrections obtained in the example solved in 2.4.3.1. when another identical triangle is added to it, figure 2.29.

$A_1 = B_1 = C_1 = 59^\circ \; 59' \; 59''.00$

$A_2 = D_2 = C_2 = 60 \; 00 \; 01.00.$

Coordinates of the base line AB are A(0.00, 0.00) and B(50,000.00, 0.00).

The calculated coordinates obtained from approximate observations are C(25,000, 43,301), and D(-25,000, 43,301).

Solution by the least squares is obtained by satisfying the angular conditions of the two triangles in separate stages. This solution gives the following corrections: $\delta A_1 = \delta B_1 = \delta C_1 = + 1''.00$

and $\delta A_2 = \delta D_2 = \delta C_2 = - 1''.00$

Procedure of computation by the mechanical analogue is exactly the same as before. The angles corrections obtained being identical after the fourth iteration, when the corrected coordinates were:

C(25,000.03 , 43,301.32), and

D(-25,000.03, 43,301.39).

Results obtained are more accurate than expected from such a heavy and complicated mechanical model.

2.4.3.3. Adjustment of a Braced Quadrilateral

As compared with the previous example two different triangles are used, figure 2.30.

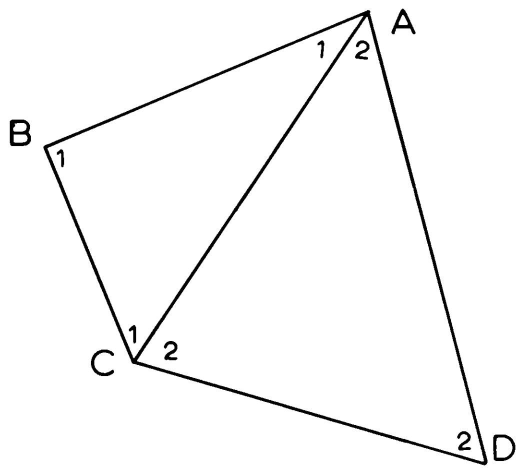


Figure 2.30

The angles used in the problem are:

A_1 =	32°	18'	19".00
B_1 =	91	01	43.00
C_1 =	56	39	57.00
Σ =	179	59	59.00

and,

A_2 =	42°	55'	02".00
D_2 =	59	50	57.00
C_2 =	73	14	02.00
Σ =	180	00	01.00

The fixed coordinates of the base line AB are (0.00, 0.00) and B(0.00 , 40,000.00). The approximate coordinates C and D from computation are C(25,582 , 40,459) and D(52,204 , 13,771).
Results obtained by the least squares solution are:

$$\delta A_1 = \delta B_1 = \delta C_1 = 0''.33, \quad \text{and } \delta A_2 = \delta D_2 = \delta C_2 = -0''.33$$

The corrections to angles when using the mechanical analogue are:

$$\delta A_1 = +0''.40, \quad \delta B_1 = +0''.70, \quad \text{and } \delta C_1 = -0''.10$$

and $$\delta A_2 = -0''.30, \quad \delta D_2 = -0''.70, \quad \text{and } \delta C_2 = 0''.00$$

obtained when the coordinates are C(25,582.70 , 40,459.43) and D(52,204.50 , 13,771.09).
Results obtained for this example are different to those obtained by the mathematical solution.

The reason for this difference is not a mechanical one at the joints as the connection between sides allows the same mechanical effect on the two sides of each angle, independent of the size of the angle itself. The only possible reason seems to be that the joints of this experimental model are too heavy to give the properly translatory movement when the angles are very large or very small. Probably very large forces will be needed to produce appreciable movement of some joints in this configuration.

2.4.3.4. Adjustment of a Doubly Braced Quadrilateral

When a doubly braced quadrilateral is adjusted, three horizontal discs have to be mounted and connected at all four points instead of the two required in the previous pair of examples. This adds to the weight and the complications of the connections. The connection necessary at a single joint is shown in figure 2.32. Obviously the difficulties over the translatory movement can only be overcome if the weight is reduced to a minimum. Only in this case would better results be achieved.

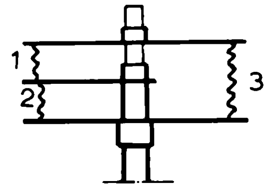


Figure 2.32

2.4.4. Possibilities of Using the Mechanical Analogue for Angular Adjustment

As can be seen from the details of the construction the mechanical analogue for angular adjustment is complicated and lacks the real representation of the observations in the field. As mentioned before, the construction of such an analogue tries to produce a model which will allow direct forces to act between the sides of an angle, so avoiding the necessity for station correction. The work with the experimental analogue recounted above showed that:

(1) Real observations are not represented. For instance, the direction observed at both ends of a side requires two sides in the mechanical representation. This necessitates the use of twice as many mechanical pieces, as would be required for the construction of an analogue which did not include such a consideration. The extra mechanical pieces mean heavier and more complicated models.

(2) As the complexity and therefore weight increases, larger forces are found necessary to translate joints and these cause excessive strain on the elastic units.

(3) A centered point figure in a net requires the following mechanical connections if all possible angles are considered at this station.

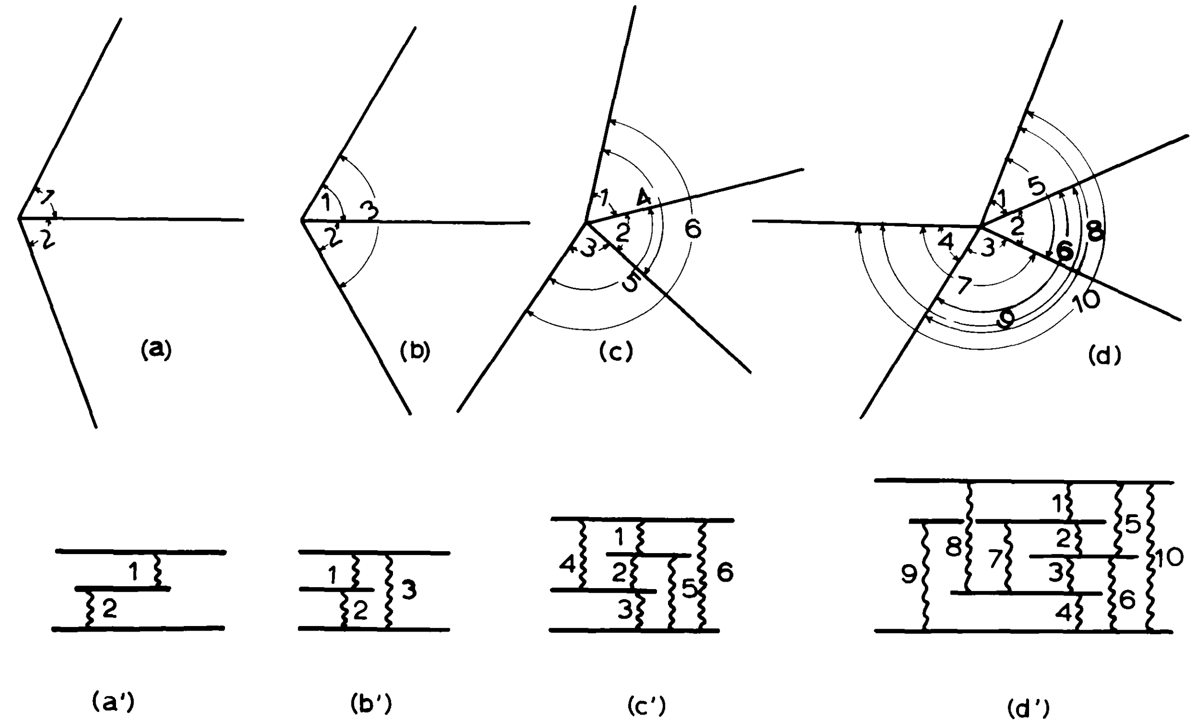


Figure 2.33

In figure 2.33 (a) two angles are derived from three observed directions, and in figure (b) three are considered. The mechanical connections for these are given in (a') and (b') respectively. In both figures the mechanical representation has the same sides but the number of elastic units will vary due to these different considerations.

If all possible angles are considered at the central point, the number of the mechanical connections required for adjustment will increase very rapidly. For instance, if all possible angles derived from four directions are considered six connections are necessary as given in figure (c). For five observed directions the number of connections which will represent all possible angles will be 10 as shown in figure (d). Table 2.4, shows the rapid increase in the number of connections as the number of directions increase.

Number of observed directions = n	Number of mechanical connections		
2	1	$= \frac{n}{2}(n-1)$	= 1
3	1+2	$= n(\frac{n-1}{2})$	= 3
4	1+2+3	$= \frac{n}{2}(n-1)$	= 6
5	1+2+3+4	$= n(\frac{n-1}{2})$	= 10
6	1+2+3+4+5	$= \frac{n}{2}(n-1)$	= 15
7	1+2+3+4+5+6	$= n(\frac{n-1}{2})$	= 21

table 2.4.

The complications in constructing the mechanical assembly required for such observations will be enormous. When fifteen or twenty one forces are applied at a single nodal point the difficulties in making the requisite connections between six or seven discs at this joint will be apparent. The weight is very great and probably different sizes of disc would have to be introduced.

If one abandons the procedure for deriving all possible angles given above and adopts the simpler procedure of considering only the angles derived from adjacent directions the number of connections required will be equal to (n-1), where n is the number of directions observed, e.g. for four rays there are three angles as shown in figure 2.34. This reduces the number of connections necessary but still requires the same number of discs as before, so that while the complications are reduced the weight is not.

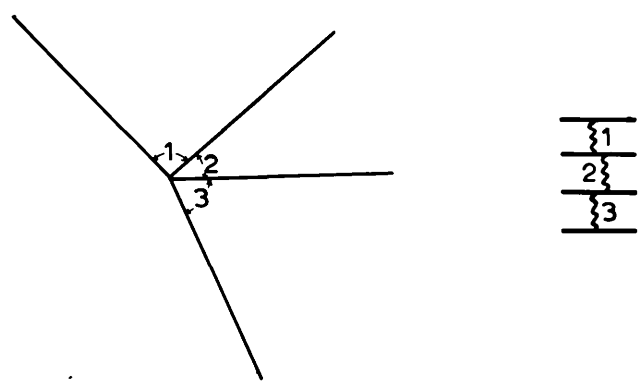


Figure 2.34

(4) If further development was carried out, especially on weight reduction, some of these difficulties would be overcome and much better results obtained. However it was felt that the use of a mechanical analogue for the type described for adjusting

angles is restricted to rather small nets and even if developed further it would not be capable of giving the required corrections for a larger survey nets especially where first or secondary order was required. It might be used for problems of tertiary nets, but such a restricted use would not justify the further effort required.

However this first analogue did prove that many of the basic ideas were sound and that a fresh solution along somewhat different lines would be more fruitful.

2.5. MECHANICAL ANALOGUE FOR THE PROBLEM OF DIRECTIONS ADJUSTMENT

The main purpose of this second construction is to investigate the possibilities of adjustment of directions which give promise of simpler connections at the joints, less complications overall and of lighter weight.

The basic distinction between the adjustment of angles and that of directions is given in 2.3.5.

2.5.1. Analogy Between the Least Squares and Mechanical Solutions

The analogy between equations (2.74) and (2.75) with equation (2.69) given in 2.3.2.1. and 2.3.2.2. is obtained after subtraction of two directions from one another for each angle. This means that the analogy made for angle adjustment is also valid for directions adjustment, e.g.,

$$d\alpha_{AC} = \frac{r_2}{l_{AC}} \cdot dx_A - \frac{r_2}{l_{AC}} \cdot dx_C - \frac{q_2}{l_{AC}} \cdot dy_A + \frac{q_2}{l_{AC}} \cdot dy_C \quad \ldots\ldots (2.60)$$

When the coordinates of station C are fixed, equation (2.60) will be rewritten as,

$$d\alpha_{AC} = \frac{r_2}{l_{AC}} \cdot dx_A - \frac{q_2}{l_{AC}} \cdot dy_A \quad \ldots\ldots\ldots\ldots (2.77)$$

Similar equations will be derived for all the other directions observed in the net. To solve this problem by the least squares, the normal equations will be formed in the usual way using the following observation equations:

$$v_{AC} = -(r_A + \varphi_{AC}) + \alpha_{AC} + d\alpha_{AC} \qquad \text{........(2.53)}$$

For a triangle ABC figure 2.12, with the base line BC the following observation equations are obtained:

$$\begin{aligned} v_{AC} &= [-(r_A + \varphi_{AC}) + \alpha_{AC}] + d\alpha_{AC} \\ v_{AB} &= [-(r_A + \varphi_{AB}) + \alpha_{AB}] + d\alpha_{AB} \end{aligned} \qquad \text{........(2.78)}$$

Substituting equation (2.77) into equation (2.78) we have:

$$v_{AC} = -k_{AC} + \frac{r_2}{l_{AC}}.dx_A - \frac{q_2}{l_{AC}}.dy_A \qquad \text{........(2.79)}$$

similarly for direction AB we have the following observation equation:

$$v_{AB} = -k_{AB} + \frac{r_1}{l_{AB}}.dx_A - \frac{q_1}{l_{AB}}.dy_A \qquad \text{........(2.80)}$$

The normal equations are therefore:

$$A'A = \left|\begin{array}{c|c} \frac{r_1}{l_{AB}}.\frac{r_1}{l_{AB}} & -\frac{r_1}{l_{AB}}.\frac{q_1}{l_{AB}} \\ \frac{r_2}{l_{AC}}.\frac{r_2}{l_{AC}} & -\frac{r_2}{l_{AC}}.\frac{q_2}{l_{AC}} \\ \hline -\frac{q_1}{l_{AB}}.\frac{r_1}{l_{AB}} & \frac{q_1}{l_{AB}}.\frac{q_1}{l_{AB}} \\ -\frac{q_2}{l_{AC}}.\frac{r_2}{l_{AC}} & \frac{q_2}{l_{AC}}.\frac{q_2}{l_{AC}} \end{array}\right| \quad K = \left|\begin{array}{c} -\frac{r_1}{l_{AB}}.k_{AB} \\ -\frac{r_2}{l_{AC}}.k_{AC} \\ \hline \frac{q_1}{l_{AB}}.k_{AB} \\ \frac{q_2}{l_{AC}}.k_{AC} \end{array}\right| \qquad \text{..(2.81)}$$

In mechanical terms equilibrium of figure 2.13 requires that $\Sigma X = 0$, and $\Sigma Y = 0$, and according to equations (2.72),

$$\sum_1^2 X = \sum_1^2 \frac{r_1}{l_{AB}}.v_{AB}\,, \quad \text{and} \quad \sum_1^2 Y = \sum_1^2 -\frac{q_1}{l_{AB}}.v_{AB} \qquad \text{........(2.82)}$$

$$\begin{aligned} \Sigma X &= \frac{r_1}{l_{AB}}\left(-k_{AB} + \frac{r_1}{l_{AB}}.dx_A - \frac{q_1}{l_{AB}}.dy_A\right) + \\ &\quad + \frac{r_2}{l_{AC}}\left(-k_{AC} + \frac{r_2}{l_{AC}}.dx_A - \frac{q_2}{l_{AC}}.dy_A\right) \\ &= \left(\frac{r_1}{l_{AB}}.\frac{r_1}{l_{AB}} + \frac{r_2}{l_{AC}}.\frac{r_2}{l_{AC}}\right).dx_A - \left(\frac{r_1}{l_{AB}}.\frac{q_1}{l_{AB}} + \frac{r_2}{l_{AC}}.\frac{q_2}{l_{AC}}\right).dy_A \\ &\quad - \left(\frac{r_1}{l_{AB}}.k_{AB} + \frac{r_2}{l_{AC}}.k_{AC}\right) \end{aligned} \qquad \text{.......(2.83)}$$

also,

$$\begin{aligned} \Sigma Y &= -\left(\frac{q_1}{l_{AB}}.\frac{r_1}{l_{AB}} + \frac{q_2}{l_{AC}}.\frac{r_2}{l_{AC}}\right).dx_A + \left(\frac{q_1}{l_{AB}}.\frac{q_1}{l_{AB}} + \frac{q_2}{l_{AC}}.\frac{q_2}{l_{AC}}\right).dy_A \\ &\quad + \left(\frac{q_1}{l_{AB}}.k_{AB} + \frac{q_2}{l_{AC}}.k_{AC}\right) \end{aligned} \qquad \text{.....(2.84)}$$

By inspection of equations (2.81), (2.83) and (2.84) the analogy between the mathematical and mechanical solutions can be seen.

2.5.2. Construction Difficulties

In constructing this new analogue the main aims were to avoid the troubles encountered during the construction of the first analogue for angular adjustment and to avoid difficulties over the special features of a direction adjustment such as the station corrections.

The points to be borne in mind were:

(1) The avoidance of the excessive weight in the joints themselves.

(2) The horizontal disc size must be chosen so that the large linear scale is avoided.

(3) The connections at joints must be as simple as possible.

(4) There must be a full representation of the observed directions.

(5) There must be a simple application of forces at joints.

(6) A mechanical analogy for a station correction should be incorporated.

(7) An improved design for elastic unit has to be devised.

2.5.2.1. Experimental Analogue for Adjustment of Directions

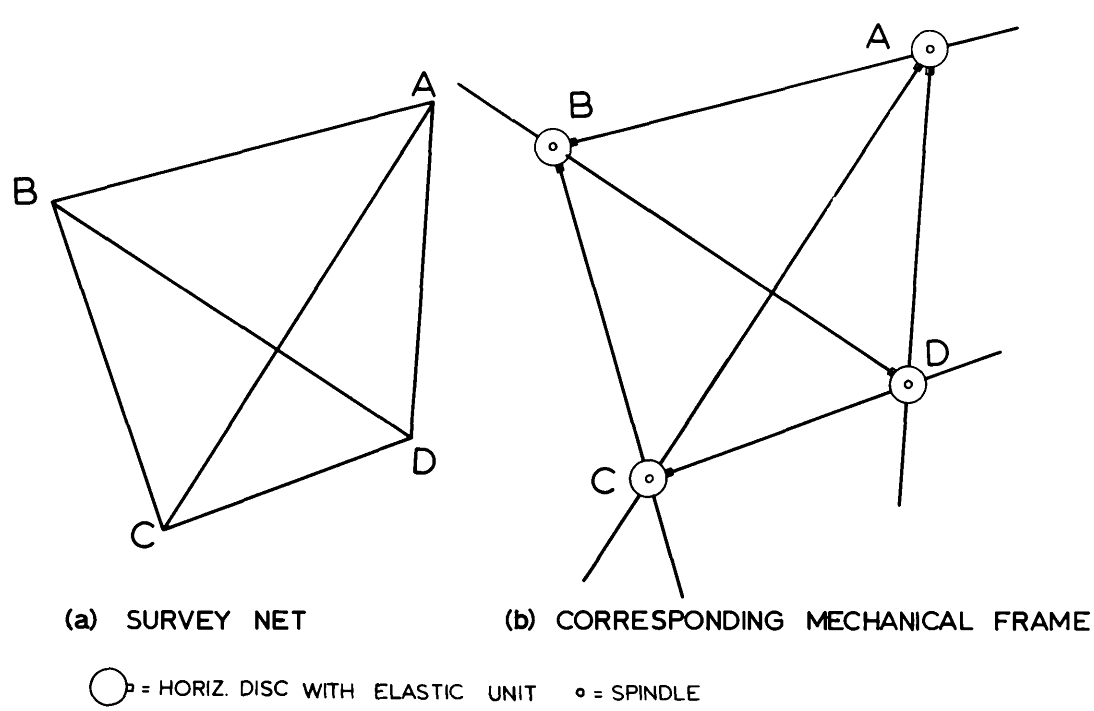


Figure 2.35

The analogue design is shown diagrammatically in figure 2.35 and as a photograph in figure 2.36, the basic components of its contruction are as follows:

(i) Horizontal disc, figure 2.37d, the diameter of which is 2.5 inches, made of light alloy, and fitted with a coller and clamping screw so that each closely fits on the vertical spindle. Sometimes when the net has small angles or very close directions, an extra horizontal disc has to be added to the same spindle, but this does not result in a large increase in the weight at this special joint.

(ii) Hollow steel spindle, figure 2.37c, having an inner diameter to take the marker pin. This spindle fits into a boss on the centre of each base plate.

(iii) An elastic unit, which is mounted on the horizontal disc to give the connection to each direction. The number mounted on a given disc equals the number of directions to be adjusted. Each unit consists of two pieces. (a) Piece A, figure 2.37e which is made from a block of aluminuim. A groove is cut into the block and the horizontal disc is inserted into this and clamped. This block is connected to a long tongue of aluminuim to form a T-shaped piece which has two vertical metal posts which are removable. The distance between these two posts is set to suit the length of the two elastic springs. (b) Piece B, figure 2.37f which is attached to the sides representing the directions. It is also T-shaped with a hollow slot cut in the cross piece into which the side can be inserted and slide freely along. A rectangular slot at the foot of the upright has a wire inserted with two eyelets, one at each end to allow the two vertical metal posts from piece A to be inserted. Along this wire two compression springs are mounted. The wire is used for keeping the springs in position so that they act always along a single line.

(iv) Marker pin, figure 2.37b, made of hard steel used to prick the positions of new stations on the working surface and to keep the control points fixed in position.

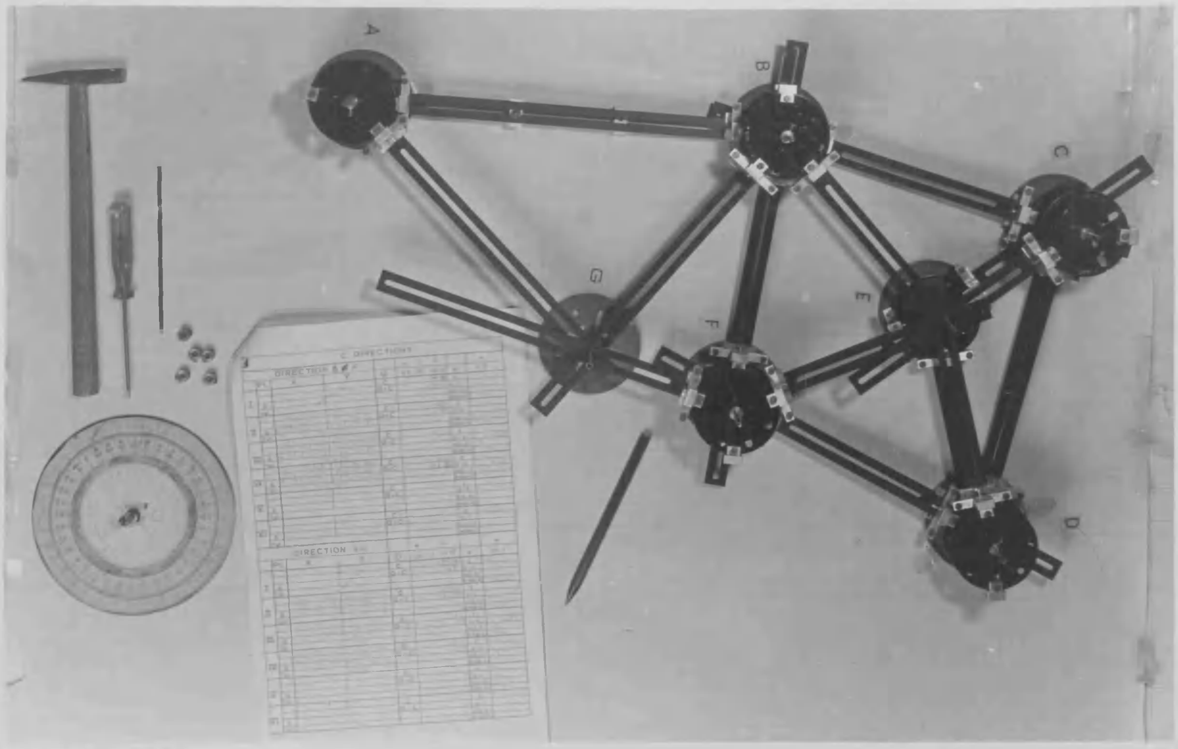


figure 2.36a

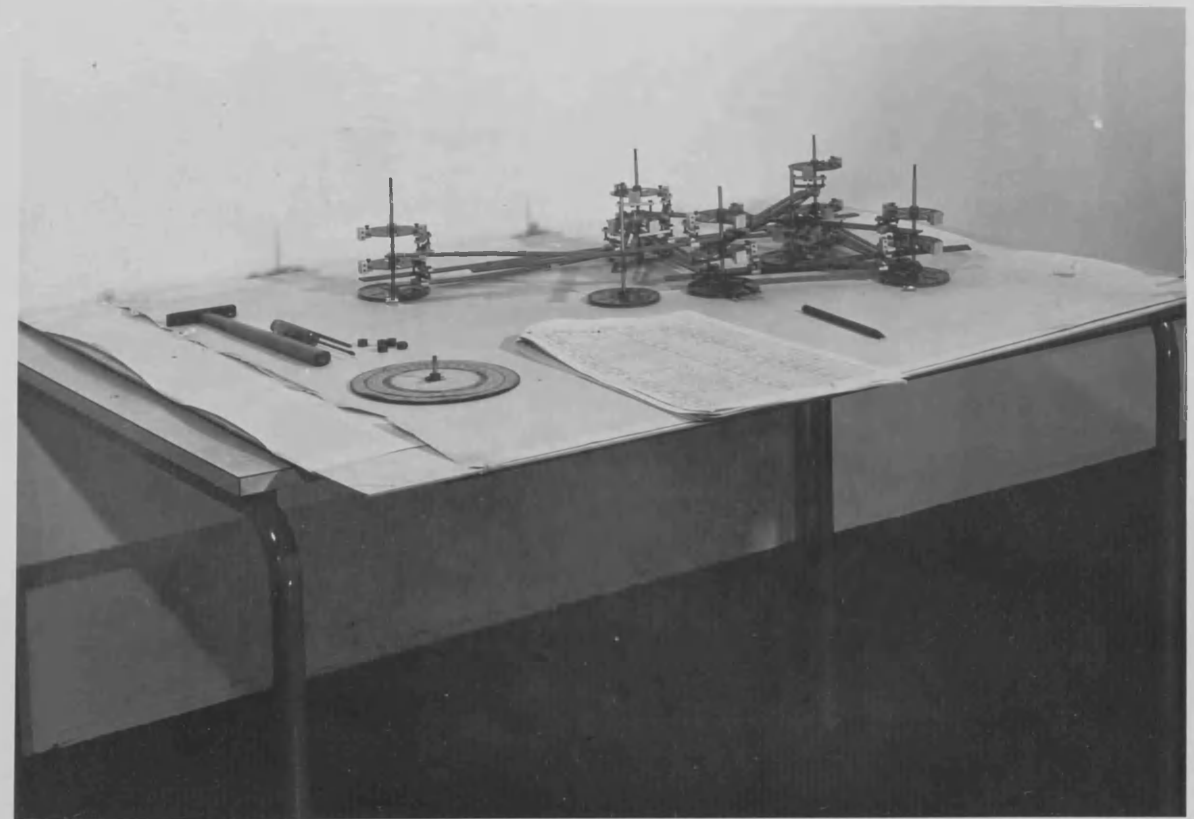


figure 2.36b

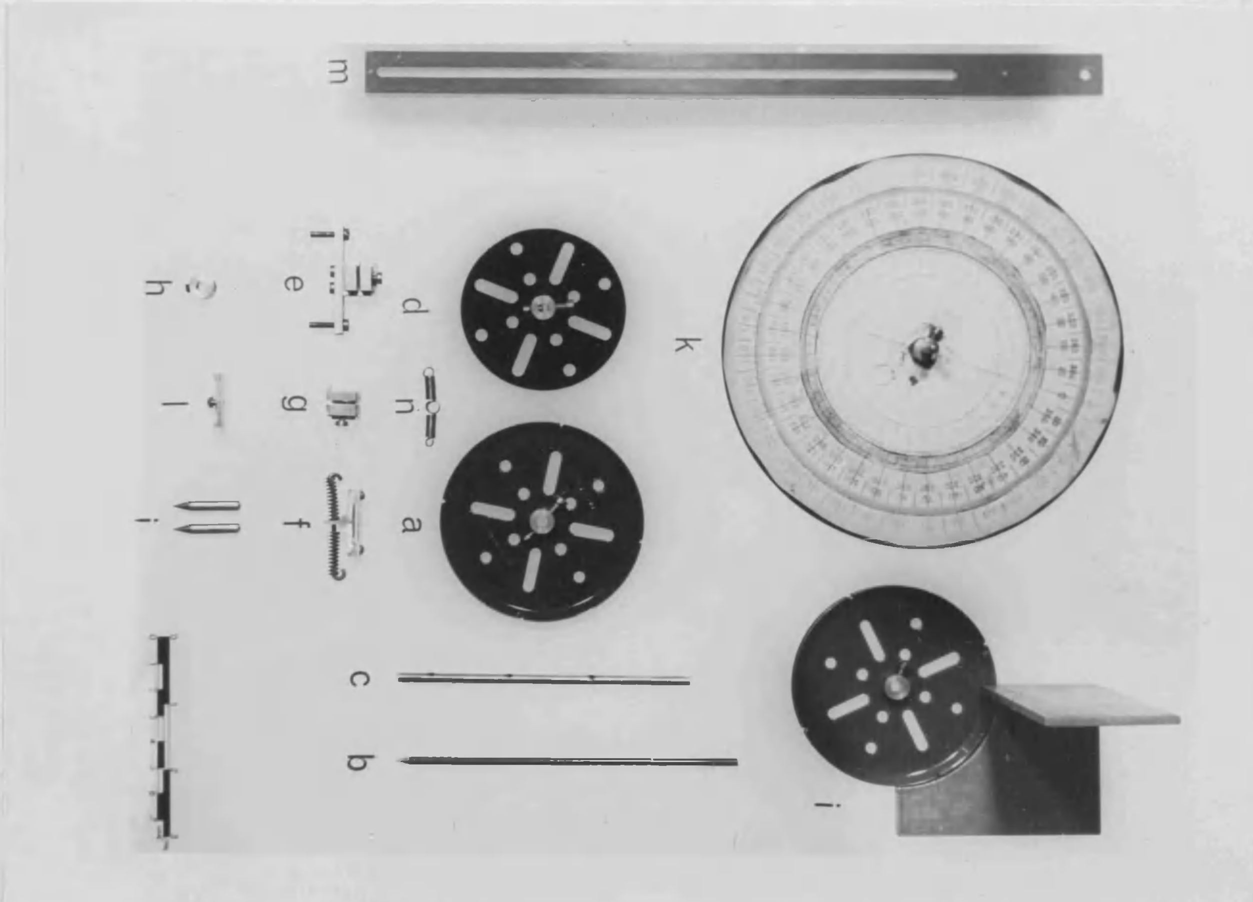


figure 2,37

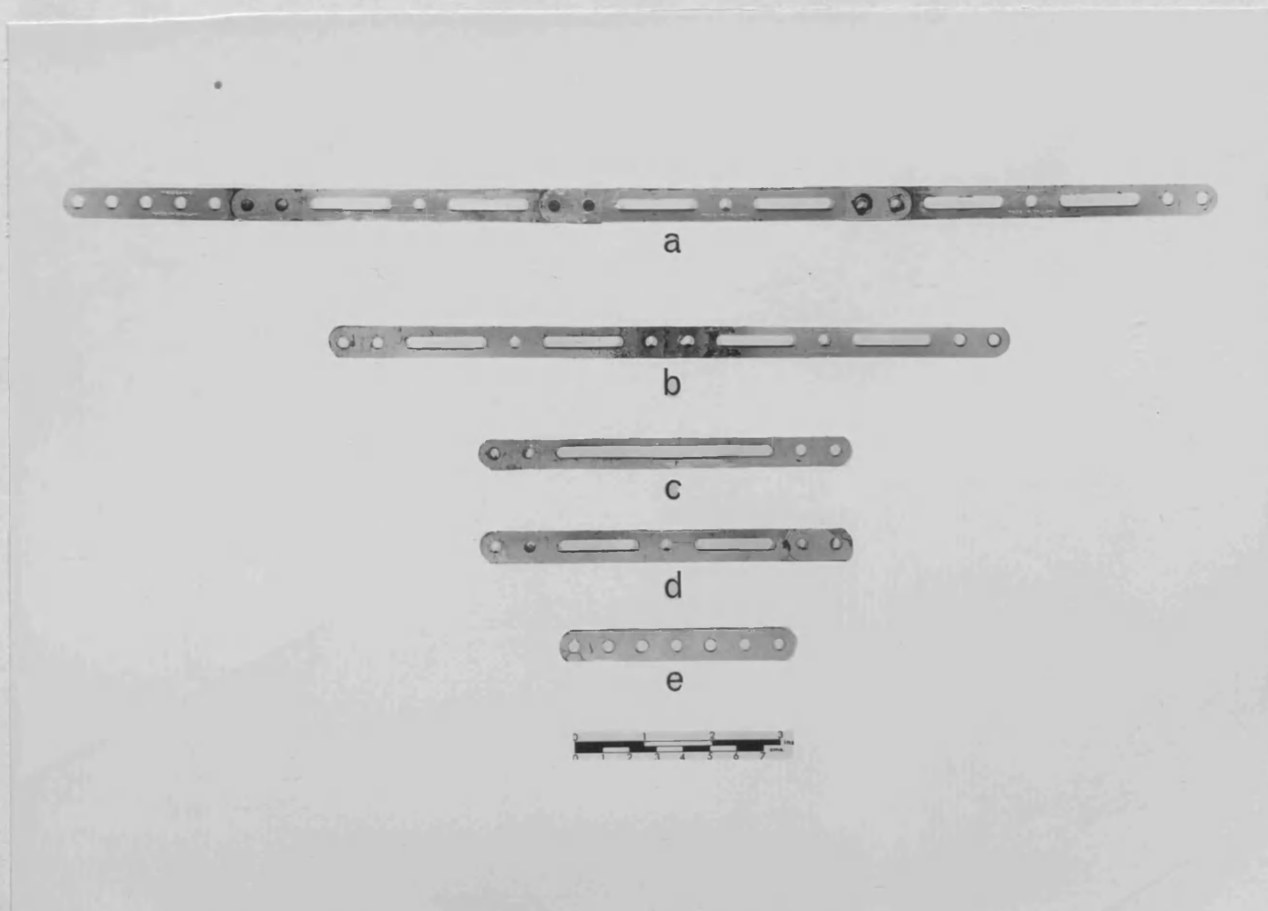


figure 2.38

figure 2.37a

(v) Base plates, figure 2.37a, of very light alloy. Each has a diameter of 3 inches with a boss (or a collar) into which fits the vertical spindle, and a screw to act as a clamp on the vertical spindle.

(vi) Sides, figure 2.37m, to achieve maximum light weight with sufficient rigidity in the lateral direction, 5/8 inches Tufnol strip has been used with a slot cut in to accept the vertical spindle and to allow it to translate along the side with minimum friction. Various lengths, 8, 11, 14, 17, and 20 inches have been fabricated to cope with the varying length of side likely to be encountered. With longer strips, aluminuim clamps are used to prevent twisting or compression or widening of the slot when lateral forces are applied. The material has been fabricated to give smooth surface which results in minimum friction when movement of the analogue is in progress.

As a first attempt metal alloy sides adopted from Meccano pieces were tried, figure 2.38, but difficulty arose in getting long enough sides. When these were constructed from individual smaller pieces, it was difficult to cut a single straight slot so recourse was made to the Tufnol, which had the additional benefit of being still lighter.

The weight of each component is, (i) 18.30 grms., (ii) 4.54 grms, (iii) 12.99 grms, and (v) 40.00 grms. Total weight of all components at each joint is 75.83 grms. This weight could be reduced still further to about three quarter of that given.

2.5.2.2. Disc Size and Linear Scale

As already discussed the linear scale chosen has a great effect on the performance of the analogue, in general the smaller the scale the better the results obtained. As the minimum linear scale attained is governed by the larger diameter of either the horizontal disc or the base-plate. These two diameters have to be chosen with a diameter which is small, yet will prevent the joint being tilted or overturned when forces are applied. A 3 inches diameter has been used for the base-plate. In this analogue the minimum side length is made = 2d, which allows adequate space for the translation of the joints, than 3r given before, which gives 2x3 inches against the 7.5 inches given before.

2.5.2.3. Connection of Sides and Joints

In the first experimental mechanical analogue for angles, a horizontal disc was provided for each side, so that when the discs are elastically connected a certain rigidity is induced. Also each disc required the provision of a separate spindle. The system in second analogue is designed such that normally one disc is sufficient to provide a connection to a number of sides. When two of these discs are used, these are connected as a single unit to the individual spindle which is simpler and allows a much freer movement than before. Obviously with two discs a common zero should be retained at all the times during the adjustment procedure. This can be obtained either by using a vertical marker, figure 2.37i, or by aligning the two by eye. In practice the latter is much easier and quicker, while still retaining the same precision. When the two are aligned they are clamped together via the spindle which is free to rotate in its boss.

The simplicity of the connection between a side and the horizontal disc makes it possible to attach as many as eight sides, if these are evenly distributed. Although the elastic joint has the rigid cross piece of the T with two posts which keeps the two directions apart by an angle of less than 60°, this angle can be made still smaller (to about 45°) by a special connection, figure 2.39. In figure 2.40 side a is connected to the underside of the disc, while side b is connected to the top side of the same disc. In this case the size of the cross piece will not have any effect on the size of the angle between the two directions. The size of the block being the deciding factor. When this special type of connection fails to give room for very small angles between pairs of directions another disc has to be

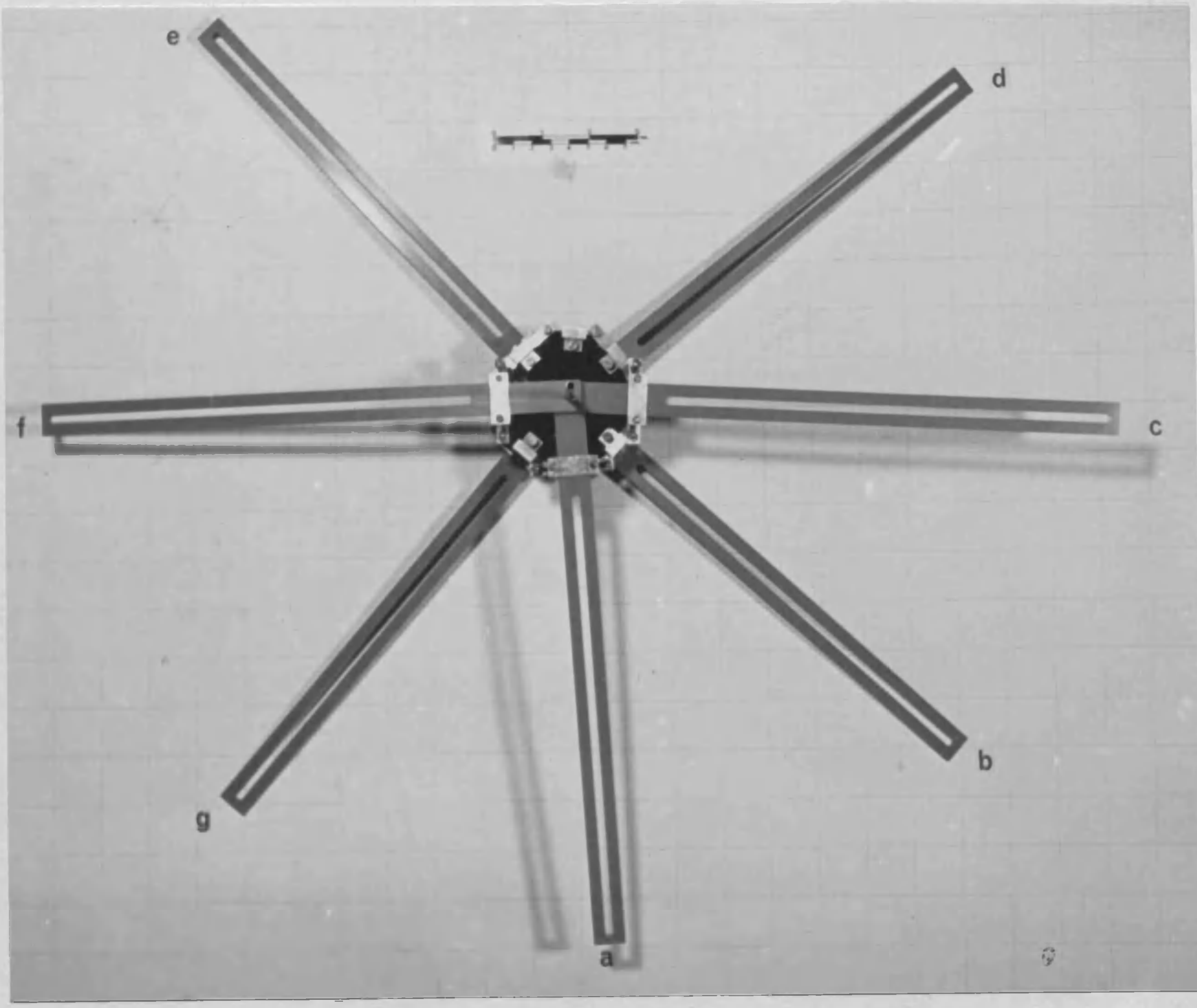


figure 2.39

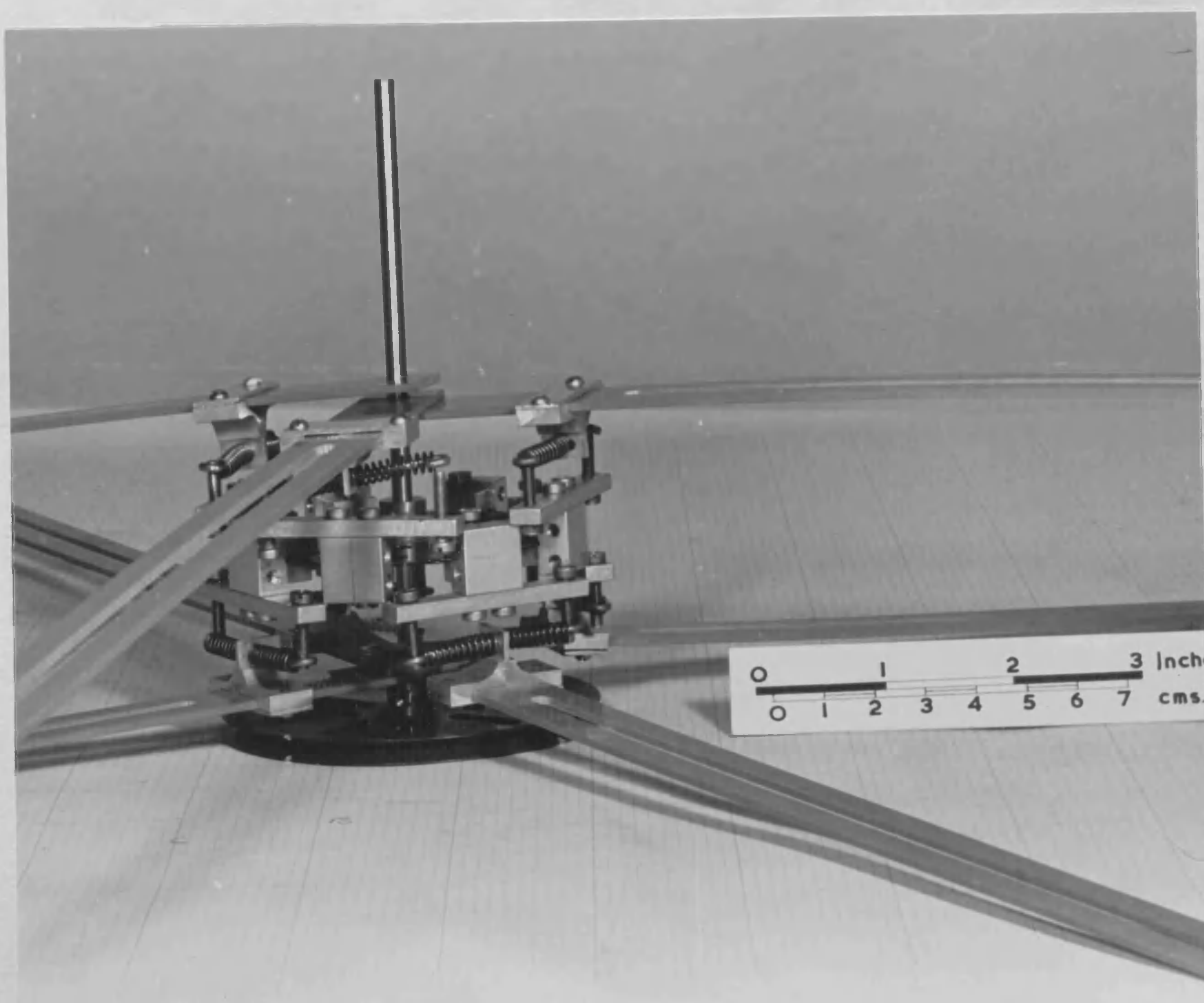


figure 2.40

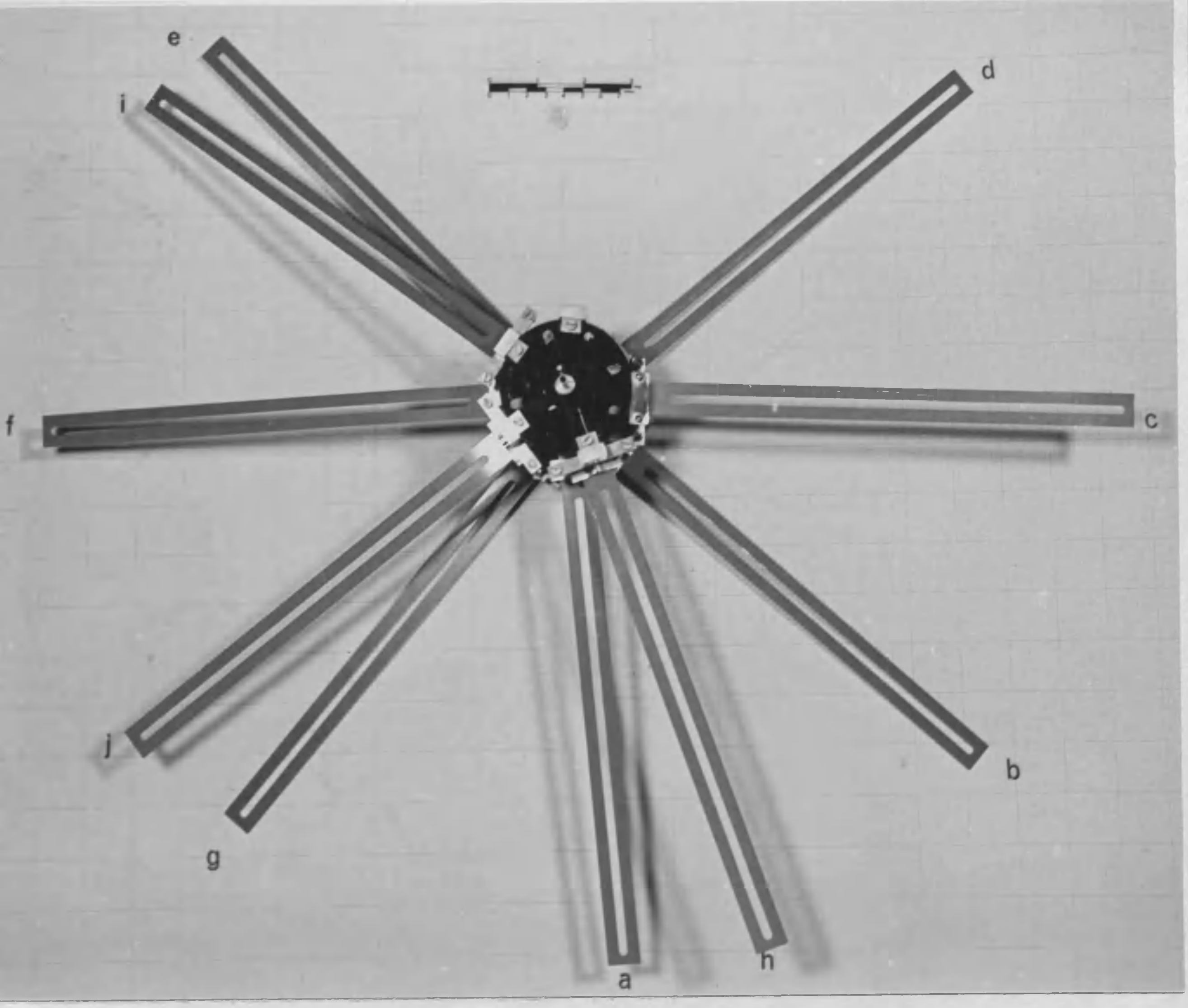


figure 2.41

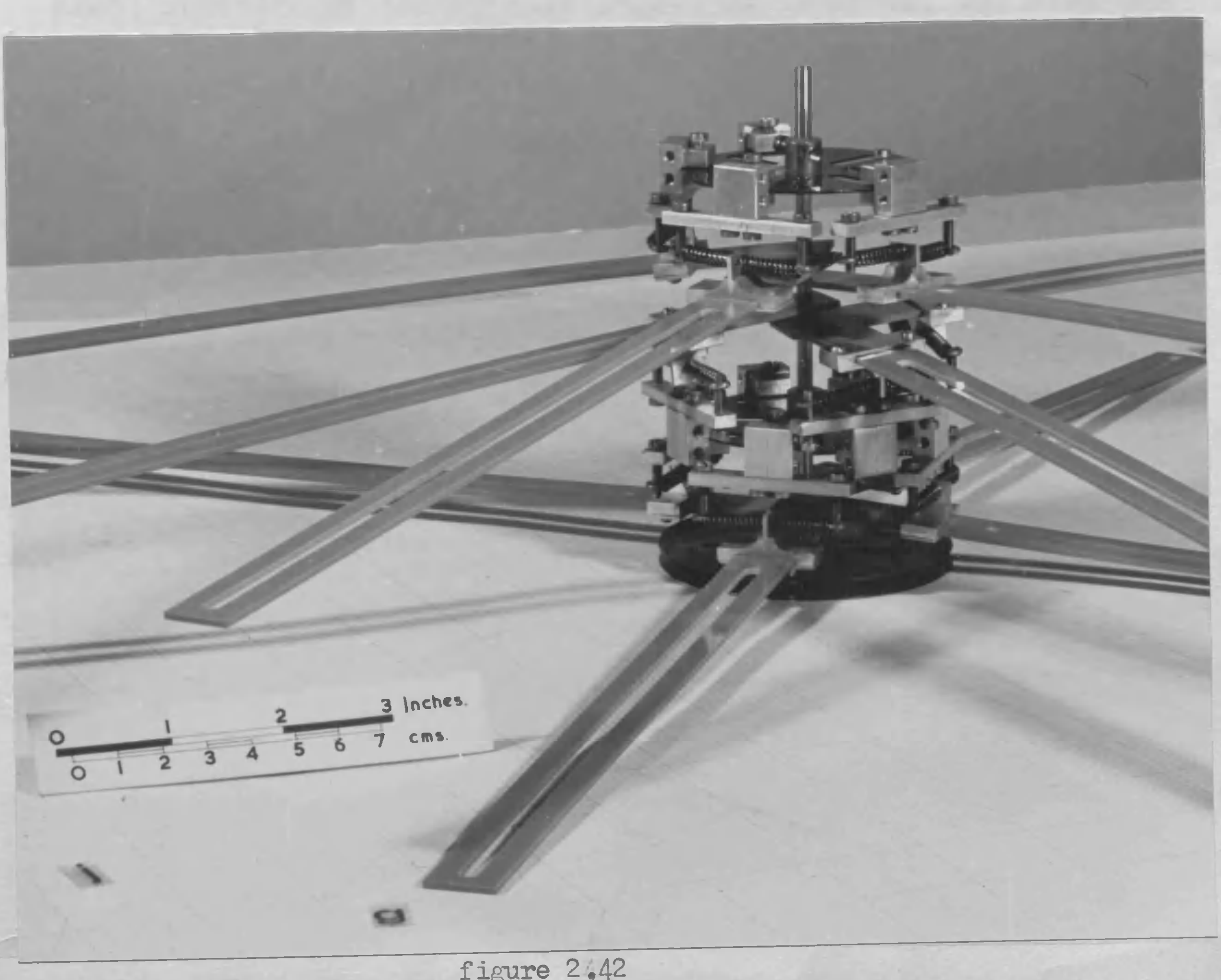


figure 2.42

added, figure 2.41, and 2.42. This would apparently allow sixteen directions (eight to each disc) but in practice a reduction to eight or ten (four or five to each disc) is advisable, because large contradictions might be introduced as angular changes at large scales. If the connections between two discs and spindle remain really rigid under the different forces applied, there is no difficulty in increasing the number of discs to three giving a total of twelve directions at one joint. In practice there is no need to provide for such a number of directions at one station, six being a practical maximum. So a third disc will not normally be needed.

Although the double disc functions just as accurately as the single disc, the following differences will exist:

1- For one disc, setting and assembling will be done very quickly but the time will obviously increase with two discs which affects the speed of this operation.

2- When a single disc is used, rotation of this disc about the spindle will be very easy. However when more than one disc is used, rotation of the spindle itself is involved, and care has to be taken to prevent this sticking in the boss.

2.5.2.4. <u>Correct Representation of the Observed Directions</u>

Solution by the least squares method is recognised as the most reliable way of adjustment, since apart from other virtues, it offers the possibility of including every observed element in the mathematical model during the computation, e.g. when a direction is observed and has two different readings at both ends. To include such observations in the solution of the same problem by using the mechanical analogue for angular adjustment, leads, as have been seen, to a very complicated and heavy analogue.

The adjustment of directions by a mechanical analogue is made simpler since all observed quantities are easily included. In this case each direction is considered to be a separate element to be adjusted. Each of the two directions at both ends of the same line will have a separate correction. The mechanical

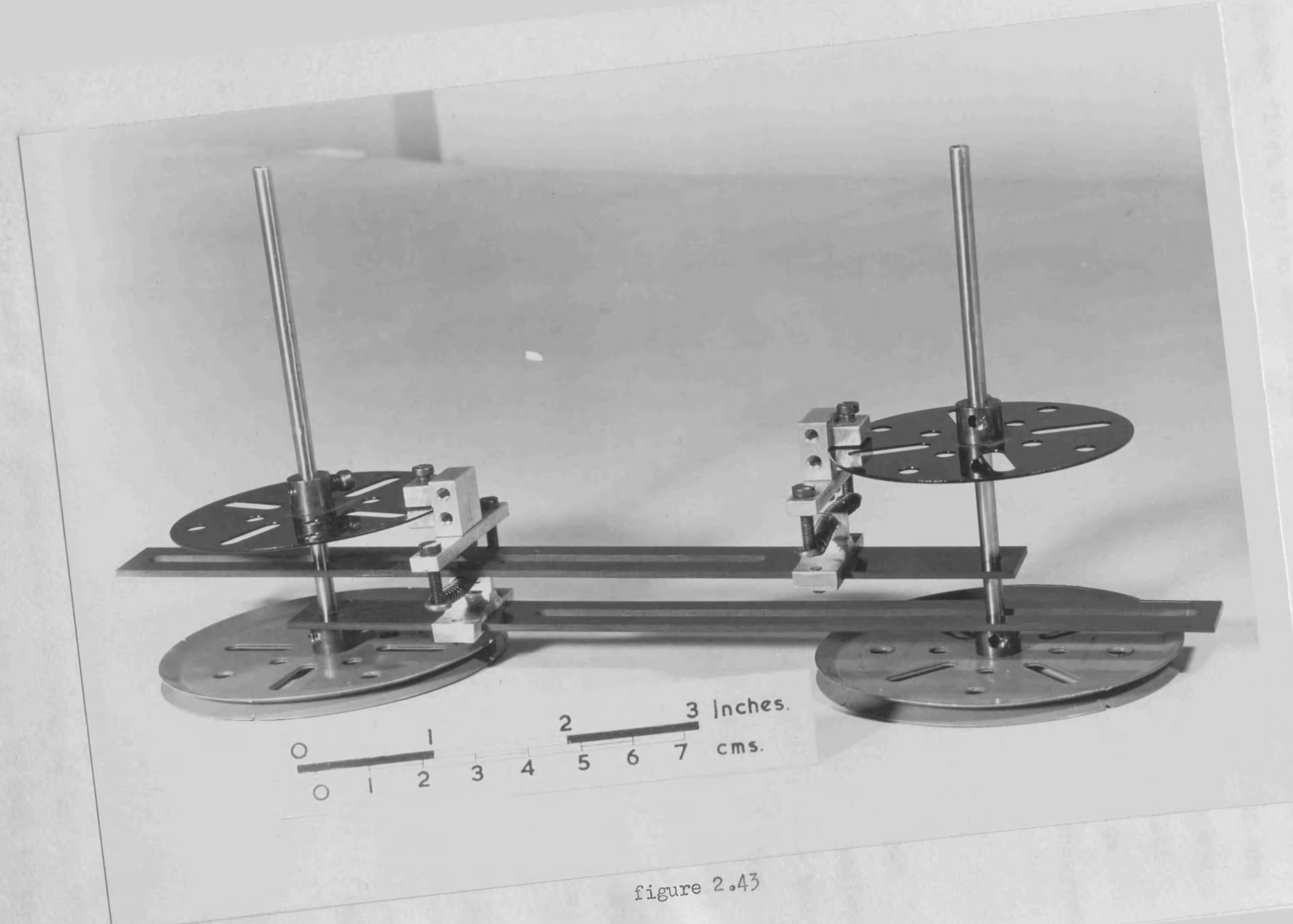


figure 2.43

arrangement to allow this is shown in figure 2.43. In this figure the side representing one direction is elastically connected at the first station and freely attached to the second station, while the other is elastically connected at the second station and freely attached to the first. The elastic connection of one side at a station should allow sufficient room for the free end of the other side to go through the spindle. The elastic unit consists of the two separate pieces A, and B to allow such an arrangement. So a correct and easy representation of the observed quantities in the field could be obtained.

2.5.2.5. Application of Forces

The springs used for adjusting the angles in the first analogue were connected to the two separate discs which transferred the forces to the two corresponding collars and then to the two component sides of the individual angle. When directions are being adjusted in the second analogue, connection will be made directly to the horizontal disc. Thus all the forces applied will be acting on the same disc and so to all the sides passing through that point. This is a more direct application of forces than in the first case.

2.5.2.6. Station Correction

As mentioned in 2.2.3.2.2. a station correction is necessary for the adjustment of triangulation net when using the direction method. Using a mechanical analogue for the same purpose requires the same treatment. This could be explained fully by the following:

The function of the horizontal disc in the mechanical analogue system is to represent the horizontal circle of the theodolite making the observations. Setting up the observed quantity for each direction on the disc can be made to an accuracy of 30' of arc. In the ordinary least squares solution, the station correction provides a specific angular shift of the zero of the theodolite circle to help provide a best fit of all the observed directions. Such a shift is necessary if a single value for each direction observed from both ends of a line is to be achieved. The possibility of a rotation of each horizontal disc in the analogue has to be provided also and must be made under the influence of all the readings at the different stations. This is done quite automatically by the arrangement which has been described

2.5.2.7. Design of Elastic System

Since the analogue must be kept as small as possible, only limited space exists for inserting the elastic units. The springs chosen therefore have to be short, figure 2.37f. Considerable work was carried out on different types of springs and on different possible arrangements.

Flat steel springs were connected vertically along the vertical sides of the aluminuim block to straddle the side, to provide the elastic change required. These springs have such a small elastic stiffness that even the light joints were not translated properly.

Tension springs showed the same difficulty in use due partly to the necessity of also keeping the two springs under tension all the time.

Compression springs are found to give suitable translation to the joints but as already seen they require the introduction of a wire to guide the line of action of each pair of springs. The wire is chosen to satisfy the purpose only, i.e. stiff enough to withstand deformation due to applied forces, and light enough to share in minimising the weight. Also they have to be soft enough to allow easy workability to give the slight curvature necessary as a route for a point moving on a circle.

2.5.3. Working Procedure

Much of the procedure given in 2.4.2. for adjustment of angles is followed with one or two additional steps due to the rather different design.

2.5.3.1. Use of a Separate Setting Device

Connecting several sides to one disc in a convenient way requires that a separate setting device should be used, figure 2.37k. This is rather similar to the disc itself and consists of a protractor mounted on the usual spindle. When only one horizontal disc is required, all the directions are set by reference to the protractor using the fine mark on the block of the elastic unit as an index. They are then clamped to the horizontal disc. With two discs it is obviously necessary to use the common zero with mark on each disc, but this plays no part in setting a single disc.

2.5.3.2. Calculation of Directions

The calculation of contradictions will be also different to that given before, and has the merit of being simpler and involving less computational work. In this case,

$$\tan\alpha_{12} = \frac{(y_2 + \Delta y_2) - (y_1 + \Delta y_1)}{(x_2 + \Delta x_2) - (x_1 + \Delta x_1)}$$

where x and y are those given in 2.4.2.7.

The contradictions will be the difference between, for example, a calculated direction from the coordinates to be corrected α_{12} and the corresponding observed direction φ_{12}.

2.5.4. Examples

An example given in Survey Adjustment and Least Squares by Rainsford, p.174, has been solved. One component triangle CDE was solved first. The given data are as follows:

Direction	°	'	"
CD	104	36	45.69
CE	142	28	18.43
DC	284	36	45.69
DE	256	57	51.14
EC	322	28	18.43
ED	76	57	51.49

Coordinates of C and D are fixed, C(343 232.36 , 656 116.53) and D(313 928.90 , 768 512.17).

Approximate coordinates of E are (296 960 , 692 200).

Solution by least squares using the variation of coordinates method yields the corrections:

$dx_E = -0.11$ ft., $dy_E = -3.47$ ft.

Using the mechanical analogue gives the following results:

Iteration	dx_E	dy_E
first	- 0.32	- 4.05
second	- 0.16	- 3.61
third	- 0.10	- 3.37
fourth	- 0.16	- 3.44

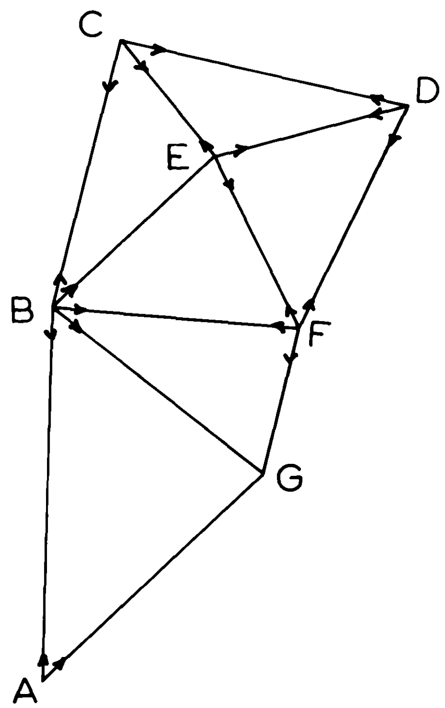


Figure 2.44

The figures given above show that a close approximation to the final results was obtained after two iterations and that there was no necessity to have any more iterations than four at most. The following figures for the successive contradictions confirm this:

Iteration in seconds	CE	EC	DE	ED
first	+ 9.80	+ 9.80	+ 1.60	+ 1.95
second	- 1.50	- 1.50	- 0.06	+ 0.29
third	- 0.77	- 0.77	- 0.22	+ 0.13
fourth	+ 0.25	+ 0.25	- 0.22	+ 0.13

The full net to be adjusted is given in figure 2.44. Arrows in the figure show the direction of observations. Stations A, B, C, and D are pre-fixed, the coordinates of these stations in feet are, A(81 511.76 , 613 085.00), B(236 406.59, 622 992.06), C(343 232.36 , 656 116.53) and D(313 928.90 , 768 512.17).

Stations E, F and G are to be fixed, their approximate coordinates are, E(296 260 , 692 200), F(224 510 , 724 290), and G(164 920 , 707 290).

Rainsford solution by least squares using variation of coordinates method including the station corrections gives the following results:

dx_E	dy_E	dx_F	dy_F	dx_G	dy_G
- 0.34	- 3.07	+ 0.81	+ 3.12	+ 3.44	- 2.23

Solution by the mechanical analogue gives the following results:

Iteration	dx_E	dy_E	dx_F	dy_F	dx_G	dy_G
first	- 0.65	- 3.07	+ 1.94	+ 3.48	+ 2.92	- 2.75
second	- 0.39	- 2.94	+ 1.04	+ 3.26	+ 3.53	- 2.29
third	- 0.30	- 3.14	+ 0.86	+ 3.15	+ 3.85	- 2.10

Difference from Rainsford solution is

0.04 0.07 0.05 0.03 0.41 0.13

There is therefore a close agreement between the results obtained by using both solutions. Further iterations were found not necessary as increase of scale of correction beyond 10°:1" does not give any significant improvement. Also compression forces due to the large contradictions at very large scale will be of such an order that they might cause failure of the individual joints.

2.5.5. Conclusions

Using the mechanical analogue for the adjustment of directions by correcting the approximate coordinates in the above examples shows that:

1- The design and construction of this second model overcomes most of the difficulties found in the first one, which means that it can be considered a much more practical solution from the user's point of view. In particular, there is a free and easy translation of the joints during adjustment.

2- The physical representation of each direction includes the possibility of representing observations from one or both ends of the ray.

3- The station or zero correction required for the best fit does not cause any difficulty, as it is obtained automatically by allowing a free rotation at each station where directions have been observed.

4- Some remarks must be made over the poor results at station G which occurred every time the example was repeated. It will be noticed that the other points to be fixed, E and F had observations made at these points whereas G had none. The difference between the analytical least squares solution and the mechanical solution seems to be due to this.

The effect of the spring at the point of insertion, e.g. close to the point E in figure 2.44 is very marked, but this will decrease as the point moves along the side away from E. When no observations are taken at point G, then its position will be located only by the effect of the spring at the far ends of the lines FG, BG and AG. This explains the importance of making observations at each station in the net which is to be adjusted mechanically by this analogue. These observations will determine the locus of the movement of this station, while the particular position will be determined by the intersection of the rays at the other stations, e.g. at point C in figure 2.45, the same angle will be obtained for different rays from A and B. In this case C will be moving along this special locus, but its position will be determined by the angles at the other two stations A, and B.

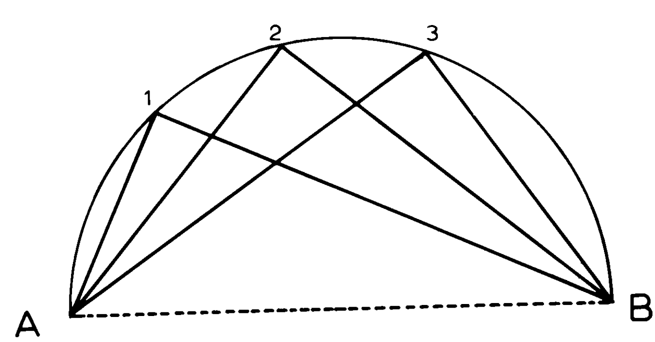


Figure 2.45

2.5.6. Further Possible Improvements

A great deal of time was spent on experimenting with the two mechanical analogues until the second one proved practicable. In the light of still further experiments, the following further improvements could be considered:

1- The weight assembled at each joint might be still further reduced by:

  i) A reduction in the size of piece A of the elastic unit to half that used.

  ii) A still lighter material could be used for the base plate, which could be drilled out to give the configuration as a wheel with spokes. The importance of such steps can be seen when it is realised that this plate is about half the total weight of the whole joint. The horizontal disc can also be treated in the same way to give a better weight reduction.

Chapter III

ELECTRICAL AND ELECTRONIC MODELS OF ADJUSTMENT

3.1 INTRODUCTION

The theory of the analogy between survey and electrical nets is the same as that for survey nets related to mechanical and structural analogues. Southwell and Black [12], applied the basic theory of the relaxation method to both survey and electrical nets. For practical reasons, a different form of analogy between survey and electrical nets was found necessary and was devised by Su [102], [105], and [106]. Successive approximations [103] has been also used by Su to demonstrate this analogy using level nets; in this context, the special systematic relaxation used in the mechanical and structural problems always involves such an iterative procedure.

As the work involved here deals with the adjustment of triangulation, trilateration, and hybrid observations, adjustment of level nets using this iterative method will not be given, and the reader has to refer to these in [103] and [104].

Many of the electrical components suitable for survey problems have been devised by Speart [96], 1947, but he does not appear to have published any practical results. However in 1958, Su made the first general statement of how electrical analogies might be applied to survey problems and produced some practical results.

3.2. ELECTRICAL ANALOGY OF THE PROBLEM

Obviously in this case electric units are the physical quantities being used to represent the variables of the survey problem to be solved. In these the voltage at a point in a circuit is directly proportional to the variable that is to be represented.

Consider a component triangle, figure 3.1, in which the six directions are observed in the usual way. The angle condition for this

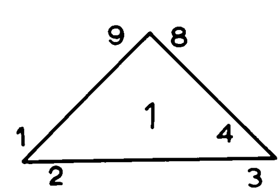


Figure 3.1

triangle will be formed by subtracting the observed directions, thus;

$$- v_1 + v_2 - v_3 + v_4 - v_8 + v_9 + k_1 = 0 \qquad \ldots\ldots\ldots(3.1)$$

For a net of n triangles, figure 3.2, n such condition equations will be formed, which can be solved by using the method of correlates.

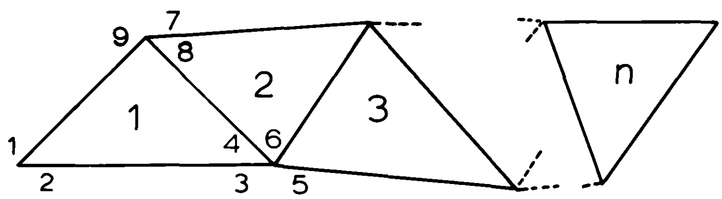


Figure 3.2

The normal equations will take the form:

$$\left.\begin{array}{l} 6C_1 - 2C_2 + k_1 = 0 \\ - 2C_1 + 6C_2 - 2C_3 + k_2 = 0 \\ \ldots \quad \ldots \quad \ldots \\ - 2C_{n-2} + 6C_{n-1} - 2C_n + k_{n-1} = 0 \\ - 2C_{n-1} + 6C_n + k_n = 0 \end{array}\right\} \ldots\ldots\ldots(3.2)$$

(where C is the correlate)

Equations (3.2) can be given in the form:

$$\left.\begin{array}{l} C_1 = \frac{1}{3}.C_2 - \frac{1}{6}.k_1 \quad \ldots\ldots \\ C_2 = \frac{1}{3}.C_1 + \frac{1}{3}.C_3 - \frac{1}{6}.k_2 \quad \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots \\ C_{n-1} = \frac{1}{3}.C_{n-2} + \frac{1}{3}.C_n - \frac{1}{6}.k_{n-1} \quad \ldots\ldots \\ C_n = \frac{1}{3}.C_{n-1} - \frac{1}{6}.k_n \quad \ldots\ldots \end{array}\right\} \ldots\ldots(3.3)$$

In the case of a simple direct current, according to figure 3.3, we have:

$$I_{QA} = \frac{V_Q - V_A + E_{QA}}{R_{QA}},$$

$$I_{QB} = \frac{V_Q - V_B + E_{QB}}{R_{QB}}, \text{ and}$$

$$I_{QC} = \frac{V_Q - V_C + E_{QC}}{R_{QC}}. \qquad \ldots\ldots(3.4)$$

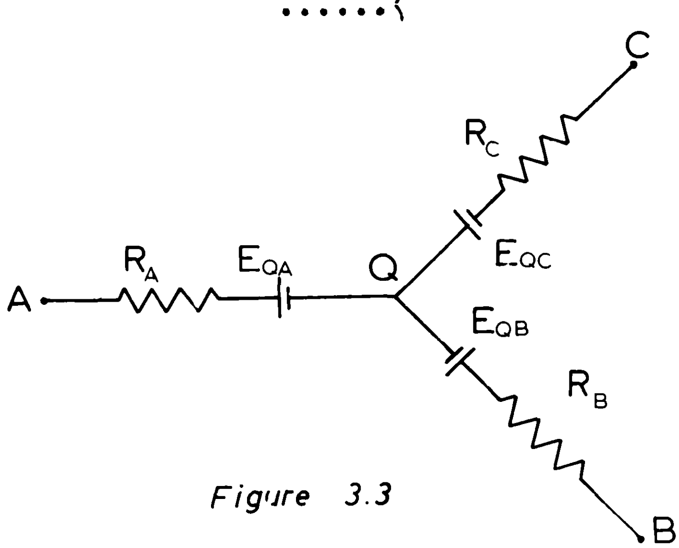


Figure 3.3

Where;

I = current in amperes,

V = potential in volts,

E = electro-motive force in volts,

R = resistance in ohms.

According to current continuity, ΣI must be zero at any point. Therefore;

$$V_Q\left(\frac{1}{R_{QA}} + \frac{1}{R_{QB}} + \frac{1}{R_{QC}}\right) = \left(\frac{V_A}{R_{QA}} + \frac{V_B}{R_{QB}} + \frac{V_C}{R_{QC}}\right) - \left(\frac{E_{QA}}{R_{QA}} + \frac{E_{QB}}{R_{QB}} + \frac{E_{QC}}{R_{QC}}\right) \quad \text{..........(3.5)}$$

If R is the same for all resistances, i.e. $R_{QA} = R_{QB} = R_{QC} = R$, and if joint C is earthed, i.e. $V_C = 0$, equation (3.5) will be:

$$V_Q = \frac{1}{3}(V_A + V_B) - \frac{1}{3}(\Sigma E) \quad \text{..........(3.6)}$$

Equation (3.6) is similar to equation (3.3), thus an analogy can be introduced between the solution of the two considered problems. The solution of any of these two analogous problems can be obtained by following the procedure of solving the other. An electrical net can be built up to solve the adjustment problem, in which case the results of the solution will be in the form of the electrical potential at the joints considered, so that the problem of adjustment is transferred to a problem of voltage measurement once the net has been set up correctly. Resemblance between electric and survey nets is given in the following:

(1) An internal triangle in the triangulation net (i.e. one completely surrounded by other triangles), is represented in the electrical net by an internal joint, which does not have any earthed path, (triangle r and joint r in figure 3.4).

(2) An external triangle (i.e. one with at least one side with no adjacent triangle), is represented by an external joint in the electrical net. If there are two external sides they are represented in the electrical net by an earthed path, (triangle r+2, and joint r+2, figure 3.4).

(3) Sides of a triangle correspond to paths of the electrical

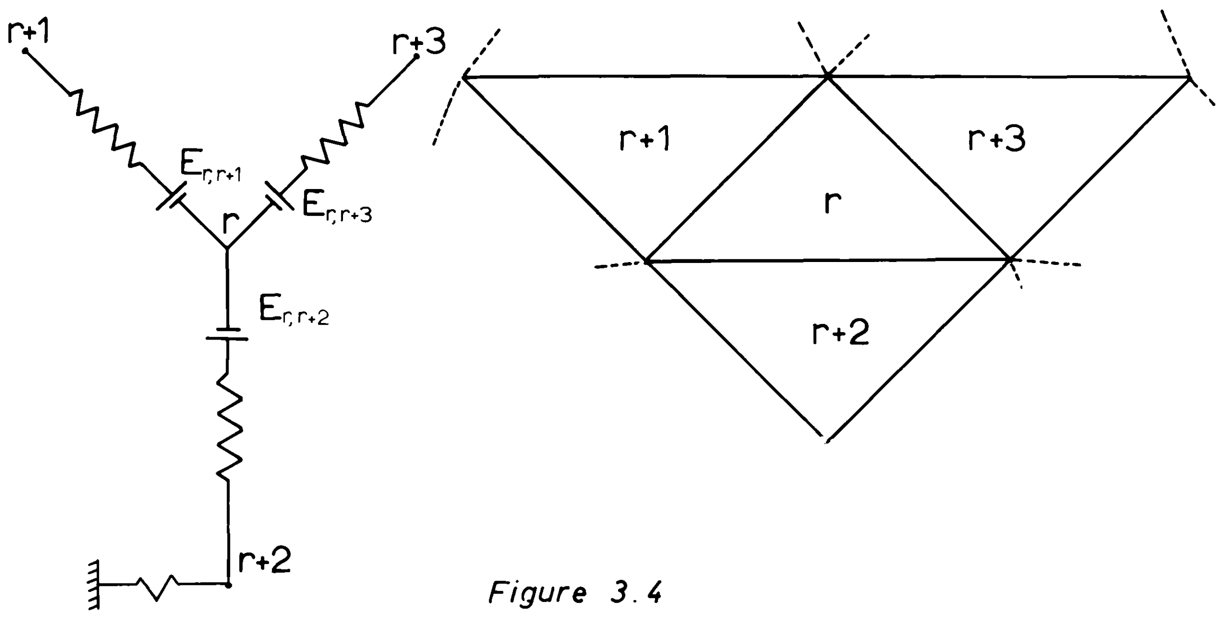


Figure 3.4

net connected into a joint which correspond to the triangle.

For simple triangles, or for small nets it may be easier to calculate the required corrections via the electrical theory than to build a net and measure these voltages.

For the calculation method we have:

$$V = IR \pm E \qquad \text{........(3.7)}$$

As both V and I are variables, therefore;

$$\Delta V = R.\Delta I, \quad \text{and} \quad \Delta I = \frac{\Delta V}{R} \qquad \text{........(3.8)}$$

Again according to the continuity law,

$$\Sigma(I + \Delta I) = \Sigma I + \Delta V.\Sigma\frac{1}{R} = 0 \qquad \text{........(3.9)}$$

and

$$\Delta V = - \frac{\Sigma I}{\Sigma\frac{1}{R}} \qquad \text{........(3.10)}$$

Substituting in (3.8) therefore;

$$\Delta I = \frac{\Delta V}{R} = - \frac{\frac{1}{R}}{\Sigma\frac{1}{R}}.\Sigma I = - s.\Sigma I \qquad \text{.......(3.11)}$$

For a joint with three paths, and corresponds to a triangle,

$$s = \frac{1}{3},$$

which is called the distributing factor.

This analogy can be extended to a net of n triangles which means n conditions to be solved for the adjustment. In this case, it could be seen that all coefficients are the same, with the same distributing factor, and the same sign (which is positive in this case).

Up till now the solution is offered for the simple triangulation net, corresponding to very simple direct current electrical net. The general problem in surveying is to adjust nets of different individual units, such as triangles, braced quadrilaterals, etc. The most common problem is to adjust quadrilaterals where side equations will be involved. Also trilateration problems could not be adjusted if the analogy is limited to such a simple case.

3.2.1. General Survey Problems

As already explained, the solution of survey problems is achieved after reducing the correction equations (observation or condition equations) to normal equations, and solving these by the ordinary methods of linear algebra. This procedure for solving observation and condition equations is also followed when using digital computers, the only difference being that the computer is carrying out the routine work while avoiding mistakes and errors in the calculations.

For solving the survey problems the least squares method requires (equation 1.5) that:

$$(V'PV) = \text{minimum} \qquad \text{........(3.12)}$$

where V is a column matrix of the residuals (v_1, v_2,...), V' is its transpose, and P is the diagonal matrix of the weights. The normal equations will also be preferred in matrix form for an electrical analogy.

From equation (3.12), $\frac{\partial(V'PV)}{\partial x_i} = 0$, (i=1,2,..,m) .....(3.13)

where x is a linear function of V.

As the independent observations are subject to the condition that,

$$V = H - A'X \qquad \text{.......(3.14)}$$

where A is the matrix with the elements $a_{ji} = \frac{\partial v_i}{\partial x_i}$, and j = 1,2, ..., n, substituting (3.14) into (3.13) gives;

$$APH + APA'X = 0 \qquad \text{.......(3.15)}$$

and, BX = K .... (3.16)

where B = APA', and K = APH

3.2.2. General Electrical Problems

Any form of analogy between the least squares and the minimum energy conserved allows the possibility of building an analogue or conservative system capable of solving the linear equations used in the least squares solution. For an electrical analogue this similarity can be shown as follows:-

In electrical networks the total energy is a positive definite quantity, and at the equilibrium position the internal energy resulting from the external disturbance (corresponding to the discrepancies of the survey nets) is a minimum.

If (E'C) is the internal energy = minimum, ...(3.12)'

where E' represents the effective potential differencies ($= e_1, e_2, \ldots$), and C the effective current corresponding to the relative differences ($=c_1, c_2, \ldots$), then differentiating with respect to the potentials,

$$\frac{\partial(E'YE)}{\partial e_i} = 0, \qquad \ldots\ldots(3.13)'$$

where Y is a symmetrical matrix of short circuit admittances with the self-admittances as its principal diagonal.

According to Ohm's law, E' = V' - C'Z ......(3.14)'

where V' is a row matrix of actual voltages, C' is a row matrix of the actual current supplied, and Z is a symmetrical matrix of open circuit impedences and equal to Y^{-1}.

Substituting equation (3.14)' into equation (3.13)' gives:

$$E' = V'Y - C'ZY = 0 \qquad \ldots\ldots(3.17)$$

since ZY = 1, therefore $V'Y - C' = 0$ ......(3.18)

and $YV = C$ ......(3.16)'

Equation (3.16)' is known to be Kirchoff's first law.

It could be seen that, equations (3.12), (3.13), (3.14), and (3.16) are analogous to equations (3.12)', (3.13)', (3.14)', and (3.16)' respectively.

Equation (3.16)' has the property that the sum of the elements in any row or any column of the matrix Y must vanish, which does not occur in matrix B of equation (3.16). This missing property can be achieved by adding another column and another row to matrix B, so completing the analogy. Thus equation (3.16) will become:

$$DX_o = K_o \qquad ....(3.19)$$

where $D = \begin{bmatrix} B & \Sigma \\ \Sigma & \infty \end{bmatrix}$, which is one rank higher than B.

X_o denotes the column matrix $[X, x_\Sigma]$,

K_o denotes the column matrix $[K, k_\Sigma]$.

Σ is a vector to make the sum of the elements in any row or any column vanish.

∞ corresponds to the admittance of the earth which is unlimited.

x_Σ is equal to zero.

k_Σ is an indefinite value (current fed into the earth, which could be any value).

The possibilities of constructing this analogue and making use of it in a practical way are discussed in 3.2.3.

3.2.3. Practicality of Su's Analogue

The use of the calculation method based on the electrical analogy gave satisfactory results which added another method to the computation methods already known. The practicality of constructing and using such an analogue electrically is discussed from the point of view of a civil engineer, who is already familiar with simple electrical nets.

An electrical analogue for the solution of linear equations consisting of a direct current circuit is very simple and accurate if positive and equal resistances are used. The net in this case consists of: (1) Resistance boxes. (2) Power supply. (3) Voltmeters.

Since readings are required to three significant figures, and different coefficients will be encountered, some additional features proved to be necessary. These have been devised by Speart [96], and a short summary is given below:

(1) Each resistance has to consist of two wire-rheostats or potentiometers in series. Their size is chosen such that one is used for coarse setting and the other for fine adjustment. Such connection is given in figure 3.5.

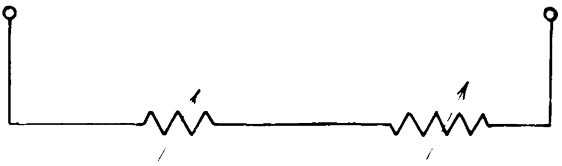


Figure 3.5

(2) Power supply, figure 3.6. This usually consists of a battery of chosen voltage, (2 volts gives good results) linked in series with a fixed resistance (a) which prevents overloading;

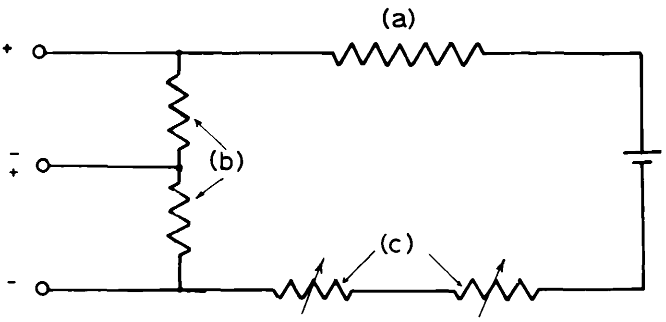


Figure 3.6

two small bleeder resistances (b), across either of which the desired voltage may be tapped, and two variable resistances (c) for adjusting the tapped voltage. For large discrepancies the voltage is tapped across the two bleeders, while for small discrepancies the voltage is tapped across one only. A separate power supply has to be supplied for each discrepancy.

(3) Voltmeter, figure 3.7.

The voltmeter is required for setting the closure (i.e. discrepancy) and measuring the corrections. Thus it is carrying out the critical part of the solution, so it has to be sensitive and not liable to errors.

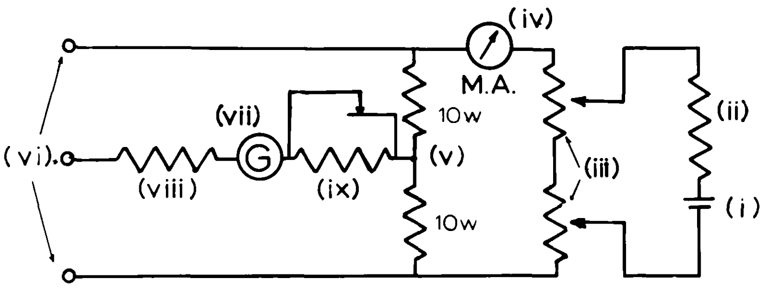


Figure 3.7

The needle of the standard type of resistance voltmeter deflects under the effect of a small amount of the current being drained from the circuit to be measured. In such a circuit where current is measured in milliamperes and voltage measured in millivolts range (to give the three significant figures) any appreciable drain of current from the circuit will make the meter reading erroneous. To overcome this a special "feed back" meter is needed to measure the voltage without any drain from the circuit. This is done by setting a voltage in the meter, equal and opposite to the voltage of the circuit to be measured. When the two voltages are exactly balanced, as indicated by a null reading on a galvanometer, the meter voltage is read. The layout of the complete voltmeter is shown in figure 3.7, where:

(i) Battery.

(ii) Fixed resistance to prevent overload.

(iii) Two potentiometers (coarse and fine) for regulating the current through the milliammeter circuit.

(iv) Milliammeter for current reading.

(v) Two accurately matched resistances.

(vi) Voltmeter terminals.

(vii) Galvanometer for volt measurement.

(viii) Fixed resistance to protect the galvanometer.

(ix) Large resistance (variable) for sensitive setting of the galvanometer.

For the use of this voltmeter, two terminals are connected to the two points between which the voltage corresponding to the discrepancy or corrections is to be taken. The galvanometer reading is brought to zero by manipulating potentiometer (iii) and the large resistance (ix). When the needle of the galvanometer indicates null the needle of the milliammeter indicates the current passing through this circuit, and hence the voltage drop of each of the two fixed resistances is known, since it is proportional to the milliammeter reading.

This electrical net does not include any negative resistors which will be required if any of the coefficients of the

problem has a negative sign as might occur if the general problem is to be introduced into the analogue.
Su [106] suggests the following negative resistor for this situation, figue 3.8.

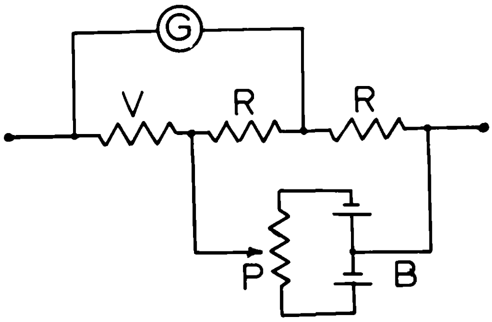


Figure 3.8

3.2.4. Discussion

In the analogue introduced in 3.2.3 the number of electrical components depends on the size of the problem to be solved, or, in other words, on the number of unknowns for which the normal equations are to be solved. It should be kept in mind that electrical preparations for solving the normal equations start after the formation of the normal equations themselves. For a small problem of about 8 unknowns (8 normal equations), formation of the normal equations takes half the time and effort necessary for solving the whole problem by the desk calculator. There appears therefore little to be gained in using the electrical analogue in place of the calculator for problems of this order. At the same time, the number of components necessary for the analogue solution increases linearly with the number of the unknowns, so that there are considerable difficulties with larger nets.

Su sees that reliability in solution by a desk calculator can be reached only after months of practice, but this ignores the fact that most of the difficulties over obtaining reliable results are found to be due to mistakes in forming the normal equations themselves. Once these are properly formed and checked, either a desk calculator, or an electrical analogue will be able to give reliable results in a comparatively short time. One realises the great effort necessary in forming the normal equations, if we consider the way in which the matrix B of the normal equation (3.16) is formed.

Here B = AA', in the case of condition equations, or A'A in the case of observation equations.

For observation equations $\underset{mm}{B} = \underset{mn}{A'}\underset{nm}{A}$, n is greater than m.

For condition equations $\underset{nn}{B} = \underset{nm}{A}\underset{mn}{A'}$, m .... n.

For example if a 6x14 rectangular matrix A with 6 unknowns anf 14 equations must be multiplied by its transpose to give a 6x6 square matrix which involves the multiplication and addition of the number of equations by itself (in this case 14x14).

A further point is that the time considered in preparing the electrical components necessary for the solution by the electrical analogue does not encourage the use of such a method, especially when they follow the long calculation processes to form the normal equations.

One point which seems to be important is the acceptance of the change. I find it difficult myself to stop the calculation after spending so much time in formation of the normal equations, only to start and work in a different field. However, one feels that one should just finish the job off as one began. The work could be split between two people, with some other person taking over the electrical part. This might not be a very different procedure however.

To have sufficient electrical components to meet the requirements of solving a medium size problem (which is considered to be laborious by a desk calculator), requires an electrical laboratory with considerable resources and skilled technical staff. The analogue suggested by Su is a simple and inexpensive one, but it may be quite difficult to implement because (a) it requires a pyramid of electrical units, the interconnection of which must be changed for every problem, which is a time consuming job; (2) adjusting every unit to give the necessary working elements promises to be quite troublesome.

If experience is taken into account, a desk calculator operated by an expert computer. will give accurate results, while the electrical analogue operated by an expert will give rather less accurate results in comaratively the same time. However in survey field it is easier to find the first type of experience, but not the second one.

The idea of constructing a D.C. circuit for the solution of linear equations will stay simple for small problems, with no negative coefficients involved. For example, for adjusting small nets of triangulation where no side condition equations are involved.

The larger and more complex survey nets which would be difficult to adjust using Su's suggested scheme may be adjusted by means of an electronic analogue which, although more sophisticated, is of a general purpose type which is more readily available. Also, the work involved will be greatly reduced. All computations will be eliminated in the solution of the correction equations, i.e. for observation and condition equations, without the extra work of forming the normal equations found necessary when solving problems of adjustment using any means, which operate along the lines of a general purpose electrical analogue system, but with substitution of electronic components for many of the electrical ones.

3.3. ELECTRONIC ANALOGUES

The analogy between the survey problems and the electronic system is similar to that given for the electrical analogue in 3.2.1. A suitable electronic analogue is readily available in in the form of electronic analogue computers. These solve mathematical equations instead of trying to achieve a physical correspondence with the problem to be solved. This is called indirect computation [50], as there is only one dependent variable, thus strictly it is an equation solver rather than a physical analogue system. As will be seen, this computer is capable of forming mathematical operations of addition, subtraction, multiplication, and integration using electronic differential analysers.

The basic electronic components of the computer are:

1) Drift corrected, high gain operational amplifiers.

2) Close-tolerance resistors and capacitors.

3) Coefficient-setting potentiometers.

4) Function generators.

5) Multipliers.

6) Resolvers.

3.3.1. Operational Amplifier and Basic Mathematical Operations

The operational amplifier: This is the heart of the analogue computer, which performs mathematical operations to a high degree of accuracy, and so deserves some detailed description. It has the following characteristics:

i- High gain, normally this will exceed 15,000 - a gain of 100,000 is quite usual in practical operations.

ii- Linearity over a wide region of operations, generally from - 100 to + 100 volts at the output.

iii- Zero output voltage for zero-input voltage.

iv- A very high input impedence; this input stage should draw negligible grid current.

A typical direct current operational amplifier used to carry out basic mathematical operations is shown diagramatically in figure 3.9.

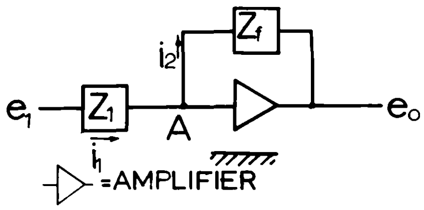


Figure 3.9

where:

Z_1 is the input impedence

Z_f is the feed-back impedence.

By this arrangement the amplifier draws a negligible grid current.

The node (A) equation for the current is $i_1 = i_2 = i$ ..(3.20)

therefore, $$\frac{e_1 - e_A}{Z_1} = \frac{e_A - e_o}{Z_f} \qquad \text{.........(3.21)}$$

Due to the high gain (-A), $$e_A = - \frac{e_o}{A} = 0 \qquad \text{.........(3.22)}$$

From equations (3.21) and (3.22) the output voltage is:

$$e_o = -\frac{Z_f}{Z_1}.e_1 \qquad \text{........(3.23)}$$

If the feed-back and input elements are resistors of the same magnitude, therefore: $e_o = - e_1$, ...(3.24)
i.e. the output voltage is equal to the input voltage with a change of sign.

By varying one of the two resistors, figure 3.10, the input voltage will be multiplied by an arbutrary constant according to:

$$e_o = - \frac{R_f}{R_1}.e \qquad \text{......(3.25)}$$

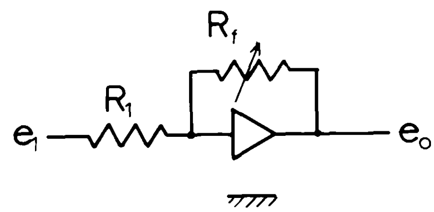


Figure 3.10

For addition a number of input resistors can be connected to the amplifier, figure 3.11, and in this case:

$$e_o = - R.[\frac{e_1}{R_1} + \frac{e_2}{R_2}] \qquad \text{.....(3.26)}$$

This is called a "sammer" circuit.

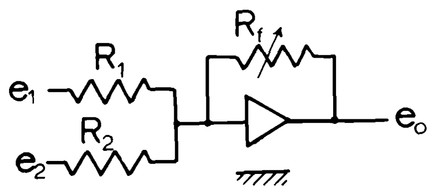


Figure 3.11

For integration, a capacitor is used in place of the feed-back resistor, as shown in figure 3.12. In this case:

$$e_o = - \frac{1}{R_1.C}\int_0^{t_1} e_1.dt + k \qquad \text{....(3.27)}$$

i.e. the output voltage is therefore proportional to the negative value of the integral with respect to the time of the input voltage. k is the output at time equal to zero, which specifies the initial voltages to which the capacitor must be changed at the beginning of the computation.

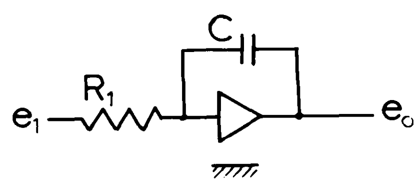


Figure 3.12

For addition and integration the connection is shown in figure 3.13. In this case:

$$e_o = - \frac{1}{C}\int_0^{t_1}[\frac{e_1}{R_1} + \frac{e_2}{R_2}] \qquad \text{...(3.28)}$$

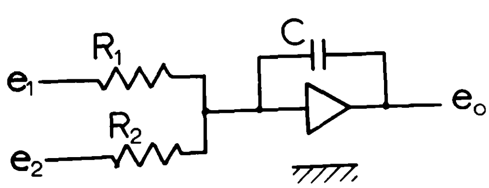


Figure 3.13

These basic mathematical operations are enough for the solution of the linear equations, and have the merit of being easy to understand as they only require the knowledge of some simple laws of electricity.

To allow the multiplication of two variables by each other several devices have been developed which are connected as additional items to the operational amplifier. However as all these devices are complicated and require details and explanations found beyond the scope of this thesis, readers may refer to [50], at p. 249. The same reference gives details of the negative resistors and capacitors, at p. 257. Also reference may be made to [42], for further explanation.

3.3.2. Pace Analogue Computer

The computer which has been used to solve examples is the PACE 231R-V, manufactured by Electronic Associates Limited.

To programme a problem a patch panel is used. This has numerous sockets which provide access to the electronic components of the computer. These components are inter-connected by wires carrying plugs which are inserted into appropriate sockets. For large problems which use many of the component units, making these interconnections takes considerable time, and if it were to be done on the machine, the computer would be out of use for a considerable period. The patch panels are therefore detachable allowing the interconnections to be made off line. The complete plugboard is then placed onto the computer in one operation. Inputs are in the form of time-varying voltage, while the solutions (obtained in similar fashion) are displayed on a digital voltmeter, or a graphical plotter.

3.3.3. Solution of Linear Equations

Since the inputs and outputs of electronic analogue computers are usually in the form of time varying voltage, the computer is most frequently used for the solution of differential equations. To solve linear algebraic equations, a special form of programming [84] is required.

Let the linear algebraic equations be given by:

$$AX = K \qquad \text{........(3.29)}$$

where A is a square matrix of order n, K and X are two column vectors, where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ x_n \end{bmatrix} \quad K = \begin{bmatrix} k_1 \\ k_2 \\ \cdot\cdot \\ k_n \end{bmatrix}$$

Consider the differential equations,

$$\dot{X} + AX = K \qquad \text{........(3.30)}$$

where $\dot{X}$ is a column vector,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot\cdot \\ \dot{x}_n \end{bmatrix}$$

A steady state or equilibrium will be reached when $\dot{X} = 0$, in which case X will satisfy equations (3.92) and (3.30). To ensure this steady state for equation (3.30), matrix A must be positive, definite (where all determinants are positive and greater than zero), which is not always the case in the survey problems. Where it is not, matrix A must be multiplied by its transpose A' to yields positive definite positive matrix A'A (=B). Equation (3.29) would then be replaced by,

$$A'AX = A'K \qquad \text{........(3.31)}$$

similarly equation (3.30) will have the form,

$$\dot{X} + A'AX = A'K \qquad \text{........(3.32)}$$

and

$$\dot{X} + A'(AX - K) = 0 \qquad \text{........(3.33)}$$

The steady state will then be reached to give the solution sought.

Simulation of equation (3.32) on the computer will be shown later in the solved examples.

3.3.3.1 Solution of Survey Problems

Electronic analogue computers may be programmed to give the least sum of the squares of the residuals using the steepest descent method.

For observation equations, where a matrix $_{n}A_{m}$ is used in equation (3.29), let e be the residual error and t the time in seconds.

If $e_j = f_j(1,2,\ldots\ldots,n)$

the sum of the least squares will be

$$S = \sum_{j=1}^{n} e_j^2 \qquad \ldots\ldots\ldots(3.34)$$

therefore;

$$\frac{dS}{dt} = 2\sum_{j=1}^{n} e_j . \frac{de_j}{dt} \qquad \ldots\ldots\ldots(3.35)$$

$$\frac{de_j}{dt} = \sum_{i=1}^{m} \frac{de_j}{dx_i} . \frac{dx_i}{dt} \qquad (j=1,2,\ldots,n)\ldots\ldots(3.36)$$

from (3.35) and (3.36) we have:

$$\frac{dS}{dt} = 2\sum_{i=1}^{m} \frac{dx_i}{dt} \sum_{j=1}^{n} e_j . \frac{de_j}{dx_i} \qquad \ldots\ldots\ldots\ldots(3.37)$$

In order that S be minimum, dS/dt must be zero, and the computer is programmed so that,

$$\frac{dx_i}{dt} = - \sum_{j=1}^{n} e_j . \frac{de_j}{dx_i} \qquad \ldots\ldots\ldots\ldots(3.38)$$

Inserting equation (3.38) into equation (3.37) yields,

$$\frac{dS}{dt} = - 2\sum_{i=1}^{m} \left[\frac{dx_i}{dt}\right]^2 \qquad \ldots\ldots\ldots\ldots(3.39)$$

Since all terms on the right hand side of equation (3.39) are negative, the computer operates to decrease S until dS/dt = 0.

For condition equations, the same procedure must be followed, but the following points have to be considered.

(a) The results have to satisfy exactly the condition equations, but since the PACE computer is not programmed to produce zero residual errors, an additional constraint must be introduced. The residual errors may be made infinitesimally small by increasing the gain of the feed-back paths from the residual errors to the integrator inputs.

(b) The increase in gain of these feed-back paths will not provide a unique solution and an infinite number of solutions could be obtained which satisfy exactly the condition equations. A further constraint must be introduced to obtain a unique solution, which will be that characterised by Σx_i^2 = minimum. Σx_i^2 may be minimised by a method similar to that used for minimising the residual errors, by introduction of an additional feed-back path.

3.3.3.2. Examples

To discover the possibilities and problems of using electronic analogues for the solution of both observation and condition equations several examples have been solved on the PACE computer. Two of these, together with the necessary simulation and programming are given in detail.

Example 1.

This solution is given for the problem solved by Rainsford in "Survey Adjustment and Least Squares", page 180. e_1, e_2, .., ,...., e_{14} are the residual errors of the corresponding equations. This problem is given in table 3.1

e	$x_1=dx_E$	$x_2=dy_E$	$x_3=dx_F$	$x_4=dy_F$	$x_5=dx_G$	$x_6=dy_G$	k
$-e_1$	- 2.12	- 2.76					- 9.80
$-e_2$	+ 2.12	+ 2.76					+ 9.45
$-e_3$	- 1.70	+ 1.48					+ 3.01
$-e_4$	+ 1.70	- 1.48	- 2.01	- 0.24			+ 0.77
$-e_5$			+ 2.01	+ 0.24	- 1.42	- 1.21	- 2.40
$-e_6$					+ 1.42	+ 1.21	- 1.68
$-e_7$					- 1.23	+ 1.09	+ 9.21
$-e_8$			- 1.10	- 3.44	- 0.91	+ 3.20	+19.91
$-e_9$	+ 1.07	+ 2.40	+ 0.94	- 2.16			+14.61
$-e_{10}$	- 1.07	- 2.40	+ 1.99	+ 0.55			- 9.69
$-e_{11}$	+ 4.68	+ 2.17					+ 7.85
$-e_{12}$	- 1.49	+ 2.99	- 1.07	- 2.40			+ 17.78
$-e_{13}$	- 2.56	+ 0.59					+ 1.60
$-e_{14}$	+ 2.56	- 0.59	- 0.92	+ 1.85			- 6.62

table 3.1

The patching diagram for this problem is givin in figure 3.14. According to equation (3.38),

$$\frac{dx_i}{dt} = -\sum_{j=1}^{14} e_j \cdot \frac{de_j}{dx_i} \quad (i = 1,2,\ldots 6)$$

$-\frac{dx_i}{dt}$ is in fact used as the integrator input because of the associated inversion in the operational amplifier.

therefore,

$$-\frac{dx_1/10}{dt} = -2.12.\frac{e_1}{10} + 2.12.\frac{e_2}{10} - 1.70.\frac{e_3}{10} + 1.70.\frac{e_4}{10} + 1.07.\frac{e_9}{10}$$
$$- 1.07\frac{e_{10}}{10} + 4.68\frac{e_{11}}{10} - 1.49\frac{e_{12}}{10} - 2.56\frac{e_{13}}{10} + 2.56\frac{e_{14}}{10}$$

$$-\frac{dx_2/10}{dt} = -2.76.\frac{e_1}{10} + 2.76.\frac{e_2}{10} + 1.48.\frac{e_3}{10} - 1.48.\frac{e_4}{10} + 2.40.\frac{e_9}{10}$$
$$- 2.40\frac{e_{10}}{10} + 2.17\frac{e_{11}}{10} + 2.99\frac{e_{12}}{10} + 0.59\frac{e_{13}}{10} - 0.59\frac{e_{14}}{10}$$

$$-\frac{dx_3/10}{dt} = -2.01.\frac{e_4}{10} + 2.01.\frac{e_5}{10} - 1.10.\frac{e_8}{10} + 0.94.\frac{e_9}{10} + 1.99\frac{e_{10}}{10}$$
$$- 1.07\frac{e_{12}}{10} - 0.92\frac{e_{14}}{10}$$

$$-\frac{dx_4/10}{dt} = -0.24.\frac{e_4}{10} + 0.24.\frac{e_5}{10} - 3.44.\frac{e_8}{10} - 2.16.\frac{e_9}{10} + 0.55\frac{e_{10}}{10}$$
$$- 2.40\frac{e_{12}}{10} + 1.85\frac{e_{14}}{10}$$

$$-\frac{dx_5/10}{dt} = -1.42.\frac{e_5}{10} + 1.42.\frac{e_6}{10} - 1.23.\frac{e_7}{10} - 0.91.\frac{e_8}{10}$$

$$-\frac{dx_6/10}{dt} = -1.21.\frac{e_5}{10} + 1.21.\frac{e_6}{10} + 1.09.\frac{e_7}{10} + 3.20.\frac{e_8}{10}$$

Since amplifiers are designed to provide linearity between - 100 and + 100 volts, it is necessary to scale the problem to ensure that the voltage corresponding to a variable never exceeds these limits. Scaling is carried out by dividing every variable by its maximum expected value. After scaling, all new variables have values between -1 and +1, and the computer voltages may then be restricted to the linear range by a voltage scaling of 100.

For this particular problem, equation 8, 9, and 12 given in table 3.1, have to be divided by 10 due to the fact that the values of k are 19.91, 14.61, and 17.78. This reduces these values to 1.991, 1.461, and 1.778, and by using gain 10, their potentiometers will be set at 0.1991, 0.1461, and 0.1778. The other equations do not need division by this factor.

The output reading should also lie between -1 and +1, which means that instead of using the full values of the unknowns $\frac{x}{x_{max}}$. is used, in this problem it is $\frac{x}{10}$.

Results were as follows:-

Amplifier	Unknown	Reading	Rainsford's solution
A 00	$x_1/10$	- 0.0340	- 0.0338
A 05	$x_2/10$	- 0.3031	- 0.3036
A 50	$x_3/10$	+ 0.1058	+ 0.1042
A 10	$x_4/10$	+ 0.3112	+ 0.3110
A 15	$x_5/10$	+ 0.3131	+ 0.3126
A 60	$x_6/10$	- 0.2135	- 0.2144

table 3.2.

<u>Example 2</u>

This is the problem solved by Clark, in "Plane and Geodetic Surveying", vol. II, page 285. The condition equations are as follows:-

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 - 4.80 = -e_1$$

$$x_1 - x_4 - x_5 + x_8 + 3.90 = -e_2$$

$$+ x_2 + x_3 - x_6 - x_7 + 7.50 = -e_3$$

$$3.58x_1 - 2.78x_2 + 1.87x_3 - 1.74x_4 + 2.18x_5 - 3.51x_6 + 1.49x_7 - 1.03x_8 + 25.10 = -e_4$$

The additional equation required for the stability of the solution is

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 = \text{minimum} = -e_5$$

The patching diagram for the problem is given in figure 3.15. For linear operations the problem is scaled for x/10, and equation 4 is divided by 10 for the same reason.

According to equation (3.38),

$$-\frac{dx_1/10}{dt} = \frac{e_1}{10} + \frac{e_2}{10} + 0.358.\frac{e_4}{10} - 2.\frac{x_1}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_2/10}{dt} = \frac{e_1}{10} + \frac{e_3}{10} - 0.278.\frac{e_4}{10} - 2.\frac{x_2}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_3/10}{dt} = \frac{e_1}{10} + \frac{e_3}{10} + 0.187.\frac{e_4}{10} - 2.\frac{x_3}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_4/10}{dt} = \frac{e_1}{10} - \frac{e_2}{10} - 0.174.\frac{e_4}{10} - 2.\frac{x_4}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_5/10}{dt} = \frac{e_1}{10} - \frac{e_2}{10} + 0.218.\frac{e_4}{10} - 2.\frac{x_5}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_6/10}{dt} = \frac{e_1}{10} - \frac{e_3}{10} - 0.351.\frac{e_4}{10} - 2.\frac{x_6}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_7/10}{dt} = \frac{e_1}{10} - \frac{e_3}{10} + 0.149.\frac{e_4}{10} - 2.\frac{x_7}{10}.\frac{e_5}{100}.100$$

$$-\frac{dx_8/10}{dt} = \frac{e_1}{10} + \frac{e_2}{10} - 0.103.\frac{e_4}{10} - 2.\frac{x_8}{10}.\frac{e_5}{100}.100$$

Results:

Amplifier	Unknown	Reading at		Clark's sol.
		gain 10	gain 100	
A 05	$x_1/10$	- 0.1751	- 0.1768	- 0.176
A 21	$x_2/10$	+ 0.0106	+ 0.0120	+ 0.012
A 25	$x_3/10$	- 0.1989	- 0.1999	- 0.200
A 10	$x_4/10$	+ 0.2128	+ 0.2137	+ 0.213
A 15	$x_5/10$	+ 0.0355	+ 0.0341	+ 0.034
A 30	$x_6/10$	+ 0.3933	+ 0.3952	+ 0.395
A 35	$x_7/10$	+ 0.1668	+ 0.1674	+ 0.167
A 00	$x_8/10$	+ 0.0334	+ 0.0343	+ 0.034

table 3.3

Comparison between the errors corresponding to each equation, for the gain employed are given in table 3.4

Amplifier	A 02	A 12	A 22	A 27	A 07=Σx^2
Gain 10	0.0003	0.0006	0.0012	0.0029	0.3008
Gain 100	0.0002	0.0001	0.0002	0.0004	0.3034

table 3.4

As shown in tables 3.3 , and 3.4, close agreement is obtained between the results obtained by this method and those obtained by Clark when a gain of 100 is employed.

Some experiments were carried out to ascertain the gain to be given to the feed-back paths corresponding to the errors of the condition equations. The amount of error at each amplifier representing a condition equation could always be reduced by increasing the gain of the feed-back paths corresponding to this special equation. However, beyond a certain point this error is virtually irreducible. For one problem the following was recorded.

Gain	10	30	100	130	200
Error	0.0043	0.0015	0.0006	0.0004	0.0004

table 3.5

In all the problems so far solved, no worthwhile improvement in the results has been observed by using a gain greater than 100. For feed-back paths corresponding to the additional equation ($\Sigma x^2 =$ minimum), gain 1 is chosen. Gain 10 is used for the loops corresponding to the original condition equations. If the error readings with this setting are not satisfactory, gain 100 should be used, and this will be satisfactory for most problems.

3.3.3.3. Constraint Necessary for the Solution of the Condition Equations

This point has already mentioned in 3.3.3.1., but it requires some more detailed discussion.

The PACE computer could be easily used for solving rectangular matrix $_{n}A_{m}$, where n is greater than m. In this case the most probable solution will be obtained according to the steepest descent method (3.3.3.1.), i.e. according to the least squares method, where the sum of the squares of the residuals is minimum.

For the rectangular matrix $_{n}A_{m}$, where m is greater than n solution of the problem will not be so easy. For this matrix an infinite number of solutions would be obtained which gives instability to the answers obtained by the computer. All these answers satisfy the condition equations. The required solution which is the most probable one is known to be that one characterised by Σx^2 = minimum. Thus a meaningful set of answers will be obtained by adding this condition as additional condition. In order to add this condition it is necessary to consider the following:

$$f_j = (x_1, x_2, \ldots, x_m) - b = 0 \qquad \ldots\ldots\ldots(3.40)$$

and $f_j = e_j$, the residual error which will go to zero for an infinite number of results.

e_{n+1} is the residual error of the additional condition which should go to minimum and not to zero.

Both e_j and e_{n+1} will be used as feed-back to the same integrators. To bring both e_j and e_{n+1} to the same order, e_{n+1} should be multiplied by a factor k, which must be close to zero. In fact using $k = 10^{-4}$ or 10^{-5} will suit the computer which can give readings = 0.0001.

To derive a value of the gain which will be applied, let,

$$e_{n+1} = \Sigma x_i^2$$

According to the steepest descent method,

$$\frac{dx_i}{dt} = - \sum_{j=1}^{n} e_j . \frac{de_j}{dx_i} - 2.e_{n+1}.\frac{de_{n+1}}{dx_i} \quad \ldots\ldots(3.41)$$

To bring e_{n+1} and e_j to the same order, equation (3.41) should be replaced by;

$$\frac{dx_i}{dt} = -\sum_{j=1}^{n} e_j . \frac{de_j}{dx_i} - 2.k.e_{n+1}.\frac{de_{n+1}}{dx_i} \quad \ldots\ldots\ldots(3.42)$$

$$= - \sum_{j=1}^{n} e_j . a_{ji} . x_i - 2.k.e_{n+1} . x_i$$

where, $e_j = \Sigma\, a_{ji}.x_i - b_j$.

If M, the maximum possible value of x_i, is larger than unity, equation (3.42) should be scaled, thus,

$$\frac{d(x_i/M)}{dt} = - \sum_{j=1}^{n} (\frac{e_j}{M}).a_{ji} - 2.k.(\frac{e_{n+1}}{M^2})(\frac{x_i}{M}).M^2 \ldots\ldots(3.43)$$

let $M = Lx10^s$, where $L \leq 1$.

As the most sensitive potentiometer setting should lie between 0.1 and 1 (= $2.L\times10^{-1}$), therefore equation (3.43) will be:

$$\frac{d(x_i/M)}{dt} = -\sum_{j=1}^{n}\left(\frac{e_i}{M}\right).a_{ji} - 2.k.L^2.10^{2s}\left(\frac{e_{n+1}}{M^2}\right)\left(\frac{x_i}{M}\right) \quad ....(3.44)$$

$$= -k.10^{2s+1}\left[\frac{10^{-2s-1}}{k}\sum_{j=1}^{n}\left(\frac{e_i}{M}\right).a_{ji} + 2.L^2.10^{-1}\left(\frac{e_{n+1}}{M^2}\right)\left(\frac{x_i}{M}\right)\right] \quad ....(3.45)$$

If the most sensitive potentiometer setting is used with gain 1 for the second term in equation (3.45), the first term in equation (3.44) will be multiplied by a corresponding factor = $\frac{10^{-2s-1}}{k}$. Since k is infinitesimally small, the factor $\frac{10^{-2s-1}}{k}$ will be much larger than unity.

In practical survey adjustment problems <u>s</u> is equal to 1, and M = 1x10 = 10,

therefore, $$\frac{10^{-2s-1}}{k} = \frac{10^{-3}}{k} \quad ....(3.46)$$

for k = 10^{-4}, this factor will be = 10, and

for k = 10^{-5}, ... = 100.

The most accurate results are obtained with the gain 100 which is derived from k = 10^{-5}.

The quantity $k.10^{2s+1}$ outside the square brackets in equation (3.45) will affect the rate of integration and will not affect the values of x_i. Thus it is found that, for solving condition equations;

(i) The most sensitive potentiometer settings are used for setting the coefficients of both original and additional condition equations.

(ii) Gain 1 is given to the loops corresponding to the additional condition equation.

(iii) Gain 100 is given to the loops corresponding to the original condition equations.

For any other problem, which might require <u>s</u> $\gtrless$ 1, k should be smaller than 10^{-5}. However in this case the gain necessary to solve this problem without causing overloading or instability to the solution has to be obtained after some experimentation. It should be noticed that, using larger gains than are really necessary will not give any more improvement to the results. The best guide is to check the value of e_j to make sure that it is zero, or as near as possible to zero. After a certain point, overloading and instability of the results given by the computer will stop further trials.

3.3.4. Accuracy and Capacity of PACE 231R-V Analogue Computer

The PACE computer found capable of solving unknowns up to the fourth decimal place. These unknowns in survey problems are corrections to angles, length of sides and coordinates. For first order work these corrections should be accurate to the second decimal place which is easily achieved on the PACE computer. Sometimes the accuracy of the unknowns given by the PACE computer is reduced by the necessity of scaling the problem. For instance, if instead of obtaining the unknowns x to 0.0001, we read x_1 (= x/10) to this accuracy, x will be obtained only to 0.001, which is still satisfactory for first order problems.

The size of the problem to be solved depends on the size of the available computer. The size of the PACE computer available for this work at Glasgow is restricted by:

(a) 100 amplifiers. (b) 150 potentiometers. (c) 48 multipliers.

(a) 100 amplifiers. These are used for:-

i- Summation; Summation of different unknowns will be obtained by using some of the amplifiers. In other words, every equation will be represented by an amplifier.

ii- Integration: This is achieved by connecting a capacitor as as the feed-back element in 30 of the operational amplifiers. Since the integrators are used to solve the unknowns the computer is limited to problems of 30 unknowns.

iii- Inversion: In most cases, it is necessary to obtain both positive and negative values for each unknown, and for residual errors of each equation. This necessity reduces the capacity of the amplifiers to half the nominal figure. Multipliers could be used as invertors, which adds 48 invertors in the case when multipliers or part of them are not used for multiplication.

(b) 150 potentiometers: These are the units which mainly restrict the size of the problem to be solved. For every coefficient in the problem, two potentiometers are necessary and an additional potentiometer is required for each equation for the inclusion of the constant term. For a problem requiring the solution of ten equations with seven unknowns, ten potentiometers are used for the constant terms, leaving 140 potentiometers (2x70) for 70 coefficients. The problem given by example 1 of 14 equations and 6 unknowns was solved very easily, due to the fact that four and two were the maximum and minimum number of unknowns in an individual equation. In practice the full number of unknowns is rarely found in an equation in survey work. Furthermore some coefficients are unity, in which case no potentiometers will be necessary for setting this value which is fed in directly. For these two reasons, observation equations of the order n.m = 100 (e.g. 10x10 or 16x6) could be solved by this computer on condition that (m+n) is not greater than 50. (m is the number of unknowns and n is the number of equations).

For condition equations the extra constraints necessary reduce the number of amplifiers available by twice the number of residual errors to be driven to zero, e.g. if n is the number of residual errors(e) to be driven to zero, and g is the gain necessary for this purpose (which is equal to 10^p), the number of amplifiers to be reduced will be n(p+1). 10 is the normal gain given to a loop from an individual amplifier, though a maximum of 33 can be obtained if required. Thus the number of amplifiers which can be used will be $(m+n) \leqslant 50-n(p+1)$.

(c) 48 multipliers: Since an additional equation ($\Sigma x^2 = e$) is necessary to give stability to the solution of problems, where the number of unknowns is always greater than the number of equations (condition equations), the number of multipliers necessary for solving these problems is another important factor affecting the size of the problem which can be tackled. For every unknown two multipliers are necessary. 48 multipliers are available so that the unmber of unknowns should not exceed 24 rather than the 30 which the number of integrators would seem to allow.

3.3.5. Conclusions

1- In an analogue computer, computation of observation equations is a linear function of the number of equations, whereas using ordinary desk calculators, it is a quadratic function of the number of equations.

2- Although a positive definite matrix (of the normal equations) is necessary reach a steady state on the PACE computer, programming of this matrix is very simple. The derivation of the dynamic form of equations is very simple and is achieved in routine fashion. In the example solved the required equations were obtained from the original linear equations. Furthermore programming and drawing the patching diagrams is carried out using these original equations, but in a special way which leads to the steady state. This is done in a manner that raises no difficulties for those continually dealing with survey computations.

3- Although for the solution of condition equations some work must be done to choose the necessary gain, unknowns will be obtained directly, without any of the substitutions which are found necessary in any other method of solution, e.g. by desk calculators or digital computers.

C O N C L U S I O N S

and

C O M P A R I S O N

I- CONCLUSIONS

Since sides as well as angles may now be observed, it has been found necessary to derive a system of conditions which deals with both types of observations at the same time. For this purpose two sets of new conditions have been reached. These proved to be simple and easy to form and to calculate. However their simplicity depends to a certain extent on the weights recommended which in these cases will be unity for both angles and sides.

As for the ways of computations there is the desk calculator, which will always be used either to solve the whole problem or to help in calculation when using any other form of computation such as these given below. As desk calculators are freely available they are particularly useful in solving small problems, e.g. the solution of 8 normal equations by the least squares method may take two hours which is not a long time when compared with the considerable time spent in preparation for programming or setting up problems in other methods. However the time consumed in solving linear equations for the purpose of adjustment is known to be a quadratic function of the number of these equations [105]. This means that the simultaneous solution of a large block of survey points by desk calculator is uneconomic from the practical point of view.

Apart from the desk calculators, the ways of computations can be classified under four main headings. The criteria for this classification is the type of equation which can be solved and the way in which the adjustment can be carried out. These four types are:

I- Analogues which are capable of simulating physically the net to be solved, including the observations made in the field, e.g. mechanical analogues. As has been seen, these do not require the formation of set of linear equations since they solve problems directly from the observed quantities.

II- Analogues capable of physically simulating the normal equations, e.g. the electrical analogue system designed by Su.

III- Digital computers, which are able to solve numerically for a square matrix, i.e. for a set of normal equations.

IV- Computers and analogues which are capable of solving directly any sort of linear equations (i.e. observation, condition, or normal equations). These are capable of solving over-determined or underdetermined equations directly, e.g. electronic analogue computers.

The order and size of problems recommended as to be adjusted by using each type can be given by the following:

Group I

These simulate the problem physically in an exact way, and so far the mechanical analogues in which angles are represented by angles, directions by directions and sides by sides, are the only examples which have been realised. Results obtained by Jerie for trilateration networks [46], and by the writer for the examples solved in 2.5.4. were identical to those of the least squares solution.

However the necessity of having frequent control points, both to fix the mechanical net to the working surface, and to give a datum to the mechanical energy at frequent intervals, restricts the use of this method to the solution of secondary and tertiary networks only, as far as angles and directions are concerned. The size of the problem which can be handled cannot be definitely given, as this depends on the weight and the standardisation in manufacture of the different joints to be translated. Both of these need to be developed further than in the prototypes which have been constructed so far. Furthermore the size of these problems, beside being dependent on the number of control points, depends also on being able to observe at every new station towards two control points.

However the size of the problem can perhaps be increased by introducing a differential adjustment of observations [122]. In this case new points would be connected to some existing control points, and using the mechanical analogue these are adjusted as a unit. The whole net is then used as control for the remaining observations, so that adjustment is made stage by stage using the analogue. To assist in these successive adjustments it would be necessary to introduce some form of spring-loaded intersection locator, figure 1. The purpose of these locators would be to allow some adjustment of the previously adjusted stations in conjunction with the newly introduced observations. The sketch given in figure 1 is based on the locator of Trorey used in photogrammetric work [124]. The springs used would have to be of a special stiffness to allow further adjustment to take place, and obviously this will be the most difficult problem to solve. This idea, if it can be realised, would assist greatly in overcoming the limitations of the mechanical analogue, where there is an insufficient density or a lack of suitably distributed control points.

The size of the secondary net recommended for the existing analogue should not be greater than that given in figure 2.44 at least to one side of the net. On the other hand, a further three stations can easily be accomodated to the west and north of the control points. The net in this case will amount to nine new stations, beside the four existing ones. Comparing it with a least squares solution solved numerically this will be equivelant to a problem of 18 unknowns with approximately 45 observations. However it should not be made any larger without some of the necessary improvements discussed in 2.5.6. being incorporated. These might allow a 50 percent increase in the size of the problem which might be tackled.

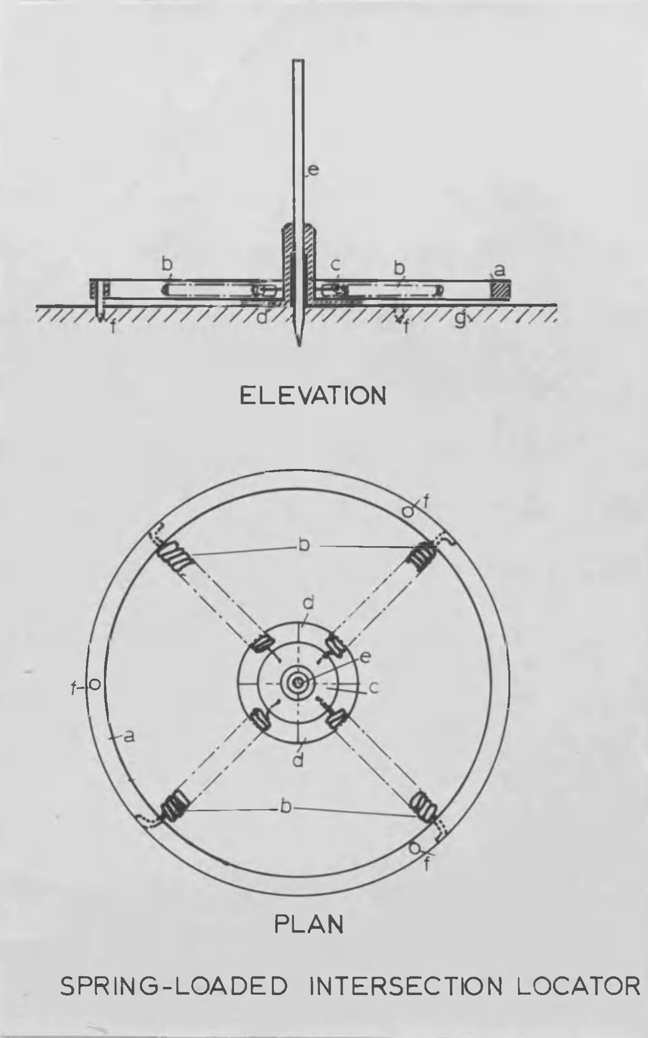


figure 1

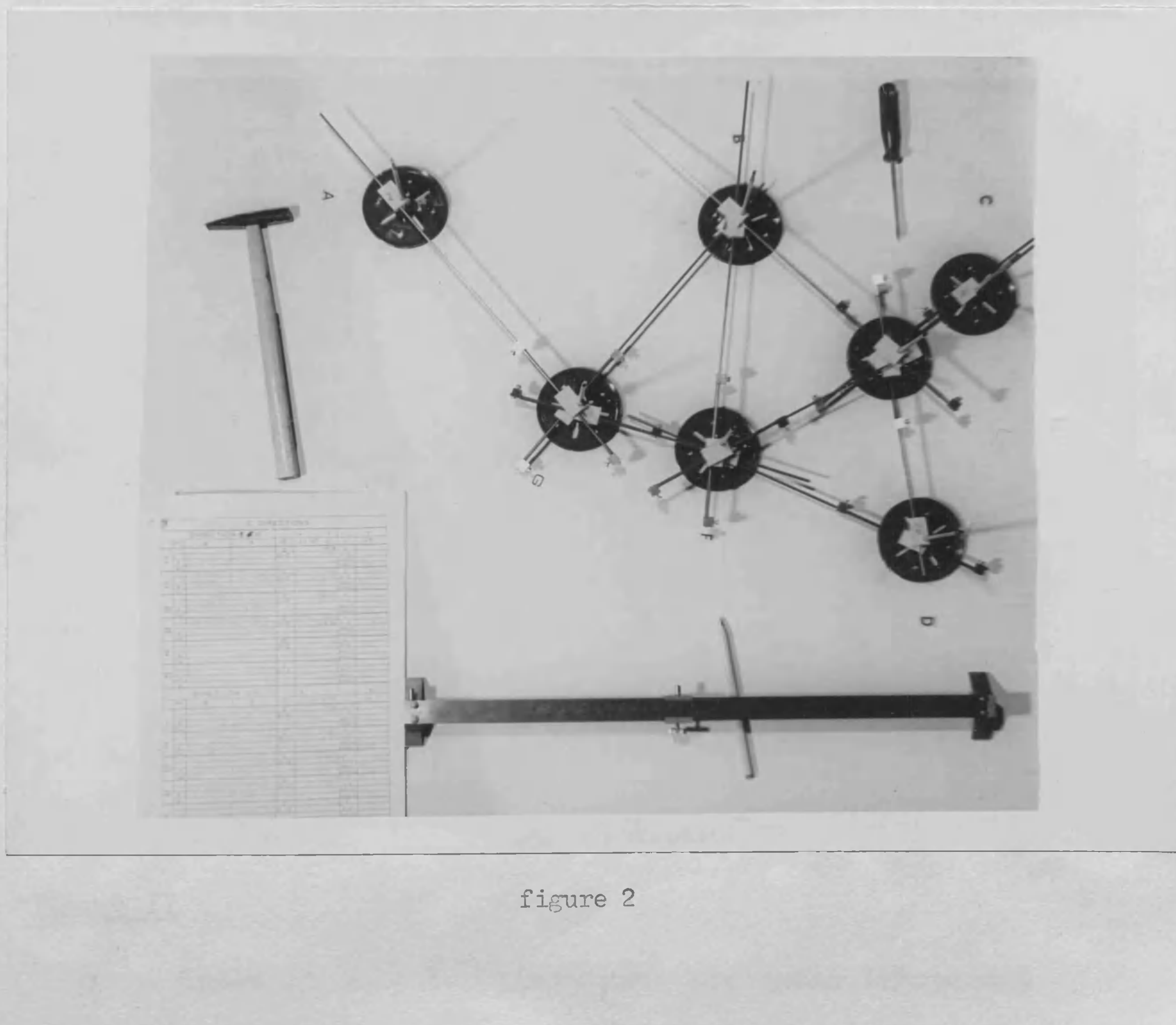


figure 2

Primary nets cannot be easily adjusted by the mechanical analogue, because of the required frequency of control points. Normally Laplace stations are not frequent enough to provide enough control points for the mechanical nets. However the use of a mechanical analogue in this case needs further consideration.

To detect gross errors, this analogue can be used for any triangulation net without the necessity for the density of control points needed for a complete adjustment. This gives the method a certain value. The accuracy in this case is less important, as the gross error shown by the excessive compression in some elastic units is not affected very much by the weight of the analogue.

The possibility of simultaneously adjusting sides and directions by this analogue has been investigated, and as a result, it is believed to be difficult. The introduction of the linear contradictions will affect the radial lengths, which result in different linear scales for angular adjustment. To introduce the angular contradictions to the model with different linear scales upsets the translation of the different joints. However figure 2, shows that the second model of the mechanical analogue for angles has been constructed in such a way that after adjusting angles, it can be used directly by substitution of the appropriate parts to allow an adjustment of the sides only using the system devised by Dr. Jerie of the I.T.C., Delft. This allows the adjustment of size after a previous adjustment of the shape. One precaution has to be considered, that this can only be applied to figures where different individual components (sides and angles) were measured with the same precision, otherwise the adjustment of sides will only result in distorting the shape which has been previously corrected.

Group II

These include the electrical analogues introduced by Su in which he obtained the correlates in the form of the physical voltages measured between different points to solve the normal equations only. It is difficult to build this type of system, which requires so many electrical elements and some sort of standardisation. So this method can be used only if the problem is small and does not involve any negative coefficients. The difficulties are such that no one has as yet constructed this type of analogue, and since electronic analogues are quite widely available and work along a similar analogy, the possibility of having electrical analogues constructed in future would appear to be very small.

This leads to a discussion of group IV, which includes the electronic analogue computers, so allowing the analogue methods to be considered together; the discussion of group III, mainly the electronic digital computers, can then take place partly as a comparison with the analogue methods.

Group IV

The fourth group (the equation solving analogues), which includes electronic analogue computers can be used to solve any problem if the latter meets the size requirements. The different elements that restrict the size of this analogue has already been discussed in 3.3.4., and the problem recommended for a medium size electronic analogue computer was 10x10 or 16x6. Quite recently much larger electronic analogue machines have been proposed and now begining to come into use. Their capacity will be much larger.

The main aim of the electronic analogue computer is to solve observation and condition equations directly, i.e. without the necessity of forming the normal equations, which does not seem possible in any other computer. In fact using this facility reduces the size of this problem considerably. For example, it was recommended that a problem 16x6 can be introduced to this analogue as maximum size, but this problem can be reduced to a 6x6 matrix if the normal equations are formed. But with the normal equations much more preliminary calculations will be necessary to form this type of equations. If this is accepted, this solution allows the use of such a medium size analogue for larger size problems. The size of the problem will then be increased to obtain 10 unknowns (i.e. 10x10 matrix). In this case the short side of the rectangular matrix will be of the order 10, while the long side of the matrix may go up to any practical number. In this case the analogue will solve a problem for fixing 5 new stations, when the solution is carried out by the variation of coordinates method. But in this case the analogue computer will loose its advantage of being able to solve the observation equations directly.

The size of the problem solvable on this analogue computer can also be greatly increased if one adopts one of the iterative methods of solution, e.g. [99]. The size of the problem solved by this method will be doubled, i.e. twenty unknowns can be obtained by using this analogue. The idea in this case is to solve for the lower diagonal matrix of the normal equations. But this will require more and more preparation and desk calculation. However even with all these computations and preparations, the efficiency of the analogue computer compared with that of the digital computers is remarkably high, especially when compared with small or even medium size digital computers. In the latter great deal of preparatory work and much time is involved in trying to introduce a large problem to these computers.

The time required for solving by the electronic analogue method is mostly that necessary for preparing the patch board, which is carried out away from the machine. For a matrix of 10x10, 4 or 5 hours will be enough to have all the connections of the patch board made ready. Another hour or less may be necessary for the physical check of the machine before running the problem. Once this is achieved then using the key board, the answers appear instantaneously when the individual keys are pressed, so that the total time spent in solving this problem is not long.

Group III

Digital computers are fairly familiar to those engaged in survey computation, at least in principle. They require the use of an input device which accepts tapes or cards to allow observational data and the programme giving the computational sequence to be entered and solved in the computer. The data is then processed in accordance with the details given in the programme and the answers are recorded in a store for output either on tape or cards, or visually printed out via a line printer.

There are now a large number of electronic computers on the market possessing widely different characteristics with regard to storage capacity, calculation speed, input/output devices, etc.. These features will affect greatly the type and size of survey adjustment problem which can be tackled and a discussion of all the relevant points would require at least a separate dissertation.

Since the computers now available are so large in capacity and computing possibilities, at first sight it would seem possible to solve all the types of survey network which might be encountered in practice. However such computers are few in number and require complicated programming particularly for the discovery and elimination of gross errors and mistakes, so much so that they have apparently not yet been used for the adjustment of survey networks. These difficulties will in time probably be overcome, but so far most survey computations and adjustments are being handled on relatively small digital computers.

The procedure generally followed can be seen from the following examples:

1- The Directorate of Overseas Surveys uses an I.C.T.(Ferranti)

Pegasus computer with an 8000 word drum store. This capacity is rather limited so that the following procedure is undertaken.

The method of variation of coordinates and observation equations is used. As a first stage the observation equations are produced from the known coordinates of the fixed stations and the provisional coordinates of the stations which are being adjusted. These observation equations are punched out on tape. The second stage is the production of normal equations. This is done from the output tape of the first stage without any alterations being made, the normal equations being punched out on tape. The third stage is the solution of the equations directly using the Cholesky method. Again this is done from the output tape of the second stage without any alterations being made. The output from this stage is the required corrections to the provisional coordinates.

The process is done in three stages in order to get as large a block as possible into the computer - the punching out of equations is time consuming and expensive, but it saves storage space. However it might be possible to speed up the solution or to increase the problem size to be tackled, e.g. by taking account of the large number of zero coefficients which occur in survey matrices and by the solution of the normal equations by iterative rather than direct methods.

2- The Photogrammetry and Mapping Division of the Iranian Oil Operating Companies utilises an IBM 1620 computer with 5000 word store. Their experience is reported by King [133]. The variation of coordinates method is also utilised and again a multiple stage procedure is necessary. As a complete maximum a problem of 22 unknowns (11 new points) can be solved. It may be noted that this is similar to the capacity of the mechanical analogue designed and constructed by the author or by the PACE 231R-V electronic analogue computer discussed in <u>3.3.</u>

3- The author has attempted a much larger problem using the larger English Electric KDF-9 computer at the University of Glasgow. This has a total storage of 32,000 words, but 18,000 words are used for the compiling programme, so that only 14,000 words are available for the programming instructions and the observational data. A programme was prepared with the assistance of Mr Saad Ben Hamid, which allowed the solution of the normal equations formed by 12x18 condition equations, so that 18 unknowns were solved.

This experience gives some experience for a comparison for analogue and digital computers.

II- COMPARISON

First of all, the time which can be spent on achieving a correct programme is very long. On the other hand if a suitable programme of a general nature can be satisfactorily achieved it can be used for a large number of networks. So if there is a great deal of work to be carried out, which will recur regularly then the investment of a considerable time in achieving a programme is amply repaid.

With analogue computation, no programming is required so that, after a small amount of preparation, the data can be introduced and the problem solved directly in short time. On the other hand since no programme exists, if another problem of the same type occurs then the problem has to be tackled completely afresh. This will happen both with electronic analogue and mechanical analogue solutions, but is not so serious for relatively small problems. A large problem would be a different question.

Again the question of expertise is important. Much less skill is needed for analogue computation and it is possible to use semi-skilled personnel after a little training. On the other hand the analysis, formulation and programming of the forward and back solution of the normal equations for electronic digital computers is quite an exacting and elaborate task requiring the use of highly skilled and trained programmers. However such skills are rather more readily available than before and as the use of digital computers becomes more widespread this point may become less important. Electronic analogue computers are not uncommon, but are nowhere as common as digital computers except in research institutions.

The question of data input is a vital one in weighing up the relative merits of these computation systems. The observations are made in the field - manually at present. These have to be reduced and tapes or cards prepared to give all observational data as well as the programming instructions. To produce error-free tapes or cards is difficult and if the errors go undetected, their effect goes unobserved until the final solution is printed out. Even then a great expertise may be required to detect errors in programming or observational data. When they are found there is a need to alter the tape and to re-run the problem. With large computers it is probable that the limit of the problem which can be tackled is more likely to be determined in practice by the difficulties of providing an error-free input tape for a large block than by the capacity available. This is particularly so with survey observations which are invariably manually recorded and do not have automatic measurement and recording as is normally encountered in photogrammetric work for example.

With analogue computing, corrections can be made immediately by inspection while the problem is being solved. There is no need to wait till the final solution appears in either the mechanical or electronic analogues. In the latter the time required for the physical checks of all the elements incorporated in the problem is very much less than the time necessary to find out the errors and repeat the programming when the digital computers fail to give any solution. However with an electronic analogue computer, if a problem of double the size requires solution, an electronic analogue computer of double the capacity(and cost)has to be provided. With digital computers those larger problems can be handled without the need to employ a computer of double the cost, so that they are more suitable for large adjustment problems from an economic point of view. However digital computers only become attractive for larger problems when the amount of storage is correspondingly large. A solution which might be attractive for adjustment work could be a hybrid link between an analogue and a digital computer. The digital machine could be used to programme the analogue computer so eliminating the need to set up the coefficients manually on the electronic analogue computer. It also provides the storage which is missing on the analogue computer while still allowing the problem to be solved on the analogue computer so retaining the advantages already discussed in 3.3.5.

The place of the mechanical analogue computer is more difficult to assess. It can be developed more highly than the prototypes constructed for this study. Whether this would be worthwhile in view of the developments in both electronic digital and analogue computers is open to question. It is a useful method in places where shortage of skilled personnel and lack of resources prevent the use of more expensive machines and it is the only one of the four computation methods which is of use in the field, especially for the detection of gross errors.

It is also possible to view the mechanical analogue method as being an extremely useful one for demonstrating the computation and adjustment processes and my belief is that they could play an extremely useful role in the way in universities and similar institutions. In particular students can understand the procedures much more easily through this seeing it happens directly. At present the mechanical analogue methods could hold their own against the competing ones having little less capacity and the attractive features of easy checking of the computation and detection of mistakes and requiring a relative small investment both in capital and skill. The method can be developed to give a greater efficiency, but it must be doubtful this can be done to the extent that seems to be possible with either of the electronic methods.

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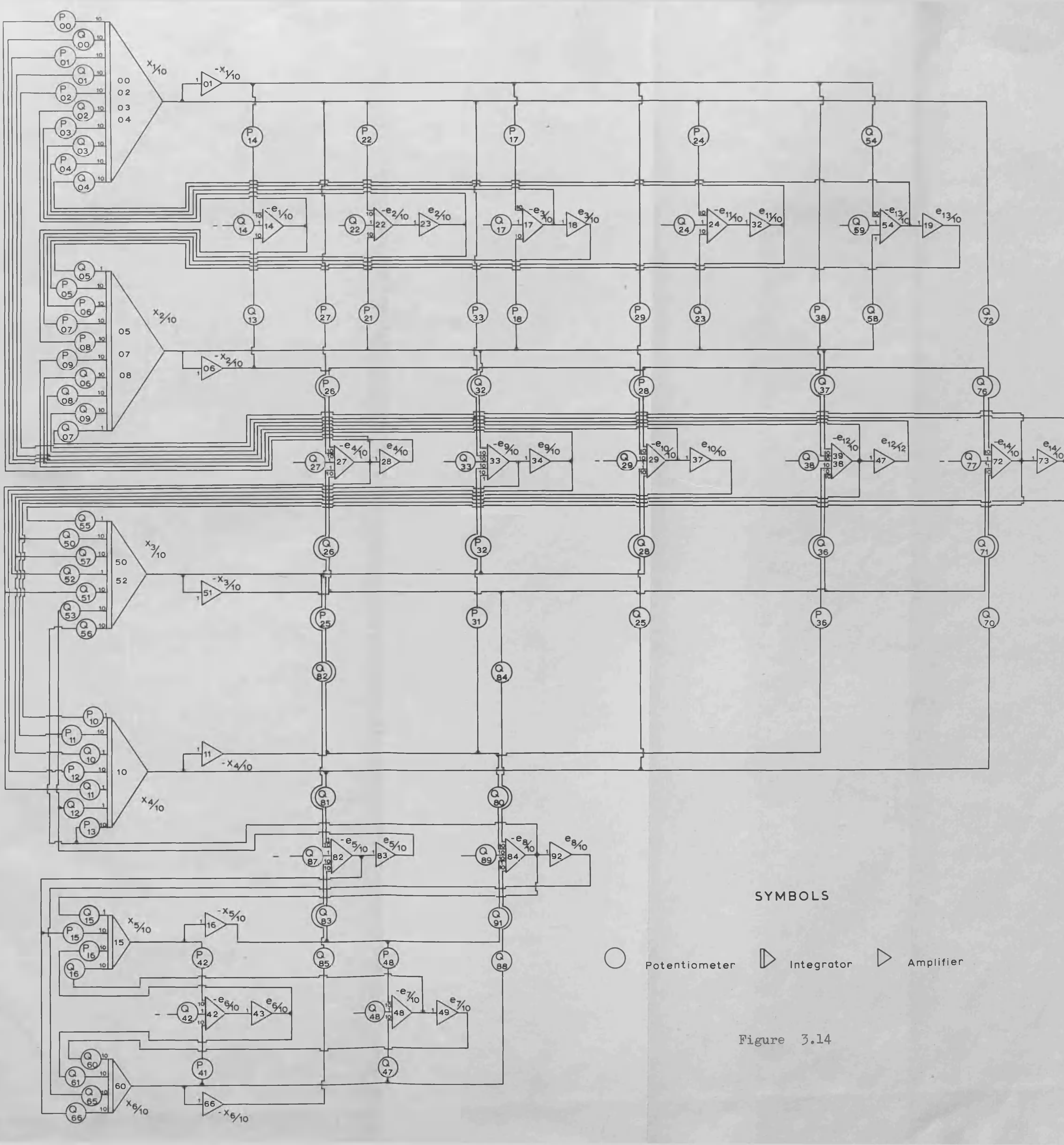

Figure 3.14

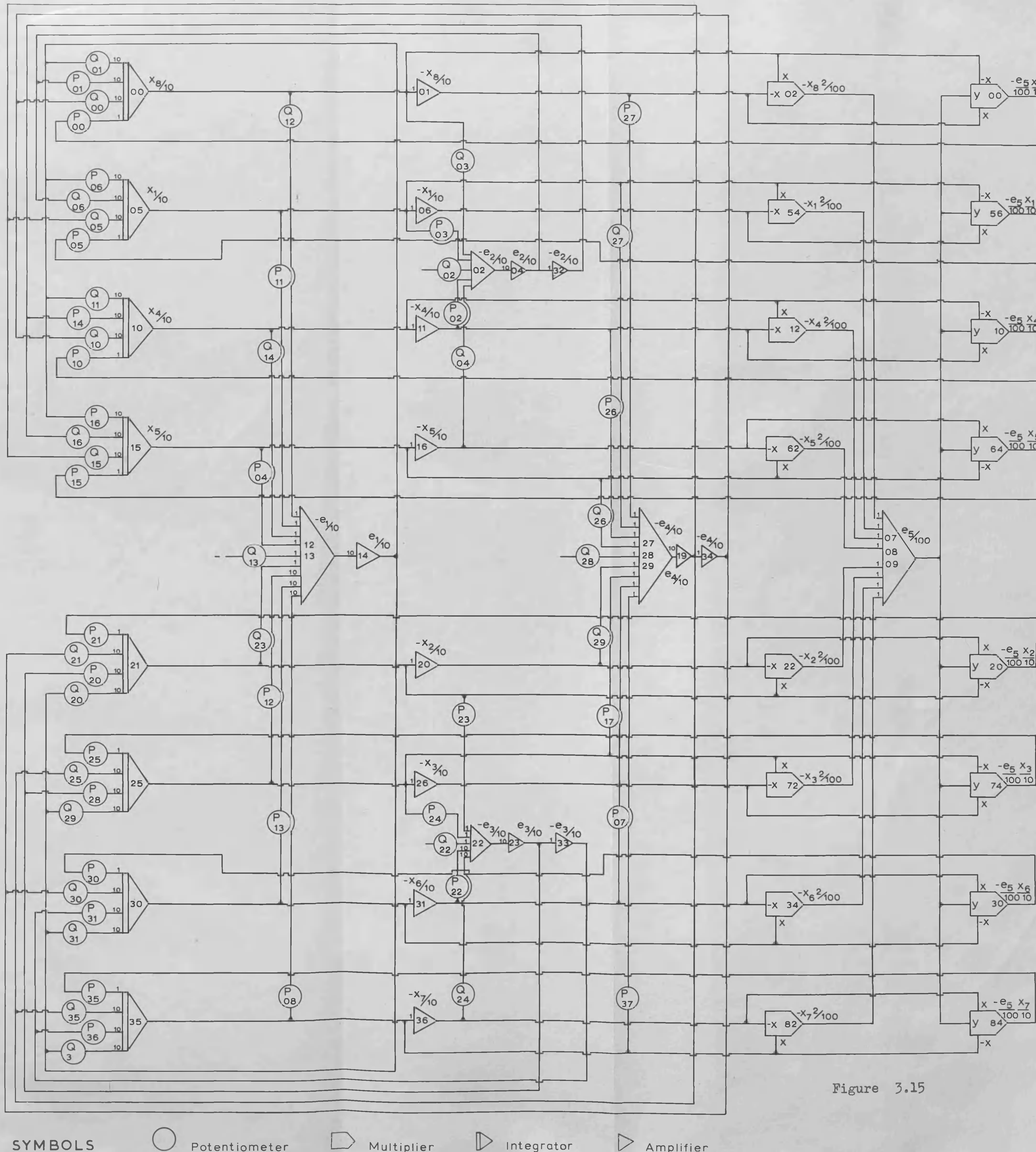

Figure 3.15