THE QUANTISATION OF CONSTRAINED SYSTEMS USING THE BATALIN, FRADKIN, VILKOVISKY FORMALISM

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Declaration

The contents of this thesis are original with the following exceptions. Chapter one contains an introductory survey of the field of constraint quantisation and none of the contents are original. The contents of chapter two are also not original with the exception of the treatment of constraint rescaling. Constraint rescaling has been discussed before in [8,9,29,30] however, the treatment given in chapter two is much more comprehensive than any previous discussion. Also, the general form of the canonical constraint rescaling transformation given in section 2.4 has never been published before.

Chapter three is all original except where references indicate otherwise. Various earlier versions of this work have appeared in the papers

"Factor ordering and ghost variables" Phys. Lett. B202 (3), 358 (1988).
"Covariant Factor Ordering of Gauge Systems Using Ghost Variables I: Constraint Rescaling" J. Math. Phys. 30, 477 (1989).
"Covariant Factor Ordering of Gauge Systems Using Ghost Variables II: States and Observables" J. Math. Phys. 30, 487 (1989).

The first two sections of chapter four are not original but the remaining sections are claimed as original work. Likewise, the first two sections of chapter five are not original but the remaining sections are.

Appendix one does not contain original work but appendix two does.

Acknowledgements

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The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact.

A. N. Whitehead

Geometry is a magic that works ...

R. Thom

Contents

Abstract

The purpose of this work is to examine the problem of quantising constrained dynamical systems within the Batalin Fradkin Vilkovisky (BFV) formalism. The work concentrates almost entirely on theories with a finite number of dimensions and constraints linear in the phase space momenta. Chapters four and five give some discussion of possible extensions of the work to more general constraints.

Chapter two will give a discussion of the classical theory of constrained systems and, in particular, will study the symmetries present in such theories. The main result in this chapter is that the constraint rescaling symmetry (this is the freedom to transform to new sets of constraints which describe the same true degrees of freedom) is a canonical transformation in the BFV phase space. An implicit definition of the most general form of this transformation will be given.

After chapter two we will study the quantisation of constrained systems. We will always work with the basic assumption that the correct constraint quantisation should give the same results as one obtains from quantising the classical true degrees of freedom.

Chapter three will examine the quantisation of finite dimensional linear constraints. It will be shown that, to obtaining the correct constraint quantisation, one must use four symmetries. These symmetries are coordinate transformations on the classical configuration space, coordinate transformations on the true configuration space (i.e. the configuration space obtained by solving the constraints), weak changes to observables (i.e. adding terms which vanish when the constraints are applied) and rescaling of constraints. The main result of chapter three is that enforcing these four symmetries is sufficient to fix the main ambiguities in the quantisation and that the resultant quantum theory is equivalent to classically solving the constraints and then quantising. These results rely on the fact that the classical

canonical rescaling transformation can, for the restricted class of rescalings which are of interest in gauge theories, be made into a unitary quantum transformation. This quantum transformation is the main tool used in chapter three and enables us to maintain a Hilbert space structure on the extended state space (i.e the state space which contains both physical and unphysical states). Previous attempts by other authors to quantise finite dimensional gauge theories without ghost variables have failed to maintain a Hilbert space structure. This is one of the main advantages of the work presented here.

In chapter four we will look at the use of the BFV method in geometric quantisation. The main motivation for this is to study constraints which depend quadratically on the phase space momenta e.g. the constraints which arise in general relativity. Chapter four does not give a proper quantisation of quadratic constraints but it does give some indication of the new features which arise in these theories. The main result seems to be the need to use polarisations, in the BFV phase space, which genuinely mix the bosonic and fermionic degrees of freedom.

Chapter five will look at some classical aspects of constraint rescaling for Yang-Mills field theories. The various possible field theory constraint rescalings will be discussed and a few results will be proven showing to what extent it is possible to simplify the conventional Yang-Mills constraints via rescalings. These simplifications consist of forming an equivalent set of constraints which commute with respect to Poisson brackets. These simplified constraints were very useful in the analysis of the finite dimensional case.

Chapter One
Introduction

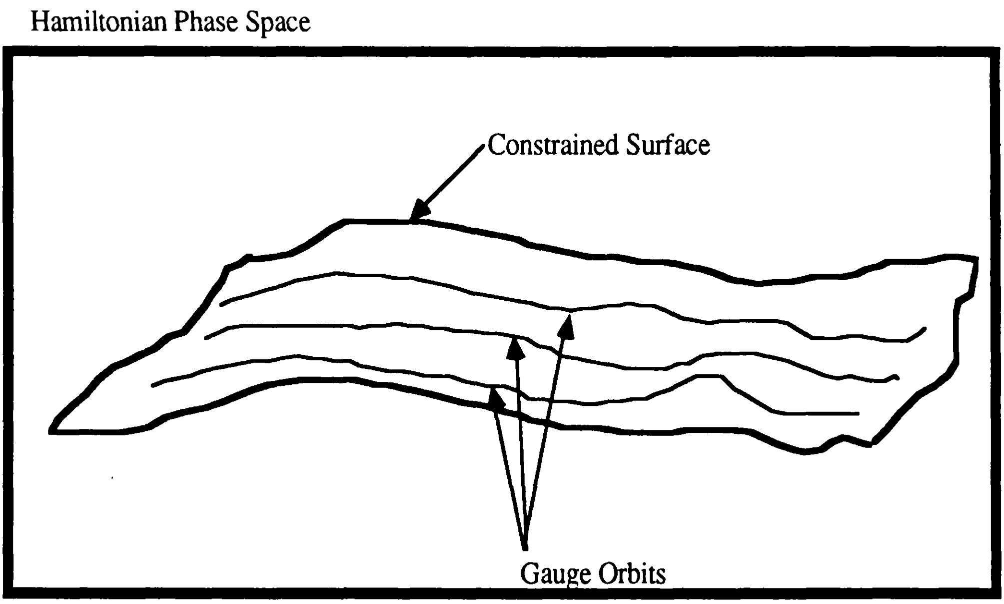
1.1 Symmetries and Constraints

During this century there have been many dramatic developments in our understanding of the fundamental interactions of nature. Throughout all these developments there has been one common theoretical principle which is now central to all the successful theories we have today. This is the principle of local invariance of the theory under some group of symmetry transformations. One of the main examples of this is the Yang-Mills theories (for a modern account of these theories see [1]) where the symmetry group is taken as the fundamental object and the field equations are derived from their invariance properties. General relativity (see for example [2]) also exhibits a local symmetry namely the freedom to change to different coordinate systems. Local symmetries also playing a major role in the more speculative modern ideas such as supergravity [3] and supersymmetric string theory [4].

Although the use of symmetries has proved very beneficial to the development of theoretical physics there is a price to pay because the description of the dynamics becomes more complex. Essentially the problem is that the presence of symmetries implies a certain amount of redundancy in the theory, i.e. there exist different mathematical configurations which describe the same physical state. In classical mechanics this redundancy leads to the presence of constraints in the phase space description of the dynamics. These constraints essentially remove the redundant degrees of freedom. The presence of constraints in theoretical physics was first analysed in detail by Dirac [5] thought there is now a deeper, though more technical, understanding of his work in terms of symplectic reduction. A good modern survey of the subject is given in [6].

Most of the work that is done in classical constrained dynamics uses the phase space methods of Hamiltonian mechanics. For this reason it is very useful to have, in one's head, the following picture of the sort of constrained Hamiltonian system that we shall be discussing (We will work exclusively with first class constraints. The technical definition of this will be given in chapter two).

Figure 1.1 Diagrammatic Representation of
The Phase Space of a First Class Constrained Dynamical System



This diagram indicates that all the physical dynamics of the theory takes place on a subset of the phase space known as the constrained surface. In addition, there is a redundancy on this constrained surface in the sense that there are different points representing the same physical state. The set of all points representing a given state is known as a "gauge orbit" (this terminology is borrowed from gauge theory where one moves along the gauge orbit via the action of the gauge group on the phase space). It is the set of these gauge orbits which represent the true, independent degrees of freedom of the constrained theory. We shall refer to the phase space with the constrained surface etc. as the extended phase space and will denote it by P. The set of gauge orbits will be referred to as the reduced phase space or the true physical phase space and will be denoted by P_{phys}.

The gauge orbits can totally change the mathematical structure of the theory. For example, in Yang-Mills theory the extended phase space is an affine space which has trivial curvature and topology associated with it. However, in the process of reducing to the true degrees of freedom, nontrivial curvature and topology appear. The most obvious way of dealing with these features is to abandon using the extended phase space and work instead with the reduced phase space. However, this has not been a very useful approach because the set of gauge orbits on the constrained surface is a rather complex and unmanageable set. Thus, in the absence of any simple characterisation of the true degrees of freedom, one is forced to work with the extended phase space though, in doing this, we must be careful not to ignore any relevant properties of the true degrees of freedom.

The previous paragraph has stated the core problem of classical constrained dynamics i.e., all calculation procedures must, for practical reasons, be given in terms of the extended phase space and the constraints. However, these calculation procedures must be equivalent to working with only the true physical degrees of freedom .

The main task we will be addressing in this thesis is the study of constraints in quantum mechanics. However, before we begin to look at this it will be useful to look at some examples of the constraints that arise in classical physics. The simplest situation occurs for theories with a *gauge type symmetry* (by a gauge type symmetry we mean that there is some Lie group acting as a symmetry group on the configuration space) where the constraints are linear in momenta. The main example of this type of theory are the Yang-Mills theories where the constraints are,

$$\Pi_{\alpha}^{\ 0} = 0, \tag{1.1}$$

and,

$$\partial_i \Pi_{\alpha}^{\ i} - q f^{\gamma}_{\ \alpha\beta} A_i^{\ \beta} \Pi_{\gamma}^{\ i} = 0. \tag{1.2}$$

In these equations $A_i^{\ \beta}$ denote the gauge fields and $\Pi_{\alpha}^{\ i}$ the canonically conjugate momentum fields in the phase space. The Greek indices represent gauge degrees of freedom and the i sum over the spacial directions (the Einstein summation convention will always be assumed unless otherwise stated). The $f^{\gamma}_{\ \alpha\beta}$ are the structure constants of the gauge group and q is a coupling constant.

All other theories in physics have, at worst, constraints which depend quadratically on the phase space momenta. For example the constraints in general relativity are (for a discussion of this see, for example, [7]),

$$G_{ab\,cd} P^{ab} P^{cd} - g^{\frac{1}{2}} R = 0, \tag{1.3}$$

and,

$$P_a^{\ b}{}_{|b} = 0. \tag{1.4}$$

In these equations we are using the standard notations of canonical relativity. That is, we use the metric components, g_{ab}, on some spatial hypersurface as the configuration space variables and have denoted the canonically conjugate fields by P^{ab}. $G_{ab\,cd}$ is the Wheeler DeWitt supermetric (defined, for example, in [7]), g = det $[g_{ab}]$, R is the scalar curvature associated with g_{ab} and the bar in equation (1.4) denotes covariant differentiation.

Quadratic constraints (i.e. quadratic in the phase space momenta) also arise in, for example, supergravity and string theory and have their origin in the fact that all these theories have a reparametrisation invariance.

It is worth noting that all the above examples of physically interesting constrained systems are field theories. This means that the phase space, constrained surface and gauge orbits, represented in figure 1.1, are going to be infinite dimensional. This leads to many technical complications and difficulties which are not necessarily intrinsic to the problem of constrained dynamics. For this reason there has been much recent work [8,9,10,11,12,13,14] studying finite dimensional examples of constrained systems. The main hope of this work is to gain an understanding of the problems intrinsic to constrained dynamics thereby leading to a better understanding of the dynamical aspects of modern physical theories. Most of the original work of this thesis will concentrate on finite dimensional problems.

1.2 Quantisation and Constraints

The standard way to construct a modern quantum theory of some physical phenomenon is to start from a classical theory and then quantise it. There are a number of ways of attempting the quantisation of a classical theory. The two standard ways are path integral quantisation [15] and canonical quantisation (see any standard quantum mechanics text such as [16]). For the present purposes we will concentrate on canonical quantisation where the basic idea is to take the classical physical observables (functions on the phase space) and replace them with hermitian operators acting on some Hilbert space (these operators are usually only defined on a dense subset of the Hilbert space but we will ignore such technicalities here). Canonical quantisation can be thought of as a mapping from some subset of $C^\infty(P)$ (the set of smooth functions on the phase space P) to the set of linear operators on the chosen Hilbert space. This mapping is often written symbolically as $f \rightarrow \hat{f}$ where $f \in C^\infty(P)$.

The simplest example of canonical quantisation occurs when the configuration space is $\mathbb{R}^N$ equipped with the standard flat metric. We will denote the cartesian coordinate system on this space by $(Q^1, .. ,Q^N)$ and the coordinates canonically conjugate to these on the phase space by $(P_1, .. ,P_N)$. The quantisation of this space is based on the canonical commutation relations,

$$[\hat{P}_A, \hat{Q}^B] = -i\hbar\, \delta_A^{\ B}, \tag{1.5a}$$

$$[\hat{P}_A, \hat{P}_B] = [\hat{Q}^A, \hat{Q}^B] = 0. \tag{1.5b}$$

The Stone-Von Neumann theorem [17,18] tells us that (modulo some technicalities [19]) all irreducible representations of equations (1.5) are unitarily equivalent to the familiar Schrödinger picture whose Hilbert space is the set of L^2 functions on $\mathbb{R}^N$ with respect to the Lebesgue measure.

Unfortunately this rather tidy procedure of quantising on $\mathbb{R}^N$ is spoiled as soon as we examine the rest of the quantisation which consists of specifying the operators $\hat{f}$ which correspond to the classical functions f. The reason for this is that the construction of the quantum operators is highly ambiguous. To see how these ambiguities arise let us look at the case where f is a polynomial function of the P_A with coefficients that depend on the Q^A. For such an f the construction of $\hat{f}$ consists of choosing a specific ordering of the $\hat{P}_A$ and $\hat{Q}^A$ in each term of f. There are many possible choices of ordering and therefore many possible choices of $\hat{f}$. One may be tempted to choose the orderings in such a way as to preserve the Dirac correspondence rule,

$$\{f, g\} \rightarrow \frac{1}{i\hbar} [\hat{f}, \hat{g}], \tag{1.6}$$

where f and g are smooth functions on the phase space. However, Van Hove [20] showed that this is impossible. Van Hove's theorem states that there exists no irreducible representation of the entire classical Poisson algebra of functions as a commutator algebra. This means that equation (1.6) must fail for some choice of functions f and g. This *Van Hove problem* is met as soon as one looks at functions with a nonlinear dependence on the momenta and is thus directly relevant to physics.

We shall return in later chapters to present possible choices of orderings for a limited set of classical functions but, for the moment, we will leave this problem and look instead at some further aspects of canonical quantisation.

There is a very important corollary to the Van Hove theorem which states that it is impossible to make the quantum theory invariant under all of the symmetries of the classical theory. This follows because (modulo some technicalities) the Lie algebra of the group of classical canonical transformations is the set of Hamiltonian vector fields on the phase space (see [21] for details of Hamiltonian mechanics). The Van Hove theorem now says that this Lie algebra structure is not preserved by the quantisation and so some of the symmetry group structure will be lost.

All of the above problems and ambiguities of quantisation become more severe when one studies configuration spaces which are curved and possess nontrivial topology. For example, a nontrivial topology would imply the nonexistence of a global canonical coordinate system. This would then imply that any quantisation based on the canonical commutation relations will be local and will hence miss some aspects of the full quantum theory.

There exist a few methods of quantisation which attempt to give a proper global approach. The two main examples of this are geometric quantisation (see for example [22]) and the group theoretic approach of Isham [19]. These approaches are conceptually much more appealing than the naive local method but we will not explore them further here.

Finally, to complete this discussion of canonical quantisation, one should realise that all the ambiguities and problems discussed above are still present, only more so, in the infinite dimensional case. For example, the Stone-Von Neumann theorem fails even for flat, topologically trivial infinite dimensional configuration spaces. In additional to all the above problems infinite dimensional systems also suffer from the presence of infinities and the need to renormalise. This feature of quantisation is so restrictive that most classical field theories do not appear to be quantisable and these theories are normally excluded from physics.

We now wish to study the problem of canonically quantising systems which possess constraints. It was noted in the previous section that the physically interesting constrained systems are infinite dimensional and that the constraints can induce nontrivial topology and curvature on the true degrees of freedom. Therefore, any full discussion of constraint quantisation should take place within the framework of some global quantisation scheme and should also address such issues as renormalisability. Not surprisingly this is very difficult and no major progress on global constraint quantisation has been made. There has been some work done on finite dimensional global constraint quantisation [23,24] but this will not be discussed further here.

In this thesis we will follow the approach of many authors [8,9,10,11,12,13, 14] and only study the local quantisation of constraints on finite dimensional phase spaces. This is not a totally satisfactory approach and it must miss some of the problems that will occur when one studies the physically relevant examples of constrained theories. However, the author feels that a sufficient number of important facts have been learned from these simpler systems to justify their study.

1.3 The Dirac Approach to Constraint Quantisation

Dirac [5] was the first to study how canonical quantisation would be modified in the presence of classical constraints. Dirac's ideas have greatly influenced most approaches to constraint quantisation so a summary of his ideas will now be given.

Let the classical constraints be denoted by φ_α where $\alpha = 1, .., k$. The basic idea developed by Dirac is to quantise the theory as usual, i.e construct a Hilbert space (e.g. the space of Schrödinger wave functions) and make all the observables, including the constraints, into operators acting on this Hilbert space. The operators corresponding to the constraints will be denoted by $\hat{\varphi}_\alpha$ and Dirac's idea was to use these operators to pick out the subset of states, Ψ, in the Hilbert space which satisfy,

$$\hat{\varphi}_\alpha \Psi = 0 \qquad \alpha = 1,.., k. \tag{1.7}$$

These states are known as the *physical states*. Dirac's next step was to restrict all the physical observables to this subset of the Hilbert space and, for consistency, the physical observables must preserve this subset i.e. acting on a state satisfying (1.7) they must produce another state satisfying (1.7). This consistency condition is normally expressed mathematically as,

$$[\hat{f}, \hat{\varphi}_\alpha] = \hat{f}_\alpha^{\beta} \hat{\varphi}_\beta \qquad \alpha = 1,.., k \tag{1.8}$$

where $\hat{f}$ is the quantum operator associated with some classical observable f and the $\hat{f}_\alpha{}^\beta$ are linear operators. It is also necessary for the constraints to satisfy a consistency condition to ensure that the solution space to (1.7) is not too small. This consistency condition is,

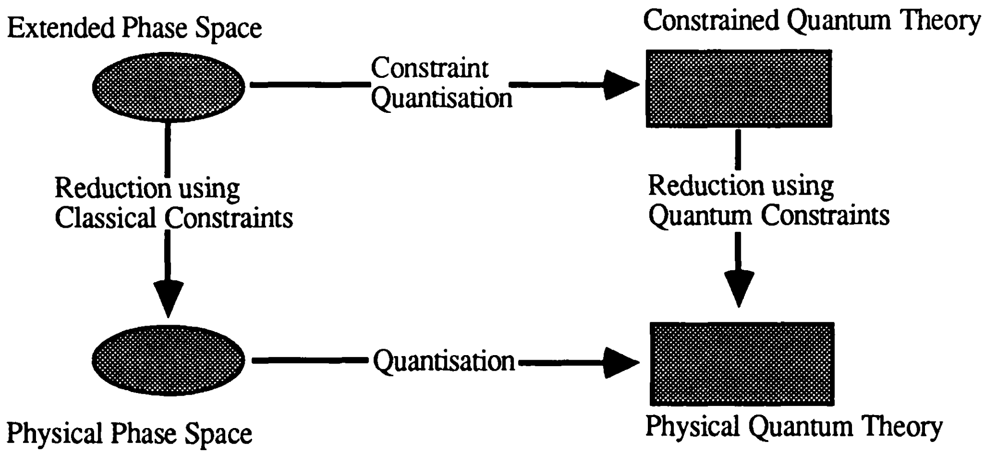
$$[\hat{\varphi}_\alpha, \hat{\varphi}_\beta] = \hat{C}^\gamma{}_{\alpha\beta} \hat{\varphi}_\gamma \qquad \alpha, \beta = 1,.., k. \tag{1.9}$$

where the $\hat{C}^\gamma{}_{\alpha\beta}$ are operators. We will elaborate much more on equations (1.8) and (1.9) in the later chapters so let us just accept them for the moment.

To obtain a quantisation that satisfies the above consistency conditions most authors try to exploit the ordering ambiguities in the constraints and the observables i.e., they try and adjust the orderings until (1.8 and 9) are satisfied. This has lead to the viewpoint that constraint quantisation only poses additional factor ordering problems over and above the normal ordering problems of quantisation. The unspoken assumption is that if a consistent factor ordering can be found then the constraints have been quantised correctly and the Dirac procedure will give the right results. This assumption clearly requires some justification but to do this some criterion is needed for deciding what the *correct quantisation procedure* is.

There is a fairly natural choice of correct quantisation procedure which is as follows. First of all take the classical theory and reduce to the true degrees of freedom thereby eliminating all the constraints. Now quantise this theory as an unconstrained system and the result will be taken to be the correct quantum theory. There will, of course, be ambiguities in quantising the true degrees of freedom but these can normally be dealt with (see section 3.2). This criterion says that the following diagram is commutative.

Figure 1.2 Diagrammatic Representation of the Relationship between the Constraint Quantisation and the Physical Quantisation



This criterion for a correct quantisation seems fairly natural but, upon further thought, it is not obviously correct. What it assumes is that the classically redundant degrees of freedom remain as such in the quantum theory. This is not necessary so, for example in the gauge theory case the criterion is equivalent to saying that the quantum theory must still be gauge invariant. To see that this may not be so one only has to look at the considerable amount of interest in anomalous gauge theories who's quantum versions do not appear to be gauge invariant (see for example [25]). Of course if one decides to relax this criterion there is no obvious replacement and one would at least hope that the criterion works for simple cases such as finite dimension gauge theories. The conventional wisdom says that anomalous behaviour is due to the infinite dimensional nature of field theory and is presumably not present in the finite dimensional case. Even in the light of these criticisms the criterion is still worth adopting, it is the obvious first choice and should be pursued to see if it works or not. Therefore, for the remainder of this thesis, when the phrase "correct quantum theory" is used it should be taken to mean correct according to the above criterion.

1.4 Recent Developments in the Dirac Approach to Constraint Quantisation

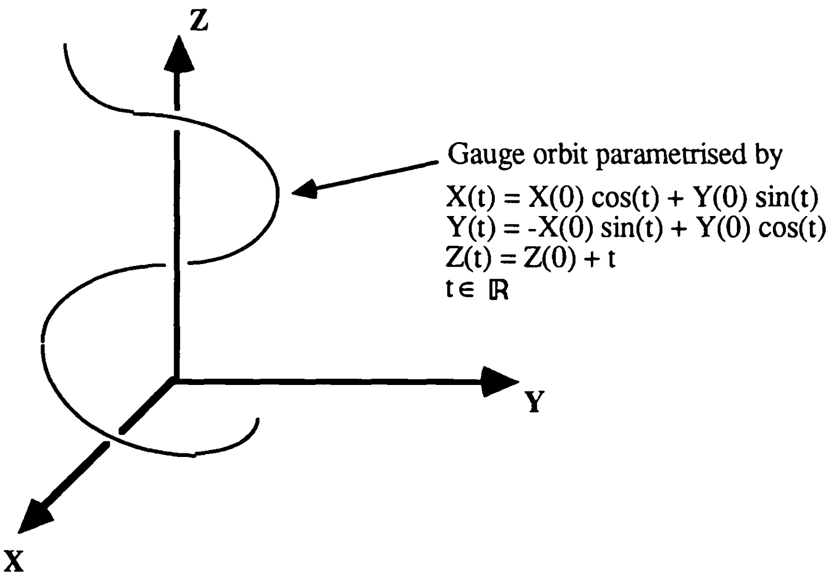
There has been some recent work by Kuchar [8,9,10] which has revealed a number of new features of constraint quantisation. These discoveries have been the main motivation for undertaking the work of this thesis so a review of them will now be given.

Kuchar studied finite dimensional theories with constraints linear in momenta. The main example of this is a gauge theory though Kuchar does not require his constraints to come from any group. Kuchar gave a detailed analysis of the factor ordering problems present in such systems. He was able to give an explicit quantisation procedure, of the Dirac form, which is equivalent to quantising the classical true degrees of freedom. His quantisation procedure needs only the information explicitly given on the extended phase space and does not require any reduction to the true degrees of freedom. This work contained a number of new features which will now be discussed.

Firstly, Kuchar showed that it is not sufficient to just find a factor ordering which satisfies the Dirac consistency conditions (1.8 and 9). A very instructive example of this is contained in an example first suggested by DeWitt [26] and developed in detail by Kuchar [8,10]. The example, christened "The Quantum Well of Orvieto" by Kuchar, consists of a nonrelativistic particle moving in flat three dimensional space subject to the gauge group of helical motions. That is all the points on a curve of the form shown in figure 1.3 are regarded as being equivalent.

Figure 1.3

An Example of a Configuration Space Gauge Orbit for the Quantum Well of Orvieto



In phase space language this problem is described by the Hamiltonian,

$$H = (P_R)^2 + \frac{1}{R^2}(P_\Theta)^2 + (P_Z)^2, \tag{1.10}$$

and one constraint,

$$\varphi = P_Z - P_\Theta , \tag{1.11}$$

where we have expressed everything in cylindrical polar coordinates (R,Θ,Z) and their canonically conjugate momenta (P_R, P_Θ, P_Z).

The constraint (1.11) is simple enough to be solved (see [10] for details). The true configuration space of the problem is shaped like a bottle with an infinitely long neck and is described by the coordinates,

$$r = R = (X^2 + Y^2)^{\frac{1}{2}} , \tag{1.12a}$$

and,

$$\theta = (\Theta + Z) \bmod 2\pi . \tag{1.12b}$$

This space is curved and with respect to the above coordinates the metric is,

$$[g^{ab}] = \mathrm{diag} [1, 1 + \frac{1}{r^2}] . \tag{1.13}$$

With this information we can now write down what we are assuming to be the correct quantum theory. The state space is the set of complex valued L^2 functions of (r,θ) where the square integrability is with respect to the following inner product,

$$\langle \Psi_1 | \Psi_2 \rangle = \int \Psi_1^* \Psi_2 \frac{r}{(1 + r^2)^{\frac{1}{2}}} dr\, d\theta. \tag{1.14}$$

The quantum hamiltonian is,

$$\hat{h} = -\hbar^2 \left\{ \frac{1}{r(1+r^2)} \partial_r + \partial_r^{\,2} + \left(1+\frac{1}{r^2}\right) \partial_\theta^2 \right\}, \tag{1.15}$$

where ∂_r denotes partial differentiation with respect to r and similarly for ∂_θ. In equation (1.15) we have used the Laplace-Beltrami ordering for the hamiltonian (more details of this quantisation procedure will be given in section 3.2).

To see the significance of the Quantum Well of Orvieto let us now look at the constraint quantisation of this system. The extended configuration space for the Quantum Well of Orvieto is $\mathbb{R}^3$ with the standard euclidean metric. If we just follow the usual method for quantising on this space we will end up with the Hilbert space of L^2 function with respect to the normal Lebesgue measure and the operators corresponding to (1.10 and 11) will be,

$$\hat{H} = -\hbar^2 \left\{ \frac{1}{R} \partial_R + \partial_R^2 + \frac{1}{R^2} \partial_\Theta^2 + \partial_Z^2 \right\}, \tag{1.16}$$

and,

$$\hat{\varphi} = -i\hbar \left\{ \partial_Z - \partial_\Theta \right\}. \tag{1.17}$$

Observe that the operators in (1.16 and 17) commute with each other so the Dirac consistency conditions are satisfied by this quantisation. However, the important point is that this quantisation is not equivalent to the correct quantisation given above. To see this look at the action of (1.16) on a "physical state" (i.e. a state which is killed by the operator (1.17)) Such a state is of the form $\Psi(R, \Theta+Z)$ which is correct for a wave function on the true degrees of freedom but unfortunately the action of (1.16) on this state is,

$$\hat{H}\Psi = -\hbar^2 \{ \frac{1}{R}\partial_R + \partial_R^2 + (1 + \frac{1}{R^2})\partial_{\Theta}^2 + Z \}\Psi, \qquad (1.18)$$

which disagrees with equation (1.15).

To summarise, it has been shown that there is a consistent constraint quantisation of the Quantum Well of Orvieto which is not equivalent to the true quantum theory. This was first pointed out by Kuchar in [10]. This is a very important observation and it is worth pausing to see why the naive constraint quantisation is wrong. Essentially the problem arises because the extended configuration space is flat whereas the true physical configuration space is curved. This curvature alters the quantum theory but the naive quantisation totally ignores this and hence gets the wrong result.

Kuchar's study of constraint quantisation [8,9] gives a method of incorporating any curvature on the true degrees of freedom into the constraint quantisation . This enables him to give an alternative quantisation of the Quantum Well of Orvieto which is equivalent to the true quantum theory [10]. To obtain his quantisation Kuchar made use of certain symmetries of the classical theory which he elevated to quantum symmetries as well. The full set of symmetries Kuchar used will be discussed in section 3.1 so, for the moment, we will concentrate only on the constraint rescaling symmetry as this appears to be the key new idea introduced by Kuchar. A constraint rescaling is a transformation from one set of constraints φ_α to a new set of constraints $\tilde{\varphi}_\alpha$ given by,

$$\tilde{\varphi}_\alpha = \Lambda_\alpha^{\ \beta}(Q^A, P_A)\, \varphi_\beta\ , \qquad (1.19)$$

where Λ is any invertible, matrix valued function on the extended phase space. The Q^A denote a coordinate system on the extended configuration space and the P_A the momenta canonically conjugate to the Q^A. Kuchar demanded that the quantum theory should be invariant under all the constraint rescalings which depend on Q^A only (this preserves the linearity of the constraints).

The use of constraint rescaling invariance is essentially saying that the classical and quantum theory should not depend on how one chooses to parametrises the constrained degrees of freedom. Kuchar's work suggests that demanding this parametrisation invariance may be sufficient to force the constraint quantisation to give the correct answers. However, Kuchar does not make this conclusion as he was not able to give an analysis of the extent to which his quantum theory is forced purely by the required symmetry invariances. We will analyse this question in detail in chapter three.

Kuchar found that it is not possible to express his constrained quantum theory using the standard formalism of Hilbert spaces. He introduces a Hilbert space structure only on the physical states (this is essential if the normal probabilistic interpretation of quantum mechanics is to be maintained) but leaves the extended quantum state space (i.e. the space that contains both the physical and unphysical quantum states) as a linear space with no inner product structure. This means that he regards as meaningless such questions as the self adjointness of the constraints. This is another new feature of Kuchar's work because it is conventional to assume that the extended state space is a Hilbert space and that the constraints are self adjoint operators. Kuchar's reason for abandoning the conventional view point is as follows. Suppose we have a set of linear constraints φ_α and we rescale then to $\tilde{\varphi}_\alpha$ given by equation (1.19) with Λ a function of the Q^A only. Then, to maintain self adjointness of the constraints the quantum operators must transform as,

$$\hat{\varphi}_\alpha \rightarrow \hat{\tilde{\varphi}}_\alpha = \frac{1}{2} [\Lambda_\alpha^\beta , \hat{\varphi}_\beta]_+$$

$$= \Lambda_\alpha^\beta \hat{\varphi}_\beta + \text{some nonzero function of } Q^A, \qquad (1.20)$$

where $[\ ,\ ]_+$ denotes an anticommutator. The second line of (1.20) follows because $\hat{\varphi}_\alpha$ will be a first order linear differential operator (the constraints are linear in momenta). The exact form of the function of Q^A in (1.20) depends on the ordering adopted for the constraints and is not relevant here.

Now, because Kuchar requires the quantum theory to be invariant under constraint rescaling a physical state must give zero when acted upon by both $\hat{\varphi}_\alpha$ and $\hat{\hat{\varphi}}_\alpha$. This then forces all physical states to give zero when multiplied by the nonzero function on the right hand side of equation (1.20) but the only state which does this is the zero state. This argument lead Kuchar to conclude that constraint rescaling invariance was inconsistent with self adjoint constraints so he abandoned both self adjointness of the constraints, and a Hilbert space structure for the full state space.

1.5 Plan of the Thesis

The original aim of this thesis was to examine the new results of Kuchar using the Batalin Fradkin Vilkovisky (BFV) method of constraint quantisation [27,28]. The BFV technique is an alternative approach to that of Dirac and a detailed discussion of the method will be given in chapter two.

In the light of Kuchar's work there were a number of interesting questions which were clearly in need of answer.

1) How can constraint rescaling, and the other symmetries considered by Kuchar, be implemented using the BFV method? This question looked very interesting due to the known fact [29, 30] that classical constraint rescaling is a canonical transformation in the BFV phase space.

2) Kuchar had to abandon a Hilbert space structure for his extended state space. Technically this is a disadvantage of Kuchar's work so it is natural to ask if the BFV formalism also has this drawback.

3) What are the BFV analogues of Kuchar's ordering prescriptions and e.g.

what is the correct BFV quantisation of the Quantum Well of Orvieto?

4) Does the BFV formalism make it any easier to analyse the uniqueness of the orderings? In particular could it be proven that constraint rescaling invariance is enough to force the correct ordering?

We have been able to answer all of these questions and the results will be presented in chapters two and three. Chapter two is mainly a technical review of the relevant theory of classical constraints and the BFV method, only some of the contents are original. Chapter three then gives a full discussion of the BFV theory of finite dimensional linear constraints. In chapter four we will look at possible extensions of our work to the quantisation of quadratically constrained theories and in chapter five some discussion of the more physically interesting field theory case will be given. We will then finish in chapter six by giving the overall conclusions.

Chapter Two
The Classical Theory of Constrained Systems

2.1 First Class Constraints

In this chapter we will develop, and summarise, the main machinery of classical constrained hamiltonian systems. The presentation will concentrate on finite dimensional phase spaces with only occasional comments on the generalisation to field theory. The field theory case will be discussed in more detail in chapter five.

We will work on an extended phase space (P, ω) i.e. P is a symplectic manifold with symplectic form ω (see [21] for details of the geometric approach to hamiltonian mechanics). It will be assumed that P is a cotangent bundle i.e. $P=T^*Q$ for some manifold Q (T^*Q denotes the cotangent bundle to Q). The symplectic form on P will always be taken to be the canonical one induced by the cotangent bundle structure. The manifold Q will be known as the extended configuration space and we will adopt the convention that Q has dimension N and therefore P has dimension 2N.

It should be noted that a symplectic manifold need not be of the form T^*Q. An example of this is the reduced phase space of general relativity [31]. However, taking $P=T^*Q$ is not unreasonable as the extended phase space of both Yang-Mills theories and general relativity are cotangent bundles. Also, since the quantisation in this thesis is local, any global properties of P are going to be ignored so we may as well assume, from the outset, that they are not present.

All the dynamics will take place on a submanifold, C, of P. This will be known as the constrained surface. We will assume that C is globally describable as the zero set of k real valued, smooth functions on P. These functions will be denoted by φ_α where α runs from 1 to k. The assumption that globally defined constraints exist covers all cases of physical interest. For example, in Yang-Mills theories the global constraints can be obtained from the equivariant momentum map derived from the gauge group action on Q [32]. We will always assume that the k<N and that the constraints are irreducible. By this we shall mean that 0 is a regular value of the map from P to $\mathbb{R}^k$ defined by,

$$x \in P \rightarrow (\varphi_1(x), \dots, \varphi_k(x)). \tag{2.1}$$

In particular this means that C is a 2N-k dimensional submanifold of P (this follows from the regular value theorem [33]). There are theories where reducible constraints occur [34,35,36] but we will not discuss this case here .

The constraints will always be first class. This means that there exist smooth functions $C^{\gamma}{}_{\alpha\beta}$ on P, called structure functions, such that,

$$\{ \varphi_\alpha, \varphi_\beta \} = C^{\gamma}{}_{\alpha\beta}(Q^A, P_A)\, \varphi_\gamma, \tag{2.2}$$

where { , } denotes the Poisson bracket on P. First class constraints implies that C is a coisotropic submanifold of P i.e. $\omega\,|_C$ (the symplectic form restricted to C) has degenerate directions all of which lie in C. These degenerate directions form an integrable distribution on C (see [37] for details on differential geometry). One way to see that the distribution is integrable is to observe that the degenerate directions are spanned by the hamiltonian vector fields of the constraints and that, on C, these hamiltonian vector fields satisfy,

$$[X_{\varphi_\alpha}, X_{\varphi_\beta}] = - C^{\gamma}{}_{\alpha\beta}\, X_{\varphi_\gamma}, \tag{2.3}$$

where [,] denotes the lie bracket of vector fields. The hamiltonian vector field, X_f, of a function f is defined by,

$$\iota_{X_f}\,\omega = df, \tag{2.4}$$

where ι denotes contraction on the first index of ω. Equation (2.3) implies that the degenerate directions are integrable (this is Frobenius' theorem [38]). This integrability property means that the degenerate directions of $\omega\,|_C$ foliate C with k dimensional smooth submanifolds and these are what we called the gauge orbits in figure 1.1. Modulo some technicalities the space of gauge orbits is a symplectic manifold [22] which we call the physical phase space and we denote it by P_{phys}. P_{phys} will have dimension 2(N-k).

For gauge theories the $C^{\gamma}{}_{\alpha\beta}$ are normally the structure constants $f^{\gamma}{}_{\alpha\beta}$ of the gauge group. Constraints with structure constants are said to form a closed algebra because the hamiltonian vector fields of the constraints close under lie brackets every where on P. Gauge theory constraints are always linear in momenta i.e. there exist smooth functions $\varphi_\alpha{}^A$ on Q such that,

$$\varphi_\alpha = \varphi_\alpha{}^A\,(Q^B)P_A\,. \tag{2.5}$$

In this equation we have adopted the convention that Q^A (A = 1, .. ,N) denotes a coordinate system on Q and P_A (A = 1, .. ,N) the corresponding canonical momenta. Unless otherwise stated this convention will be adopted for the rest of the thesis.

We will not find it useful to distinguish between gauge theory constraints and more general *linear constraints* (i.e. linear in momenta). For this reason we will refer to all linear constraints as gauge theory constraints.

The constraints for theories which incorporate gravity are more complex than the linear case. These theories have genuine structure functions. Such constraints are said to form an open algebra as the hamiltonian vector fields of the constraints only close, under lie brackets, on the constrained surface. In addition, some of the constraints in gravitational theories depend quadratically on the phase space momenta. In this chapter we will work with general first class constraints but in later chapters additional assumptions will be made depending on the situation under study.

The dynamical objects of interest on P are the true physical observables (i.e. the smooth functions on P_{phys}). These can be represented by smooth functions on P which are constant along the gauge orbits on the constraint surface. A function F on P which satisfies this invariance property is called a physical observable and the invariance property can be expressed mathematically as,

$$\{ F, \varphi_\alpha \} = F_\alpha{}^\beta \varphi_\beta \qquad (\alpha = 1, \dots , k) \tag{2.6}$$

for some smooth functions $F_\alpha{}^\beta$. There is no physical distinction between two functions on P which agree on the constraint surface and such functions are said to be weakly equivalent. The true physical observables (i.e. the functions on P_{phys}) are in one to one correspondence with the equivalence classes of physical observables on P with respect to weak equivalence.

The following result gives a simple characterisation of weak equivalence.

Theorem 2.1

Two functions F_1 and F_2 are weakly equivalent if and only if there exists smooth functions F^α, on P, such that $F_1 = F_2 + F^\alpha \varphi_\alpha$.

The proof can be found in [39]. This theorem relies on the finite dimensional assumption and it is not known if it works in the field theory case though most authors assume that it does.

2.2 The Symmetries of Constrained Theories

The theory described above has a number of symmetries which do not affect the true dynamical content of the theory. These are,

(a) The group of canonical transformations on the phase space P.

(b) The group of canonical transformations on the physical phase space P_{phys} (this is a special case of (a) but it is useful to single it out for special attention).

(c) Weak changes to observables i.e.,

$$F \rightarrow F + F^{\alpha}\varphi_{\alpha} \tag{2.7}$$

(d) Invariance under rescaling of the constraints i.e.,

$$\varphi_{\alpha} \rightarrow \tilde{\varphi}_{\alpha} = \Lambda_{\alpha}{}^{\beta}\ \varphi_{\beta}. \tag{2.8}$$

where, $\Lambda_{\alpha}{}^{\beta}$ is any smooth, invertible matrix which depends on Q^A and P_A.

Symmetry (d) reflects the fact that it is the constraint surface and the gauge orbits that are important, not how one has chosen to describe them. For finite dimensional systems the symmetry (d) fully incorporates all *equivalent* sets of constraints in the neighbourhood of the constrained surface (two sets of constraints are said to be equivalent if they have identical zero sets). That is, the transformation (d) acts transitively, at least in the neighbourhood of the constrained surface, on the space of all constraints. To see this let φ_{α} and $\tilde{\varphi}_{\alpha}$ be two equivalent sets of constraints and apply theorem 2.1 to get,

$$\tilde{\varphi}_{\alpha} = \Lambda_{\alpha}{}^{\beta}\ \varphi_{\beta}, \tag{2.9}$$

and,

$$\varphi_{\alpha} = \Gamma_{\alpha}{}^{\beta}\ \tilde{\varphi}_{\beta}. \tag{2.10}$$

It is tempting to deduce from this that $\Lambda_\alpha{}^\beta$ is invertible everywhere on P. However, one must be careful because of the following result.

Theorem 2.2

$$F^{\alpha_1 .. \alpha_m} \varphi_{\alpha_m} = 0 \quad \Rightarrow F^{\alpha_1 .. \alpha_m} = G^{\alpha_1 .. \alpha_m \alpha_{m+1}} \varphi_{\alpha_{m+1}} ,$$

where, the $F^{\alpha..\alpha}$'s and the $G^{\alpha..\alpha}$'s are functions on P that are totally antisymmetric in their indices. The proof of this is given in [40] and the result underlies much of the classical Batalin, Fradkin and Vilkovisky approach to constrained systems [41, 42].

To use this result for our problem notice that equations (2.9) and (2.10) imply that,

$$(\delta_\alpha{}^\gamma - \Lambda_\alpha{}^\beta \Gamma_\beta{}^\gamma) \tilde{\varphi}_\gamma = 0,$$

where $\delta_\alpha{}^\gamma$ denotes the Kronecker delta. Theorem 2.2 now implies that,

$$\Lambda_\alpha{}^\beta \Gamma_\beta{}^\gamma = \delta_\alpha{}^\gamma + G_\alpha{}^{\gamma\delta} \tilde{\varphi}_\delta .$$

Thus, Λ is invertible on the constrained surface and hence, because Λ is continuous as a matrix valued function on P, Λ is invertible on some neighbourhood of the constrained surface.

An obvious question to ask about the above proof is whether Λ is globally invertible. This question is rather more complex than it appears because, using theorem 2.2, we see that Λ defined by equation (2.9) is not unique. Λ can always be modified by a transformation of the form,

$$\Lambda_\alpha{}^\beta \to \Lambda_\alpha{}^\beta + H_\alpha{}^{\beta\gamma}\,\varphi_\gamma,$$

where $H_\alpha{}^{\beta\gamma}$ is antisymmetric in its upper two indices. The real question that should be asked is whether Λ in equation (2.9) can be made globally invertible via such a transformation. The author has been unable to answer this question for general constraints. However, it is possible to give a global result for linear constraints and this will be done in chapter three.

It should be noted that the above discussion may not be valid for field theories because theorem 2.1 could fail. Therefore, it is possible that these theories may possess alternative sets of constraints which are not related by a rescaling transformation, even in the neighbourhood of C. The possible concequencies of this will be discussed briefly in chapter five.

Transformation (d) is not very easy to implement in its present form. The main reason for this is that it is not consistent with the canonical structure of the phase space. To illustrate this, and also to give a very important result that will be used later, we state the following theorem.

Theorem 2.3

For every point x in the phase space there exists an invertible matrix $\Lambda(Q^A,P_A)$ such that, in some neighbourhood of x,

$$\{\, \tilde{\varphi}_\alpha \,,\, \tilde{\varphi}_\beta \,\} = 0, \tag{2.11}$$

where $\tilde{\varphi}_\alpha$ is given by (2.8). The proof of this can be found in [29]. This will be referred to as abelianisation of constraints. It is possible to go further and change the phase space coordinates to make the $\tilde{\varphi}_\alpha$ into the first k momenta (this follows from standard theorems in hamiltonian mechanics [21]). This will be referred to as trivialisation of constraints and this situation is particularly simple because the coordinates split into Q^α ($\alpha = 1, .. k$) and Q^a ($a = k + 1, .. N$) . The Q^a are the physical degrees of freedom and the Q^α are the unphysical constraint directions.

It will be useful at this point to introduce a notational convention which will be adopted, unless otherwise stated, for the rest of the thesis. That is, capital Latin indices will denote the degrees of freedom on Q and will run over the range 1 to N. Greek indices will denote the constraint degrees of freedom and will normally run from 1 to k. Finally, lower case Latin indices will denote the true physical degrees of freedom and will normally run from k+1 to N.

We will now give a proof of theorem 2.3 for the special case of linear constraints. This is the case we will study most and the following proof will be generalised to cover Yang-Mills theories in chapter five. Note also that the following proof shows that linear constraints can be abelianised using a rescaling matrix which depends only on the Q^A.

Proof of theorem 2.3 for Linear Constraints

The first step is to realise that the constraints can be used to construct a set of k vector fields, v_α, on Q given locally by,

$$v_\alpha = \varphi_\alpha{}^A \, \partial_A \, , \tag{2.12}$$

where $\partial_A = \partial / \partial Q^A$. These vector fields satisfy,

$$[\, v_\alpha \, , v_\beta \,] = C^\gamma{}_{\alpha\beta} \, v_\gamma \, , \tag{2.13}$$

where [,] denotes the Lie bracket between vector fields on Q. In coordinate free notation this construction consists of first forming the hamiltonian vector fields associated with the φ_α and then projecting these fields onto Q using the push forward of the projection map from T^*Q to Q.

Equation (2.13) tells us that the v_α are surface forming i.e. they generate a foliation of Q by k dimensional submanifolds (this follows from Frobenius' theorem [38]). We proceed now by introducing a local coordinate system adapted to this foliation i.e. a coordinate system in which the first k coordinates follow the foliating leaves (the gauge directions) and the others describe the physical degrees of freedom. The trick is to form the basis of vector fields associated with this coordinate system and restrict attention to the vector fields associated with the gauge directions. Call these $\tilde{v}_\alpha$. Since these vector fields are part of a coordinate base they satisfy,

$$[\tilde{v}_\alpha, \tilde{v}_\beta] = 0, \tag{2.14}$$

and, since they form a basis of the tangent plane along the gauge directions, they satisfy,

$$\tilde{v}_\alpha = \Lambda_\alpha{}^\beta \, v_\beta , \tag{2.15}$$

for some invertible $\Lambda\,(Q^A)$. The $\tilde{v}_\alpha$ will be expressible in the form,

$$\tilde{v}_\alpha = \tilde{\Phi}_\alpha{}^A \, \partial_A,$$

which enables us to define a new set of linear constraints in P via.,

$$\tilde{\Phi}_\alpha = \tilde{\Phi}_\alpha{}^A \, P_A.$$

Equations (2.14) implies that the $\tilde{\Phi}_\alpha$ are abelian with respect to the Poisson bracket which completes the proof.

When we come to quantise these theories we will not attempt to preserve all of the symmetries (a) to (d). Indeed such an attempt would be doomed to failure because of the Van Hove theorem [20]. Only a limited subset of the symmetries will be used though which subset will depend on the specific situation. As the main purpose of this chapter is to set up the necessary classical formalism for all the cases studied later, we will not limit the set of symmetries at present.

2.3 The Batalin Fradkin Vilkovisky Formalism

The Batalin Fradkin Vilkovisky (BFV) method of constraint quantisation gives an alternative approach to that of Dirac. This approach consists of adding new anticommuting (or grassmann) variables to the theory and has had a rather complex history. The subject began with Feynman [43] who observed that, for some gauge theories, a one loop unitary S-matrix could only be obtained if one added *ghost particles* i.e. scalar particles with fermionic statistics. DeWitt [44] then showed that Feynman's trick gives a unitary S-matrix to all orders in perturbation theory and Faddeev and Popov [45] gave an analysis of these ideas using functional techniques. The next step was taken by Becchi, Rouet and Stora [46] and independently by Tyutin [47] who observed that the introduction of ghosts leads to a new, nilpotent supersymmetry now called the BRST symmetry.

The Russian workers Batalin, Fradkin and Vilkovisky [27, 28] gave a new analysis of the BRST method and extended the method to Gravity. The BFV formalism is now regarded as the most general formulation of the BRST method and the standard review of the method was given by Henneaux in [29].

As we have said the central idea of the BFV formalism is to enlarge the extended phase space P by introducing k anticommuting (or grassmann) variables η^α and their conjugate variables ρ_α. The η^α are called ghosts and the ρ_α

conjugate ghosts. This space will be denoted by SP (for superphase space). We shall adopt the convention of denoting functions on SP by bold letters to distinguish them from functions on P. SP has a superpoisson bracket defined on it. There are many possible sign conventions associated with this bracket structure. We shall follow the sign conventions of [6] and all the relevant definitions are given in appendix one.

For the present we will regard this extension of P as a purely algebraic procedure and, grassmann differentiation and integration [48] as algebraic operations on SP. For most of what follows this is the easiest way to regard the formalism. However, this simplistic view point is not totally satisfactory and, we will later need to adopt a more sophisticated approach using supermanifold techniques [49, 50, 51, 52]. By this we mean that SP is a topological space which looks locally like the product of 2N copies of the even part of some grassmann algebra with 2k copies of the odd part of the same grassmann algebra. We will delay further discussion of this until the formal development of the BFV technique is completed. It will then be clearer why the mathematical machinery of supermanifold theory is needed.

The main object of interest on SP is the BRST charge $\mathbf{\Omega}$. This is a function on SP which takes values in a grassmann algebra (the specific grassmann algebra will be the one on which the supermanifold SP is modelled) and is defined by the following properties,

1) $\mathbf{\Omega}$ can be written as a sum of terms all of which have ghost number one (ghost number equals the number of ghosts minus the number of conjugate ghosts).

2) The first term in $\mathbf{\Omega}$ (i.e. the term involving no ρ_α) is $\varphi_\alpha \eta^\alpha$.

3) $\{ \mathbf{\Omega} , \mathbf{\Omega} \} = 0.$

Such an $\mathbf{\Omega}$ is proved to exist in [29] but it is not unique. $\mathbf{\Omega}$ can be written in the form,

$$\Omega = \varphi_\alpha \eta^\alpha + \sum_{m=1}^{k} \frac{1}{m!} C^{\alpha_1 .. \alpha_{m-1}}{}_{\beta_1 .. \beta_m} \eta^{\beta_1} .. \eta^{\beta_m} \rho_{\alpha_1} .. \rho_{\alpha_{m-1}} , \qquad (2.16)$$

The $C^{\alpha .. \alpha}{}_{\beta .. \beta}$ in this equation are referred to as higher order structure functions. The nonuniqueness of $\mathbf{\Omega}$ occurs because the $C^{\alpha .. \alpha}{}_{\beta .. \beta}$ are not fixed uniquely by conditions 1) .. 3) (see [29] for details). Most of the time we will be working with constraints linear in momenta and in this case the BRST charge simplifies to,

$$\Omega = \varphi_\alpha \eta^\alpha + \frac{1}{2} C^\gamma{}_{\alpha\beta} \eta^\alpha \eta^\beta \rho_\gamma , \qquad (2.17)$$

where $C^\gamma{}_{\alpha\beta}$ is the same function as appears on the left hand side of equation (2.2). We shall only need the general form for the BRST charge when we look at quadratic constraints in chapter four.

$\mathbf{\Omega}$ is used to construct an operator $\boldsymbol{\delta}$ which acts on the functions on SP via.,

$$\boldsymbol{\delta} \mathbf{F} = \{ \mathbf{\Omega} , \mathbf{F} \}. \qquad (2.18)$$

Property 3) of the BRST charge implies that $\boldsymbol{\delta}$ is nilpotent i.e.,

$$\boldsymbol{\delta}^2 = 0. \qquad (2.19)$$

The next step in the BFV approach is to extend all the gauge invariant functions on P to grassmann valued functions on SP. Let F be a gauge invariant function, its extension **F** is defined by;

1) **F** is a sum of terms each with ghost number zero.
2) The first term in **F** (i.e. the term with no ghosts or conjugate ghosts) is F.

3) $\boldsymbol{\delta}\mathbf{F} = 0.$

Such an **F** is proved to exist in [29] but it is not unique. **F** can be written in the form,

$$\mathbf{F} = \mathrm{F} + \sum_{m=1}^{k} \mathrm{F}_{\alpha_1 .. \alpha_m}{}^{\beta_1 .. \beta_m} \eta^{\alpha_1} .. \eta^{\alpha_m} \rho_{\beta_1} .. \rho_{\beta_m}. \tag{2.20}$$

The ambiguities in **F** occur because the $\mathrm{F}_{\alpha .. \alpha}{}^{\beta .. \beta}$ appearing in the summation are not uniquely determined by conditions 1) .. 3).

If one starts with two weakly equivalent functions F_1 and F_2 then their BRST extensions satisfy,

$$\mathbf{F}_1 = \mathbf{F}_2 + \boldsymbol{\delta}\mathbf{G}, \tag{2.21}$$

for some function **G**, on SP, with ghost number minus one [29].

Equations (2.19) and (2.21) suggest that we have a homological construction of the physical observables. This is, in fact, what is happening and the BFV method can be elegantly stated by introducing the following complex of function spaces.

$$\ldots \xrightarrow{\boldsymbol{\delta}} \Gamma^{-2} \xrightarrow{\boldsymbol{\delta}} \Gamma^{-1} \xrightarrow{\boldsymbol{\delta}} \Gamma^{0} \xrightarrow{\boldsymbol{\delta}} \Gamma^{1} \xrightarrow{\boldsymbol{\delta}} \Gamma^{2} \ldots \tag{2.22}$$

Here, Γ^r denotes the set of functions defined on SP with ghost number r. The operator $\boldsymbol{\delta}$ acts as shown because $\boldsymbol{\Omega}$ has ghost number one. In this language the main result is that the physical observables (functions on $\mathrm{P}_{\mathrm{phys}}$) are in one to one correspondence with the zeroth cohomology class of this complex i.e. the space,

$$\frac{\{\ \mathbf{F} \in \Gamma^0 : \boldsymbol{\delta}\mathbf{F} = 0\ \}}{\{\ \mathbf{F} \in \Gamma^0 : \mathbf{F} = \boldsymbol{\delta}\mathbf{G} \text{ for some } \mathbf{G} \in \Gamma^{-1}\}}\ . \tag{2.23}$$

It was noted above that there are ambiguities in the construction of Ω and, in the extension of gauge invariant functions to SP. None of these ambiguities matter because they do not affect the cohomology classes of the above complex [29].

2.4 The Classical Symmetries in the BFV Formalism

The main power of the BFV method, for the contents of this thesis anyway, is that the symmetry properties of the theory become much neater in the ghost language. What happens is that weak changes to observables, and rescaling of constraints, can be described by a canonical transformation in the superphase space. This is very important because it means that, when we come to quantise the theory, these symmetries can be described by unitary transformations. One is then able to demand invariance of the quantum theory under these transformations to fix the ambiguities in the quantisation. Such an approach can also be adopted in the more conventional approach without ghosts but there the symmetry transformations are much more awkward to work with [8, 9].

We shall not discuss the canonical transformations which gives weak changes to observables as we will not need to use them at any later stage. Any interested reader is referred to [29] for details. We will, however, give a fairly comprehensive treatment of constraint rescaling as this is the key tool that will be used in the quantisation.

The main result follows by looking at the following generating function (see [53] for details on hamiltonian mechanics) which gives an implicitly defined, active, even canonical transformation on SP.

$$F(\tilde{Q}^A, \tilde{\eta}^\alpha, P_B, \rho_\beta) = - \tilde{Q}^A P_A - (\Lambda^{-1})_\alpha{}^\beta (\tilde{Q}^A, \tilde{\eta}^\alpha, P_B, \rho_\beta) \, \tilde{\eta}^\alpha \rho_\beta, \qquad (2.24)$$

where Λ is an arbitrary, grassmann valued, invertible, matrix valued function of the variables shown. Λ must also have ghost number zero. The transformation generated by this function is,

$$\tilde{Q}^A = Q^A - (\Lambda^{-1})_\alpha{}^{\beta,A}\, \tilde{\eta}^\alpha \rho_\beta . \tag{2.25a}$$

$$\tilde{P}_A = P_A + (\Lambda^{-1})_\alpha{}^\beta{}_{,\tilde{A}}\, \tilde{\eta}^\alpha \rho_\beta . \tag{2.25b}$$

$$\tilde{\eta}^\alpha = (\Lambda)_\beta{}^\alpha\, \eta^\beta + (\Lambda)_\delta{}^\alpha (\Lambda^{-1})_\beta{}^{\gamma,\delta}\, \tilde{\eta}^\beta \rho_\gamma . \tag{2.25c}$$

$$\tilde{\rho}_\alpha = (\Lambda^{-1})_\alpha{}^\beta\, \rho_\beta + (\Lambda^{-1})_\beta{}^\gamma{}_{,\tilde{\alpha}}\, \tilde{\eta}^\beta \rho_\gamma . \tag{2.25d}$$

The notation being used is ${}^{,A} = \partial/\partial P_A$; ${}_{,\tilde{A}} = \partial/\partial \tilde{Q}^A$; ${}^{,\alpha} = \partial/\partial \rho_\alpha$ and ${}_{,\tilde{\alpha}} = \partial/\partial \tilde{\eta}^\alpha$. The latter two derivatives are grassmannian [48]. In these equations the Λ and Λ^{-1} have the same functional dependence as Λ^{-1} has in the generating function so (2.25) only defines the transformation implicitly. Before proceeding to look at the detailed properties of this transformation some technical comments are in order.

To understand equations (2.25) it is no longer sufficient to regard the use of grassmann variables on a purely formal, algebraic level. It is necessary to give a proper meaning to equations (2.25) which mix fermionic and bosonic objects. It is also necessary to show that (2.25) actually gives a single valued and suitably differentiable transformation.

To make sense of the mixed transformation it is necessary to regard SP as a supermanifold. As we said earlier this means that SP is a topological space which looks locally like the cartesian product of 2N copies of the even part of some grassmann algebra with 2k copies of the odd part of the same grassmann algebra. These local regions where SP looks simple are the supermanifold equivalent of coordinate patches. Equations (2.25) can now be understood to be the local coordinate representation of the active transformation which sends the point

$(Q^A, \eta^{\alpha}, P_A, \rho_{\alpha})$ to the point $(\tilde{Q}^A, \tilde{\eta}^{\alpha}, \tilde{P}_A, \tilde{\rho}_{\alpha})$. This transformation then maps the functions on SP via its pull back.

To show that equations (2.25) give a well defined, and well behaved, transformation we need a version of the implicit function theorem which will work for grassmann functions and, we need to check that transformation (2.25) satisfies the conditions of this theorem. To do this one must be careful in defining differentiation on the space that the supermanifold SP is modelled on. There exist a number of ways of defining superdifferentiation. So far we have regarded it as a purely algebraic procedure which is the approach adopted in the supermanifold theory developed by DeWitt [49]. However, it is hard to see how an implicit function theorem could be proved using this formalism.

Rather than using DeWitt's ideas we shall adopt the approach of Rogers which she developed in [51] i.e. we now regard SP as a supermanifold of the Rogers type. The main advantage of Rogers' work is that she relates grassmann differentiation to differentiation on a Banach space where there is a well developed theory of calculus [54]. In particular the implicit function theorem works for Banach calculus and it is possible to use Rogers' work to develop an implicit function theorem for grassmann functions (see appendix two). The superimplicit function theorem will not be discussed further here as it is rather technical and would disrupt the main flow of the argument. All we need know from appendix two is that equations (2.25) do give a locally well defined, single valued, invertible, differentiable transformation.This transformation will be denoted by R and its pull back by R^*.

We will now study the properties of the transformation R and will show that it can be interpreted as a rescaling of constraints with the matrix equal to the body of Λ (the body of a grassmann valued object, **F**, is the part of **F** left when all the grassmann variables are set equal to zero). To do this we must first look at the behaviour of Ω under R.

Theorem 2.4

Let Ω be the BRST charge constructed from a set of constraints φ_α and apply the pull back transformation R^* to obtain $\tilde{\Omega}$. Then $\tilde{\Omega}$ has ghost number one and,

$$\tilde{\Omega} = \tilde{\varphi}_\alpha \eta^\alpha + \text{terms with higher numbers of ghosts.} \tag{2.26}$$

Also,

$$\{ \tilde{\Omega} , \tilde{\Omega} \} = 0. \tag{2.27}$$

In equation (2.26) the $\tilde{\varphi}_\alpha$ are the set of constraints constructed by rescaling the φ_α using the body of the matrix Λ which defines the transformation R.

Proof

$\tilde{\Omega}$ has ghost number one because the transformation R has ghost number zero. To prove the rest of the theorem we must obtain the explicit transformation equations for the new coordinates under the transformation R. Fortunately to prove (2.26) we only need the terms with fewest ghosts. These are,

$$\tilde{Q}^A = Q^A + \text{terms with higher numbers of ghosts.} \tag{2.28a}$$

$$\tilde{P}_A = P_A + \text{terms with higher numbers of ghosts.} \tag{2.28b}$$

$$\tilde{\eta}^\alpha = \epsilon(\Lambda)_\beta{}^\alpha \, \eta^\beta + \text{terms with higher numbers of ghosts.} \tag{2.28c}$$

$$\tilde{\rho}_\alpha = \epsilon(\Lambda^{-1})_\alpha{}^\beta \, \rho_\beta + \text{terms with higher numbers of ghosts.} \tag{2.28d}$$

In these equations $\epsilon(\Lambda)$ denotes the body of Λ (similarly for Λ^{-1}). To obtain these equations we have iterated equations (2.25) and Taylor expanded Λ and φ_α. This then means that,

$$\begin{aligned}\tilde{\Omega} &= \varphi_\alpha(\tilde{Q}^A, \tilde{P}_A)\, \tilde{\eta}^\alpha + \text{terms with higher numbers of ghosts} \\ &= \varphi_\alpha(Q^A, P_A)\, \epsilon(\Lambda)_\beta{}^\alpha \, \eta^\beta + \text{terms with higher numbers of ghosts}\end{aligned}$$

$$= \tilde{\varphi}_{\alpha}(Q^A, P_A)\, \eta^{\alpha} + \text{terms with higher numbers of ghosts,}$$

which is equation (2.26). Equation (2.27) follows because the transformation is canonical (it is derived from a generating function). This completes the proof.

This theorem guarantees that $\tilde{\mathbf{\Omega}}$ satisfies all the requirements of the BRST charge associated with the rescaled set of constraints. Hence, $\tilde{\mathbf{\Omega}}$ transforms as if the constraints had been rescaled and it is in this sense that the transformation R can be interpreted as a constraint rescaling. This is a rather strange situation because R does not transform the constraints via a rescaling. R only transforms the ghosts via a rescaling and it is only in the BRST charge that this can be interpreted as a transformation of the constraints.

Next, we must examine the action of the transformation R on the BRST extension of gauge invariant observables.

Theorem 2.5

Let **F** be a BRST extension of a gauge invariant observable F and apply the pull back transformation R^* to obtain $\tilde{\mathbf{F}}$. Then $\tilde{\mathbf{F}}$ has ghost number zero and,

$$\tilde{\mathbf{F}} = F(Q^A, P_A) + \text{terms with higher numbers of ghosts,} \qquad (2.29)$$

and,

$$\{\tilde{\mathbf{F}}, \tilde{\mathbf{\Omega}}\} = 0. \qquad (2.30)$$

In these equations $\tilde{\mathbf{\Omega}}$ has the same meaning as in theorem 2.4.

Proof

The proof is almost identical to that of theorem 2.4. $\tilde{\mathbf{F}}$ has ghost number zero because the transformation R has ghost number zero. To get the other results we use equations (2.28) to get,

$$\tilde{F} = F(\tilde{Q}^A, \tilde{P}_A) + \text{terms with higher numbers of ghosts}$$
$$= F(Q^A, P_A) + \text{terms with higher numbers of ghosts.}$$

F has been Taylor expanded to get this. This gives equation (2.29). Equation (2.30) follows because the transformation is canonical. This completes the proof.

This theorem now guarantees that all the relevant classical observables transform correctly. This follows because theorem 2.5 tells us that $\tilde{F}$ satisfies all the required conditions for an extension of F with respect to the rescaled constraints.

To summarise, it has now been shown that the transformation R^* acts as a constraint rescaling on Ω and gauge invariant observables. It can also be shown that this transformation does not alter any of the cohomology classes of the complex (2.22). To do this one shows that R^* is a chain mapping between the complexes constructed from the old and new constraints. A chain mapping has the property of sending closed elements to closed elements and exact elements to exact elements i.e. it preserves the cohomological structures. The proof of this is simple because, as we have already noted, R^* does not alter the ghost number and so it acts on (2.22) as shown below.

$$\begin{array}{ccccccccc}
\rightarrow \tilde{\Gamma}^{-1} & \overset{\tilde{\delta}}{\rightarrow} & \tilde{\Gamma}^{0} & \overset{\tilde{\delta}}{\rightarrow} & \tilde{\Gamma}^{1} & \overset{\tilde{\delta}}{\rightarrow} & \tilde{\Gamma}^{2} & \rightarrow \\
\downarrow R^* & & \downarrow R^* & & \downarrow R^* & & \downarrow R^* & \\
\rightarrow \Gamma^{-1} & \overset{\delta}{\rightarrow} & \Gamma^{0} & \overset{\delta}{\rightarrow} & \Gamma^{1} & \overset{\delta}{\rightarrow} & \Gamma^{2} & \rightarrow .
\end{array}$$

The upper complex is constructed using the rescaled constraints and the bottom one using the original constraints. It is now claimed that R^* is a chain mapping i.e. that,

$$\delta R^* = R^* \tilde{\delta}. \tag{2.31}$$

This follows easily by the following argument. Let $\tilde{\mathbf{F}} \in \tilde{\Gamma}$ and notice that,

$\delta R^*(\tilde{\mathbf{F}})$	$= \{\, \Omega, R^*\tilde{\mathbf{F}} \,\}$	by definition of δ.
	$= \{\, R^*\tilde{\Omega}, R^*\tilde{\mathbf{F}} \,\}$	since Ω transforms correctly under R.
	$= R^* \,\{\tilde{\Omega}, \tilde{\mathbf{F}} \,\}$	since R is a canonical transformation.
	$= R^* \,\tilde{\delta}(\tilde{\mathbf{F}})$	by definition of $\tilde{\delta}$.

In the general rescaling transformation R the grassmann matrix Λ can be written in the form,

$$\Lambda_\alpha{}^\beta = \varepsilon(\Lambda)_\alpha{}^\beta + \sum_{m=1}^{k} (\Lambda)_{\alpha\alpha_1 \cdots \alpha_m}{}^{\beta\beta_1 \cdots \beta_m} \tilde{\eta}^{\alpha_1} \cdots \tilde{\eta}^{\alpha_m} \rho_{\beta_1} \cdots \rho_{\beta_m} \,, \tag{2.32}$$

where the $\Lambda_{\alpha \cdots \alpha}{}^{\beta \cdots \beta}$ on the right hand side depend only on $\tilde{Q}^A$ and P_A. Up till now we have not needed to use the additional ghost terms that are present in (2.32) so it is interesting to know what such terms do. These terms do not alter the value of the rescaling matrix which multiplies the constraints in the BRST charge. In fact, they are free to be chosen in any way one wishes (Λ is invertible if and only if $\varepsilon(\Lambda)$ is invertible [49]) and they only change the higher ghost terms in Ω and $\mathbf{F}$. This means that the second term in equation (2.32) represents the ambiguities that exist in constructing the BRST charge and, in extending gauge invariant observables to SP. For this reason it is normally easier to set the second term in (2.32) to zero and work only with the simpler transformation that arises. We will adopt this simplification most of the time.

2.5 Mathematical Background to the BFV Formalism

To finish this chapter we will give a brief survey of the attempts to understand where the cohomological ideas of the BFV approach come from. This will not be directly relevant to the later chapters but it is of considerable intrinsic interest and should be included in any modern discussion of the BFV method.

The main way of understanding the cohomological ideas arises in gauge theories where there is much more mathematical machinery available. Here one starts from the configuration space Q and a gauge group G which has a free action on Q. This action then lifts to a hamiltonian action of G on T^*Q.

There have been a number of attempts to understand the ghosts as Maurer-Cartan forms (left invariant one forms) on G [55,56,57,58]. The most comprehensive discussion of this approach is given in [41]. In that paper it was shown how one can construct a Lie algebra cohomology taking values in $C^\infty(T^*Q)$ (the vector space of smooth functions on T^*Q). This cohomological structure does not, however, properly pick out the weak equivalence classes of gauge invariant observables on T^*Q. It is necessary to modify the Lie algebra cohomology by introducing new grassmann variables which extend the complex into a second direction. This extended structure now does pick out the physical observables and gives an algebraic understanding of the BFV technique. The new grassmann variables are, of course, interpreted as the conjugate ghosts and the whole algebraic structure is known as a Koszul resolution of the Lie algebra cohomology.

What is surprising is that this algebraic procedure can be extended to incorporate more general first class constraints which do not come from any group action. This extension is developed in [42] and basically shows how the higher order structure functions in the BRST charge are constructed.

There is an alternative way of interpreting the BFV method which is, in some ways, better. The only criticism of the above construction is that it does come from gauge theory ideas. This is rather odd because, in sections 2.3 and 2.4 we did not need to use any gauge group. The BFV construction needs only a set of first class constraints and the associated constrained surface and gauge orbits. Indeed, as has been emphasised in [8], constraint rescaling destroys any structure functions in the BRST charge and so weakens the link with any group structure. It would therefore be more pleasing to have an interpretation of the BRST cohomology which does not need any group techniques such as Lie algebra cohomology. An attempt to do this has been made in [59]. In this paper a "restricted" de Rham cohomology is constructed on P. It is restricted in the sense that only a limited set of differential forms are used. These forms are constructed using forms which are tangent to the gauge orbits on the constrained surface. The intuitive idea behind this is that the cohomology spaces which are constructed measure the topology of the gauge orbits and so contains information about the true degrees of freedom. The authors of [59] show that their cohomology groups are isomorphic to the BRST ones and so give a possible interpretation of the ghosts in terms of their restricted differential forms. There is one odd features about the construction in [59]. They do not need to use all the higher order structure functions in the BRST charge, only the constraints and $C^{\gamma}{}_{\alpha\beta}$ are used. The work of [59] has also been studied in [39, 60].

As a final point about the above constructions, no one has managed to show, either geometrically or algebraically, why constraint rescaling becomes a canonical transformation in the superphase space. It would be interesting to see if any of the above approaches could give a deeper understanding of the rescaling transformation R.

Chapter Three
The Theory of Constraints Linear in Momenta

3.1 Classical Aspects of Linear Constraints

In this chapter we will study the classical and quantum theory of constraints linear in momenta. The linearity of the constraints only holds in a coordinate system that is the *lift* of a coordinate system on Q. These lifted coordinate systems are constructed as follows. Let $(Q^1, .. ,Q^N)$ be a coordinate system on Q then, any one form on Q, that lies in this coordinate patch, can be written in the form $\sigma = P_A dQ^A$ for a unique set of numbers P_A. Thus $(Q^1, .. ,Q^N, P_1, .. P_N)$ is a valid coordinate system on $P=T^*Q$ and in such a coordinate system the constraints will be assumed to have the form,

$$\varphi_\alpha = \varphi_\alpha{}^A(Q^B)\, P_A. \tag{3.1}$$

This linearity structure is not preserved by arbitrary canonical transformations on P. We will therefore limit ourselves to the transformations which do preserve the linearity structure. These are the point transformations i.e. lifts of coordinate transformations on Q. Such a transformation is of the form,

$$Q^A \to Q^{A'}(Q^A), \tag{3.2a}$$

$$P_A \to P_{A'} = Q^A{}_{A'}\, P_A \qquad \text{where } Q^A{}_{A'} = \partial Q^A / \partial Q^{A'}. \tag{3.2b}$$

The linear form of the constraints is also not going to be preserved by arbitrary constraint rescalings. We will therefore restrict ourselves to transformations of the form,

$$\varphi_\alpha \to \tilde{\varphi}_\alpha = \Lambda_\alpha{}^\beta\,(Q^A)\,\varphi_\beta, \tag{3.3}$$

which will preserve the linearity in momenta. We will refer to this restricted class of rescalings as configuration space rescalings.

The above restrictions in the symmetries of the theory are essentially saying that we will take the cotangent bundle structure of P as fundamental i.e., we will not study any transformations which do not project down onto Q. This is a physically reasonable thing to do because linear constraints project nicely onto Q. In fact, they can be represented on Q by a set of vector fields v_α given by,

$$v_\alpha = \varphi_\alpha{}^A\,\partial_A. \tag{3.4}$$

It was shown, in the proof of theorem 2.3, that these vector fields are integrable and so give a foliation on Q. In fact, for gauge theories, one starts from a gauge group action on Q which gives this foliation and the constraints are then derived from this rather than vice versa. Given this foliation of Q one can form the factor space Q_{phys} which is always a smooth manifold in the physically interesting cases. It can be shown that the physical phase space P_{phys}, obtained by factoring off the gauge orbits on the constrained surface, is equal to T^*Q_{phys} [21]. It is this fact which really justifies using only the restricted set of coordinate and rescaling transformations introduced above.

The rescaling transformations (3.3) have a number of simplifying properties which are not possessed by the general transformations examined in the previous chapter. The first of these properties is contained in the following theorem.

Theorem 3.1

Let φ_α and $\tilde{\varphi}_\alpha$ be any two equivalent sets of linear constraints (by this we mean that they have identical zero sets). Then, there exists a globally defined and everywhere invertible matrix $\Lambda(Q^A)$ such that,

$$\tilde{\varphi}_\alpha = \Lambda_\alpha{}^\beta \varphi_\beta.$$

Proof

Construct the two sets of vector fields v_α and $\tilde{v}_\alpha$ on Q. These sets of vectors both form a basis to the tangent planes of the foliation on Q and so there must exist a globally defined, everywhere invertible matrix $\Lambda(Q^A)$ which relates the two sets of vectors via,

$$v_\alpha = \Lambda_\alpha{}^\beta \tilde{v}_\beta.$$

The result follows immediately from this.

This result means that configuration space constraint rescalings will globally incorporate all the linear constraints. We will not take advantage of this result because all our quantisation will be local. However, the result should make any attempt at global quantisation easier. It is also worth noting that the result is not quite as obvious as it may seem. For example, the corresponding result for quadratic constraints fails i.e., there exist two globally defined equivalent sets of quadratic constraints which are not related by a globally defined rescaling which depends on the Q^A only. This will be shown in chapter four.

We will now look at the form of the rescaling transformation (2.25) for configuration space rescalings. For this case the transformation becomes much simpler and can be written explicitly in the form,

$$\tilde{Q}^A = Q^A, \tag{3.5a}$$

$$\tilde{P}_A = P_A + (\Lambda^{-1})_\alpha{}^\beta{}_{,A} \Lambda_\gamma{}^\alpha \eta^\gamma \rho_\beta, \tag{3.5b}$$

$$\tilde{\eta}^\alpha = \Lambda_\beta{}^\alpha \eta^\beta, \tag{3.5c}$$

$$\tilde{\rho}_\alpha = (\Lambda^{-1})_\alpha{}^\beta \rho_\beta. \tag{3.5d}$$

It is possible to think of this transformation as a point transformation on the superphase space SP. To see this look at the following coordinate transformation on SQ (the superconfiguration space),

$$Q^A \to Q^A, \tag{3.6a}$$

$$\eta^\alpha \to \Lambda_\beta{}^\alpha \eta^\beta, \tag{3.6b}$$

and apply equation (3.2b) to recover equations (3.5).

There is one more restriction to be made on the classical theory before we start to discuss the quantisation. Not all the gauge invariant smooth functions on P are of physical relevance so we will only look at a restricted subset of them. This restricted set of functions will be called the special physical observables [8] and will consist of functions of Q^A only, functions linear in momenta and functions quadratic in momenta. These will be denoted by Y, Z; U, V; and H, K respectively. The equivalent lower case letters will denote functions with the same momenta dependence but now on P_{phys} instead of P. The quantisation that follows will also cover functions with inhomogeneous momenta dependence. One simply has to add together the operators for the individual homogeneous terms.

For future reference, the symmetries that will be studied and used in the quantum theory are,

(a) The point transformations (3.2) on the phase space P.

(b) The point transformations on the physical phase space P_{phys}.

(c) Weak changes to special observables which do not change their momentum dependence.

(d) Invariance under configuration space rescalings of the constraints.

This more or less completes the classical discussion apart from a rather technical point that has to be made regarding SP and SQ. In the remaining sections of this chapter we will often be using integrals of the form $\int F(Q^A)\, dQ^1 .. dQ^N$. It would be much easier if this integral could be interpreted in its normal Lebesgue sense. Unfortunately, it is not possible to do this if we stick to the interpretation of SP as a supermanifold. This is because, on a supermanifold, Q^A is not a real number, it is an even vector in a grassmann algebra. We can, however, arrange for Q^A to be a real number because we have no use for transformations which mix the Q^A with the fermionic elements (transformation (3.5) does not alter Q^A and transformation (3.2) does not mix any P_A dependence into Q^A). It will be assumed from now on that Q^A is a real number. This means that we are dealing with a hybrid object, partly normal manifold and partly supermanifold. This new interpretation of SP will be used only in this chapter.

3.2 The Physical Quantisation

The main task we now face is the quantisations of the classical theory described above. The principle requirement that we put on this quantisation procedure is that the resulting quantum theory should be equivalent to that arising from the true degrees of freedom. It is therefore necessary to investigate the quantisation of unconstrained theories which we do in this section.

The main difficulty in constructing a quantum theory on P_{phys} is that Q_{phys} is normally going to be a curved Riemannian manifold, even if Q is flat. An example showing this is the quantum well of Orvieto discussed in sections 1.4 and 3.9. The metric, g, which gives this curvature comes from the kinetic energy part of the hamiltonian, h, on P_{phys} i.e. h is of the form,

$$h = g^{ab}(q^c)\, p_a p_b + u^a(q^c)\, p_a + y\,(q^c) \tag{3.7}$$

The lower case Latin indices are running from 1 to n (=N - k). The classical theory is invariant with respect to point transformations on P_{phys} i.e. transformations of the form,

$$q^a \rightarrow q^{a'}\,(q^a), \tag{3.8a}$$

$$p_a \rightarrow p_{a'} = q^a{}_{a'}\, p_a \qquad \text{where } q^a{}_{a'} = \partial q^a / \partial q^{a'}. \tag{3.8b}$$

To construct the quantum theory we will demand that it is also invariant with respect to these transformations. We will also work locally ignoring any global properties of Q_{phys}. This means that we can base the quantum theory on the operators $\hat{q}^a$ and $\hat{p}_a$ which satisfy the canonical commutation relations,

$$[\hat{q}^a, \hat{p}_b] = i\hbar\,\delta^a_b\,, \tag{3.9}$$

$$[\hat{q}^a, \hat{q}^b] = [\hat{p}_a, \hat{p}_b] = 0. \tag{3.10}$$

The assumption of locality, i.e. ignoring any global topological properties of Q_{phys}, is being made purely to make the analysis tractable. The problem of global quantisation has been studied by various authors, the two main schools of thought being geometric quantisation (see for example [22]) and canonical group quantisation [19]. It is, unfortunately, very difficult to solve the problem of constraint quantisation using either of these global methods and global quantisation of constraints remains a topic for further study. We will discuss in chapter six what the physical limitations of the locality assumption are.

The easiest way to realise the algebra (3.9 and 10) is to use the Schrödinger picture where the state space is the set of complex valued functions on Q_{phys} which are square integrable with respect to the following pairing,

$$\langle \Psi | X \rangle = \int \Psi^*(q^a)\ X(q^a)\ |g|^{\frac{1}{2}}\ dq^1 \,..\, dq^n\ , \tag{3.11}$$

where $|g| = \det[g_{ab}]$. This pairing is invariant under (3.8a) because $|g|$ transforms as a scalar density of weight two i.e.,

$$|g|^{\frac{1}{2}} \rightarrow |g'|^{\frac{1}{2}} = |q|\ |g|^{\frac{1}{2}} \quad \text{where } |q| = \det[\, q^a_{\ a'}]\ . \tag{3.12}$$

On this Hilbert space the representation of (3.9) and (3.10) is,

$$\hat{q}^a = q^a, \tag{3.13}$$

$$\hat{p}_a = -i\hbar \left(\partial_a + \frac{1}{2} \frac{|g|^{\frac{1}{2}}_{\ ,a}}{|g|^{\frac{1}{2}}} \right) . \tag{3.14}$$

These operators are covariant with respect to the quantum point transformations which are,

$$\hat{q}^a \rightarrow \hat{q}^{a'} = q^{a'}(q^a), \tag{3.15}$$
$$\hat{p}_a \rightarrow \hat{p}_{a'} = \frac{1}{2}\, [\, q^a_{\ a'}\ ,\ \hat{p}_a\,]_+ \ . \tag{3.16}$$

The only remaining thing required to specify the quantum theory is to give the orderings for the special observables. These orderings are required to be covariant under the transformations (3.15) and (3.16), self adjoint with respect the pairing (3.11) and reduce to the correct classical limit. These conditions do not uniquely fix the orderings but the simplest choice is,

$$y(q^a) \rightarrow \hat{y} = y(q^a), \qquad (3.17)$$

$$u = u^a(q^b)\, p_a \rightarrow \hat{u} = \frac{1}{2}\, [\, u^a , \hat{p}_a\,]_+ \,, \qquad (3.18)$$

$$k = k^{ab}(q^c)\, p_a p_b \rightarrow \hat{k} = |g|^{-\frac{1}{4}}\, \hat{p}_a\, k^{ab}\, |g|^{\frac{1}{2}}\, \hat{p}_b\, |g|^{-\frac{1}{4}}. \qquad (3.19)$$

These equations do not completely implement the classical Poisson algebra as a commutator algebra i.e. the Dirac prescription,

$$\{f , g\} \rightarrow -\frac{i}{\hbar}\, [\, \hat{f} , \hat{g}\,], \qquad (3.20)$$

is not valid for all the above orderings. Equation (3.20) does work for Poisson brackets which involve only the configuration space observables or observables linear in momenta. The failure of (3.20) for observables quadratic in momenta is an example of the well known Van Hove theorem [20] which says that such obstructions to equation (3.20) are unavoidable. More details of the Van Hove obstructions to the quantisation above can be found in [9].

The orderings (3.17), (3.18) and (3.19) are not unique. They are the simplest orderings in the sense that the other orderings various authors have considered differ from the ones above by $\hbar$ corrections which are scalars under coordinate transformations on Q_{phys}. In order to maintain generality we will look at these terms but it should be emphasised there is no simple theoretical reason for including or excluding them. Therefore, by the principle of Ockham's razor one would be tempted not to add them.

The alteration to the orderings that is most common is to add a term to (3.19) of the form $\xi \hbar^2 R$ where ξ is a number and R is the scalar curvature of the metric. The original motivation for this came from the attempt by various authors [61,62,63] to construct a covariant path integral equivalent to the above canonical quantisation. The earliest attempts to do this produced path integrals whose equivalent hamiltonian was (3.19) plus a scalar curvature term, though different authors found different coefficients ξ. The most comprehensive discussion of this can be found in [64]. In this paper it is shown that one can freely vary the coefficient ξ by choosing different definitions of the path integral. In particular, it is possible to have $\xi = 0$ while using a fairly natural path integral definition. We shall return to these scalar curvature terms in section 3.8.

This completes the discussion of the physical quantum theory. The task of constraint quantisation is to find a practical quantisation on Q (or in the BFV case SQ) which reproduces the above physical quantum theory. By a "practical quantisation" we mean one which does not require an explicit reduction to the true degrees of freedom. This being an effectively impossible task for real physical theories.

3.3 Outline of the Constrained Quantum Theory using Ghosts

In this section we will outline the basic philosophy behind the use of ghosts in quantisation. The idea is to quantise the theory in the ghost equivalent of the Schrödinger picture i.e the ghosts will become operators by the following prescription.

$$\eta^{\alpha} \rightarrow \hat{\eta}^{\alpha} = \eta^{\alpha}, \qquad (3.21a)$$

$$\rho_{\alpha} \rightarrow \hat{\rho}_{\alpha} = -i\hbar \frac{\partial}{\partial \eta^{\alpha}} . \qquad (3.21b)$$

These operators satisfy the canonical anticommutation relations i.e.,

$$[\hat{\eta}^{\alpha}, \hat{\rho}_{\beta}] = -i\hbar\delta_{\beta}^{\alpha},$$

$$[\hat{\eta}^{\alpha}, \hat{\eta}^{\beta}] = [\hat{\rho}_{\alpha}, \hat{\rho}_{\beta}] = 0.$$

Here we have adopted the notational convention of not distinguishing between commutators and anticommutators. One can always work out from the context which is the relevant one i.e [,] will always denote a commutator unless both operators are fermionic in which case it will be an anticommutator. There will be occasional exceptions to this convention and in these cases a more explicit notation will be used.

The quantum state space that will be used consists of functions on SQ of the form,

$$\Psi(Q^A, \eta^{\alpha}) = \Psi_0 + \sum_{m=1}^{k} \Psi_{\alpha_1 .. \alpha_m} \eta^{\alpha_1} .. \eta^{\alpha_m} . \qquad (3.22)$$

The $\Psi_{\alpha .. \alpha}$ are totally antisymmetric in their indices and will be square integrable with respect to some measure $d\mu = \mu(Q^A)\, dQ^1 .. dQ^N$ on Q. There is no unique choice of the density function μ, different ones being used in [65] and [66,67] where the contents of this chapter were originally presented. In [65] a Riemannian integration measure was used. This was done by assuming that the metric in the kinetic energy piece of the hamiltonian (i.e the quadratic term $G^{AB} P_A P_B$ in H) was nonsingular and using the determinant of this for μ. In general G^{AB} will not be invertible it is only the physical metric that has to be nondegenerate. It is, however, always locally possible to patch up G^{AB} and make it nonsingular by adding a term to H which vanishes on the constrained surface. To see this it is sufficient to consider the case where the constraints have been trivialised to are the first k momenta. In this case G^{AB} will be of the form,

$$\begin{bmatrix} G^{\alpha\beta} & G^{\alpha a} \\ G^{a\alpha} & G^{ab} \end{bmatrix},$$

where G^{ab} is the physical metric which is nondegenerate. In this matrix α and β runs from 1 to k while a and b run from k+1 to N. If one subtracts the term $2G^{\alpha a}P_aP_\alpha + (G^{\alpha\beta} - \delta^{\alpha\beta})\,P_\alpha P_\beta$ from H this will make the metric invertible.

There is nothing wrong with this approach. It does give a workable quantum theory but it is rather clumsy, the point being that the dynamics does not give Q a natural Riemannian structure. Instead the dynamics gives Q the structure of a fibre bundle whose base space, Q_{phys}, is Riemannian. It would be preferable to have a measure function μ which does not force unnatural geometric structures on Q. This is not just an aesthetic objection since it may not be possible to make G^{AB} invertible globally i.e. we could be introducing a further obstruction to global quantisation. It should be pointed out, though, that G^{AB} can be made globally invertible for Yang-Mills theories by adding a term involving the primary constraints.

In this thesis we will use the measure of [66,67]. This takes the measure function μ to be $\|\varphi\|$ which is defined by the following two equations,

$$\varphi_{A_1 .. A_n} = \delta_{A_1 .. A_n B_1 .. B_k}\, \varphi_1^{\ B_1} .. \varphi_k^{\ B_k}\,, \tag{3.23}$$

and,

$$\|\varphi\|^{-2} = \frac{1}{n!}\, \varphi_{A_1 .. A_n}\, G^{A_1B_1} .. G^{A_nB_n}\, \varphi_{B_1 .. B_n}\,. \tag{3.24}$$

where $\delta_{A .. A}$ is the totally antisymmetric tensor density defined by $\delta_{1 .. N} = 1$. This object was first introduced in [9] though the definition above has been modified slightly to avoid unnecessary minus signs later. The $\|\varphi\|$ has the following three properties (these guarantee that $\|\varphi\|$ is well defined i.e. that the right hand side of (3.24) is never zero).

(1). When the constraints are trivialised i.e. when the constraints are the first k momenta,

$$\|\varphi\| = |g|^{\frac{1}{2}}, \qquad (3.25)$$

where $|g|$ is the determinant of the physical metric i.e. the metric in the directions Q^{k+1} to Q^N.

(2). Under general coordinate transformations on Q $\|\varphi\|$ transforms as a scalar density of weight one i.e.,

$$\|\varphi\| \rightarrow \|\varphi'\| = |Q|\,\|\varphi\|, \qquad (3.26)$$

where $|Q| = \det[\, Q^{A}{}_{A'}]$.

(3). Under a rescaling of the constraints $\|\varphi\|$ has the transformation law,

$$\|\varphi\| \rightarrow \|\tilde{\varphi}\| = |\Lambda|^{-1}\,\|\varphi\|, \qquad (3.27)$$

where $|\Lambda| = \det[\, \Lambda_{\alpha}{}^{\beta}\,]$.

Property (1) above indicates that $\|\varphi\|$ is related to the physical volume form on Q_{phys}. The relationship between these two quantities was established in [9] and is contained in the following theorem.

Theorem 3.2

Let π: $Q \rightarrow Q_{phys}$ be the projection map from Q to Q_{phys} and π^* its pull back. Then,

$$\|\varphi\|\ \varphi_{A_1 .. A_n}\, dQ^{A_1} .. dQ^{A_n} = \pi^*(\, |g|^{\frac{1}{2}}\, \delta_{a_1 .. a_n}\, dq^{a_1} .. dq^{a_n}\,) \qquad (3.28)$$

Proof

This is taken directly from [9]. It is being repeated here because we will later attempt to generalise it. The key step is to notice that the form on the left hand side of (3.28) is perpendicular to the orbits of the gauge directions in the sense that,

$$\varphi_{\alpha}^{A}\ \varphi_{A A_2 .. A_n} = 0. \qquad (3.29)$$

This property is clearly also possessed by the differential form on the right hand side of (3.28). Thus, these two forms are proportional to each other although the proportionality factor could depend on the position in Q. It is easy to check that this proportionality factor must be unity because both forms have unit norm (norm being used in the sense of equation (3.24)). This completes the proof. For future reference it should be noted that there are two essential ingredients in this argument. Firstly, it is vital that the tensors are antisymmetric and secondly it is vital that the tensor (3.23) saturates the constraints so that property (3.29) holds. We will return to this in section 3.8.

Let us now return to the main theme of this section. The ghost quantum theory will be built on the states of the form (3.22) on which we need to introduce some inner product structure. The pairing that is traditionally taken for the ghosts is the Berezin one so that the pairing is,

$$\langle \Psi \,|\, X \rangle = (i)^{\frac{1}{2}k(k-1)} \int \Psi^* \, X \, \|\varphi\| \, dQ^1 \,..\, dQ^N \, d\eta^1 \,..\, d\eta^k , \qquad (3.30)$$

where $d\eta^{\alpha}$ denotes Berezin integration. Complex conjugation is defined on ghosts by $(\eta^{\alpha})^* = \eta^{\alpha}$ and * acts antilinearly on sums and satisfies a reversal rule on products. The combinatorial factor in front of (3.30) is there to insure that,

$$\langle \Psi | X \rangle^* = \langle X | \Psi \rangle.$$

It is not a priori obvious that (3.30) is the most useful pairing to use. This will be examined critically in section 3.4 where it will be deduced that (3.30) is really the only workable choice of inner product.

The above pairing gives the state space the structure of an indefinite Hilbert space though, for brevity, it will be referred to (incorrectly) as a Hilbert space. The physical states in this space are projected out by the quantum version of the BRST charge i.e the physical states are those satisfying,

$$\hat{\Omega}\Psi = 0. \tag{3.31}$$

There is a certain amount of redundance in this because the BRST charge normally satisfies,

$$\hat{\Omega}^2 = 0, \tag{3.32}$$

and so states of the form $\hat{\Omega}\Psi$ will automatically satisfy (3.31). This redundancy can be ignored provided that $\hat{\Omega}$ is hermitian because then all these states will have zero norm and will be orthogonal to the physical states. This means that, in BFV quantisation, the physical state space is,

$$\frac{\{\ \Psi : \hat{\Omega}\Psi = 0\ \}}{\{\ \Psi : \Psi = \hat{\Omega}X \text{ for some state } X\ \}}, \tag{3.33}$$

which can clearly be interpreted as the zeroth cohomology group of the complex constructed from the different ghost number states. We will refer to the set (3.33) as H^0.

The quantum, physical observable must preserve the BRST invariance of the states. This will occur if the operator $\hat{\mathbf{F}}$ satisfy,

$$[\,\hat{\mathbf{F}}, \hat{\Omega}\,] = 0. \tag{3.34}$$

The problem of BFV quantisation is to give explicit expressions for the BRST charge and the gauge invariant observables such that the conditions (3.32) and (3.34) are satisfied. More over, the resultant quantum theory must be equivalent to the theory described in section 3.2 though, for practical reasons, the solution must not require any knowledge of the reduction of the classical theory to the true degrees of freedom.

The solution to this problem will be presented in the remainder of this chapter. The key to solving the problem is to demand invariance of the quantum theory with respect to the symmetries (a) to (d) listed at the end of section 3.1. It will be shown that these invariances force a unique "simplest" theory which satisfies equations (3.32) and (3.34). It will also be shown that this theory is equivalent to the physical theory described in section 3.2.

Before presenting this solution some more preliminaries must be disposed of. In section 3.4 the Berezin pairing (3.30) will be critically examined. In section 3.5 the quantum version of the constraint rescaling transformation (3.5) will be given. In section 3.6 some important technical points about the quantum state space will discussed. The previous discussion was too naive because it ignores the fact that the nontrivial functions satisfying (3.31) will not be square integrable on Q and so do not lie within the state space we have defined. The remaining sections of this chapter will then give the full quantum version of a classical system with linear constraints.

3.4 The Choice of Measure for Integrating the Ghosts

As was said in the previous section, the standard ghost measure used is the Berezin one. We will first of all give a simple example to show that this not obviously correct.

Consider $Q = \mathbb{R}^N$ with the normal euclidean metric and let the phase space constraints be $P_1, .. P_k$. Then,

$$\hat{\Omega} = \hat{P}_\alpha \eta^\alpha, \tag{3.35}$$

because there are no ordering ambiguities. The states projected out by equation (3.31) will be of the form,

$$\Psi^{phys} = \Psi_0(Q^{k+1}, .. ,Q^N) + \text{higher ghost number terms.} \tag{3.36}$$

The higher ghost terms do not have any simple dependence on the physical directions $Q^{k+1}, .., Q^N$. Thus, it appears to be Ψ_0 which contains the physical information in Ψ^{phys} and so, it would appear natural for the pairing $\langle\Psi | X\rangle$ to contain a term of the form $\Psi_0^* X_0$. This might lead one to suggest a ghost measure which gives, for example,

$$\langle \Psi | X \rangle = \int [\, \Psi_0^* X_0 + .. + \Psi_{1..k}^* X_{1..k} \,]\, d\mu. \tag{3.37}$$

We have introduced the notation $d\mu = \|\varphi\|\, dQ^1 .. dQ^N$.

However, the Berezin pairing has no such terms, instead,

$$\langle \Psi | X \rangle = (i)^{\frac{1}{2}k(k-1)} \int k!\, [\, (-1)^{\frac{1}{2}k(k-1)} \Psi_0^* X_{1..k} + \Psi_{1..k}^* X_0 \,]\, d\mu$$

$$+ \text{ terms independent of } \Psi_0 \text{ or } X_0, \tag{3.38}$$

which makes it hard to see how the physical Hilbert space structure can emerge. It is therefore perfectly reasonable to examine other possible pairings such as (3.37). To do this we will start from the most general form for the inner product which can be parametrised by [68],

$$\langle \Psi | X \rangle = \sum_{m,n=0}^{k} \int \Psi^*_{\alpha_1 .. \alpha_m} X_{\beta_1 .. \beta_n} I^{\alpha_1 .. \alpha_m, \beta_1 .. \beta_n} \, d\mu, \tag{3.39}$$

where the $I^{\alpha .. \beta}$ are arbitrary functions on Q which are antisymmetric in both their α and β indices. We will now show that, because certain operators must be hermitian with respect to (3.39), the I are restricted to the values which correspond to the Berezin pairing. The hermiticity conditions that we demand are,

$$(\eta^\alpha)^\dagger = \eta^\alpha, \tag{3.40}$$

and,

$$(\hat{\rho}_\alpha)^\dagger = -\hat{\rho}_\alpha . \tag{3.41}$$

Condition (3.40) follows from (3.35) and the condition that $\hat{\Omega}$ be hermitian (due to theorem 2.3 there is no loss of generality in considering (3.35)). Equation (3.41) follows because the BRST extension of most physical observables contain terms of the form $\eta^\alpha \rho_\beta$ $(\alpha \neq \beta)$. The corresponding quantum operators will anticommute and so (3.41) must be satisfied if physical observables are to be hermitian.

The following two technical theorems show that (3.40) and (3.41) are only satisfied by the Berezin pairing (3.30).

Theorem 3.3

$$(\eta^{\alpha})^{\dagger} = \eta^{\alpha} \Leftrightarrow$$

$$I^{\beta_1 .. \beta_m , \alpha\gamma_1 .. \gamma_n} = (-1)^m I^{\beta_1 .. \beta_m \alpha , \gamma_1 .. \gamma_n} \quad \forall m,n \qquad (3.42)$$

Proof

It is easy to show that, using (3.39),

$$\langle X | \eta^{\alpha}\Psi \rangle = \sum_{m,n=0}^{k} \int X^{*}_{\beta_1 .. \beta_m} \Psi_{\gamma_1 .. \gamma_n} I^{\beta_1 .. \beta_m , \alpha\gamma_1 .. \gamma_n} d\mu ,$$

and,

$$\langle \eta^{\alpha} X | \Psi \rangle = \sum_{m,n=0}^{k} \int X^{*}_{\beta_1 .. \beta_m} \Psi_{\gamma_1 .. \gamma_n} I^{\beta_1 .. \beta_m \alpha , \gamma_1 .. \gamma_n} (-1)^m d\mu .$$

Therefore,

$$(\eta^{\alpha})^{\dagger} = \eta^{\alpha} \Leftrightarrow \langle X | \eta^{\alpha}\Psi \rangle = \langle \eta^{\alpha} X | \Psi \rangle \quad \forall X, \Psi$$

$$\Leftrightarrow I^{\beta_1 .. \beta_m , \alpha\gamma_1 .. \gamma_n} = (-1)^m I^{\beta_1 .. \beta_m \alpha , \gamma_1 .. \gamma_n} \quad \forall m,n.$$

Theorem 3.4

$(\eta^{\alpha})^{\dagger} = \eta^{\alpha}$ and $(\hat{p}_{\alpha})^{\dagger} = -\hat{p}_{\alpha} \Leftrightarrow$ the pairing (3.39) is maximal in the sense that the only I allowed are those with exactly k indices.

Proof

The previous theorem excludes the possibility of I with greater than k indices (simply take all the indices to one side of the comma and use antisymmetry). To eliminate the case of less than k indices observe that,

$$\langle \Psi | \hat{p}_{\alpha} X \rangle = -i\hbar \sum_{m,n=0}^{k} (n+1) \int \Psi^{*}_{\beta_1 .. \beta_m} X_{\alpha\gamma_1 .. \gamma_n} I^{\beta_1 .. \beta_m , \gamma_1 .. \gamma_n} d\mu ,$$

and,

$$\langle \hat{p}_\alpha \Psi | X \rangle = +i\hbar \sum_{m,n=0}^{k} (m+1) \int \Psi^*_{\alpha\beta_1 .. \beta_m} X_{\gamma_1 .. \gamma_n} I^{\beta_1 .. \beta_m , \gamma_1 .. \gamma_n} d\mu .$$

Thus,

$$(\hat{p}_\alpha)^\dagger = - \hat{p}_\alpha \Leftrightarrow \langle \Psi | \hat{p}_\alpha X \rangle = - \langle \hat{p}_\alpha \Psi | X \rangle \;\; \forall \; \Psi, X .$$

$$\Leftrightarrow m \, \delta_\alpha^{[\beta_1} I^{\beta_2 .. \beta_m], \gamma_1 .. \gamma_n} =$$

$$n \, I^{\beta_1 .. \beta_m , [\gamma_2 .. \gamma_n} \delta_\alpha^{\gamma_1]} \;\; \forall \; m, n. \qquad (3.43)$$

where [ ] denotes antisymmetrisation of the enclosed indices. After some manipulation the above condition reduces to,

$$(\hat{p}_\alpha)^\dagger = - \hat{p}_\alpha \Leftrightarrow \sum_{h=1}^{m} (-1)^{h+1} \delta_\alpha^{\beta_h} I^{, \beta_1 .. \hat{\beta}_h .. \beta_m \gamma_1 .. \gamma_n}$$

$$= \sum_{h=1}^{n} (-1)^{m+h} \delta_\alpha^{\gamma_h} I^{, \beta_1 .. \beta_m \gamma_1 .. \hat{\gamma}_h .. \gamma_n} \;\; \forall \; m,n. \qquad (3.44)$$

The notation $\beta_1 .. \hat{\beta}_h .. \beta_m$ means that the hatted index $\hat{\beta}_h$ is excluded from the list. Now, equation (3.44) must be satisfied for all possible choices of the β and γ so let us take $\beta_1 = \alpha$ and all the other β and γ different from α. With this choice (3.44) becomes,

$$I^{, \beta_2 .. \beta_m \gamma_1 .. \gamma_n} = 0. \qquad (3.45)$$

Now, if $m + n - 1 \leq k - 1$ (i.e the number of indices on I is less than k) it is possible to choose all the $\beta_2, .. , \gamma_n$ different and so (3.45) is a nontrivial equation. Thus, if I has less than k indices it must be zero.

It now only remains to show that when I is maximal, i.e when it has exactly k indices (m+n-1=k in (3.44)), the right hand side of condition (3.44) is satisfied. Note that the only possible maximal I is $B\sigma(\alpha_1 .. \alpha_k)$ where B is some function of Q^A and $\sigma(\alpha_1 .. \alpha_k)$ is the totally antisymmetric object defined by $\sigma(1, ..,k) = 1$. The function B corresponds to changing $d\mu$ and so can be ignored. Therefore we have to show that,

$$\sum_{h=1}^{m} (-1)^{h+1} \delta_\alpha^{\beta_h} \sigma(\beta_1 .. \hat{\beta}_h .. \beta_m \gamma_1 .. \gamma_n) =$$

$$\sum_{h=1}^{n} (-1)^{h+m} \delta_\alpha^{\gamma_h} \sigma(\beta_1 .. \beta_m \gamma_1 .. \hat{\gamma}_h .. \gamma_n) \qquad (3.46)$$

where m + n = k + 1. This is true by the following argument. There are k + 1 indices so at least two of them must be the same. If more than two are the same both sides are zero and so there is nothing to prove. Therefore assume that only two indices are the same and consider the following three cases.

<u>Case One</u>

Two of the β, say β_i and β_j, are the same. The condition then reduces to,

$$(-1)^{i+1} \delta_\alpha^{\beta_i} \sigma(\beta_1 .. \hat{\beta}_i .. \beta_m \gamma_1 .. \gamma_n) +$$
$$(-1)^{j+1} \delta_\alpha^{\beta_j} \sigma(\beta_1 .. \hat{\beta}_j .. \beta_m \gamma_1 .. \gamma_n) = 0.$$

Take $\beta_i = \beta_j = \alpha$ as the equation is trivial otherwise. Suppose, without loss of generality, that i < j so that the above equation becomes,

$$(-1)^{i+1} \sigma(\beta_1 .. \beta_{i-1} \beta_{i+1} .. \beta_{j-1} \alpha \beta_{j+1} .. \beta_m \gamma_1 .. \gamma_n) +$$
$$(-1)^{j+1} \sigma(\beta_1 .. \beta_{i-1} \alpha \beta_{i+1} .. \beta_{j-1} \beta_{j+1} .. \beta_m \gamma_1 .. \gamma_n) = 0.$$

By permuting α through one of these terms it is easy to show that the above equation is true. The remaining two cases to deal with are,

Case Two

Two of the γ are equal.

Case Three

One of the β equals one of the γ.

The arguments for both of these are very similar to case one and will not be given here. This completes the proof of theorem 3.4.

These two theorems tell us that the conditions (3.40) and (3.41) are sufficient to force the pairing to be maximal. It is easy to show that the Berezin pairing is maximal so theorems 3.3 and 3.4 force us to use the pairing (3.30). This does, however, leave us with the problem pointed out at the beginning of this section i.e., the Berezin pairing does not look as if it can recover the inner product on the physical Hilbert space. We will show that it is possible to get the physical Hilbert space from the Berezin pairing but only by introducing a duality condition on the quantum states. The results of this section show that it is not possible to get the BFV quantisation to work without introducing this duality idea.

3.5 Quantum Rescaling of Constraints

We know that the transformation (3.5) makes the classical rescaling of constraints into a canonical transformation on SP. This result leads one to the expectation that this transformation will be a unitary transformation in the quantum theory. We will show, in this section, that (modulo some technicalities) this is indeed the case. It is this transformation that is the key to solving the factor ordering problems in the quantisation.

The transformation (3.5) involves Q^A and P_A so it is necessary to know what their corresponding quantum operators are. These operators must be hermitian with respect to (3.30) so this leads one to suggest,

$$\hat{Q}^A = Q^A, \tag{3.47}$$

and,

$$\hat{P}_A = -i\hbar\left(\partial_A + \frac{1}{2}\frac{\|\varphi\|_{,A}}{\|\varphi\|}\right). \tag{3.48}$$

The latter equation is equivalent to taking (3.14) as the momentum operator associated with (3.11). To make this clear we state the following result.

Theorem 3.5

Assume that $\hat{P}_A$ is of the form,

$$\hat{P}_A = -i\hbar\left(\partial_A + f(Q^A)\right),$$

for some function $f(Q^A)$. Then (3.48) is the unique choice that makes $\hat{P}_A$ hermitian with respect to (3.30). The proof is straight forward.

The above choice of operators $\hat{P}_A$ and $\hat{Q}^A$ satisfy the canonical commutation relations,

$$[\hat{Q}^A, \hat{P}_B] = i\hbar\delta_B{}^A, \tag{3.49}$$

$$[\hat{Q}^A, \hat{Q}^B] = [\hat{P}_A, \hat{P}_B] = 0. \tag{3.50}$$

These operators are also covariant with respect to the transformations,

$$\hat{Q}^A \rightarrow \hat{Q}^{A'} = Q^{A'}(Q^A), \tag{3.51}$$

$$\hat{P}_A \rightarrow \hat{P}_{A'} = \frac{1}{2}[Q^A_{A'}, \hat{P}_A]_+ . \tag{3.52}$$

These are the quantum analogues of the point transformations on Q.

To implement the quantum constraint rescaling some care is needed because the $\|\varphi\|$ is not invariant under rescalings and therefore the pairing will change. To deal with this properly it is necessary to regard the state spaces, before and after the transformation, as different (they contain the same states but have a different pairing). We will denote the initial state space by H_{BFV} and the final state space by $(H_{BFV})_R$. In addition we denote the pairing on H_{BFV} by $<\Psi\,|\,X>$ and the pairing on $(H_{BFV})_R$ by $<\Psi\,|\,X>_R$. With this notation the quantum version of (3.5) will take the form of a bijective mapping, $\hat{\mathbf{R}}$, from H_{BFV} to $(H_{BFV})_R$ which satisfies,

$$\hat{\mathbf{R}} Q^A \hat{\mathbf{R}}^{-1} = Q^A, \tag{3.53a}$$

$$\hat{\mathbf{R}}\,\hat{P}_A\,\hat{\mathbf{R}}^{-1} = \hat{P}_A + \frac{1}{2}(\Lambda^{-1})_{\alpha}{}^{\beta}{}_{,A}\,\Lambda_{\gamma}{}^{\alpha}\,(\eta^{\gamma}\hat{\rho}_{\beta} - \hat{\rho}_{\beta}\eta^{\gamma}), \tag{3.53b}$$

$$\hat{\mathbf{R}}\eta^{\alpha}\hat{\mathbf{R}}^{-1} = \Lambda_{\beta}{}^{\alpha}\eta^{\beta}, \tag{3.53c}$$

$$\hat{\mathbf{R}}\,\hat{\rho}_{\alpha}\,\hat{\mathbf{R}}^{-1} = (\Lambda^{-1})_{\alpha}{}^{\beta}\,\hat{\rho}_{\beta}, \tag{3.53d}$$

where $\hat{\mathbf{R}}^{-1}$ is the inverse mapping to $\hat{\mathbf{R}}$. On the right hand side of (3.53b) we have taken the commutator ordering of $\eta^{\gamma}\rho_{\beta}$ which is necessary for this term to be hermitian. It is also important to note that, because the momentum operators are of the form (3.48) they are going to change under rescaling transformations. This means that, in (3.53b), the $\hat{P}_A$ on the left hand side is constructed using the old constraints whereas the $\hat{P}_A$ on the right hand side is constructed using the rescaled constraints.

We also require $\hat{\mathbf{R}}$ to be norm preserving i.e for any two $\Psi, X \in H_{BFV}$,

$$\langle\hat{\mathbf{R}}\Psi\,|\,\hat{\mathbf{R}}X\rangle_R = \langle\Psi\,|\,X\rangle\,. \tag{3.54}$$

Such a transformation is not, strictly speaking, unitary because it maps between different spaces. It would be more technically correct to call $\hat{\mathbf{R}}$ a Hilbert space isomorphism. However, we will not do this and will refer to $\hat{\mathbf{R}}$ (incorrectly) as a unitary mapping.

We will now show that, up to a multiple of a complex number, there is a unique mapping satisfying (3.53) and (3.54). This operator is most easily defined by its action on an arbitrary state Ψ, of the form (3.22),

$$\hat{\mathbf{R}}\Psi = \sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} \Lambda_{\beta_1}^{\ \alpha_1} .. \Lambda_{\beta_m}^{\ \alpha_m} \eta^{\beta_1} .. \eta^{\beta_m} . \qquad (3.55)$$

It is easy to show that this operator is invertible and the inverse operator is defined by,

$$\hat{\mathbf{R}}^{-1}\Psi = \sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} (\Lambda^{-1})_{\beta_1}^{\ \alpha_1} .. (\Lambda^{-1})_{\beta_m}^{\ \alpha_m} \eta^{\beta_1} .. \eta^{\beta_m} . \qquad (3.56)$$

We will now prove that (3.55) and (3.56) satisfy all the required conditions.

Proof that (3.55) satisfies (3.53)

Equation (3.53a) is trivial to prove. To prove (3.53b) let Ψ be an arbitrary state of the form (3.22) and observe that,

$$\hat{\mathbf{R}}\,\hat{\mathrm{P}}_A \hat{\mathbf{R}}^{-1}\,\Psi = \hat{\mathbf{R}}\,\hat{\mathrm{P}}_A \sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} (\Lambda^{-1})_{\beta_1}^{\ \alpha_1} .. (\Lambda^{-1})_{\beta_m}^{\ \alpha_m} \eta^{\beta_1} .. \eta^{\beta_m}$$

$$= \hat{\mathbf{R}} \{ \sum_{m=0}^{k} (\hat{P}_A \Psi_{\alpha_1 .. \alpha_m}) (\Lambda^{-1})_{\beta_1}^{\alpha_1} .. (\Lambda^{-1})_{\beta_m}^{\alpha_m} \eta^{\beta_1} .. \eta^{\beta_m}$$

$$- i\hbar \sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} [(\Lambda^{-1})_{\beta_1}^{\alpha_1}{}_{,A} .. (\Lambda^{-1})_{\beta_m}^{\alpha_m} + ..$$

$$+ (\Lambda^{-1})_{\beta_1}^{\alpha_1} .. (\Lambda^{-1})_{\beta_m}^{\alpha_m}{}_{,A}] \eta^{\beta_1} .. \eta^{\beta_m} \}$$

$$= \hat{P}_A \Psi - i\hbar \hat{\mathbf{R}} \{ \sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} [(\Lambda^{-1})_{\beta_1}^{\alpha_1}{}_{,A} .. (\Lambda^{-1})_{\beta_m}^{\alpha_m} + ..$$

$$+ (\Lambda^{-1})_{\beta_1}^{\alpha_1} .. (\Lambda^{-1})_{\beta_m}^{\alpha_m}{}_{,A}] \eta^{\beta_1} .. \eta^{\beta_m} \}.$$

This expression can be reduced, via some straight forward but technical manipulations, to give the following result.

$$\hat{\mathbf{R}} \hat{P}_A \hat{\mathbf{R}}^{-1} = \hat{P}_A + (\Lambda^{-1})_{\alpha}^{\beta}{}_{,A} \Lambda_{\gamma}^{\alpha} \eta^{\gamma} \hat{p}_{\beta} . \quad (3.57)$$

In this equation it is important to realise that $\hat{P}_A$ is constructed using the old $\|\varphi\|$. Thus, written more fully (3.57) reads,

$$\hat{\mathbf{R}} \hat{P}_A \hat{\mathbf{R}}^{-1} = -i\hbar \left(\partial_A + \frac{1}{2} \frac{\|\varphi\|_{,A}}{\|\varphi\|} \right) + (\Lambda^{-1})_{\alpha}^{\beta}{}_{,A} \Lambda_{\gamma}^{\alpha} \eta^{\gamma} \hat{p}_{\beta} .$$

This equation can be rewritten in the form,

$$\hat{\mathbf{R}} \hat{P}_A \hat{\mathbf{R}}^{-1} = -i\hbar \left(\partial_A + \frac{1}{2} \frac{\|\tilde{\varphi}\|_{,A}}{\|\tilde{\varphi}\|} \right) + (\Lambda^{-1})_{\alpha}^{\beta}{}_{,A} \Lambda_{\gamma}^{\alpha} (\eta^{\gamma} \hat{p}_{\beta} - \hat{p}_{\beta} \eta^{\gamma}),$$

which proves (3.53b).

The proof of equations (3.53c and d) are similar so we will only give one of them. As before, let Ψ be an arbitrary state of the form (3.22) and observe that,

$$\hat{R}\hat{p}_\alpha\hat{R}^{-1}\Psi = \hat{R}\hat{p}_\alpha \{\sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} (\Lambda^{-1})_{\beta_1}^{\alpha_1} .. (\Lambda^{-1})_{\beta_m}^{\alpha_m} \eta^{\beta_1} .. \eta^{\beta_m}$$

$$= -i\hbar\hat{R} \{\sum_{m=1}^{k} m\, \Psi_{\alpha_1 .. \alpha_m} (\Lambda^{-1})_{\alpha}^{\alpha_1} .. (\Lambda^{-1})_{\beta_{m-1}}^{\alpha_m} \eta^{\beta_1} .. \eta^{\beta_{m-1}} \}$$

$$= -i\hbar \sum_{m=1}^{k} m\, \Psi_{\alpha_1 .. \alpha_m} (\Lambda^{-1})_{\alpha}^{\alpha_1} \eta^{\alpha_2} .. \eta^{\alpha_m} .$$

It is straight forward to show that,

$$(\Lambda^{-1})_{\alpha}^{\beta}\, \hat{p}_\beta\, \Psi = -i\hbar \sum_{m=1}^{k} m\, \Psi_{\alpha_1 .. \alpha_m} (\Lambda^{-1})_{\alpha}^{\alpha_1} \eta^{\alpha_2} .. \eta^{\alpha_m} ,$$

and hence (3.53d) follows.

Proof that (3.55) satisfies (3.54)

Let Ψ, X be arbitrary states of the form (3.22). Observe that,

$$\langle \hat{R}\Psi \,|\, \hat{R}X \rangle_R = (-i)^{\frac{1}{2}k(k-1)} \int \{ (\sum_{m=0}^{k} \Psi^*_{\alpha_1 .. \alpha_m} \Lambda_{\beta_1}^{\alpha_1} .. \Lambda_{\beta_m}^{\alpha_m} \eta^{\beta_1} .. \eta^{\beta_m})$$

$$\times (\sum_{n=0}^{k} X_{\gamma_1 .. \gamma_n} \Lambda_{\delta_1}^{\gamma_1} .. \Lambda_{\delta_n}^{\gamma_n} \eta^{\delta_1} .. \eta^{\delta_n})$$

$$d\eta^1 .. d\eta^k \frac{\|\varphi\|}{|\Lambda|} dQ^1 .. dQ^N \}$$

$$= (-i)^{\frac{1}{2}k(k-1)} \sum_{m=0}^{k} \int \Psi^*_{[\alpha_1 .. \alpha_m} X_{\alpha_{m+1} .. \alpha_k]} \frac{\Lambda_{\beta_1}^{\alpha_1} .. \Lambda_{\beta_k}^{\alpha_k}}{|\Lambda|}$$

$$\eta^{\beta_m} .. \eta^{\beta_1} \eta^{\beta_{m+1}} .. \eta^{\beta_k} \, d\eta^1 .. d\eta^k \, \|\varphi\| \, dQ^1 .. dQ^N ,$$

where [ ] denotes antisymmetrisation of the enclosed indices. This equation reduces easily to,

$$\langle \hat{\mathbf{R}}\Psi \,|\, \hat{\mathbf{R}}X \rangle_R = (-i)^{\frac{1}{2}k(k-1)} \sum_{m=0}^{k} \int \Psi^*_{[1 .. m} X_{m+1 .. k]} \, k! \, \eta^m .. \eta^1 \eta^{m+1} .. \eta^k$$

$$d\eta^1 .. d\eta^k \, \|\varphi\| \, dQ^1 .. dQ^N .$$

By a similar method it follows that,

$$\langle \Psi \,|\, X \rangle = (-i)^{\frac{1}{2}k(k-1)} \sum_{m=0}^{k} \int \Psi^*_{[1 .. m} X_{m+1 .. k]} \, k! \, \eta^m .. \eta^1 \eta^{m+1} .. \eta^k$$

$$d\eta^1 .. d\eta^k \, \|\varphi\| \, dQ^1 .. dQ^N ,$$

and so (3.54) is established.

It has now been shown that (3.55) does indeed have all the properties required of the quantum rescaling transformation. It will now be shown that (3.55) is essentially the only operator to satisfy (3.53) and (3.54).

Theorem 3.6

$\hat{\mathbf{R}}$, defined by equation (3.55), is the unique solution of equations (3.53) and (3.54) apart from a constant phase factor.

Proof

Assume that $\hat{\mathbf{R}}$ satisfies (3.53) and (3.54). Then (3.53a and c) can be rewritten more conveniently as,

$$\hat{\mathbf{R}}\, Q^A = Q^A \hat{\mathbf{R}}, \tag{3.58a}$$

and,

$$\hat{\mathbf{R}}\eta^{\alpha} = \Lambda_{\beta}^{\ \alpha}\, \eta^{\beta}\, \hat{\mathbf{R}}. \tag{3.58b}$$

These relations can now be used to commute $\hat{\mathbf{R}}$ through (3.22) to get,

$$\hat{\mathbf{R}}\Psi = \sum_{m=0}^{k} \Psi_{\alpha_1 .. \alpha_m} \Lambda_{\beta_1}^{\ \alpha_1} .. \Lambda_{\beta_m}^{\ \alpha_m} \eta^{\beta_1} .. \eta^{\beta_m} \hat{\mathbf{R}}(1), \tag{3.59}$$

where 1 is being thought of as the state with $\Psi_0 = 1$ and the other Ψ zero. To determine the allowed values of $\hat{\mathbf{R}}(1)$ observe that, because of (3.53c and d), $\hat{\mathbf{R}}$ commutes with the ghost number operator (The ghost number operator is defined by,

$$\hat{g} = \frac{i}{\hbar}\eta^{\alpha}\hat{\rho}_{\alpha}. \tag{3.60}$$

and the eigenspace of $\hat{g}$, corresponding to eigenvalue $r \in \mathbb{Z}$, is Γ^r). This means that $\hat{\mathbf{R}}(1)$ is, at most, a function of the Q^A. Let,

$$\hat{\mathbf{R}}(1) = f(Q^A).$$

The condition (3.54) puts major restrictions on f as we now show. Let Ψ_k be an arbitrary state with ghost number k i.e.,

$$\Psi_k = \Psi_{\alpha_1 .. \alpha_k} \eta^{\alpha_1} .. \eta^{\alpha_k}.$$

Observe that,

$$\left\langle \Psi_k \,|\, \hat{\mathbf{R}}(1) \right\rangle_R = k! \int \Psi^*_{1..k} f(Q^A) \frac{\|\varphi\|}{|\Lambda|} \, dQ^1 .. dQ^N ,$$

and,

$$\left\langle \hat{\mathbf{R}}^{-1}(\Psi_k) \,|\, 1 \right\rangle = k! \int \Psi^*_{1..k} [\hat{\mathbf{R}}^{-1}(1)]^* \frac{\|\varphi\|}{|\Lambda|} \, dQ^1 .. dQ^N .$$

These two expressions must be equal for all Ψ_k because of (3.54). We can thus conclude that,

$$f(Q^A)^* = \hat{\mathbf{R}}^{-1}(1).$$

From this it follows that,

$$|f(Q^A)|^2 = 1,$$

i.e $f(Q^A)$ is a phase factor. It has already been proven that f = 1 satisfies all the required conditions. From this it follows trivially that f equal to any constant phase factor will satisfy (3.53 and 54). To show that only a constant phase factor is allowed observe that, if $\hat{\mathbf{R}}$ is given by (3.55),

$$\exp[-i\, s(Q^A)]\, \hat{\mathbf{R}}\, \hat{P}_A \exp[i\, s(Q^A)] \hat{\mathbf{R}}^{-1} = \hat{\mathbf{R}}\, \hat{P}_A\, \hat{\mathbf{R}}^{-1} + \hbar\, s_{,A} .$$

This is sufficient to prove that $f(Q^A)$ must be a constant and this proves the theorem.

3.6 The "Rigged Hilbert Space" of States

There is one technical problem with all constraint quantisation schemes which we have so far ignored. The problem is simply that the physical states are only going to depend on the physical directions and are therefore not going to be square integrable on Q i.e., the nontrivial physical states do not lie in the Hilbert space H that we have defined.

This problem is very similar to the situation that arises in ordinary quantum mechanics when operators with continuous spectra are used. The eigenstates of these operators do not lie in the Schrödinger picture Hilbert space. For example, the eigenstates of momenta in one dimensional quantum theory are complex exponentials which are not square integrable on $\mathbb{R}$. The solution to this problem is to use a Rigged Hilbert space [69,70] rather than a Hilbert space [71]. We will very briefly present the relevant details of this construction. The key observation is that, when one talks of a Hilbert space H in quantum theory, one must realise that H is obtained as the completion of some space S. This gives rise to the following triplet of densely nested spaces,

$$S \subset H \subset S^*, \tag{3.61}$$

where S^* is the topological dual to S (i.e the space of continuous, scalar valued linear maps on S). The chain of sets (3.61) is a Rigged Hilbert space or Gelfand triplet. The object is important because S^* contains the eigenstates of the operators with continuous spectra. Such states are often called generalised eigenstates.

To see how the above ideas help to solve the problems that arise in constraint quantisation let us look at the case where the constraints are the first k momenta. In this situation S will be taken to be the set of smooth functions, on Q, with suitably fast fall off rate. Then, in the Dirac analysis, the solutions to $\hat{P}_\alpha \,|\, \mathrm{phys} \rangle = 0$ are just the zero eigenstates of the constraint momenta and are therefore going to lie in S^*, rather than in the Hilbert space H.

To implement these ideas in the BFV approach we will introduce the *ghost Gelfand triplet.* Let H be the Hilbert space of L^2 functions on Q (with respect to the $\|\varphi\|$ measure) and let S be a dense subset of suitably fast fall off functions e.g the Schwartz space (this consists of all C^∞ functions with the property that the function, and all its derivatives, decrease faster than any polynomial at infinity).This gives rise to a Gelfand triplet of the form (3.61) and we can use this to introduce the following triplet of states.

$$S_{BFV} \subset H_{BFV} \subset (S_{BFV})^*, \tag{3.62}$$

where Ψ belongs to an element of this triplet if it has an expansion (3.22) where all the coefficient functions lie in S, H or S^* accordingly as Ψ lies in S_{BFV}, H_{BFV} or $(S_{BFV})^*$. We shall refer to (3.62) as the ghost Gelfand triplet though, strictly speaking, it is not a Gelfand triplet since the pairing on H_{BFV} is not positive definite.

All the operators that were defined earlier in this chapter are also defined on the sets in (3.62). The only problem is that the self adjointness properties of the operators on H_{BFV} will not, in general, hold on $(S_{BFV})^*$. This is an important subtlety that shall arise later.

There has been another attempt to implement Rigged Hilbert space ideas in the ghost formalism [14]. The philosophy there was to put the different parts of the Gelfand triplet (3.61) into the different ghost number parts of the quantum states. This philosophy was also used in a previous presentation of this work [65]. At first sight this approach has some very attractive features but care is needed because one wants to work with operators such as $\hat{\eta}^\alpha$, $\hat{\rho}_\alpha$ and $\hat{\Omega}$ and these change the ghost number of the states. If the coefficient spaces of (3.22) are not all the same these operators will take us out of the state space, a situation that is clearly undesirable. This is the reason why the previous formulation of Rigged Hilbert spaces has been abandoned in favour of the one above which was first presented in [66].

3.7 The Projection of Physical States

All the necessary mathematical machinery has now been set up so we will now turn our attention to explicitly producing the constrained quantum theory. The key step that will be used is to demand that the quantum theory is invariant under the transformations (a) to (d) listed at the end of section 3.1. It will be shown that this invariance is sufficient to eliminate all the ambiguities in the factor ordering of the operators and moreover, the resultant theory is equivalent to the physical one described in section 3.2. In this section we will present a prescription for projecting the physical states of the quantum theory and in the section 3.8 we will give explicit orderings for all the special physical observables.

To project the physical states from (3.62) we will use the BRST charge $\hat{\Omega}$ which we require to satisfy the following three conditions

1) $\hat{\Omega}^\dagger = \hat{\Omega}$.

2) $\hat{\Omega}^2 = 0$.

3) $\hat{\Omega}$ is covariant with respect to all the symmetries a) .. d) of section 3.1.

We will show that the classical BRST charge Ω (2.17) has a unique ordering which satisfies these requirements. The orderings of the classical expression (2.17) that we will allow are of the form,

$$\hat{\Omega} = \{ \varepsilon_1 \hat{\varphi}_\alpha^A \hat{P}_A + \varepsilon_2 \hat{P}_A \hat{\varphi}_\alpha^A \} \eta^\alpha$$

$$+ \frac{1}{2} C^\alpha_{\beta\gamma} \{ \lambda_1 \eta^\beta \eta^\gamma \hat{\rho}_\alpha - \lambda_2 \eta^\beta \hat{\rho}_\alpha \eta^\gamma + \lambda_3 \hat{\rho}_\alpha \eta^\beta \eta^\gamma \}, \qquad (3.63)$$

where the ε and λ are real numbers which satisfy,

$$\varepsilon_1 + \varepsilon_2 = 1, \tag{3.64}$$

and,

$$\lambda_1 + \lambda_2 + \lambda_3 = 1. \tag{3.65}$$

Equation (3.63) is easily reducible to the form,

$$\hat{\Omega} = \varphi_\alpha^A \hat{P}_A \eta^\alpha - i\hbar\varepsilon_2 \varphi_{\alpha,A}^A \eta^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \eta^\beta \eta^\gamma \hat{\rho}_\alpha + \frac{i\hbar}{2} a_1 C^\alpha_{\beta\alpha} \eta^\beta, \tag{3.66}$$

where $a_1 = \lambda_2 + 2\lambda_3$. The adjoint of the BRST charge can be computed from the above expression and it is,

$$\hat{\Omega}^\dagger = \varphi_\alpha^A \hat{P}_A \eta^\alpha + i\hbar(\varepsilon_2 - 1) \varphi_{\alpha,A}^A \eta^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \eta^\beta \eta^\gamma \hat{\rho}_\alpha + \frac{i\hbar}{2}(2 - a_1) C^\alpha_{\beta\alpha} \eta^\beta. \tag{3.67}$$

Thus the BRST charge is self adjoint if and only if,

$$(2\varepsilon_2 - 1) \varphi_\alpha{}^A{}_{,A} \eta^\alpha + (1 - a_1) C^\beta{}_{\alpha\beta} \eta^\alpha = 0. \tag{3.68}$$

From this one can conclude that both terms in (3.68) are zero and therefore that $\varepsilon_2 = 1/2$ and $a_1 = 1$. It is not possible to arrange for the two terms in (3.68) to cancel each other while neither is zero. This is because we require the formalism to be invariant under rescaling transformations and these do not preserve such a cancellation (this is straightforward, but tedious, to check from (3.72) which gives the transformation law for $C^{\alpha}{}_{\beta\gamma}$ under a rescaling).

In conclusion we have shown that the requirement of self adjointness alone is sufficient to determine the factor ordering of the BRST charge. This ordering can be written in the form,

$$\hat{\Omega} = \frac{1}{2} [\varphi^{A}_{\alpha} , \hat{P}_{A}]_{+} \eta^{\alpha} + \frac{1}{4} C^{\alpha}_{\beta\gamma} (\eta^{\gamma} \hat{\rho}_{\alpha} \eta^{\beta} - \eta^{\beta} \hat{\rho}_{\alpha} \eta^{\gamma}). \qquad (3.69)$$

It is now necessary to check that this ordering also satisfies the requirements 2) and 3) listed above. The nilpotency of $\hat{\Omega}$ can be checked directly but is an extremely messy calculation which will not be given as a much slicker proof is available once the invariance properties of $\hat{\Omega}$ are established.

To establish the coordinate covariance of $\hat{\Omega}$ it is only necessary to show the invariance of the anticommutator in the first term. This follows from the general result that the anticommutator ordering of an arbitrary function, linear in momenta, is coordinate covariant. That is,

$$\frac{1}{2} [U^{A}, \hat{P}_{A}]_{+} = \frac{1}{2} [U^{A} Q^{A'}_{A} , \hat{P}_{A'}]_{+} , \qquad (3.70)$$

where the U^{A} are arbitrary functions of the Q^{A} and the momenta are transforming according to (3.52). This result is proved by direct substitution of the transformation law (3.52).

The proof of the invariance of $\hat{\Omega}$ under the rescaling transformation (3.53) goes as follows.

$$\begin{aligned} \hat{\mathbf{R}}\hat{\Omega}\hat{\mathbf{R}}^{-1} &= \frac{1}{2} \hat{\mathbf{R}} [\varphi^{A}_{\alpha} , \hat{P}_{A}]_{+} \eta^{\alpha} \hat{\mathbf{R}}^{-1} \\ &+ \frac{1}{4} C^{\alpha}_{\beta\gamma} \hat{\mathbf{R}} (\eta^{\gamma} \hat{\rho}_{\alpha} \eta^{\beta} - \eta^{\beta} \hat{\rho}_{\alpha} \eta^{\gamma}) \hat{\mathbf{R}}^{-1} \end{aligned}$$

$$= \frac{1}{2} \varphi^A_\alpha \{ \hat{P}_A + \frac{1}{2} (\Lambda^{-1})_\mu^{\ \beta}{}_{,A} \Lambda_\gamma^{\ \mu} (\eta^\gamma \hat{\rho}_\beta - \hat{\rho}_\beta \eta^\gamma) \} \Lambda_\delta^{\ \alpha} \eta^\delta$$

$$+ \frac{1}{2} \{ \hat{P}_A + \frac{1}{2} (\Lambda^{-1})_\mu^{\ \beta}{}_{,A} \Lambda_\gamma^{\ \mu} (\eta^\gamma \hat{\rho}_\beta - \hat{\rho}_\beta \eta^\gamma) \} \varphi^A_\alpha \Lambda_\delta^{\ \alpha} \eta^\delta$$

$$+ \frac{1}{4} C^\alpha_{\beta\gamma} \Lambda_\mu^{\ \gamma} \Lambda_\delta^{\ \beta} (\Lambda^{-1})_\alpha^{\ \sigma} \{ \eta^\mu \hat{\rho}_\sigma \eta^\delta - \eta^\delta \hat{\rho}_\sigma \eta^\mu \}.$$

This reduces to,

$$\hat{R}\hat{\Omega}\hat{R}^{-1} = \frac{1}{2} [\Lambda_\alpha^{\ \beta} \varphi^A_\beta , \hat{P}_A]_+ \eta^\alpha + \frac{i\hbar}{2} \varphi^A_\beta \Lambda_{\alpha ,A}^{\ \beta} \eta^\alpha$$

$$- \frac{i\hbar}{2} \varphi^A_\alpha \frac{|\Lambda|_{,A}}{|\Lambda|} \Lambda_\beta^{\ \alpha} \eta^\beta + \varphi^A_\alpha (\Lambda^{-1})_{\mu ,A}^{\ \beta} \Lambda_\gamma^{\ \mu} \Lambda_\delta^{\ \alpha} \eta^\delta \eta^\gamma \hat{\rho}_\beta$$

$$+ \frac{1}{4} C^\alpha_{\beta\gamma} \Lambda_\delta^{\ \beta} \Lambda_\mu^{\ \gamma} (\Lambda^{-1})_\alpha^{\ \sigma} (\eta^\mu \hat{\rho}_\sigma \eta^\delta - \eta^\delta \hat{\rho}_\sigma \eta^\mu). \qquad (3.71)$$

This expression must be compared with $\hat{\tilde{\Omega}}$ which is constructed, using the ordering (3.69), from the rescaled constraints. To construct $\hat{\tilde{\Omega}}$ we need to know how the structure functions transform under rescalings. This is easy to compute by working out the Poisson bracket of $\tilde{\varphi}_\alpha$ with $\tilde{\varphi}_\beta$ and the result is,

$$\tilde{C}^\sigma_{\alpha\beta} = C^\zeta_{\gamma\delta} \Lambda_\alpha^{\ \gamma} \Lambda_\beta^{\ \delta} (\Lambda^{-1})_\zeta^{\ \sigma} + \Lambda_\beta^{\ \gamma} (\Lambda^{-1})_\zeta^{\ \sigma} \Lambda_{\alpha ,A}^{\ \zeta} \varphi^A_\gamma$$

$$- \Lambda_\alpha^{\ \gamma} (\Lambda^{-1})_\zeta^{\ \sigma} \Lambda_{\beta ,A}^{\ \zeta} \varphi^A_\gamma . \qquad (3.72)$$

With this result available one can write down $\hat{\tilde{\Omega}}$ and manipulate it into the form (3.71) thereby giving the result,

$$\hat{R}\hat{\Omega}\hat{R}^{-1} = \hat{\tilde{\Omega}}, \qquad (3.73)$$

which proves the invariance of the BRST charge under rescalings. As promised above we can now give a slick proof of the nilpotency of (3.69). To do this we will look at the case where the constraints have been trivialised to the first k momenta. In this situation the momenta operators are,

$$\hat{P}_\alpha = -i\hbar\partial_\alpha \qquad (\alpha = 1, ..,k) \tag{3.74a}$$

and,

$$\hat{P}_a = -i\hbar\left(\partial_a + \frac{1}{2}\frac{|g|^{\frac{1}{2}}_{,a}}{|g|^{\frac{1}{2}}}\right) \qquad (a = k+1, .. ,N) \tag{3.74b}$$

This means that the BRST charge (3.69) has the form,

$$\hat{\Omega} = -i\hbar\partial_\alpha\eta^\alpha. \tag{3.75}$$

This operator is trivially nilpotent and this nilpotency will be preserved by the transformation $\hat{\mathbf{R}}$ and by coordinate transformations therefore (3.69) is nilpotent.

We will now look at the physical states that are projected by $\hat{\Omega}$ and show that they are equivalent to the states in the physical Hilbert space described in section 3.2. To do this it is sufficient to look at the case of trivialised momenta as the rescaling and coordinate transformations do not change the cohomology structure (the quantum rescaling transformation is a chain mapping for the quantum cohomology because of equation (3.73)).

To analyse the cohomology of (3.75) it is important to work on the correct state space which, as was pointed out in section 3.6, is $(S_{BFV})^*$, the distributional part of the ghost triplet. If we let Γ^{*r} denote the ghost number r states in $(S_{BFV})^*$ then $(\Gamma^*,\hat{\Omega})$ defines a complex given by,

$$\Gamma^{*0} \stackrel{\hat{\Omega}}{\rightarrow} \Gamma^{*1} \stackrel{\hat{\Omega}}{\rightarrow} \ldots \quad \ldots \stackrel{\hat{\Omega}}{\rightarrow} \Gamma^{*k}. \tag{3.76}$$

This complex is almost identical to the de Rham cohomology complex familiar from differential geometry (see for example [37]) (just make the formal identification of η^{α} with dQ^{α}). Indeed, if the coefficient space of distributions on Q was replaced by the smooth functions on Q the complexes would be identical and, the only nonvanishing cohomology group in (3.76) would be the zeroth one (i.e Poincare's lemma which is valid because all the quantisation is local). Now, because $C^{\infty}(Q) \subset S^{*}(Q)$ one would expect at least some remnant of the de Rham cohomology in the distributional complex above. In fact, it turns out that Poincare's lemma still holds for distributional forms (see, for example, chapter three in [72]) so the only nonvanishing cohomology in (3.76) is the first one (we denote this by H^0) which, for these trivialised constraints, consists of those elements of $(S_{BFV})^{*}$ with ghost number zero and no dependence on $Q^1, \ldots, Q^k$. Thus, H^0 is isomorphic to $(S_{phys})^{*}$, the distributional part of the physical Gelfand triplet. Given this it is possible to construct the rest of the physical Gelfand triplet because S_{phys} is reflexive and H_{phys} is the completion of S_{phys}.

In summary, the operator (3.69) does project the correct physical states. However, we have been forced to make essential use of a local result, Poincare's lemma, indicating that there may, for some theories, be global obstructions to the BFV method. To avoid such obstructions it may be necessary to restrict attention to ghost number zero states thereby actively removing any nontrivial higher ghost cohomology. Alternatively any nontrivial higher ghost cohomology may be encoding important information about the global properties of Q_{phys} and should be included in the quantum theory. We cannot say anything more about this until a global version of the quantum BFV method is available.

We now have the situation where we know that the physical Gelfand triplet is isomorphic to the BRST cohomology group (3.33) which means that we can regard H_{phys}, and $(S_{phys})^*$, as being embedded in $(S_{BFV})^*$. This, however, is not sufficient to solve the problem of projecting physical states because we also wish to recover, from $(S_{BFV})^*$, the inner product structure on the physical Gelfand triplet. To do this we will have to extend the inner product structure on the Ghost Gelfand triplet to enable some of the distributional elements, in $(S_{BFV})^*$, to be paired together. We will also have to be more careful with the embedding of H_{phys} into $(S_{BFV})^*$. This will now be studied in detail for trivialised constraints and the discussion will then be generalise to arbitrary linear constraints.

There seems to be an obvious way to embed H_{phys} into $(S_{BFV})^*$. Simply let $\Psi_{phys} \in H_{phys}$ be represented by $\Psi = \Psi_0 = \Psi_{phys}$ (a ghost number zero element of $(S_{BFV})^*$). Clearly $\hat{\Omega}\Psi = 0$ so we have apparently solved the problem of projecting the physical states. However, as was pointed out at the beginning of section 3.4, this embedding does not recover the physical inner product because the Berezin measure pairs a ghost number r state with a ghost number k-r state. What we have to do is introduce the concept of duality where the physical ghost number zero state is related to a ghost number k state. Once this is done the Berezin pairing will enable the physical inner product to be recovered on $(S_{BFV})^*$.

The definition of duality on H_{BFV} is as follows. Let $\Psi \in H_{BFV}$ have ghost number r then the dual state to Ψ, denoted by Ψ', is

$$\Psi' = \Psi'_{\alpha_{r+1} \cdots \alpha_k} \eta^{\alpha_{r+1}} \cdots \eta^k , \tag{3.77}$$

where,

$$\Psi'_{\alpha_{r+1} \cdots \alpha_k} = \frac{(i)^{\frac{3}{2}k(k-1)}}{r!} \Psi_{\alpha_1 \cdots \alpha_r} \varepsilon^{\alpha_k \cdots \alpha_{r+1} \alpha_1 \cdots \alpha_r} . \tag{3.78}$$

This now enables us to define an inner product (,) on the ghost number r elements of H_{BFV} by $(\Psi_1, \Psi_2) = < \Psi'_1, \Psi_2 >$. Duality on a mixed ghost number state can be defined by taking the duals of the separate ghost number terms. Obviously, if one has a self dual state then its Berezin pairing will give its norm. So, if Ψ_0 is a ghost number zero element of H_{BFV} it has dual coefficients,

$$\Psi_{k\,..\,1} = (i)^{\frac{3}{2}k\,(k-1)} \Psi_0 . \tag{3.79}$$

We can thus construct a self dual state Ψ by adding this ghost number k state to Ψ_0. The norm of the resulting state will be precisely the norm of Ψ_0 considered as a square integrable function on Q (up to a normalisation factor).

The above ideas of duality are fine except that we require to self dualise the solutions to (3.31) and these lie in $(S_{BFV})^*$ not in H_{BFV}. Hence, we need to extend the duality to distributional states. Let $\Psi_0 \in H_{phys}$, we define its distributional dual in $(S_{BFV})^*$ to be the ghost number k state with coefficients,

$$\Psi_{k\,..\,1} = (i)^{\frac{3}{2}k\,(k-1)} \delta(Q^1) \,..\, \delta(Q^k)\, \Psi_0 . \tag{3.80}$$

Hence we can embed H_{phys} into $(S_{BFV})^*$ by constructing the self dual state to Ψ_0 which is,

$$\Psi = \Psi_0 + k!\,(i)^{\frac{3}{2}k\,(k-1)} \delta(Q^1) \,..\, \delta(Q^k)\, \Psi_0\, \eta^k \,..\, \eta^1 . \tag{3.81}$$

The Berezin pairing can be extended to $(S_{BFV})^*$ and allows two such states to be paired in such a way that the resulting pairing agrees with the inner product on H_{phys}. This is easy to see because, in the case of trivialised constraints, the Berezin pairing has the form,

$$\langle \Psi | X \rangle = (i)^{\frac{1}{2}k(k-1)} \int \Psi^* X \, d\eta^1 .. d\eta^k \; |g|^{\frac{1}{2}} dQ^1 .. dQ^N, \qquad (3.82)$$

and if $\Psi_0, X_0 \in H_{phys}$ the associated self dual elements in $(S_{BFV})^*$ pair to give,

$$\langle \Psi | X \rangle = 2(k!) \int \Psi_0^* \, X_0 \, |g|^{\frac{1}{2}} dQ^{k+1} .. dQ^N, \qquad (3.83)$$

which, up to a normalisation, is the desired physical result. Thus, the self dual BRST invariant states correctly characterise the physical states of the system.

It might be thought that this elaborate definition of self dual embeddings of the physical states into $(S_{BFV})^*$ has not solved anything since the ghost number k part of the self dual state is also a solution to (3.31). Hence, by the vanishing cohomology argument, this term must be of the form $\hat{\Omega}\chi$ for some $\chi \in \Gamma^{*k-1}$ (indeed such a χ is easy to write down and involves step functions). Therefore, by the self adjointness of $\hat{\Omega}$, such a term will give zero when paired with Ψ_0. Clearly this is not the case as (3.83) is definitely not identically zero. The flaw in the above argument is that $\hat{\Omega}$ is only self adjoint with respect to the pairing on H_{BFV} or, the pairing between S_{BFV} and $(S_{BFV})^*$. We are working with a pairing between two distributional objects and these do not necessarily vanish at infinity so, the momentum operators will not be self adjoint and therefore the BRST charge will not be self adjoint. In fact, if one works out the surface terms, in the pairing, which arising from partially integrating the BRST charge off of the ghost number k part of (3.81), one recovers (3.83).

The nonhermiticity of $\hat{\Omega}$ means that we can no longer assume that coboundary states (ones of the form $\hat{\Omega}\chi$) will decouple from the inner products, and hence never contribute to physical results. This is a good thing for the ghost number k part of

(3.81) but we must be careful that other, unwanted coboundaries, do not start altering the results via surface terms. Thus, in practise, one must be careful that the states one uses consist only of a self dual state (3.81) plus coboundaries of the form $\hat{\Omega}\chi$ where χ vanishes at infinity.

This completes the solution to the problem of projecting the physical states in the case of trivialised constraints. To obtain a general solution we need only work out the form of the self dual condition when the constraints have been rescaled back to their original form. To do this we apply the rescaling transformation to (3.81) and get,

$$\hat{\mathbf{R}}\Psi = \Psi_0 + k!\,(i)^{\frac{3}{2}k\,(k-1)}\,\delta(Q^1)\,..\,\delta(Q^k)\,|\Lambda|\,\Psi_0\eta^k\,..\,\eta^1, \qquad (3.84)$$

which we need to write in a coordinate and rescaling covariant form. The coordinate covariance is easily dealt with by letting $X^\alpha = Q^\alpha$ and regarding this as a local gauge fixing condition on Q. To get a rescaling invariant form of (3.84) observe that,

$$|\Lambda| = \det[\{X^\alpha, \varphi_\beta\}] = |\{X^\alpha, \varphi_\beta\}|.$$

This alternative form for $|\Lambda|$ is coordinate covariant. Thus, the general form of the duality condition is,

$$\Psi = \Psi_0 + k!\,(i)^{\frac{3}{2}k\,(k-1)}\,\delta(X^1)\,..\,\delta(X^k)\,|\{X^\alpha, \varphi_\beta\}|\,\Psi_0\eta^k\,..\,\eta^1. \qquad (3.85)$$

The term,

$$\delta(X^1)\,..\,\delta(X^k)\,|\{X^\alpha, \varphi_\beta\}|, \qquad (3.86)$$

is familiar from the phase space path integral description of constrained systems [73] where it enters as a modification to the measure. It is a standard result that (3.86) is invariant under infinitesimal changes to the gauge fixing conditions [6]. One may wonder if it is possible for us to insert (3.86) into the measure of the pairing and so avoid having to introduce distributional duals. This does not work because, e.g. in the trivialised case, (3.86) would destroy the hermiticity of the constraint momenta.

This completes the solution to projecting physical states. To summarise, equations (3.69) and (3.85) are the solution to the kinematic aspects of constraint quantisation. Together they project out the space of BRST invariant, self dual states. When endowed with the pairing (3.30) these states are isomorphic to the Rigged Hilbert space of physical states.

3.8 The Ordering of the Quantum Observables

In this section the factor ordering of the special, gauge invariant observables will be derived. If **F** is such an observable we require its quantum version to satisfy,

1) $[\hat{\mathbf{F}}, \hat{\Omega}] = 0$.
2) $\hat{\mathbf{F}}^{\dagger} = \hat{\mathbf{F}}$.
3) $\hat{\mathbf{F}}$ is covariant under all the symmetries a) .. d) of section 3.1.

Condition 1) is essential for the consistence of the theory. If it failed to be true $\hat{\mathbf{F}}$ would map physical states to unphysical states. For similar reasons, it is necessary for $\hat{\mathbf{F}}$ to preserve the self dual condition, at least up to zero norm states.

Conditions 2) and 3) are not logically necessary as Kuchar pointed out [8,9]. It is only strictly necessary for $\hat{\mathbf{F}}$ to be hermitian on physical states and covariant with respect to symmetries that are lifts of symmetries from the true degrees of freedom. Having the full properties 2) and 3) is, none the less, very convenient and the ghost methods allow them to be achieved. Before presenting the orderings there are a few aspects of the quantum theory that must be discussed.

As with the classical observables, the quantum observables have an equivalence class structure where $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$ are equivalent if they differ by a coboundary i.e.,

$$\hat{\mathbf{F}} = \hat{\mathbf{G}} + [\,\hat{\mathbf{K}}, \hat{\mathbf{\Omega}}\,], \tag{3.87}$$

where $\hat{\mathbf{K}}$ is some ghost number minus one operator (It is standard to refer to commutators of the form $[\hat{\mathbf{K}}, \hat{\mathbf{\Omega}}]$ as coboundaries. The terminology is borrowed from the geometrical uses of cohomology theory). Coboundaries never contribute to any physical results i.e., they vanish when paired with BRST invariant states. The equivalence class structure of the quantum observables is consistent with the commutator algebra i.e if,

$$\hat{\mathbf{F}} = \hat{\mathbf{F}}' + [\,\hat{\mathbf{A}}, \hat{\mathbf{\Omega}}\,],$$

and,

$$\hat{\mathbf{G}} = \hat{\mathbf{G}}' + [\,\hat{\mathbf{B}}, \hat{\mathbf{\Omega}}\,],$$

then,

$$[\,\hat{\mathbf{F}}, \hat{\mathbf{G}}\,] = [\,\hat{\mathbf{F}}', \hat{\mathbf{G}}'\,] + [\,\hat{\mathbf{C}}, \hat{\mathbf{\Omega}}\,],$$

for some $\hat{\mathbf{C}}$. This result will be important later.

The strategy that will be used to solve the factor ordering problems in the observables is similar to that used in arriving at the quantum BRST charge. That is, we will first of all parametrise all the possible orderings that are regarded as reasonable and then enforce the conditions 1) .. 3) to fix the parameters. To verify that the orderings obtained are equivalent to the physical quantum theory of section 3.2 we will then look at the solution for the case of trivialised constraints. In this simple case we will be able to show that our operators differ from the physical operators by, at most, a coboundary and are hence equivalent to each other.

The above technique is not the only way of obtaining the quantum observables.

In earlier versions of this work [67] an alternative strategy was used. This strategy consisted of firstly looking at the case of trivialised constraints where it is easy to write the classical, special observables in the form,

$$\mathbf{F} = \mathbf{F}_{\text{phys}} + \{ \mathbf{F}', \mathbf{\Omega} \}, \tag{3.88}$$

where $\mathbf{F}_{\text{phys}}$ is a special observable on Q_{phys}. One can then quantise this via,

$$\mathbf{F}_{\text{phys}} \rightarrow \hat{\mathbf{F}}_{\text{phys}}, \tag{3.89a}$$

and,

$$\{ \mathbf{F}', \mathbf{\Omega} \} \rightarrow \frac{-i}{\hbar} [\hat{\mathbf{F}}', \hat{\mathbf{\Omega}}], \tag{3.89b}$$

where $\hat{\mathbf{F}}_{\text{phys}}$ is the required operator taken from (3.17, 18 or 19) and $\hat{\mathbf{F}}'$ is ordered to be antihermitian. There are no Van Hove obstructions to (3.89b) because $\mathbf{F}'$ and $\mathbf{\Omega}$ are, at most, linear in the momenta. These orderings can be generalised to nonabelian constraints by boosting (3.89) with the rescaling operator $\hat{\mathbf{R}}$.

This procedure guarantees that the resulting quantum operators are hermitian, commute with the BRST charge, preserve the self dual condition and give the required answers when paired with physical states. It is also found that the operators can be written in a form which is covariant with respect to all the symmetries a) .. d) of section 3.1. The details of this method can be found in [67] and will not be given here.

The method that will be used here to derive the orderings has a number of advantages over the older one. Firstly, it shows that the trivialisation step is only a technical tool and it is the invariances of the theory that control the orderings. There is an analogy here with Riemannian geometry where the trivialisation procedure plays the role of Riemann normal coordinates i.e. trivialisation is a local representation that is easy to use in calculations. Another advantage of the new approach is that it enables us to show that the orderings are more or less unique. They could always be supplemented by $\hbar$ corrections which are invariant under all transformations but, by Ockham's razor, such terms will be ignored. It is also an advantage to have two separate methods of deriving the results because it gives a check that the algebra is correct.

The orderings for each of the special observables will now be presented.

3.8.1 Configuration Space Observables

This case is trivial and is included for completeness only. Let $Y(Q^A)$ be an arbitrary gauge invariant, configuration space observable. Being gauge invariant it must satisfy,

$$\{ Y, \varphi_\alpha \} = Y_\alpha{}^\beta \varphi_\beta, \qquad (3.90)$$

for some $Y_\alpha{}^\beta$. However, the left hand side of (3.90) depends only on Q^A whereas the right hand side has momentum dependence. Thus, to avoid contradiction, $Y_\alpha{}^\beta$ must be zero. This means that the BRST extension, $\mathbf{Y}$, of Y is trivial i.e.,

$$\mathbf{Y} = Y. \qquad (3.91)$$

There are no ordering ambiguities in this expression so the only natural choice for $\hat{\mathbf{Y}}$ is,

$$\hat{\mathbf{Y}} = \mathbf{Y}. \qquad (3.92)$$

This is trivially invariant under coordinate and rescaling transformations. We need not consider invariance under weak changes to Y as there are no nontrivial transformations, of this type, which preserve the momentum dependence of Y. It is also trivial to check that (3.92) is equivalent to the physical quantisation of section 3.2.

3.8.2 Linear Observables

Let $U = U^A(Q^B)\, P_A$ be a general linear, gauge invariant observable. It will satisfy,

$$\{U, \varphi_\alpha\} = U_\alpha{}^\beta\, \varphi_\beta, \tag{3.93}$$

where $U_\alpha{}^\beta$ depends only on the Q^A. Therefore, the BRST extension of U is,

$$\mathbf{U} = U + U_\alpha{}^\beta \eta^\alpha \rho_\beta. \tag{3.94}$$

The orderings that we will allow for $\hat{\mathbf{U}}$ are of the form,

$$\hat{\mathbf{U}} = \{\varepsilon_1 U^A \hat{P}_A + \varepsilon_2 \hat{P}_A U^A\} + U_\alpha^\beta \{\lambda_1 \eta^\alpha \hat{\rho}_\beta - \lambda_2 \hat{\rho}_\beta \eta^\alpha\}, \tag{3.95}$$

where the ε and λ are real numbers satisfying,

$$\varepsilon_1 + \varepsilon_2 = 1, \tag{3.96}$$

and,

$$\lambda_1 + \lambda_2 = 1. \tag{3.97}$$

Equation (3.95) reduces to,

$$\hat{\mathbf{U}} = U^A \hat{P}_A - i\hbar\varepsilon_2 U^A{}_{,A} + U_\alpha^\beta \eta^\alpha \hat{\rho}_\beta + i\hbar\lambda_2 U_\alpha^\alpha. \tag{3.98}$$

From this it follows that,

$$\hat{U}^{\dagger} = U^A \hat{P}_A - i\hbar(1-\varepsilon_2)U^A_{,A} + U_\alpha^\beta \eta^\alpha \hat{\rho}_\beta + i\hbar(1-\lambda_2)U_\alpha^\alpha . \qquad (3.99)$$

Hence, $\hat{U}$ is self adjoint if and only if,

$$(2\varepsilon_2 - 1)\, U^A_{,A} - (2\lambda_2 - 1)\, U_\alpha^\alpha = 0, \qquad (3.100)$$

from which we can conclude that $\varepsilon_2 = 1/2$ and $\lambda_2 = 1/2$. It is not possible for the two terms in (3.100) to cancel while neither is zero because we require the ordering to be invariant under weak changes to U and, these transformations would not preserve such a cancellation (the proof of this is easy but rather tedious and will not be given). This means that self adjointness requires the following ordering for linear observables.

$$\hat{U} = \frac{1}{2}[U^A, \hat{P}_A]_+ + \frac{1}{2}U_\alpha^\beta(\eta^\alpha \hat{\rho}_\beta - \hat{\rho}_\beta \eta^\alpha). \qquad (3.101)$$

We must now check that this ordering satisfies all the other requirements. The fact that (3.101) commutes with the BRST charge can be checked by direct calculation. This will not be given as there is a much easier way of proving this once the invariance properties of (3.101) are established. Likewise, it is easier to prove that (3.101) preserves the self duality condition using the invariance properties.

The coordinate covariance of (3.101) follows immediately from (3.70). To check the invariance of (3.101) under weak changes to U we must examine such a transformation and show that the resultant operator differs from the original one by, at most, a coboundary. Let $\tilde{U} = U + B^\alpha(Q^A)\varphi_\alpha$ be an arbitrary weak transformation. Then it is easy to see that the BRST extension of U will transform as,

$$U \to \tilde{U} = (U^A + \varphi^A_\alpha B^\alpha)P_A + [U^\beta_\alpha + B^\beta_{,A}\varphi^A_\alpha + C^\beta_{\gamma\alpha} B^\gamma]\eta^\alpha\rho_\beta.$$

From this one can compute the quantum operator $\hat{\tilde{U}}$ and, after some manipulation, the following result is obtained.

$$\begin{aligned}\hat{\tilde{U}} &= \hat{U} + \varphi^A_\alpha B^\alpha \hat{P}_A - \frac{i\hbar}{2}\varphi^A_{\alpha,A} B^\alpha + \frac{i\hbar}{2} C^\alpha_{\gamma\alpha} B^\gamma \\ &\qquad + [B^\beta_{,A}\varphi^A_\alpha + C^\beta_{\gamma\alpha}B^\gamma]\eta^\alpha\hat{\rho}_\beta \\ &= \hat{U} + \frac{i}{\hbar}[B^\alpha\hat{\rho}_\alpha, \hat{\Omega}],\end{aligned} \tag{3.102}$$

which proves the weak invariance of (3.101).

To prove the rescaling invariance of (3.101) observe that,

$$\begin{aligned}\hat{R}\hat{U}\hat{R}^{-1} &= \frac{1}{2}\{U^A[\hat{P}_A + \frac{1}{2}(\Lambda^{-1})^\beta_{\alpha,A}\Lambda^\alpha_\gamma(\eta^\gamma\hat{\rho}_\beta - \hat{\rho}_\beta\eta^\gamma)] \\ &\qquad + [\hat{P}_A + \frac{1}{2}(\Lambda^{-1})^\beta_{\alpha,A}\Lambda^\alpha_\gamma(\eta^\gamma\hat{\rho}_\beta - \hat{\rho}_\beta\eta^\gamma)]U^A\} \\ &\qquad + \frac{1}{2}U^\beta_\alpha\Lambda^\alpha_\gamma(\Lambda^{-1})^\delta_\beta(\eta^\gamma\hat{\rho}_\delta - \hat{\rho}_\delta\eta^\gamma). \\ &= \frac{1}{2}[U^A, \hat{P}_A]_+ + \\ &\qquad \frac{1}{2}\{U^\beta_\alpha\Lambda^\alpha_\gamma(\Lambda^{-1})^\delta_\beta + U^A\Lambda^\alpha_\gamma(\Lambda^{-1})^\delta_{\alpha,A}\}(\eta^\gamma\hat{\rho}_\delta - \hat{\rho}_\delta\eta^\gamma).\end{aligned}$$

This last expression should be compared with the quantisation of the classically rescaled **U** which is,

$$\tilde{U} = U + \{U^\beta_\alpha\Lambda^\alpha_\gamma(\Lambda^{-1})^\delta_\beta + U^A\Lambda^\alpha_\gamma(\Lambda^{-1})^\delta_{\alpha,A}\}\eta^\gamma\rho_\delta.$$

and it follows straight forwardly that,

$$\hat{\mathbf{R}}\hat{\mathbf{U}}\hat{\mathbf{R}}^{-1} = \hat{\mathbf{U}},$$

which proves the rescaling invariance of expression (3.101).

Now that we know (3.101) is invariant under all relevant transformations we need only prove the remaining results in the case of trivialised constraints. The results will then automatically hold for the general case.

When the constraints are trivial expression (3.101) takes the form,

$$\hat{U} = \frac{1}{2}[\, U^A, \hat{P}_A\,]_+ + \frac{1}{2} U^{\beta}{}_{.\alpha}(\eta^{\alpha}\hat{\rho}_{\beta} - \hat{\rho}_{\beta}\eta^{\alpha}), \tag{3.103}$$

where it should be remembered that, for trivial constraints, the coordinates on Q break naturally into Q^{α} and Q^a, the gauge and physical directions respectively. The trick we now employ is to notice that (3.103) can be rewritten in the form,

$$\hat{U} = \frac{1}{2}[\, U^a, \hat{P}_a\,]_+ + \frac{i}{\hbar}[\, U^{\alpha}\hat{\rho}_{\alpha}, \hat{\Omega}\,]\,. \tag{3.104}$$

From this equation we can read off the remaining properties of $\hat{U}$ that we desire to prove. The BRST charge trivially commutes with the first term of (3.104) and, because the BRST charge is nilpotent, it also commutes with the second term.

The operator (3.104) will preserve the self duality condition. The first term clearly preserves the duality condition and the second term contributes a coboundary which can be ignored (this coboundary is of the form $\hat{\Omega}\chi$ with χ vanishing at infinity so there are no potential problems that we need worry about).

The only remaining observation to make is that (3.104) differs only by a coboundary from the physical operator (3.18). Hence the operator (3.104) gives the correct physical results.

3.8.3 Quadratic Observables

So far we have managed to solve the ordering ambiguities without having to invoke rescaling invariance. We have only used rescaling invariance to check that the ordering behave as required. For quadratic observables we will see that things are nowhere near as simple. It will be shown that rescaling invariance forces the addition of $\hbar$ corrections to the obvious orderings.We will then see that these $\hbar$ corrections are essential to getting the correct physical results. The calculations below are not technically difficult but they are long, messy and not very illuminating so most of the intermediate steps have not been given.

Let $K = K^{AB}(Q^C)P_A P_B$ be a general, gauge invariant quadratic observable. Then it satisfies,

$$\{ K, \varphi_\alpha \} = K_\alpha{}^\beta \varphi_\beta. \tag{3.105}$$

Unlike linear observables $K_\alpha{}^\beta$ has momentum dependence. In fact, it is linear in momenta and so can be written as,

$$K_\alpha{}^\beta = K_\alpha{}^{\beta A}(Q^B) P_A. \tag{3.106}$$

It is important to note that $K_\alpha{}^{\beta A}$ is not unique. One can always add to $K_\alpha{}^\beta$ a term of the form $C_\alpha{}^{\beta\gamma} \varphi_\gamma$ and, provided C is antisymmetric in its top two indices, the new $K_\alpha{}^\beta$ will still satisfy (3.105). We did not have to worry about this for linear or configuration observables because such transformations would then have altered the momentum dependence of the structure functions. In the present case, provided C is a function of the Q^A only, we will not alter the momentum dependence of $K_\alpha{}^\beta$ so we must consider such transformations, and we must insure that the quantum ordering will work for any choice of $K_\alpha{}^\beta$. The BRST extension of K is,

$$\mathbf{K} = K + K_\alpha{}^\beta \eta^\alpha \rho_\beta + K_{\alpha\beta}{}^{\gamma\delta} \eta^\alpha \eta^\beta \rho_\gamma \rho_\delta. \tag{3.107}$$

The $K_{\alpha\beta}{}^{\gamma\delta}$ are functions of the Q^A only and are defined by,

$$\begin{aligned} &\{ K, C^\gamma{}_{\alpha\beta} \} - \{ K_\alpha{}^\gamma, \varphi_\beta \} + \{ K_\beta{}^\gamma, \varphi_\alpha \} + \\ &\quad C^\gamma{}_{\beta\varepsilon} K_\alpha{}^\varepsilon + C^\gamma{}_{\varepsilon\alpha} K_\beta{}^\varepsilon + C^\varepsilon{}_{\alpha\beta} K_\varepsilon{}^\gamma = K_{\alpha\beta}{}^{\gamma\delta} \varphi_\delta . \end{aligned} \tag{3.108}$$

The details of this can be found in [29]. Once $K_\alpha{}^\beta$ has been chosen there is no ambiguity in $K_{\alpha\beta}{}^{\gamma\delta}$ because we insist that it should depend on the Q^A only. However, if we change $K_\alpha{}^\beta$ we will also have to adjust $K_{\alpha\beta}{}^{\gamma\delta}$ to keep $\{\mathbf{K},\Omega\} = 0$.

We must now parametrise all the possible orderings of $\hat{\mathbf{K}}$. Care must be taken in doing this because an ordering of $\hat{K}$ of the form,

$$\varepsilon_1 K^{AB} \hat{P}_A \hat{P}_B + \varepsilon_2 \hat{P}_A K^{AB} \hat{P}_B + \varepsilon_3 \hat{P}_A \hat{P}_B K^{AB},$$

is not going to produce a coordinate covariant expression. This is similar to the situation in physical quantisation where one uses the Laplace-Beltrami operator rather than an ordering of the form above. Having examined the form of the Laplace-Beltrami ordering (3.19) and noticed that, in our situation, the $\|\varphi\|^2$ has the analogous role to $|g|$ one is naturally lead to the following conjecture,

$$\hat{K} = \|\varphi\|^{-\frac{1}{2}} \hat{P}_A K^{AB} \|\varphi\| \hat{P}_B \|\varphi\|^{-\frac{1}{2}}. \tag{3.109}$$

We will take this as a basic assumption as there seems to be no obvious way of proving (3.109) from more basic principles. This means that the obvious orderings to allow for (3.107) are of the form,

$$\hat{K} = \|\varphi\|^{-\frac{1}{2}} \hat{P}_A K^{AB} \|\varphi\| \hat{P}_B \|\varphi\|^{-\frac{1}{2}}$$
$$+ (\varepsilon_1 K_\alpha^{\beta A} \hat{P}_A + \varepsilon_2 \hat{P}_A K_\alpha^{\beta A})(\lambda_1 \eta^\alpha \hat{p}_\beta - \lambda_2 \hat{p}_\beta \eta^\alpha)$$
$$+ K_{\alpha\beta}^{\gamma\delta} [\mu_1 \eta^\alpha \eta^\beta \hat{p}_\gamma \hat{p}_\delta - \mu_2 \eta^\alpha \hat{p}_\gamma \eta^\beta \hat{p}_\delta + \mu_3 \eta^\alpha \hat{p}_\gamma \hat{p}_\delta \eta^\beta$$
$$+ \mu_4 \hat{p}_\gamma \eta^\alpha \eta^\beta \hat{p}_\delta - \mu_5 \hat{p}_\gamma \eta^\alpha \hat{p}_\delta \eta^\beta + \mu_6 \hat{p}_\gamma \hat{p}_\delta \eta^\alpha \eta^\beta]. \qquad (3.110)$$

where the ε, λ and μ are real number parameters satisfying,

$$\sum_{j=1}^{2} \varepsilon_j = \sum_{j=1}^{2} \lambda_j = \sum_{j=1}^{6} \mu_j = 1. \qquad (3.111)$$

We will show that it is impossible to get a rescaling invariant ordering of the form (3.110) and will thus have to modify (3.110) but, for the moment, let us work with it as it is. It is possible to rewrite (3.110) in the form,

$$\hat{K} = \|\varphi\|^{-\frac{1}{2}} \hat{P}_A K^{AB} \|\varphi\| \hat{P}_B \|\varphi\|^{-\frac{1}{2}} + i\hbar\lambda_2 K_\alpha^{\alpha A} \hat{P}_A + \hbar^2 \varepsilon_2 \lambda_2 K_{\alpha\ ,A}^{\alpha A}$$
$$+ K_\alpha^{\beta A} \hat{P}_A \eta^\alpha \hat{p}_\beta - i\hbar\varepsilon_2 K_{\alpha\ ,A}^{\beta A} \eta^\alpha \hat{p}_\beta$$
$$+ K_{\alpha\beta}^{\gamma\delta} \eta^\alpha \eta^\beta \hat{p}_\gamma \hat{p}_\delta + i\hbar b_1 K_{\alpha\gamma}^{\gamma\beta} \eta^\alpha \hat{p}_\beta + \hbar^2 b_2 K_{\alpha\beta}^{\alpha\beta} \qquad (3.112)$$

where,

$$b_1 = \mu_2 + 2\,\mu_3 + 2\,\mu_4 + 3\,\mu_5 + 4\,\mu_6, \qquad (3.113)$$

and,

$$b_2 = \mu_5 + 2\,\mu_6. \tag{3.114}$$

The first thing to do is work out the adjoint of (3.112) and invoke the requirement of self adjointness. The details of this are easy and one arrives at the following result,

$$\hat{K}^{\dagger} - \hat{K} = i\hbar\,(2\lambda_2 - 1)K_{\alpha}^{\alpha A}\hat{P}_A + \hbar^2(\varepsilon_2 + \lambda_2 - 1)K_{\alpha}^{\alpha A}{}_{,A} - \hbar^2(2 - b_1)K_{\alpha\beta}{}^{\alpha\beta}$$

$$+\, i\hbar(1 - 2\varepsilon_2)K_{\alpha}^{\beta A}{}_{,A}\eta^{\alpha}\hat{\rho}_{\beta} + 2i\hbar(b_1 - 2)K_{\alpha\beta}{}^{\beta\delta}\eta^{\alpha}\hat{\rho}_{\delta}. \tag{3.115}$$

Thus, we can deduce that $\varepsilon_2 = \lambda_2 = 1/2$ and $b_1 = 2$. As with linear observables there is no point in trying to cancel the terms in (3.115) while keeping them nonzero. Any attempt to do this would fail because the cancellation would not be preserved by weak changes to K.

With the choice of parameters above (3.112) can be written in the form,

$$\hat{K} = \|\varphi\|^{-\frac{1}{2}}\hat{P}_A K^{AB}\|\varphi\|\,\hat{P}_B\|\varphi\|^{-\frac{1}{2}} + \frac{1}{4}[K_{\alpha}^{\beta A}, \hat{P}_A]_{+}(\eta^{\alpha}\hat{\rho}_{\beta} - \hat{\rho}_{\beta}\eta^{\alpha})$$

$$+ \frac{1}{2}K_{\alpha\beta}{}^{\gamma\delta}(\hat{\rho}_{\gamma}\eta^{\alpha}\eta^{\beta}\hat{\rho}_{\delta} - \hat{\rho}_{\delta}\eta^{\alpha}\eta^{\beta}\hat{\rho}_{\gamma}) + \hbar^2 b_2\,K_{\alpha\beta}{}^{\alpha\beta}. \tag{3.116}$$

The parameter b_2 is still free and it is easy to check that (3.116) is coordinate covariant for all choices of b_2. The next step is to investigate the rescaling invariance of (3.116) which is, unfortunately, extremely messy. After a considerable amount of algebra one arrives at the following result,

$$\hat{R}\hat{K}\hat{R}^{-1} - \hat{K} = \frac{\hbar^2}{2} K^{AB} \frac{|\Lambda|_{,A}}{|\Lambda|} \frac{\|\tilde{\varphi}\|_{,B}}{\|\tilde{\varphi}\|} + \frac{\hbar^2}{4} K_{\alpha}{}^{\alpha A} \frac{|\Lambda|_{,A}}{|\Lambda|}$$

$$+\frac{\hbar^2}{2}\Big[K^{AB}{}_{,A}\frac{|\Lambda|_{,B}}{|\Lambda|} + K^{AB}\frac{|\Lambda|_{,AB}}{|\Lambda|} - K^{AB}\frac{|\Lambda|_{,A}\,|\Lambda|_{,B}}{|\Lambda|^2}\Big]$$

$$-\frac{\hbar^2 b_2}{2}\Big[K_{\alpha}{}^{\alpha A}\frac{|\Lambda|_{,A}}{|\Lambda|} + K_{\alpha}{}^{\beta A}(\Lambda^{-1})_{\beta}{}^{\mu}{}_{,A}\Lambda_{\mu}{}^{\alpha} - K^{AB}\frac{|\Lambda|_{,A}\,|\Lambda|_{,B}}{|\Lambda|^2}$$

$$- K^{AB}(\Lambda^{-1})_{\alpha}{}^{\mu}{}_{,A}\Lambda_{\mu}{}^{\alpha}{}_{,B}\Big]. \tag{3.117}$$

In this expression $\tilde{K}$ is obtained by first rescaling the classical expression (3.107) and then ordering the result by the prescription (3.116). The $\|\tilde{\varphi}\|$ in (3.117) is constructed using the rescaled constraints and Λ is the rescaling matrix.

Observe that it is impossible to make the right hand side of (3.117) vanish by choosing b_2 in some special way. To see this remember that (3.117) must vanish for all possible choices of constraints and, for all possible weak changes to K. Also remember that it is possible to vary $K_{\alpha}{}^{\beta A}$ independently of K^{AB} which means that the terms in (3.117) which only involve $K_{\alpha}{}^{\beta A}$ must cancel independently of the other terms. However these other terms cannot vanish as the first term can be varied independently of the others by starting from a different set of constraints.

Thus, it has been shown that (3.116) cannot be rescaling invariant. Therefore we need to consider more general orderings than (3.110). There are many ways of generalising (3.110) but we shall try and choose the simplest one which we will take to be the addition of an $\hbar$ scalar correction i.e., the orderings we will allow are,

$$\hat{K} = (3.116) + \hbar^2 f(Q^A, \varphi_{\alpha}). \tag{3.118}$$

The notation means that f is a function on Q which depends on the particular choice of constraints. Using $\hbar^2$ rather than $\hbar$ in front of f is purely for convenience later. The idea now is to arrange for f to transform in such a way as to cancel the right hand side of (3.117) and make (3.118) rescaling invariant. It is also necessary for f to be coordinate covariant since (3.116) is already known to be coordinate covariant. To analyse the possible values of f let us rewrite (3.117) in the following way.

$$\hat{\mathbf{R}}\hat{\mathbf{K}}\hat{\mathbf{R}}^{-1} - \hat{\mathbf{K}} = \hbar^2 \Big\{ \frac{1}{4} \frac{(K_\alpha^{\alpha A}\|\varphi\|)_{,A}}{\|\varphi\|} + b_2 K_{\alpha\beta}{}^{\alpha\beta}$$

$$- \frac{1}{4} \frac{(\tilde{K}_\alpha^{\alpha A}\|\tilde{\varphi}\|)_{,A}}{\|\tilde{\varphi}\|} - b_2 \tilde{K}_{\alpha\beta}{}^{\alpha\beta} \Big\}, \tag{3.119}$$

where, as before, the symbols with a ~ on top are constructed using the rescaled constraints. The proof of (3.119) follows by direct substitution of the following, easily derived, expressions,

$$\tilde{K}_\alpha^{\beta A} = K_\gamma^{\delta A} \Lambda_\alpha^\gamma (\Lambda^{-1})_\delta^\beta + 2K^{AB}(\Lambda^{-1})_{\gamma,B}^\beta \Lambda_\alpha^\gamma , \tag{3.120}$$

and,

$$\tilde{K}_{\alpha\beta}{}^{\gamma\delta} = K_{\mu\sigma}{}^{\varepsilon\tau} \Lambda_\alpha^\mu \Lambda_\beta^\sigma (\Lambda^{-1})_\varepsilon^\gamma (\Lambda^{-1})_\tau^\delta - K_\mu^{\sigma A} (\Lambda^{-1})_{\varepsilon,A}^{[\gamma} (\Lambda^{-1})_\sigma^{\delta]} \Lambda_{[\alpha}^\varepsilon \Lambda_{\beta]}^\mu$$

$$- K^{AB} (\Lambda^{-1})_{\mu,A}^{[\gamma} (\Lambda^{-1})_{\sigma,B}^{\delta]} \Lambda_{[\alpha}^\mu \Lambda_{\beta]}^\sigma . \tag{3.121}$$

The form of (3.119) suggests the following value of f.

$$f = -\frac{1}{4} \frac{(K_\alpha^{\alpha A}\|\varphi\|)_{,A}}{\|\varphi\|} - b_2 K_{\alpha\beta}{}^{\alpha\beta} . \tag{3.122}$$

This expression is also coordinate covariant and so is definitely a reasonable choice for f. In fact, we are now in a position to state the following result. Any f which makes (3.118) coordinate and rescaling covariant must be of the form (3.122) plus a term which is a scalar under coordinate and rescaling transformations. We will show that the choice of f in (3.122) gives a $\hat{\mathbf{K}}$ which is invariant under weak changes to K and which commutes with the BRST charge. Thus, there is no need to add extra terms to (3.122) so, by Ockham's razor, we will use (3.122) from now on.

Substituting (3.122) into (3.118) gives the following ordering for $\hat{\mathbf{K}}$.

$$\hat{\mathbf{K}} = \|\varphi\|^{-\frac{1}{2}} \hat{P}_A K^{AB} \|\varphi\| \hat{P}_B \|\varphi\|^{-\frac{1}{2}} - \frac{\hbar^2}{4} \frac{(K_\alpha^{\alpha A}\|\varphi\|)_{,A}}{\|\varphi\|}$$
$$+ \frac{1}{4}[K_\alpha^{\beta A}, \hat{P}_A]_+ (\eta^\alpha \hat{\rho}_\beta - \hat{\rho}_\beta \eta^\alpha)$$
$$+ \frac{1}{2} K_{\alpha\beta}{}^{\gamma\delta} (\hat{\rho}_\gamma \eta^\alpha \eta^\beta \hat{\rho}_\delta - \hat{\rho}_\delta \eta^\alpha \eta^\beta \hat{\rho}_\gamma). \qquad (1.123)$$

It is possible to show, by direct calculation, that this expression commutes with the BRST charge and is invariant under weak changes to K. These calculations are very messy and fortunately we can exploit rescaling invariance to prove the results in a far simpler way.

Let us look at the case of trivialised constraints for which (3.107) can be written in the form,

$$\mathbf{K} = K + [2K^{a\beta}{}_{,\alpha} P_a + K^{\gamma\beta}{}_{,\alpha} P_\gamma] \eta^\alpha \rho_\beta. \qquad (3.124)$$

The quantum version of this is,

$$\hat{K} = |g|^{-\frac{1}{4}} \hat{P}_A K^{AB} |g|^{\frac{1}{2}} \hat{P}_B |g|^{-\frac{1}{4}} - \frac{\hbar^2}{4} \frac{(2K^{a\alpha}_{,\alpha} |g|^{\frac{1}{2}})_{,a}}{|g|^{\frac{1}{2}}}$$

$$- \frac{\hbar^2}{4} \frac{(K^{\beta\alpha}_{,\alpha} |g|^{\frac{1}{2}})_{,\beta}}{|g|^{\frac{1}{2}}}$$

$$+ \frac{1}{4} (2[K^{a\beta}_{,\alpha}, \hat{P}_a]_+ + [K^{\gamma\beta}_{,\alpha}, \hat{P}_\gamma]_+) (\eta^\alpha \hat{\rho}_\beta - \hat{\rho}_\beta \eta^\alpha). \qquad (3.125)$$

This expression can be written in the more useful form,

$$\hat{K} = |g|^{-\frac{1}{4}} \hat{P}_a K^{ab} |g|^{\frac{1}{2}} \hat{P}_b |g|^{-\frac{1}{4}}$$

$$+ \frac{i}{2\hbar} \left[(2[K^{a\beta}, \hat{P}_a]_+ + [K^{\gamma\beta}, \hat{P}_\gamma]_+) \hat{\rho}_\beta, \hat{\Omega} \right]. \qquad (3.126)$$

From this we can read off the remaining properties of $\hat{K}$ that have to be proved.

Firstly, (3.126) commutes with the BRST charge (to see this remember that the BRST charge has the simple form (3.75) for trivialised constraints) and therefore (3.123) commutes with the BRST charge. The operator (3.126) also preserves the self dual condition on states, up to ignorable states of the form $\hat{\Omega}\chi$.

The expression (3.126) differs from the physical operator (3.19) by a coboundary and hence the ordering (3.123) is equivalent to that of the physical quantum theory.

To establish the weak invariance of (3.123) it is sufficient to prove weak invariance for trivialised constraints. Therefore, let us consider a weak change to K i.e.,

$$K \rightarrow \tilde{K} = K + C^{\alpha}\varphi_{\alpha}, \tag{3.127}$$

where, to preserve the momentum dependence of K, C^{α} must be of the form $C^{\alpha A}P_A$. This weak transformation does not alter the physical part of K (i.e K^{ab}) and therefore the transformation will only change the second term in (3.126). As the second term in (3.126) is a coboundary it follows that the weak transformation alters the quantum operator by a coboundary. This is sufficient to prove the weak invariance of (3.125) and hence the weak invariance of (3.123).

It now only remains to show that (3.123) changes by a coboundary when one chooses a different choice of $K_{\alpha}{}^{\beta}$. Again it is sufficient to look at this in the trivialised case. The general form of (3.124) is,

$$\begin{aligned} \mathbf{K} = K + [\, 2K^{a\beta}{}_{,\alpha}P_a + K^{\gamma\beta}{}_{,\alpha} + E_{\alpha}{}^{\beta\gamma}P_{\gamma}\,]\,\eta^{\alpha}\rho_{\beta} \\ + \frac{1}{2}E_{\alpha}{}^{\delta\mu}{}_{,\beta}\,\eta^{\alpha}\eta^{\beta}\rho_{\mu}\rho_{\delta}, \end{aligned} \tag{3.128}$$

where $E_{\alpha}{}^{\beta\gamma}$ is an arbitrary function of the Q^A which is antisymmetric in its top two indices. When this expression is ordered according to (3.123) one can manipulate the operator into the form,

$$\begin{aligned} \hat{\mathbf{K}} = |g|^{-\frac{1}{4}}\hat{P}_a K^{ab}|g|^{\frac{1}{2}}\hat{P}_b|g|^{-\frac{1}{4}} \\ + \frac{i}{2\hbar}\big[(\, 2[\, K^{a\beta}, \hat{P}_a]_+ + [\, K^{\gamma\beta}, \hat{P}_{\gamma}]_+)\hat{\rho}_{\beta}, \hat{\Omega}\,\big] \\ - \frac{1}{4i\hbar}\Big[\, E_{\beta}^{\delta\gamma}(\hat{\rho}_{\gamma}\eta^{\beta}\hat{\rho}_{\delta} - \hat{\rho}_{\delta}\eta^{\beta}\hat{\rho}_{\gamma}), \hat{\Omega}\,\Big]. \end{aligned} \tag{3.129}$$

This proves that (3.123) changes only by a coboundary when $K_{\alpha}{}^{\beta}$ is altered.

We have now proved that the ordering (3.123) does everything required. It is invariant under all the relevant symmetries and it is equivalent to the physical operator (3.19).

We pointed out in section 3.2 that the operator (3.19) is sometimes supplemented by terms of the form $\zeta\hbar^2 R$ where R is the scalar curvature on Q_{phys}. For completeness we should now find a modified form of $\hat{K}$ which incorporates these scalar curvature terms. The phrase "we should find" is used because it is extremely difficult to explicitly find the modification to $\hat{K}$ that will do what we require. The author has failed to solve this problem though it is easy to prove that a suitable modification to $\hat{K}$ does exist. The proof goes as follows. In the case of trivial constraints we will use the operator,

$$\hat{K} = (3.126) + \zeta\hbar^2 R. \qquad (3.130)$$

For more general constraints we define $\hat{K}$ to be the rescaled version of (3.130). Similarly, for other coordinate systems we define $\hat{K}$ to be the coordinate transform of (3.130). These two sets of transformations commute with each other so the definition is consistent. In fact, rescaling transformations do not alter the second term in (3.130) so we can ignore them anyway.

The problem with this ordering prescription is that it can only be used if we know how to explicitly trivialise the constraints. To obtain a useful solution we must rewrite R in a form that does not require a knowledge of the trivialisation. This is the problem that the author has not been able to solved. The problem can be stated in the following geometrical way. Find a way of writing the function $\pi^* R: Q \rightarrow \mathbb{R}$ (π denotes the projection map from Q to Q_{phys} and π^* its pull back) in a way that only requires a knowledge of the constraints and does not require a knowledge of the projection π. $\|\varphi\|$ does not help in solving this problem as it depends only on the determinant of the physical metric whereas the scalar curvature can be changed without altering $|g|$. It is unlikely that we are going to stumble on the solution to this problem so some systematic approach is required. The only way forward that the author can see is to reexamine the proof of theorem 3.2 (this gave the geometrical interpretation of $\|\varphi\|$) in the hope that the techniques used there can be extended to construct the pull back of R. This, however, does not appear very hopeful because the proof of theorem 3.2 relies critically on two facts. Firstly that antisymmetric tensors are being used and secondly the ability to saturate constraints. None of these features are present here so some new insight is required. It should also be born in mind that the problem could be insoluble. There is no reason why $\pi^* R$ should be expressible in a way that only requires a knowledge of the constraints.

3.9 The Quantum Well of Orvieto

We have now completed our solution to quantising linear constraints. We will now finish this chapter by returning to the example discussed in the introduction and will present our quantisation of it.

Remember that this problem is described by $Q = \mathbb{R}^3$ and in cylindrical polar coordinates (R, Θ, Z) the hamiltonian is,

$$H = (P_R)^2 + \frac{1}{R^2}(P_\Theta)^2 + (P_Z)^2, \qquad (3.131)$$

and the one constraint is,

$$\varphi = P_Z - P_\Theta. \qquad (3.132)$$

The BFV formulation of this problem is,

$$\Omega = (P_Z - P_\Theta)\eta, \qquad (3.133)$$

and,

$$\mathbf{H} = \mathrm{H}. \tag{3.134}$$

To do the quantisation it is necessary to compute $\|\varphi\|$. This is straight forward and the result is,

$$\|\varphi\| = \frac{R}{[1+R^2]^{\frac{1}{2}}}. \tag{3.135}$$

The ordering prescriptions can now be applied and the results are,

$$\hat{\Omega} = -i\hbar\left[\partial_Z - \partial_\Theta\right]\eta, \tag{3.136}$$

and,

$$\hat{\mathbf{H}} = -\hbar^2\left(\frac{1}{R^2}\partial_\Theta{}^2 + \partial_Z{}^2 + \partial_R{}^2 + \frac{1}{R[1+R^2]}\partial_R\right). \tag{3.137}$$

We will choose the gauge fixing condition $\Theta = 0$ so that the self dual, BRST invariant states are of the form,

$$\Psi = \Psi_0(R, \Theta+Z) + \delta(\Theta)\,\Psi_0\eta. \tag{3.138}$$

We can explicitly check that we have the correct quantum theory by letting Ψ^1 and Ψ^2 be two states of the form (3.138) and observing that,

$$\langle\Psi^1|\hat{\mathbf{H}}|\Psi^2\rangle = -\hbar^2\int \Psi_0^1\left[\frac{\partial}{\partial R^2} + \left(1+\frac{1}{R^2}\right)\frac{\partial^2}{\partial(\Theta+Z)^2} + \right.$$

$$\left.\frac{1}{R(1+R^2)}\frac{\partial}{\partial R}\right]\Psi_0^2\frac{R}{(1+R^2)^{\frac{1}{2}}}\,dR\,d(\Theta+Z). \tag{3.139}$$

This result agrees with the pairing (1.14), and the hamiltonian (1.15), on the true degrees of freedom.

Chapter Four
Geometric Quantisation and Quadratic Constraints

4.1 Introduction

In this chapter we will attempt to extend the ideas of chapter three to more general constraints and, in particular, to those which depend quadratically on the phase space momenta. One way of doing this is to formulate the previous ideas in a manner which is invariant under all constraint rescalings, not just those which depend on the configuration space. If this can be done one could then rescale the quantum theory for trivial constraints to get a valid, local quantisation for any set of constraints.

We will show that it is possible, in principle, to formulate the contents of chapter three in a totally rescaling invariant way. However, this is very much an existence proof, there appears to be little hope of actually using this result for any real theory. To prove these results we will use the techniques of geometric quantisation.

There has recently been a few studies [11] of the uses of geometric quantisation for constraint systems and there appear to be a few problems in using this technique [12,13]. However, none of these authors used BFV techniques and we will show that the problems pointed out in [12,13] do not arise once ghosts are used. After this work was done [74] appeared and also discusses the BFV method using geometric quantisation.

We will conclude this chapter by discussing some further properties of quadratic constraints, but we will not be able to give a proper quantisation. We will begin by quickly summarising the ideas of geometric quantisation.

4.2 Geometric Quantisation

Geometric quantisation provides a neat way of quantising a classical system using the symplectic structures on the phase space. The standard references on the subject are [22,75]. The main advantages of geometric quantisation are that it is global and that it does not need the phase space to possess a cotangent bundle structure (i.e. there need not be a globally defined configuration space). This latter point makes geometric quantisation particularly suitable for studying quantum gravity as the true, physical phase space of General Relativity is not a cotangent bundle [31].

Geometric quantisation proceeds in two stages. The first step is known as prequantisation and this consists of forming a complex line bundle over the phase space P. The cross sections, Ψ, of this bundle form a complex vector space which is the state space for the prequantum theory. This state space can be given an inner product structure using the natural Liouville measure on P (to do this the line bundle also requires a hermitian form but we have not introduce this as we will soon simplify to a case where the hermitian form is unnecessary). The *prequantisation line bundle* is required to have a connection, ∇, whose curvature is related to the symplectic form, ω, on P via,

$$-i\hbar(\nabla_a\nabla_b - \nabla_b\nabla_a) = \omega_{ab}, \qquad (4.1)$$

where ω_{ab} are the components of ω (for this section we have temporarily abandoned the previous conventions on indices and lower case Latin indices will run over all the degrees of freedom on P). Not all phase spaces will admit such a bundle and even if they do it may not be unique [22].

The connection ∇ enables us to introduce a prequantum operator for every smooth function on P. To do this let $f \in C^\infty(P)$ and denote its Hamiltonian vector field by X_f. (Our convention for defining a hamiltonian vector field is,

$$\iota_{X_f}\omega = df. \tag{4.2}$$

where ι denotes contraction on the first index of ω.) The prequantum operator, O_f, associated with f is,

$$O_f = \frac{\hbar}{i} X_f^{\ b} \nabla_b + f , \tag{4.3}$$

where $X_f^{\ b}$ denotes the components of X_f in some coordinate system. Due to (4.1) this quantisation procedure obeys the Dirac quantisation rule i.e.,

$$[O_f, O_g] = i\hbar \, O_{\{f,g\}} . \tag{4.4}$$

This does not give a contradiction with the Van Hove theorem because the prequantum operators act reducibly on the prequantum Hilbert space.

The prequantum states depend on all the 2N variables that describe P and so they cannot be regarded as a viable choice for the correct quantum states. To get from the prequantum to the proper quantisation one introduces the second step in the geometric quantisation method. Basically we must eliminate half the variables on P and this is done by introducing a polarisation. A polarisation, Γ, is a choice of an N dimensional subspace to the tangent space of P at every point of P. These subspaces are required to be integrable and the symplectic form must vanish when restricted to Γ. This means that Γ foliates P with n dimensional Lagrangian submanifolds. The space of these Lagrangian submanifolds will be denoted by P/Γ and can be roughly thought of as the configuration space associated with Γ. The states in the full quantum theory are the elements of the prequantum state space that are constant along Γ i.e. the states Ψ which satisfy,

$$L_V \Psi = 0, \tag{4.5}$$

for every vector field V that is tangent to Γ. In this equation L denotes Lie differentiation. The states, Ψ, which satisfy (4.5) can be thought of as functions on P/Γ and by introducing an N form on P/Γ these states can be given an inner product structure. It can be shown that a different choice of N form will give an equivalent quantum theory [11] indeed, it is possible to formulate the quantisation without introducing any N form [22].

To complete the geometric quantisation procedure it is necessary to restrict the prequantum operators to act on the states which satisfy (4.5). This is easy for functions f whose prequantum operator preserve the polarisation (i.e. the prequantum operator acts on states satisfying (4.5) to give states which also satisfy (4.5)). One quantises these functions as,

$$\hat{f} = O_f + \frac{\hbar}{2i}(\text{div } X_f), \tag{4.6}$$

where div X_f is defined by,

$$L_{X_f} \mu = (\text{div } X_f)\, \mu, \tag{4.7}$$

where μ is the N form introduced on P/Γ. The $\hbar$ correction to the prequantum operator in (4.6) is analogous to the term involving the determinant of the metric in equation (3.14) (it is necessary to make $\hat{f}$ hermitian). This quantisation procedure obeys the Dirac corespondence rule.

It is usually necessary to quantise some functions whose prequantum operators do not preserve the polarisation (for example, the kinetic energy part of the hamiltonian does not preserve the Schrödinger polarisation). When the polarisation is not preserved it is not sufficient to work with some simple modification to the prequantum operator. One way of dealing with this situation is to use the method of Blattner, Kostant, and Sternberg [76,77,78]. This method does not always work and when it does it can fails to preserve the Dirac corespondence rule.

There are further technicalities in geometric quantisation such as the use of metaplectic corrections that we will not discuss here (see [22]).

4.3 Prequantisation with Ghosts

We will now attempt to write down an analogous version of the ideas in section 4.2 for the superphase space, SP, rather than P. Some of this has been done, in a slightly different context, by Kostant [50]. On SP we have a supersymplectic form $\omega = dQ^A dP_A + d\eta^\alpha d\rho_\alpha$ (our conventions for differential forms on supermanifolds are summarised in appendix one and follow that of [49] which the reader should consult for more details). The form ω enables us to define Hamiltonian vector fields on SP using the superphase space version of (4.2). We will normally work locally where the Hamiltonian vector field associated with a function **F** is,

$$X_{\mathbf{F}} = \frac{\partial F}{\partial P_A}\frac{\partial}{\partial Q^A} - \frac{\partial F}{\partial Q^A}\frac{\partial}{\partial P_A} - (-1)^{\mathbf{F}}\frac{\partial F}{\partial \rho_\alpha}\frac{\partial}{\partial \eta^\alpha} - (-1)^{\mathbf{F}}\frac{\partial F}{\partial \eta^\alpha}\frac{\partial}{\partial \rho_\alpha}\ . \tag{4.8}$$

These vector fields satisfy the graded identities,

$$X_{\mathbf{F}}\mathbf{G} = (-1)^{\mathbf{FG}}\{\ \mathbf{G}, \mathbf{F}\ \}, \tag{4.9}$$

and,

$$[\, X_{\mathbf{F}}, X_{\mathbf{G}} \,] = - X_{\{\mathbf{F},\, \mathbf{G}\}}. \qquad (4.10)$$

It should be remembered that, in (4.10), the commutator is actually an anticommutator if **F** and **G** are both fermionic.

As with normal geometric quantisation we must introduce a line bundle over SP. However, we want the prequantum states to be grassmann valued, rather than complex valued, so the typical fibre of the bundle must be a grassmann algebra rather than $\mathbb{C}$. The prequantum state space will be the sections of this grassmann bundle. This bundle must be equipped with a graded connection ∇ (see [49] for mathematical details) satisfying the analogous expression to (4.1) which is,

$$-i\hbar\, (\nabla_a \nabla_b - (-1)^{ab} \nabla_b \nabla_a) = \omega_{ab}. \qquad (4.11)$$

We have again temporarily abandoned our notational conventions and in (4.11) the lower case Latin indices run over all the degrees of freedom on SP. This is also true of the lower case Latin indices in the next equation. The connection ∇ enables us to introduce prequantum operators which are defined by,

$$O_{\mathbf{F}} = -i\hbar\, X^a_{\mathbf{F}} \nabla_a + \mathbf{F}\,. \qquad (4.12)$$

These prequantum operators satisfy the following hermiticity relation,

$$O^{\dagger}_{\mathbf{F}} = (-1)^{\frac{1}{2} g(g-1)}\, O_{\mathbf{F}}\,, \qquad (4.13)$$

where g is the ghost number of **F** (this equation assumes that the coefficient functions in the expansion of **F** in powers of the ghosts are all real). In particular, if **F** has ghost number zero or one, $O_{\mathbf{F}}$ is hermitian and so the prequantum BRST charge and prequantum physical observables are all hermitian.

The prequantum operators satisfy,

$$[\, O_{\mathbf{F}} \,, O_{\mathbf{G}} \,] \;=\; i\hbar \, O_{\{\mathbf{F},\mathbf{G}\}} \,. \tag{4.14}$$

This enables us to conclude that the prequantum theory will automatically satisfy the following consistency conditions.

$$O_{\mathbf{\Omega}}^{\;2} \;=\; 0, \tag{4.15}$$

and, for all physical observables **F**,

$$[\, O_{\mathbf{\Omega}}, O_{\mathbf{F}} \,] \;=\; 0. \tag{4.16}$$

It is worthwhile comparing these equations with the analogous equations that occur in geometric quantisation when ghosts are not used. To do this let us introduce the notation O_α to denote the prequantum operator associated with the constraint φ_α. The analogue of equation (4.15) is,

$$\begin{aligned} [\, O_\alpha \,, O_\beta \,] &= i\hbar \; O_{\{\varphi_\alpha,\, \varphi_\beta\}} \\ &= i\hbar \, O_{C^{\gamma}_{\alpha\beta}\,\varphi_\gamma} \\ &= i\hbar \, C^{\gamma}_{\alpha\beta} \, O_\gamma + \; i\hbar\varphi_\gamma \, O_{C^{\gamma}_{\alpha\beta}} \; - i\hbar \, C^{\gamma}_{\alpha\beta}\varphi_\gamma \,. \end{aligned} \tag{4.17}$$

Likewise, the analogue of equation (4.16) is,

$$[\, O_F , O_\alpha \,] = i\hbar F_\alpha^{\ \beta} O_\beta + i\hbar \varphi_\beta \, O_{F_\alpha^{\ \beta}} - i\hbar F_\alpha^{\ \beta} O_\beta, \qquad (4.18)$$

where F is a physical observable on P and $F_\alpha{}^\beta$ is defined by equation (2.6). It is clear from equation (4.17) that the prequantum constraints are not, in general, going to satisfy the Dirac consistency conditions. Equation (4.18) also indicates that the prequantum physical observables are not going to be consistently ordered with the operators O_α. These problems were pointed out in [12,13] and can cause considerable difficulties in using geometric quantisation for constrained systems.

In the light of these remarks we can see that equations (4.15 and 16) indicate the BFV formalism to be much better suited to the prequantisation of constrained systems than the Dirac method is.

Before proceeding further with the theory we can make use of the fact that SP is a cotangent bundle to simplify the prequantisation procedure. Firstly, the fact that SP is a cotangent bundle guarantees that it will admit a prequantisation line bundle and more over we can take this bundle to be SP×L where L denotes the grassmann algebra generated by the η^α and ρ_α. The proof of this fact is identical to the proof of the analogous result for standard bosonic geometric quantisation [22]. All we have to do is give a connection on the trivial bundle which satisfies the required condition on the curvature. This is easily done using a symplectic one form on SP i.e. a one form η such that $\omega = - d\eta$. Such a one form is guaranteed to exist because we are using the canonical symplectic form on SP, one possible choice is the canonical one form,

$$\eta = P_A \, dQ^A - \rho_\alpha \, d\eta^\alpha. \qquad (4.19)$$

Using η one can introduce the connection ∇ defined by,

$$\nabla_V \Psi = V(\Psi) - \frac{i}{\hbar} \eta(V) \Psi , \qquad (4.20)$$

where V is some vector on SP and Ψ is a grassmann valued function on SP (because the prequantisation bundle is trivial we can represent any section by a function and will do this from now on). Although we have introduced a specific one form it should be noted that the prequantisation is independent of this choice [22].

If the topology of the super configuration space is nontrivial and, in particular, if it is not simply connected there could exist alternative prequantisations which do not use the trivial bundle. We shall ignore this as the quantisation we give will be local.

Equations (4.12) and (4.20) give the prequantisation of the classical observables that we are going to use. These operators will act on grassmann valued functions on SP that are square integrable with respect to the Liouville pairing,

$$\langle \Psi_1 | \Psi_2 \rangle = \int \Psi_1^* \, \Psi_2 \, dQ^1 .. dP_N d\eta^1 .. \, d\rho_k . \qquad (4.21)$$

We will now show that the above prequantisation is invariant under arbitrary rescaling transformations. This proof follows the standard procedure in geometric quantisation for implementing a canonical transformation as a unitary prequantum transformation. Let us quickly review this procedure.

Suppose that on a phase space (P, ω) we have a canonical transformation C which is expressible as the exponential of an infinitesimal transformation (i.e. C lies in the component of the group of all canonical transformations which contains the identity). Modulo some technicalities any infinitesimal transformation can be generated by the hamiltonian vector field X_h of some function h. Let us form the prequantum operator O_h corresponding to h and then exponentiate $(i/\hbar)O_h$ to obtain the operator $\hat{C}$ (we need not worry about the fact that h is only defined up to the addition of a real number because these ambiguities only change $\hat{C}$ by a constant phase factor). To first order in h the following relationship holds,

$$(1 - \frac{i}{\hbar} O_h)\, O_f\, (1 + \frac{i}{\hbar} O_h) = 1 + O_{\{h,f\}}, \tag{4.22}$$

where f is an arbitrary real function on P. Upon exponentiation this gives us the result,

$$\hat{C}^{-1} O_f \hat{C} = O_{C^* f}, \tag{4.23}$$

where C^* denotes the pull back of C. Equation (4.23) guarantees that $\hat{C}$ acts on the prequantum state space in the manner required. $\hat{C}$ is also automatically unitary because O_h is hermitian.

We will now apply the above procedure to the rescaling transformations on the superphase space. The procedure will go through without any obstructions provided that an arbitrary rescaling can be obtained by exponentiating an infinitesimal one. Unfortunately, there is a potential problem here because an arbitrary invertible real rescaling matrix cannot be expressed as the exponential of another real matrix. Only matrices with positive determinant can be written as exp(M) for some real matrix M. We have to allow M to be complex if we are to obtain the negative determinant matrices. To examine this problem let $\Lambda_\alpha{}^\beta = \delta_\alpha{}^\beta + \varepsilon_\alpha{}^\beta$ be an infinitesimal rescaling transformation where ε is possibly complex. To first order in ε the canonical transformation on SP is,

$$\tilde{Q}^A = Q^A + \varepsilon_\alpha^{\;\beta,A}\,\eta^\alpha \rho_\beta\,, \tag{4.24a}$$

$$\tilde{P}_A = P_A - \varepsilon_{\alpha\;,A}^{\;\beta}\,\eta^\alpha \rho_\beta\,, \tag{4.24b}$$

$$\tilde{\eta}^\alpha = \eta^\alpha + \varepsilon_\beta^{\;\alpha}\eta^\beta - \varepsilon_\beta^{\;\gamma,\alpha}\,\eta^\beta \rho_\gamma\,, \tag{4.24c}$$

$$\tilde{\rho}_\alpha = \rho_\alpha - \varepsilon_\alpha^{\;\beta}\rho_\beta - \varepsilon_{\beta\;,\alpha}^{\;\gamma}\,\eta^\beta \rho_\gamma\,, \tag{4.24d}$$

where the notation being used for derivatives is the same as in equation (2.25) but now all derivatives are with respect to the old coordinates and ε depends only on the old coordinates.

The transformation (4.24) is generated by the hamiltonian vector field of the function $\varepsilon_\alpha^{\;\beta}\eta^\alpha\rho_\beta$ so we must form the prequantum operator, O_ε, associated with this function and then exponentiate $(i/\hbar)O_\varepsilon$ to obtain the prequantum rescaling operator $\hat{R}$. This operator will automatically satisfy the equation,

$$\hat{R}^{-1} O_{\mathbf{F}} \hat{R} = O_{\mathbf{R^*F}}\,, \tag{4.25}$$

which basically says that constraint rescaling commutes with the operation of prequantisation.

Unfortunately, because ε can be complex, it is not a priori obvious that $\hat{R}$ will be unitary. There is no problem for finite rescalings with positive determinant since these come from real infinitesimal transformations. To show that the finite rescalings with negative determinant also give a unitary $\hat{R}$ we will use a trick that was introduced in [59]. Basically we need to observe that any negative determinant matrix can be written as a positive determinant matrix times the matrix diag(-1, 1, .. ,1). Note also that diag(-1, 1, .., 1) can be written as the exponential of ε defined by,

$$\varepsilon = \mathrm{diag}\,(\,i\pi,\,1,\,..,\,1\,)\,. \tag{4.26}$$

Thus, if it can be shown that the operator $\exp\{\,i/\hbar\; O_\varepsilon\}$ is unitary for the specific ε in equation (4.26) it will follow that all finite, invertible rescalings become unitary transformations in the prequantum theory. Let ε be as in (4.26) and observe that,

$$\frac{i}{\hbar}\, O_\varepsilon = X_\varepsilon\,,$$

where X_ε is the hamiltonian vector field associated with ε. How X_ε exponentiates to give a real transformation namely, the rescaling given by the matrix diag (-1, 1, .. ,1) and so $(i/\hbar)O_\varepsilon$ exponentiates to give a unitary transformation. This completes the proof that the ghost prequantum theory is fully rescalings invariant.

4.4 Graded Polarisations

Having set up the prequantisation of ghost variables we would now like to see if the BFV formalism can also give a tractable way of introducing a polarisation and hence a full quantum theory.

The concept of a polarisation on SP is defined similarly to the bosonic case. A polarisation is a choice of (N,k) dimensional (N bosonic directions and k fermionic directions) subspace of the tangent space at every point of SP. These subspaces are required to be integrable and the supersymplectic form must vanish when restricted to these subspaces. Any wave function that is invariant along the directions of a given polarisation will depend on N even and k odd variables.

It is essential to ensure that the polarisation is *compatible* with the particular set of constraints being used. By this we mean that any wave function that is BRST invariant, and constant along the direction of the polarisation, must depend on only

N-k even variables. The question of compatibility of polarisations has been examined, for Dirac constraint quantisation, in [11] and we will not pursue it further here.

An interesting question to look at is the behaviour of polarisations under constraint rescalings. Because classical rescalings are canonical they will automatically transform a polarisation to another polarisation (this will not happen in the Dirac theory). This fact gives us a method of locally quantising any set of constraints. All we have to do is trivialise them and then quantise as in chapter three using the vertical polarisation (i.e. the polarisation which is equivalent to the Schrödinger picture). This trivialised quantisation can then be rescaled back to give a local quantisation of the original set of constraints. In the process of rescaling back we are going to transform the polarisation and mix the fermionic and bosonic directions. This suggests that the quantisation of nonlinear (in momenta) constraints may require the use of polarisations which genuinely mix the ghost directions and the physical directions.

There are a number of obvious criticism of the above local quantisation. Firstly, we cannot actually implement it without knowing how to trivialise the constraints. We do not know how to do this in the physically interesting cases. The second criticism is that for nonlinear constraints local trivialisation is almost certainly going to introduce much more severe global problems. It is all very well to say that, quadratic constraints can be made to look linear in a local region but this does not alter the fact that globally linear and quadratic constraints are different.

We should not be too critical of the above naive quantisation. It was only meant to indicate the sort of new features that could arise for nonlinear constraints. The main features seem to be the need for polarisations which mix the fermionic and bosonic directions and the possibility that polarisations will change when the constraints are rescaled.

4.5 Quadratic Constraints

In the discussion above we have made no restriction on the rescalings that will be used in the quantum theory. It may be that no restrictions are necessary but, since we restricted the rescalings in the linear case, it is natural to examine a restricted class of rescalings for quadratic constraints as well. The obvious question though is which class of rescaling should we use?

The most obvious way to proceed is to continue the philosophy of chapter three and restrict attention to configuration space dependent rescalings. It would then be hoped that, via such a rescaling, the constraints could be reduced to some standard, relatively simple quadratic form which would play the role of "trivialised constraints". This approach is based on the tacit assumption that the configuration space should still play the same fundamental role that it played for linear constraints. Unfortunately this assumption may not be valid because one cannot, for quadratic constraints, take the full configuration space and factor off the redundant degrees of freedom to get the true configuration space. Indeed, as we have already mentioned, the true phase space of general relativity is not a cotangent bundle and so has no globally defined configuration space.

There is another, more concrete, argument against using only configuration space dependent rescalings when treating quadratic constraints. This argument goes as follows.

Remember that when we demand constraint rescalings invariance we are basically saying that we should be free to use any set of constraints within a given class. Constraint rescaling invariance makes this possible because it shows that any two given equivalent sets of constraints give equivalent quantum theories. To insure that this last statement is correct it is essential that we can move between any two given sets of constraints using the allowed rescalings. That is, it is essential that the allowed rescalings act transitively on the chosen set of constraints.

It is now going to be shown, via. an example, that the set of configuration space dependent rescalings does not act transitively on the set of all equivalent quadratic constraints (i.e. all quadratic constraints which have an identical zero set). The example is as follows. Let the configuration space $Q = \mathbb{R}^2$ and let us start with the following quadratic constraint,

$$\varphi = (P_R)^2 + R\, P_R + 1. \tag{4.27}$$

In this equation we are using polar coordinates (R, Θ). The key observation is that this constraint does not have a solution for all values of (R, Θ). For φ to be zero it is necessary for $R \geq 2$. This is a generic feature of nonlinear constraints and it is a feature that occurs in physics so we cannot ignore it. For example, in general relativity the region on the configuration space permitted by the constraints is the region where the ADM mass [2] is positive. This is the well known positive mass theorem [79].

We will now exploit the fact that φ no solutions when $R < 2$ to construct a quadratic constraint φ' with identical zero set to (4.27). Let $B: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function which is nonzero only when $R \leq 1$ and we will use this to define φ' via.

$$\varphi' = (1 + B)(P_R)^2 + (R - B)P_R + 1. \tag{4.28}$$

The constraint φ' will have an identical zero set to φ provided B is chosen such that $(R - B)^2 - 4(1 - B) < 0$ in the region where $R < 1$.

The key point to notice is that φ' cannot be written in the form $\Lambda\varphi$ for any Λ that depends only on (R, Θ). In fact, to globally rescale between φ and φ', it is necessary to use a Λ which is a rational function of the momenta.

The above construction can easily be extended to theories where there is more than one constraint. To see this just note that there are more coefficients functions in the constraints than there are entries in a rescaling matrix.

In summary, the above example has shown that, if one works with the class of all quadratic constraints, it may not be sufficient to demand invariance under only the configuration space rescalings. It may be necessary to include some rescalings which depend rationally on the momenta. This observation could lead to some difficult problems because rescaling matrices with entries which are rational functions of the momenta are not going to be quantised in any simple way.

We will not discuss quadratic constraints further. We have only been able to illustrate some of the problems that can occur and more work is required before any proper quantisation can be done.

Chapter Five
Discussion of the Field Theory Case

> Unfortunately what is little recognised is that the most worthwhile scientific books are those in which the author clearly indicates what he does not know; for an author most hurts his readers by concealing difficulties.
>
> E. Galois

5.1 Introduction

In order to apply the work presented in the previous chapters to real physical theories it is essential to work with infinite dimensional phase spaces. In this chapter we will discuss some of the classical aspects of constraints in infinite dimensions. We will concentrate almost exclusively on Yang-Mills constraints as this is the simplest case and will begin the discussion by briefly setting up the relevant aspects of the Yang-Mills phase space.

5.2 The Yang-Mills Phase Space

An infinite dimensional manifold is a topological space that looks locally like some infinite dimensional vector space. This *model* vector space is normally a Banach space or, even better, a Hilbert space and a large part of the theory of finite dimensional manifolds can be extended to the infinite dimensional case (see for example [38]). It is sometimes necessary to use a model vector space which is only a locally convex topological vector space an example of this being the diffeomorphism group of a manifold [80]. However, for Yang-Mills theory it is possible to work with Hilbert manifolds and we shall concentrate on this simpler situation.

In order to ensure that the configuration space and the phase space of Yang-Mills theories are well behaved smooth manifolds modelled on a Hilbert space some care is needed. Firstly, we must specify the space-time that the Yang-Mills fields are defined on. We will concentrate on the case where space-time is $S^3 \times \mathbb{R}$ as this is well adapted to studying the canonical evolution and hence the constraints. More general space-times will not be discussed as we are only trying to illustrate the problems and questions that must be addressed for field theories. The case when space-time is $S^3 \times \mathbb{R}$ will be quite sufficient for this purpose.

We will take the gauge group to be $G = SU(N)$ (the previous use of N to denote the dimension of the configuration space should be forgotten) and the set of gauge fields on S^3 is most neatly defined as the space of smooth connections on a principle G bundle over S^3 (see e.g. [81]). We will denote this space of connections by **A** and, for simplicity, we will only work with connections on the trivial G bundle. **A** has a natural action under the infinite dimensional gauge group $\mathbf{G} = C^\infty(S^3 \to G)$. This gauge group action enables us to define the factor space **A/G** which represents the physical degrees of freedom for Yang-Mills theories.

From now on we shall assume that the center of G has been removed (i.e. we always work with G/(center G) rather than G) and that **A** has been restricted to the space of irreducible connections. With these precautions all the spaces **A**, **G** and **A/G** are smooth Hilbert manifolds [82,83]. In addition, the space **A** is a smooth principle bundle over the true degrees of freedom **A/G** and provided $N>1$ the bundle is nontrivial [82] (this is the Gribov ambiguity).

The phase spaces that we are interested in are $T^*\mathbf{A}$ and $T^*(\mathbf{A/G})$ where the L^2 dual has been used to define the cotangent bundle [31,84]. This choice of dual is fairly conventional as it gives a well posed Cauchy problem [31,84]. $T^*\mathbf{A}$ and $T^*(\mathbf{A/G})$ both come equipped with a canonical symplectic form which is weakly nondegenerate [84] (this means that the mapping induced by the symplectic form between tangent and cotangent vectors is one to one but not onto).

5.3 Constraint Rescaling in Yang-Mills Theories

The Yang-Mills constraints are,

$$\Pi_\alpha{}^0 = 0, \tag{5.1}$$

and,

$$\partial_i \Pi_\alpha{}^i - q f^\gamma{}_{\alpha\beta} A_i{}^\beta \Pi_\gamma{}^i = 0. \tag{5.2}$$

where the $A_i{}^\beta$ denote the gauge fields and the $\Pi_\alpha{}^i$ the canonically conjugate momentum fields in the phase space. The Greek indices represent gauge degrees of freedom and the i sum over the spacial directions. The $f^\gamma{}_{\alpha\beta}$ are the structure constants of the gauge group and q is a coupling constant. To simplify the notation we shall denote these constraints by $\varphi_\alpha(x)$ where α is a discrete index and $x \in S^3$.

We now wish to consider constraint rescalings of the $\varphi_\alpha(x)$. As in the case of finite dimensional linear constraints it is natural to restrict attention to rescalings which only depend on the configuration space degrees of freedom which, in this case, means the $A_i{}^\alpha(x)$. Within this class of rescalings there are many possibilities which do not occur in the finite dimensional case. For example, the constraint rescalings could be nonlocal in space, an example of this would be a rescaling of the form,

$$\tilde{\varphi}_\alpha(x) = \int_{S^3} \Lambda_\alpha^\beta(x, x') \, \varphi_\beta(x') \, dx', \tag{5.3}$$

where $\Lambda_\alpha{}^\beta(x,x')$ is chosen to make the transformation invertible. This should be compared with a spatially local rescaling transformation (we are in the rather unfortunate position of wanting to use the word "local" to mean two different things. On the one hand it means locally defined on S^3 and on the other it refers to some local region of the Yang-Mills phase space. It will normally be clear from the context which meaning is the correct meaning but if there is an ambiguity we will use the phrase "spatially local" to refer to locality on S^3) which would be of the form,

$$\tilde{\varphi}_\alpha (x) = \Lambda_\alpha^{\ \beta}(x)\, \varphi_\beta (x), \tag{5.4}$$

where $\Lambda_\alpha{}^\beta(x)$ is some invertible, locally defined functional of the gauge fields $A_i^\alpha(x)$.

In addition to the above rescalings we could consider the case where the matrix Λ is a linear differential operator on S^3. However, care would be required with these transformations if we are to be sure that they are invertible.

To decide which of the above classes of rescalings we should consider we have to first of all decide which class of constraints we want to work with. The most natural choice of constraints would be the sets of spatially local functionals of the $A_i^\alpha(x)$ and $\Pi_\alpha^{\ i}(x)$ which vanish only when the Yang-Mills constraints (5.1 and 2) vanish (we take it as read that all the constraint functionals are linear in the momenta). The restriction to local functionals is fairly standard in field theory; it is physically fairly reasonable and, at a more practical level, nonlocal constraints are going to be much harder to quantise.

We now must find out which set of constraint rescalings are sufficient to generate the local constraints. That is, which set of rescalings preserve the locality of the constraints and act transitively on the set of all possible local constraints. If we can find such a set of rescalings and insure that the quantisation is invariant under these transformations then hopefully (by analogy with the finite dimensional case) we will have the correct physical quantisation.

Clearly, the nonlocal rescalings of the form (5.3) are not going to preserve locality of the constraints so we will restrict attention to rescalings of the form (5.4) with the possible addition of rescalings where Λ is a local, linear differential operator. It is not clear that this set of rescalings will act transitively on the set of all local constraints. The basic problem is that the finite dimensional result theorem 2.1 may not generalise to the field theory case. If this theorem is not true then there could exist two perfectly valid sets of local constraints which are not related by any linear rescaling even in the neighbourhood of the constrained surface. To overcome this and still have a transitive set of rescalings it may be necessary to investigate nonlinear transformations of the constraints. This is obviously going to be difficult and will pose considerable additional problems in the quantisation. We will not explore this further here.

We will now address the question of constraint trivialisation for Yang-Mills theories. Basically we wish to know if there exists local constraints which commute with respect to the Poisson bracket in some local region of the phase space. The answer is that there do exist such sets of constraints and we will now prove this.

Let us choose a particular point p in the space **A** and let us use the local triviality of **A**, regarded as a bundle over **A/G**, to introduce a diffeomorphism $\vartheta: U \to B_1 x B_2$ where U is an open neighbourhood of p and B_1, B_2 are Banach spaces with the property that B_2 corresponds to the gauge directions on **A** and B_1 to the physical directions. It is always possible to choose ϑ in a spatially local way. This follows because we can always find a spatially local trivialisation of the bundle **A** by using a standard gauge fixing condition such as the Coulomb gauge (Such a gauge fixing gives a section of the bundle **A**, defined on some local neighbourhood of the base space. This section then gives a trivialisation of the bundle over that neighbourhood since **A** is a principle bundle).

We can now use ϑ to construct a set of vector fields on U which span the gauge orbits and commute with respect to the Lie bracket on **A**. To do this let us choose a Hamel basis v_i (i a member of some infinite index set I) for the Banach space B_2 (the axiom of choice guarantees that it is always possible to choose such a basis within conventional set theory). Let p' be a point in U and let us define the vectors V_i at p' to be the tangent vectors at t=0 to the curves C_i: $(-\varepsilon, \varepsilon) \rightarrow \mathbf{A}$ ($\varepsilon \in \mathbb{R}$) defined by,

$$C_i(t) = \vartheta^{-1}(\vartheta(p') + t\, v_i).$$

The vector fields V_i are automatically smooth and spatially local on U because ϑ is. In addition it is straight forward to show that the V_i commute with respect to the Lie bracket on **A**.

We can now define the set of constraints φ_i on $T^*\mathbf{A}$ by $\varphi_i(p, \sigma) = \sigma(V_i)$ where $p \in$ **A** and σ is a one form on **A** at p. The φ_i are the set of constraints that we wished to construct. They Poisson commute because the V_i commute with respect to Lie brackets and they are, by construction, local functionals of the gauge fields and their conjugate momenta. Finally, the φ_i are a set of constraints, i.e. they vanish only when (5.1 and 2) vanish, because the V_i span the gauge directions on **A**.

Having established the existence of a locally abelian set of constraints it is natural to ask is if there are any obstructions to globally abelianising the Yang-Mills constraints. The global obstructions that exist on **A** normally arises from the fact that **A** is a nontrivial principle bundle over **A**/**G**, for example this is the origin of the Gribov ambiguity. The author has not been able to construct a proof of the nonglobal abelianisability of Yang-Mills constraints using the bundle properties of **A**. A sketch of an attempted proof will now be given to indicate where the problems occur.

Suppose we have a globally trivial set of constraints φ_α and let us return to the finite dimensional case to make things easier. These abelian constraints enable us to introduce a set of globally commuting, linearly independent vector fields, V_α, on the configuration space, Q, which span the gauge directions (i.e. the the vector fields span the fibres of Q regarded as a bundle over Q_{phys}). We will now try and use the

V_α to contradict the result [82] that the principle bundle Q does not admit a globally defined flat connection.

The V_α can be used to construct a *connection form* σ (we shall see shortly that σ is not actually a connection form). Let $p \in Q$ and, because the V_α commute there exists a coordinate system around p of the form (q^a, q^α) such that $\partial_\alpha = V_\alpha$. Any vector U at p can be written in the form $U = U^a \partial_a + U^\alpha V_\alpha$ and we can define the one form σ by,

$$\sigma(U) = U^\alpha \lambda_\alpha,$$

where λ_α is a basis of the Lie algebra of the gauge group. It is easy to show that σ is well defined in that it does not depend on the specific choice of coordinate system (q^a, q^α).

The curvature of the one form σ is zero because the V_α commute. Unfortunately we cannot conclude from this that we have a flat connection because a connection is required to be invariant under the group action along the fibres. The form σ will only be invariant under this action if the V_α are and this will only occur if the constraints are invariant under the gauge group. This is the problem that has prevented the author from showing that Yang-Mills constraints cannot be globally abelianised. There is no reason why any given set of constraints have to be gauge invariant. For example, if one does a gauge transformation to the nonabelian Gauss law (5.2) one does not recover the nonabelian Gauss, instead one gets a rescaled version of the constraint.

This concludes the discussion of constraint rescalings for field theories. In the concluding chapter we will briefly discuss how one could attempt to implement our ideas in quantum field theory.

Chapter Six
Discussion and Conclusions

The main results of this thesis have been proven for finite dimensional gauge theories. For these relatively simple theories a number of advantages of the BFV techniques over the traditional Dirac constraint quantisation procedure have been found. The most obvious advantage is that BFV quantisation retains a Hilbert space structure on the extended state space whereas rescaling invariance forces the Dirac approach to have an inner product structure on the physical states only. Another advantage of the BFV method is that it has enabled us to show that demanding invariance of the quantum theory under point transformation, weak changes to observables and constraint rescalings is sufficient to fix all the ordering ambiguities. In addition, the unique quantum theory that the symmetries permit is exactly the one which incorporates all the local curvature properties of Q_{phys}. To prove this result we have made use of the reasonably simple transformation (3.53) for constraint rescaling. In the Dirac approach the transformation analogous to (3.53) is much more complex which makes it much harder to establish the above results.

All the advantages of the BFV method stem from the fact that classically ghost variables give a formulation of constrained theories that is consistent with the symplectic structure of SP. In particular all the relevant symmetries are canonical transformations which become unitary transformations in the quantum theory. There is an interesting analogy between the BFV method and the use of complex numbers in classical physics. In both cases one introduces some extra unphysical degrees of freedom with unfamiliar mathematical properties. These extra degrees of freedom give one the ability to construct a more tractable mathematical formalism but the extra degrees of freedom never appear in the final answers.

The main criticism of the work presented in chapter three is that it is local and ignores any topological properties of Q_{phys}. The assumption of locality was used firstly to insure that the constraints could be abelianised and secondly to insure that the quantum BRST charge picks out the correct physical states (Poincare's lemma was used to show that there are no nontrivial BRST invariant states with nonzero ghost number). It remains an interesting open question to see if ghosts incorporate the global properties of Q_{phys} as efficiently as they incorporate the local curvature. It is not unreasonable to expect a global BFV quantisation to work as the classical BFV method has a strong topological origin (see section 2.5). At a more practical level the neglect of topological structures means that our quantum theory, if applied to a more realistic field theory, will not be able to examine any nonperturbative properties and could only be expected to work for perturbation theory involving small changes to the fields.

When the work of chapter three was first presented [65,66,67] the operator orderings were constructed by first abelianising the constraints and quantising this simpler problem. This procedure is always going to be limited to local quantisation and one of the improvements that has been made in this thesis is to derive the operator orderings using only symmetry arguments. The quantisation is still local but at least the abelianisation step is only used as a tool to examine the properties of the quantum theory. This gives one hope that the abelianisation step may be unnecessary and could be removed. The author feels that the results of chapter three strongly indicate that constraint rescaling invariance is an essential tool in constraint quantisation and this invariance principle should certainly play a role in any attempted global quantisation.

Probably the most important question to examine is the relevance of the results in chapter three to real gauge field theories. It is certainly true that none of the main features of our quantisation such as constraint rescaling, the use of the $\|\varphi\|$ measure and the Hamiltonian ordering (3.123) have ever appeared in e.g QCD. Indeed, the normal quantisation of gauge field theories looks to have more in common with the naive quantisation of the Quantum Well problem than with what we now believe to be the correct quantisation. It is also important to remember that the true degrees of freedom of nonabelian gauge theories do possess nontrivial curvature [82] so that the problems arising in the Quantum Well are possibly present in real physical theories. It therefore appears that an examination of our work in field theory is very important though the present author has not pursued these questions in any detail. The standard complications of quantum field theory such as renormalisation and anomalies may prevent the methods from working. However, the author feels that some aspects of the work presented here should be relevant to real physics.

There are a number of possible ways of making progress in the field theory case. One possibility would be to use the Vilkovisky-DeWitt ideas on the effective action [85]. This approach is particularly attractive because the Vilkovisky-DeWitt effective action is normally studied using geometrical principles and pays particular attention to the symmetries of the theory.

An alternative way of approaching the field theory case would be to examine lattice gauge theory rather than the continuum version. There are versions of lattice gauge theory which discretise space but leave time continuous [86] and this effectively approximates the Yang-Mills phase space by a finite dimensional manifold. Obviously these lattice approaches lose some of the structure of the continuum theory but they do give a more tractable starting point.

Both of the above approaches to field theories would be aided if one could reformulate the quantum theory of this thesis in a path integral form. This is also an interesting question in its own right and the author has spent some time thinking about it but has, unfortunately, failed to make any significant progress.

Appendix One
Summary of Conventions on Grassmann Variables and Supermanifolds

The purpose of this appendix is to summarise our conventions on Grassmann variables and supermanifolds.

We shall always work with right grassmann derivatives [48] which are defined via,

$$\frac{\partial \eta^{\alpha}}{\partial \eta^{\beta}} = \delta_{\beta}{}^{\alpha}, \tag{A1.1}$$

and,

$$\frac{\partial}{\partial \eta^{\alpha}} (\mathbf{F}_1 \mathbf{F}_2) = \frac{\partial \mathbf{F}_1}{\partial \eta^{\alpha}} \mathbf{F}_2 + (-1)^{\varepsilon_{F_1}} \mathbf{F}_1 \frac{\partial \mathbf{F}_2}{\partial \eta^{\alpha}} . \tag{A1.2}$$

If **F** is a grassmann object ε_F is 0 or 1 according as **F** is even or odd.

Grassmann integration is defined using the standard Berezin measure [48] which is,

$$\int d\eta^{\alpha} = 0. \tag{A1.3}$$

and,

$$\int \eta^{\alpha} d\eta^{\beta} = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta \end{cases} . \tag{A1.4}$$

The conventions we use for tensors on supermanifolds are the same as [49]. The most important type of tensor is a super differential form which is defined as a totally antisymmetric tensor. Two objects are said to be antisymmetric if,

$$F_1 F_2 = (-1)^{\varepsilon_{F_1} \varepsilon_{F_2}} F_2 F_1 . \qquad (A1.5)$$

When two super differential forms are multiplied together it will be taken for granted that the product is the wedge product. The mixed grading between fermionic and differential form structure is such that, if τ (respectively μ) is a graded m (respectively n) form then $\tau\mu = (-1)^{nm}(-1)^{\tau\mu}\mu\tau$. We will always use the right exterior derivative which, for a supermanifold with coordinates x^A is defined by,

$$dF = dx^A \frac{\partial F}{\partial x^A} , \qquad (A1.6)$$

and,

$$d(\tau\mu) = (d\tau)\mu + (-1)^m \tau(d\mu), \qquad (A1.7)$$

where **F** is a function on the supermanifold and τ, μ are differential forms with τ an m form. The one forms dx^A in equation (A1.6) are defined by the fact that, acting on an arbitrary vector $\mathbf{V} = V^A \, \partial/\partial x^A$, they give,

$$dx^A(V) = (-1)^{VA} V^A. \qquad (A1.8)$$

Our main application of super differential forms will be to the superphase space SP introduced in chapter two. Let $(Q^A, P_A, \eta^\alpha, \rho_\alpha)$ be a canonical coordinate system on this space i.e., the super symplectic form, ω, is expressible in this coordinate system as,

$$\omega = dQ^A dP_A + d\eta^\alpha d\rho_\alpha . \qquad (A1.9)$$

This symplectic form defines the superpoisson brackets as follows. Let $\mathbf{B}_1$, $\mathbf{B}_2$ denote even functions on SP and $\mathbf{F}_1$, $\mathbf{F}_2$ odd functions on SP then,

$$\{\mathbf{B}_1, \mathbf{B}_2\} = - \{\mathbf{B}_2, \mathbf{B}_1\} = \frac{\partial \mathbf{B}_1}{\partial Q^A}\frac{\partial \mathbf{B}_2}{\partial P_A} - \frac{\partial \mathbf{B}_2}{\partial Q^A}\frac{\partial \mathbf{B}_1}{\partial P_A} +$$

$$\frac{\partial \mathbf{B}_1}{\partial \eta^\alpha}\frac{\partial \mathbf{B}_2}{\partial \rho_\alpha} - \frac{\partial \mathbf{B}_2}{\partial \eta^\alpha}\frac{\partial \mathbf{B}_1}{\partial \rho_\alpha}. \qquad \text{(A1.10)}$$

$$\{\mathbf{F}_1, \mathbf{B}_1\} = - \{\mathbf{B}_1, \mathbf{F}_1\} = \frac{\partial \mathbf{F}_1}{\partial Q^A}\frac{\partial \mathbf{B}_1}{\partial P_A} - \frac{\partial \mathbf{B}_1}{\partial Q^A}\frac{\partial \mathbf{F}_1}{\partial P_A} -$$

$$\frac{\partial \mathbf{F}_1}{\partial \eta^\alpha}\frac{\partial \mathbf{B}_1}{\partial \rho_\alpha} - \frac{\partial \mathbf{B}_1}{\partial \eta^\alpha}\frac{\partial \mathbf{F}_1}{\partial \rho_\alpha}. \qquad \text{(A1.11)}$$

$$\{\mathbf{F}_1, \mathbf{F}_2\} = \{\mathbf{F}_2, \mathbf{F}_1\} = \frac{\partial \mathbf{F}_1}{\partial Q^A}\frac{\partial \mathbf{F}_2}{\partial P_A} + \frac{\partial \mathbf{F}_2}{\partial Q^A}\frac{\partial \mathbf{F}_1}{\partial P_A} -$$

$$\frac{\partial \mathbf{F}_1}{\partial \eta^\alpha}\frac{\partial \mathbf{F}_2}{\partial \rho_\alpha} - \frac{\partial \mathbf{F}_2}{\partial \eta^\alpha}\frac{\partial \mathbf{F}_1}{\partial \rho_\alpha}. \qquad \text{(A1.12)}$$

These superpoisson brackets satisfy modified versions of the standard identities for Poisson brackets (see appendix C of [6]).

Appendix Two
The Implicit Function Theorem for Superdifferentiable Functions

In this appendix we will sketch the proof of the superimplicit function theorem which justifies the use of equations (2.25). The theorem is an extension of the implicit function theorem for Banach spaces (see, for example, chapter six of [54]).

To the knowledge of the author the form of the super implicit function theorem that is going to be presented here has not been discussed in the literature. There is an implicit function theorem for graded manifolds (Throughout the thesis we have use the terms graded, grassmann, and super interchangeably but, for this appendix only, it should be realised that there is a difference between graded and super manifolds. Graded manifolds were introduced by Kostant in [50] where, rather than generalising the concept of a manifold, Kostant generalises the concept of a function or, more technically, he generalises the concept of a sheaf of smooth functions. A good review of the different approaches to super/graded manifolds is [52]). Kostant presented a superimplicit function theorem for graded manifolds in [50]. This theorem is stated using rather different concepts to the ones we are using, and it is not obvious to the author that Kostant's theorem is relevant to the present problem.

We will use the definition of superdifferentiation introduced by Rogers [51]. To give the definition we need the following notation. Let B_L be a grassmann algebra with L generators, and let $B_L{}^{m,n}$ be the Cartesian product of m copies of the even part of B_L with n copies of the odd part of B_L. These two spaces can be equipped with a norm and made into Banach spaces (see [51]). A function $f{:}U \rightarrow B_L$ (U an open set in $B_L{}^{m,n}$) is said to be G^1 at $(a,b)\in U$ (here we are regarding $B_L{}^{m,n}$ as $B_L{}^{m,0}xB_L{}^{0,n}$) if there exists m+n grassmann numbers $(G_af)(a,b)$ such that, if $(a+h,b+k)\in U$, then

$$\frac{\left\| f(a+h,b+k) - f(a,b) - \sum_{i=1}^{m} h_i(G_i f)(a,b) - \sum_{j=1}^{n} k_j(G_{j+m} f)(a,b) \right\|}{\left\| (h,k) \right\|} \to 0, \quad \text{(A2.1)}$$

as $\|(h,k)\| \to 0$. Higher derivatives, i.e. G^p functions, can be defined recursively as in normal differential calculus. Partial derivatives are also introduced in a manner similar to normal differential calculus. It is important to realise that a G^p function is also C^p [51]. For example, the Banach space derivative at (a,b) of the function f above is given by Df: $B_L{}^{m,n} \to B_L{}^{m,n}$ where,

$$Df(h,k) = \sum_{i=1}^{m} h_i(G_i f)(a,b) + \sum_{j=1}^{n} k_j(G_{j+m} f)(a,b). \quad \text{(A2.2)}$$

The only difference between C^1 and G^1 is that in the latter the derivative is linear with respect to grassmann scalars whereas, in the former, the derivative need only be linear with respect to real or complex scalars. With the above definitions the superimplicit function theorem that we require is as follows.

Theorem A2.1

Let U and V be open sets in $B_L{}^{m,n}$ and let f:UxV $\to$ $B_L{}^{m,n}$ be a G^p mapping. Let (a,b)$\in$ UxV be such that,

$$f(a,b) = 0. \quad \text{(A2.3)}$$

Assume also that, at (a,b), the partial derivative in the second coordinate is an isomorphism between $B_L{}^{m,n}$ and itself.

Then, there exists a G^p function $g:U_0 \to V$ defined on an open neighbourhood U_0 of a such that g(a) = b and,

$$f(x, g(x)) = 0. \tag{A2.4}$$

If U_0 is taken sufficiently small g is unique.

Sketch of Proof

The theorem is identical to the normal implicit function theorem except that we use G^p functions rather than C^p functions. However, we know that G^p functions are automatically C^p, so the theorem follows from the standard result except for the claim that g is G^p (the standard theorem only says that it is C^p). If one consults a proof of the standard implicit function theorem (e.g. [54]) it will be seen that, to prove g is G^p, it is sufficient to establish the following result which is essentially an inverse function theorem for G^1 functions.

If $f: B_L{}^{m,n} \to B_L{}^{m,n}$ is G^1 and Df is an isomorphism at (a,b) then the local inverse of f (f is locally invertible by the inverse function theorem) is also G^1 at f(a,b). To prove this note that if (A2.2) is an isomorphism each of the $(G_a f)(a,b)$ must have a multiplicative inverse which we will denote by $(G_a f)^{-1}(a,b)$. It then follows that, at f(a,b),

$$Df^{-1}(h,k) = \sum_{i=1}^{m} h_i (G_i f)^{-1}(a,b) + \sum_{j=1}^{n} k_j (G_{j+m} f)^{-1}(a,b). \tag{A2.5}$$

and so f^{-1} is G^1 at f(a,b). This completes the sketch proof of the theorem.

We now wish to show that the transformation (2.25) satisfies the conditions of theorem A2.1. Equations (2.25) can be recast into the required form by defining a mapping $f: B_L{}^{2N,2k} \times B_L{}^{2N,2k} \to B_L{}^{2N,2k}$ by,

$$f(Q^A, P_A, \eta^\alpha, \rho_\alpha, \tilde{Q}^A, \tilde{P}_A, \tilde{\eta}^\alpha, \tilde{\rho}_\alpha) = (\tilde{Q}^A - Q^A + (\Lambda^{-1})_\alpha^{\ \beta,A}\, \tilde{\eta}^\alpha \rho_\beta,$$
$$\tilde{P}_A - P_A - (\Lambda^{-1})_{\alpha\ ,\tilde{A}}^{\ \beta}\, \tilde{\eta}^\alpha \rho_\beta\ ,\ \tilde{\eta}^\alpha - (\Lambda)_\beta^{\ \alpha} \eta^\beta - (\Lambda)_\delta^{\ \alpha} (\Lambda^{-1})_\beta^{\ \gamma,\delta}\, \tilde{\eta}^\beta \rho_\gamma,$$
$$\tilde{\rho}_\alpha - (\Lambda^{-1})_\alpha^{\ \beta} \rho_\beta - (\Lambda^{-1})_{\beta\ ,\tilde{\alpha}}^{\ \gamma}\, \tilde{\eta}^\beta \rho_\gamma) \qquad \text{(A2.6)}$$

We must compute the matrix representing the partial derivative of this function, with respect to the ~ coordinates, and confirm that this matrix is invertible. At first sight this looks like a very messy calculation but fortunately it simplifies greatly because of the following result. A Grassmann matrix is invertible if and only if its body is invertible (the body of a matrix being the part left when all the Grassmann variables are set to zero) [49]. This means that, when computing the partial derivative of f, we need only retain the terms with no ghost dependence and the calculation now becomes trivial. The body of partial derivative matrix is, in fact, the identity matrix.

It only remains to show that, at an arbitrary point $(Q^A, P_A, \eta^\alpha, \rho_\alpha)$ of SP, there is a point $(\tilde{Q}^A, \tilde{P}_A, \tilde{\eta}^\alpha, \tilde{\rho}_\alpha)$ which makes the right hand side of (A2.6) zero. This can be shown by solving equations (2.25) iteratively obtaining the next highest ghost term at each stage. For example the first iterate would be,

$$\tilde{Q}^A = Q^A, \qquad \text{(A2.7a)}$$
$$\tilde{P}_A = P_A, \qquad \text{(A2.7b)}$$
$$\tilde{\eta}^\alpha = (\Lambda)_\beta^{\ \alpha} \eta^\beta, \qquad \text{(A2.7c)}$$
$$\tilde{\rho}_\alpha = (\Lambda^{-1})_\alpha^{\ \beta} \rho_\beta, \qquad \text{(A2.7d)}$$

where Λ and Λ^{-1} depend on the Q^A and P_A and not on the $\tilde{Q}^A$ and $\tilde{P}_A$. The second iterate would give the two ghost corrections to (A2.7a and b) and the three ghost corrections to (A2.7c and d). To confirm that this iterative construction of $(\tilde{Q}^A, \tilde{P}_A, \tilde{\eta}^\alpha, \tilde{\rho}_\alpha)$ will work it is only necessary to observe that, in equations (2.25), the next highest ghost terms are uniquely specified by the lower ghost terms. Unfortunately, this iterative solution of (2.25) has not been of much practical use because the higher ghost corrections rapidly become very complicated, and there appears to be no simple way of writing the full solution.

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