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INEQUALITIES AND ASYMPTOTIC EXPANSIONS RELATED TO THE VOLUME OF THE UNIT BALL IN $\mathbb{R}^n$

CHAO-PING CHEN* AND RICHARD B. PARIS

Abstract. Let $\Omega_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)\,(n \in \mathbb{N})$ denote the volume of the unit ball in $\mathbb{R}^n$. In this paper, we present asymptotic expansions and inequalities related to Ω_n and the quantities:

$$\frac{\Omega_{n-1}}{\Omega_n}, \quad \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} \quad \text{and} \quad \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}}.$$

1. Introduction

In the recent past, several researchers have established interesting properties of the volume Ω_n of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \qquad n \in \mathbb{N} := \{1, 2, \ldots\},$$

including monotonicity properties, inequalities and asymptotic expansions.

Böhm and Hertel [7, p. 264] pointed out that the sequence $\{\Omega_n\}_{n\geq 1}$ is not monotonic for $n \geq 1$. Indeed, we have

$$\Omega_n < \Omega_{n+1} \quad \text{if} \quad 1 \leq n \leq 4 \quad \text{and} \quad \Omega_n > \Omega_{n+1} \quad \text{if} \quad n \geq 5.$$

Anderson *et al.* [5] showed that $\{\Omega_n^{1/n}\}_{n\geq 1}$ is monotonically decreasing to zero, while Anderson and Qiu [4] proved that the sequence $\{\Omega_n^{1/(n\ln n)}\}_{n\geq 2}$ decreases to $e^{-1/2}$. Guo and Qi [14] proved that the sequence $\{\Omega_n^{1/(n\ln n)}\}_{n\geq 2}$ is logarithmically convex. Klain and Rota [16] proved that the sequence $\{n\Omega_n/\Omega_{n-1}\}_{n\geq 1}$ is increasing.

Diverse sharp inequalities for the volume of the unit ball in $\mathbb{R}^n$ have been established [2,3,6, 8,11,19,21–24,28]. For example, Alzer [2] proved that for $n \in \mathbb{N}$,

$$a_1\Omega_{n+1}^{n/(n+1)} \leq \Omega_n < b_1\Omega_{n+1}^{n/(n+1)}, \tag{1.1}$$

$$\sqrt{\frac{n+a_2}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b_2}{2\pi}}, \tag{1.2}$$

$$\left(1+\frac{1}{n}\right)^{a_3} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1+\frac{1}{n}\right)^{b_3}, \tag{1.3}$$

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with the best possible constants

$$a_1 = \frac{2}{\sqrt{\pi}} = 1.1283\ldots, \quad b_1 = \sqrt{e} = 1.6487\ldots,$$
$$a_2 = \frac{1}{2}, \quad b_2 = \frac{\pi}{2} - 1 = 0.5707\ldots,$$
$$a_3 = 2 - \frac{\ln \pi}{\ln 2} = 0.3485\ldots, \qquad b_3 = \frac{1}{2}.$$

The double inequality (1.2) refines a result due to Borgwardt [8, p. 253], who proved (1.2) with $a_2 = 0$ and $b_2 = 1$. Mortici [22, Theorem 3] obtained the following inequality:

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi n}}, \qquad n \in \mathbb{N}, \tag{1.4}$$

which improves the right-hand side of (1.2). Ban and Chen [6, Theorem 3.1] improved (1.4) and obtained the following double inequality:

$$\sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi(n+\vartheta_1)}} \le \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16\pi(n+\vartheta_2)}}, \qquad n \in \mathbb{N}, \tag{1.5}$$

with best possible constants

$$\vartheta_1 = \frac{13-4\pi}{4\pi - 12} = 0.7656283\ldots \quad \text{and} \quad \vartheta_2 = \frac{1}{2}.$$

Merkle [21] improved the left-hand side of (1.3) and obtained the following result:

$$\left(1 + \frac{1}{n+1}\right)^{1/2} \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}}, \qquad n \in \mathbb{N}. \tag{1.6}$$

Chen and Lin [11, Theorem 3.1] developed (1.6) to produce the following symmetric double inequality:

$$\left(1 + \frac{1}{n+1}\right)^{\alpha} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \le \left(1 + \frac{1}{n+1}\right)^{\beta}, \qquad n \in \mathbb{N}, \tag{1.7}$$

with the best possible constants

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2\ln 2 - \ln \pi}{\ln 3 - \ln 2} = 0.5957713\ldots,$$

Ban and Chen [6, Theorem 3.2] proved, for $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n+\theta_1}\right)^{1/2} \le \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n+\theta_2}\right)^{1/2}, \tag{1.8}$$

with best possible constants

$$\theta_1 = \frac{2\pi^2 - 16}{16 - \pi^2} = 0.60994576\ldots \quad \text{and} \quad \theta_2 = \frac{1}{2}.$$

Alzer [3] continued the work on this subject and offered new inequalities. For example, Alzer [3, Theorem 4] proved that for $n \ge 2$,

$$\frac{\alpha^*}{\sqrt{n}} \le \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \frac{\beta^*}{\sqrt{n}}, \tag{1.9}$$

with best possible constants

$$\alpha^* = \frac{3\sqrt{2}\pi}{6+4\pi} = 0.7178\ldots \quad \text{and} \quad \beta^* = \sqrt{2\pi} = 2.5066\ldots,$$

while Chen and Lin [11, Theorem 3.3] proved that for $n \in \mathbb{N}$,

$$\sqrt{\frac{2\pi}{n+a^*}} \le \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \sqrt{\frac{2\pi}{n+b^*}}, \tag{1.10}$$

with best possible constants

$$a^* = \frac{\pi(1+\pi)^2}{2} - 1 = 25.94353\ldots, \quad b^* = \frac{1}{2} + 4\pi = 13.06637\ldots.$$

Chen and Lin [11, Theorem 3.4] showed that for $n \in \mathbb{N}$,

$$\frac{1}{\sqrt{\pi(n+\theta)}}\left(\frac{2\pi e}{n}\right)^{n/2} \le \Omega_n < \frac{1}{\sqrt{\pi(n+\vartheta)}}\left(\frac{2\pi e}{n}\right)^{n/2}, \tag{1.11}$$

with best possible constants

$$\theta = \frac{e}{2} - 1 = 0.3591409\ldots, \quad \vartheta = \frac{1}{3}.$$

Recently, Mortici [24] constructed asymptotic series associated with some expressions involving the volume of the n-dimensional unit ball. New refinements and improvements of some old and recent inequalities for Ω_n are also presented. Lu and Zhang [19] established a general continued fraction approximation for the nth root of the volume of the unit n-dimensional ball, and then obtained related inequalities.

In this paper, we present asymptotic expansions and inequalities related to Ω_n and the quantities:

$$\frac{\Omega_{n-1}}{\Omega_n}, \quad \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} \quad \text{and} \quad \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}}.$$

The numerical values given in this paper have been calculated via the computer program MAPLE 17.

2. Lemmas

The following lemmas are required in our present investigation.

Lemma 2.1 ([10, Theorem 5]). *Let*

$$A(x) = \sum_{n=1}^{\infty} a_n x^{-n}, \quad x \to \infty$$

be an asymptotical expansion. Then the composition $\exp(A(x))$ *has the asymptotic expansion given by*

$$\exp(A(x)) = \sum_{n=0}^{\infty} b_n x^{-n}, \quad x \to \infty,$$

where

$$b_0 = 1, \quad b_n = \frac{1}{n}\sum_{k=1}^{n} k a_k b_{n-k}, \quad n \ge 1.$$

Lemma 2.2 ([1, Theorem 8]). *For every $m \in \mathbb{N}_0$, the function*

$$R_m(x) = (-1)^m \left[\ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right] \tag{2.1}$$

is completely monotonic on $(0,\infty)$, where here and elsewhere in this paper an empty sum is understood to be zero. Here B_n ($n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{2.2}$$

In 2006, Koumandos [17] presented a simpler proof of complete monotonicity of the functions $R_m(x)$, which was further strengthened by Koumandos and Pedersen in [18, Theorem 2.1].

Recall that a function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for} \quad x \in I \quad \text{and} \quad n \in \mathbb{N}_0. \tag{2.3}$$

Dubourdieu [13, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a,\infty)$, then strict inequality holds true in (2.3); see also [15] for a simpler proof of this result.

The gamma function $\Gamma(x)$ is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi (or digamma) function. It is known that

$$\Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

From the inequality $R_m(x) > 0$ for $x > 0$ and $m \in \mathbb{N}_0$, we obtain that for $x > 0$ and $n \in \mathbb{N}_0$,

$$\sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} < \ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right) < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \tag{2.4}$$

In particular, the choice $n = 1$ in (2.4) yields

$$x\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} < \ln\Gamma(x+1) - \ln\sqrt{2\pi x} < x\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}, \quad x > 0. \tag{2.5}$$

From the inequality $R'_m(x) < 0$ for $x > 0$ and $m \in \mathbb{N}_0$, we obtain that for $x > 0$ and $n \in \mathbb{N}_0$,

$$\sum_{j=1}^{2n} \frac{B_{2j}}{2jx^{2j}} < \ln x + \frac{1}{2x} - \psi(x+1) < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2jx^{2j}} \tag{2.6}$$

In particular, the choice $n = 1$ in (2.6) yields

$$\frac{1}{12x^2} - \frac{1}{120x^4} < \ln x + \frac{1}{2x} - \psi(x+1) < \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6}, \qquad x > 0. \tag{2.7}$$

Let $\Omega_x = \pi^{x/2}/\Gamma(\frac{x}{2}+1)$. We find that (2.1) can be written as

$$R_m(x) = (-1)^m \left[\ln\left(\frac{1}{\sqrt{2\pi x}}\left(\frac{\pi e}{x}\right)^x\right) - \ln\Omega_{2x} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right]. \tag{2.8}$$

From the inequality $R_m(x) > 0$ for $x > 0$ and $m \in \mathbb{N}_0$, we then obtain

$$\frac{1}{\sqrt{2\pi x}}\left(\frac{\pi e}{x}\right)^x \exp\left(-\sum_{i=1}^{2m+1}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right) < \Omega_{2x} < \frac{1}{\sqrt{2\pi x}}\left(\frac{\pi e}{x}\right)^x \exp\left(-\sum_{i=1}^{2m}\frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right). \tag{2.9}$$

Replacement of x by $n/2$ in (2.9) then produces the following double inequality for Ω_n:

$$\frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\sum_{j=1}^{2m+1}\frac{2^{2j}B_{2j}}{4j(2j-1)n^{2j-1}}\right) < \Omega_n < \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\sum_{j=1}^{2m}\frac{2^{2j}B_{2j}}{4j(2j-1)n^{2j-1}}\right) \tag{2.10}$$

for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

Mortici [23, Theorem 2] has given the following double inequality:

$$\frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\frac{1}{6n}+\frac{1}{45n^3}-\frac{8}{315n^5}+\frac{8}{105n^7}-\frac{128}{297n^9}\right) < \Omega_n < \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2} \exp\left(-\frac{1}{6n}+\frac{1}{45n^3}-\frac{8}{315n^5}+\frac{8}{105n^7}\right), \qquad n \in \mathbb{N}, \tag{2.11}$$

which can be seen to follow from (2.10) when $m = 2$.

Lemma 2.3 (see [12, Corollary 1]). *Let $m \in \mathbb{N}_0$. Then for $x > 0$,*

$$\sqrt{x}\exp\left(\sum_{j=1}^{2m}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) < \frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \sqrt{x}\exp\left(\sum_{j=1}^{2m+1}\left(1-\frac{1}{2^{2j}}\right)\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right). \tag{2.12}$$

In particular, the choice $m = 1$ in (2.12) yields

$$\frac{1}{2}\ln x + \frac{1}{8x} - \frac{1}{192x^3} < \ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) < \frac{1}{2}\ln x + \frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5}, \quad x > 0. \tag{2.13}$$

3. Asymptotics and inequalities for Ω_n

The asymptotic expansion of Ω_n as $n \to \infty$ is given by

$$\Omega_n = \frac{2\pi^{n/2}}{n}\frac{1}{\Gamma(\frac{n}{2})} \sim \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}\sum_{k=0}^{\infty}\frac{2^k\gamma_k}{n^k}$$
$$= \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}\left\{1-\frac{1}{6n}+\frac{1}{72n^2}+\frac{139}{6480n^3}-\frac{571}{155520n^4}-\frac{163879}{6531840n^5} + \frac{5246819}{1175731200n^6}+\frac{534703531}{7054387200n^7}-\cdots\right\}, \tag{3.1}$$

which follows from the well-known result

$$\frac{1}{\Gamma(z)} \sim \frac{e^z z^{\frac{1}{2}-z}}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \gamma_k z^{-k} \qquad (z \to \infty, \quad |\arg z| < \pi);$$

see, for example, [26, p. 32], [27, p. 70]. The γ_k denote the Stirling coefficients defined by the recurrence (with $d_0 = 1$)

$$\gamma_k = \frac{(-2)^k \Gamma(k+\frac{1}{2})}{\sqrt{\pi}} d_{2k}, \qquad d_n = \frac{n+1}{n+2} \left\{ \frac{d_{n-1}}{n} - \sum_{j=1}^{n-1} \frac{d_j d_{n-j}}{j+1} \right\} \quad (n \geq 1).$$

The first few values of γ_k are

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320}, \quad \gamma_5 = -\frac{163879}{209018880},$$
$$\gamma_6 = \frac{5246819}{75246796800}, \quad \gamma_7 = \frac{534703531}{902961561600}, \ldots.$$

The expansion (3.1) was also given by Mortici in [24, Theorem 1].

Motivated by (3.1) we establish the following theorem.

Theorem 3.1. *Let*[1]

$$V(x) = \frac{\Omega_{2x}}{\frac{1}{\sqrt{2\pi x}} \left(\frac{\pi e}{x}\right)^x}. \tag{3.2}$$

Then, for $x \geq 1/2$, we have

$$\sum_{k=0}^{5} \gamma_k x^{-k} < V(x) < \sum_{k=0}^{7} \gamma_k x^{-k} \tag{3.3}$$

and

$$V'(x) < \sum_{k=1}^{5} (-k) \gamma_k x^{-k-1}. \tag{3.4}$$

Proof. Here we only prove the left-hand side of (3.3) as the proof of the right-hand side is analogous. The inequality (2.9) can be written as

$$\exp\left(-\sum_{j=1}^{2m+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right) < V(x) < \exp\left(-\sum_{j=1}^{2m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right). \tag{3.5}$$

The choice $m = 1$ on the left-hand side of (3.5) then yields,

$$\exp\left(-\frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5} \right) < V(x).$$

In order prove the left-hand side of (3.3), it suffices to show for $x \geq 1/2$ that

$$\sum_{k=0}^{5} \gamma_k x^{-k} < \exp\left(-\frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5} \right). \tag{3.6}$$

[1] In terms of the scaled gamma function $\Gamma^*(x) = (e^x x^{\frac{1}{2}-x}/\sqrt{2\pi})\Gamma(x)$, the function $V(x) = 1/\Gamma^*(x)$. Consequently, the double inequality (3.3) supplies bounds on $1/\Gamma^*(x)$.

The inequality (3.6) is obtained by considering the function $P(x)$ defined, for $x \geq 1/2$, by

$$P(x) := \ln \sum_{k=0}^{5} \gamma_k x^{-k} - \left(-\frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5} \right).$$

Differentiation yields

$$P'(x) = \frac{P_4(x - \frac{1}{2})}{20160x^6 P_5(x - \frac{1}{2})},$$

where

$$P_4(x) = 98855357 + 763346136x + 2472851136x^2 + 3459743616x^3 + 1762931184x^4,$$

$$P_5(x) = 5486171 + 57666084x + 236795328x^2 + 488436480x^3 + 505128960x^4 + 209018880x^5.$$

Thus, we have $P'(x) > 0$ for $x \geq 1/2$. So, $P(x)$ is strictly increasing for $x \geq 1/2$, and we have

$$P(x) < \lim_{t\to\infty} P(t) = 0 \quad \text{for} \quad x \geq \frac{1}{2}.$$

Hence, (3.6) holds for $x \geq 1/2$.

Since the function $R_m(x)$ defined by (2.8) is completely monotonic on $(0, \infty)$, we have, for $x > 0$ and $m \in \mathbb{N}_0$,

$$R'_m(x) = (-1)^{m+1} \left[\frac{V'(x)}{V(x)} - \sum_{j=1}^{m} \frac{B_{2j}}{2jx^{2j}} \right] < 0, \quad \text{so that} \quad V'(x) < V(x) \sum_{j=1}^{2m+1} \frac{B_{2j}}{2jx^{2j}}.$$

Using the right-hand side of (3.3), we obtain that

$$\begin{aligned} V'(x) &< V(x) \sum_{j=1}^{3} \frac{B_{2j}}{2jx^{2j}} \\ &< \left(1 - \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3} - \frac{571}{2488320x^4} - \frac{163879}{209018880x^5} \right. \\ &\quad \left. + \frac{5246819}{75246796800x^6} + \frac{534703531}{902961561600x^7} \right) \left(\frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} \right) \\ &= \frac{1}{12x^2} - \frac{1}{144x^3} - \frac{139}{17280x^4} + \frac{571}{622080x^5} + \frac{163879}{41803776x^6} + \mathcal{R}(x), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \mathcal{R}(x) = -\frac{1}{2275463135232000x^{13}} &\left\{ \frac{25799409925}{8} + \frac{215497370169}{2} \left(x - \frac{1}{2} \right) \right. \\ &+ 627676689198 \left(x - \frac{1}{2} \right)^2 + 1958303670708 \left(x - \frac{1}{2} \right)^3 \\ &+ 3296254174710 \left(x - \frac{1}{2} \right)^4 + 2807022408120 \left(x - \frac{1}{2} \right)^5 \\ &\left. + 951982839360 \left(x - \frac{1}{2} \right)^6 \right\} < 0 \quad \text{for} \quad x \geq \frac{1}{2}. \end{aligned}$$

Hence, upon identifying the coefficients of $x^{-k-1} (1 \leq k \leq 5)$ in (3.7) as $(-k)\gamma_k$, we see that (3.4) holds for $x \geq 1/2$. This completes the proof. □

Remark 3.1. *Noting that*

$$\sum_{k=4}^{7}\gamma_k x^{-k} = -\frac{571}{2488320x^4} - \frac{163879}{209018880x^5} + \frac{5246819}{75246796800x^6} + \frac{534703531}{902961561600x^7}$$
$$= -\frac{117938613 + 800398812(x-\frac{1}{2}) + 1018764000(x-\frac{1}{2})^2 + 207204480(x-\frac{1}{2})^3}{902961561600x^7} < 0$$

holds for $x \geq 1/2$, we obtain from the right-hand side of (3.3) that

$$V(x) < 1 - \frac{1}{12x} + \frac{1}{288x^2} + \frac{139}{51840x^3}, \qquad x \geq \frac{1}{2}. \tag{3.8}$$

Remark 3.2. *Replacement of x by $n/2$ in (3.3) yields the following double inequality for Ω_n:*

$$\frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1 - \frac{1}{6n} + \frac{1}{72n^2} + \frac{139}{6480n^3} - \frac{571}{155520n^4} - \frac{163879}{6531840n^5}\right) < \Omega_n$$
$$< \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1 - \frac{1}{6n} + \frac{1}{72n^2} + \frac{139}{6480n^3} - \frac{571}{155520n^4} - \frac{163879}{6531840n^5} + \frac{5246819}{1175731200n^6} + \frac{534703531}{7054387200n^7}\right). \tag{3.9}$$

From the right-hand side of (1.11) it follows that

$$\Omega_n \approx \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}, \qquad n \to \infty. \tag{3.10}$$

Theorem 3.2 develops the approximation formula (3.10) to produce a complete asymptotic expansion.

Theorem 3.2. *The following asymptotic expansion holds:*

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\exp\left(\sum_{j=2}^{\infty}\frac{a_j}{n^j}\right), \qquad n \to \infty, \tag{3.11}$$

where the coefficients a_j $(j \geq 2)$ are given by

$$a_j = \frac{(-1)^{j-1}}{2j\cdot 3^j} - \frac{2^j B_{j+1}}{j(j+1)} \tag{3.12}$$

and B_j denote the Bernoulli numbers.

Proof. Noting that $\Omega_{2x} = \pi^x/\Gamma(x+1)$, we find that

$$x\ln x - x + \ln(\sqrt{2\pi x}) - \ln\Gamma(x+1) + \frac{1}{2}\ln\left(1 + \frac{1}{6x}\right) \sim \sum_{j=2}^{\infty}\frac{a_j}{(2x)^j} \tag{3.13}$$

as $x \to \infty$, where a_j $(j \geq 2)$ are real numbers to be determined.

Stirling's series for the gamma function is given (see [25, p. 140]) by

$$\ln\Gamma(x+1) \sim x\ln x - x + \ln(\sqrt{2\pi x}) + \sum_{j=1}^{\infty}\frac{B_{j+1}}{j(j+1)x^j}, \qquad x \to \infty. \tag{3.14}$$

The Maclaurin expansion of $\ln(1+t)$ with $t = 1/(6x)$ gives

$$\ln\left(1+\frac{1}{6x}\right) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{j\cdot 6^j x^j}. \tag{3.15}$$

Substitution of (3.14) and (3.15) into (3.13) yields

$$\sum_{j=2}^{\infty}\left(\frac{(-1)^{j-1}}{2j\cdot 6^j} - \frac{B_{j+1}}{j(j+1)}\right)\frac{1}{x^j} \sim \sum_{j=2}^{\infty}\frac{a_j}{2^j}\frac{1}{x^j}. \tag{3.16}$$

Equating coefficients of equal powers of x in (3.16) we obtain (3.12). This completes the proof. □

From (3.11), we find the following explicit asymptotic expansion:

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\exp\left(-\frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{648n^4} - \frac{85}{3402n^5} - \frac{1}{8748n^6} + \frac{1667}{21870n^7} - \frac{1}{104976n^8} - \frac{1679605}{3897234n^9} - \frac{1}{1180980n^{10}} + \cdots\right). \tag{3.17}$$

By Lemma 2.1, we then obtain the following asymptotic expansion:

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(\sum_{j=0}^{\infty}\frac{b_j}{n^j}\right), \qquad n\to\infty, \tag{3.18}$$

where the coefficients b_j are given by

$$b_0 = 1, \quad b_j = \frac{1}{j}\sum_{k=1}^{j} k a_k b_{j-k}, \quad j \geq 1, \tag{3.19}$$

and a_j are given in (3.12). This produces the expansion

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left\{1 - \frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{864n^4} - \frac{5261}{204120n^5} + \frac{6889}{20995200n^6} + \frac{125549}{1632960n^7} - \cdots\right\}. \tag{3.20}$$

The expansion in (3.20) motivated us to establish the following double inequality for Ω_n.

Theorem 3.3. *For $n \in \mathbb{N}$,*

$$\frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1 - \frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{864n^4} - \frac{5261}{204120n^5}\right) < \Omega_n < \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1 - \frac{1}{36n^2} + \frac{23}{810n^3}\right). \tag{3.21}$$

Proof. It suffices to show that

$$F(x) > 0 \quad \text{and} \quad f(x) < 0 \quad \text{for} \quad x \geq \frac{1}{2},$$

where

$$F(x) = x\ln x - x + \ln(\sqrt{2\pi x}) - \ln\Gamma(x+1) + \frac{1}{2}\ln\left(1+\frac{1}{6x}\right) - \ln\left(1 - \frac{1}{36(2x)^2} + \frac{23}{810(2x)^3} - \frac{1}{864(2x)^4} - \frac{5261}{204120(2x)^5}\right)$$

and

$$f(x) = x\ln x - x + \ln(\sqrt{2\pi x}) - \ln\Gamma(x+1) + \frac{1}{2}\ln\left(1+\frac{1}{6x}\right) - \ln\left(1 - \frac{1}{36(2x)^2} + \frac{23}{810(2x)^3}\right).$$

Differentiating $f(x)$ and applying the left-hand side of (2.7), we obtain that for $x \ge 1/2$,

$$\begin{aligned} f'(x) &= \ln x + \frac{1}{2x} - \psi(x+1) - \frac{6480x^3 - 693x - 115 + 1080x^2}{2x(6x+1)(6480x^3 - 45x + 23)} \\ &> \frac{1}{12x^2} - \frac{1}{120x^4} - \frac{6480x^3 - 693x - 115 + 1080x^2}{2x(6x+1)(6480x^3 - 45x + 23)} \\ &= \frac{1350x^3 + 500x^2 - 93x - 23}{120x^4(6x+1)(6480x^3 - 45x + 23)} > 0. \end{aligned}$$

Hence, $f(x)$ is strictly increasing for $x \ge 1/2$, and we have

$$f(x) < \lim_{t\to\infty} f(t) = 0 \quad \text{for} \quad x \ge \frac{1}{2}.$$

Differentiating $F(x)$ and applying the right-hand side of (2.7), we obtain that for $x \ge 1/2$,

$$\begin{aligned} F'(x) &= \ln x + \frac{1}{2x} - \psi(x+1) \\ &\quad - \frac{3(4354560x^5 - 465696x^3 - 62160x^2 + 212645x + 31566 + 725760x^4)}{2x(6x+1)(13063680x^5 - 90720x^3 + 46368x^2 - 945x - 10522)} \\ &< \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} \\ &\quad - \frac{3(4354560x^5 - 465696x^3 - 62160x^2 + 212645x + 31566 + 725760x^4)}{2x(6x+1)(13063680x^5 - 90720x^3 + 46368x^2 - 945x - 10522)} \\ &= -\frac{P_5(x-\frac{1}{2})}{2520x^6(6x+1)Q_5(x-\frac{1}{2})}, \end{aligned}$$

where

$$P_5(x) = \frac{9402445}{16} + \frac{29999799}{8}x + 14897652x^2 + 28984704x^3 + 23697723x^4 + 6076098x^5,$$

$$Q_5(x) = \frac{794995}{2} + 4059783x + 16239888x^2 + 32568480x^3 + 32659200x^4 + 13063680x^5.$$

Hence, $F'(x) < 0$ for $x \ge 1/2$. So, $F(x)$ is strictly decreasing for $x \ge 1/2$, and we have

$$F(x) > \lim_{t\to\infty} f(t) = 0 \quad \text{for} \quad x \ge \frac{1}{2}.$$

This completes the proof. □

Mortici [24, Theorem 3] improved (1.11) and obtained the following double inequality, for every integer $n \geq 3$ on the left-hand side and $n \geq 1$ on the right-hand side:

$$\frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\left(n+\frac{1}{3}+\frac{1}{18n}-\frac{31}{810n^2}\right)}} < \Omega_n < \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\left(n+\frac{1}{3}+\frac{1}{18n}-\frac{31}{810n^2}-\frac{139}{9720n^3}\right)}}. \tag{3.22}$$

This last inequality implies

$$\Omega_n = \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\Big(n+\frac{1}{3}+\frac{1}{18n}-\frac{31}{810n^2}+O\left(\frac{1}{n^3}\right)\Big)}}. \tag{3.23}$$

If we now define v_n by the equality

$$\Omega_n = \frac{1}{\sqrt{\pi\left(n+v_n\right)}}\left(\frac{2\pi e}{n}\right)^{n/2}, \tag{3.24}$$

we find $v_n = \theta\left(n/2\right)$, where

$$\theta(x) = 2x\left\{\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^2 - 1\right\}. \tag{3.25}$$

Theorem 3.4 presents the asymptotic expansion and inequality for $\theta(x)$.

Theorem 3.4. (i) *The function $\theta(x)$ has the following asymptotic expansion:*

$$\theta(x) \sim \sum_{j=0}^{\infty}\frac{c_j}{x^j}, \qquad x \to \infty, \tag{3.26}$$

with the coefficients c_j given by

$$c_j = 2\lambda_{j+1}, \qquad j \in \mathbb{N}_0, \tag{3.27}$$

where

$$\lambda_0 = 1, \quad \lambda_j = \frac{1}{j}\sum_{k=1}^{j}\frac{2B_{k+1}}{k+1}\lambda_{j-k}, \qquad j \in \mathbb{N}. \tag{3.28}$$

Namely,

$$\begin{aligned}\theta(x) \sim \frac{1}{3} &+ \frac{1}{36x} - \frac{31}{3240x^2} - \frac{139}{77760x^3} + \frac{9871}{3265920x^4} + \frac{324179}{587865600x^5} - \frac{8225671}{3527193600x^6} \\ &- \frac{69685339}{169305292800x^7} + \frac{1674981058019}{502836719616000x^8} + \frac{24279707153761}{42238284447744000x^9} - \cdots .\end{aligned} \tag{3.29}$$

(ii) *For $x > 0$ and $m \in \mathbb{N}_0$,*

$$2x\left\{\exp\left(\sum_{j=1}^{2m}\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) - 1\right\} < \theta(x) < 2x\left\{\exp\left(\sum_{j=1}^{2m+1}\frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) - 1\right\}. \tag{3.30}$$

Proof. It follows from [9, (3.6)] that

$$\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(x/e\right)^x}\right)^2 \sim \sum_{j=0}^{\infty}\lambda_j x^{-j}, \qquad x \to \infty, \tag{3.31}$$

with the coefficients λ_j given by

$$\lambda_0 = 1, \quad \lambda_j = \frac{1}{j}\sum_{k=1}^{j} \frac{2B_{k+1}}{k+1}\lambda_{j-k}, \qquad j \ge 1.$$

Combination of (3.25) and (3.31) then gives (3.26).

Write (2.4) as

$$\exp\left(\sum_{j=1}^{2m} \frac{B_{2j}}{j(2j-1)x^{2j-1}}\right) < \left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)^2 < \exp\left(\sum_{j=1}^{2m+1} \frac{B_{2j}}{j(2j-1)x^{2j-1}}\right). \tag{3.32}$$

Combination of (3.25) and (3.32) then gives (3.30). This completes the proof. □

Corollary 3.1. (i) *The sequence $v_n = \theta(n/2)$ has the following asymptotic expansion:*

$$v_n \sim \sum_{j=0}^{\infty} \frac{d_j}{n^j}, \qquad n \to \infty, \tag{3.33}$$

with the coefficients d_j given by

$$d_j = 2^{j+1}\lambda_{j+1}, \qquad j \in \mathbb{N}_0,$$

where λ_j are given in (3.28)*. Namely,*

$$v_n \sim \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2} - \frac{139}{9720n^3} + \frac{9871}{204120n^4} + \frac{324179}{18370800n^5} - \frac{8225671}{55112400n^6} - \frac{69685339}{1322697600n^7} + \frac{1674981058019}{1964205936000n^8} + \frac{24279707153761}{82496649312000n^9} - \cdots. \tag{3.34}$$

(ii) *For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, we have*

$$n\exp\left(\sum_{j=1}^{2m} \frac{2^{2j-1}B_{2j}}{j(2j-1)n^{2j-1}}\right) < n + v_n < n\exp\left(\sum_{j=1}^{2m+1} \frac{2^{2j-1}B_{2j}}{j(2j-1)n^{2j-1}}\right). \tag{3.35}$$

It follows from (3.24) and (3.33) that

$$\Omega_n \sim \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi\left(n + \sum_{j=0}^{\infty} \frac{d_j}{n^j}\right)}} \tag{3.36}$$

which develops (3.23) to produce a complete asymptotic expansion.

From (3.24) and (3.35) we see that for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$\frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp\left(\sum_{j=1}^{2m+1} \frac{2^{2j-1}B_{2j}}{j(2j-1)n^{2j-1}}\right)}} < \Omega_n < \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp\left(\sum_{j=1}^{2m} \frac{2^{2j-1}B_{2j}}{j(2j-1)n^{2j-1}}\right)}}. \tag{3.37}$$

In particular, the choice $m = 1$ on the left-hand side and $m = 2$ on the right-hand side yields

$$\frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp\left(\frac{1}{3n} - \frac{2}{45n^3} + \frac{16}{315n^5}\right)}} < \Omega_n < \frac{(2\pi e/n)^{n/2}}{\sqrt{\pi n \exp\left(\frac{1}{3n} - \frac{2}{45n^3} + \frac{16}{315n^5} - \frac{16}{105n^7}\right)}} \tag{3.38}$$

for $n \in \mathbb{N}$.

The inequalities (3.38) are sharper than the inequalities (3.22) for $n \ge 4$.

Based on the formula (3.1) we now establish a sharp inequality for Ω_n.

Theorem 3.5. *For $n \in \mathbb{N}$, we have*

$$\frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1-\frac{1}{6n+p}\right) < \Omega_n \le \frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1-\frac{1}{6n+q}\right) \tag{3.39}$$

with the best possible constants

$$p = \frac{1}{2} \quad \text{and} \quad q = \frac{6\sqrt{2}-5\sqrt{e}}{\sqrt{e}-\sqrt{2}} = 1.03056322\ldots. \tag{3.40}$$

Proof. If we write (3.39) as

$$p < x_n \le q, \qquad x_n = \left(1-\frac{\Omega_n}{\frac{1}{\sqrt{n\pi}}\left(\frac{2\pi e}{n}\right)^{n/2}}\right)^{-1} - 6n,$$

we find that

$$x_1 = \frac{6\sqrt{2}-5\sqrt{e}}{\sqrt{e}-\sqrt{2}} \quad \text{and} \quad \lim_{n\to\infty} x_n = \frac{1}{2}. \tag{3.41}$$

The limit in (3.41) is obtained by using the asymptotic expansion (3.1).

In order prove Theorem 3.5, it suffices to show that the sequence $\{x_n\}$ is strictly decreasing for $n \ge 1$. The monotonicity property of $\{x_n\}$ is obtained by considering the function $U(x)$ defined by

$$U(x) = \frac{1}{1-V(x)} - 12x,$$

where $V(x)$ is given in (3.2). Differentiating $U(x)$ and applying (3.4) and (3.8), we obtain that for $x \ge 2$,

$$\begin{aligned}
-(1-V(x))^2U'(x) &= 12(1-V(x))^2 - V'(x)\\
&> 12\left(\frac{1}{12x}-\frac{1}{288x^2}-\frac{139}{51840x^3}\right)^2\\
&\quad -\left(\frac{1}{12x^2}-\frac{1}{144x^3}-\frac{139}{17280x^4}+\frac{571}{622080x^5}+\frac{163879}{41803776x^6}\right)\\
&= \frac{19066289+33264000(x-2)+8860320(x-2)^2}{3135283200x^6} > 0.
\end{aligned}$$

Hence, $U(x)$ is strictly decreasing for $x \ge 2$. We then obtain that the sequence $\{x_n\} = \{U(n/2)\}$ is strictly decreasing for $n \ge 4$.

Direct computation yields

$$x_1 = 1.030563\ldots, \quad x_2 = 0.843071\ldots \quad x_3 = 0.748041\ldots \quad x_4 = 0.692684\ldots.$$

Consequently, the sequence $\{x_n\}$ is strictly decreasing for all $n \ge 1$. This completes the proof. □

4. Asymptotics and inequalities for $\frac{\Omega_{n-1}}{\Omega_n}$ and $\frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}}$

It is easy to see that

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}, \qquad \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} = \frac{\sqrt{\pi}(n+1)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{(n+1+2\pi)\Gamma\left(\frac{n}{2}+1\right)}. \tag{4.1}$$

The asymptotic expansion of these two ratios follows immediately from the well-known expansion for the ratio of two gamma functions [25, p. 141]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b} \sum_{k=0}^{\infty} \binom{a-b}{k} B_k^{(a-b+1)}(a) z^{-a} \qquad (z\to\infty, \quad |\arg z| < \pi),$$

where $B_k^{(\nu)}(x)$ denote the generalized Bernoulli polynomials defined by

$$\left(\frac{t}{e^t-1}\right)^{\nu} e^{xt} = \sum_{k=0}^{\infty} B_k^{(\nu)}(x) \frac{t^k}{k!}, \qquad |t| < 2\pi. \tag{4.2}$$

Hence we obtain the expansions

$$\begin{aligned} \frac{\Omega_{n-1}}{\Omega_n} &\sim \sqrt{\frac{n}{2\pi}} \sum_{k=0}^{\infty} \binom{1/2}{k} B_k^{(3/2)}(1) \left(\frac{2}{n}\right)^k \\ &= \sqrt{\frac{n}{2\pi}} \left\{ 1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6} \right. \\ &\qquad \left. - \frac{39325}{262144n^7} - \frac{334477}{8388608n^8} + \cdots \right\} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} &\sim \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \sum_{k=0}^{\infty} \binom{-1/2}{k} B_k^{(1/2)}(\tfrac{1}{2}) \left(\frac{2}{n}\right)^k \\ &= \frac{n+1}{n+1+2\pi} \sqrt{\frac{2\pi}{n}} \left\{ 1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3} - \frac{21}{2048n^4} - \frac{399}{8192n^5} + \cdots \right\}. \end{aligned} \tag{4.4}$$

Formula (4.3) was presented by Mortici in [24, Theorem 8].

Upper and lower bounds for the ratio Ω_{n-1}/Ω_n can be obtained by replacement of x by $n/2$ in (2.12) to find

$$\sqrt{\frac{n}{2\pi}} \exp\left(\sum_{j=1}^{2m} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}} \right) < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n}{2\pi}} \exp\left(\sum_{j=1}^{2m+1} \frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}} \right). \tag{4.5}$$

Mortici [24, Theorem 7] established the following inequality:

$$\begin{aligned} \sqrt{\frac{n}{2\pi}} \exp\left(\frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{17}{112n^7} \right) &< \frac{\Omega_{n-1}}{\Omega_n} \\ &< \sqrt{\frac{n}{2\pi}} \exp\left(\frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{17}{112n^7} + \frac{31}{36n^9} \right), \qquad n\in\mathbb{N}. \end{aligned} \tag{4.6}$$

We observe that the choice $m = 2$ in (4.5) yields (4.6).

Based on (4.3), Mortici [24, Theorem 9] established the following inequality:

$$\begin{aligned} \sqrt{\frac{n}{2\pi}} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} \right) &< \frac{\Omega_{n-1}}{\Omega_n} \\ < \sqrt{\frac{n}{2\pi}} \left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6} \right), &\quad n\in\mathbb{N}. \end{aligned} \tag{4.7}$$

Remark 4.1. *The left-hand side of* (4.6) *is sharper than the left-hand side of* (4.7) *for* $n \geq 10$. *The right-hand side of* (4.6) *is sharper than the right-hand side of* (4.7) *for* $n \geq 3$.

Replacement of x by $n/2$ in (2.12) yields

$$\sqrt{\frac{2}{n}}\exp\left(-\sum_{j=1}^{2m}\frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) > \frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)} > \sqrt{\frac{2}{n}}\exp\left(-\sum_{j=1}^{2m+1}\frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) \tag{4.8}$$

for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. We then find form (4.1) and (4.8) that, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$,

$$\begin{aligned}\frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n}}\exp\left(-\sum_{j=1}^{2m+1}\frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right) &< \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} \\ &< \frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n}}\exp\left(-\sum_{j=1}^{2m}\frac{(2^{2j}-1)B_{2j}}{2j(2j-1)n^{2j-1}}\right).\end{aligned} \tag{4.9}$$

In particular, the choice $m = 0$ on the left-hand side and $m = 1$ on the right-hand side yields

$$\frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n}}\exp\left(-\frac{1}{4n}\right) < \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n}}\exp\left(-\frac{1}{4n}+\frac{1}{24n^3}\right) \tag{4.10}$$

for $n \in \mathbb{N}$.

Mortici [24, Theorem 12] improved the bounds in (1.9) and (1.10) as follows:

$$\sqrt{\frac{2\pi}{n+4\pi+\frac{1}{2}}+\varepsilon_1(n)} < \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \sqrt{\frac{2\pi}{n+4\pi+\frac{1}{2}}+\varepsilon_2(n)}, \tag{4.11}$$

where

$$\varepsilon_1(n) = -\frac{\frac{1}{4}\pi - 4\pi^2 + 8\pi^3}{n^3} \quad \text{and} \quad \varepsilon_2(n) = \varepsilon_1(n) + \frac{\frac{3}{8}\pi - 7\pi^2 - 12\pi^3 + 64\pi^4}{n^4}.$$

We remark that the left-hand side of (4.11) holds for $n \geq 10$, because

$$\frac{2\pi}{n+4\pi+\frac{1}{2}} + \varepsilon_1(n) \begin{cases} < 0, & 1 \leq n \leq 9, \\ > 0, & n \geq 10. \end{cases}$$

The right-hand side of (4.11) is valid for all $n \in \mathbb{N}$.

Remark 4.2. *The inequalities* (4.10) *are sharper than the inequalities* (4.11) *and, moreover,* (4.10) *is valid for all* $n \in \mathbb{N}$ *and has a simple form.*

Noting that

$$\exp\left(-\frac{1}{4n}\right) > 1 - \frac{1}{4n} \quad \text{and} \quad \exp\left(-\frac{1}{4n}+\frac{1}{24n^3}\right) < \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{5}{128n^3}\right)$$

holds for $n \in \mathbb{N}$, we obtain the following alternative form of the double inequality in (4.10):

$$\begin{aligned}\frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n}}\left(1-\frac{1}{4n}\right) &< \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} \\ &< \frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n}}\left(1-\frac{1}{4n}+\frac{1}{32n^2}+\frac{5}{128n^3}\right)\end{aligned} \tag{4.12}$$

for $n \in \mathbb{N}$.

The following theorem presents a sharp inequality for $\Omega_n/(\Omega_{n-1}+\Omega_{n+1})$.

Theorem 4.1. *For $n \in \mathbb{N}$, we have*

$$\frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n+\frac{\pi}{2}-1}} \le \frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}} < \frac{n+1}{n+1+2\pi}\sqrt{\frac{2\pi}{n+\frac{1}{2}}}. \tag{4.13}$$

The constants $\frac{\pi}{2}-1 = 0.5707963\ldots$ and $\frac{1}{2}$ are the best possible.

Proof. First, we establish the left-hand inequality of (4.13). Elementary calculations show that this is valid for $n=1$ and $n=2$. We now prove that the left-hand inequality of (4.13) holds for $n \ge 3$. It suffices to show by appeal to (4.12) that

$$\frac{1}{\sqrt{n}}\left(1-\frac{1}{4n}\right) > \frac{1}{\sqrt{n+c}} \quad \text{with} \quad c = \frac{\pi}{2}-1.$$

We find, for $n \ge 3$,

$$\left(\frac{1}{\sqrt{n}}\left(1-\frac{1}{4n}\right)\right)^2 - \frac{1}{n+c} = \frac{121c-69+(88c-47)(n-3)+(16c-8)(n-3)^2}{16n^3(n+c)} > 0.$$

This proves the left-hand inequality of (4.13) for $n \ge 1$.

We now establish the right-hand inequality of (4.13). Elementary calculations show that this is valid for $n=1$. We now prove that the right-hand inequality of (4.13) holds for $n \ge 2$. It suffices to show by a similar appeal to (4.12) that

$$\frac{1}{\sqrt{n}}\left(1-\frac{1}{4n}+\frac{1}{32n^2}+\frac{5}{128n^3}\right) < \frac{1}{\sqrt{n+\frac{1}{2}}}.$$

We find, for $n \ge 2$, that

$$\begin{aligned}\left(\frac{1}{\sqrt{n}}\left(1-\frac{1}{4n}+\frac{1}{32n^2}+\frac{5}{128n^3}\right)\right)^2 - \frac{1}{n+\frac{1}{2}} \\ = -\frac{1}{16384n^7(2n+1)}\Big(62899+192422(n-2)+227104(n-2)^2+130656(n-2)^3 \\ +36864(n-2)^4+4096(n-2)^5\Big) < 0.\end{aligned}$$

This proves the right-hand inequality of (4.13) for $n \ge 1$.

If we write (4.13) as

$$\frac{\pi}{2}-1 \ge y_n > \frac{1}{2}, \qquad y_n = 2\pi\left(\frac{n+1+2\pi}{n+1}\frac{\Omega_n}{\Omega_{n-1}+\Omega_{n+1}}\right)^{-2} - n,$$

we find that

$$y_1 = \frac{\pi}{2}-1 \quad \text{and} \quad \lim_{n\to\infty} y_n = \frac{1}{2}. \tag{4.14}$$

The limit in (4.14) is obtained by using the asymptotic expansion (4.4). Hence, the inequality (4.13) holds for $n \ge 1$, and the constants $\frac{\pi}{2}-1$ and $\frac{1}{2}$ are the best possible. □

Remark 4.3. *The inequalities* (4.13) *are sharper than the inequalities* (1.9) *and* (1.10) *for $n \ge 3$.*

5. Asymptotics and inequalities for $\Omega_n^{1/n}/\Omega_{n+1}^{1/(n+1)}$

It is easy to see that

$$\frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} = \frac{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)^{1/(n+1)}}{\Gamma\left(\frac{n}{2}+1\right)^{1/n}}.$$

We first establish the asymptotic expansion for $\Gamma\left(x+3/2\right)^{1/(2x+1)}/\Gamma(x+1)^{1/(2x)}$.

Theorem 5.1. *As $x\to\infty$, we have*

$$\frac{\Gamma\left(x+\frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} \sim \left(\sqrt{2\pi x}\right)^{-\frac{1}{2x(2x+1)}} \exp\left\{\sum_{j=1}^{\infty}\frac{\mu_j}{x^j}\right\}, \tag{5.1}$$

with the coefficients μ_j given by $\mu_1 = 1/4$ and

$$\mu_j = (-1)^j\left\{\frac{(-1)^{j+1}B_j}{2j(j-1)} - \frac{1}{2^{j+1}} + \sum_{k=0}^{j-2}\frac{k+2-(2^{k+1}-1)B_{k+2}}{(k+1)(k+2)2^j}\right\} \qquad (j\ge 2). \tag{5.2}$$

Proof. The logarithm of gamma function has asymptotic expansion (see [20, p. 32]):

$$\ln\Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}\frac{1}{x^n} \tag{5.3}$$

as $x\to\infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by (4.2) with $\nu = 1$. Using (5.3), we find as $x\to\infty$

$$\begin{aligned}
&\frac{1}{2x+1}\ln\Gamma\left(x+\frac{3}{2}\right) - \frac{1}{2x}\ln\Gamma(x+1) + \frac{1}{2x(2x+1)}\ln\sqrt{2\pi x}\\
&\sim \frac{1}{2x+1}\left\{(x+1)\ln x - x + \ln(\sqrt{2\pi}) + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(3/2)}{j(j+1)}\frac{1}{x^j}\right\}\\
&\quad - \frac{1}{2x}\left\{\left(x+\frac{1}{2}\right)\ln x - x + \ln(\sqrt{2\pi}) + \sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(1)}{j(j+1)}\frac{1}{x^j}\right\}\\
&\quad + \frac{1}{2x(2x+1)}\ln\sqrt{2\pi x}\\
&\sim \frac{1}{2x+1}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(3/2)}{j(j+1)}\frac{1}{x^j} - \sum_{j=2}^{\infty}\frac{(-1)^{j}B_{j}(1)}{2j(j-1)}\frac{1}{x^j} + \frac{1}{2(2x+1)}.
\end{aligned} \tag{5.4}$$

Noting that (when $x\ge 1/2$)

$$\frac{1}{2x+1} = \frac{1}{2x}\sum_{j=0}^{\infty}(-1)^j\frac{1}{(2x)^j} = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{2^j x^j} \tag{5.5}$$

holds, we obtain, as $x\to\infty$,

$$\begin{aligned}
\frac{1}{2x+1}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(3/2)}{j(j+1)}\frac{1}{x^j} &\sim \sum_{j=1}^{\infty}\frac{(-1)^{j+1}B_{j+1}(3/2)}{j(j+1)}\frac{1}{x^j}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2^k x^k}\\
&\sim \sum_{j=2}^{\infty}\left\{\sum_{k=0}^{j-2}\frac{(-1)^j B_{k+2}(3/2)}{(k+1)(k+2)2^{j-k-1}}\right\}\frac{1}{x^j}.
\end{aligned} \tag{5.6}$$

Substitution of the expressions (5.5) and (5.6) into (5.4) then yields

$$\frac{\Gamma\left(x+\frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} \sim \left(\sqrt{2\pi x}\right)^{-\frac{1}{2x(2x+1)}} \exp\left\{\sum_{j=1}^{\infty} \frac{\mu_j}{x^j}\right\},$$

with the coefficients μ_j given by

$$\mu_1 = \frac{1}{4}, \quad \mu_j = (-1)^j \left\{-\frac{B_j(1)}{2j(j-1)} - \frac{1}{2^{j+1}} + \sum_{k=0}^{j-2} \frac{B_{k+2}(3/2)}{(k+1)(k+2)2^{j-k-1}}\right\} \quad \text{for} \quad j \geq 2. \tag{5.7}$$

Noting that

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad B_n(1) = (-1)^n B_n, \quad B_n(1/2) = -(1-2^{1-n})B_n$$

holds (see [25, p. 590]), we find that (5.7) can be written as (5.2). This completes the proof. □

From (5.1), we obtain the following explicit asymptotic expansion:

$$\frac{\Gamma\left(x+\frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} \sim \left(\sqrt{2\pi x}\right)^{-\frac{1}{2x(2x+1)}} \exp\left\{\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4} - \frac{547}{11520x^5} + \frac{601}{23040x^6} - \frac{4691}{322560x^7} + \cdots\right\}. \tag{5.8}$$

Replacement of x by $n/2$ in (5.8) yields

$$\frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} \sim \left(\sqrt{n\pi}\right)^{-\frac{1}{n(n+1)}} \exp\left\{\frac{1}{2n} + \frac{1}{4n^2} - \frac{11}{12n^3} + \frac{31}{24n^4} - \frac{547}{360n^5} + \frac{601}{360n^6} - \frac{4691}{2520n^7} + \cdots\right\} \tag{5.9}$$

and, by Lemma 2.1, we finally obtain the following asymptotic expansion as $n \to \infty$:

$$\frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} \sim \left(\sqrt{n\pi}\right)^{-\frac{1}{n(n+1)}} \left\{1 + \frac{1}{2n} + \frac{3}{8n^2} - \frac{37}{48n^3} + \frac{115}{128n^4} - \frac{13781}{11520n^5} + \cdots\right\}. \tag{5.10}$$

Formula (5.8) motivated us to establish the following double inequality for $\Gamma\left(x+3/2\right)^{1/(2x+1)}/\Gamma(x+1)^{1/(2x)}$.

Theorem 5.2. *For $x \geq 1/2$,*

$$\left(\sqrt{2\pi x}\right)^{-\frac{1}{2x(2x+1)}} \exp\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) < \frac{\Gamma\left(x+\frac{3}{2}\right)^{1/(2x+1)}}{\Gamma(x+1)^{1/(2x)}} < \left(\sqrt{2\pi x}\right)^{-\frac{1}{2x(2x+1)}} \exp\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4}\right). \tag{5.11}$$

Proof. In order prove the left-hand side of (5.11), it suffices to show that for $x \geq 1/2$,

$$\frac{1}{2x+1}\ln\left(x+\frac{1}{2}\right) + \frac{1}{2x+1}\ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2x}\ln\Gamma(x+1) + \frac{1}{2x(2x+1)}\ln\sqrt{2\pi x} - \left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) > 0;$$

that is,

$$u(x) := \ln\left(x+\frac{1}{2}\right) - \ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) - \frac{1}{2x}\Big(\ln\Gamma(x+1) - \ln\sqrt{2\pi x}\Big) - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) > 0.$$

Similarly, to prove the right-hand side of (5.11), it suffices to show that for $x \geq 1/2$,

$$v(x) := \ln\left(x+\frac{1}{2}\right) - \ln\left(\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right) - \frac{1}{2x}\Big(\ln\Gamma(x+1) - \ln\sqrt{2\pi x}\Big) - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4}\right) < 0.$$

Using the inequalities (2.5) and (2.13), we obtain, for $x \geq 1/2$,

$$\begin{aligned} u(x) &> \ln x + \ln\left(1+\frac{1}{2x}\right) - \left(\frac{1}{2}\ln x + \frac{1}{8x} - \frac{1}{192x^3} + \frac{1}{640x^5}\right) \\ &\quad - \frac{1}{2x}\left(x\ln x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}\right) - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3}\right) \\ &= \ln\left(1+\frac{1}{2x}\right) - \frac{20160x^5 - 5040x^4 - 4830x^3 - 56x^2 + 63x + 16}{40320x^6} \\ &> \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} - \frac{1}{64x^4} - \frac{20160x^5 - 5040x^4 - 4830x^3 - 56x^2 + 63x + 16}{40320x^6} \\ &= \frac{\frac{2491}{4} + \frac{8491}{2}(x-\frac{1}{2}) + 9191(x-\frac{1}{2})^2 + 6510(x-\frac{1}{2})^3}{40320x^6} > 0 \end{aligned}$$

and

$$\begin{aligned} v(x) &< \ln x + \ln\left(1+\frac{1}{2x}\right) - \left(\frac{1}{2}\ln x + \frac{1}{8x} - \frac{1}{192x^3}\right) \\ &\quad - \frac{1}{2x}\left(x\ln x - x + \frac{1}{12x} - \frac{1}{360x^3}\right) - (2x+1)\left(\frac{1}{4x} + \frac{1}{16x^2} - \frac{11}{96x^3} + \frac{31}{384x^4}\right) \\ &= \ln\left(1+\frac{1}{2x}\right) - \frac{2880x^3 - 720x^2 + 240x + 457}{5760x^4} \\ &< \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} - \frac{2880x^3 - 720x^2 + 240x + 457}{5760x^4} = -\frac{457}{5760x^4} < 0. \end{aligned}$$

This completes the proof. □

Replacement of x by $n/2$ in (5.11) yields

$$\begin{aligned} \left(\sqrt{n\pi}\right)^{-\frac{1}{n(n+1)}} \exp\left(\frac{1}{2n} + \frac{1}{4n^2} - \frac{11}{12n^3}\right) &< \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} \\ &< \left(\sqrt{n\pi}\right)^{-\frac{1}{n(n+1)}} \exp\left(\frac{1}{2n} + \frac{1}{4n^2} - \frac{11}{12n^3} + \frac{31}{24n^4}\right), \qquad n \in \mathbb{N}. \end{aligned} \tag{5.12}$$

The expansion (5.10) motivated us to establish the following sharp inequality for $\Omega_n^{1/n}/\Omega_{n+1}^{1/(n+1)}$.

Theorem 5.3. *For $n \in \mathbb{N}$, we have*

$$\left(\sqrt{n\pi}\right)^{-\frac{1}{n(n+1)}}\left(1+\frac{1}{2n-\frac{4-3\pi^{1/4}}{2-\pi^{1/4}}}\right) \le \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} < \left(\sqrt{n\pi}\right)^{-\frac{1}{n(n+1)}}\left(1+\frac{1}{2n-\frac{3}{2}}\right). \tag{5.13}$$

The constants $\frac{4-3\pi^{1/4}}{2-\pi^{1/4}} = 0.008963\ldots$ *and* $\frac{3}{2}$ *are the best possible.*

Proof. First, we establish the left-hand inequality of (5.13). Elementary calculations show that this is valid for for $n = 1, 2$ and 3. We now prove the left-hand inequality of (5.13) for $n \ge 4$. For this it suffices to show by appeal to (5.12) that

$$G(n) > 0 \quad \text{for} \quad n \ge 4,$$

where

$$G(x) = \frac{1}{2x} + \frac{1}{4x^2} - \frac{11}{12x^3} - \ln\left(1+\frac{1}{2x-a}\right) \quad \text{with} \quad a = \frac{4-3\pi^{1/4}}{2-\pi^{1/4}}.$$

Differentiation yields

$$G'(x) = -\frac{P_3(x-4)}{4x^4(2x+1-a)(2x-a)},$$

where

$$P_3(x) = 29a^2 - 493a + 40 + (18a^2 - 422a + 234)x \\ + (2a^2 - 106a + 104)x^2 + (12-8a)x^3 > 0 \quad \text{for} \quad x \ge 0.$$

We then obtain $G'(x) < 0$ for $x \ge 4$. So, the sequence $\{G(n)\}$ is strictly decreasing for $n \ge 4$, and we have

$$G(n) > \lim_{m\to\infty} G(m) = 0 \quad \text{for} \quad n \ge 4.$$

This proves the left-hand inequality of(5.13) for $n \ge 1$.

Now, we establish the right-hand inequality of (5.13). Elementary calculations show that this is valid for $n = 1$. We now prove that the right-hand inequality of (5.13) holds for $n \ge 2$. It suffices to show by appeal to (5.12) that

$$g(n) < 0 \quad \text{for} \quad n \ge 2,$$

where

$$g(x) = \frac{1}{2x} + \frac{1}{4x^2} - \frac{11}{12x^3} + \frac{31}{24x^4} - \ln\left(1+\frac{1}{2x-\frac{3}{2}}\right).$$

Differentiation yields

$$g'(x) = \frac{692 + 2211(x-2) + 2098(x-2)^2 + 606(x-2)^3}{12x^5(4x-1)(4x-3)} > 0 \qquad (x \ge 2).$$

Hence, the sequence $\{g(n)\}$ is strictly increasing for $n \ge 2$, and we have

$$g(n) < \lim_{m\to\infty} g(m) = 0 \quad \text{for} \quad n \ge 2.$$

This proves that the right-hand inequality o(5.13) for $n \ge 1$.

If we write (5.13) as

$$\frac{4-3\pi^{1/4}}{2-\pi^{1/4}} \le x_n < \frac{3}{2}, \qquad x_n = 2n - \left(\frac{\left(\sqrt{n\pi}\right)^{\frac{1}{n(n+1)}}\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} - 1\right)^{-1},$$

we find that

$$x_1 = \frac{4-3\pi^{1/4}}{2-\pi^{1/4}} \quad \text{and} \quad \lim_{n\to\infty} x_n = \frac{3}{2}. \tag{5.14}$$

The limit in (5.14) is obtained by using the asymptotic expansion (5.10). Hence, the inequality (5.13) holds for $n \geq 1$, and the constants $\frac{4-3\pi^{1/4}}{2-\pi^{1/4}}$ and $\frac{3}{2}$ are the best possible. □

Remark 5.1. *Write* (1.1) *as*

$$\left(\frac{4}{\pi}\right)^{1/(2n)} \leq \frac{\Omega_n^{1/n}}{\Omega_{n+1}^{1/(n+1)}} < e^{1/(2n)}. \tag{5.15}$$

The inequalities (5.13) *are sharper than the inequalities* (5.15) *for* $n \geq 2$.

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