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REDUCTION FOR NATURAL OPERATORS ON PROJECTABLE CONNECTIONS

Abstract. We present a very simple proof of a general reduction for natural operators on torsion free projectable classical linear connections.

A reduction for natural operators on classical linear connections on manifolds is a very old and known fact, but the known for the authors proof of it (see e.g [1]) is very complicated. In the present paper we propose a very simple proof of it in a more general than in [1] situation (namely, for natural operators on projectable torsion free classical linear connections on fibred manifolds).

We start with the following standard notions and definitions, see e.g. [1].

A fibred manifold is a surjective submersion $p : Y \to M$ between manifolds. Given another fibred manifold $p_1 : Y_1 \to M_1$ a map $f : Y \to Y_1$ is called fibred iff there exists a (unique) underlying map $\underline{f} : M \to M_1$ such that $p_1 \circ f = \underline{f} \circ p$. All fibred manifolds and their fibred maps form a category which is denoted by $\mathcal{FM}$. A fibred manifold $p : Y \to M$ is of dimension (m, n) iff Y is of dimension $m+n$ and M is of dimension m. All fibred manifolds of dimension (m, n) and their fibred embeddings (fibred maps which are diffeomorphisms onto open subsets) form a category which is denoted by $\mathcal{FM}_{m,n}$.

We recall that a classical linear connection on a manifold N is an $\mathbf{R}$-bilinear map $\nabla : \mathcal{X}(N) \times \mathcal{X}(N) \to \mathcal{X}(N)$ such that (1) $\nabla_{fX}Y = f\nabla_X Y$ and (2) $\nabla_X fY = XfY + f\nabla_X Y$ for any map $f : N \to \mathbf{R}$ and any vector fields $X, Y \in \mathcal{X}(N)$ on N. A classical linear connection ∇ on N is called torsion-free iff its torsion tensor field T vanishes. (We recall that T is given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ for any $X, Y \in \mathcal{X}(N)$.)

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Let $p : Y \to M$ be a fibred manifold. A classical linear connection ∇ on Y is called projectable (with respect to p) iff there exists a (unique) classical linear connection $\underline{\nabla}$ on M such that ∇ is p-related to $\underline{\nabla}$. (The last condition means that if $X, Z \in \mathcal{X}(Y)$ and $\underline{X}, \underline{Z} \in \mathcal{X}(M)$ are such that $Tp \circ X = \underline{X} \circ p$ and $Tp \circ Z = \underline{Z} \circ p$ then $Tp \circ \nabla_X Z = \underline{\nabla}_{\underline{X}} \underline{Z} \circ p$.)

A general concept of bundle functors and natural operators can be found in [1]. In the present note we need only the following partial concepts.

Definition 1. A bundle functor on (m, n)-fibred manifolds is a covariant functor $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ satisfying the following properties:

(1) (*Base preservation*) $B \circ F = \mathcal{T}_{|\mathcal{FM}_{m,n}}$, where $B : \mathcal{FM} \to \mathcal{M}f$ is the base functor ($B(q : Z \to N) = N$, $B(g) = \underline{g}$) and $\mathcal{T} : \mathcal{FM} \to \mathcal{M}f$ is the total space functor ($\mathcal{T}(p : Y \to M) = Y$, $\mathcal{T}(f) = f$). Hence the induced projections $\pi : F(Y) \to Y$ form the functor transformation $\pi : F \to \mathcal{T}_{|\mathcal{FM}_{m,n}}$;

(2) (*Locality*) If $U \subset Y$ is an open subset then $F(U) = \pi_Y^{-1}(U)$ and $F(i_U)$ is the inclusion of $F(U)$ onto $\pi_Y^{-1}(U)$.

Definition 2. Let $F, G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be bundle functors. An $\mathcal{FM}_{m,n}$-natural operator $D : G \times Q_{\tau-proj} \rightsquigarrow F$ is a system of $\mathcal{FM}_{m,n}$-invariant regular operators (functions)

$$D : \underline{GY} \times Q_{\tau-proj}Y \to \underline{FY}$$

for any $\mathcal{FM}_{m,n}$-object $p : Y \to M$, where $\underline{FY}$ (or $\underline{GY}$) is the set of all sections on $FY \to Y$ (or $GY \to Y$) and $Q_{\tau-proj}(Y)$ is the set of all torsion free projectable classical linear connections on $Y \to M$. The invariance means that if $\sigma_1 \in \underline{GY_1}$ and $\sigma_2 \in \underline{GY_2}$ are Ψ-related by an $\mathcal{FM}_{m,n}$-map $\Psi : Y_1 \to Y_2$ (i.e. $G\Psi \circ \sigma_1 = \sigma_2 \circ \Psi$) and $\nabla_1 \in Q_{\tau-proj}(Y_1)$ and $\nabla_2 \in Q_{\tau-proj}(Y_2)$ are also Ψ-related (i.e. Ψ is (∇_1, ∇_2)-affine), then so are $D(\sigma_1, \nabla_1)$ and $D(\sigma_2, \nabla_2)$ (i.e. $F\Psi \circ D(\sigma_1, \nabla_1) = D(\sigma_2, \nabla_2) \circ \Psi$). The regularity means that D transforms smoothly parametrized families of pairs of sections and connections into smoothly parametrized families of sections.

Definition 3. We say that a natural operator $D : G \times Q_{\tau-proj} \rightsquigarrow F$ is of finite order s if for any $y \in Y$ and any $(\sigma, \nabla) \in \underline{GY} \times Q_{\tau-proj}(Y)$ the value $D(\sigma, \nabla)(y)$ depends only on the s-jet $j^s_y(\sigma, \nabla)$ of (σ, ∇) at y.

We have the following simple examples of bundle functors and natural operators of finite order in the sense of Definitions 1–3.

Example 1. A very simple example of a bundle functor on (m, n)-fibred manifolds is the identity functor $id_{m,n} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ sending any $\mathcal{FM}_{m,n}$-object $p : Y \to M$ into $\pi = id_Y : Y \to Y$ and any $\mathcal{FM}_{m,n}$-map

$f : Y \to Y_1$ into $f : Y \to Y_1$. A simple example of a bundle functor on (m, n)-fibred manifolds is the tangent bundle functor $T : \mathcal{FM}_{m,n} \to \mathcal{FM}$ sending any $\mathcal{FM}_{m,n}$-object $p : Y \to M$ into the tangent bundle $\pi : TY \to Y$ of Y and any $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ into the tangent map $Tf : TY \to TY_1$ of f. Another simple example is the cotangent bundle functor $T^* : \mathcal{FM}_{m,n} \to \mathcal{FM}$ given by $T^*Y = (TY)^*$ and $T^*f = (Tf^{-1})^*$ for any $p : Y \to M$ and $f : Y \to Y_1$ as above. A more complicated example of a bundle functor on (m, n)-fibred manifolds is a bundle functor $T^{1,3} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ of tensors of type $(1, 3)$ sending any $\mathcal{FM}_{m,n}$-object $p : Y \to M$ into the bundle $T^{1,3}Y = \otimes^3 T^*Y \otimes TY$ of tensors of type $(1, 3)$ and any $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ into the induced map $T^{1,3}f := \otimes^3 T^*f \otimes Tf : T^{1,3}Y \to T^{1,3}Y_1$.

Example 2. Let $G = id_{m,n} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ and $F = T^{1,3} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be as in Example 1. A standard (well-known) example of an $\mathcal{FM}_{m,n}$-natural operator $D : G \times Q_{\tau-proj} \rightsquigarrow F$ is the curvature operator $\mathcal{R}$ sending a projectable classical linear connection ∇ on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$ into the well-known curvature tensor $\mathcal{R}(\nabla)$ of ∇ (we note that in our situation $\underline{G}Y$ is the one-element set). Clearly, $\mathcal{R}$ is of order $s = 1$.

Example 3. Let $G = T : \mathcal{FM}_{m,n} \to \mathcal{FM}$ and $F = T^{1,3} : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be as in Example 1. Let $p : Y \to M$ be an $\mathcal{FM}_{m,n}$-object. We define $D : \underline{GY} \times Q_{\tau-proj}(Y) \to \underline{FY}$ by $D(X, \nabla) := \nabla_X \mathcal{R}(\nabla)$, the covariant ∇-derivative of the curvature tensor $\mathcal{R}(\nabla)$ of ∇ in direction $X \in \mathcal{X}(Y) = \underline{GY}$. The family $D : G \times Q_{\tau-proj} \rightsquigarrow F$ of functions $D : \underline{GY} \times Q_{\tau-proj}(Y) \to \underline{FY}$ for any $\mathcal{FM}_{m,n}$-object $p : Y \to M$ is an $\mathcal{FM}_{m,n}$-natural operator of order $s = 2$.

To present a general example of an $\mathcal{FM}_{m,n}$-natural operator $D : G \times Q_{\tau-proj} \rightsquigarrow F$ of finite order for arbitrary bundle functors $F, G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ we use the following lemma (being an obvious generalization of the well-known fact on the construction of normal coordinates for classical linear connections on manifolds).

Lemma 1. *Let ∇ be a torsion free projectable classical linear connection on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$ covering $\underline{\nabla}$ on M. Let $y \in Y_x$, $x \in M$.*

(a) *There exists a fibred normal coordinate system $\Psi^y : (U, y) \to (\mathbf{R}^m \times \mathbf{R}^n, (0, 0))$ on Y of ∇ with center y covering a normal coordinate system $\underline{\Psi}^x : (\underline{U}, x) \to (\mathbf{R}^m, 0)$ on M of $\underline{\nabla}$ with center x.*

(b) *If Ψ^y_1 is another such fibred normal coordinate system then there exists $\Phi \in GL(m, n)$ (=the group of all fibred linear isomorphisms $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$) such that $\Psi^y_1 = \Phi \circ \Psi^y$ near y.*

Proof. The proof of Lemma 1 is standard and is presented in some previous papers (e.g. by the first author). (We propose to use the well-known construction (from the exponent) of normal coordinates of classical linear connections on manifolds and next to to use a simple observation that if ∇ is a projectable torsion-free classical linear connection on $p : Y \to M$ with the underlying connection $\underline{\nabla}$ on M then the exponent Exp_y of ∇ at y is fibred over the exponent $Exp_{p(x)}$ of $\underline{\nabla}$ (this is a simple consequence of the definition of the exponent and the fact that the p-projections of ∇-geodesics are $\underline{\nabla}$-geodesics as p is a surjective submersion).) ∎

We are in position to present the following general (maybe known) example of $\mathcal{FM}_{m,n}$-natural operator $D : G \times Q_{\tau-proj} \rightsquigarrow F$.

Example 4 (A general construction). Let s be a positive integer. Let $S : (J^sG\mathbf{R}^{m,n})_{(0,0)} \times Q^s \to F_{(0,0)}\mathbf{R}^{m,n}$ be a $GL(m,n)$-invariant map, where $(J^sG\mathbf{R}^{m,n})_{(0,0)}$ is the space of all s-jets at $(0,0) \in \mathbf{R}^m \times \mathbf{R}^n$ of sections from $\underline{G\mathbf{R}^{m,n}}$, $\mathbf{R}^{m,n}$ is the standard $\mathcal{FM}_{m,n}$-object $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ (the usual projection) and Q^s is the vector space of all s-jets $j^s_{(0,0)}(\nabla^i_{jk})$ of torsion free projectable classical linear connections ∇ on $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ with the Christoffell symbols ∇^i_{jk} $(i,j,k = 1,...,m+n)$ with respect to the usual fiber coordinates $x^1,...,x^m,x^{m+1},...,x^{m+n}$ on $\mathbf{R}^m \times \mathbf{R}^n$ satisfying $\sum_{j,k=1}^{m+n} \nabla^i_{jk}(x)x^jx^k = 0$ for $i = 1,...,m+n$ or equivalently the usual coordinates $x^1,...,x^{m+n}$ are ∇-normal with center $(0,0)$. (The equivalence of the last two conditions can be obtained by applying the well-known differential equations on geodesics and by using the fact that in normal coordinates the geodesics passing by the centrum are straight lines.) Let $p : Y \to M$ be an $\mathcal{FM}_{m,n}$-object. Let $y \in Y_x$, $x \in M$. Let $\sigma \in \underline{GY}$ and $\nabla \in Q_{\tau-proj}(Y)$. Let Ψ^y be a fibred normal coordinate system from Lemma 1. We define

$$D^S(\sigma,\nabla)(y) = F(\Psi^y)^{-1}(S(j^s_{(0,0)}(\Psi^y_*\sigma), j^s_{(0,0)}(\Psi^y_*\nabla))) \in FY,$$

where $\Psi^y_*\sigma = G(\Psi^y) \circ \sigma \circ (\Psi^y)^{-1}$ is the image of σ by Ψ^y and similarly $\Psi^y_*\nabla$ is the image of ∇ by Ψ^y. If Ψ^y_1 is another such fibred normal coordinate system then (by Lemma 1 (2)) $\Psi^y_1 = C \circ \Psi^y$ for some $C \in GL(m,n)$. By the $GL(m,n)$-invariance of S, the definition of $D^S(\sigma,\nabla)(y)$ is correct (independent of the choice of Ψ_y). Then we have the resulting section $D^S(\sigma,\nabla) \in \underline{FY}$. Then we have the resulting operator (function) $D^S : \underline{GY} \times Q_{\tau-proj}(Y) \to \underline{FY}$. The family $D^S : G \times Q_{\tau-proj} \rightsquigarrow F$ of the above functions $D^S : \underline{GY} \times Q_{\tau-proj} \to \underline{FY}$ for any $\mathcal{FM}_{m,n}$-object $p : Y \to M$ is an $\mathcal{FM}_{m,n}$-natural operator of order s.

We have

Theorem 2. *Any $\mathcal{FM}_{m,n}$-natural operator $D : G \times Q_{\tau-proj} \rightsquigarrow F$ of finite order s is of the form $D = D^S$ for some uniquely determined $GL(m,n)$-invariant map $S : (J^sG\mathbf{R}^{m,n})_{(0,0)} \times Q^s \to F_{(0,0)}\mathbf{R}^{m,n}$.*

Proof. We must define $S : (J^s G\mathbf{R}^{m,n})_{(0,0)} \times Q^s \to F_{(0,0)}\mathbf{R}^{m,n}$ by

$$S(j^s_{(0,0)}\sigma, j^s_{(0,0)}\nabla) = D(\sigma, \nabla)(0,0) \in F_{(0,0)}\mathbf{R}^{m,n} \ ,$$

$j^s_{(0,0)}\nabla \in Q^s$, σ a section of $G\mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$. The definition of S is correct because D is of order s. By the invariance of D and D^S and Lemma 1(1) we immediately see that $D = D^S$. ∎

Remark 1. For $n = 0$, we reobtained (in some another more geometrical form) the main technical result from Section 28 in [1] (namely Proposition 28.9). It seem that our general proof is less complicated than the one of Section 28 in [1]. This "trick" generalizes similar partial "tricks" applied in previous papers, e.g. [2], [3] and others.

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