DEMONSTRATIO MATHEMATICA
Vol. XLII No 1 2009
10.1515/dema-2013-0145

W. M. Mikulski

RIEMANNIAN VECTOR BUNDLES HAVE NO CANONICAL LINEAR CONNECTIONS

Abstract. We prove that Riemannian vector bundles have no canonical linear connections.

Introduction

Given a vector bundle $E \to M$, a Riemannian structure on $E \to M$ is a map $G : E \times_M E \to \mathbf{R}$ such that for any $x \in M$ the restriction $G_x : E_x \times E_x \to \mathbf{R}$ of G is an inner product on the fiber E_x of $E \to M$ over x (i.e. it is symmetric bilinear and positive define). For example, if $E = TM \to M$ is a tangent bundle of a manifold M, then a Riemannian structure on $TM \to M$ is called a Riemannian structure on M.

Given a vector bundle $E \to M$, by a linear connection D on $E \to M$ we mean an $\mathbf{R}$-bilinear map $D : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$ such that

(i) $D_{fX}\sigma = fD_X\sigma$ and
(ii) $D_X f\sigma = Xf\sigma + fD_X\sigma$

for any vector field $X \in \mathcal{X}(M)$ on M, any map $f : M \to \mathbf{R}$ and any section $\sigma \in \Gamma(E)$ of $E \to M$. For example, if $E = TM \to M$ is the tangent bundle of a manifold M, then a linear connection on $TM \to M$ is called a classical linear connection on M.

Example 1. Let g be a Riemannian structure on a manifold M. It is well-known that there exist many classical linear connections ∇ on M such that

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{1}$$

Key words and phrases: vector bundle, Riemannian vector bundle, classical linear connection, linear connection.

1991 *Mathematics Subject Classification*: 58A20, 58A32.

for any vector fields X, Y, Z on M. However, if ∇ satisfying the above property (1) satisfies also an additional condition (depending canonically on ∇ and g) saying that

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \tag{2}$$

for any vector fields X, Y on M, then such connection ∇ is unique. This is the well-known Levi-Civita connection of g.

Example 2. Let G be a Riemannian structure on a vector bundle $E \to M$. Similarly as in the Riemannian manifold case, there exist many linear connections D on a $E \to M$ such that

$$XG(\sigma, \eta) = G(D_X \sigma, \eta) + G(\sigma, D_X \eta) \tag{3}$$

for any vector field $X \in \mathcal{X}(M)$ and any sections $\sigma, \eta \in \Gamma(E)$, see [4].

So, we have the following natural question.

Question 1. Whether there exists a condition

$$C(G, D) \tag{4}$$

(canonically determined by G and D) such that D satisfying (3) and this additional condition (4) is uniquely determined? In other words, whether do Riemannian structures G on a vector bundle have (induce canonically) linear connections (like Levi-Civita one)?

In this note we prove that the answer to the above question is negative. In fact, we prove a more general result that there is no canonical condition

$$C(G, D, \nabla) \tag{5}$$

determined by G, D and an additional classical linear connection ∇ on M such that D satisfying (3) and condition (5) is uniquely determined.

All manifolds and maps are assumed to be smooth (of class C^∞).

1. The main result

To present a mathematical formulation of the main result of the paper we need the following definition being a particular case of a definition of natural operators from [3].

Let $\mathcal{VB}_{m,n}$ be the category of vector bundles with m-dimensional bases and n-dimensional fibres and their (local) vector bundle isomorphisms.

Definition 1. A $\mathcal{VB}_{m,n}$-gauge natural operator $A : C \times Riem \rightsquigarrow Q$ is a $\mathcal{VB}_{m,n}$-invariant family

$$A : Con_{clas}(M) \times Riem(E) \to Con(E)$$

of operators for any $\mathcal{VB}_{m,n}$-object $E \to M$, where $Con_{clas}(M)$ is the set of all classical linear connections on M, $Riem(E)$ is the set of all Riemannian structures on $E \to M$ and $Con(E)$ is the set of all linear connections on $E \to M$. The invariance means that if $(\nabla_1, G_1) \in Con_{clas}(M_1) \times Riem(E_1)$ and $(\nabla_2, G_2) \in Con_{clas}(M_2) \times Riem(E_2)$ are Φ-related by an $\mathcal{VB}_{m,n}$-map $\Phi : E_1 \to E_2$ then so are $A(\nabla_1, G_1)$ and $A(\nabla_2, G_2)$.

Now, a negative answer of Question 1 follows (obviously) from the following theorem (which is the main result of the present note).

Theorem 1. *There is no $\mathcal{VB}_{m,n}$-gauge natural operator $A : C \times Riem \rightsquigarrow Q$ transforming Riemannian structures $G : E \times_M E \to \mathbf{R}$ on vector bundles $E \to M$ and classical linear connections ∇ on M into linear connections $A(\nabla, G)$ on $E \to M$.*

2. Preparations to the proof of Theorem 1

In the proof of Theorem 2 we will use the following well-known facts.

Proposition 1. ([2]) *Let ∇ be a classical linear connection on a connected manifold N. Then the group $Aff(\nabla)$ of all ∇-affine isomorphisms is a Lie group.*

Proposition 2. ([4; Proposition 2.116]) *Let ∇ be a classical linear connection on a connected manifold N. Let $f, g : N \to N$ be ∇-affine maps. If $j^1_x f = j^1_x g$ at some point $x \in N$ then $f = g$.*

We will also use the following fact.

Proposition 3. ([1], [3]) *Let D be a linear connection on a vector bundle $E \to M$ and ∇ be a classical linear connection on M. Then there exists a unique classical linear connection $\Gamma = \Gamma(D, \nabla)$ on the total space E with the following property*

$$\Gamma_{X^D} Y^D = (\nabla_X Y)^D,\ \Gamma_{X^D} s^V = (D_X s)^V,$$
$$\Gamma_{s^V} X^D = 0,\ \Gamma_{s^V} \sigma^V = 0,$$

for all vector fields X, Y on M and all sections s, σ of $E \to M$. Here $X^D \in \mathcal{X}(E)$ denotes the D-horizontal lift of X and $s^V \in \mathcal{X}(E)$ means the vertical lift of s, $s^V(e) = [e + ts(x)]$, $e \in E_x$, $x \in M$.

3. Proof of Theorem 1

Suppose that $A : C \times Riem \rightsquigarrow Q$ is such a $\mathcal{VB}_{m,n}$-gauge natural operator. Let $E = \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ be the trivial vector bundle. Let $G^o \in Riem(E)$ be the trivial Riemannian structure , i.e. $G^o_x = <,>: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ for any $x \in \mathbf{R}^m$, where $<,>$ is the standard scalar multiple on $\mathbf{R}^n$. Let ∇^o be the usual flat classical linear connection on $\mathbf{R}^m$. Then on E we can define a classical linear connection

$$\Theta = \Gamma(A(\nabla^o, G^o), \nabla^o)\ ,$$

where operator Γ is defined in Proposition 3. We have a group monomorphism (injection) $I : C^\infty(\mathbf{R}^m, O(n)) \to Aut(\mathbf{R}^m \times \mathbf{R}^n)$, $I(B) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$,

$$I(B)(x, y) = (x, B(x)y) ,$$

$(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$. Given $B \in C^\infty(\mathbf{R}^m, O(n))$, $I(B)$ preserves ∇^o and B^o. Then $I(B)$ preserves $A(\nabla^o, G^o)$ (because of the invariance of A) and consequently $I(B)$ preserves Θ (because of the invariance of the construction Γ). Then (in fact) $I : C^\infty(\mathbf{R}^m, O(n)) \to Aff(\Theta)$ is a group inclusion. This is a contradiction because $Aff(\Theta)$ is a Lie group (see Proposition 1) and $C^\infty(\mathbf{R}^m, O(n))$ is not finite dimensional.

4. Another proof of Theorem 1

Suppose that such operator A exists. We use the notations of Section 3. In particular, Θ and I be as in Section 3. Consider $B, C \in C^\infty(\mathbf{R}^m, O(n))$ such that $B(0) = C(0)$ and $B \neq C$. Then $I(B)$ and $I(C)$ are Θ-affine maps such that $j^1_{(0,0)}(I(B)) = j^1_{(0,0)}(I(C))$ and $I(B) \neq I(C)$. Contradiction because of Proposition 2. ■

References

[1] J. Gancarzewicz, I. Kolář, *Some gauge-natural operators on linear connections*, Monats. Math. 111 (1991), 23–33.

[2] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, New York, 1963.

[3] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[4] W. A. Poor, *Differential Geometric Structures*, McGraw-Hill Book Company, New York, 1981.

INSTITUTE OF MATHEMATICS
JAGIELLONIAN UNIVERSITY
Łojasiewicza 4
KRAKÓW, POLAND
E-mail: Wlodzimierz.Mikulski@im.uj.edu.pl

Received December 17, 2007; revised version March 13, 2008.