Supplementary information

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DERIVATION OF THE LANGEVIN EQUATION

The microscopic dynamics of the driven-dissipative Rydberg ensemble is described by the master equation Eq. (1). For realistic system sizes required for SOC, it becomes, however, intractable due to the fast growth of the Hilbert space. In order to reduce this theoretical complexity, we eliminate irrelevant degrees of freedom and map the dynamics to the Langevin equation Eq. (2).

Adiabatic elimination of atomic coherences — In the presence of strong dephasing $\gamma_{\mathrm{de}} \gg \Omega$ the evolution of the atomic coherences $\hat{\sigma}_l^{gr}, \hat{\sigma}_l^{0r}$ is dominated by a rapid dissipative decay towards their time averaged expectation values $\langle \hat{\sigma}_l^{\alpha,r} \rangle_T = \frac{1}{T} \int_0^T \mathrm{Tr}\, [\hat{\sigma}_l^{\alpha,r} \hat{\rho}]\, dt$, where $T \approx \Omega^{-1}$ is the typical time scale for a facilitated Rabi oscillation, Tr is the trace over the many-body Hilbert space and $\alpha = g, 0$. The coherences are static on many-body time scales and will be adiabatically eliminated by solving

$$0 \overset{!}{=} \partial_t \langle \hat{\sigma}_l^{\alpha,r} \rangle = \mathrm{Tr}\, [\hat{\sigma}_l^{\alpha,r} \partial_t \hat{\rho}]. \tag{S1}$$

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Here $\partial_t\hat{\rho}$ is set by Eq. (1M). After eliminating the coherences, the system evolution is governed by the remaining degrees of freedom, i.e. the average densities

$$m_l \equiv \mathrm{Tr}\left[\hat{\sigma}_l^{rr}\hat{\rho}\right], \;\; n_l \equiv \mathrm{Tr}\left[(\hat{\sigma}_l^{rr} + \hat{\sigma}_l^{gg})\hat{\rho}\right]. \tag{S2}$$

Their equation of motion is

$$\partial_t m_l = \mathrm{Tr}\left[\hat{\sigma}_l^{rr}\partial_t\hat{\rho}\right], \;\; \partial_t n_l = \mathrm{Tr}\left[(\hat{\sigma}_l^{rr} + \hat{\sigma}_l^{gg})\partial_t\hat{\rho}\right] \tag{S3}$$

and $\partial_t\hat{\rho}$ is set by Eq. (1M) and constrained to configurations that fulfill Eq. (S1). Explicit evaluation yields

$$\partial_t n_l = -\Gamma m_l + \xi_l^n, \tag{S4}$$

$$\partial_t m_l = \mathrm{Tr}\left(\frac{\Omega^2(\Gamma + \gamma_{\mathrm{de}})(\hat{\sigma}_l^{gg} - \hat{\sigma}_l^{rr})}{(\Gamma + \gamma_{\mathrm{de}})^2 + 4(\hat{V}_l - \Delta)^2}\hat{\rho}\right) - \Gamma m_l + \xi_l^m. \tag{S5}$$

The Markovian noise fields $\xi_l^{m,n}$ enforce the non-equilibrium fluctuation relation, which is imprinted by the dissipative environment (S2). The statistics of n_l, m_l, imprinted by drive and dissipation, are expressed by the vanishing mean $\langle\xi_l^{n,m}\rangle = 0$ and non-vanishing variance $\mathrm{var}(\xi_l^{n,m}) \neq 0$, and the Markovianity, i.e. locality in time and space, of the noise. Their variance is determined by the generalized Einstein relation

$$\mathrm{var}(\xi_l^m) = \partial_t\langle(\hat{\sigma}_l^{rr})^2\rangle - 2\langle\hat{\sigma}_l^{rr}\partial_t\hat{\sigma}_l^{rr}\rangle,$$

$$= \tau n_l + (\Gamma + 2\tau)m_l + O(m_l^2), \tag{S6}$$

and similar for ξ_l^n with $\hat{\sigma}_l^{rr} \to \hat{\sigma}_l^{rr} + \hat{\sigma}_l^{gg}$. In the limit $\tau \to 0$, the variance of ξ_l^m is multiplicative in m_l which is a necessary condition for a robust absorbing phase.

Equation (S5) is a coupled set of differential equations describing the Rydberg population and total remaining population for each atom, satisfying the completeness relation $\hat{\sigma}_l^{rr} + \hat{\sigma}_l^{gg} + \hat{\sigma}_l^{00} = \mathbb{1}$. Interatomic interactions enter as an effective detuning $\hat{V}_l = C_6\sum_{l'\neq l}\hat{\sigma}_{l'}^{rr}/|\mathbf{r}_{l,l'}|^6$, where $|\mathbf{r}_{l,l'}| = |\mathbf{r}_l - \mathbf{r}_{l'}|$ is the distance between atom l and l' [1].

In order to expand the trace in Eq. (S5) in the projection operators $\hat{\sigma}_{l'}^{rr}$, one exploits the fact that $\hat{\sigma}_{l'}^{rr} = (\hat{\sigma}_{l'}^{rr})^2$. For an arbitrary function f of the projectors $\hat{\sigma}_{l'}^{rr}$, up to linear order in the projection operators one finds $f(\{\hat{\sigma}_{l'}^{rr}\}) = f(0) + \sum_m[f(\hat{\sigma}_{l'\neq m}^{rr} = 0, \hat{\sigma}_m^{rr} = 1) - f(0)]\hat{\sigma}_m^{rr}$.

Consequently, the operator acting on atom l

$$\frac{\Omega^2(\Gamma + \gamma_{\rm de})}{(\Gamma + \gamma_{\rm de})^2 + 4(\hat{V}_l - \Delta)^2} = \underbrace{\frac{\Omega^2(\Gamma + \gamma_{\rm de})}{(\Gamma + \gamma_{\rm de})^2 + 4\Delta^2}}_{=\tau} +$$

$$\sum_{l'} \left[\frac{\Omega^2(\Gamma + \gamma_{\rm de})}{(\Gamma + \gamma_{\rm de})^2 + 4(\Delta - {\rm C}_6 |\mathbf{r}_{{\rm l},l'}|^{-6})^2} - \tau \right] \hat{\sigma}_{l'}^{rr} + ..., \tag{S7}$$

followed by higher order products of projectors (i.e. terms $\sim \hat{\sigma}_{l'}^{rr} \hat{\sigma}_m^{rr}$). The first term on the right hand side describes single particle excitations with rate τ, while the second term describe the facilitated (de-)excitation of atom l by another atom l' in the Rydberg state. This describes a Lorentzian peaked at the facilitation radius $|\mathbf{r}_{l,l'}| = ({\rm C}_6/\Delta)^{1/6} \equiv r_{\rm fac}$ and deviates considerably from zero only for $|\mathbf{r}_{l,l'}| \in [r_{\rm fac} - \Delta r_{\rm fac}, r_{\rm fac} + \Delta r_{\rm fac}]$ with $\Delta r_{\rm fac} = r_{\rm fac} \frac{\Gamma + \gamma_{\rm de}}{12\Delta}$. Introducing a projector $\Pi_{ll'}$ with $\Pi_{ll'} = 1$ if $|\mathbf{r}_{l,l'}| \in [r_{\rm fac} - \Delta r_{\rm fac}, r_{\rm fac} + \Delta r_{\rm fac}]$ and zero elsewhere, Eqs. (S5)-(S7) yield

$$\partial_t m_l = \left(\tau + \sum_{l'} \frac{\Omega^2 \Pi_{ll'} m_{l'}}{\Gamma + \gamma_{\rm de}} \right)(n_l - 2m_l) - \Gamma m_l + \xi_l^m. \tag{S8}$$

Equation (S8) provides a good approximation to the facilitation rate assuming the excitation density is small, but overestimates the true facilitation rate when there is several Rydberg excitations in proximity to state l (due to truncating the expansion (S7) at first order). An exact computation of the facilitation radius for $w \geq 1$ excited states inside a single shell shows that it grows as $r_R^{(w)} = w^{1/6} r_R$ (in $d = 3$ dimensions). For a homogenous distribution of atoms, this yields a facilitation rate that grows proportional to $\sqrt{w}$, which is not a severe correction compared to the $\propto w$ growth predicted by Eq. (S8) if one bears in mind the largely suppressed off-resonant excitation rate. More than a single excitation inside the facilitation radius, i.e., $w > 1$, can only be realized via additional spontaneous excitation events. The probability for $w > 1$ is a factor of O(10^{-4}) smaller compared to $w \leq 1$ and will have no impact on the dynamics.

Continuum Limit — For the reported experiments the atoms are free to move on the timescale of the slow SOC dynamics. However, the diffusion time scale (set by the temperature) for distances of the order of the facilitation radius $O(r_{fac})$ is about one order of magnitude slower than the inverse Rabi frequency. This justifies an effectively static model for the external degrees of freedom while ensuring that the the ground state density remains approximately homogenous on length scales compared to the facilitation radius. Thus we

can coarse grain the dynamics by averaging the densities over facilitation shells

$$\rho(\mathbf{r}, t) \equiv \mathcal{N} \sum_{|\mathbf{r}_l - \mathbf{r}| \leq r_{\text{fac}}} m_l, \quad n(\mathbf{r}, t) \equiv \mathcal{N} \sum_{|\mathbf{r}_l - \mathbf{r}| \leq r_{\text{fac}}} n_l, \tag{S9}$$

where $\mathcal{N} = (\frac{4\pi}{3} r_{\text{fac}}^3)^{-1}$ is the normalization volume.

This coarse-graining procedure modifies the completeness relation compared to the single atom case. An excited atom facilitates excitations at the border of the facilitation shell but blocks the excitation for any atoms within the shell. Decay of a Rydberg excitation to a removed state thus removes the blockade constraint on the remaining ground state atoms. At the scale of the facilitation radius, the averaging procedure (S9) yields the effective rate of decay into removed states is $\Gamma \to b\Gamma$ and adds an effective decay rate back to the ground state $\Gamma(1 - b)$, where $b = \rho(\mathbf{r}, t)/n(\mathbf{r}, t) \approx \text{const}$. Defining $n_t \equiv n(\mathbf{r}, t), \rho_t \equiv \rho(\mathbf{r}, t)$, the averaged densities evolve as

$$\partial_t n_t = -b\Gamma\rho_t + \xi_t, \tag{S10}$$

$$\partial_t \rho_t = -\Gamma\rho_t + \xi_t + (n_t - 2\rho_t)\left(\tau + \frac{\Omega^2}{\Gamma + \gamma_{\text{de}}}\mathcal{M}(\rho_t)\right). \tag{S11}$$

The averaged noise ξ_t remains Markovian in time and space with variance $\text{var}(\xi_t) = \tau n_t + (\Gamma + 2\tau)\rho_t$. The nonlinearity $\mathcal{M}(\rho_t)$ is obtained from the execution of the density averaging (S9) in the sum $\sim \sum_{l'} \Pi_{ll'} m_{l'}$ in Eq. (S8). It is a non-local function in space and has to be read as $\rho_t \mathcal{M}(\rho_t) = \rho(\mathbf{r}, t) \int_{\mathbf{r} - \mathbf{r}' \in S_{\text{fac}}} \rho(\mathbf{r}', t)$ with $S_{\text{fac}} = [r_{\text{fac}} - \Delta r_{\text{fac}}, r_{\text{fac}} + \Delta r_{\text{fac}}]$ being the facilitation shell. Taking advantage of the smooth densities for $|\mathbf{r}_{ll'}| = r_{\text{fac}}$ we can perform a Taylor expansion of $\rho(\mathbf{r}', t)$, yielding

$$\mathcal{M}(\rho_t) = \mathcal{M}(1)\rho_t + \frac{\mathcal{M}(\mathbf{r}^2)}{2}\nabla^2\rho_t + O(\nabla^4\rho_t), \tag{S12}$$

where odd derivative terms vanished due to isotropy in space. The factors $\mathcal{M}(1) = \int_{\mathbf{r} \in S_{\text{fac}}} 1$ and $\mathcal{M}(\mathbf{r}^2) = \int_{\mathbf{r} \in S_{\text{fac}}} \mathbf{r}^2$ are the averages of $1, \mathbf{r}^2$ along the facilitation shell.

Including thermal diffusion with diffusion constant D_T caused by the thermal motion of the atoms this yields the final form of the Langevin equation

$$\partial_t n_t = D_T\nabla^2 n_t - b\Gamma\rho_t + \xi_t, \tag{S13}$$

$$\partial_t \rho_t = (D\nabla^2 + \kappa n_t - \Gamma - 2\tau)\rho_t - 2\kappa\rho_t^2 + \tau n_t + \xi_t. \tag{S14}$$

In the experiment $\Gamma/\tau \approx 10^4$ and $n_t/\rho_t \approx 20$, justifying $\Gamma + 2\tau \to \Gamma$ and $n_t + \rho_t \to n_t$. Together with the numerical simulations this yields the estimate $b \approx 0.05$. Furthermore, the

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theoretical derivation predicts $\kappa = \frac{\Omega^2}{\Gamma+\gamma_{\text{de}}}\mathcal{M}(1) \approx \frac{2\pi\Omega^2}{3\Delta}r_{\text{fac}}^3$ and $D \approx D_T + n_t\frac{\pi\Omega^2}{3\Delta}r_{\text{fac}}^5$. Assuming van der Waals interactions $r_{\text{fac}} = (C_6/\Delta)^{1/6} \approx 1.7\,\mu$m. The diffusion of excitations $\sim D$ is primarily governed by fast facilitation of neighboring atoms and only marginally affected by temperature, i.e. $D_T \ll n_t\frac{\pi\Omega^2}{3\Delta}r_{\text{fac}}^5$. The numerical simulations of Eq.2 in the manuscript presented in Fig. 1 of the manuscript for $D_T = 0$ show that for experimentally relevant parameters n_t remains mostly homogeneous during the evolution since the diffusion of ρ_t is sufficiently fast compared to the effective loss rate $b\Gamma$. Thus for the conditions of the experiment atomic motion has very little impact on the qualitative SOC behaviour. However this might change in some lattice systems for example, where additional geometric constraints could have a more dramatic effect on the SOC dynamics.

The structure of the Langevin equations (S13), (S14) is obtained from the discussed, controlled coarse graining procedure and is insensitive to minor variations of the microscopic details. The effective parameters $D, \Gamma, \kappa, b, \tau$, however, can be influenced by such variations, that may include disorder, atomic motion and cooperative excitation processes, in the present setup. For this reason, the predicted values above only serve as a rough guide and in order to compare theoretical predictions with the experimental results we fit the data to mean field solutions of Eq. (S13) to consistently determine the relevant parameters of the model.

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