THE SUBSAMPLED POINCARÉ INEQUALITY FOR FUNCTIONAL RECOVERY

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Abstract. In this paper, we study the Poincaré inequality with subsampled measurement functions and apply it to functional recovery problems. The optimality of the inequality with respect to the subsampled length scale is demonstrated. The approximation accuracy of the recovery using different basis functions and under different regularity assumptions is established by using the subsampled Poincaré inequality. The error bound blows up as the subsampled length scale approaches 0 if the underlying function is not regular enough. We discuss a weighted version of the Poincaré inequality to address this problem.

1. Introduction

The Poincaré inequality, in one of its forms, states that for a bounded domain Ω in $\mathbb{R}^d$, there exists a constant $C(d,p)$, depending on d and p only, such that for every function u in the Sobolev space $W^{1,p}(\Omega)$, it holds

$$\|u-(u)_\Omega\|_{L^p(\Omega)} \le C(d,p)\mathrm{diam}(\Omega)\|Du\|_{L^p(\Omega)}\,,$$

where, $(u)_\Omega$ is the average of u in Ω, i.e. $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u(x)\,\mathrm{d}x$. Here, $\|\cdot\|_{L^p(\Omega)}$ represents the L^p norm of a function in the domain Ω, and $\mathrm{diam}(\Omega)$ is the diameter of Ω.

From a functional recovery perspective, we can interpret it in the following way. Suppose we have the knowledge that the function u is in $W^{1,p}(\Omega)$ and $\|Du\|_{L^p(\Omega)} \le M$ for some $M>0$, and we measure the average data $(u)_\Omega$. Our target is to recover u as accurate as possible. Simple recovery of u can be chosen as the constant function $(u)_\Omega$. Despite being so simple, guaranteed error control in the L^p norm of the recovery, due to the Poincaré inequality, is given by $C(d,p)\mathrm{diam}(\Omega)M$, in the worst case.

Now, one starts to place more sensors in the physical domain to look for more refined measurement data. For simplicity, let us assume $\Omega=[0,1]^d$ and the domain is partitioned evenly into $1/H^d$ cubes each with a length scale H. We denote by $\Omega=\bigcup_{i\in I}\omega_i^H$ where ω_i^H is the cube for the index $i\in I$ and $|I|=1/H^d$. The measuring strategy is that for each i, we acquire the data $(u)_{\omega_i^H}$, which is the average of u in ω_i^H. With these data, a recovery of u is taken as a piecewise constant function u^H, which attains the value $(u)_{\omega_i^H}$ in the patch ω_i^H for every $i\in I$. We have the following error control of this recovery:

$$\|u-u^H\|^p_{L^p(\Omega)} = \sum_{i\in I}\|u-(u)_{\omega_i^H}\|^p_{L^p(\omega_i^H)} \le C(d,p)^pH^p\sum_{i\in I}\|Du\|^p_{L^p(\omega_i^H)} = C(d,p)^pH^p\|Du\|^p_{L^p(\Omega)}\,.$$

So, we get $\|u-u^H\|_{L^p(\Omega)} \le C(d,p)MH$ by using the bound on Du. From the estimate, we see that the worst case error decreases with the rate of $O(H)$ as we refine the measurements. In fact, it is the best error rate one can achieve when we only know that the function satisfies $\|Du\|_{L^p(\Omega)} \le M$, in the perspective of the Kolmogorov n-width [16].

The above example implies the usefulness of the Poincaré inequality for estimating recovery residues. Indeed, many of the estimates in approximation theory, finite element analysis rely

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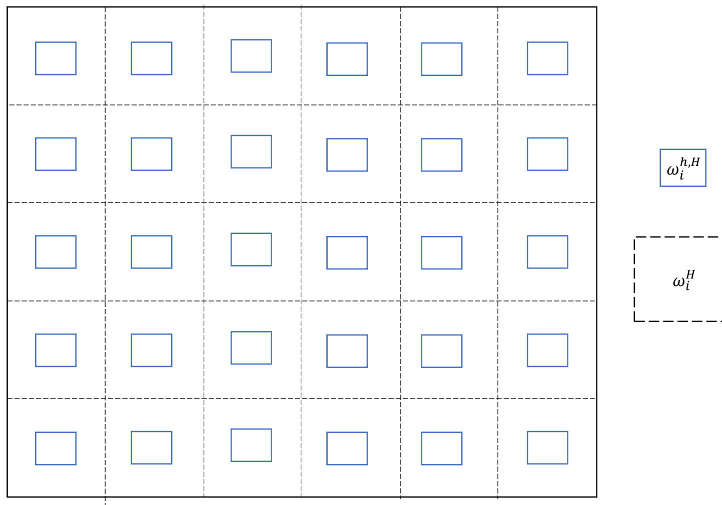

FIGURE 1. Domain $\Omega = [0,1]^2$; the local cube ω_i^H and the subsampled cube $\omega_i^{h,H}$

on similar ideas, where many local basis functions with error control are constructed, and a suitable global coupling scheme glues these basis functions to get the final recovery. Inspecting the process, we see there may be two potential places to generalize: 1) the measurement data type, which is the average in the local patches in the above example; 2) the local recovery basis, which is taken to be a constant function in each patch in the example. In this paper, we are interested in the subsampled data, which, under the above context, is an average of u in the set $\omega_i^{h,H} \subset \omega_i^H$ that has a possibly smaller length scale compared to that of the patch ω_i^H for each i, i.e. $h \leq H$, see Figure 1 for a demonstration in the case $d = 2$. Thus, it is a generalization for the $h = H$ case. Physically, the measurement data of a field function are often the macroscopic averaged quantities and represented by integration over a small region. The subsampled measurements match this context and are also more general than the Diracs type of measurements (i.e., $h = 0$ case), which may not be well-defined if the function does not have enough regularity according to the Sobolev embedding theorem [6]. It is useful to understand the behavior of these subsampled data in such a setting. We will also discuss the scenario that the measurement data is integration against some low dimensional slices of the domain.

Given the subsampled data, we discuss different local basis functions for the recovery that can attain desired approximation accuracy when u is in different functional classes. The approach relies on a generalized Poincaré inequality for subsampled measurement data with an optimal rate on the small scale parameter h. To improve from the piecewise constant recovery, we borrow ideas in the spline approximation theory to obtain basis functions with better regularity. This has connections to the multiscale PDEs context, see the work of rough polyharmonic splines [15] and Gamblets [13]. We will discuss the implication of our subsampled setting in the multiscale PDEs problem and other applications in our subsequent paper [3].

In the functional recovery setting, when the underlying function is not regular enough, we observe that the error bound of the recovery blows up as we decrease the subsampled scale h. This is due to the fact that the point-wise value of a $W^{1,p}$ function is not well-defined if $d \geq p$. However, if we put more structures on u, for example, $\int_\Omega w(x)|Du(x)|^p \,\mathrm{d}x < \infty$ for some singular weight function, then we can obtain improved accuracy. We discuss a weighted Poincaré inequality to analyze the error of the recovery in such a case.

Related works. Many people have considered extending the constant $(u)_\Omega$ in the Poincaré inequality to a general linear functional on u. In [9][10], the authors analyzed the condition of the functional in a great depth. In Chapter 4 of [18], a unified approach of the Poincaré inequality was discussed. In [1], the linear constraints in Poincaré and Korn type inequalities were investigated. Our subsampled measurements can be seen as a special case of their linear functional or linear constraints. However, the motivation is different, and their results do not directly lead to the optimal rate of h here. In the literature, we found a result similar to ours in Corollary 2.7 of [17] with a different proof strategy. Their rate on h is a little tighter than ours up to a log term in the critical $p = d$ case. We prove the rate is indeed optimal with respect to h in Proposition 2.1.

The optimal recovery problem has been framed in [11]. In a recent book [14], the authors discussed the game-theoretical and Bayesian ways of optimal recovery and numerical homogenization. When $h = 0$ and the coefficient a used in the improved basis functions is constant, our improved basis functions reduce to the polyharmonic splines [7][5]. When $h = 0$ or H, and the coefficient a is in $L^\infty(\Omega)$, then the improved basis functions reduce to Gamblets [13] and rough polyharmonic splines [15]. We remark that in [14], the discussion of the measurement function entails a great generality, and some general conditions on the measurements were proposed to guarantee the approximation accuracy. Our $h \in (0, H)$ case does satisfy their condition, but the results there do not cover the optimal dependence regarding h. In the finite element context, the case $h \neq 0$ also relates to the Clément interpolation [4].

There has been a vast literature on the weighted Sobolev space and weighted Poincaré inequality. To the best of our knowledge, most of them focus on the case in which both the left-hand side and the right-hand side of the inequality are weighted. In our case, we only set the right-hand side gradient norm to be weighted. In [2], a similar degeneracy issue regarding the graph Laplacian approach [12] for semi-supervised learning was discussed. Our weight function shares a form similar to theirs.

Notations. We present our notations here. We use $\chi_A(x)$ for the characteristic function of the set A. The diameter of a set $\Omega \subset \mathbb{R}^d$ is denoted by $\mathrm{diam}(\Omega)$. For a function in Euclidean space $\mathbb{R}^d$ with variable x, i.e. $f(x)$, the integration on a measurable set A against the Lebesgue measure will be denoted by $\int_A f(x)\,\mathrm{d}x$, while the integration with respect to a measure λ will be written as $\int_A f(x)\,\mathrm{d}\lambda(x)$. When there is no ambiguity, the variable name "x" in the integration may be omitted for simplicity. $L^p(\Omega)$ stands for the space of pth power summable functions over Ω with the corresponding norm $\|\cdot\|_{L^p(\Omega)}$, and $W^{1,p}(\Omega)$ represents the standard Sobolev space on the domain Ω. We use $|\cdot|$ for both the absolute value of a scalar and the modulus of a vector. When we say a set Ω is a domain, it refers to a connected, open set. The d dimensional Lebesgue measure of $\Omega \subset \mathbb{R}^d$ (i.e. the volume) is written as $\mu_d(\Omega)$. For $k < d$, we use $\mu_k(\Gamma)$ to represent the k dimensional Hausdorff measure of a k dimensional measurable subset $\Gamma \subset \mathbb{R}^d$.

Throughout the paper, $C(d, p)$ (resp. $C(d)$) stands for a positive generic constant which only depends on d, p (resp. d) and may attain different values at different places.

Organization. In Section 2, we discuss a generalized version of the Poincaré inequality, and establish the optimality of the subsampled Poincaré inequality. The sliced measurement data is also mentioned here. In Section 3, we consider an improvement of the basis function using ideas from the spline approximation theory, motivated by the works on rough polyharmonic splines [15] and Gamblets [13]. In Section 4, we present a weighted Poincaré inequality. Finally, we conclude the paper in Section 5.

2. The Poincaré inequality with general measurement function

We prove a generalized version of the Poincaré inequality here, which allows a general measure as measurement functional. First, we give the assumption on the domain under consideration.

Assumption 2.1. *Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded convex domain with a Lipschitz boundary. λ is a non-negative measure with unit mass on $\overline{\Omega}$.*

The convexity assumption could be relaxed, see Remark 2.1. We start with a Poincaré inequality for $W^{1,1}(\Omega)$ in Theorem 2.1, and then generalize it to $W^{1,p}(\Omega)$ for $1 < p < \infty$ in Theorem 2.2 through a special weighted Hölder inequality.

Theorem 2.1. *Under Assumption 2.1, the following inequality holds for every $u \in W^{1,1}(\Omega)$:*

$$\|u - \int_{\overline{\Omega}} u \,\mathrm{d}\lambda\|_{L^1(\Omega)} \leq \operatorname{diam}(\Omega) \int_{\overline{\Omega}} \left(\int_0^1 \frac{1}{t^d} \lambda(\frac{z - t\overline{\Omega}}{1-t} \cap \overline{\Omega}) \,\mathrm{d}t \right) |Du(z)| \,\mathrm{d}z \,. \tag{2.1}$$

Proof. We only need to prove the result for $u \in C^\infty(\overline{\Omega}) \cap W^{1,1}(\Omega)$ since this set is dense in $W^{1,1}(\Omega)$. A direct calculation gives

$$\begin{aligned} &\|u - \int_{\overline{\Omega}} u \,\mathrm{d}\lambda\|_{L^1(\Omega)} \\ =& \int_{\overline{\Omega}} \int_{\overline{\Omega}} (u(x) - u(y)) \,\mathrm{d}\lambda(x)\mathrm{d}y \\ \leq& \int_{\overline{\Omega}} \int_{\overline{\Omega}} |u(x) - u(y)| \,\mathrm{d}\lambda(x)\mathrm{d}y \,. \end{aligned} \tag{2.2}$$

We express the difference $u(x) - u(y)$ through its derivative Du using the Newton-Leibniz rule:

$$\begin{aligned} |u(x) - u(y)| &= | \int_0^1 (x-y) \cdot Du((1-t)x + ty) \,\mathrm{d}t| \\ &\leq \operatorname{diam}(\Omega) \int_0^1 |Du((1-t)x + ty)| \,\mathrm{d}t \,. \end{aligned}$$

Plugging the above formula into the integral in (2.2) and using Fubini's theorem, we obtain

$$\int_{\overline{\Omega}} \int_{\overline{\Omega}} |u(x) - u(y)| \,\mathrm{d}\lambda(x)\mathrm{d}y \leq \operatorname{diam}(\Omega) \int_0^1 \,\mathrm{d}t \int_{\overline{\Omega}} \int_{\overline{\Omega}} |Du((1-t)x + ty)| \,\mathrm{d}\lambda(x)\mathrm{d}y \,. \tag{2.3}$$

For any $0 \leq t \leq 1$, we have

$$\begin{aligned} &\int_{\overline{\Omega}} \int_{\overline{\Omega}} |Du((1-t)x + ty)| \,\mathrm{d}\lambda(x)\mathrm{d}y \\ =& \int_{\overline{\Omega}} \mathrm{d}\lambda(x) \int_{\overline{\Omega}} |Du((1-t)x + ty)| \,\mathrm{d}y \\ \overset{(a)}{=}& \int_{\overline{\Omega}} \mathrm{d}\lambda(x) \int_{\overline{\Omega}} |Du(z)| \chi_{(1-t)x + t\overline{\Omega}}(z) \frac{1}{t^d} \,\mathrm{d}z \\ =& \frac{1}{t^d} \int_{\overline{\Omega}} |Du(z)| \,\mathrm{d}z \int_{\overline{\Omega}} \chi_{\frac{z - t\overline{\Omega}}{1-t}}(x) \,\mathrm{d}\lambda(x) \\ =& \frac{1}{t^d} \int_{\overline{\Omega}} \lambda(\frac{z - t\overline{\Omega}}{1-t} \cap \overline{\Omega}) |Du(z)| \,\mathrm{d}z \,. \end{aligned} \tag{2.4}$$

where we have used the change of variables $z = (1-t)x + ty$ in step (a). Since the set Ω is assumed to be convex, the whole line will lie inside Ω, a fact which is employed in the above calculation. Combining (2.3) and (2.4) leads to

$$\int_{\overline{\Omega}}\int_{\overline{\Omega}}|u(x)-u(y)|\,\mathrm{d}\lambda(x)\mathrm{d}y \leq \operatorname{diam}(\Omega)\int_{\overline{\Omega}}\left(\int_0^1 \frac{1}{t^d}\lambda(\frac{z-t\overline{\Omega}}{1-t}\cap\overline{\Omega})\,\mathrm{d}t\right)|Du(z)|\,\mathrm{d}z\,. \tag{2.5}$$

This implies:

$$\|u-\int_{\overline{\Omega}} u\,\mathrm{d}\lambda\|_{L^1(\Omega)} \leq \operatorname{diam}(\Omega)\int_{\overline{\Omega}}\left(\int_0^1 \frac{1}{t^d}\lambda(\frac{z-t\overline{\Omega}}{1-t}\cap\overline{\Omega})\,\mathrm{d}t\right)|Du(z)|\,\mathrm{d}z\,. \tag{2.6}$$

The proof is completed. □

We give an assumption on the upper bound of the measure. This assumption will be satisfied for our subsampled measurements, see Corollary 2.1 and 2.2.

Assumption 2.2. *There exists $\alpha(t)$ such that for every $t\in[0,1]$ and $z\in\overline{\Omega}$ it holds that $\lambda(\frac{z-\overline{\Omega}}{1-t}\cap\overline{\Omega})\leq\alpha(t)$.*

Given the above assumption, the generalized Poincaré inequality for $1\leq p<\infty$ is stated as follows.

Theorem 2.2. *Under Assumptions 2.1 and 2.2, the following Poincaré type inequality is true for every $u\in W^{1,p}(\Omega)$ and $1\leq p<\infty$:*

$$\|u-\int_{\overline{\Omega}} u\,\mathrm{d}\lambda\|_{L^p(\Omega)} \leq \operatorname{diam}(\Omega)\left(\int_0^1 \frac{\alpha(t)^{\frac{1}{p}}}{t^{\frac{d}{p}}}\,\mathrm{d}t\right)\|Du\|_{L^p(\Omega)}\,. \tag{2.7}$$

Proof. The result of the case $p=1$ is a direct combination of Theorem 2.1 and Assumption 2.2. For the case $1<p<\infty$, we obtain by using Jensen's inequality that,

$$\begin{aligned}
&\|u-\int_{\overline{\Omega}} u\,\mathrm{d}\lambda\|^p_{L^p(\Omega)}\\
&=\int_{\overline{\Omega}}\left(\int_{\overline{\Omega}}(u(x)-u(y))\,\mathrm{d}\lambda(x)\right)^p\mathrm{d}y\\
&\leq\int_{\overline{\Omega}}\int_{\overline{\Omega}}|u(x)-u(y)|^p\,\mathrm{d}\lambda(x)\mathrm{d}y\,.
\end{aligned}$$

Similarly, we use the Newton-Leibniz rule to express the term $u(x)-u(y)$:

$$\begin{aligned}
|u(x)-u(y)|^p &= |\int_0^1 (x-y)\cdot Du((1-t)x+ty)\,\mathrm{d}t|^p\\
&\overset{(b)}{\leq} \operatorname{diam}(\Omega)^p\left(\int_0^1 w(t)^{-\frac{1}{p-1}}\,\mathrm{d}t\right)^{p-1}\int_0^1 w(t)|Du((1-t)x+ty)|^p\,\mathrm{d}t\,.
\end{aligned}$$

Here, the step (b) is due to the Hölder inequality, in which we introduce a weight function $w(t)\geq 0$, which will be determined in the subsequent calculations. We remark that without a correct choice of the weight function, we would not be able to obtain an inequality with a constant that has an optimal scaling property with respect to h, for case $d\neq p$, as in Corollary 2.1 and Corollary 2.2.

Then, by the same change of variables as in (2.4), we get

$$\begin{aligned}&\int_{\overline{\Omega}}\int_{\overline{\Omega}}|Du((1-t)x+ty)|^p\,\mathrm{d}\lambda(x)\mathrm{d}y\\ =&\frac{1}{t^d}\int_{\overline{\Omega}}\lambda(\frac{z-t\overline{\Omega}}{1-t}\cap\overline{\Omega})|Du(z)|^p\,\mathrm{d}z\\ \le&\frac{\alpha(t)}{t^d}\int_{\overline{\Omega}}|Du(z)|^p\,\mathrm{d}z\,.\end{aligned}$$

Following the same argument as in (2.5) and (2.6), we obtain

$$\|u-\int_{\overline{\Omega}}u\,\mathrm{d}\lambda\|_{L^p(\Omega)}^p\le \mathrm{diam}(\Omega)^p\left(\int_0^1 w(t)^{-\frac{1}{p-1}}\,\mathrm{d}t\right)^{p-1}\left(\int_0^1\frac{w(t)\alpha(t)}{t^d}\,\mathrm{d}t\right)\|Du\|_{L^p(\Omega)}^p\,.$$

Now, we optimize the choice of the weight function $w(t)$. Let

$$w(t)^{-\frac{1}{p-1}}=\frac{w(t)\alpha(t)}{t^d}\,,$$

which is the condition for the corresponding Hölder inequality to become an equality. Under such a choice, we obtain

$$\|u-\int_{\overline{\Omega}}u\,\mathrm{d}\lambda\|_{L^p(\Omega)}\le \mathrm{diam}(\Omega)\left(\int_0^1\frac{\alpha(t)^{\frac{1}{p}}}{t^{\frac{d}{p}}}\,\mathrm{d}t\right)\|Du\|_{L^p(\Omega)}\,.$$

This completes the proof. □

We remark that some requirements in Assumption 2.1 can be relaxed, such as the convexity of the domain and the regularity of the boundary, see Remark 2.1 and 2.2. That being said, the present version is enough for our purpose of applications in the functional recovery and multiscale PDEs with subsampled data.

Remark 2.1. *The convexity assumption of the domain Ω can be relaxed. For general non-convex domains, we can use the Sobolev extension theorem to extend the function to a larger convex domain, for example, a ball. Then the results for the convex case can be applied. In this regard, we only need the assumption of the domain that allows the Sobolev extension theorem to hold.*

Remark 2.2. *In Assumption 2.1, we require the regularity of the boundary of the domain. However, when λ has no mass in the boundary, this requirement can be removed. The reason is that the density argument of Meyers-Serrin can be applied to any generic domain. That is to say, $C^\infty(\Omega)\cap W^{1,p}(\Omega)$ is always dense in $W^{1,p}(\Omega)$ and all the arguments follow in the same way. When λ has mass in the boundary, we need $C^\infty(\overline{\Omega})\cap W^{1,p}(\Omega)$ to be dense, which puts the regularity requirements on the boundary.*

Using Theorem 2.2, we can get the following inequalities with subsampled data as a special case.

Corollary 2.1 (Subsampled Poincaré inequality)**.** *Consider a bounded convex domain $\Omega\subset\mathbb{R}^d$ with Lipschitz boundary and a measurable subset $D\subset\Omega$. Let $\mu_d(\Omega)=H^d,\mu_d(D)=h^d$, then for any $1\le p<\infty$ and $u\in W^{1,p}(\Omega)$, the following inequality holds:*

$$\|u-\frac{1}{h^d}\int_D u\|_{L^p(\Omega)}\le C(d,p)\mathrm{diam}(\Omega)\rho_{p,d}(\frac{H}{h})\|Du\|_{L^p(\Omega)}\,,$$

where

$$\rho_{p,d}(x) = \begin{cases} 1, & d < p \\ \ln(x+1), & d = p \\ x^{\frac{d-p}{p}} & d > p \end{cases}$$

and $C(d,p)$ is a constant that depends on d and p only.

Proof of Corollary 2.1. Let the measure λ in Theorem 2.1 be supported on D and uniform in D. Then, $\frac{1}{h^d}\int_D u = \int_\Omega u \, d\lambda$. Hence, we have

$$\|u - \frac{1}{h^d}\int_D u\|_{L^p(\Omega)} \leq \text{diam}(\Omega) \left(\int_0^1 \frac{\alpha(t)^{\frac{1}{p}}}{t^{\frac{d}{p}}} \, dt \right) \|Du\|_{L^p(\Omega)} ,$$

where $\alpha(t)$ is an upper bound on $\lambda(\frac{z-t\Omega}{1-t} \cap \Omega)$. A trivial bound is $\alpha(t) \leq 1$. On the other hand, since λ is supported on D, we have

$$\lambda(\frac{z-t\Omega}{1-t} \cap \Omega) = \lambda(\frac{z-t\Omega}{1-t} \cap D) \leq \frac{1}{h^d}\mu_d(\frac{z-t\Omega}{1-t}) \leq \frac{H^d}{h^d}(\frac{t}{1-t})^d,$$

where we have used the fact that the density of λ on D is $\frac{1}{h^d}$. Thus, we choose

$$\alpha(t) = \min\{1, \frac{H^d}{h^d}(\frac{t}{1-t})^d\} = \frac{H^d}{h^d}(\frac{t}{1-t})^d \cdot \chi_{[0,\frac{h}{H+h})}(t) + 1 \cdot \chi_{[\frac{h}{H+h},1]}(t) \, .$$

We then calculate the integral:

$$\int_0^1 \frac{\alpha(t)^{\frac{1}{p}}}{t^{\frac{d}{p}}} \, dt = (\frac{H}{h})^{\frac{d}{p}} \int_0^{\frac{h}{H+h}} \frac{1}{(1-t)^{\frac{d}{p}}} \, dt + \int_{\frac{h}{H+h}}^1 \frac{1}{t^{\frac{d}{p}}} \, dt \, . \tag{2.8}$$

When $d < p$, the integral in (2.8) becomes

$$\frac{p}{p-d}\left((\frac{H}{h})^{\frac{d}{p}}(1 - (\frac{H}{H+h})^{1-\frac{d}{p}}) + 1 - (\frac{h}{H+h})^{1-\frac{d}{p}} \right) . \tag{2.9}$$

Since $-1 < \frac{d}{p} - 1 < 0$, by Bernoulli's inequality, we have

$$(\frac{H}{h})^{\frac{d}{p}}(1 - (\frac{H}{H+h})^{1-\frac{d}{p}}) = (\frac{H}{h})^{\frac{d}{p}}(1 - (1+\frac{h}{H})^{\frac{d}{p}-1}) \leq (\frac{H}{h})^{\frac{d}{p}}\frac{h}{H}(1-\frac{d}{p}) \leq 1 - \frac{d}{p} \, ,$$

where we have used the fact $(\frac{H}{h})^{\frac{d}{p}}\frac{h}{H} = (\frac{h}{H})^{1-\frac{d}{p}} \leq 1$. Thus, we have the quantity in (2.9) bounded by

$$\frac{p}{p-d}(1 - \frac{d}{p} + 1) = \frac{2p-d}{p-d} \leq C(d,p) \, .$$

When $d = p$, the integral in (2.8) is

$$\frac{H}{h}\ln(1+\frac{h}{H}) + \ln(1+\frac{H}{h}) \leq 1 + \ln(1+\frac{H}{h}) \leq C\ln(1+\frac{H}{h}) \, . \tag{2.10}$$

When $d > p$, the integral in (2.8) becomes

$$\frac{p}{d-p}\left((\frac{H}{h})^{\frac{d}{p}}((1+\frac{h}{H})^{\frac{d-p}{p}} - 1) + (1+\frac{H}{h})^{\frac{d-p}{p}} - 1 \right) \leq C(d,p)(\frac{H}{h})^{\frac{d-p}{p}} \, . \tag{2.11}$$

The proof is completed. □

In the literature, we found that in Corollary 2.7 of [17], a similar rate on h is obtained through a different approach from ours. In the critical case $p = d$, they use the Orlicz norm to get the log dependence of H/h. Indeed, their result is a little tighter in the power of log than ours. Based on their results, we can improve the rate function to be

$$\tilde{\rho}_{p,d}(x) = \begin{cases} 1, & d < p \\ (\ln(x+1))^{\frac{d-1}{d}}, & d = p \\ x^{\frac{d-p}{p}} & d > p \end{cases} \tag{2.12}$$

We show the optimality of the rate above for Ω, D being balls. However, this choice of domain to be balls here is just for the sake of the construction of critical examples. The optimality shall hold for more general domains by following similar ideas here.

Proposition 2.1 (Sharpness of the rate). *Let $\Omega = B_1(0), D_h = B_h(0)$ be the balls centered at 0 with radius 1 and $0 \le h \le 1/4$ respectively. Then, for $d \ge p$, there exists a constant $C(d,p)$ that depends on d and p only, such that we can find a sequence of functions $u_h \in W^{1,p}(\Omega)$ which satisfy*

$$\frac{\|u_h - \frac{1}{\mu_d(D_h)}\int_{D_h} u_h\|_{L^p(\Omega)}}{\|Du_h\|_{L^p(\Omega)}} \ge C(d,p)\tilde{\rho}_{p,d}(\frac{1}{h})\,.$$

Proof of Proposition 2.1. We construct the sequence u_h explicitly. For $d = p$, we take

$$u_h(x) = \frac{\max\{0, \ln(1+\frac{|x|}{h}) - \ln 2\}}{\ln(1+\frac{1}{h})}\,.$$

Then $u_h(x)$ equals 0 in D_h. Thus,

$$\begin{aligned}\|u_h - \frac{1}{\mu_d(D_h)}\int_{D_h} u_h\|^p_{L^p(\Omega)} &= \int_{B_1(0)\backslash B_h(0)} u_h^p \\ &= \mu_{d-1}(\mathbb{S}^d)\int_h^1 \frac{\max\{0, \ln(1+\frac{r}{h}) - \ln 2\}^p}{\ln(1+\frac{1}{h})^p} r^{d-1}\,\mathrm{d}r \\ &\ge \mu_{d-1}(\mathbb{S}^d)\frac{(\ln(1+\frac{1}{2h}) - \ln 2)^p}{\ln(1+\frac{1}{h})^p}\int_{1/2}^1 r^{d-1}\,\mathrm{d}r \\ &\ge C(d,p)\end{aligned}$$

for some $C(d,p) > 0$ independent of h, since $\lim_{h\to 0}\frac{\ln(1+\frac{1}{2h})}{\ln(1+\frac{1}{h})} = 1$. Here we use $\mathbb{S}^d$ to represent the d dimensional unit sphere. On the other hand,

$$\begin{aligned}\|Du_h\|^p_{L^p(\Omega)} &= \frac{1}{(\ln(1+\frac{1}{h}))^p}\int_{B_1(0)\backslash B_h(0)} \frac{1}{(h+|x|)^p}\,\mathrm{d}x \\ &= \mu_{d-1}(\mathbb{S}^d)\frac{1}{(\ln(1+\frac{1}{h}))^p}\int_h^1 \frac{r^{d-1}}{(h+r)^p}\,\mathrm{d}r \\ &\le C(d,p)\frac{1}{(\ln(1+\frac{1}{h}))^{d-1}}\end{aligned}$$

for some $C(d,p)$ dependent of d,p. In the last step, we have used the inequality $h + r \ge r$ and the fact that $\lim_{h\to 0}\frac{\ln(1+\frac{1}{h})}{\ln(\frac{1}{h})} = 1$.

Hence, for this sequence u_h, we get

$$\frac{\|u_h - \frac{1}{\mu_d(D_h)}\int_{D_h} u_h\|_{L^p(\Omega)}}{\|Du_h\|_{L^p(\Omega)}} \ge C(d,p)(\ln(1+\frac{1}{h}))^{\frac{d-1}{d}} = C(d,p)\tilde{\rho}_{p,d}(\frac{1}{h})\,.$$

For $d > p$, we construct

$$u_h(x) = \min \{\frac{\max \{|x| - h, 0\}}{h}, 1\}.$$

Then, $u_h(x)$ vanishes in D_h, and

$$\begin{aligned}\|u_h - \frac{1}{\mu_d(D_h)} \int_{D_h} u_h\|^p_{L^p(\Omega)} &= \int_{B_1(0)\setminus B_h(0)} u_h^p \\ &\geq \int_{B_1(0)\setminus B_{1/2}(0)} u_h^p = C(d,p)\,,\end{aligned}$$

where $C(d,p)$ is independent of h. Here we have used the fact $h \leq 1/4$ and $u_h = 1$ when $|x| \geq 1/2$. In the meanwhile, we get

$$\|Du_h\|^p_{L^p(\Omega)} = \int_{B_{2h}(0)\setminus B_h(0)} \frac{1}{h^p}\,\mathrm{d}x = C(d,p)h^{d-p}.$$

Hence, we conclude that

$$\frac{\|u_h - \frac{1}{\mu_d(D_h)} \int_{D_h} u_h\|_{L^p(\Omega)}}{\|Du_h\|_{L^p(\Omega)}} \geq C(d,p)h^{\frac{p-d}{p}} = C(d,p)\tilde{\rho}_{p,d}(\frac{1}{h}).$$

The proof is completed. □

Remark 2.3. *In the functional recovery context, suppose we have the measurement data* $\{(u)_{\omega_i^{h,H}}\}_{i\in I}$*, then following the same argument in the introduction, we get the error bound of the piecewise constant recovery*

$$C(d,p)H\tilde{\rho}_{p,d}(\frac{H}{h})\|Du\|_{L^p(\Omega)}.$$

Inspecting this formula, we see that if the ratio $H/h > 0$ *is fixed, then the error still achieves the* $O(H)$ *rate for functions in the space* $W^{1,p}(\Omega)$*. If* $p \leq d$*, then taking* $h \to 0$ *the error bound will blow up. This is due to the fact that the Sobolev embedding theorem fails to embed* $W^{1,p}(\Omega)$ *to the functional space consisting of continuous functions.*

We also consider the sliced version of the subsampled data, in the following Corollary 2.2.

Corollary 2.2 (Subsampled Poincaré inequality with sliced data)**.** *Consider a bounded convex domain* $\Omega \subset \mathbb{R}^d$ *with Lipschitz boundary and a hyperplane* $\Gamma \subset \overline{\Omega}$ *with dimension* $d-1$*. Let* $\mu_{d-1}(\Gamma) = h^{d-1}$*, and suppose that for every hyperplane contained in* Ω *that is parallel to* Γ*, its* $d-1$ *dimensional Hausdorff measure is bounded by* H^{d-1}*. Then for any* $1 < p < \infty$ *and* $u \in W^{1,p}(\Omega)$*, the following inequality holds:*

$$\|u - \frac{1}{h^{d-1}} \int_\Gamma u\|_{L^p(\Omega)} \leq C(d,p)\mathrm{diam}(\Omega)\rho_{p,d}(\frac{H}{h})\|Du\|_{L^p(\Omega)}\,,$$

where

$$\rho_{p,d}(x) = \begin{cases} 1, & d < p \\ \ln(x+1), & d = p \\ x^{\frac{d-p}{p}} & d > p \end{cases}$$

and $C(d,p)$ *is a constant that depends on* d *and* p *only.*

Proof of Corollary 2.2. Similar to the proof of Example 2.1, we first characterize $\alpha(t)$, and then calculate the related integral. Since λ is supported on Γ, we have

$$\lambda(\frac{z - t\Omega}{1-t} \cap \Omega) = \lambda(\frac{z - t\Omega}{1-t} \cap \Gamma) \leq \frac{H^{d-1}}{h^{d-1}}(\frac{t}{1-t})^{d-1},$$

where we have used the fact that the density of λ on the $d-1$ dimensional Γ is $\frac{1}{h^{d-1}}$. Hence, we choose

$$\alpha(t)=\min\{1,\frac{H^{d-1}}{h^{d-1}}(\frac{t}{1-t})^{d-1}\}=\frac{H^{d-1}}{h^{d-1}}(\frac{t}{1-t})^{d-1}\cdot\chi_{[0,\frac{h}{H+h})}(t)+1\cdot\chi_{[\frac{h}{H+h},1]}(t)\,.$$

The corresponding integral is

$$\int_0^1\frac{\alpha(t)^{\frac{1}{p}}}{t^{\frac{d}{p}}}\,\mathrm{d}t=(\frac{H}{h})^{\frac{d-1}{p}}\int_0^{\frac{h}{H+h}}\frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{d-1}{p}}}\,\mathrm{d}t+\int_{\frac{h}{H+h}}^1\frac{1}{t^{\frac{d}{p}}}\,\mathrm{d}t\,. \tag{2.13}$$

For the first term in (2.13),

$$\begin{aligned}(\frac{H}{h})^{\frac{d-1}{p}}\int_0^{\frac{h}{H+h}}\frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{d-1}{p}}}\,\mathrm{d}t\leq&(\frac{H}{h})^{\frac{d-1}{p}}(1+(1+\frac{h}{H})^{\frac{d-1}{p}})\int_0^{\frac{h}{H+h}}\frac{1}{t^{\frac{1}{p}}}\,\mathrm{d}t\\ =&\frac{p}{p-1}(\frac{H}{h})^{\frac{d-1}{p}}(1+(1+\frac{h}{H})^{\frac{d-1}{p}})(1+\frac{H}{h})^{\frac{1}{p}-1}\\ =&\frac{H^{\frac{d-1}{p}}(H+h)^{\frac{1}{p}-1}}{h^{\frac{d-p}{p}}}(1+(1+\frac{h}{H})^{\frac{d-1}{p}})\\ \leq&(2^{\frac{1}{p}-1}+1)(\frac{H}{h})^{\frac{d-p}{p}}(2+2^{\frac{d-1}{p}})\,,\end{aligned}$$

where in the last step we have used the estimate

$$(H+h)^{\frac{1}{p}-1}\leq H^{\frac{1}{p}-1}+(2H)^{\frac{1}{p}-1}\quad\text{and}\quad(1+\frac{h}{H})^{\frac{d-1}{p}}\leq1+2^{\frac{d-1}{p}}\,.$$

This is due to $0\leq h\leq H$ and the fact that, the value of a one dimensional non-negative monotone function will be not larger than the summation values of its two endpoints in an interval. Observe that the last term in the above calculation will be bounded by a constant $C(d,p)$ if $d\leq p$ and by $C(d,p)(H/h)^{\frac{d-p}{p}}$ if $d>p$. Moreover, the second term in (2.13) is the same as in (2.8). Thus, the same argument there can be applied here. Finally, we obtain the Poincaré inequality with the same $\rho_{p,d}$ dependence on H/h. □

It is possible to combine arguments in Corollary 2.7 of [17] (the Orlicz norm) to improve the rate in the critical case $d=p$, i.e., to gain a log factor.

Remark 2.4. *Similar to the case in Corollary 2.1, if we use the sliced data to make the piecewise constant recovery, the error bound is given by* $C(d,p)H\rho_{p,d}(\frac{H}{h})\|Du\|_{L^p(\Omega)}$.

3. Improve the regularity of the recovery

The discussion above only concerns piecewise constant function recovery. In this section, we consider an improvement of the regularity of the basis function for $p=2$. To achieve so, we apply $\mathcal{L}^{-1}$ on it, where $\mathcal{L}=-\nabla\cdot(a\nabla\cdot)$ is an elliptic operator with homogeneous Dirichlet boundary condition. The coefficient a in this operator is assumed to satisfy $0<a_{\min}\leq a(x)\leq a_{\max}$ for all $x\in\Omega\subset\mathbb{R}^d$ such that its inverse is well-defined.

Let $\Omega=[0,1]^d$ and its decomposition into patches follows the previous routine. Given $a(x)$, the associated energy norm of u is defined by $\|u\|^2_{H^1_a(\Omega)}=\int_\Omega a(x)|\nabla u(x)|^2\,\mathrm{d}x$. The induced inner product is denoted by $\langle\cdot,\cdot\rangle_a$ such that for $u,v\in H^1_0(\Omega)$, we have

$$\langle u,v\rangle_a=\int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\,.$$

We write the subsampled measurement functions by $\{\phi_i^{h,H}\}_{i\in I}$ where each $\phi_i^{h,H}$ is the L^1-normalized indicator function of the patch $\omega_i^{h,H}$. The improved basis functions, as stated above, will span the space $\text{span}_{i\in I}\ \{\mathcal{L}^{-1}\phi_i^{h,H}\}$. A set of basis functions for this space can be obtained through the following optimization problem:

$$\psi_i^{h,H} = \text{argmin}_{\psi\in H_0^1(\Omega)} \|\psi\|^2_{H_a^1(\Omega)} \quad \text{s.t.} \quad \int_\Omega \psi\phi_j^{h,H} = \delta_{i,j} \quad \text{for} \quad j\in I\,. \tag{3.1}$$

We have $\text{span}_{i\in I}\ \{\psi_i^{h,H}\} = \text{span}_{i\in I}\ \{\mathcal{L}^{-1}\phi_i^{h,H}\}$, and $\psi_i^{h,H}$ is given by a linear combination of $\mathcal{L}^{-1}\phi_i^{h,H}$ for $i\in I$, see the following Proposition 3.1. The proof follows the same strategy as Theorem 3.1 in [13].

Proposition 3.1. *The solution $\psi_i^{h,H}$ has the form*

$$\psi_i^{h,H} = \sum_{j\in I}\Theta^{-1}_{i,j}\mathcal{L}^{-1}\phi_j^{h,H}\,,$$

where $\Theta\in\mathbb{R}^{|I|\times|I|}$ with entries $\Theta_{i,j} = [\phi_j^{h,H}, \mathcal{L}^{-1}\phi_i^{h,H}]$ and Θ^{-1} is the inverse of Θ.

Given the basis functions $\psi_i^{h,H}$, the recovered function is defined by

$$u^{h,H} = \sum_{i\in I}[u, \phi_i^{h,H}]\psi_i^{h,H}\,,$$

which is the projection of u onto $\text{span}_{i\in I}\ \{\psi_i^{h,H}\}$ under $\langle\cdot,\cdot\rangle_a$, see Proposition 3.2.

Proposition 3.2. *The function $u^{h,H}$ is the projection of u into the space spanned by $\{\psi_i^{h,H}\}_{i\in I}$ under the inner product $\langle\cdot,\cdot\rangle_a$.*

Proof. It suffices to show $u-u^{h,H}$ is orthogonal to $\psi_i^{h,H}$ for any $i\in I$ under the inner product $\langle\cdot,\cdot\rangle_a$. Equivalently, we need to show $\left\langle u-u^{h,H}, \psi_i^{h,H}\right\rangle_a = 0$. Since $\psi_i^{h,H}\in\text{span}_{i\in I}\ \{\mathcal{L}^{-1}\phi_i^{h,H}\}$, this is equivalent to $[u-u^{h,H}, \phi_i^{h,H}] = 0$. Observing that

$$[u-u^{h,H}, \phi_i^{h,H}] = [u, \phi_i^{h,H}] - \sum_{j\in I}[u, \phi_j^{h,H}][\phi_i^{h,H}, \psi_j^{h,H}] = 0\,,$$

we complete the proof. □

Now, we derive the approximation accuracy of the above recovery. We discuss two scenarios: 1) $u\in H_0^1(\Omega)$, the same setting as the last section; 2) we further have $\mathcal{L}u\in L^2(\Omega)$, i.e. an improved regularity assumption on u. Below Theorem 3.1 shows the error estimate.

Theorem 3.1. *Suppose $u\in H_0^1(\Omega)$, then the following error estimate holds:*

$$\begin{aligned}
&\|u-u^{h,H}\|_{H_a^1(\Omega)} \le \|u\|_{H_a^1(\Omega)}\,,\\
&\|u-u^{h,H}\|_{L^2(\Omega)} \le \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|u\|_{H_a^1(\Omega)}\,,
\end{aligned}$$

where $C(d)$ is a constant that depends on d only.

Moreover, if it holds that $\mathcal{L}u\in L^2(\Omega)$, then, we have the improved $H_a^1(\Omega)$ estimate:

$$\|u-u^{h,H}\|_{H_a^1(\Omega)} \le \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|\mathcal{L}u\|_{L^2(\Omega)}\,,$$

which then leads to the improved $L^2(\Omega)$ estimate

$$\|u-u^{h,H}\|_{L^2(\Omega)} \le \frac{1}{a_{\min}}C(d)^2H^2\tilde{\rho}_{2,d}(\frac{H}{h})^2\|\mathcal{L}u\|_{L^2(\Omega)}\,.$$

Proof. The first estimate is trivial since $u^{h,H}$ is the projection of u under the energy norm $H_a^1(\Omega)$. For the second inequality, introduce functions v, w such that $v = u - u^{h,H}$ and $\mathcal{L}w = v$. Then we have

$$\|u-u^{h,H}\|^2_{L^2(\Omega)} = [v,v] = [v,\mathcal{L}w] = \langle v,w\rangle_a \,. \tag{3.2}$$

Since v is orthogonal to every $\psi_i^{h,H}$ under the inner product $\langle\cdot,\cdot\rangle_a$, we get

$$\langle v,w\rangle_a = \left\langle v, w - \sum_{i\in I}[w,\phi_i^{h,H}]\psi_i^{h,H}\right\rangle_a \leq \|v\|_{H_a^1(\Omega)}\|w - \sum_{i\in I}[w,\phi_i^{h,H}]\psi_i^{h,H}\|_{H_a^1(\Omega)} \,. \tag{3.3}$$

For the second term in the above right-hand side, we know from orthogonality:

$$\|w - \sum_{i\in I}[w,\phi_i^{h,H}]\psi_i^{h,H}\|_{H_a^1(\Omega)} = \min_{c_i}\|w - \sum_{i\in I}c_i\mathcal{L}^{-1}\phi_i^{h,H}\|_{H_a^1(\Omega)} \,. \tag{3.4}$$

To obtain an upper bound of this term, we choose $c_i = \frac{1}{\|\phi_i^{h,H}\|_{L^1(\Omega)}}\int_{\omega_i^H} v$. Denote $w_0 = \sum_{i\in I}c_i\mathcal{L}^{-1}\phi_i^{h,H}$, then

$$\begin{aligned}
\|w-w_0\|^2_{H_a^1(\Omega)} &= \int_\Omega (w-w_0)(v - \sum_{i\in I}c_i\phi_i^{h,H})\\
&= \sum_{i\in I}\int_{\omega_i^H}(w-w_0)(v - c_i\phi_i^{h,H})\\
&= \sum_{i\in I}\int_{\omega_i^H}\left(w - w_0 - \frac{1}{\|\phi_i^{h,H}\|_{L^1(\Omega)}}\int_{\omega_i^H}(w-w_0)\phi_i^{h,H}\right)v\\
&\overset{c)}{\leq} \sum_{i\in I} C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|D(w-w_0)\|_{L^2(\omega_i^H)}\|v\|_{L^2(\omega_i^H)}\\
&\leq \frac{1}{\sqrt{a_{\min}}}C(d)\tilde{\rho}_{2,d}(\frac{H}{h})\|w-w_0\|_{H_a^1(\Omega)}\|v\|_{L^2(\Omega)}
\end{aligned} \tag{3.5}$$

where $c)$ is due to the subsampled Poincaré inequality. We get

$$\|w - \sum_{i\in I}[w,\phi_i^{h,H}]\psi_i^{h,H}\|_{H_a^1(\Omega)} \leq \|w-w_0\|_{H_a^1(\Omega)} \leq C(d)\frac{1}{\sqrt{a_{\min}}}H\tilde{\rho}_{2,d}(\frac{H}{h})\|v\|_{L^2(\Omega)} \,. \tag{3.6}$$

Returning to (3.3), we obtain

$$\langle v,w\rangle_a \leq \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|v\|_{H_a^1(\Omega)}\|v\|_{L^2(\Omega)} \,.$$

Recalling $v = u - u^{h,H}$ and equation (3.2), we get

$$\|u-u^{h,H}\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|u-u^{h,H}\|_{H_a^1(\Omega)} \leq \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|u\|_{H_a^1(\Omega)} \,. \tag{3.7}$$

If $\mathcal{L}u \in L^2(\Omega)$, we follow the strategy (3.4), (3.5) and (3.6) (apply all the operations on w to the function u), which give us

$$\|u-u^{h,H}\|_{H_a^1(\Omega)} \leq \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|\mathcal{L}u\|_{L^2(\Omega)} \,.$$

To get the improved L^2 estimate, we apply the argument in (3.7), which leads to

$$\|u-u^{h,H}\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{a_{\min}}}C(d)H\tilde{\rho}_{2,d}(\frac{H}{h})\|u-u^{h,H}\|_{H_a^1(\Omega)} \leq \frac{1}{a_{\min}}C(d)^2H^2\tilde{\rho}_{2,d}(\frac{H}{h})^2\|\mathcal{L}u\|_{L^2(\Omega)} \,.$$

The proof is completed. □

Theorem 3.1 implies that under the assumption $\|u\|_{H^1_a(\Omega)} \le M$, using piece-wise constant functions for recovery and the improved basis $\psi_i^{h,H} (i \in I)$ achieve the same optimal accuracy rate on H, if the ratio H/h is kept fixed. When we know additional information that $\|\mathcal{L}u\|_{L^2(\Omega)}$ is finite, we can improve to $O(H)$ accuracy in the energy norm and $O(H^2)$ accuracy in the L^2 norm.

On the other hand, the construction of the basis $\psi_i^{h,H} (i \in I)$ requires more computational efforts, since the optimization is on the global domain Ω. The difficulty is addressed by observing that $\psi_i^{h,H}$ exhibits exponential decaying property in energy norm [8][13]. Thus, the computation of the basis can be localized, i.e., we can replace the global domain Ω in the constraint $\psi \in H_0^1(\Omega)$ in (3.1) by some localized oversampling domain around ω_i^H. We include the discussion of this issue in our companion paper [3] together with the multiscale PDEs problem.

We note that the above results also hold for the subsampled measurements with sliced type since Corollary 2.1 and 2.2 share the same form.

4. Weighted Poincaré inequality and non-degenerate recovery

In previous sections, we have seen that when $d \ge p$, the error will blow up when h goes to 0. In general, it is not improvable if we only know the information that u belongs to $H_0^1(\Omega)$. Since in practice, we often encounter recovery problems in high dimensions, we are led to ask if this degeneracy problem could be fixed by imposing more structures on u. A natural extension of previous estimates is to consider the weighted Poincaré inequality, which we will present below.

Here, we work under general p that may not equal 2, and we assume $d \ge p$ such that space $W^{1,p}(\Omega)$ does not embed into the functional space consisting of continuous functions.

The weighted norm $\|\cdot\|_{L^p_w(\Omega)}$ is defined by $\|u\|_{L^p_w(\Omega)} := (\int_\Omega w(x)|u(x)|^p \,\mathrm{d}x)^{1/p}$. The distance of x to a set D is denoted by $d(x, D)$, and the distance between two sets A and B in Euclidean space is denoted by $d(A, B)$.

Assumption 4.1. *There exist positive constants $C_1(d, p)$ and $C_2(d, p)$, such that for the domain $M = \Omega$ or D, it holds that $C_1^d \mathrm{diam}(M)^d \le \mu_d(M) \le C_2^d \mathrm{diam}(M)^d$.*

Theorem 4.1. *Let $D \subset \Omega$ satisfy Assumptions 2.1 and 4.1, with $\mu_d(\Omega) = H^d$ and $\mu_d(D) = h^d$. For every $u \in W^{1,1}(\Omega)$, the following inequality holds:*

$$\|u - \frac{1}{h^d}\int_D u\|_{L^1(\Omega)} \le C(d, p)H\|Du\|_{L^1_w(\Omega)}$$

where the weight function is chosen to be

$$w(x) = \left(\frac{H}{\max\{h, d(x, D)\}}\right)^{d-1}$$

and $C(d, p)$ is a constant that depends on d and p only.

Proof. Assumption 4.1 implies $C_1\mathrm{diam}(\Omega) \le H \le C_2\mathrm{diam}(\Omega)$ and $C_1\mathrm{diam}(D) \le h \le C_2\mathrm{diam}(D)$. We use the result in our Theorem 2.1:

$$\|u - \int_\Omega u \,\mathrm{d}\lambda\|_{L^1(\Omega)} \le \mathrm{diam}(\Omega)\int_\Omega \left(\int_0^1 \frac{1}{t^d}\lambda(\frac{z - t\Omega}{1-t} \cap D)\,\mathrm{d}t\right)|Du(z)|\,\mathrm{d}z\,, \tag{4.1}$$

where $\lambda = \frac{1}{h^d}\mu_d$ in D. Now we characterize $\lambda(\frac{z-t\Omega}{1-t} \cap D)$ in more details rather than just using a uniform bound $\alpha(t)$ as before. We look at when the intersection $\frac{z-t\Omega}{1-t} \cap D$ becomes empty, i.e. $d(\frac{z-t\Omega}{1-t}, D) > 0$. Without loss of generality we assume $0 \in D$, otherwise we can shift the domain to contain the origin. Then, $0 \in D \cap \frac{t\Omega}{1-t}$. If $|z|$ is large then $\frac{z-t\Omega}{1-t}$ will be separated from D. A sufficient condition can be

$$\frac{|z|}{1-t} > \mathrm{diam}(\frac{t\Omega}{1-t}) + \mathrm{diam}(D) \geq \frac{1}{C_2}(\frac{tH}{1-t} + h)\,.$$

This is equivalent to $t \leq \frac{C_2|z|-h}{H-h}$. Thus we obtain

$$t \leq \frac{C_2|z|-h}{H-h} \quad \Rightarrow \quad \lambda(\frac{z-t\Omega}{1-t} \cap D) = 0\,. \tag{4.2}$$

We decompose the integral on the right-hand side of equation (4.1) into two parts (the integrand is abbreviated as I):

$$\int_\Omega I \mathrm{d}z = \int_{\{C_2|z|<2h\}\cap\Omega} I \mathrm{d}z + \int_{\{C_2|z|\geq 2h\}\cap\Omega} I \mathrm{d}z\,.$$

For the first part, we use the result in Corollary 2.1:

$$\begin{aligned}\int_{\{C_2|z|<2h\}\cap\Omega} I \mathrm{d}z &\leq C(d,p)(\frac{H}{h})^{d-1}\int_{\{C_2|z|<2h\}\cap\Omega}|Du(z)|\,\mathrm{d}z\\ &\leq C(d,p)\int_{\{C_2|z|<2h\}\cap\Omega}\left(\frac{H}{\max\{h,|z|\}}\right)^{d-1}|Du(z)|\,\mathrm{d}z\\ &\leq C(d,p)\int_{\{C_2|z|<2h\}\cap\Omega} w(z)|Du(z)|\,\mathrm{d}z\end{aligned} \tag{4.3}$$

where the last line is due to $d(z,D) \leq |z|$.

For the second part, we have $C_2|z| \geq 2h$. Due to equation (4.2), for $z \in \{C_2|z| \geq 2h\} \cap \Omega$ and at the same time $z \in \{C_2|z| \leq H\}$, we have

$$\begin{aligned}\int_0^1 \frac{1}{t^d}\mu_d(\frac{z-t\Omega}{1-t}\cap D)\,\mathrm{d}t &\leq \int_{\frac{C_2|z|-h}{H-h}}^1 \frac{1}{t^d}\,\mathrm{d}t\\ &\leq \frac{1}{d-1}\left((\frac{C_2|z|-h}{H-h})^{1-d}-1\right)\\ &\leq C(d,p)(\frac{H}{|z|})^{d-1} \leq C(d,p)w(z)\,,\end{aligned} \tag{4.4}$$

where the last two lines are due to the relation $0 \leq h \leq \frac{C_2}{2}|z|$ and $d(z,D) \leq |z|$. For $z \in \{C_2|z| \geq 2h\} \cap \Omega$ and also $z \in \{C_2|z| > H\}$, the integral vanishes due to equation (4.2). Combining all these together, we arrive at

$$\|u - \frac{1}{h^d}\int_D u\|_{L^1(\Omega)} \leq C(d,p)H\|Du\|_{L^1_w(\Omega)}\,.$$

This completes the proof. $\square$

Theorem 4.2. *Let $D \subset \Omega$ satisfy Assumptions 2.1 and 4.1, with $\mu_d(\Omega) = H^d$ and $\mu_d(D) = h^d$. For every $u \in W^{1,p}(\Omega)$ with $p > 1$, the following inequality holds:*

$$\|u - \frac{1}{h^d}\int_D u\|_{L^p(\Omega)} \leq C(d,p)H\|Du\|_{L^p_w(\Omega)}$$

if the weight function satisfies the condition

$$\int_\Omega \left(\frac{H}{\max\{h,d(z,D)\}}\right)^{\frac{p(d-1)}{p-1}} w(z)^{-\frac{1}{p-1}}\,\mathrm{d}z \leq C_w(d,p)H^d\,,$$

where $C(d,p)$ and $C_w(d,p)$ are constants that depend on d and p only.

Proof.

$$\begin{aligned}\|u - \frac{1}{h^d}\int_D u\|_{L^p(\Omega)} &\leq \|u - \frac{1}{H^d}\int_\Omega u\|_{L^p(\Omega)} + H^{\frac{d}{p}}|\frac{1}{H^d}\int_\Omega u - \frac{1}{h^d}\int_D u| \\ &\leq C(d,p)H\|Du\|_{L^p(\Omega)} + H^{\frac{d}{p}-d}\int_\Omega\int_D \frac{1}{h^d}|u(x)-u(y)|\,\mathrm{d}x\mathrm{d}y\end{aligned} \tag{4.5}$$

where we have used the standard Poincaré inequality for the first part. For the second part, due to the proof in Theorem 2.1 and Theorem 4.1, we have

$$H^{\frac{d}{p}-d}\int_\Omega\int_D \frac{1}{h^d}|u(x)-u(y)|\,\mathrm{d}x\mathrm{d}y \leq C(d,p)H^{\frac{d}{p}-d+1}\int_\Omega \left(\frac{H}{\max\{h, d(z,D)\}}\right)^{d-1}|Du(z)|\,\mathrm{d}z\,.$$

Using the Hölder inequality, we get

$$\begin{aligned}&\int_\Omega \left(\frac{H}{\max\{h, d(z,D)\}}\right)^{d-1}|Du(z)|\,\mathrm{d}z \\ =&\int_\Omega \left(\frac{H}{\max\{h, d(z,D)\}}\right)^{d-1} w(z)^{-\frac{1}{p}}\cdot w(z)^{\frac{1}{p}}|Du(z)|\,\mathrm{d}z \\ \leq&\left(\int_\Omega \left(\frac{H}{\max\{h, d(z,D)\}}\right)^{\frac{p(d-1)}{p-1}} w(z)^{-\frac{1}{p-1}}\,\mathrm{d}z\right)^{\frac{p-1}{p}}\|Du\|_{L^p_w(\Omega)} \\ \leq& C_w^{1-\frac{1}{p}}H^{d-\frac{d}{p}}\|Du\|_{L^p_w(\Omega)}\,.\end{aligned}$$

Plugging this into equation (4.5) gives

$$\|u - \frac{1}{h^d}\int_D u\|_{L^p(\Omega)} \leq C(d,p)H\|Du\|_{L^p(\Omega)} + C_w^{1-\frac{1}{p}}C(d,p)H\|Du\|_{L^p_w(\Omega)} \leq C(d,p)H\|Du\|_{L^p_w(\Omega)}$$

where $C(d,p)$ represents a generic constant that depends on d and p only. □

Example 4.1. *The weight function*

$$w(x) = \left(\frac{H}{\max\{h, d(x,D)\}}\right)^{d-p+\beta}$$

for any $\beta > 0$ satisfies the condition in Theorem 4.2.

Example 4.2. *The weight function*

$$w(x) = \left(\frac{H}{\max\{h, d(x,D)\}}\right)^{d-p}(\log(\frac{H}{\max\{h, d(x,D)\}})+1)^\beta$$

for $\beta > p-1$ satisfies the condition in Theorem 4.2.

A sufficient condition for the assumption on w in Theorem 4.2 is

$$\int_\Omega \left(\frac{H}{|x-x_0|}\right)^{\frac{p(d-1)}{p-1}} w(x)^{-\frac{1}{p-1}}\,\mathrm{d}z \leq C_w(d,p)H^d\,,$$

for a selected point $x_0 \in D$. This condition does not involve h. Similar to Example 4.1, one candidate function that satisfies this condition is

$$w(x) = \left(\frac{H}{|x-x_0|}\right)^{d-p+\beta}$$

where $\beta > 0$. Hence, for this w, due to Theorem 4.2, for any $h > 0$, and $x_0 \in D \subset \Omega$, we have

$$\|u - \frac{1}{h^d}\int_D u\|_{L^p(\Omega)} \leq C(d,p)H\|Du\|_{L^p_w(\Omega)}\,.$$

If we have the assumption that $\|Du\|_{L^p_w(\Omega)} \leq M$, i.e.

$$\int_\Omega \left(\frac{H}{|x-x_0|}\right)^{d-p+\beta} |Du(x)|^p \,\mathrm{d}x \leq M\,,$$

then, for any small h, we have the guaranteed non-degenerate accuracy $\|u - \frac{1}{h^d}\int_D u\|_{L^p(\Omega)} \leq C(d,p)HM$. Indeed, due to [2], the trace of u at x_0 is well-defined. Thus, we even have the inequality with the pointwise measurements by taking h to 0:

$$\|u - u(x_0)\|_{L^p(\Omega)} \leq C(d,p)H\|Du\|_{L^p_w(\Omega)}\,.$$

This inequality implies recovery from a pointwise value is possible if $\|Du\|_{L^p_w(\Omega)} < \infty$.

5. Discussion

In this paper, we discussed the subsampled Poincaré inequality with applications to functional recovery problems. When applied to the recovery problem, the Poincaré inequality naturally connects to the piecewise constant basis functions. The optimality with respect to the subsampled length scale is demonstrated in such a case. We can use ideas from the spline approximation theory to improve the regularity of the basis, which will improve the accuracy when the underlying function has better regularity. Inspired by [15][13], these basis functions can be used to solve the multiscale PDEs problem when their exponential decay and localization property are established. We will discuss it in our companion numerical paper [3] regarding the trade-off between the subsampled scale h, the exponential decay rate of the basis function, and the accuracy of the approximate solution.

Our discussion on the weighted Poincaré inequality connects to the degeneracy issue in graph Laplacian based semi-supervised learning problems [12], which are formulated as (possibly discrete) functional recovery problems. Adjusting the weights of the Laplacian to achieve good recovery performance is essential. Recently [2] established the consistency of the properly weighted graph Laplacian approach. The weight function there has the same form as our Example 4.1. These Sobolev critical functions help regularize the process to obtain a non-degenerate recovery. We will present more examples in the work [3].

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