Positive Solutions of Nonlinear Elliptic Eigenvalue Problems

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1. Introduction. We shall study a class of mildly nonlinear elliptic eigenvalue problems which are suggested by several recently occurring problems concerning the steady state temperature distribution of a physical medium in which heat is being generated nonlinearly. (See [12] for references to the physics literature.) Specifically, we investigate the general problem

$$(1.1)\qquad \begin{aligned} Lu &= \lambda f(x, u), & x &\in D, \\ Bu &= 0, & x &\in \partial D, \end{aligned}$$

where $x = (x_1, x_2, \cdots, x_N)$ and L is the uniformly elliptic, second order operator

$$(1.2)\qquad Lu \equiv -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{k=1}^{N} a_k(x)\frac{\partial u}{\partial x_k} + c(x)u,$$

and the boundary operator B is given by

$$(1.3)\qquad Bu \equiv b_0(x) + \sum_{k=1}^{N} b_k(x)\frac{\partial u}{\partial x_k}.$$

We denote by $C^{m+\alpha}(R)$ the space of functions which are m times continuously differentiable on a point set R and have Hölder continuous m^{th} derivatives on R with Hölder exponent α. We assume that D is a bounded domain in N dimensions with ∂D of class $C^{2+\alpha}$; that the coefficients $a_{ij}(x)$, $a_k(x)$, $c(x)$ are in $C^{\alpha}(D)$ and $b_0(x)$, $b_k(x)$ are in $C^{1+\alpha}(\partial D)$ for some $\alpha \in (0, 1)$. Taking $n_i(x)$ as the components of the outer normal to ∂D at x, we assume that

$$\text{(i)}\qquad \sum_{k=1}^{N} b_k(x)n_k(x) > 0 \quad \text{unless} \quad \sum_{k=1}^{N} b_k(x)b_k(x) = 0,$$

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(ii) $$\max \left\{ b_0(x), \sum_{k=1}^{N} b_k(x) b_k(x) \right\} > 0,$$

(iii) either $c(x) > 0, \quad x \in D, \quad b_0(x) \geqq 0, \quad x \in \partial D,$
or $c(x) \geqq 0, \quad x \in D, \quad b_0(x) > 0, \quad x \in \partial D.$

Under these conditions, a solution of the linear problem, for $F(x) \in C^\alpha(D)$,

$$Lu(x) = F(x), \qquad x \in D,$$
$$Bu(x) = 0, \qquad x \in \partial D, \tag{1.4}$$

exists in $C^{2+\alpha}(D)$ ([2], pp. 134–136), is unique, and enjoys the positivity property that $u(x) > 0$ for $x \in D$ if $F(x) \geqq 0$, $F(x) \not\equiv 0$ for $x \in D$ ([1] pp. 4-7). Alternatively, L^{-1} is a continuous, positive operator from $C^\alpha(D)$ to $C^{2+\alpha}(D) \cap \{w(x) \mid w(x) \in C^{2+\alpha}(D), Bw(x) = 0, x \in \partial D\}$.

The nonlinearity is assumed to satisfy the following conditions, hereafter referred to as conditions (H):

(H) $f(x, u)$ is defined on $D \times \{u \mid u \geqq 0\} \equiv D_1$, $f(x, u) \in C^\alpha(D_1)$,

$f(x, u)$ is assumed to be monotonic in u and concave in u in the sense that for all $\tau \in (0, 1)$, $u > 0$, $(x, u) \in D_1$ we have

$$f(x, \tau u) - \tau f(x, u) > 0. \tag{1.5}$$

(A useful geometrical interpretation of (1.5) is that it implies that $f(x, u)$, when graphed as a function of u for fixed x, has the property that any line segment from the origin to the function lies below the graph of the function.)

We assume further that $f(x, 0) \geqq 0$ for $x \in D$ and define

Case a. The forced case: $f(x_0, 0) > 0$ for some $x_0 \in D$.
Case b. The unforced (or free) case: $f(x, 0) = 0$ for all $x \in D$.

This terminology was clearly suggested by the usual physical interpretation of the right-hand side $\lambda f(x, u)$ in (1.1) as a forcing function. With this interpretation it will not be surprising when we show that the two different cases exhibit markedly different behavior in some respects.

In keeping with the physical problems suggesting our study, we shall look for positive solutions of (1.1) in both the forced and unforced cases. In Section 2 we shall establish that there exists a unique positive solution of (1.1) for all values of λ in some interval whose left endpoint depends on the behavior of $f(x, u)$ near $u = 0$ and whose right endpoint depends on the behavior of $f(x, u)$ for large u. Furthermore, these endpoints will be given precisely, and the positive solutions are characterized constructively by iteration schemes, involving only linear equations, yielding monotonically non-decreasing converging sequences.

Various properties of the positive solution and the values of λ for which it exists are derived in Section 3. The main results are a study of the positive solution and the interval of λ for which it exists in the forced case in the limit as the forcing vanishes. These results clearly show the similarities and differences of the problem in the two cases.

In Section 4, in addition to strengthening the requirements on $f(x, u)$, we impose the further condition of self-adjointness on L. Then, two different variational principles are derived, and the implication with regard to stability of the positive solution is discussed. Finally, iteration procedures different from those of Section 2 are introduced which yield monotonically non-increasing converging sequences. Hence, combining these with the results of Section 2, we obtain sequences of pointwise upper and lower bounds on the positive solution.

2. Existence of positive solutions. The purpose of this section is to establish that under the conditions of hypothesis (H) there exists a unique positive solution of the boundary value problem (1.1) for all values of λ in some interval whose left endpoint depends on the behavior of $f(x, u)$ near $u = 0$ and whose right endpoint depends on the behavior of $f(x, u)$ for large u. Furthermore, these endpoints will be given precisely.

It is well known that the lowest eigenvalues and eigenfunctions of the problem (1.1) linearized about $u = 0$ and $u = \infty$ play an important role in discussing (1.1). Hence, for each $m > 0$ we define $\mu(m)$ and $\varphi_{(m)}(x)$ to be the lowest eigenvalue and normalized eigenfunction of

$$\begin{aligned} L\varphi_{(m)}(x) &= \mu(m)\left(\frac{f(x, m)}{m}\right)\varphi_{(m)}(x), && x \in D, \\ B\varphi_{(m)}(x) &= 0, && x \in \partial D, \end{aligned} \tag{2.1}$$

such that $\varphi_{(m)}(x) > 0$ for $x \in D$ and $\int_D \varphi_{(m)}^2(x)\, dx = 1$. From [2] we conclude that $\varphi_{(m)}(x) \in C^{2+\alpha}(D)$. If $m_1 > m_2 > 0$, it follows from (1.5) that $f(x, m_1)/m_1 < f(x, m_2)/m_2$. Consequently, for a given $\psi(x) \geqq 0, \not\equiv 0$, in $C^{2+\alpha}(D)$ we have

$$L^{-1}\left(\frac{f(x, m_1)}{m_1}\right)\psi(x) \leqq L^{-1}\left(\frac{f(x, m_2)}{m_2}\right)\psi(x),$$

and from [5], p. 94, we conclude that $\mu(m_1) > \mu(m_2) > 0$.

We define

$$\begin{aligned} \mu(0) &= \lim_{m \to 0} [\mu(m)], \\ \mu(\infty) &= \lim_{m \to \infty} [\mu(m)], \end{aligned} \tag{2.2}$$

allowing $\mu(\infty) = \infty$ if the latter limit is not finite. Note that in the event that $f(x, u) \in C^1(D \times (-\epsilon, \epsilon))$ for some $\epsilon > 0$, then in the unforced case (*i.e.*, $f(x, 0) = 0$) we have $\lim_{m \to 0} f(x, m)/m = f_u(x, 0) > 0$ for $x \in D$, and thus $\mu(0) > 0$ is the primary eigenvalue of (1.1) linearized about $u = 0$.

The main results of this section are embodied in the

Theorem 2.1. *Let $f(x, u)$ satisfy the conditions (H). Then, there exists a unique positive solution $u(x; \lambda) \in C^{2+\alpha}(D)$ of (1.1)*

(a) *in the forced case for $\lambda \in (0, \mu(\infty))$, and*
(b) *in the unforced case for $\lambda \in (\mu(0), \mu(\infty))$.*

Theorem 2.1 will be proven in three main steps, (i) the generation of a monotone non-decreasing sequence of successive approximations, (ii) the proof in Lemma 1.1 of a uniform bound for the sequence and (iii) the proof in Lemma 1.2 that the regularity of the members of the sequence is inherited by the limit which consequently is a solution of (1.1). Comparable existence theorems for the forced case in [3] and in [5] use a similar first step. However the abstract theory is non-constructive, and cannot be applied directly to spaces giving the regularity results given here. Furthermore, Theorem 2.1 relaxes somewhat the smoothness requirements on $f(x, u)$ of previous theorems.

For the *unforced* case (b) we define our sequence of successive approximations as follows: Choose an m such that $\lambda > \mu(m)$, and then define

$$\left.\begin{aligned} Lu_n(x) &= \lambda f(x, u_{n-1}(x)), \qquad & x \in D,\\ Bu_n(x) &= 0, & x \in \partial D,\end{aligned}\right\} n = 1, 2, 3, \cdots . \tag{2.3}$$

Take $u_0(x) \equiv \epsilon\varphi_{(m)}(x)$ where ϵ is chosen so as to ensure that the sequence is monotone non-decreasing. This is accomplished as follows: Equations (2.1) and (2.3) imply that

$$\begin{aligned} L(u_1 - u_0) &= \lambda f(x, u_0) - \mu(m)\left[\frac{f(x, m)}{m}\right]u_0 \\ &= [\lambda - \mu(m)]f(x, u_0) + \mu(m)\left[f(x, u_0) - \frac{f(x, m)}{m}u_0\right] \\ &= [\lambda - \mu(m)]f(x, \epsilon\varphi_{(m)}) + \mu(m)\left[f(x, \epsilon\varphi_{(m)}) - \frac{f(x, m)}{m}\epsilon\varphi_{(m)}\right]. \end{aligned} \tag{2.4}$$

Choose ϵ such that $\epsilon\varphi_{(m)}(x) < m$. Then, (1.5) implies

$$f(x, \epsilon\varphi_{(m)}) - \frac{f(x, m)}{m}\epsilon\varphi_{(m)} = \epsilon\varphi_{(m)}\left[\frac{f(x, \epsilon\varphi_{(m)}}{\epsilon\varphi_{(m)}} - \frac{f(x, m)}{m}\right] \geqq 0.$$

Hence,

$$\begin{aligned} L(u_1 - u_0) &\geqq 0, \qquad & x \in D, \\ B(u_1 - u_0) &= 0, & x \in \partial D. \end{aligned}$$

Thus, from the Maximum Principle [11] for uniformly elliptic second order equations, we conclude that $u_1(x) \geqq u_0(x)$ on D. It follows by a simple induction argument that $\{u_n(x)\}$ is monotone non-decreasing in n.

For the *forced* case (a) we define our sequence of successive approximations by taking $u_0(x) \equiv 0$ and defining $u_n(x)$ for $n = 1, 2, 3, \cdots$ by (2.3). An induction argument shows here also that $\{u_n(x)\}$ is monotone non-decreasing in n.

Lemma 2.1. Let $\{u_n(x)\}$ be the sequence defined by (2.3) with $u_0(x) \geqq 0$. Then, there is a constant M such that for $x \in \bar{D}$,

$$0 \leqq u_n(x) \leqq M.$$

Proof. Let m be chosen sufficiently large so that

$$\lambda < \mu(m), \quad \text{and} \quad u_0(x) \leqq m. \tag{2.5}$$

Then, it follows from the Positivity Lemma of H. B. Keller and D. S. Cohen [3] that a function $h(x) > 0$ can be defined in D as the solution of

$$\begin{aligned} Lh(x) - \lambda\left[\frac{f(x, m)}{m}\right]h(x) &= \lambda f(x, m), \qquad x \in D, \\ Bh(x) &= 0, \qquad x \in \partial D. \end{aligned} \tag{2.6}$$

We shall show that

$$u_n(x) \leqq h(x) + m. \tag{2.7}$$

Let

$$\begin{aligned} w_n(x) &= \max\{u_n(x) - m, 0\}, \qquad x \in \bar{D}, \\ S_n &= \{x \mid x \in D, u_n(x) > m\}. \end{aligned} \tag{2.8}$$

Since $\{u_n(x)\}$ is monotone increasing in n, for $x \in S_n$,

$$\begin{aligned} Lw_n(x) &= Lu_n(x) - c(x)m \\ &= \lambda f(x, u_{n-1}(x)) - c(x)m \\ &\leqq \lambda f(x, w_{n-1}(x) + m) - c(x)m \\ &\leqq \lambda(f(x, m)/m)(w_{n-1}(x) + m) - c(x)m, \end{aligned} \tag{2.9}$$

the last inequality following from (1.5). Using (2.6), we obtain

$$L(h(x) - w_n(x)) \geqq \lambda(f(x, m)/m)(h(x) - w_{n-1}(x)) + c(x)m. \tag{2.10}$$

From the Minimum Principle for L, we can conclude that if

$$h(x) \geqq w_{n-1}(x) \quad \text{for} \quad x \in S_{n-1}, \tag{2.11}$$

the minimum of $h(x) - w_n(x)$ is assumed on ∂S_n. Let $\xi \in \partial S_n$ be a point at which this minimum occurs; then either $\xi \in D$, in which case $w_n(\xi) = 0$, or $\xi \in \partial D$, in which case

$$B(h(\xi) - w_n(\xi)) = b_0(\xi)m,$$

that is,

$$b_i(\xi) \frac{\partial(h(\xi) - w_n(\xi))}{\partial x_i} = -b_0(\xi)(h(\xi) - w_n(\xi)) + b_0(\xi)m. \tag{2.12}$$

At a minimum on ∂D the left side of (2.12) must be negative ([1], pp. 4–7) although $b_0(x)m \geqq 0$. Hence on whichever part of ∂S_n, ξ lies, we can conclude that

$$h(\xi) - w_n(\xi) \geqq 0.$$

Consequently,

$$h(x) - w_n(x) \geqq 0 \quad \text{for} \quad x \in S_n . \tag{2.13}$$

Hence from (2.11) we conclude that (2.13) is true. However, our choice of m, (2.5), ensures that $w_0(x) \equiv 0$. Hence, by induction we have $w_n(x) \leqq h(x)$ for all n, which by the definition (2.8) implies (2.7). Q.E.D.

On the basis of this lemma we conclude that the sequence $\{u_n(x)\}$ of iterates converges at least pointwise in $\bar{D}$, and for $x \in \bar{D}$ we let

$$u(x) = \lim_{n\to\infty} [u_n(x)]. \tag{2.14}$$

In order to establish that this limit $u(x)$ is the positive solution $u(x; \lambda)$ of the Theorem 2.1 we need several *á priori* inequalities which we derive in the

Lemma 2.2. *Let $\{u_n(x)\}$ be the sequence of successive approximations defined by (2.3) such that $u_0(x) \in C^{2+\alpha}(D)$. Then, the functions $u_n(x)$ are Hölder continuous with exponent μ and Hölder constants uniform in n for $0 < \mu < 1$.*

Proof. We employ a technique used by Morrey [7], p. 79, to prove a Sobolev Lemma. Let $H^m_p(D)$ be the space of functions defined on D, with distributional derivatives up to order m in $L_p(D)$, and let $||w(x)||_{m,p,D}$ be the norm on $H^m_p(D)$. Then if D' is a sphere containing $\bar{D}$ in its interior, $w(x) \in H^m_p(D)$ can be extended to $w(x) \in H^m_p(D')$ such that

$$||w(x)||_{m,p,D'} \leqq K \, ||w(x)||_{m,p,D} , \tag{2.15}$$

where K depends only on m, p, D and D' ([7], p. 74). Each iterate $u_n(x)$ belongs to $C^{2+\alpha}(D)$ and hence $H^2_p(D)$ for every $p \geqq 1$; we now consider $u_n(x)$ as extended to D' so that (2.15) holds for $m = 2$ and p sufficiently large. By Theorem 3.6.6 of [7], the extended functions will be in $C^{1+\beta}(D')$ for $\beta = 1 - N/p$. For arbitrary μ, $0 < \mu < 1$, and some $\epsilon_0 > 0$ depending on D, we shall exhibit a constant, C, independent of n such that

$$|u_n(x) - u_n(\xi)| \leqq C \, |x - \xi|^\mu \tag{2.16}$$

for x, ξ in D and $|x - \xi| < \epsilon_0$.

For x and ξ in D, let $\bar{x} = (x + \xi)/2$, $|x - \xi| = 2\rho$ and let $B(\bar{x}, \rho)$ be the sphere of radius ρ centered on $\bar{x}$. Since $D \subset D'$, we can choose ϵ_0 such that if $\rho < \epsilon_0/2$, $B(\bar{x}, \rho) \subset D'$. For any $\eta \in B(\bar{x}, \rho)$, since $u_n(x) \in C^{1+\beta}(D')$, we can write

$$
\begin{aligned}
|u_n(\eta) - u_n(\xi)| &= \left|(\eta_k - \xi_k) \int_0^1 u_{n,k}(\xi + t(\eta - \xi))\, dt\right| \\
&\leqq |\eta - \xi| \int_0^1 |\operatorname{grad} u_n(\xi + t(\eta - \xi))|\, dt.
\end{aligned} \tag{2.17}
$$

Denoting the volume of $B(\bar{x}, \rho)$ by $|B(\bar{x}, \rho)|$, we can average (2.17) over $B(\bar{x}, \rho)$ with respect to η to obtain

$$
\begin{aligned}
&\frac{1}{|B(\bar{x}, \rho)|} \int_{B(\bar{x},\rho)} |u_n(\eta) - u_n(\xi)|\, d\eta \\
&\qquad \leqq \frac{2\rho}{|B(\bar{x}, \rho)|} \int_{B(\bar{x},\rho)} \int_0^1 |\operatorname{grad} u_n(\xi + t(\eta - \xi))|\, dt\, d\eta \\
&\qquad = \frac{2\rho}{|B(\bar{x}, \rho)|} \int_0^1 \int_{B_t(\bar{x},\rho)} |\operatorname{grad} u_n(y)|\, t^{-N}\, dy\, dt,
\end{aligned} \tag{2.18}
$$

where $y = \xi + t(\eta - \xi)$ and $B_t(\bar{x}, \rho)$ is the sphere of radius $t\rho$ tangent to $B(\bar{x}, \rho)$ at ξ. Using Hölder's Inequality and replacing $B_t(\bar{x}, \rho)$ by $B(\bar{x}, \rho)$ for $q = p/(p - 1)$, we obtain further that

$$
\begin{aligned}
&\frac{1}{|B(\bar{x}, \rho)|} \int_{B(\bar{x},\rho)} |u_n(\eta) - u_n(\xi)|\, d\eta \\
&\leqq \frac{2\rho}{|B(\bar{x}, \rho)|} \int_0^1 \left(\int_{B_t(\bar{x},\rho)} |\operatorname{grad} u_n(y)|^p\, dy\right)^{1/p} \left(\int_{B_t(\bar{x},\rho)} 1\, dy\right)^{1/q} t^{-N}\, dt \\
&= \frac{2\rho}{|B(\bar{x}, \rho)|} \int_0^1 \left(\int_{B_t(\bar{x},\rho)} |\operatorname{grad} u_n(y)|^p\, dy\right)^{1/p} (C_N (t\rho)^N)^{1/q} t^{-N}\, dt \\
&\leqq \frac{2C_N^{1/q} \rho^{1+N/q}}{|B(\bar{x}, \rho)|} \left(\int_{B(\bar{x},\rho)} |\operatorname{grad} u_n(y)|^p\, dy\right)^{1/p} \int_0^1 t^{N(1/q-1)}\, dt \\
&\leqq K\rho^{1-N(1-1/q)}\, \|u_n(y)\|_{1,p,D'}\,,
\end{aligned} \tag{2.19}
$$

where K is a constant that depends only on N and q. For $\mu \in (0, 1)$ we can choose p so that $1 - N(1 - 1/q) = \mu$. With p fixed we can conclude from the extension of $u_n(x)$ to D' that

$$
\begin{aligned}
\|u_n(y)\|_{1,p,D'} &\leqq \|u_n(y)\|_{2,p,D'} \leqq K_1\, \|u_n(y)\|_{2,p,D} \\
&\leqq K_2\{\|f(y, u_{n-1}(y))\|_{0,p,D} + \|u_n(y)\|_{0,p,D}\}
\end{aligned} \tag{2.20}
$$

for constants K_1 and K_2. This last estimate comes from the L_p estimates for the elliptic operator L with the given boundary conditions found in [8] coupled with the definition of the sequence $\{u_n(x)\}$. However, by Lemma 2.1, we know that the successive approximations are uniformly bounded, hence the right hand side of (2.20) is bounded independently of n. From (2.19) we find, for a constant c, that

$$
(1/|B(\bar{x}, \rho)|) \int_{B(\bar{x},\rho)} |u_n(\eta) - u_n(\xi)|\, d\eta \leqq (c/2)\, |x - \xi|^{\mu}. \tag{2.21}
$$

Clearly (2.21) holds with ξ replaced by x on the left hand side; hence, a use of the Triangle Inequality gives the result (2.16). Q.E.D.

With this lemma proved we can now complete the proof of Theorem 2.1 as follows: We pass to the limit with n in (2.16) to conclude that $u(x) \in C^{\mu}(\bar{D})$ for $0 < \mu < 1$. Since $\{u_n(x)\}$ converges monotonically to the uniformly continuous function $u(x)$, we conclude from Dini's Theorem, that the convergence is uniform. These lemmas show that $\{f(x, v_n(x))\}$ is a sequence of uniformly bounded functions in $C^{\gamma}(D)$ for $0 < \gamma < \alpha$ with Hölder constants depending on γ but uniform in n. Using the Schauder Estimate of Theorem 7.3, [8], we have

$$|u_n(x)|_{2+\gamma} \leqq K\{|f(x, u_{n-1}(x)|_{\gamma} + |u_n(x)|_0\} \leqq M,$$

where $| \ |_{m+\gamma}$ is the Schauder norm on $C^{m+\gamma}(D)$, $| \ |_0$ is the maximum norm, and M is a uniform bound. Thus we have satisfied the hypotheses of Theorem 12.2 of [8], from which we conclude that $u(x) \in C^{2+\gamma}(D)$ satisfies (1.1). However, $u(x) \in C^{2+\gamma}(D)$ implies $\lambda f(x, u(x)) \in C^{\alpha}(D)$, so that $u(x) \in C^{2+\alpha}(D)$. The uniqueness of $u(x; \lambda)$ follows from Theorem 6.3 of [5], completing the proof of Theorem 2.1.

Corollary 2.1. *The sequence* $\{u_n(x)\}$ *defined in Lemma* 2.2 *converges uniformly with its first and second derivatives to the solution of* (1.1).

This corollary follows easily from Theorem 12.2 of [8].

In the case $f(x, u)$ is differentiable we have a result on non-existence given in

Theorem 2.2. *If* $\mu(0) > 0$ (*that is, in the unforced case*), *then* (1.1) *has no positive solutions for* $\lambda \in (0, \mu(0))$.

Proof. Let $v(x; \lambda)$ be a positive solution of (1.1) for $\lambda \in (0, \mu(0))$, *i.e.*

$$v(x; \lambda) = \lambda L^{-1} f(x, v(x; \lambda)). \tag{2.22}$$

We can interpret (2.22) as implying that λ^{-1} is the primary eigenvalue, with eigenfunction $v(x; \lambda)$, of the linear operator A, defined, for $w(x) \in C^{2+\alpha}(D)$ and $Bw(x) = 0$ for $x \in \partial D$, by

$$Aw(x) = L^{-1}([f(x, v(x; \lambda))/v(x; \lambda)]w(x)). \tag{2.23}$$

Now (1.5) implies that for $0 < \epsilon < 1$

$$f(x, \epsilon v(x; \lambda))/\epsilon v(x, \lambda) > f(x, v(x; \lambda))/v(x; \lambda).$$

If we define A_ϵ by

$$A_\epsilon w(x) = L^{-1}([f(x, \epsilon v(x; \lambda))/\epsilon v(x; \lambda)]w(x)), \tag{2.24}$$

then for all positive functions $w(x) \in C^{2+\alpha}(D)$, $A_\epsilon w(x) \geqq Aw(x)$. Hence, denoting the primary eigenvalue of A_ϵ by λ_ϵ^{-1},

$$\lambda_\epsilon^{-1} \geqq \lambda^{-1}, \quad i.e. \quad \lambda_\epsilon \leqq \lambda$$

([5], p. 94). However, λ_ϵ converges to $\mu(0)$ as ϵ tends to zero, which contradicts $\lambda \in (0, \mu(0))$. Q.E.D.

In the event that $\mu(0) > 0$, we shall extend the definition of $u(x; \lambda)$ by defining $u(x, \lambda) \equiv 0$ for $x \in D$, $\lambda \in (0, \mu(0)]$.

3. Properties of the positive solution. The iteration schemes and characterization of the positive solution given in Section 2 lead to a variety of results for the positive solution $u(x, \lambda)$ of (1.1). Bounds and estimates on $\mu(0)$ and $\mu(\infty)$ and various comparison theorems can be given analogous to those derived in Section 3 of [3] for the self-adjoint operator L with slightly more restrictive conditions on $f(x, u)$. We restrict ourselves here to new results, one of which is the proof of a conjecture made in [3]. One of our principal results is the study in Theorem 3.3 of the behavior of $u(x, \lambda)$ and $\mu(0)$ in the forced case as the forcing vanishes (that is, the behavior as the forced case approaches the unforced case).

Our first result is the

Theorem 3.1. *If* $\mu(0) < \lambda_1 < \lambda_2 < \mu(\infty)$, *then* $u(x, \lambda_1) < u(x, \lambda_2)$, $x \in D$. This easily follows from Theorem 2.1 by forming the sequence of successive approximations for $\lambda = \lambda_2$ with $v_0(x) = u(x, \lambda_1)$.

We now establish a conjecture made in [3] in our

Theorem 3.2.

$$\lim_{\lambda \to \mu(\infty)-} [\max_{x \in D} |u(x, \lambda)|] = \infty. \tag{3.1}$$

Proof. In the event that $\mu(\infty) = \infty$, the result follows from the fact that $\lambda f(x, u(x; \lambda))$ becomes infinite, uniformly on compact subsets of D, as λ becomes infinite. We assume that $\mu(\infty) < \infty$. If (3.1) is not true, then we can find a monotone increasing sequence $\{\lambda_n\}$ for which $\lim_{n\to\infty} \lambda_n = \mu(\infty)$ and $\{u(x, \lambda_n)\}$ is a monotone increasing sequence uniformly bounded above. Let

$$\lim_{n\to\infty} [u(x, \lambda_n)] \equiv w(x) \leqq M, \quad \text{for} \quad x \in D, \tag{3.2}$$

for some constant M. Then, $\{\lambda_n f(x, u(x, \lambda_n))\}$ converges in $L_p(D)$ to $\mu(\infty) f(x, w(x))$. If we let $H_{2,p}^{(B)}(D)$ be the closed subspace of $H_{2,p}(D)$ satisfying the boundary conditions of (1.1) (in the sense of [8]), then it follows from the L_p estimates of [8] that L^{-1} is continuous from $L_p(D)$ to $H_{2,p}^{(B)}(D)$, *i.e.* that $\{u(x; \lambda_n)\}$ is a Cauchy sequence in $H_{2,p}^{(B)}(D)$. Hence $w(x) \in H_{2,p}^{(B)}(D)$ and

$$\begin{aligned} w(x) &= \mu(\infty) L^{-1} f(x, w(x)), \qquad && x \in D, \\ Bw(x) &= 0, && x \in \partial D. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3) a contradiction can be derived. We can regard $w(x)$ as an eigenfunction of the linear problem

$$\begin{aligned} w(x) &= \mu(\infty) L^{-1}(\rho(x) w(x)), \qquad && x \in D, \\ Bw(x) &= 0, && x \in \partial D, \end{aligned} \tag{3.4}$$

where

$$\rho(x) = f(x, w(x))/w(x) > f(x, M)/M \tag{3.5}$$

for $x \in D$. However, from (3.5) and the comparison theorem of [5] (p. 94), we conclude that $\mu(\infty) < \mu(M)$, contradicting the definition of $\mu(\infty)$. Q.E.D.

We now wish to show that the positive solution in the free case for $\lambda \in (\mu(0), \mu(\infty))$ is the limit of the forced case for vanishing 'load.' To do this we introduce the family of functions $f(x, u; \sigma)$ depending on a parameter $\sigma > 0$ such that for each $\sigma > 0$ $f(x, u; \sigma)$ satisfies the conditions for (1.1) and $f(\bar{x}, 0; \sigma) > 0$ for some $\bar{x} \in D$. Let $f(x, u; \sigma_1) \geqq f(x, u; \sigma_2)$ if $\sigma_1 \geqq \sigma_2$ and $\lim_{\sigma \to 0} f(x, u; \sigma) = F(x, u)$ exist uniformly on compact subsets of $D_1 = D \times [0, \infty)$. We assume that $F(x, 0) \equiv 0$, and $F(x, u) \in C^\alpha(D_1)$. Hence, we have simply embedded our original function $f(x, u)$ in a one-parameter family $f(x, u; \sigma)$ such that when σ tends to zero the forcing tends to zero. Let $\{u(x; \lambda, \sigma)\}$ be the solutions in the forced cases of (1.1) with $\lambda f(x, u; \sigma)$ on the right-hand side, and let $w(x; \lambda)$ be the solution in the unforced case of (1.1) with $\lambda F(x, w)$ on the right-hand side.

Theorem 3.3. *For $\lambda \in (0, \mu(\infty))$ we have*

$$u(x; \lambda, \sigma_1) \geqq u(x; \lambda, \sigma_2) \quad \textit{if} \quad \sigma_1 > \sigma_2 > 0. \tag{3.6}$$

Furthermore, as $\sigma \to 0$, $u(x; \lambda, \sigma)$ converges to $w(x; \lambda)$ uniformly for $x \in D$.

Proof. For $\sigma_1 > \sigma_2 > 0$, consider the successive approximations $\{v_n(x)\}$ generated by (2.3) from the initial choice of $v_0(x) = u(x; \lambda, \sigma_2)$ with $\lambda f(x, v_{n-1}(x); \sigma_1)$ on the right side. This is a monotone increasing sequence and hence by Corollary 2.1 of Theorem 2.1 converges to $u(x; \lambda, \sigma_1)$, which proves (3.6). If we now consider the successive approximations $\{y_n(x)\}$ for the same equation generated from the initial choice $y_0(x) = w(x; \lambda)$ it can be seen that this also is a monotone increasing sequence which converges to $u(x; \lambda, \sigma_1)$, *i.e.*

$$u(x; \lambda, \sigma) \geqq w(x; \lambda) \tag{3.7}$$

for $x \in D$, $\sigma > 0$. Hence the following limit exists in $L_p(D)$ for $1 \leqq p < \infty$, and satisfies

$$\lim_{\sigma \to 0} u(x; \lambda, \sigma) = \tilde{w}(x) \geqq w(x; \lambda). \tag{3.8}$$

Furthermore, $\lambda f(x, u(x; \lambda, \sigma); \sigma)$ converges in $L_p(D)$ to $\lambda F(x, \tilde{w}(x))$. From (2.20), we conclude that $\tilde{w}(x) \in H_{2,p}^{(B)}(D)$ and is a weak solution of (1.1) with $\lambda F(x, \tilde{w}(x))$ on the right-hand side. However, by the uniqueness of the solution of (1.1), $\tilde{w}(x) = w(x; \lambda)$, (see [5], Theorem 6.3), and so, by Dini's Theorem the convergence of $u(x; \lambda, \sigma)$ to $w(x, \lambda)$ is uniform for $x \in D$. Q.E.D.

As an immediate consequence of this theorem and Theorem 2.2 we have the

Corollary 3.3. *For $\lambda \in (0, \mu(0))$*

$$\lim_{\sigma \to 0} [u(x; \lambda, \sigma)] \equiv 0 \quad \textit{for} \quad x \in D.$$

4. The self-adjoint case and variational principles. The object of this section is to present a variety of properties of the unique positive solution, which, with the exception of Theorem 4.1, one way or another stem from the basic lemma of the section, Lemma 4.1, to the effect that $\lambda < \mu(u(x; \lambda))$. The properties themselves, of differentiability of $u(x; \lambda)$ with respect to λ, two variational characterizations, and a representation of $u(x; \lambda)$ as a uniform limit of monotone decreasing upper bounds have no clear common connection; however critical in our proof of each has been the use of Lemma 4.1. In this section we shall require the following additional hypothesis on $f(x, u)$:

$(H')f(x, u)$ is continuously differentiable with respect to u and $f_u(x, u) \in C^\alpha(D_1)$. In addition, *throughout this section we shall require that L be self-adjoint with boundary operator B.* In particular, this implies that $a_k(x) \equiv 0$, $x \in D$, for $k = 1, \cdots, N$, and

$$b_j(x) = \sum_{i=1}^{N} a_{ij}(x)n_i(x), \qquad x \in \partial D, \quad \text{for} \quad j = 1, \cdots, N.$$

We can now introduce $\mu(h(x))$ for a smooth, positive function $h(x)$ as the lowest eigenvalue of the problem (1.1) linearized about $h(x)$; that is,

$$\begin{aligned} Lv &= \mu f_u(x, h(x))v, \qquad && x \in D, \\ Bv &= 0, && x \in \partial D. \end{aligned} \tag{4.1}$$

We wish to note here for future reference that the concavity condition (1.5) implies that $f(x, u)/u$ is a decreasing function of u for positive u. Therefore, with $f(x, u)/u$ continuously differentiable on $(0, \infty)$ the condition (1.5) is equivalent to

$$\frac{d}{du}\left[\frac{f(x, u)}{u}\right] = \frac{1}{u}\left[f_u(x, u) - \frac{f(x, u)}{u}\right] < 0. \tag{4.2}$$

We shall now demonstrate a result complimentary to Theorem 3.2, *i.e.* in the unforced case max $|u(x; \lambda)| \to 0$ as $\lambda \to \mu(0)+$. It seems to us that this should hold under the conditions (H') on $f(x, u)$. However, we have been able to prove this result only under the stronger restrictions on $f(x, u)$ given in the

Theorem 4.1. *If, in addition to satisfying (H), the function $f(x, u)$ is twice continuously differentiable with $f_{uu}(x, u) < 0$ for $u > 0$, then in the unforced case,*

$$\lim_{\lambda \to \mu(0)+} [\max_{x \in D} |u(x; \lambda)|] = 0. \tag{4.3}$$

Proof. Since by Theorem 3.1 $u(x; \lambda)$ is monotone decreasing as $\lambda \to 0$, then $\lim_{\lambda \to \mu(0)+} [u(x; \lambda)]$ exists and is equal to $w(x)$, say, and $w(x)$ satisfies

$$\begin{aligned} Lw &= \mu(0)f(x, w), \qquad && x \in D, \\ Bw &= 0, && x \in \partial D. \end{aligned} \tag{4.4}$$

This follows from the application of Theorem 12.2 of [8] to the one parameter family of inhomogeneous terms $\{\lambda f(x, u(x; \lambda))\}$. Recall that we showed in Section 2 (before the statement of Theorem 2.1) that in the unforced case $\mu(0)$ is the primary eigenvalue of (1.1) linearized about $u = 0$. Now, using Taylor's Theorem, we can write (4.4) as

$$\begin{aligned} Lw - \mu(0)f_u(x, 0)w &= \mu_1 f_{uu}(x, \xi)\frac{w^2}{2}, \qquad x \in D, \\ Bw &= 0, \qquad x \in \partial D, \end{aligned} \tag{4.5}$$

where $0 \leqq \xi(x) \leqq w(x)$. Hence, $w(x) \equiv 0$ for $x \in D$ because if there exists an $\bar{x} \in D$ such that $w(\bar{x}) > 0$, then by the Fredholm Alternative Theorem $\mu(0)f_{uu} \cdot (x, \xi)w^2/2$ must be orthogonal to $\nu(x)$, the principal eigenfunction of $L\nu - \mu(0)f_u(x, 0)\nu = 0$, $B\nu = 0$. However, $\nu(x)$ and $\mu(0)f_{uu}(x, \xi)w^2/2$ each have only one sign in D, and thus, this latter situation is impossible. Q.E.D.

Lemma 4.1. *If $u(x; \lambda)$ is the unique positive solution of* (1.1), *then*

$$\lambda < \mu(u(x; \lambda)) \tag{4.6}$$

(a) *in the forced case for $\lambda \in (0, \mu(\infty))$, and*
(b) *in the unforced case for $\lambda \in (\mu(0), \mu(\infty))$.*

Proof. Let $p(x) \equiv \lambda\{f(x, u(x; \lambda)) - f_u(x, u(x; \lambda))u(x; \lambda)\}$. Then, from (4.2), $p(x) \geqq 0$ for $x \in D$ with inequality for some $x \in D$. Thus the problem, for λ as specified in the above statement,

$$\begin{aligned} Lw(x) - \lambda f_u(x, u(x; \lambda))w(x) &= p(x), \qquad x \in D, \\ Bw(x) &= 0, \qquad x \in \partial D, \end{aligned} \tag{4.7}$$

has the positive solution $w(x) = u(x; \lambda)$ and, by the Positivity Lemma of [3], $\lambda < \mu(u(x; \lambda))$. Q.E.D.

Corollary 4.1. *$u(x; \lambda)$ is continuously differentiable with respect to λ, for $\lambda \in (\mu(0), \mu(\infty))$ in the free case; and $\lambda \in (0, \mu(\infty))$ in the forced case, and $\partial u(x; \lambda)/\partial\lambda \equiv y(x)$ satisfies*

$$Ly(x) - \lambda f_u(x, u(x; \lambda))y(x) = f(x, u(x; \lambda)).$$

Proof. This statement follows as an application of the abstract Implicit Function Theorem ([9], p. 265) applied to the operator

$$T(u; \lambda) \equiv Lu(x) - \lambda f(x, u(x))$$

mapping $H_{2,p}^{(B)}(D)x(-\infty, \infty)$ onto $L_p(D)$. (Here $f(x, u)$ is extended to negative arguments in u in anyway convenient.) Q.E.D.

In this self adjoint case, (1.1) is the Euler Equation for the functional

$$\begin{aligned} J(w) \equiv \int_D a_{ij}(x)\frac{\partial w(x)}{\partial x_i}\frac{\partial w(x)}{\partial x_j} + c(x)w^2(x) \\ - \lambda\left(\int_0^{w(x)} f(x, \xi)\, d\xi\right) dx + \oint_{\partial D} b_0(x)w^2(x)\, dx. \end{aligned} \tag{4.8}$$

Regarding $J(w)$ as the potential energy of a system which has $u(x; \lambda)$ as an equilibrium position, a common 'rule of thumb' for the stability of the equilibrium position $u(x; \lambda)$ is that $\lambda < \mu(u(x; \lambda))$, *i.e.* the linearized eigenvalue represents the instability threshold for λ. We show that this criterion, which is satisfied according to Lemma 4.1, implies that $u(x; \lambda)$ is a local minimum for $J(w(x))$. Other aspects of the stability of $u(x; \lambda)$ are discussed in [3].

Theorem 4.2. *For* $\|u(x; \lambda) - v(x)\|_\infty \neq 0$ *and sufficiently small,*

$$J(v(x)) > J(u(x; \lambda)). \tag{4.9}$$

Proof. We can expand $J(v(x))$ about $u(x; \lambda)$ using Taylor's Formula ([9], p. 186)

$$\begin{aligned} J(v(x)) = J(u(x; \lambda)) &+ J'(u)\cdot(u - v) \\ &+ \int_0^1 (1 - s)J''(u + s(v - u))\cdot(u - v, u - v)\, ds. \end{aligned} \tag{4.10}$$

However, since $u(x; \lambda)$ satisfies the Euler Equation for $J(u)$, $J'(u) = \theta$, the zero mapping. Writing the last term of (4.10) explicitly, we have

$$\begin{aligned} J(v) = J(u) + \int_0^1 (1 - s) \int_D \Big[& a_{ij}(x) \frac{\partial(u - v)}{\partial x_i} \frac{\partial(u - v)}{\partial x_j} \\ & + c(x)(u - v)^2 - \lambda f_u(x, u + s(u - v))(u - v)^2 \Big]\, dx\, ds. \end{aligned} \tag{4.11}$$

For $\|u(x; \lambda) - v(x)\|_\infty$ sufficiently small $\lambda \leqq \mu(u(x; \lambda) + s[u(x; \lambda) - v(x)])$ for $s \in [0, 1]$, by Lemma 4.1. Hence by the Rayleigh Quotient Inequality for $\mu(u(x; \lambda) + s[u(x; \lambda) - v(x)])$, the integrand in the last term of (4.11) is positive for all $s \in [0, 1)$, which yields (4.6). Q.E.D.

For the forced problem we can give an interesting alternate variational characterization of the position solution. For $\lambda \in (0, \mu(\infty))$, let $w(x; \lambda)$ be the solution of (1.1) for a forced case, and let $u(x; \lambda)$ be the solution as defined above for the corresponding free case, *i.e.*

$$\begin{aligned} Lu(x; \lambda) &= \lambda[f(x, u(x; \lambda)) - f(x, 0)], && x \in D, \\ Bu(x; \lambda) &= 0, && x \in \partial D. \end{aligned} \tag{4.12}$$

For a non-negative function $v(x) \in C^\alpha(D)$ we can define $h(x)$ as the solution of the linear problem

$$\begin{aligned} Lh(x) - \lambda f_u(x, v(x))h(x) &= \lambda[f(x, v(x)) - v(x)f_u(x, v(x))], && x \in D, \\ Bh(x) &= 0, && x \in \partial D. \end{aligned} \tag{4.13}$$

To indicate its dependence on the function $v(x)$, we write $h(x)$ as $h(x; v)$.

Theorem 4.3. *Let* $f(x, u)$ *satisfy* (1.5), (*H*) *and* (*H'*). *In addition, let*

$$f_u(x, u_1) \leqq f_u(x, u_2) \quad \text{if} \quad u_1 > u_2 > 0. \tag{4.14}$$

Then,

$$w(x, \lambda) = \min_{v(x) \geqq u(x;\lambda)} [h(x, v)]. \tag{4.15}$$

Remark 1. For $\lambda \in (0, \mu(0))$, the representation (4.15) reduces to

$$w(x; \lambda) = \min_{v(x) \geqq 0} [h(x, v)]. \tag{4.16}$$

Remark 2. The concavity condition (4.14) implies that

$$f(x, u) = \min_{v \geqq 0} [f(x, v) + f_u(x, v)(u - v)]. \tag{4.17}$$

Hence, Theorem 4.3 gives a sense in which the minimizing operation of (4.17) can be interchanged with the solution operator for the linear problem (4.13), *i.e.* a form of quasilinearization, [10].

Proof. For $v(x) \geqq u(x; \lambda)$, $v(x) \in C^\alpha(D)$, we have

$$Lw(x; \lambda) = \lambda[f(x, v(x)) + (w(x; \lambda) - v(x))f_u(x, v(x))] - \lambda p(x), \qquad x \in D,$$

where

$$p(x) = f(x, v(x)) + (w(x; \lambda) - v(x))f_u(x, v(x)) - f(x, w(x; \lambda)) \geqq 0,$$

for $x \in D$, in consequence of (4.17). Then

$$L(h(x, v) - w(x; \lambda)) - \lambda f_u(x, v(x))(h(x, v) - w(x; \lambda)) = \lambda p(x) \geqq 0, \qquad x \in D,$$

$$B(h(x, v) - w(x; \lambda) = 0, \qquad x \in \partial D.$$

However, from Lemma 4.1 and (4.14), we have $\lambda < \mu(u(x; \lambda)) \leqq \mu(v(x))$. Hence, by the Positivity Lemma of [3], we conclude that $h(x; v) \geqq w(x; \lambda)$. Since $w(x; \lambda) \geqq u(x; \lambda)$, we can set $v(x) = w(x; \lambda)$, in which case, $h(x, v) \equiv w(x; \lambda)$, which proves (4.15). Q.E.D.

It would be convenient if the class of admissible functions in (4.15) could be extended to all non-negative, smooth functions on D as in (4.16) for the whole range of λ. The following argument shows that, in general, this is not possible. Consider a case in which λ_1 and λ may be chosen to satisfy $\mu(0) < \lambda_1 < \mu(u(x; \lambda_1)) < \lambda$; *e.g.*, $\mu(\infty) = \infty$. Then, taking $v(x) = u(x; \lambda_1)$, we have

$$L(h(x; v) - u(x; \lambda_1)) - \lambda f_u(x, u(x; \lambda_1))(h(x; v) - u(x; \lambda_1)) = (\lambda/\lambda_1 - 1)f(x, u(x; \lambda_1)).$$

Then $h(x; v) - u(x; \lambda_1) \geqq 0$ for $x \in D$ would imply that $\lambda < \mu(u(x; \lambda_1))$ by the Positivity Lemma of [3], hence a contradiction. Thus, there is $\xi \in D$ for which $u(\xi; \lambda) \geqq u(x; \lambda_1) > h(\xi, v)$, and $v(x) = u(x; \lambda_1)$ cannot be an admissible function for the variational representation (4.15).

The successive approximation procedure of Section 2 provides a convergent sequence of lower bounds for $u(x; \lambda)$. With the help of Lemma 4.1, we can indicate how Newton's Method can be used to provide a convergent sequence of upper bounds for $u(x; \lambda)$. Newton's Method for approximating $u(x; \lambda)$ is to calculate successively the sequence $\{u_n(x)\}$ as solutions of the linear problems

$$(4.18)\qquad \begin{aligned} Lu_n(x) - \lambda f_u(x, u_{n-1}(x))u_n(x) &\\ = \lambda[f(x, u_{n-1}(x)) - f_u(x, u_{n-1}(x))u_{n-1}(x)], &\qquad x \in D, \\ Bu_n(x) = 0, \qquad x \in \partial D, \quad N = 1, 2, \cdots, & \end{aligned}$$

for arbitrary $u_0(x) \in C^{2+\alpha}(D)$, $u_0(x) \geqq 0$, $x \in D$.

In [4], a suitable starting value, $u_0(x)$, for Newton's Method is constructively defined and the monotone decreasing convergence of the sequence is proven for the forced case, for $\lambda \in (0, \mu(0))$.

Theorem 4.4. *For $f(x, u)$ satisfying* (4.2) *and* (4.14), *for $\lambda \in (0, \mu(\infty))$, let $\{u_n(x)\}$ be the sequence of approximations calculated by* (4.18). *If, for all $x \in D$,*

$$(4.19)\qquad u_0(x) \geqq u_1(x) \quad \text{and} \quad u_0(x) \geqq u(x; \lambda),$$

then $\{u_n(x)\}$ converges monotonically nonincreasing, and uniformly to $u(x; \lambda)$. (It, in fact, converges in $H^{(B)}_{2,p}(D)$ for $1 \leqq p < \infty$. See proof of Theorem 3.2 for $H^{(B)}_{2,p}(D)$.)

Proof. Hypotheses (4.19) are the first steps in two induction proofs. First we show that if $u_{n-1}(x) \geqq u(x; \lambda)$ then $u_n(x) \geqq u(x; \lambda)$.

$$(4.20)\qquad \begin{aligned} &L(u_n(x) - u(x; \lambda)) - \lambda f_u(x, u_{n-1}(x))(u_n(x) - u(x; \lambda)) \\ &\quad = \lambda\{f(x, u_{n-1}(x)) - f(x, u(x; \lambda)) - f_u(x, u_{n-1}(x))(u_{n-1}(x) - u(x; \lambda))\} \\ &\quad = \lambda\{(f_u(x, \xi(x)) - f_u(x, u_{n-1}(x))(u_{n-1}(x) - u(x; \lambda))\}, \end{aligned}$$

where, by the Mean Value Theorem, $\xi(x) \leqq u_{n-1}(x)$. The induction hypothesis provides, with (4.14), that the right side of (4.20) is now negative, and that $\lambda < \mu(u(x; \lambda)) \leqq \mu(u_{n-1}(x))$. Consequently $u_n(x) \geqq u(x; \lambda)$.

We now deduce that if $u_n(x) \leqq u_{n-1}(x)$, then $u_{n+1}(x) \leqq u_n(x)$. By an argument similar to the preceding one, we have for some $\zeta(x)$ satisfying

$$u_n(x) \leqq \zeta(x) \leqq u_{n-1}(x),$$

$$(4.21)\qquad \begin{aligned} L(u_{n+1}(x) - u_n(x)) - \lambda f_u(x, u_n(x))(u_{n+1}(x) - u_n(x)) &\\ = \lambda\{[f_u(x, \zeta(x) - f_u(x, u_{n-1}(x))](u_n(x) - u_{n-1}(x))\}, &\qquad x \in D, \\ B(u_{n+1}(x) - u_n(x)) = 0, &\qquad x \in \partial D. \end{aligned}$$

As above, using the fact that $\mu(u_n(x)) \geqq \mu(u(x; \lambda))$ for all n, we conclude that $u_{n+1}(x) - u_n(x) \geqq 0$ for $x \in D$. Since $\{u_n(x)\}$ is monotone non-increasing and is bounded below by $u(x; \lambda)$, the sequence converges in $L_p(D)$ to a non-negative limit, $\tilde{u}(x)$, in $L_p(D)$ for $1 < p < \infty$. We can rewrite (4.18) as

$$(4.22)\qquad u_{n+1}(x) = \lambda L^{-1}\{f_u(x, u_n(x))(u_{n+1}(x) - u_n(x)) + f(x, u_n(x))\},$$

where L^{-1} is continuous from $L_p(D)$ to $H_{2,p}^{(B)}(D)$. As $n \to \infty$, the argument of L^{-1} converges to $f(x, \tilde{u}(x))$ in $L_p(D)$; hence the left side of (4.24) converges to $\tilde{u}(x)$ in $H_{2,p}^{(B)}(D)$. However, by the uniqueness in $L_p(D)$ of the positive solution of $u(x, \lambda) = \lambda L^{-1} f(x, u(x; \lambda))$ ([5], Theorem 6.3), we conclude that $\tilde{u}(x) = u(x; \lambda)$. Q.E.D.

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