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Numerical solution of fuzzy equations with Z-numbers using neural networks

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In this paper, the uncertainty property is represented by the Z-number as the coefficients of the fuzzy equation. This modification for the fuzzy equation is suitable for nonlinear system modeling with uncertain parameters. We also extend the fuzzy equation into dual type, which is natural for linear-in-parameter nonlinear systems. The solutions of these fuzzy equations are the controllers when the desired references are regarded as the outputs. The existence conditions of the solutions (controllability) are proposed. Two types of neural networks are implemented to approximate solutions of the fuzzy equations with Z-number coefficients. Type or paste your abstract here.

Keywords: fuzzy equation; Z-number; fuzzy control

Introduction

Uncertainties are inevitable in real systems. Control of uncertain system is classified in two methodologies: direct and indirect techniques (Feng, 2006). The methodology involves the direct control incorporates uncertain system as a controlling mechanism, whereas the indirect uncertain model is used to approximate the nonlinear system as a first step, then proceeds controller design based on uncertain model. The indirect fuzzy controller works on the principle of generalized topological structure as well as universal approximation capacity associated to fuzzy model. It has been utilized primarily, considering the case of uncertain nonlinear system control. This paper utilizes the indirect control method.

Since the uncertainty in parameters can be transformed into fuzzy set theory (Zadeh, 2005), fuzzy set and fuzzy system theory are good tools to deal with uncertain systems. Fuzzy models are applied for a large class of uncertain nonlinear systems. Fuzzy method is a highly favorable tool for uncertain nonlinear system modeling. The fuzzy models approximate uncertain nonlinear systems with several linear piecewise systems

(Takagi-Sugeno method) (Takagi & Sugeno, 1985). Mamdani models use fuzzy rules to achieve a good level of approximation of uncertainties (Mamdani, 1976). In recent days, many methods involving uncertainties have used fuzzy numbers (Buckley & Qu, 1990) (Jafari & Yu, 2015) (Jafarian & Jafari, 2012) (Jafarian, Jafari, Mohamed Al Qurashi, & Baleanu, 2016), where the uncertainties of the system are represented by fuzzy coefficients.

The application of the fuzzy equations is in direct connection with the nonlinear control. Given a fuzzy equation, the control incorporated in the equation is in fact a solution of the equation. There are number of techniques to study the solutions of fuzzy equations. (Friedman, Ming, & Kandel, 1998) used the fuzzy number on parametric shapes and replaced the original fuzzy equations with crisp linear systems. A survey on the extension principle is proposed by (Buckley & Qu, 1990) and it suggests that the coefficients can be either real or complex fuzzy numbers. Nevertheless, there will be no guarantee that the solution exists. (Abbasbandy, 2006) proposed the homeotypic analysis technique. (Abbasbandy & Ezzati, 2006) used the Newton methodology. In (Allahviranloo, Otadi, & Mosleh, 2007) the solution associated to the fuzzy equations is studied by the fixed point technique. One of the most popular methods is the α -level (Goetschel & Voxman, 1986). By applying the technique of overlay of sets, fuzzy numbers can be resolved (Mazandarani & Kamyad, 2013). The fuzzy fractional differential and integral equations have been investigated extensively in (Agarwal, Lakshmikantham, & Nieto, 2010) (Arshad & Lupulescu, 2011) (Salahshour, Allahviranloo, & Abbasbandy, 2012) (Wang & Liu, 2011). In (Khastan, Nieto, & Rodriguez-lopez, 2013), the first-order fuzzy differential equation with periodic boundary conditions is analyzed. Then, higher order linear fuzzy differential equations is studied. In (Allahviranloo, Kiani, & Barkhordari, 2009), the analytical solutions of the second-order fuzzy differential equation is obtained. The analytical solutions of third-order linear fuzzy differential equations are found in (Hawrra & Amal, 2013), while (Buckley & Feuring, 2001) proposed analytical approach to resolve nth-order linear fuzzy differential equations. Nevertheless, the analytical solutions of fuzzy equations are difficult to obtain and the aforementioned techniques involve greater complexity.

The numerical solution associated to the fuzzy equation and the fuzzy differential equations (Lupulescu, 2009)] can be extracted by iterative technique (Kajani, Asady, & Vencheh, 2005), interpolation technique (Waziri & Majid, 2012) and the Runge-Kutta technique (Pederson & Sambandham, 2008). However, the implementation of these techniques are difficult. Both neural networks as well as fuzzy logic are considered to be the universal estimators which can estimate any nonlinear function to any notified precision (Cybenko, 1989). Recent results show that the fusion of the neural networks and the fuzzy logic gives remarkable success in nonlinear system modeling (Yu & Li, 2004). The neural networks may also be used to solve fuzzy equations. (Buckley & Eslami, 1997) used a neural network with three neurons to estimate the second degree fuzzy equation. (Jafarian, Jafari, Khalili, & Baleanu, 2015) and (Jafarian & Measoomynia, 2011) extended the result of (Buckley & Eslami, 1997) to fuzzy polynomial equations. In (Jafarian & Jafari, 2012), the solution of dual fuzzy equation is obtained by neural networks. (Mosleh, 2013) gave a matrix form of the neuronal learning. By extending classical fuzzy set theory, (Hullermeier, 1997) obtained a numerical solution for an fuzzy differential equation. The predictor-corrector approach is applied in (Allahviranloo, Ahmadi, & Ahmadi, 2007). The Euler numerical technique is used in (Tapaswini & Chakraverty, 2014) to solve fuzzy differential equations. Whatsoever, these techniques are not general, they cannot give the fuzzy coefficients directly with neural networks (Tahavvor & Yaghoubi, 2012).

The decisions are carried out based on knowledge. In order to make the decision fruitful, the knowledge acquired must be credible. Z-numbers connect to the reliability of knowledge (Zadeh, 2006). Many fields related to the analysis of the decisions use the ideas of Z-numbers. Z-numbers are much less complex to calculate when compared to nonlinear system modeling methods. The Z-number is abundantly adequate number than the fuzzy number. Although Z-numbers are implemented in many literatures, from theoretical point of view this approach is not certified completely. There are few structure based on the theoretical concept of Z-numbers (Gardashova, 2014). (Aliev, Alizadeh, & Huseynov, 2015) gave an inception which results in the extension of the Z-numbers. (Kang, Wei, Li, & Deng, 2012) proposed a theorem to transfer the Z-numbers to the usual fuzzy sets. In (Zadeh, 2006) a novel approach was followed for the conversion of Z-number into age old fuzzy number.

Normal fuzzy equations contain fuzzy numbers just on one side of the equation. Nevertheless, dual fuzzy equations contain fuzzy numbers on both sides of the equation. Whereas the fuzzy numbers are not able to move between the sides of the equation (Kajani, Asady, & Venchech, 2005), dual fuzzy equations can be considered to be more general and complicated.

In this paper, we use dual fuzzy equations (Waziri & Majid, 2012) to model the uncertain nonlinear systems, where the coefficients are Z-numbers and the Z-numbers are on both sides of the equation. The Z-number is a novel idea that is subjected to a higher potential in order to illustrate the information of the human being as well as to use in information processing (Zadeh, 2006). Z-numbers can be regarded as to answer questions and carry out the decisions (Kang, Wei, Li, & Deng, 2012).

This paper is one of the first attempts in finding the solution of dual fuzzy equations based on Z-numbers. We first discuss the existence of the solutions of the dual fuzzy equations. It corresponds to controllability problem of the fuzzy control (Chen, 1994). After that, we use two types of neural networks, feed-forward and feedback networks, to approximate the solutions (control actions) of the dual fuzzy equation. At the end several examples are utilized in order to demonstrate the affectivity of our fuzzy control design methods.

Nonlinear system modeling with dual fuzzy equations and Z-numbers

In order to utilize dual fuzzy equations and Z-numbers, we first introduce some concepts of discrete-time nonlinear system and Z-numbers.

A general discrete-time nonlinear system can be described as

$$\bar{x}_{k+1} = \bar{f}\left[\bar{x}_k, u_k\right], \quad y_k = \bar{g}\left[\bar{x}_k\right] \tag{1}$$

Here we consider $u_k \in \Re^u$ as the input vector, $\bar{x}_k \in \Re^l$ is regarded as an internal state vector and $y_k \in \Re^m$ is the output vector. $\bar{f}$ and $\bar{g}$ are noted as generalized nonlinear smooth functions $\bar{f}, \bar{g} \in C^\infty$. Define $Y_k = \left[y_{k+1}^T, y_k^T, \cdots\right]^T$ and $U_k = \left[u_{k+1}^T, u_k^T, \cdots\right]^T$. Suppose $\frac{\partial Y}{\partial \bar{x}}$ is non-singular at the instance $\bar{x}_k = 0,\ U_k = 0,$ this will create a path towards the following model

$$y_k = \Psi[y_{k-1}^T, y_{k-2}^T, \cdots u_k^T, u_{k-1}^T, \cdots] \tag{2}$$

where $\Psi(\cdot)$ is an nonlinear difference equation exhibiting the plant dynamics, u_k and y_k are computable scalar input and output respectively, d is noted to be time delay.

The nonlinear system which is represented by (2) is implied as a NARMA model. The input of the system with incorporated nonlinearity is considered to be as

$x_k = [y_{k-1}^T, y_{k-2}^T, \cdots u_k^T, u_{k-1}^T, \cdots]^T$

Taking into consideration the nonlinear systems as mentioned in (2), it can be simplified as the following linear-in-parameter model

$$z_k = \sum_{i=1}^{n} a_i f_i(x_k) \quad (3)$$

or

$$z_k + \sum_{i=1}^{m} b_i g_i(x_k) = \sum_{i=1}^{n} a_i f_i(x_k) \quad (4)$$

here a_i and b_i are considered to be the linear parameters, $f_i(x_k)$ and $g_i(x_k)$ are nonlinear functions. The variables related to these functions are quantifying input and output. A popular example of this pattern of model is considered to be a robot manipulator (Spong & Vidyasagar, 1989)

$$M(p)\ddot{p} + C(p, \dot{p})\dot{p} + B\dot{p} + g(p) = \tau \quad (5)$$

(5) can be explained as

$$\sum_{i=1}^{n} Y_i(p, \dot{p}, \ddot{p})\theta_i = \tau \quad (6)$$

The modeling of uncertain nonlinear systems can be achieved by utilizing the linear-in-parameter models linked to fuzzy parameters. We assume the model of the nonlinear systems (3) and (4) have uncertainties in the parameters a_i and b_i . These uncertainties are in the sense of Z-numbers (Zadeh, 2011).

Definition 1: A fuzzy number A is a function $A \in E : \Re \to [0,1]$, in such a way, 1) A is normal, (there prevail $x_0 \in \Re$ in such a way $A(x_0) = 1$; 2) A is convex, $A(\lambda x + (1-\lambda) y) \geq \min \{A(x), A(y)\}$, $\forall x, y \in \Re, \forall \lambda \in [0,1]$; 3) A is upper semicontinuous on $\Re$, i.e., $A(x) \le A(x_0) + \varepsilon$, $\forall x \in N(x_0)$, $\forall x_0 \in \Re$, $\forall \varepsilon > 0$, $N(x_0)$ is a neighborhood; 4) The set $A^+ = \{x \in \Re, A(x) > 0\}$ is compact.

Definition 2: A Z-number has two components $Z = [A(x), p]$. The primary component $A(x)$ is termed as a restriction on a real-valued uncertain variable x. The secondary component p is a measure of reliability of A. p can be reliability, strength of belief, probability or possibility. When $A(x)$ is a fuzzy number and p is the probability distribution of x , the Z-number is defined as Z^+-number. When both $A(x)$ and p are fuzzy numbers, the Z-number is defined as Z^--number.

The Z^+-number carries more information than the Z^--number. In this paper, we use the definition of Z^+-number, i.e., $Z = [A, p]$, A is a fuzzy number, p is a probability distribution.

In order to demonstrate the fuzzy numbers, the membership functions are utilized. The most widely discussed membership functions are noted to be the triangular function

$$\mu_A = F(a,b,c) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ \frac{c-x}{c-b} & b \le x \le c \end{cases} \text{otherwise } \mu_A = 0 \tag{7}$$

as well as trapezoidal function

$$\mu_A = F(a,b,c,d) = \begin{cases} \frac{x-a}{b-a} & a \le x \le b \\ \frac{d-x}{d-c} & c \le x \le d \\ 1 & b \le x \le c \end{cases} \text{otherwise } \mu_A = 0 \tag{8}$$

The probability measure is expressed as

$$P = \int_R \mu_A(x)\, p(x) dx \tag{9}$$

where p is the probability density of x and R is the restriction on p. For discrete Z-numbers, we have

$$P(A) = \sum_{i=1}^{n} \mu_A(x_i)\, p(x_i) \quad (10)$$

The space of discrete fuzzy sets is denoted by $\tilde{E}$. $\tilde{E}_{[a,b]}$ denotes the space of discrete fuzzy sets of $[a,b] \subset R$. Signifying $\hat{Z}$ the space of discrete Z-numbers as

$$\hat{Z} = \{Z = (A, p) \mid A \in \tilde{E},\ p \in \tilde{E}_{[0,1]}\} \quad (11)$$

Definition 3: The α -level associated to a fuzzy number A is stated as

$$[A]^{\alpha} = \{x \in \Re :\ A(x) \geq \alpha\} \quad (12)$$

also, $0 < \alpha \leq 1$. Or

$$[A]^{\alpha} = \left(\underline{A}^{\alpha}, \overline{A}^{\alpha}\right)$$

In order to operate the Z-number, we propose the following definition.

Definition 4: The α -level of the Z-number $Z = (A, p)$ is demonstrated as

$$[Z]^{\alpha} = \left([A]^{\alpha}, [p]^{\alpha}\right) \quad (13)$$

where $0 < \alpha \leq 1$. $[p]^{\alpha}$ is calculated by the Nguyen's theorem

$$[p]^{\alpha} = p([A]^{\alpha}) = p([\underline{A}^{\alpha}, \overline{A}^{\alpha}]) = \left[\underline{P}^{\alpha}, \overline{P}^{\alpha}\right]$$

where $p([A]^{\alpha}) = \{p(x) \mid x \in [A]^{\alpha}\}$. So $[Z]^{\alpha}$ can be expressed as the form α -level of a fuzzy number

$$[Z]^{\alpha} = \left(\underline{Z}^{\alpha}, \overline{Z}^{\alpha}\right) = \left(\left(\underline{A}^{\alpha}, \underline{P}^{\alpha}\right), \left(\overline{A}^{\alpha}, \overline{P}^{\alpha}\right)\right) \quad (14)$$

where $\underline{P}^{\alpha} = \underline{A}^{\alpha} p(\underline{x_i}^{\alpha})$, $\overline{P}^{\alpha} = \overline{A}^{\alpha} p(\overline{x_i}^{\alpha})$, $[x_i]^{\alpha} = (\underline{x_i}^{\alpha}, \overline{x_i}^{\alpha})$.

Similarly with the fuzzy numbers (Jafari & Yu, 2015), the Z-numbers are also incorporated with four primary operations: $\oplus,\ \ominus,\ \odot\ and\ \oslash$. These operations are exhibited by: sum, subtract, multiply and division. The operations in this paper are different from that mentioned in (Zadeh, 2011). The α -level of Z-numbers is applied to simplify the operations.

Let us consider $Z_1 = (A_1, p_1)$ and $Z_2 = (A_2, p_2)$ be two discrete Z-numbers illustrating the uncertain variables x_1 and x_2, also $\sum_{k=1}^{n} p_1(x_{1k}) = 1$ and $\sum_{k=1}^{n} p_2(x_{2k}) = 1$.

The operations are defined as

$Z_{12} = Z_1 * Z_2 = (A_1 * A_2, p_1 * p_2)$

where $* \in \{\oplus, \ominus, \odot, \oslash\}$.

The operations for the fuzzy numbers are defined as (Jafari & Yu, 2015)

$$[A_1 \oplus A_2]^\alpha = \left[\underline{A}_1^\alpha + \underline{A}_2^\alpha, \overline{A}_1^\alpha + \overline{A}_2^\alpha\right] \tag{15}$$
$$[A_1 \ominus A_2]^\alpha = [\underline{A}_1^\alpha - \underline{A}_2^\alpha, \overline{A}_1^\alpha - \overline{A}_2^\alpha]$$
$$[A_1 \odot A_2]^\alpha = [\underline{A}_1^\alpha \underline{A}_2^\alpha + \underline{A}_1^\alpha \underline{A}_2^\alpha - \underline{A}_1^\alpha \underline{A}_2^\alpha, \overline{A}_1^\alpha \overline{A}_2^\alpha + \overline{A}_1^\alpha \overline{A}_2^\alpha - \overline{A}_1^\alpha \overline{A}_2^\alpha]$$

For all $p_1 * p_2$ operations, we use convolutions for the discrete probability distributions

$$p_1 * p_2 = \sum_i p_1(x_{1,i}) p_2(x_{2,(n-i)}) = p_{12}(x)$$

The above definitions satisfy the Hukuhara difference (Alieva, Pedryczb, Kreinovich, & Huseynov, 2016)

$$Z_1 \ominus_H Z_2 = Z_{12}$$

$$Z_1 = Z_2 \oplus Z_{12}$$

Here if $Z_1 \ominus_H Z_2$ prevail, the α -level is

$$[Z_1 \ominus_H Z_2]^\propto = [\underline{Z}_1^\alpha - \underline{Z}_2^\alpha, \overline{Z}_1^\propto - \overline{Z}_2^\propto]$$

Obviously, $Z_1 \ominus_H Z_1 = 0$, $Z_1 \ominus Z_1 \neq 0$.

If A is a triangle function, the absolute value of the Z-number $Z = (A, p)$ is

$$|Z(x)| = (|a_1| + |b_1| + |c_1|, p(|a_2| + |b_2| + |c_2|)) \tag{16}$$

Now we utilize fuzzy equations (3) or (4) to model the uncertain nonlinear system (2). The parameters of the fuzzy equations (3) or (4) are in the form of Z-numbers

$$y_k = a_1 \odot f_1(x_k) \oplus a_2 \odot f_2(x_k) \oplus \ldots \oplus a_n \odot f_n(x_k) \tag{17}$$

or

$$a_1 \odot f_1(x_k) \oplus a_2 \odot f_2(x_k) \oplus \ldots \oplus a_n \odot f_n(x_k) =$$

$$b_1 \odot g_1(x_k) \oplus b_2 \odot g_2(x_k) \oplus \ldots \oplus b_m \odot g_m(x_k) \oplus y_k \tag{18}$$

where a_i and b_i are Z-numbers. (18) is considered to be more general as compared to (17), it is termed as dual fuzzy equation.

Taking into consideration a particular case, $f_i(x_k)$ has polynomial pattern,

$$(a_1 \odot x_k) \oplus \ldots \oplus (a_n \odot x_k^n) = (b_1 \odot x_k) \oplus \ldots \oplus (b_n \odot x_k^n) \oplus y_k \tag{19}$$

(19) is termed as dual polynomial based on Z-number.

The main intention associated with the modeling is to diminish error in midst of two output y_k and z_k. As y_k is noted as a Z-number and z_k is considered to be crisp Z-number, hence we apply the minimum of every points as the model mentioned below

$$\begin{aligned} &\max_k |y_k - z_k| = \max_k |\beta_k| \\ &y_k = ((u_1(k), u_2(k), u_3(k)), p(v_1(k), v_2(k), v_3(k))) \\ &\beta_k = ((\rho_1(k), \rho_2(k), \rho_3(k)), p(\varphi_1(k), \varphi_2(k), \varphi_3(k))) \end{aligned} \tag{20}$$

By the definition of absolute value (abs), we conclude

$$\begin{aligned} &\max_k |\beta_k| = \max_k [(| u_1(k) - f(x_k) | + | u_2(k) - f(x_k) | \\ &+ | u_3(k) - f(x_k) |), (| p(v_1(k)) - f(x_k) | + | p(v_2(k)) - f(x_k) | + | p(v_3(k)) - f(x_k) |)] \\ &\rho_1(k) = \max_k | u_1(k) - f(x_k) |, \quad \rho_2(k) = \max_k | u_2(k) - f(x_k) |, \\ &\rho_3(k) = \max_k | u_3(k) - f(x_k) | \\ &p(\varphi_1(k)) = \max_k | p(v_1(k)) - f(x_k) |, \quad p(\varphi_2(k)) = \max_k | p(v_2(k)) - f(x_k) |, \\ &p(\varphi_3(k)) = \max_k | p(v_3(k)) - f(x_k) | \end{aligned} \tag{21}$$

The modelling constraint (20) is to uncover $u_1(k)$, $u_2(k)$, $u_3(k)$, $p(v_1(k))$, $p(v_2(k))$ and $p(v_3(k))$ in such a manner

$$\min_{u_i(k),\, p(v_i(k))} \left\{ \max_k |\beta_k| \right\} = \min_{u_i(k),\, p(v_i(k))} \left\{ \max_k | y_k - f(x_k) | \right\}, \quad i = 1, 2, 3 \tag{22}$$

Considering (21), we have

$$\rho_1(k) \geq |u_1(k) - f(x_k)|, \quad \rho_2(k) \geq |u_2(k) - f(x_k)|, \quad \rho_3(k) \geq |u_3(k) - f(x_k)|$$
$$p(\varphi_1(k)) \geq |p(v_1(k)) - f(x_k)|, \quad p(\varphi_2(k)) \geq |p(v_2(k)) - f(x_k)|$$
$$p(\varphi_3(k)) \geq |p(v_3(k)) - f(x_k)|$$

(22) can be resolved by the application of linear programming methodology

$$\begin{cases} \min \rho_1(k) \\ subject: \begin{array}{l} \rho_1(k) + \{\sum_{j=0}^{n} a_j \odot x_k^j\} \ominus_H (\sum_{j=0}^{n} b_j \odot x_k^j) \geq f(x_k) \\ \rho_1(k) - \{\sum_{j=0}^{n} a_j \odot x_k^j\} \ominus_H \left(\sum_{j=0}^{n} b_j \odot x_k^j\right) \geq -f(x_k) \end{array} \\ \min \varphi_1(k) \\ subject: \begin{array}{l} p(\varphi_1(k)) + [\sum_{j=0}^{n} a_j \odot x_k^j\} \ominus_H (\sum_{j=0}^{n} b_j \odot x_k^j) \geq f(x_k) \\ p(\varphi_1(k)) - \{\sum_{j=0}^{n} a_j \odot x_k^j\} \ominus_H \left(\sum_{j=0}^{n} b_j \odot x_k^j\right) \geq -f(x_k) \end{array} \end{cases} \tag{23}$$

$$\begin{cases} \min \rho_2(k) \\ \text{subject}: \begin{array}{l} \rho_2(k) - \left[\sum_{j=0}^{n} \underline{a}_j \underline{x}_k^j - \sum_{j=0}^{n} \underline{b}_j \underline{x}_k^j\right] \geq f(x_k) \\ \rho_2(k) \geq 0 \end{array} \\ \min \varphi_2(k) \\ \text{subject}: \begin{array}{l} p(\varphi_2(k)) - \left[\sum_{j=0}^{n} \underline{a}_j \underline{x}_k^j - \sum_{j=0}^{n} \underline{b}_j \underline{x}_k^j\right] \geq f(x_k) \\ p(\varphi_2(k)) \geq 0 \end{array} \end{cases} \tag{24}$$

$$\begin{cases} \min \rho_3(k) \\ \text{subject}: \begin{array}{l} \rho_3(k) - \left[\sum_{j=0}^{n} \bar{a}_j \bar{x}_k^j - \sum_{j=0}^{n} \bar{b}_j \bar{x}_k^j\right] \geq f(x_k) \\ \rho_3(k) \geq 0 \end{array} \\ \min \varphi_3(k) \\ \text{subject}: \begin{array}{l} p(\varphi_3(k)) - \left[\sum_{j=0}^{n} \bar{a}_j \bar{x}_k^j - \sum_{j=0}^{n} \bar{b}_j \bar{x}_k^j\right] \geq f(x_k) \\ p(\varphi_3(k)) \geq 0 \end{array} \end{cases} \tag{25}$$

here $\underline{a}_j$, $\underline{b}_j$, $\underline{x}_k$, $\bar{a}_j$, $\bar{b}_j$ and $\bar{x}_k$ are explained as mentioned in (13). Henceforth, the superior way of approximating $f(x_k)$ at the juncture x_k is y_k. The minimization of the approximation error which is termed as β_k is achieved.

The process involved in order to design the controller is to obtain u_k, in such a manner that the output related to the plant y_k can approach to the desired output y_k^*, or the trajectory tracking error diminishes

$$\min_{u_k} \left\| y_k - y_k^* \right\| \tag{26}$$

This control entity can be regarded as to detect a solution u_k for the following dual equation on the basis of Z-number

$$\big(a_1 \odot f_1(x_k)\big) \oplus \big(a_2 \odot f_2(x_k)\big) \oplus \ldots \oplus \big(a_n \odot f_n(x_k)\big) =$$

$$(b_1 \odot g_1(x_k)) \oplus (b_2 \odot g_2(x_k)) \oplus \ldots \oplus (b_m \odot g_m(x_k)) \oplus y_k^* \tag{27}$$

where $x_k = [y_{k-1}^T, y_{k-2}^T, \cdots u_k^T, u_{k-1}^T, \cdots]^T$.

Controllability of uncertain nonlinear systems via dual fuzzy equations and Z-numbers

As the primary concern of control is finding out u_k as mentioned in (18) which is relied on Z-number, the controllability constraint signifies that the dual fuzzy equation (18) involves solution.

We need the following lemmas for displaying the solution of (18)

Lemma 1: If the coefficients of the dual equation (18) are Z-numbers, then the solution u_k satisfies

$$\left\{\cap_{j=1}^{n} \text{domain}\left[f_j(x)\right]\right\} \cap \left\{\cap_{j=1}^{m} \text{domain}\left[g_j(x)\right]\right\} \neq \phi \tag{28}$$

Proof. Assume $u_0 \in \hat{Z}$ is considered to be a solution of (18), the dual equation which relies on Z-numbers turns out to be

$$\big(a_1 \odot f_1(u_0)\big) \oplus \ldots \oplus \big(a_n \odot f_n(u_0)\big) = (b_1 \odot g_1(u_0)) \oplus \ldots \oplus (b_m \odot g_m(u_0)) \oplus y_k^*$$

As $f_j(u_0)$ and $g_j(u_0)$ prevails, $u_0 \in$ domain $\left\lfloor f_j(x) \right\rfloor$, $u_0 \in$ domain $\left\lfloor g_j(x) \right\rfloor$.

Subsequently, it can be inferred that $u_0 \in \cap_{j=1}^{n}$ domain $\left\lfloor f_j(x) \right\rfloor = C_1$ and $u_0 \in \cap_{j=1}^{m}$ domain $\left\lfloor g_j(x) \right\rfloor = C_2$. Hence there prevail u_0, in such a manner $u_0 \in C_1 \cap C_2 \neq \phi$. ■

Let two Z-numbers p_0, $q_0 \in \hat{Z}$, $p_0 < q_0$. We define a set $D(x) = \{x \in \hat{Z}, p_0 \leq x \leq q_0\}$ and an operator $W : D \rightarrow D$ as

$$W(p_0) \geq p_0, \quad W(q_0) \leq q_0 \tag{29}$$

in which W is condensing and continuous, also it is bounded as $W(z) < r(z),\ \ z \subset D$ and $r(z) > 0.$ $r(z)$ can be considered as the evaluation of z.

Lemma 2: We define $q_i = W(q_{i-1})$ and $p_i = W(p_{i-1})$, $i = 1, 2, ...$, and the upper and lower bounds of W are $\overline{w}$ and $\underline{w}$, then

$$\overline{w} = \lim_{i \to +\infty} q_i\,,\ \ \underline{w} = \lim_{i \to +\infty} p_i\,, \tag{30}$$

and

$$p_0 \le p_1 \le ... \le p_n \le ... \le q_n \le ... \le q_1 \le q_0. \tag{31}$$

Proof. As long as W is uprising, it is quite obvious from (29) that (31) prevail. In this case we verify that $\{p_i\}$ conjoins to some $\underline{w} \in \hat{Z}$ and $W(\underline{w}) = \underline{w}$. The set $B = \{p_0, p_1, p_2, ...\}$ is enclosed and $B = W(B) \bigcup \{p_0\}$, thus $r(B) = r(W(B))$, here $r(B)$ denotes the quantification of non-compactness of B . It is observed from W that $r(B) = 0$, i.e., B is a proportionally compact set. Thus, there prevail an outflow of $\{p_{n_k}\} \subset \{p_n\}$ in such a manner that $p_{n_k} \to \underline{w}$ for any $\underline{w} \in \hat{Z}$ (take into consideration that $\hat{Z}$ is complete). Distinctly, $p_n \le \underline{w} \le q_n$ $(n = 1, 2, ...)$. As in case $m > n_k$, according to (Aliev, Huseynov, Aliyev, & Alizadeh, 2015) the supremum metrics $D(\underline{w}, p_m) \le D(\underline{w}, p_{n_k})$. Hence, $p_m \to \underline{w}$ as $m \to \infty$. Considering limit $n \to \infty$ on either sides of the equality $p_n = W(p_{n-1})$, we find $\underline{w} = W(\underline{w})$, as a result W is continuous and D is closed.

Similarly, we can conclude that $\{q_n\}$ converges to some $\overline{w} \in \hat{Z}$ and $W(\overline{w}) = \overline{w}$. So we confirm that $\overline{w}$ and $\underline{w}$ are the maximal and minimal fixed point related to W in D respectively. Assume $\tilde{w} \in D$ and $W(\tilde{w}) = \tilde{w}$. As W is in the increasing tend, it is obvious from $p_0 \le \tilde{w} \le q_0$ that $W(p_0) \le W(\tilde{w}) \le W(q_0)$, i.e., $p_1 \le \tilde{w} \le q_1$. Utilizing the similar logic, we obtain $p_2 \le \tilde{w} \le q_2$, and formally, $p_n \le \tilde{w} \le q_n$ $(n = 1, 2, 3, \ldots)$. Here, considering limit $n \to \infty$, we extract $\underline{w} \le \tilde{w} \le \overline{w}$.

The fixed point will result in x_0 inside D, the consecutive iterates $x_i = W(x_{i-1})$, $i = 1, 2, \ldots$ will result in convergency towards x_0, i.e., $\lim_{i\to\infty} D(x_i, x_0) = 0$. ■

Theorem 1: Let us consider $Z = \left(\underline{Z}^{\alpha}, \overline{Z}^{\alpha}\right)$, where $\underline{Z}^{\alpha} = (d_{M_1}(\alpha), d_{M_2}(\alpha))$, $\overline{Z}^{\alpha} = (d_{U_1}(\alpha), d_{U_2}(\alpha))$, $\alpha \in [0,1]$. If a_i and b_j ($i = 1 \cdots n$, $j = 1 \cdots m$) in (18) are Z-numbers and they suffice the Lipschitz condition

$$\begin{aligned} &\left|(d_{M_1}(a_i), d_{M_2}(a_i)) - (d_{M_1}(a_k), d_{M_2}(a_k))\right| \le H\left|a_i(M_1) - a_k(M_1)\right| + H\left|a_i(M_2) - a_k(M_2)\right| \\ &\left|(d_{U_1}(a_i), d_{U_2}(a_i)) - (d_{U_1}(a_k), d_{U_2}(a_k))\right| \le H\left|a_i(U_1) - a_k(U_1)\right| + H\left|a_i(U_2) - a_k(U_2)\right| \end{aligned} \quad (32)$$

also, the upper bounds of the functions f_i and g_j are $|f_i| \le \overline{f}$, $|g_j| \le \overline{g}$, then the dual fuzzy equation (18) has a solution u in the set mentioned below

$$K_H = \left\{u \in \tilde{Z}, \left|\overline{u}^{(\alpha_1,\beta_1)} - \underline{u}^{(\alpha_2,\beta_2)}\right| \le (n\overline{f} \oplus m\overline{g})(H|\alpha_1 - \alpha_2| + H|\beta_1 - \beta_2|)\right\} \quad (33)$$

Proof. Since a_i and b_j are Z-numbers and from (32) we have

$$d_M(\alpha,\beta) = \left(\left(a_{1M_1}(\alpha), a_{1M_2}(\beta)\right) \odot f_1(x)\right) \oplus \ldots \oplus \left(\left(a_{nM_1}(\alpha), a_{nM_2}(\beta)\right) \odot f_n(x)\right) \ominus_H (b_{1M_1}(\alpha), b_{1M_2}(\beta)) \odot g_1(x)) \ominus_H \ldots \ominus_H ((b_{mM_1}(\alpha), b_{mM_2}(\beta)) \odot g_m(x))$$

Hence

$$\begin{aligned} |d_M(\alpha,\beta) - d_M(\varphi,\rho)| &= \left(|f_1(x)| \odot \left|(a_{1M_1}(\alpha), a_{1M_2}(\beta) \ominus_H (a_{1M_1}(\varphi), a_{1M_2}(\rho))\right|\right) \oplus \\ &\ldots \oplus \left(|f_n(x)| \odot \left|(a_{nM_1}(\alpha), a_{nM_2}(\beta) \ominus_H (a_{nM_1}(\varphi), a_{nM_2}(\rho))\right|\right) \oplus \\ &\left(|g_1(x)| \odot \left|(b_{1M_1}(\alpha), b_{1M_2}(\beta) \ominus_H (b_{1M_1}(\varphi), b_{1M_2}(\rho))\right|\right) \oplus \ldots \oplus \\ &\left(|g_m(x)| \odot \left|(b_{mM_1}(\alpha), a_{mM_2}(\beta)) \ominus_H (b_{mM_1}(\varphi), b_{mM_2}(\rho))\right|\right) \end{aligned} \quad (34)$$

With respect to the Lipschitz condition (32), (34) is

$$\left|d_M(\alpha,\beta)-d_M(\varphi,\rho)\right|\le \overline{f}(H\sum_{i=1}^{n}|\alpha-\varphi|+H\sum_{i=1}^{n}|\beta-\rho|)\oplus \overline{g}(H\sum_{i=1}^{m}|\alpha-\varphi| + H\sum_{i=1}^{m}|\beta-\rho|)=\left(n\bar{f}\oplus m\overline{g}\right)(H|\alpha-\varphi|+H|\beta-\rho|)$$

In the same manner, the upper limits suffice

$$\left|d_U(\alpha,\beta)-d_U(\varphi,\rho)\right|\le \left(n\bar{f}\oplus m\overline{g}\right)(H|\alpha-\varphi|+H|\beta-\rho|)$$

As the lower limit $\left|d_M(\alpha,\beta)-d_M(\varphi,\rho)\right|\ge 0,$ with respect to Lemma 2 the solution contains in K_H and is defined in (33).

Lemma 3: Let us consider the data number to be m and also we suggest the order of the equation to be n in (19) that suffices

$$m\ge 2n+1 \tag{35}$$

considering $k=1\cdots m,$ hence the solutions of (24) and (25) are $\rho_2(k)=p(\varphi_2(k))=\rho_3(k)=p(\varphi_3(k))=0$

Proof. Since

$$\sum_{i=0}^{n}\underline{a}_i\,\underline{x}_k^i-\sum_{j=0}^{m}\underline{b}_j\,\underline{x}_k^j\le -f(x_k) \tag{36}$$

Let us opt $2n+1$ points through $(x_k,\ f(x_k))$ and find the following interpolation dual polynomial based on Z-number on these data

$$h(k)=\sum_{i=0}^{n}\underline{a}_i\,\underline{x}_k^i-\sum_{j=0}^{m}\underline{b}_j\,\underline{x}_k^j \tag{37}$$

Let $j=\max_k\{h(k)+f(x_k)\}$ and $j>0,$ as a result we can transform the dual polynomial (19) to the other form of new dual polynomial $h(k)-j$. This suggested recent dual polynomial based on Z-number suffices (36). Since the presumable spot of (24) are $\rho_2(k)\ge 0$ and $p(\varphi_2(k))\ge 0,$ so it should be zero. In the similar manner, outcome can be extracted for (25).■

x_k and $f(x_k)$ are crisp Z-numbers. In case of $k = 1 \cdots n,$ there should be a validated solution for the equation approximation (Mhaskar & Pai, 2000).

Theorem 2: If there is a big amount of data number as (35) and the dual polynomial based on Z-number (19) satisfies

$$D\left[j\left(x_{k1}, u_{k1}\right), j\left(x_{k2}, u_{k2}\right)\right] \le rD\left[u_{k1}, u_{k2}\right] \quad 0 < r < 1 \tag{38}$$

where $j(\cdot)$ exhibits a dual polynomial based on Z-number,

$$j(x_{k1}, u_{k1}): (a_1 \odot x_{k1}) \oplus \dots \oplus (a_n \odot x_{k1}^n) = (b_1 \odot x_{k1}) \oplus \dots \oplus (b_n \odot x_{k1}^n) \oplus y_{k1} \tag{39}$$

$D[u, v]$ is the Hausdorff distance related to Z-numbers u and v,

$$D[u,v] = \max\left\{ \sup_{(x_1, y_1)\in u} \inf_{(x_2, y_2)\in v} \left(d\left(x_1, x_2\right) + d\left(y_1, y_2\right)\right), \sup_{(x_1, y_1)\in v} \inf_{(x_2, y_2)\in u} \left(d\left(x_1, x_2\right) + d\left(y_1, y_2\right)\right) \right\}$$

$d(x, y)$ is the supremum metrics considering fuzzy sets, then (19) contains a distinct solution u.

Proof. According to lemma 2, there exist solutions for (23)-(25), if there are numerous data which satisfy (35). Neglecting deficit of generality, let we consider the solutions for (23)-(25) are at par with $x_k = 0,$ which tends to u_0. (38) signifies $j(\cdot)$ in (39) is continuous. If we select $\delta > 0$ in such a manner that $D\left[y_k, u_0\right] \le \delta,$ hence

$$D\left[j(x_k, u_0), u_0\right] \le (1 - r)\delta$$

where $j(0, u_0) = u_0$. Taking into our account we choose x close to $0,$ $x_k \in [0, s],$ $s > 0,$ so

$$\mathbf{S}_0 \ : \ \vartheta = \sup_{x_k \in [0,s]} D\left[y_{k_1}, y_{k_2}\right]$$

Assume $\{y_{k_m}\}$ be a succession in $\mathbf{S}_0,$ for any $\varepsilon > 0,$ the computation of $N_0(\varepsilon)$ can be done in such a manner $\vartheta < \varepsilon$, $m, n \ge N_0$. Hence $y_{k_m} \to y_k$ for $x_k \in [0, s]$. Henceforth

$$D\left[y_k, u_0\right] \le D\left[y_k, y_{k_m}\right] + D\left[y_{k_m}, u_0\right] < \varepsilon + \delta \tag{40}$$

for all $x \in [0, s]$, $m \geq N_0(\varepsilon)$. As $\varepsilon > 0$ is randomly minute,

$$D[y_k, u_0] \leq \delta \tag{41}$$

for all $x \in [0, s]$. Now we validate that y_k is continuous at $x_0 = 0$. Taking into consideration $\delta > 0$, there prevails $\delta_1 > 0$ in such a manner

$$D[y_k, u_0] \leq D[y_k, y_{k_m}] + D[y_{k_m}, u_0] \leq \varepsilon + \delta_1$$

for every $m \geq N_0(\varepsilon)$, by means of (41), while $| x - x_0 | < \delta_1$, y_k is continuous at $x_0 = 0$.

As a result (19) contains a distinct solution u_0 . ■

The necessary circumstance in order to establish the controllability (existence of solution) related to the dual equation (27) is (28), the sufficient condition related to the controllability is (32). For majority of membership functions, such as triangular functions and the trapezoidal function, the Lipschitz condition (32) is fulfilled. In this case it is considered to be controllable.

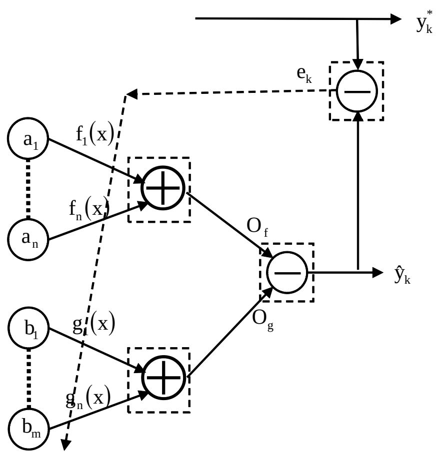

Figure 1: A feed-forward neural network (NN) approximates the solutions of fuzzy equation

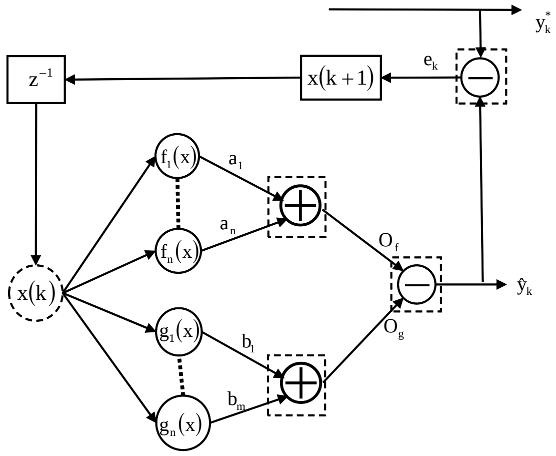

Figure 2: A feedback neural network (FNN) approximates the solutions of fuzzy equation

Utilization of neural networks for fuzzy controller design

It is not possible to acquire an analytical solution for (27). Here, neural networks are utilized to approximate the solution (control). In order to fit the neural networks, (27) is written as

$$\left(a_1 \odot f_1(x)\right) \oplus \ldots \oplus (a_n \odot f_n(x)) \ominus_H \left(b_1 \odot g_1(x)\right) \ominus_H \ldots \ominus_H (b_m \odot g_m(x)) = y_k^* \qquad (42)$$

We use two types of neural networks, feed-forward and feedback neural networks to approximate the solution of (42), see Figure 1 and Figure 2. The Z-numbers a_i and b_i represents the inputs of the neural network, the Z-number y_k represent the output. $f_i(x)$ and $g_j(x)$ are the Z-number weights.

The main idea is to detect appropriate weights of neural networks in such a manner that the output of the neural network $\hat{y}_k$, approaches the desired output y_k^*. From the view point of control, it is utter necessity to find out a suitable controller u_k which is a function of x, in such a manner that the plant (1) y_k (crisp value) estimates the Z-number y_k^*.

In the control point of view, we want to find a controller u_k which is a function of x, such that the output of the plant (1) y_k (crisp value) approximates the Z-number y_k^*.

The input Z-numbers a_i and b_i are primarily implemented to α -level as (13)

$$[a_i]^\alpha = \left(\underline{a}_i^\alpha, \overline{a}_i^\alpha\right) \quad i = 1 \cdots n$$
$$[b_j]^\alpha = \left(\underline{b}_j^\alpha, \overline{b}_j^\alpha\right) \quad j = 1 \cdots m \tag{43}$$

The next step is initiated by multiplying the above relations with the Z-number weights $f_i(x)$ and $g_j(x)$ and summarized as

$$\left[O_f\right]^\alpha = \left(\sum_{i\varepsilon M_f} \underline{f}_i^\alpha(x)\underline{a}_i^\alpha + \sum_{i\varepsilon C_f} \underline{f}_i^\alpha(x)\overline{a}_i^\alpha, \sum_{i\varepsilon M_f'} \overline{f}_i^\alpha(x)\overline{a}_i^\alpha, \sum_{i\varepsilon C_f'} \overline{f}_i^\alpha(x)\underline{a}_i^\alpha\right)$$
$$\left[O_g\right]^\alpha = \left(\sum_{j\varepsilon M_g} \underline{g}_j^\alpha(x)\underline{b}_j^\alpha + \sum_{j\varepsilon C_g} \underline{g}_j^\alpha(x)\overline{b}_j^\alpha, \sum_{j\varepsilon M_g'} \overline{g}_j^\alpha(x)\overline{b}_j^\alpha, \sum_{j\varepsilon C_g'} \overline{g}_j^\alpha(x)\underline{b}_j^\alpha\right) \tag{44}$$

Here $M_f = \{i \mid \underline{f}_i^\alpha(x) \geq 0\}$, $C_f = \{i \mid \underline{f}_i^\alpha(x) < 0\}$, $M_f' = \{i \mid \overline{f}_i^\alpha(x) \geq 0\}$,

$C_f' = \{i \mid \overline{f}_i^\alpha(x) < 0\}$, $M_g = \{j \mid \underline{g}_j^\alpha(x) \geq 0\}$, $C_g = \{j \mid \underline{g}_j^\alpha(x) < 0\}$, $M_g' = \{j \mid \overline{g}_j^\alpha(x) \geq 0\}$

, $C_g' = \{j \mid \overline{g}_j^\alpha(x) < 0\}$.

The neural network output is

$$\left[\hat{y}_k\right]^\alpha = \left(\underline{O}_f^\alpha - \underline{O}_g^\alpha, \overline{O}_f^\alpha - \overline{Q}_g^\alpha\right) \tag{45}$$

The error of the training is

$$e_k = y_k^* \ominus \hat{y}_k$$

here $\left[y_k^*\right]^\alpha = \left(\underline{y}_k^{*\alpha}, \overline{y_k^*}^\alpha\right)$, $\left[\hat{y}_k\right]^\alpha = \left(\underline{\hat{y}}_k^\alpha, \overline{\hat{y}_k}^\alpha\right)$, $\left[e_k\right]^\alpha = \left(\underline{e}_k^\alpha, \overline{e}_k^\alpha\right)$.

A cost function, which is generated on the basis of Z-numbers is implemented for the training of the weights as mentioned below

$$J_k = \underline{J}^\alpha + \overline{J}^\alpha, \quad \underline{J}^\alpha = \frac{1}{2}\left(\underline{y}_k^{*\alpha} - \underline{y}_k^\alpha\right)^2, \quad \overline{J}^\alpha = \frac{1}{2}\left(\overline{y_k^*}^\alpha - \overline{\hat{y}_k}^\alpha\right)^2 \tag{46}$$

It is quite obvious, $J_k \to 0$ when $\left[\hat{y}_k\right]^\alpha \to \left[y_k^*\right]^\alpha$.

The vital positiveness lies within the least mean square (46) is that it has a self-correcting feature that makes it suitable to function for arbitrarily vast duration without shifting from its constraints. The mentioned gradient algorithm is subjected to cumulative series of errors and is convenient for long runs in absence of an additional error rectification procedure.

The gradient technique is now utilized to train the Z-number weights $f_i(x)$ and $g_j(x)$. The solution x_0 is the function of $f_i(x)$ and $g_j(x)$. We compute $\frac{\partial J_k}{\partial \underline{x}_0}$ and $\frac{\partial J_k}{\partial \bar{x}_0}$ which are mentioned as

$$\begin{aligned} \frac{\partial J_k}{\partial \underline{x}_0} &= \frac{\partial \underline{J}^{\alpha}}{\partial \underline{x}_0} + \frac{\partial \bar{J}^{\alpha}}{\partial \underline{x}_0} \\ \frac{\partial J_k}{\partial \bar{x}_0} &= \frac{\partial \underline{J}^{\alpha}}{\partial \bar{x}_0} + \frac{\partial \bar{J}^{\alpha}}{\partial \bar{x}_0} \end{aligned} \tag{47}$$

According to the chain rule

$$\frac{\partial \underline{J}^{\alpha}}{\partial \underline{x}_0} = \frac{\partial \underline{J}^{\alpha}}{\partial \underline{y}_k^{\alpha}} \frac{\partial \underline{y}_k^{\alpha}}{\partial \underline{O}_f^{\alpha}} \frac{\partial \underline{O}_f^{\alpha}}{\partial \underline{f}_i^{\alpha}(x)} \frac{\partial \underline{f}_i^{\alpha}(x)}{\partial \underline{x}_0} - \frac{\partial \underline{J}^{\alpha}}{\partial \underline{y}_k^{\alpha}} \frac{\partial \underline{y}_k^{\alpha}}{\partial \underline{O}_g^{\alpha}} \frac{\partial \underline{O}_g^{\alpha}}{\partial \underline{g}_j^{\alpha}(x)} \frac{\partial \underline{g}_j^{\alpha}(x)}{\partial \underline{x}_0}$$

So

$$\frac{\partial \underline{J}^{\alpha}}{\partial \underline{x}_0} = \sum_{i=1}^{n} -\left(\underline{y}_k^{*\alpha} - \underline{y}_k^{\alpha}\right)\underline{a}_i^{\alpha}\, \underline{f}_i^{\prime\alpha} + \sum_{j=1}^{m}\left(\underline{y}_k^{*\alpha} - \underline{y}_k^{\alpha}\right)\underline{b}_j^{\alpha}\, \underline{g}_j^{\prime\alpha}$$

Or

$$\frac{\partial \underline{J}^{\alpha}}{\partial \underline{x}_0} = \sum_{i=1}^{n} -\left(\underline{y}_k^{*\alpha} - \underline{y}_k^{\alpha}\right)\bar{a}_i^{\alpha}\, \underline{f}_i^{\prime\alpha} + \sum_{j=1}^{m}\left(\underline{y}_k^{*\alpha} - \underline{y}_k^{\alpha}\right)\bar{b}_j^{\alpha}\, \underline{g}_j^{\prime\alpha}$$

$\frac{\partial J_k}{\partial \bar{x}_0}$ can be calculated the same as above.

The solution x_0 is upgraded as

$$\begin{aligned} \underline{x}_0(k+1) &= \underline{x}_0(k) - \eta \frac{\partial J_k}{\partial \underline{x}_0} \\ \bar{x}_0(k+1) &= \bar{x}_0(k) - \eta \frac{\partial J_k}{\partial \bar{x}_0} \end{aligned}$$

Here η is the rate of the training $\eta > 0$.

For the requirement of increasing the training process, the adding of the momentum term is mentioned as

$$\underline{x}_0(k+1)=\underline{x}_0(k)-\eta\frac{\partial J_k}{\partial \underline{x}_0}+\gamma\left[\underline{x}_0(k)-\underline{x}_0(k-1)\right]$$
$$\overline{x}_0(k+1)=\overline{x}_0(k)-\eta\frac{\partial J_k}{\partial \overline{x}_0}+\gamma\left[\overline{x}_0(k)-\overline{x}_0(k-1)\right]$$

Here $\gamma>0.$ After the updating of x_0 , it is necessary to substitute it to the weights $f_i(x_0)$ and $g_j(x_0)$.

The solution related to the dual equation (27) can also be estimated by feedback neural network, as Figure 2. In this case, the inputs are the nonlinear Z-number functions $f_i(x)$ and $g_j(x),$ the concerned weights are taken to be as Z-numbers a_i and b_j. The training error e_k has been utilized here in order to update x . Once the nonlinear operations $f_i(x)$ and $g_j(x)$ are performed, O_f and O_g are considered to be similar to (44). The output related to the neural network is taken as similar to (45).

Simulations

In this section, we use several applications to show how to use the fuzzy equation with Z-number to design the fuzzy controller.

Example 1: The main intention of the chemical reaction between the poly ethylene (PE) and poly propylene (PP) is to produce a preferred substance (PS). If x is considered to be the material cost, then the cost of PE is taken to be x and x^2 is considered to be the cost of PP. The PE and PP weights which are uncertain, are sufficed by the triangle function (7). It is our requirement to generate two different types of PS. If we urge the cost in the midst is $[(360.50,400.55,421.37),\ p(0.8,0.9,1)]=y^*,$ what can be the cost x? The PE weights are stated as

$$a_1=[(2.7951,3.35412,3.9131),\ p(0.7,0.8,1)]$$
$$b_1=[(1.5811,2.1081,2.6352),\ p(0.8,0.9,1)]$$

The PP weights are stated as

$$a_2 = [(4.8107, 5.3452, 5.8797),\ p(0.7, 0.875, 1)]$$
$$b_2 = [(3.9131, 4.4721, 5.0311),\ p(0.6, 0.8, 1)]$$

The modeling of the above mentioned relation can be carried out using the dual equation and Z-numbers

$$[(2.79,3.354,3.91), p(0.7,0.8,1)]\odot x \oplus [(4.81,5.34,5.8797), p(0.7,0.875,1)]\odot x^2 =$$
$$[(1.58,2.10,2.6352), p(0.8,0.9,1)]\odot x \oplus [(3.91,4.47,5.0311), p(0.6,0.8,1)]\odot x^2 \oplus$$
$$[(360,50,400.55,421.37), p(0.8,0.9,1)]$$

In this case $f_1(x) = g_1(x) = x,\ \ f_2(x) = g_2(x) = x^2$. The exact solution is demonstrated by

$$x^* = [(18.3712, 19.3919, 19.9022),\ p(0.8, 0.96, 1)]$$

We utilize feedforward (NN) and feedback (FNN) neural networks to estimate the solution x. The learning rate is $\eta = 0.02$. The initial state is $x(0) = [(22.66, 23.71, 24.24),\ p(0.8, 0.9, 1)]$. The approximation outcomes are exhibited in Table 1. The modeling errors are displayed in Figure 3.

Table 1. Neural networks approximate the Z-numbers

k	x(k) with NN	k	x(k) with FNN
1	[(22.53, 23.68, 24.10), p(0.6, 0.8, 0.85)]	1	[(22.33, 23.38, 23.99), p(0.7, 0.8, 0.85)]
2	[(21.79, 22.83, 23.20), p(0.7, 0.8, 0.85)]	2	[(20.98, 22.13, 22.761), p(0.7, 0.85, 0.9)]
⋮	⋮	⋮	⋮
35	[(18.67, 19.71, 20.23), p(0.8, 0.92, 1)]	18	[(18.49, 19.51, 20.13), p(0.8, 0.92, 1)]
36	[(18.38, 19.40, 19.91), p(0.8, 0.96, 1)]	19	[(18.37, 19.39, 19.90), p(0.8, 0.96, 1)]

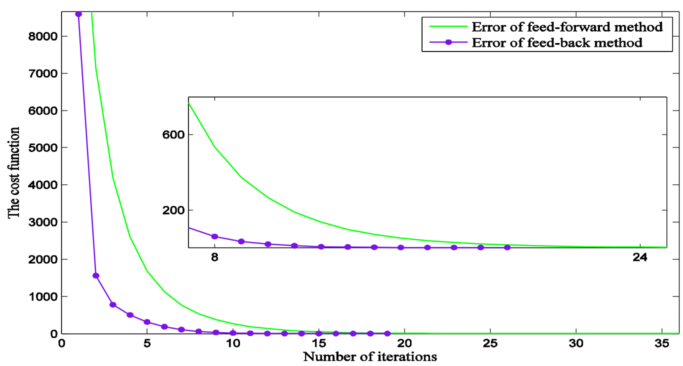

Figure 3: Approximation errors of the neural networks

We can see that both neural networks give worthy performance. We use the following to transfer the Z-numbers to fuzzy numbers,

$$\alpha = \frac{\int x\pi_{\tilde{p}}(x)dx}{\int \pi_{\tilde{p}}(x)dx}$$

Consider $Z = (A, p) = [(22.33, 23.38, 23.99), p(0.6, 0.8, 0.85)]$, then

$Z^{\alpha} = \left(22.31, 23.38, 23.99; 0.77\right)$ and so

$Z' = (\sqrt{0.77}\ 22.331, \sqrt{0.77}\ 23.384, \sqrt{0.77}\ 23.993)$. The results of neural networks approximation for the fuzzy numbers are displayed in Table 2.

Table 2. Neural networks approximate the fuzzy numbers

k	x(k) with NN	k	x(k) with FNN
1	(19.358, 20.349, 20.709)	1	(19.593, 20.517, 21.051)
2	(19.205, 20.125, 20.448)	2	(18.844, 19.878, 20.442)
⋮	⋮	⋮	⋮
35	(17.720, 18.710, 19.203)	18	(17.548, 18.517, 19.107)
36	(17.541, 18.513, 19.000)	19	(17.538, 18.509, 18.996)

The Z-numbers increase degree of reliability of the information. The crucial factor is that Z -information is not only the most generalized depiction of real-world uncomplicated information but also is the highest narrative power extracted from human cognition outlook as compared to fuzzy number. The comparison between the Z-number $Z = [(18.38, 19.40, 19.91), p(0.8, 0.96, 1)]$ and fuzzy number $(17.54, 18.51, 19.00)$ for $k = 36$ is shown in Figure 4. We see that the Z-number incorporates with various information and the solution of the Z-number is more accurate. The membership function for the restriction in the Z-number is $\mu_{A_Z} = (18.38, 19.40, 19.91)$. It can be in probability form.

Example 2: The insulating materials center is considered to be the source of heat. The materials width are not precise and hence they suffice the trapezoidal function (8),

$$A = [(0.131, 0.153, 0.164, 0.197),\ p(0.7, 0.83, 0.9)] = a_1$$
$$B = [(0.084, 0.105, 0.210, 0.527),\ p(0.8, 0.9, 1)] = a_2$$
$$C = [(0.096, 0.107, 0.214, 0.428),\ p(0.7, 0.87, 0.9)] = b_1$$
$$D = [(0.021, 0.032, 0.054, 0.086),\ p(0.8, 0.85, 0.92)] = b_2$$

see Figure 5. The coefficient associated with conductivity materials are $K_A = x = f_1$, $K_B = x\sqrt{x} = f_2$, $K_C = x^2 = g_1$, $K_D = \sqrt{x} = g_2$, where x is considered to be as the elapsed time. The control object is to reveal the time in case the thermal resistance at the right side attains $R = [(0.0162, 0.0293, 0.0424, 0.1241),\ p(0.75, 0.8, 0.9)] = y^*$. The thermal balance model is (Holman, 1997):

$$\frac{A}{K_A} \oplus \frac{B}{K_B} = \frac{C}{K_C} \oplus \frac{D}{K_D} \oplus R$$

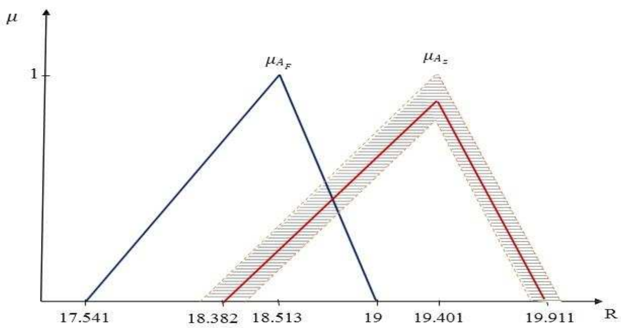

Figure 4: Z-number and fuzzy number

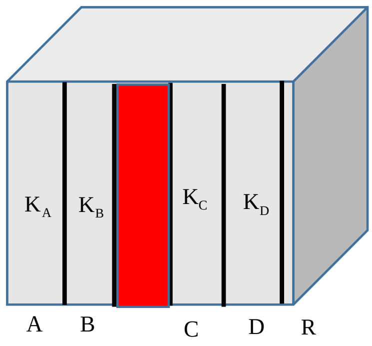

Figure 5: Heat source

The exact solution is $x^* = [(2.0519, 3.0779, 4.1039, 6.1559), p(0.8, 0.95, 1)]$ (Holman, 1997). The learning rate is $\eta = 0.1$ (NN) and $\eta = 0.005$ (FNN). The neural networks approximation results are displayed in Table 3 and Table 4. The modeling errors are displayed in Figure 6.

Table 3. Neural networks approximate the Z-numbers

k	x(k) with NN	k	x(k) with FNN
1	$[(5.97, 6.98, 7.93, 9.98), p(0.6, 0.8, 0.85)]$	1	$[(5.98, 6.99, 7.97, 9.98), p(0.7, 0.85, 0.87)]$
2	$[(5.43, 6.38, 7.35, 9.302), p(0.75, 0.8, 0.9)]$	2	$[(5.37, 6.10, 7.12, 9.16), p(0.7, 0.85, 0.87)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
61	$[(2.11, 3.170, 4.22, 6.33), p(0.8, 0.9, 1)]$	45	$[(2.08, 3.14, 4.14, 6.29), p(0.8, 0.96, 1)]$
62	$[(2.06, 3.08, 4.11, 6.17), p(0.8, 0.94, 1)]$	46	$[(2.05, 3.08, 4.11, 6.16), p(0.8, 0.94, 1)]$

Table 4. Neural networks approximate the fuzzy numbers

k	x(k) with NN	k	x(k) with FNN
1	$(5.13, 5.99, 6.841, 8.576)$	1	$(5.36, 6.25, 7.14, 8.939)$
2	$(4.93, 5.79, 6.671, 8.440)$	2	$(4.81, 5.46, 6.37, 8.199)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
61	$(2.00, 3.00, 4.007, 6.008)$	45	$(1.99, 3.00, 3.95, 6.004)$
62	$(1.96, 2.934, 3.915, 5.870)$	46	$(1.95, 2.93, 3.90, 5.864)$

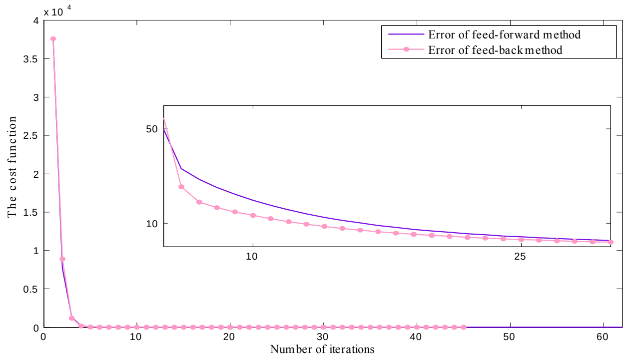

Figure 6: Approximation errors of the neural networks

Example 3: The pipe d_1 which is carrying water is subdivided into three different pipes d_2 , d_3 , d_4 , refer Figure 7. The areas of the pipes are uncertain, they suffice the trapezoidal function (8),

$A_1 = [(0.421, 0.632, 0.737, 0.843),\ p(0.75, 0.9, 1)]$
$A_2 = [(0.052, 0.104, 0.209, 0.419),\ p(0.8, 0.91, 1)]$
$A_3 = [(0.031, 0.084, 0.105, 0.210),\ p(0.8, 0.9, 0.95)]$

The velocities of water flowing through the pipes are controlled with the help of valves parameter x, $v_1 = x^3$, $v_2 = \frac{e^x}{2}$, $v_3 = x$ (Streeter,1999). The flow in pipe d_4 is initiated using the control object which is represented by

$Q = [(11.478, 40.890, 93.332, 293.056),\ p(0.8, 0.87, 0.95)]$

We need to find the valve control parameter x. By mass balance

$A_1 v_1 = A_2 v_2 \oplus A_3 v_3 \oplus Q$

The exact solution is demonstrated by $x = [(3.127, 4.170, 5.212, 7.298),\ p(0.8, 0.92, 1)]$ (Streeter, Wylie, & Benjamin, 1999). The learning rate of NN is $\eta = 0.08.$ The neural networks approximation results are displayed in Table 5 and Table 6.

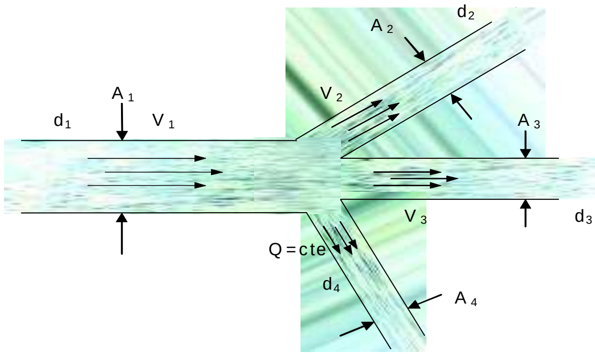

Figure 7: Water channel system

Table 5. Neural networks approximate the Z-numbers

k	x(k) with NN	k	x(k) with FNN
1	$[(5.75, 6.77, 7.74, 9.76), p(0.6, 0.8, 0.85)]$	1	$[(5.87, 6.88, 7.86, 9.877), p(0.7, 0.81, 0.85)]$
2	$[(5.32, 6.26, 7.13, 9.20), p(0.7, 0.8, 0.87)]$	2	$[(5.15, 6.00, 7.00, 9.002), p(0.7, 0.85, 0.9)]$
⋮	⋮	⋮	⋮
55	$[(3.14, 4.19, 5.22, 7.32), p(0.8, 0.9, 1)]$	20	$[(3.13, 4.18, 5.22, 7.312), p(0.85, 0.9, 1)]$
56	$[(3.13, 4.18, 5.22, 7.31), p(0.8, 0.93, 1)]$	21	$[(3.13, 4.178, 5.21, 7.305), p(0.8, 0.92, 1)]$

Table 6. Neural networks approximate the fuzzy numbers

k	x(k) with NN	k	x(k) with FNN
1	$(4.94, 5.81, 6.65, 8.38)$	1	$(5.18, 6.07, 6.94, 8.71)$
2	$(4.72, 5.54, 6.31, 8.15)$	2	$(4.63, 5.39, 6.28, 8.08)$
⋮	⋮	⋮	⋮
55	$(2.98, 3.97, 4.96, 6.94)$	20	$(2.97, 3.97, 4.95, 6.93)$
56	$(2.97, 3.97, 4.95, 6.94)$	21	$(2.97, 3.96, 4.95, 6.93)$

We can see that FNN is much faster and more robust compared with NN. After converting the Z-numbers to fuzzy numbers, it is possible to extract the fuzzy rules.

Now we compare our method with the other existing algorithms.

● In (Noorani, Kavikumar, Mustafa, & Nor, 2011), the ranking methodology is suggested in order to extract the real roots associated to a dual fuzzy polynomial equation. It is $C_1x + C_2x^2 + ... + C_nx^n = D_1x + D_2x^2 + ... + D_nx^n + q$

where $x \in R$, $C_1,...,C_n, D_1,..., D_n$, and q are fuzzy numbers. The dual fuzzy polynomial equation is converted to the system associated to the crisp dual polynomial equations. This conversion is carried out by utilizing ranking methodology on the basis of three parameters namely value, ambiguity and fuzziness. This method is applicable only when the variables are crisp, e.g., this method is not able to find the solution of dual fuzzy equations. Also the solutions of the three parameters, value, ambiguity and fuzziness, are not related to generate solutions.

● In (Mosleh, 2013), the modified Adomian decomposition method is applied for solving the following dual polynomial equations

$$a_1 x + a_2 x^2 + \ldots + a_n x^n = b_1 x + b_2 x^2 + \ldots + b_n x^n + c$$

where x, c, and all coefficients are fuzzy numbers. Figure 8 shows the comparison results. We can see that our neural networks based algorithm and the modified Adomian decomposition method can approximate the solutions of the dual fuzzy equations. However, the convergence speeds of the neural network based algorithms are faster.

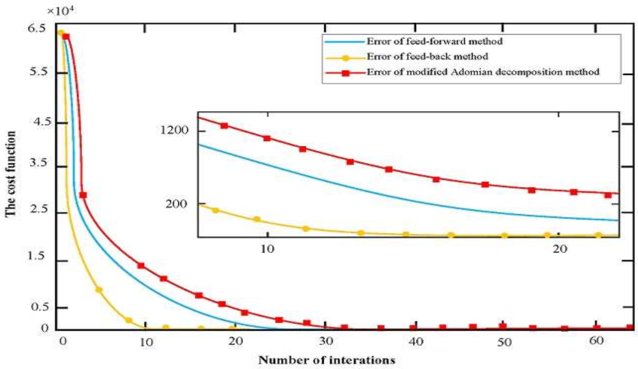

Figure 8: Approximation errors of the neural networks and modified Adomian decomposition method

Conclusions

In this paper, the classical fuzzy equation is modified such that its coefficients are Z-numbers. The dual type of this fuzzy model is applied to model uncertain nonlinear systems. We give the relation between the solution of the fuzzy equations and the nonlinear system control. The controllability of the fuzzy system is proposed. Two types of neural networks are applied to approximate the solutions of the fuzzy equations. Modified gradient descent algorithms are used to train the neural networks. The novel methods are validated by several benchmark examples. The future works are the application of the mentioned methodologies for fuzzy differential equations.

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