Distance-based Control of $\mathcal{K}_n$ Formations in General Space with Almost Global Convergence

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***Abstract*—In this paper, we propose a distance-based formation control strategy for a group of mobile agents to achieve almost global convergence to a target formation shape provided that the formation is represented by a complete graph, and each agent is governed by a single-integrator model. The fundamental idea of achieving almost global convergence is to use a virtual formation of which the dimension is augmented with some virtual coordinates. We define a cost function associated with the virtual formation and apply the gradient-descent algorithm to the cost function so that the function has a global minimum at the target formation shape. We show that all agents finally achieve the target formation shape for almost all initial conditions under the proposed control law.**

***Index Terms*—Distance-based formation control, almost global convergence, gradient control.**

I. Introduction

Based on use of *graph rigidity* theory [1]–[4], various techniques and convergence results have been developed on distance-based formation control [5]–[18]. It is known that for any undirected rigid formation, we can achieve local asymptotic stability [12] and local exponential stability [17] under the distance-based control algorithm proposed in [8] and its generalizations. Although there are many other publications dealing with distance-based formation stabilization problems, most of them are focused on local stability analysis with a few exceptions (e.g., [6], [9]–[11], [13], [15], [18]) handling global stability issues for some special formation shapes.

To be more specific about those exceptions, undirected polygonal formations in the plane are studied in [6] based on a distance-based formation control strategy, but the authors successfully show almost global convergence to only a *triangular* target formation. Regarding four-agent formations, there is a formation that is termed a $\mathcal{K}_4$ formation because it is represented by the four-vertex complete graph. References [9], [11] contribute to showing that any *rectangular* $\mathcal{K}_4$ formation in $\mathbb{R}^2$ is achieved almost globally under the control law proposed in [8]. Nevertheless, it is still not known whether a general rather than rectangular $\mathcal{K}_4$ formation can be achieved under the same control law almost globally; obviously, more complicated general rigid formations consisting of more than four agents have not been studied well. A control law to stabilize rigid formations in $\mathbb{R}^2$ almost globally was proposed in [19], but it has been shown that there exists a counter example formation which cannot be stabilized almost globally under the same control law [20]. Only partial analyses of particular classes of rigid formations are reported recently in [15], [18]. Reference [15] provides a discontinuous control law to achieve a *universally rigid* target formation, and [18] shows that a *triangulated formation* can be stabilized almost globally.[1] Consequently, the ultimate goal of providing a global stability analysis for a general rigid formation is a challenging task which still remains an open problem. Likewise, we cannot provide a complete solution to the problem at this stage, but we want to explore another branch related to the ultimate goal.

One of the aforementioned exceptions can be noted, as it motivates much of this paper. It is known that, under the control law proposed in [8] (with an extension of the dimension), we can obtain almost global convergence for a $\mathcal{K}_4$ formation in $\mathbb{R}^3$ [13]. However, whether the same control law can be used for a $\mathcal{K}_4$ formation in $\mathbb{R}^2$ to achieve global convergence has yet to be established for other than special cases. In our previous works [16], we proposed an alternative approach to treating the $\mathcal{K}_4$ formation in $\mathbb{R}^2$ by having it mimic the $\mathcal{K}_4$ formation in $\mathbb{R}^3$, thereby taking advantages of the global convergence results for the $\mathcal{K}_4$ formation in $\mathbb{R}^3$ in solving the problem of $\mathcal{K}_4$ formation in $\mathbb{R}^2$. But, the work in [16] is confined to $\mathcal{K}_4$ formation control in $\mathbb{R}^2$. In this paper, we further generalize the works of [16] in both dimensions and the size of graphs. Thus, as the main contribution of this paper, we first generalize the results so that they could be applied to general $\mathcal{K}_n$ formations in $\mathbb{R}^2$ with $n \geq 4$. Then, as the second contribution, we remove the restriction that the realization space is $\mathbb{R}^2$. More precisely, we can seek to realize a $\mathcal{K}_n$ formation in $\mathbb{R}^d$, for any $d \leq n-1$, as opposed to the case $d = 2$, $n = 4$ of [16].

The rest of the paper is organized as follows. In Section II, we provide background knowledge on formation graphs and the control law proposed in [8]. The notation used throughout the paper is summarized at the beginning of the section. In Section III, we explore $\mathcal{K}_n$ formations in $\mathbb{R}^{n-1}$, which provides some results generalized from [13] based on the results of [14]. We establish the main results in Section IV by providing a method to construct a virtual formation and analyzing the

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[1] One can refer to [15] and [18] for the definitions of universal rigidity and a triangulated formation, respectively.

convergence of the virtual formation. Some examples and related simulations are given in Section V. We summarize the paper in Section VI.

II. Notations and motivation

We first introduce the notation used in the rest of the paper.

- $\mathbb{R}^d$: d-dimensional Euclidean space.
- $\mathbb{R}_{\geq a} = \{x \in \mathbb{R} \mid x \geq a\}$.
- $|\mathcal{S}|$: the cardinality of a set $\mathcal{S}$.
- $\|x\|$: the Euclidean norm of a real vector x.
- For some real vectors $v_1, \ldots, v_n$, $(v_1, \ldots, v_n) = [v_1^\top \ \ldots \ v_n^\top]^\top$.
- $1_k = (1, \ldots, 1) \in \mathbb{R}^k$.
- $0_k = (0, \ldots, 0) \in \mathbb{R}^k$.
- $\mathbb{RF}_m^d = \mathbb{R}^m \times \{0_{d-m}\} \subseteq \mathbb{R}^d$ for $d > m$, and $\mathbb{RF}_d^d = \mathbb{R}^d$.[2]
- $\operatorname{dist}(x, y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$:

$$\operatorname{dist}(x, y) = \begin{cases} \|(x, 0_{m-n}) - y\|, & (n < m), \\ \|x - (y, 0_{n-m})\|, & (n > m), \\ \|x - y\|, & (m = n). \end{cases}$$

- $\operatorname{dist}(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} \operatorname{dist}(x, y)$, $x \in \mathbb{R}^n$, $\mathcal{S} \subseteq \mathbb{R}^m$.

A. Formation graph

Let $\mathcal{G}$ denote a graph defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of all vertices representing the agents, and $\mathcal{E} = \{\ldots, \{i, j\}, \ldots\}$ is the set of all edges representing certain pairs of the vertices. Let $p_i \in \mathbb{R}^d$ be a column vector denoting the position vector of vertex i. The j^{th} component of p_i is represented by p_i^j, i.e., $p_i = (p_i^1, \ldots, p_i^j, \ldots, p_i^d)$. We call $p = (p_1, \ldots, p_n) \in \mathbb{R}^{dn}$ a *realization* of $\mathcal{G}$ in $\mathbb{R}^d$. A *framework (formation)* is defined by a pair of a graph $\mathcal{G}$ and its realization p, and denoted by $(\mathcal{G}, p)$. Two realizations p and z are said to be *congruent* if $\|p_i - p_j\| = \|z_i - z_j\|$ for all $i, j \in \mathcal{V}$, and two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, z)$ are said to be *equivalent* if $\|p_i - p_j\| = \|z_i - z_j\|$ for all $\{i, j\} \in \mathcal{E}$. We use $\mathcal{K}_n$ to denote the *complete graph* of n vertices. A formation $(\mathcal{G}, p)$ such that $\mathcal{G} = \mathcal{K}_n$ is called a $\mathcal{K}_n$ formation.

B. Rigidity and infinitesimal rigidity

Since the notion of rigidity of a framework is essential for understanding distance-based formation control, we provide a brief introduction to rigidity and infinitesimal rigidity. One can refer to [1], [2], [4], [7], [21] for more detailed explanations. For a given framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a function $r_\mathcal{G} : \mathbb{R}^{d|\mathcal{V}|} \to \mathbb{R}^{|\mathcal{E}|}$,

$$r_\mathcal{G}(p) = \frac{1}{2}(\ldots, \|p_i - p_j\|^2, \ldots), \ \{i, j\} \in \mathcal{E},$$

is called the *rigidity function* of $(\mathcal{G}, p)$. Thus for two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, z)$, they are equivalent if and only if $r_\mathcal{G}(p) = r_\mathcal{G}(z)$ by definition.

Definition 1 ([1],[21]). *Consider a framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the associated rigidity function $r_\mathcal{G}$. Then the framework $(\mathcal{G}, p)$ is* rigid *in $\mathbb{R}^d$ if there exists a neighborhood $\mathcal{U} \subseteq \mathbb{R}^{d|\mathcal{V}|}$ of p such that*

$$r_\mathcal{G}^{-1}(r_\mathcal{G}(p)) \cap \mathcal{U} = r_{\mathcal{K}_{|\mathcal{V}|}}^{-1}(r_{\mathcal{K}_{|\mathcal{V}|}}(p)) \cap \mathcal{U}. \quad (1)$$

Furthermore, if (1) *holds for $\mathcal{U} = \mathbb{R}^{d|\mathcal{V}|}$, then $(\mathcal{G}, p)$ is said to be* globally rigid *in $\mathbb{R}^d$.*

From Definition 1, we can notice that any framework $(\mathcal{G}, z)$ which is equivalent to $(\mathcal{G}, p)$ with $z \in \mathcal{U}$ results in congruence of p and z if $(\mathcal{G}, p)$ is rigid.

In addition to rigidity, there is a concept called *infinitesimal rigidity* which is more conservative than rigidity. Although the definition of infinitesimal rigidity is given in [2] rigorously, we provide a theorem which can be taken as a definition of infinitesimal rigidity instead.

Theorem 1 ([2]–[4]). *With the same notation in Definition 1, let*

$$R(\mathcal{G}, p) = \frac{\partial r_\mathcal{G}(p)}{\partial p} \in \mathbb{R}^{|\mathcal{E}| \times d|\mathcal{V}|}.$$

The framework $(\mathcal{G}, p)$ is infinitesimally rigid in $\mathbb{R}^d$ if and only if

$$\operatorname{rank} R = \begin{cases} d|\mathcal{V}| - d(d+1)/2 & \text{if } |\mathcal{V}| \geq d, \\ |\mathcal{V}|(|\mathcal{V}| - 1)/2 & \text{otherwise.} \end{cases}$$

The matrix R in Theorem 1 is called the *rigidity matrix* of $(\mathcal{G}, p)$.

C. Steepest descent flow under the single-integrator model

Consider n agents evolving in $\mathbb{R}^d$ under the following single-integrator model:

$$\dot{p}_i = u_i, \ \forall i \in \mathcal{V}, \quad (2)$$

where $u_i = (u_i^1, \ldots, u_i^d) \in \mathbb{R}^d$ denotes the control input for agent i. Let $\bar{p}$ be a representative realization of the target formation shape. The goal of distance-based formation control is to achieve a formation which is congruent to $(\mathcal{G}, \bar{p})$ by adjusting the inter-agent distances corresponding to the edges of the underlying formation graph. Depending on the graph structure and the characteristics of the target formation, we may or may not achieve the target formation shape even if all the desired inter-agent distances are satisfied. For example, we can see two *non-congruent* formations having the same inter-agent distance set in Fig. 1. The agent 4 can satisfy the distance constraints to agents 2 and 3 at the both positions denoted by 4 and 4′. In this particular case, if there were a

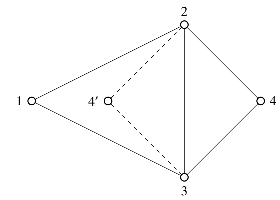


Figure 1: Two *non-congruent* formations induced from the same inter-agent distance set.

[2]For instance, $\mathbb{RF}_1^3 \subseteq \mathbb{R}^3$ is a subspace spanned by $(1, 0, 0)$, and $\mathbb{RF}_2^3 \subseteq \mathbb{R}^3$ is a subspace spanned by $(1, 0, 0)$ and $(0, 1, 0)$.

distance constraint between agents 1 and 4, there would not be such an ambiguity. From the definition of global rigidity [21], a necessary and sufficient condition to guarantee that a formation satisfying the desired inter-agent distances, which are induced from a target formation, is congruent to the target formation is that the target formation is globally rigid.

However, even if the target formation is globally rigid, finding an effective control law for (2) to achieve a target formation from almost all initial conditions is a different and challenging task. For example, consider the control law proposed by Krick et al. in [8], which is equivalent to a steepest descent flow of a potential function defined by

$$v(p) = \frac{1}{4} \sum_{\{i,j\} \in \mathcal{E}} \left(\|p_i - p_j\|^2 - d_{ij}^2 \right)^2, \quad (3)$$

where $d_{ij} = \|\bar{p}_i - \bar{p}_j\|$. The control law can be written as

$$u = -\left[\frac{\partial v}{\partial p}\right]^\top, \quad (4)$$

where $u = (u_1, \ldots, u_n)$. Since we are using a single-integrator model, the overall closed-loop system with (4) is represented by

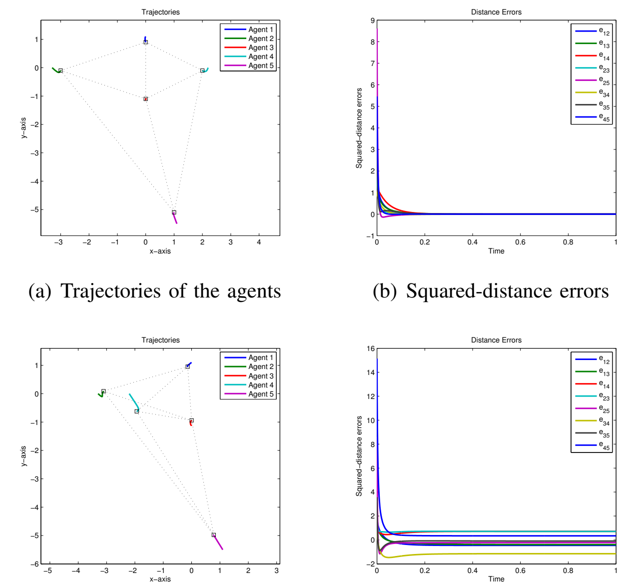
$$\dot{p} = -\left[\frac{\partial v}{\partial p}\right]^\top. \quad (5)$$

Under the control law in (4), we can find an example showing the convergence (from selected initial conditions) to an incorrect equilibrium formation (i.e. an equilibrium formation which does not make $v = 0$), even if the target formation is globally rigid. Let us consider a globally rigid five-agent target formation in $\mathbb{R}^2$ where a representative realization is given by $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{p}_5)$ with $\bar{p}_1 = (0, 1)$, $\bar{p}_2 = (-3, 0)$, $\bar{p}_3 = (0, -1)$, $\bar{p}_4 = (2, 0)$, and $\bar{p}_5 = (1, -5)$. Fig. 2(a) shows a simulation result representing the trajectories generated from an initial condition $1.1\bar{p}$ under (5). On the other hand, we obtain Fig. 2(c), which shows convergence to an incorrect equilibrium formation, if we use $1.1(\bar{p}_1, \bar{p}_2, \bar{p}_3, -\bar{p}_4, \bar{p}_5)$ as an initial condition.

A different indication of the difficulties with the closed-loop system is provided by a $\mathcal{K}_4$ formation in $\mathbb{R}^2$, which is the simplest form of globally rigid formation with more than three agents. It is still an open problem to determine whether a general $\mathcal{K}_4$ formation in $\mathbb{R}^2$ can achieve the target formation almost globally for (5). Some papers offer a partial analysis of a $\mathcal{K}_4$ formation in $\mathbb{R}^2$ to show the almost global convergence property [9], [11], but there are only limited results that can be applied to some particular examples, e.g., rectangular formations. On the other hand, it is shown in [13] that a $\mathcal{K}_4$ formation in $\mathbb{R}^3$ can achieve the target formation shape almost globally by showing the instability of degenerate[3] incorrect equilibrium formations, and the results on instability of the degenerate incorrect equilibrium formations are extended to more general formation cases in [14].

In [16], the almost global convergence property of a $\mathcal{K}_4$ formation in $\mathbb{R}^3$ is used to achieve almost global convergence of a $\mathcal{K}_4$ formation in $\mathbb{R}^2$ under a modified control law. In that paper, the actual agents are considered as agents moving in a 2-dimensional space, but one virtual variable is used as a virtual additional coordinate of an agent, thereby allowing the whole formation to be viewed as a pseudo or virtual formation in 3-dimensional space with three agents locked on the original 2-dimensional plane. A conventional control law is constructed for the virtual formation, and used to motivate a control law for the real formation, which is of course restricted to the 2-dimensional plane. This strategy can be used to guarantee almost global convergence of a $\mathcal{K}_4$ formation in $\mathbb{R}^2$. We seek to generalize this approach in this paper.

[3]Definitions and exact meaning will be provided in Section III-A.



(a) Trajectories of the agents (b) Squared-distance errors

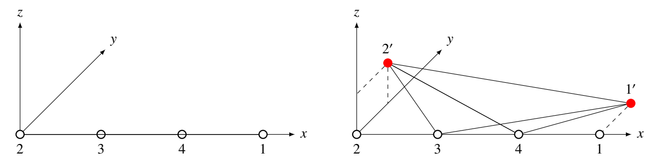
(c) Trajectories of the agents (d) Squared-distance errors

Figure 2: A 5-agent formation which is globally rigid in $\mathbb{R}^2$: (a) and (b) show the convergence to the target formation shape while (c) and (d) show the convergence to an incorrect formation shape.

Our starting point is that if we want to describe a $\mathcal{K}_n$ formation in $\mathbb{R}^d$, $1 \le d \le n-2$, we can usefully introduce a virtual formation in $\mathbb{R}^{n-1}$ and associated control problem with a minimal number of virtual variables; the number of required virtual variables is given by

$$(1 + 2 + \cdots + n - 1 - d) = \frac{(n-d)(n-1-d)}{2}. \quad (6)$$

To understand (6), let us consider a $\mathcal{K}_4$ formation in $\mathbb{R}^1$. Let the position of each agent be represented by 1-vector, e.g., $p_1 = (3)$, $p_2 = (0)$, $p_3 = (1)$, and $p_4 = (2)$ as shown in Fig. 3(a). Suppose that we want to describe the $\mathcal{K}_4$ formation in $\mathbb{R}^1$ as a virtual $\mathcal{K}_4$ formation in $\mathbb{R}^3$ as shown in Fig. 3(b). To obtain the formation in Fig. 3(b) from the formation in Fig. 3(a), we first need to augment *one* virtual coordinate variable for one agent, e.g., agent 1, so that the agent virtually lives in the *x*-*y* plane. Next, we need *two* virtual coordinate variables for another agent, e.g., agent 2, so that the agent virtually lives in the *x*-*y*-*z* space. The $\mathcal{K}_4$ formation in $\mathbb{R}^3$ is defined by agents at 1′, 2′, 3 and 4. In general, if we want to describe a $\mathcal{K}_n$



(a) $\mathcal{K}_4$ formation in $\mathbb{R}^1$. (b) $\mathcal{K}_4$ formation in $\mathbb{R}^1$ and virtually in $\mathbb{R}^3$.

Figure 3: In (b), the actual agents live in $\mathbb{RF}_1^3$ like (a), but they are treated as if they live in $\mathbb{R}^3$ with the virtual coordinates augmented to the position vectors of agent 1 and agent 2.

formation in $\mathbb{R}^d$, $1 \leq d \leq n-2$, the total number of required virtual variables is the sum of an arithmetic sequence as given in (6). The projection of the virtual formation into the $\mathbb{R}^d$ space together with a control law derived from that for the virtual formation yields convergence to the $\mathcal{K}_n$ formation of interest.

In this paper, we extend and use the control strategy proposed in [16] to enable a $\mathcal{K}_n$ formation in $\mathbb{R}^d$, $1 \leq d \leq n-2$, to achieve a target formation shape with almost global convergence. For this purpose, we first review the results in [14] showing that any degenerate incorrect equilibrium formation of (5) is unstable provided that the target formation is non-degenerate (the definition of (non-)degenerate formation is given in the next section).

III. Almost global convergence of a $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$

A. Degenerate formation

Let $\mathcal{S}$ be a set of k vectors in $\mathbb{R}^d$ such that $\mathcal{S} = \{s_1, \ldots, s_k\}$. The affine hull of $\mathcal{S}$ is defined by

$$\operatorname{aff\,hull} \mathcal{S} = \left\{ w \in \mathbb{R}^d \;\middle|\; w = \sum_{i=1}^{k} a_i s_i,\ s_i \in \mathcal{S},\ a_i \in \mathbb{R},\ \sum_{i=1}^{k} a_i = 1 \right\}.$$

Consider a formation in $\mathbb{R}^d$ represented by $p = (p_1, \ldots, p_n) \in \mathbb{R}^{dn}$. The *dimension of the formation* is defined as the dimension of $\operatorname{aff\,hull}\{p_1, \ldots, p_n\}$. The formation is said to be *degenerate* if the dimension of the formation is less than $\min\{d, n-1\}$, i.e., the formation is degenerate if the number of linearly independent vectors in $\{p_i - p_j \mid i \in \mathcal{V}, i \neq j\}$ is less than $\min\{d, n-1\}$ for some $j \in \mathcal{V}$. For instance, if three agents in $\mathbb{R}^2$ form a line formation, or four agents in $\mathbb{R}^3$ form a planar formation, then those formations are degenerate. On the other hand, if four agents in $\mathbb{R}^2$ form a planar formation, the formation is non-degenerate.

Note that infinitesimal rigidity of a formation implies its non-degeneracy by the following proposition.

Lemma 1 ([2, p. 174])**.** *For a given framework $(\mathcal{G}, p)$ in $\mathbb{R}^d$ with $p = (p_1, \ldots, p_n) \in \mathbb{R}^{dn}$, if $(\mathcal{G}, p)$ is infinitesimally rigid in $\mathbb{R}^d$, then the dimension of* $\operatorname{aff\,hull}\{p_1, \ldots, p_n\}$ *is equal to* $\min\{d, n-1\}$.

B. $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$

Let us consider a $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$ represented by $P = (P_1, \ldots, P_n) \in \mathbb{R}^{n(n-1)}$. Unlike $\mathcal{K}_n$ formations existing in $\mathbb{R}^d$ with $1 \leq d \leq n-2$, a $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$ has special properties so we use P instead of p to distinguish it. Analogously to the potential function in (3), let us define a potential function $V\colon \mathbb{R}^{n(n-1)} \to \mathbb{R}_{\geq 0}$ as follows.

$$V(P) = \frac{1}{4} \sum_{1 \leq i < j \leq n} \left(\|P_i - P_j\|^2 - D_{ij}^2 \right)^2,$$

where $D_{ij} = \|\bar{P}_i - \bar{P}_j\|$, and $\bar{P} = (\bar{P}_1, \ldots, \bar{P}_n) \in \mathbb{R}^{n(n-1)}$ is a representative of the target formation with desired inter-agent distances. Then we can consider the gradient system given by

$$\dot{P} = -\left[\frac{\partial V}{\partial P}\right]^\top = -R^\top e, \tag{7}$$

where R is the rigidity matrix of the framework $(\mathcal{K}_n, P)$, $e = (e_{12}, \ldots, e_{1n}, e_{23}, \ldots, e_{2n}, \ldots, e_{(n-1)n})$, and $e_{ij} = \|P_i - P_j\|^2 - D_{ij}^2$ for all $\{i,j\} \in \mathcal{E}$. Note that we have $e_{ij} = e_{ji}$ for all $\{i,j\} \in \mathcal{E}$ by definition. Obviously, the target formation corresponds to a correct equilibrium point of (7), i.e., an equilibrium point such that $V = 0$. In general, there may exist an incorrect equilibrium point at which we cannot achieve the target formation. Such an incorrect equilibrium point is defined by an equilibrium point of (7) with $V \neq 0$. Based on this understanding, we can state the following proposition on the relation between an incorrect equilibrium and degeneracy of the corresponding formation.

Lemma 2. *For an arbitrary $n \geq 2$, consider the $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$ governed by* (7)*. Suppose that the target formation is infinitesimally rigid in $\mathbb{R}^{n-1}$. Then, for any incorrect equilibrium point of* (7)*, the corresponding formation is degenerate.*

Proof. Consider an incorrect equilibrium point P^* of (7) at which $V \neq 0$ (equivalently $e \neq 0$ because $V = \frac{1}{4}e^\top e$). Since we are considering a $\mathcal{K}_n$ formation, we have

$$\sum_{j \in \mathcal{V} \setminus \{i\}} (P_i^* - P_j^*) e_{ij} = 0,\ \forall i \in \mathcal{V}.$$

From $e \neq 0$, we have $e_{ij} \neq 0$ for some $i, j \in \mathcal{V}$. For such i, we know that $(n-1)$ vectors in $\{P_i^* - P_j^* \mid j \in \mathcal{V} \setminus \{i\}\}$ are linearly dependent so the maximum number of the linearly independent vectors is at most $n-2$. Therefore, the dimension of $\operatorname{aff\,hull}\{P_1^*, \ldots, P_n^*\}$ is at most $n-2$, which means that the formation corresponding to P^* is degenerate. However, this conclusion contradicts the assumption that the target formation is infinitesimally rigid because infinitesimal rigidity of the formation implies its non-degeneracy from Lemma 1. □

From Lemma 2, we can state the following proposition on the Hessian matrix of V.

Lemma 3. *Consider the $\mathcal{K}_n$ formation as in Lemma 2 with the same assumption on the target formation. Then the Hessian matrix of V has at least one negative eigenvalue at any incorrect equilibrium point of* (7)*. Thus, each incorrect equilibrium point is unstable and is not a local minimizer of V.*

Proof. Let $H(P)$ be the Hessian matrix of V at P and $J(P)$ the Jacobian matrix of the right side of (7) at P. Consider an incorrect equilibrium point P^* of (7) with the same assumption on the target formation mentioned in Lemma 2. From Lemma 2, we know that the formation corresponding to P^* is degenerate. Then, from Lemma 6 in [14], $H(P^*)$ has at least one negative eigenvalue, which means that P^* is not a local minimizer of V. Moreover, since we have $H(P^*) = -J(P^*)$, P^* is an unstable equilibrium point of (7). □

IV. $\mathcal{K}_n$ FORMATION WITH VIRTUAL VARIABLES

A. Introduction of virtual variables

Consider a $\mathcal{K}_n$ formation in $\mathbb{R}^d$ represented by $p = (p_1, \ldots, p_n) \in \mathbb{R}^{dn}$ with $1 \leq d \leq n-2$. For a given target formation shape represented by $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n) \in \mathbb{R}^{dn}$, the goal of distance-based formation control can be interpreted as achieving the following control goal:

$$\lim_{t\to\infty} \|p_i(t) - p_j(t)\| \to d_{ij},\ \forall \{i,j\} \in \mathcal{E},$$

and $p(t)$ converges to a fixed point, where $d_{ij} = \|\bar{p}_i - \bar{p}_j\|$. It is not guaranteed that one can achieve the target formation shape under (5) because there might exist a non-degenerate incorrect equilibrium formation which is not guaranteed to be unstable as shown in Fig. 2. Instead of the control law (4), we are going to propose a modified control law using virtual variables to imitate the $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$.

Let $\eta = n-1-d$ and $\omega = \frac{(n-d)(n-1-d)}{2}$, where ω is determined from (6). We assign a total of ω scalar virtual variables denoted w^i_j, $1 \leq i \leq j \leq \eta$, to the first η agents so that

$$\begin{aligned} q_1 &= (p_1, w^1_1) \in \mathbb{R}^{d+1}, \\ q_2 &= (p_2, w^1_2, w^2_2) \in \mathbb{R}^{d+2}, \\ &\vdots \\ q_\eta &= (p_\eta, w^1_\eta, \ldots, w^\eta_\eta) \in \mathbb{R}^{d+\eta}, \\ q_{\eta+1} &= p_{\eta+1} \in \mathbb{R}^d, \\ &\vdots \\ q_n &= p_n \in \mathbb{R}^d. \end{aligned}$$

By doing so, we pretend that agent i lives in $\mathbb{R}^{d+i}$ for all $i \in \{1, \ldots, \eta\}$. For such augmented vectors, we now define the target formation in terms of the augmented vectors by letting $\bar{q}_i = (\bar{p}_i, \alpha 1_i)$ for all $i \in \{1, \ldots, \eta\}$ with arbitrary but fixed $\alpha > 0$, and $\bar{q}_j = \bar{p}_j$ for all $j \in \{\eta+1, \ldots, n\}$. Thus, the $\bar{q}_i$ vectors represent a virtual target formation shape in terms of the augmented vectors. Based on the augmented vectors, the desired inter-agent distances in a virtual ambient space are determined by

$$D_{ij} = \mathrm{dist}(\bar{q}_i, \bar{q}_j),\ \forall \{i,j\} \in \mathcal{E}.$$

Let $q = (q_1, \ldots, q_n)$. Then we can define a potential function $\bar{V}\colon \mathbb{R}^{dn+\omega} \to \mathbb{R}_{\geq 0}$ by

$$\bar{V}(q) = \frac{1}{4} \sum_{1\leq i<j\leq n} \left(\left[\mathrm{dist}(q_i, q_j)\right]^2 - (D_{ij})^2 \right)^2,$$

and propose

$$\dot{q} = -\left[\frac{\partial \bar{V}}{\partial q}\right]^\top, \tag{8}$$

under the assumption that the virtual variables are governed by

$$\dot{w}^j_i = s^j_i,\ \forall i \in \{1, \ldots, \eta\}, \forall j \in \{1, \ldots, i\},$$

where s^j_i are virtual control inputs. Note that since w^j_i are not physical states, and are updated in software, we can simply assume that the values can be transmitted by wireless communication, rather than being obtained with physical sensors.

B. Interpretation of $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$ space

We can view the $\mathcal{K}_n$ formation represented by q in Section IV-A as a $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$ with some constraints. Let

$$Q_i = \begin{cases} (q_i, 0_{n-1-d-i}), & \forall i \in \{1, \ldots, \eta-1\}, \\ q_i, & i = \eta, \\ (q_i, 0_{n-1-d}), & \forall i \in \{\eta+1, \ldots, n\}, \end{cases}$$
$$Q = (Q_1, \ldots, Q_n) \in \mathbb{R}^{n(n-1)}.$$

From such constructions, we know that $Q_i \in \mathbb{RF}^{n-1}_{d+i}$ for all $i \in \{1, \ldots, \eta\}$, and $Q_i \in \mathbb{RF}^{n-1}_d$ for all $i \in \{\eta+1, \ldots, n\}$. Thus, $(d+1)$ agents corresponding to the indices $(\eta+1)$ through n are locked on $\mathbb{RF}^{n-1}_d$, and they cannot escape from $\mathbb{RF}^{n-1}_d$. Similarly, agent $(\eta+1)$ through agent n and agent 1 are locked on $\mathbb{RF}^{n-1}_{d+1}$. Generally, agent $(\eta+1)$ through agent n together with agent 1 through agent i are locked on $\mathbb{RF}^{n-1}_{d+i}$ for each $i \in \{1, \ldots, \eta\}$. In terms of the target formation, we can also define $\bar{Q}$ as

$$\bar{Q}_i = \begin{cases} (\bar{q}_i, 0_{n-1-d-i}), & \forall i \in \{1, \ldots, \eta-1\}, \\ \bar{q}_i, & i = \eta, \\ (\bar{q}_i, 0_{n-1-d}), & \forall i \in \{\eta+1, \ldots, n\}, \end{cases}$$
$$\bar{Q} = (\bar{Q}_1, \ldots, \bar{Q}_n) \in \mathbb{R}^{n(n-1)}.$$

For example, consider the $\mathcal{K}_4$ formation in $\mathbb{R}^1$ illustrated in Fig. 3. In terms of Q_i vectors, agents 3 and 4 are locked on the x-axis. Thus, Q_3 and Q_4 can evolve only in $\mathbb{RF}^3_1$. Since agent 1 has one virtual variable, Q_1 can evolve in $\mathbb{RF}^3_2$, but cannot be taken out of $\mathbb{RF}^3_2$. Thus, agents 3, 4, and 1 are considered as being locked on $\mathbb{RF}^3_2$ in that aff hull$\{Q_1, Q_2, Q_3\} = \mathbb{RF}^3_2$ in general. On the other hand, agent 2 has two virtual variables so agent 2 can evolve in $\mathbb{R}^3$. By doing so, we can view the $\mathcal{K}_4$ formation in $\mathbb{R}^1$ as a virtual $\mathcal{K}_4$ formation in $\mathbb{R}^3$ with constraints.

Lemma 4. *Consider two realizations $P \in \mathbb{R}^{n(n-1)}$ and $q \in \mathbb{R}^{dn+\omega}$ which are congruent*[4]*. Assume that P and q are critical points of V and $\bar{V}$, respectively, and V and $\bar{V}$ are generated by the same target distances. Then P is not a local minimizer of V if and only if q is not a local minimizer of $\bar{V}$.*

Proof. Suppose that P is not a local minimizer of V. Then, for any $\delta > 0$, there exists $P' \in \{X \in \mathbb{R}^{n(n-1)} \mid \|P - X\| < \delta\}$ such that $V(P') < V(P)$. Consider arbitrarily small $\bar{\delta} > 0$. Then there always exists $q' \in \{x \in \mathbb{R}^{dn+\omega} \mid \|q - x\| < \bar{\delta}\}$ such that $\bar{V}(q') < \bar{V}(q)$ because we can choose q' so that q' and P' are congruent and that $V(P') < V(P)$ with arbitrarily small $\delta > 0$. Consequently, q is not a local minimizer of $\bar{V}$ if P is not a local minimizer of V.

[4] We use an extended notion of congruence of two realizations with different dimensions. Two realizations P and q are said to be congruent if $\mathrm{dist}(P_i, P_j) = \mathrm{dist}(q_i, q_j)$ for all $i, j \in \mathcal{V}$.

Conversely, suppose that q is not a local minimizer of $\bar{V}$. Then, for any $\bar{\delta} > 0$, there exists $q' \in \{x \in \mathbb{R}^{dn+\omega} \mid \|q - x\| < \bar{\delta}\}$ such that $\bar{V}(q') < \bar{V}(q)$. Now, for arbitrarily small $\delta > 0$, we can always find $P' \in \{X \in \mathbb{R}^{n(n-1)} \mid \|P - X\| < \delta\}$ such that $V(P') < V(P)$ from the fact that we can take P' so that P' and q' are congruent and that $\bar{V}(q') < \bar{V}(q)$. Thus, we can conclude that P is not a local minimizer of V if q is not a local minimizer of $\bar{V}$. □

C. Convergence analysis

In this section, we are going to first show that the solution of (8) converges to a point. Then we prove that for almost all initial conditions, the agents achieve the target formation shape under (8). For convergence analysis, let us invoke the following proposition on the gradient flow of a real analytic function.

Lemma 5 (Theorem 2.2 in [22]). *Let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be a real analytic function and let $x(t)$ be a continuously differentiable curve in $\mathbb{R}^n$. Assume that there exist a $\delta > 0$ and a real τ such that for $t > \tau$, $x(t)$ satisfies the angle condition*

$$\dot{\phi} = \frac{\partial \phi}{\partial x}\dot{x} \le -\delta\left\|\frac{\partial \phi}{\partial x}\right\|\|\dot{x}\|,$$

and a weak decrease condition

$$\dot{\phi} = 0 \text{ implies } \dot{x} = 0.$$

Then, either $\lim_{t\to+\infty}\|x(t)\| = \infty$ or there exists $x^* \in \mathbb{R}^n$ such that $\lim_{t\to+\infty} x(t) = x^*$.[5]

Theorem 2. *The solution of* (8) *converges to a limit point.*

Proof. For the real analytic function $\bar{V}$, we have

$$\dot{\bar{V}} = \frac{\partial \bar{V}}{\partial q}\dot{q} = -\left\|\frac{\partial \bar{V}}{\partial q}\right\|^2 \le 0. \tag{9}$$

In light of Lemma 5, we can choose $\delta = 1$, $\tau = 0$; then and $\dot{\bar{V}} = 0$ implies that $\frac{\partial \bar{V}}{\partial q} = 0$ which is equivalent to $\dot{q} = 0$. Thus, we can conclude that either $\lim_{t\to+\infty}\|q(t)\| = \infty$ or there exists q^* such that $\lim_{t\to+\infty} q(t) = q^*$. Then, we can rule out the case of diverging $\|q(t)\|$ by showing that the solution is bounded. Note that boundedness of q is implied by boundedness of Q mentioned in Section IV-B. Let $e_{ij} = [\mathrm{dist}(q_i, q_j)]^2 - (D_{ij})^2 = \|Q_i - Q_j\|^2 - (D_{ij})^2$. We can show that $\frac{\partial e_{ij}}{\partial p_i} = -\frac{\partial e_{ij}}{\partial p_j}$ for each edge, which results in that $\sum_{i=1}^n \dot{p}_i = -\sum_{i=1}^n \left[\frac{\partial \bar{V}}{\partial p_i}\right]^\top = 0$. Therefore, the centroid of $Q_1, \dots, Q_n$ projected onto $\mathbb{RF}_d^{n-1}$ is stationary under (8). Moreover, we know that $\|Q_i - Q_j\|$ cannot diverge for any $i, j \in \mathcal{V}$ from (9), and $Q_\eta, \dots, Q_n$ are locked on $\mathbb{RF}_d^{n-1}$. Consequently, Q is bounded, which implies that q is bounded so q converges to a limit point. □

In spite of Theorem 2, we cannot be assured that $q(t)$ will converge to the equilibrium set of (8) because convergence of a function does not imply convergence of its derivative in general. Convergence of $q(t)$ to the equilibrium set can be shown from the fact that $\dot{\bar{V}}$ is uniformly continuous[6]. Since $q(t)$ converges to a point, $\bar{V}(q(t))$ also converges. Then, from Barbalat's lemma [23, Lemma 8.2], $\dot{\bar{V}}(q(t))$ converges to 0, which guarantees that $q(t)$ converges to the equilibrium set. Consequently, we can state that $q(t)$ converges to either the desired equilibrium set or the incorrect equilibrium set of (8). Therefore, if we show that any incorrect equilibrium point of (8) is unstable, then we can conclude that $q(t)$ converges to the desired equilibrium set, in which the target inter-agent distances are achieved for almost all initial conditions. This conclusion is formalized in the following theorem.

Theorem 3. *Suppose that the target formation represented by $\bar{Q}$ is non-degenerate and infinitesimally rigid in $\mathbb{R}^{n-1}$. Then, any incorrect equilibrium point of* (8) *is unstable.*

Proof. Let q^* be an incorrect equilibrium point of (8). Suppose, in order to show a contradiction, that q^* is stable. Since $\bar{V}$ is real analytic, q^* must be a local minimizer of $\bar{V}$ [24]. Consider a realization P^* representing a $\mathcal{K}_n$ formation in $\mathbb{R}^{n-1}$ such that q^* and P^* are congruent. Then, from Lemma 4, P^* is a local minimizer of V. However, since P^* is an incorrect equilibrium point, it must be unstable, and it cannot be a local minimizer of V from Lemma 3. Thus, we reach a contradiction, which means that any incorrect equilibrium point of (8) is unstable. □

From Theorem 3, we can finally conclude that $q(t)$ converges to the desired equilibrium set, and we can achieve the target inter-agent distances for almost all initial conditions. We summarize the final conclusion in the following theorem.

Theorem 4. *Under the proposed control law* (8)*, the agents achieve the target formation shape in $\mathbb{R}^d$ for almost all initial conditions if the target formation represented by $\bar{Q}$ is non-degenerate and infinitesimally rigid in $\mathbb{R}^{n-1}$.*

D. Ingredients of the control input

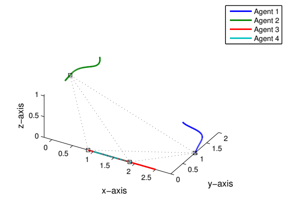
We emphasize that the modified control input for the actual (as opposed to virtual) formation can be calculated based on the relative position measurements and exchange of the information of the virtual variables. To explain this, let us consider the $\mathcal{K}_4$ formation in $\mathbb{R}^1$ shown in Fig. 3. The system equations, which are spread out from (8), in terms of q are given by

$$\begin{aligned}
\dot{p}_1 &= (p_2 - p_1)e_{12} + (p_3 - p_1)e_{13} + (p_4 - p_1)e_{14},\\
\dot{w}_1^1 &= (w_2^1 - w_1^1)e_{12} + (0 - w_1^1)e_{13} + (0 - w_1^1)e_{14},\\
\dot{p}_2 &= (p_1 - p_2)e_{12} + (p_3 - p_2)e_{23} + (p_4 - p_2)e_{24},\\
\dot{w}_2^1 &= (w_1^1 - w_2^1)e_{12} + (0 - w_2^1)e_{23} + (0 - w_2^1)e_{24},\\
\dot{w}_2^2 &= (0 - w_2^2)e_{12} + (0 - w_2^2)e_{23} + (0 - w_2^2)e_{24},\\
\dot{p}_3 &= (p_1 - p_3)e_{13} + (p_2 - p_3)e_{23} + (p_4 - p_3)e_{34},\\
\dot{p}_4 &= (p_1 - p_4)e_{14} + (p_2 - p_4)e_{24} + (p_3 - p_4)e_{34}.
\end{aligned}$$

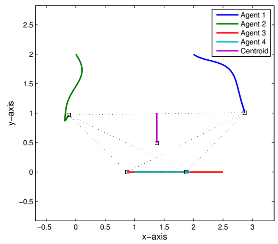
Note that $p_i - p_j$ are relative position measurements and w_i^j are supposed to be transmitted among the neighboring agents. Moreover, the e_{ij} terms consist of relative position measurements and/or the virtual variables. For example, we have $e_{34} = (p_3 - p_4)^2 - (\bar{p}_3 - \bar{p}_4)^2$, which means that the

[5] We recovered $x(t)$ omitted by mistake in Theorem 2.2 of [22].

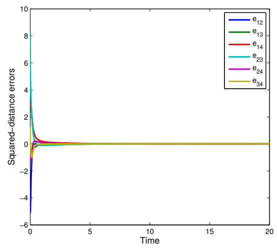
[6] Uniform continuity of $\dot{\bar{V}}$ can be concluded from boundedness of the second derivative of $\bar{V}$.



(a) Trajectories of the agents in $\mathbb{R}^3$.

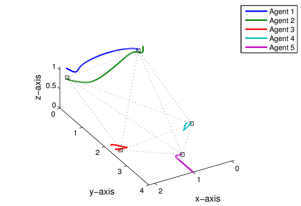


(b) Trajectories of the agents projected onto $\mathbb{RF}_2^3$.

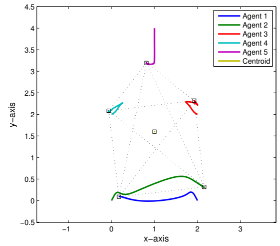


(c) Squared-distance errors.

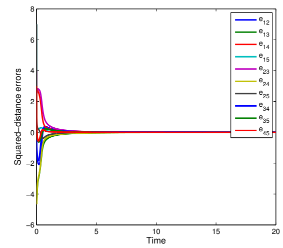
Figure 4: Simulation for a $\mathcal{K}_4$ formation in $\mathbb{R}$.



(a) Trajectories of the agents projected onto $\mathbb{RF}_3^4$.



(b) Trajectories of the agents projected onto $\mathbb{RF}_2^4$.



(c) Squared-distance errors.

Figure 5: Simulation for a $\mathcal{K}_5$ formation in $\mathbb{R}^2$.

calculation of e_{34} requires the relative position measurement $(p_3 - p_4)$ only. On the other hand, $e_{12} = (p_1 - p_2)^2 + (w_1^1 - w_2^1)^2 + (w_2^2)^2 - [(\bar{p}_1 - \bar{p}_2)^2 + \alpha^2]$ so the calculation of e_{12} needs a relative position measurement and exchange of virtual variables.

Remark. *Since the underlying formation graph used in our problem is a complete graph, one may consider the proposed control strategy as a centralized formation control strategy. However, the proposed control strategy does not require a centralized coordinator which is supposed to collect the information of the whole system, calculate and distribute the control inputs for local agents. In our problem formulation, each agent is supposed to measure the relative positions to its neighbor(s) using local sensors based on its independent local reference frame, and communicate with its neighbors to exchange the information of the virtual coordinate variables. Thus, the control law proposed in this paper is a distributed formation control law.*

V. Examples with simulation

We introduce some examples to support our results. A particular example of general $\mathcal{K}_n$ formations is a $\mathcal{K}_4$ formation in $\mathbb{R}^2$. The analysis on the $\mathcal{K}_4$ formation under the control law proposed in this paper can be found in [16]. In this section, we introduce two more examples representing situations more general than the $\mathcal{K}_4$ formation in $\mathbb{R}^2$ in terms of the number of virtual variables.

A. $\mathcal{K}_4$ formation in $\mathbb{R}$

Consider a $\mathcal{K}_4$ formation in $\mathbb{R}$. Since the number of agents of interest is 4, the virtual realization space should be $\mathbb{R}^3$, and the number of required virtual variables is 3 from (6). Let $\bar{p} = (3, 0, 1, 2)$, which represents the formation in Fig. 3(a). Then with $\alpha = 1$, we have

$$\bar{Q} = \left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right).$$

By applying the proposed control law, we can obtain the results shown in Fig. 4. The virtual $\mathcal{K}_4$ formation in $\mathbb{R}^3$ achieves the target shape represented by $\bar{Q}$. As a result, we can obtain the target formation shape in $\mathbb{R}$ as well. Note that the centroid of the formation is stationary on the x–axis in Fig. 4(b), which coincides with our analysis in the proof of Theorem 2.

B. $\mathcal{K}_5$ formation in $\mathbb{R}^2$

Consider a $\mathcal{K}_5$ formation in $\mathbb{R}^2$. In this case, the number of required virtual variables is also 3 according to (6). Let

$$\bar{p} = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right).$$

Then with $\alpha = 1$, we have

$$\bar{Q} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right).$$

By applying the proposed control law, we can obtain the results shown in Fig. 5. Remark that we cannot illustrate the trajectories of the agents in virtual $\mathbb{R}^4$ space. Thus, Fig. 5(a) represents only the trajectories projected onto $\mathbb{RF}_3^4$. Fig. 5(b) shows that we finally achieve the target formation shape in the original space, and the centroid of the agents in the original space is stationary.

VI. Conclusions

In this paper, we showed that achieving (almost) global convergence to a target formation shape under the distance-based formation control law in [8] is impossible for a particular example of a globally rigid formation. Formations represented by complete graphs form a particular set of globally rigid formations. For those formations, we proposed a new control law motivated from the existing one in [8], and showed that we can achieve almost global convergence under the proposed control law.

However, whether the proposed control law can be applied to general globally rigid formations, thereby achieving global convergence has yet to be proved. Thus, achieving global convergence for an arbitrary globally rigid formation will be our ultimate goal. Of course, one obvious approach to achieving this would be to determine a formation with a complete graph in which the target formation was embedded. There are potential difficulties with doing this however. First, if it is necessary to compute those inter-agent distances not given as part of the target formation with a globally rigid graph, especially if it is the task of the agents themselves to compute the missing inter-agent distances, the computation may be constrained to be distributed and the computational burden is simply unclear. However, it may well be that the missing distances are readily available. There remains however a second potential difficulty. Operation of the algorithm requires agents to sense relative positions. It may be that the extra relative position sensing required for the complete graph approach overloads agents (the sensed variables scale with the size of the formation) or demands sensing outside the range of the sensors through some agents.

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